

Concerns with Functional Depth

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Abstract. We study some problems inherent with certain forms of functional depth, in particular, zero depth and lack of consistency.

1. Introduction

The use of a variety of depth functions to provide a center-outward ordering of finite dimensional data is well established, and more recently a number of interesting papers have considered analogues of such depths to study multivariate data in the infinite dimensional setting. These depths apply to data given in terms of infinite sequences, as functions defined on some interval, and also as points in some infinite dimensional Banach space. The papers [López-Pintado and Romo \(2009, 2011\)](#); [Dutta et al. \(2011\)](#); [Chakraborty and Chaudhuri \(2013\)](#); [Mosler and Polyakova \(2012\)](#); [Cuevas and Fraiman \(2009\)](#) introduce interesting examples of such depths, and also include many additional references, but the focus here centers on the problem that the natural analogue of some commonly used depths in \mathbb{R}^d may well be zero "most of the time" in the infinite dimensional setting. By "most of the time" it is meant that the depth is zero on a set whose probability is one with respect to the probability the depth is based on. This was pointed out in Theorem 3 of [Dutta et al. \(2011\)](#) for Tukey's half-space depth with respect to certain probability laws on the Hilbert space ℓ_2 , and also for the band depth and half-region depth of [López-Pintado and Romo \(2009, 2011\)](#) in [Chakraborty and Chaudhuri \(2013\)](#). The paper [Kuelbs and Zinn \(2012\)](#) also examined this problem for half-region depth showing it not only vanishes most of the time, but in many examples it vanishes everywhere. Moreover, in [Kuelbs and Zinn \(2012\)](#) it is also shown how one can smooth such data so as to regain positive half-region depth, and then establish consistency for the empirical half-region depth of the smoothed data. In some cases one can also show \sqrt{n} consistency. In a related, but slightly different context, smoothing was

Received by the editors February 20, 2013; accepted September 9, 2013.

2010 Mathematics Subject Classification. Primary 60F05; Secondary 60F17, 62E20.

Key words and phrases. half-space depth, simplicial depth, functional depth, consistency.

used to guarantee various limit theorems (see [Kuelbs et al. \(2013\)](#); [Kuelbs and Zinn \(2013\)](#)).

In the present work our goal is to better understand both the problem of zero depth as well as questions of consistency for infinite dimensional data. Our results are in the context of Tukey half-space depth, and also for the simplicial depth in the infinite dimensional setting obtained as in [Mosler and Polyakova \(2012\)](#). Of course, there are other possible choices, but the importance of our choices in the finite dimensional setting made them attractive to study for functional data. Moreover, it is clear that for data symmetric about zero, the half-space depth of the zero vector will typically be $\frac{1}{2}$, but the result in [Dutta et al. \(2011\)](#) shows this depth may also be zero with probability one. Hence an immediate question to ask is for what other points might this depth be strictly positive? Also, perhaps the set of points at which the depth is non-zero is “special” and one can prove consistency for these special points.

In this paper, in a number of cases, we’ll describe the precise set of points at which the half-space depth is strictly positive and when it is zero. This is accomplished via Theorems 2.2 and 2.6 of section 2, and we also provide some explicit formulas for half-space depth in special cases in section 3. Theorem 2.7 of section 2 shows that in many situations the empirical half-space depth is zero with probability one. Combined with Theorems 2.2 and 2.6, this last result is particularly bad news, as often the empirical depth is zero with probability one at all the points where the true depth is strictly positive. Of course, if the true half-space depth is zero, in this case we would have consistency, but very little information. Also, the remark following the statement of Theorem 2.7 below points out several aspects of such problems, and how the results of Theorem 2.7 differ from those on Tukey functional depth on page eleven of [Mosler and Polyakova \(2012\)](#).

Proposition 3.6 of section 3 shows how these results combine for Gaussian measures with infinite dimensional support on a separable Banach space, and, although we can identify the precise set of positivity of the half space depth based on B^* , there still is a lack of consistency. In the last part of section 3 we see the simplicial depth of Liu (see [Liu \(1988, 1990\)](#)) extended to \mathbb{R}^∞ using definition one in section 7 of [Mosler and Polyakova \(2012\)](#) is also subject to the very same problems. This is an interesting fact in its own right, but also because this depth is quite different (see [Zuo and Serfling \(2000\)](#)) than Tukey’s half-space depth. It is also interesting to note that the context in which zero empirical depth appears for these particular depths seems to occur when there are a lot of independence-like properties in the data. Hence, it may be possible to modify or smooth either the random variable or the depth to ensure consistency or even a central limit type theorem, but that is off in the future.

There are several other interesting aspects of this study that should be mentioned. These include the fact that the results of Theorem 2.6 connect with admissible translates of probability measures on \mathbb{R}^∞ (see [Kakutani \(1948\)](#), [Shepp \(1965\)](#)), and in subsection 3.2 we need some delicate tail estimates of Rademacher series to estimate the magnitude of the actual half-space depth at a point (see [Montgomery-Smith \(1990\)](#)).

Finally, it seems to be clear that each example of functional depth brings with it its own difficulties. Some may be more immune to various difficulties than others. This can be seen in the following example of the dual integrated depth of [Cuevas and](#)

Fraiman (2009), where positivity of the depth holds, and we also have an immediate link to positivity of the modified band depth of López-Pintado and Romo (2009). We start with the one-dimensional version of the band depth of López-Pintado and Romo (2009). That is, if $r \geq 2$ and for $\{\xi_j\}_{j=1}^r$ iid with distribution μ ,

$$BD_{r,1}(b, \mu) = P(\min_{1 \leq j \leq r} \xi_j \leq b \leq \max_{1 \leq j \leq r} \xi_j).$$

Of course, $BD_{2,1}$ is the univariate version of the simplicial depth, and $BD_{2,1}(b, \mu) \leq BD_{r,1}(b, \mu)$. Hence, it is positive provided b is in the interior of the closed convex hull of the support of μ , or at a boundary point, if the boundary point has positive μ -probability. Assume that Q is the measure on the set of point evaluations, $e_t(a) = a(t)$, say for a in $C[0, 1]$, given by Lebesgue measure, m , on $[0, 1]$. (For A a Borel subset in the weak-star topology of $C^*[0, 1]$, the dual space of $C[0, 1]$, define $Q(A) = m(t \in [0, 1] : e_t \in A)$). Now, consider a stochastic process, $\{X(t) : t \in [0, 1]\}$ with distribution, P , on $C[0, 1]$ and i.i.d. copies, $\{X_j(t) : t \in [0, 1]\}_{j=1}^\infty$, and a function, $a \in C[0, 1]$. Then the definition of the dual integrated depth associated with the depth $BD_{r,1}$ gives

$$\begin{aligned} IDD(a, P) &= \int_0^1 P(\min_{1 \leq j \leq r} X_j(t) \leq a(t) \leq \max_{1 \leq j \leq r} X_j(t)) dt & (1.1) \\ &= (\text{by Fubini}) \mathbb{E} [m(t \in [0, 1] : \min_{1 \leq j \leq r} X_j(t) \leq a(t) \leq \max_{1 \leq j \leq r} X_j(t))] \\ &= MBD_{r,1}(a, P), \end{aligned}$$

where $MBD_{r,1}(a, P)$ is the modified band depth of López-Pintado and Romo (2009). In particular, the positivity of the integrand in (1.1) on some open subinterval of $[0, 1]$ implies that for each such a , the quantities in (1.1) are positive. For example, if $s_{1,t} = \inf\{x : P(X(t) \leq x) > 0\}$ and $s_{2,t} = \sup\{x : P(X(t) \leq x) < 1\}$ for $t \in [0, 1]$, and $E = \{t \in [0, 1] : s_{1,t} = -\infty, s_{2,t} = \infty\}$ has positive Lebesgue measure, then for all $a \in C[0, 1]$ and $t \in E$ the quantity

$$P(\min_{1 \leq j \leq r} X_j(t) \leq a(t) \leq \max_{1 \leq j \leq r} X_j(t))$$

is strictly positive. Also, for consistency results related to dual integrated depths one should examine Theorem 2 in Cuevas and Fraiman (2009), and for (unmodified) band depth López-Pintado and Romo (2009).

2. Infinite dimensional half space depth.

Here we formulate some results on half space depth in infinite dimensional, real, topological vector spaces B , whose topology is metrizable, complete and separable via a translation invariant metric. They are the so-called F-spaces in Rudin (1966), and include the real separable Banach spaces, as well as Fréchet spaces such as \mathbb{R}^∞ , and many other topological vector spaces.

Throughout X, X_1, X_2, \dots are i.i.d. B -valued random vectors on the probability space (Ω, \mathcal{F}, P) which are measurable from \mathcal{F} to the Borel sets \mathcal{B}_B of B , and μ denotes the law of X on (B, \mathcal{B}_B) . We also assume \mathcal{T} is a collection of Borel measurable functionals on B . Then, we define the half space depth of $a \in B$ with respect to \mathcal{T} and μ to be

$$HD_{\mathcal{T}}(a, \mu) = \inf_{t \in \mathcal{T}} P(\omega \in \Omega : t(X(\omega)) \geq t(a)). \tag{2.1}$$

We usually denote the right hand term by $\inf_{t \in \mathcal{T}} P(t(X) \geq t(a))$, and observe that we also have

$$HD_{\mathcal{T}}(a, \mu) = \inf_{t \in \mathcal{T}} \mu(x \in B : t(x) \geq t(a)). \quad (2.2)$$

If $\mu_n(\omega) = \frac{1}{n} \sum_{j=1}^n \delta_{X_j(\omega)}$, $n \geq 1$, then for each $\omega \in \Omega$ we have μ_n a probability measure on B and the empirical half space depth for $a \in B$ with respect to \mathcal{T} and μ_n is defined to be

$$HD_{\mathcal{T}}(a, \mu_n) = \inf_{t \in \mathcal{T}} \mu_n(x \in B : t(x) \geq t(a)), \quad (2.3)$$

and hence we also have

$$HD_{\mathcal{T}}(a, \mu_n) = \inf_{t \in \mathcal{T}} \frac{1}{n} \sum_{j=1}^n I(t(X_j) \geq t(a)). \quad (2.4)$$

If $B = \mathbb{R}^d$ and \mathcal{T} is the linear functionals on \mathbb{R}^d in (2.1), this is Tukey half space depth. Of course, since the linear functionals on \mathbb{R}^d are given by inner products, we denote this by $HD_{\mathbb{R}^d}(a, \mu)$, and observe that

$$HD_{\mathbb{R}^d}(a, \mu) = \inf_{t \in \mathbb{R}^d} P(t(X) \geq t(a)) = \inf_{t \in \mathbb{R}^d} \mu(x \in \mathbb{R}^d : t(x) \geq t(a)).$$

When B is a real separable Banach or Fréchet space, with the dual space of continuous linear functionals on B denoted by B^* , then a natural definition of Tukey half space depth with $\mathcal{T} = B^*$ is given by

$$HD_{B^*}(a, \mu) = \inf_{t \in B^*} P(t(X) \geq t(a)).$$

It should be observed that in the literature the half space depth we defined on \mathbb{R}^d or B is likely to be written as $HD_E(a, \mu)$ where E is \mathbb{R}^d or B , respectively. We chose our terminology to emphasize that the functionals in \mathcal{T} need not be continuous or linear on B .

The point to be seen here is that in the infinite dimensional setting the class B^* is frequently much too large to provide positive depth on much of the space B . Moreover, problems of consistency emerge even if the half space depth is positive at a point and \mathcal{T} is countably infinite. Hence it is useful to formulate depth as in (2.1) where the class of functionals \mathcal{T} allows more flexibility. It is also important to note that the functionals $t \in \mathcal{T}$ need not be linear or continuous on B , and there are good reasons for this. First, in the generality we are considering there are examples, such as L^p for $0 < p < 1$, where the only continuous linear functional on B will be the functional that is identically zero, and of more importance, the functionals of interest need not be linear to start with. For example, in section 3.3 we show a lack of consistency for the Mosler-Polyakova version of Liu's simplicial depth using maps that are probabilities (which are highly non-linear functions of the data).

Example 2.1. To see the effect that different choices of \mathcal{T} can have on half-space depth, let X take values in $C[0, 1]$, the space of continuous functions on $[0, 1]$, where $\mathcal{L}(X(s)) = N(0, 1 + s)$ and $E(X(s)X(t)) = 1 + \min\{s, t\}$ for $s, t \in (0, 1)$. Then, $\{X(s) : 0 \leq s \leq 1\}$ is a standard sample continuous Brownian motion started randomly at time zero with a $N(0, 1)$ distribution, and if \mathcal{T} consists of the evaluation maps

$$e_t(a) = a(t), t \in [0, 1], a \in C[0, 1],$$

we have

$$HD(a, \mu) = \inf_{t \in [0,1]} P(X(t) \geq a(t)),$$

where μ is the law of X on $C[0, 1]$. Therefore,

$$HD(a, \mu) = \inf_{t \in [0,1]} P\left(\frac{X(t)}{\sqrt{1+t}} \geq \frac{a(t)}{\sqrt{1+t}}\right) = \inf_{t \in [0,1]} [1 - \Phi\left(\frac{a(t)}{\sqrt{1+t}}\right)],$$

and since $\sup_{t \in [0,1]} \frac{|a(t)|}{\sqrt{1+t}} < \infty$ for all $a \in C[0, 1]$ we have $HD(a, \mu) > 0$ for all $a \in C[0, 1]$. On the other hand, if \mathcal{T} consists of the differences of two evaluation maps, then as we will now see the half-space depth will be zero with μ probability one. In fact, we need not consider all differences, but only that \mathcal{T} consists of the sequence of differences

$$\theta_k(a) \equiv a\left(\frac{1}{k}\right) - a\left(\frac{1}{k+1}\right), k \geq 1.$$

Then, the half-space depth of a function $a \in C[0, 1]$ with respect to μ and this choice of \mathcal{T} is

$$HD(a, \mu) = \inf_{k \geq 1} P(\theta_k(X) \geq \theta_k(a)).$$

Now $G_k = \sqrt{k(k+1)}\theta_k(X), k \geq 1$, are i.i.d. $N(0, 1)$ random variables and hence for $a \in C[0, 1]$

$$HD(a, \mu) = \inf_{k \geq 1} P(G_k \geq \sqrt{k(k+1)}\theta_k(a)) = 1 - \Phi(\sup_{k \geq 1} \sqrt{k(k+1)}\theta_k(a)).$$

If

$$A = \{a \in C[0, 1] : \sup_{k \geq 1} \sqrt{k(k+1)}\theta_k(a) = \infty\},$$

then $\mu(A) = P(\sup_{k \geq 1} G_k = \infty) = 1$, and hence with μ -probability one $HD(a, \mu) = 0$ when \mathcal{T} consists of these differences.

Now we turn to three theorems and the additional notation used in the remainder of the paper. The first theorem obtains sufficient conditions for half space depth to be zero in the infinite dimensional setting, and following its statement there are a couple of remarks indicating how Theorem 3 of Dutta et al. (2011) for half space depth in ℓ_2 follows as a special case. These remarks also examine other aspects of the theorem. A second theorem, when combined with Theorem 2.2, establishes necessary and sufficient conditions that the depth be positive, and the third examines when the empirical version of this depth in (2.3) and (2.4) approximates the true distributional depth. There are also corollaries and remarks pertaining to these results, which indicate how they fit together. The proofs of the theorems appear at the end of the section.

Theorem 2.2. *Let X be a random vector with values in a real separable F -space B , and assume the functionals $\{t_k : k \geq 1\}$ are measurable from the Borel subsets of B to the reals. In addition, assume $\{t_k(X) : k \geq 1\}$ are mean zero random variables with variances $\sigma_k^2 \in (0, \infty), k \geq 1$, and that $\mathbb{E}(t_i(X)t_j(X)) = 0, 1 \leq i < j < \infty$. If \mathcal{T} consists of all finite linear combinations of the maps $\{t_k : k \geq 1\}$, then*

$$HD_{\mathcal{T}}(a, \mu) = 0 \tag{2.5}$$

for all $a \in B$ such that

$$\sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} = \infty. \quad (2.6)$$

Furthermore, (2.5) holds with μ probability one if

$$P\left(\sum_{k \geq 1} t_k^2(X)/\sigma_k^2 = \infty\right) = 1. \quad (2.7)$$

Since \mathcal{T} contains the finite linear combinations of the $\{t_k : k \geq 1\}$ it will be convenient to parameterize these functionals by letting ℓ_0 denote the sequences in \mathbb{R}^∞ which have only finitely many non-zero terms, and for $\alpha = (\alpha_1, \alpha_2, \dots) \in \ell_0$ let

$$t_\alpha = \sum_{k \geq 1} \alpha_k t_k. \quad (2.8)$$

Then, the right hand term in (2.8) is a finite sum, t_α is a typical functional in \mathcal{T} , and the half space depth of $a \in B$ with respect to \mathcal{T} and μ satisfies

$$HD_{\mathcal{T}}(a, \mu) = \inf_{\alpha \in \ell_0} P(t_\alpha(X) \geq t_\alpha(a)). \quad (2.9)$$

Remark 2.3. Let $\{t_k(X) : k \geq 1\}$ be as in Theorem 2.2. Hence, if for some sequence $\{a_n : n \geq 1\}$ increasing to infinity, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n t_k^2(X)/\sigma_k^2 > 0 \quad (2.10)$$

with probability one, then (2.7) holds and the final conclusion of Theorem 2.2 implies (2.5) with μ -probability one. Furthermore, since $\mathbb{E}(t_k^2(X)/\sigma_k^2) = 1$ for $k \geq 1$, if $a_n = n$ the stability result in (2.10) would immediately hold from the ergodic theorem if the sequence $\{t_k^2(X)/\sigma_k^2\}$ is stationary and ergodic. It also follows without the ergodicity assumption provided we have stationarity and $P(t_1(X) = 0) = 0$. Of course, if the random variables $\{t_k(X) : k \geq 1\}$ are assumed independent, then (2.10) holds with $a_n = n$ and limit one in a variety of situations by applying a law of large numbers. For example, under the independence assumption and that

$$\sum_{k \geq 1} \frac{\mathbb{E}(t_k^4(X))}{k^2 \sigma_k^4} < \infty, \quad (2.11)$$

this is the case. However, the condition (2.10) also follows with $a_n = n$ and the limit being one provided (2.11) holds and that $\{t_k^2(X)/\sigma_k^2 : k \geq 1\}$ are uncorrelated. That is, under these conditions it is easy to check that

$$\mathbb{E}\left(\left[\frac{1}{n} \sum_{k=1}^n \frac{t_k^2(X)}{\sigma_k^2} - 1\right]^2\right) = \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}\left(\frac{t_k^4(X)}{\sigma_k^4}\right) - \frac{1}{n}. \quad (2.12)$$

Hence, (2.11) and Kronecker's Lemma combine to imply

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}\left(\frac{t_k^4(X)}{\sigma_k^4}\right) = 0. \quad (2.13)$$

Therefore, $\frac{1}{n} \sum_{k=1}^n \frac{t_k^2(X)}{\sigma_k^2}$ converges in L^2 to one, and (2.10) holds with $a_n = n$ and limit one with probability one.

Remark 2.4. If X takes values in the real separable Banach space ℓ_2 , then Theorem 3 of Dutta et al. (2011) shows that under certain conditions on the distribution of X the Tukey half space depth with $\mathcal{T} = \ell_2^*$ is zero. In that result the maps $t_k(X)$ are assumed to be mean zero independent random variables such that $t_k(X) = \langle X, e_k \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on ℓ_2 and $\{e_k : k \geq 1\}$ is the canonical basis of ℓ_2 . Furthermore, (2.11) is assumed to hold, and the half space depth is defined in terms of all continuous linear functionals on ℓ_2 . In our terminology, the finite linear combinations of the $\{t_k(X) : k \geq 1\}$ we denote by $\{t_\alpha : \alpha \in \ell_0\}$ would be replaced by $\{t_\alpha : \alpha \in \ell_2\}$, and hence that depth is less than or equal the depth we use. Since zero is the minimal possible depth, our result in Theorem 2.2 therefore implies the result in Dutta et al. (2011). Moreover, it implies similar results in any F-space B without an independence or a linearity assumption on the mappings $\{t_k : k \geq 1\}$.

If X is symmetric about the vector $a \in B$ and the maps $\{t_k : k \geq 1\}$ are linear, then for each $\alpha \in \ell_0$ we have $P(t_\alpha(X) \geq t_\alpha(a)) \geq 1/2$, and hence $HD_{\mathcal{T}}(a, \mu) \geq 1/2$. Furthermore, it will equal $1/2$ if $P(t_k(X) = t_k(a)) = 0$ for all $k \geq 1$. Thus certain vectors have positive half space depth, and our next proposition examines for which vectors in B this might be the case. However, in order to provide sufficient conditions for positive half space depth we require some additional assumptions. They are:

Assumptions. (A-I) For $a \in B$ and all integers $d \geq 1$

$$HD_{\mathbb{R}^d}(\Pi_d(a), \mu^{\Pi_d}) > 0, \tag{2.14}$$

where $\Pi_d(a) = (t_1(a), \dots, t_d(a))$ and μ^{Π_d} is the image of μ on \mathbb{R}^d via the map $\Pi_d(\cdot) : B \rightarrow \mathbb{R}^d$,

(A-II) For some constant $c < \infty$, $\mathbb{E}(t_k^4(X)) \leq c[\mathbb{E}(t_k^2(X))]^2$ for all $k \geq 1$,

and

(A-III) $\{t_k(X)/\lambda_k : k \geq 1\}$, $0 < \lambda_k < \infty$, are i.i.d. with probability density ϕ that is positive a.s., (locally) absolutely continuous on \mathbb{R} (i.e., ϕ is absolutely continuous on every compact interval of \mathbb{R}) satisfying

$$\mathcal{I}(\phi) = \int_{\mathbb{R}} \frac{(\phi')^2(x)}{\phi(x)} dx < \infty, \tag{2.15}$$

and

$$\sigma^2 = \int_{\mathbb{R}} x^2 \phi(x) dx < \infty. \tag{2.16}$$

Remark 2.5. The condition (2.14) may be difficult to check in some situations, but it is a necessary condition for $HD_{\mathcal{T}}(a, \mu)$ to be strictly positive since

$$\begin{aligned} HD_{\mathbb{R}^d}(\Pi_d(a), P^{\Pi_d}) &= \inf_{\alpha \in \ell_0} P\left(\sum_{k=1}^d \alpha_k t_k(X) \geq \sum_{k=1}^d \alpha_k t_k(a)\right) \\ &\geq \inf_{\alpha \in \ell_0} P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) = HD_{\mathcal{T}}(a, \mu). \end{aligned}$$

The quantity in (2.15) is often called the Fisher information. It appeared in Fisher (1973) and was used in Shepp (1965) in connection with admissible translates (see Definition 2.13 below). While the uses of the Fisher information are ubiquitous in Statistics, at this point we only use the connection to admissible translates. Furthermore, if μ is a probability measure on \mathbb{R}^d with probability density that is strictly positive a.s. with respect to Lebesgue measure, then every vector $b \in \mathbb{R}^d$ is an admissible translate. Therefore, by the proof of Lemma 2.15 every vector in \mathbb{R}^d has positive half space depth with respect to μ for μ symmetric, and under these conditions (2.14) holds. In addition, the conclusion of Theorem 2.6 given by the assumptions in (A-III) would then follow from the assumptions in (A-I) and (A-II) provided we also assume $\int_{\mathbb{R}} x^4 \phi(x) dx < \infty$. Of course, the conditions in (A-I) and (A-II) do not require that $\{t_k(X) : k \geq 1\}$ have densities, and they apply without the $\{t_k(X) : k \geq 1\}$ being scaled i.i.d. random variables. Hence, in that sense they are more general than what can be obtained from the assumptions in (A-III), but it is also of interest that the conditions in (A-III) yield results without a fourth moment assumption as in (A-II).

Condition (2.15) and Lemma 2.15 below allow us to link half space depth to admissible translates of product measures and the results of Kakutani (1948) and Shepp (1965).

Theorem 2.6. *Let X be a random vector with values in a real separable F -space B , and assume $\{t_k(X) : k \geq 1\}$ are independent, symmetric random variables with variances $\sigma_k^2 \in (0, \infty), k \geq 1$. In addition, assume (A-I) and (A-II) hold, or (A-III) holds. Then, for $a \in B$ and \mathcal{T} all finite linear combinations of the maps $\{t_k : k \geq 1\}$,*

$$HD_{\mathcal{T}}(a, \mu) > 0 \quad (2.17)$$

if and only if

$$\sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} < \infty. \quad (2.18)$$

Our next theorem examines empirical half space depth, and the corollary and remark following its statement clarify consistency for the empirical depth in the setting of Theorems 2.2 and 2.6 provided (2.20) holds.

Theorem 2.7. *Let $\{X_j : j \geq 1\}$ be i.i.d. copies of X where X is a random vector taking values in a real F -space B and $\{t_k(X) : k \geq 1\}$ are independent mean zero random variables with variances $\sigma_k^2 \in (0, \infty), k \geq 1$. Furthermore, assume for all $c_k, k \geq 1$, such that $\lim_{k \rightarrow \infty} c_k / \sigma_k = 0$,*

$$\liminf_{k \rightarrow \infty} P(t_k(X) < c_k) > 0, \quad (2.19)$$

and $\mathcal{T} \supseteq \{t_k(\cdot) : k \geq 1\}$. Then, for all $n \geq 1$ and $a \in B$ with

$$\lim_{k \rightarrow \infty} \frac{t_k(a)}{\sigma_k} = 0, \quad (2.20)$$

the empirical half space depth

$$HD_{\mathcal{T}}(a, \mu_n) \equiv \inf_{t \in \mathcal{T}} \frac{1}{n} \sum_{j=1}^n I(t(X_j) \geq t(a)) = 0 \quad (2.21)$$

with probability one.

Remark 2.8. The empirical half space depth, $HD_{\mathcal{T}}(a, \mu_n)$, can be thought to have two "random quantities" on different probability spaces. One, say, ω with corresponding probability, P , is through the random variables in μ_n and the other with respect to the induced measure, μ , on the space, B . In Mosler and Polyakova (2012) Mosler and Polyakova give a result that holds quite generally for Tukey-like depths as long as $\mathcal{T} = B^*$. Namely that fixing ω and computing with respect to μ the empirical depth equals zero with μ -probability one provided finite dimensional subspaces have μ probability zero. On the other hand, in special, but interesting circumstances (as in the above Theorem 2.7), we fix a in a large class and show, that the empirical depths are zero with P -probability one. So, in particular, we can say for each point in this "large class" that consistency fails or fails to give any information.

Corollary 2.9. *Assume the conditions in Theorem 2.7 with \mathcal{T} all finite linear combinations of $\{t_k(\cdot) : k \geq 1\}$, and that (A-I) and (A-II) hold, or (A-III) holds. Then, for each $a \in B$ such that*

$$\sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} < \infty, \tag{2.22}$$

the empirical half space depth

$$HD_{\mathcal{T}}(a, \mu_n) = 0 \tag{2.23}$$

with probability one for all $n \geq 1$, and the half space depth $HD_{\mathcal{T}}(a, \mu) > 0$. Hence the natural empirical half space depth fails to approximate the true half space depth at such points $a \in B$, i.e. consistency fails at all such points. However, we do have consistency for those $a \in B$ where (2.18) fails, but (2.20) holds.

Remark 2.10. The proof of Corollary 2.9 follows immediately from Theorems 2.2, 2.6 and 2.7. Furthermore, if the condition in (2.19) is replaced by the assumption

$$\liminf_{k \rightarrow \infty} P(t_k(X) < t_k(a)) > 0,$$

then the proof of Theorem 2.7 implies that $HD_{\mathcal{T}}(a, \mu_n) = 0$ with probability one for all $n \geq 1$ without assuming the variances σ_k^2 exist. However, the condition (2.20) allows us to relate empirical half space depth to the true half space depth as indicated in Corollary 2.9. Moreover, it is only under the assumptions in (A-I) and (A-II) where the condition (2.19) is something extra. That is, the assumptions in (A-III) imply (2.19). This can be seen by observing that $P(t_k(X) < c_k) = \int_{-\frac{c_k}{\lambda_k}}^{\frac{c_k}{\lambda_k}} \phi(x) dx$, and hence $\sigma_k = \sigma \lambda_k$, $\lim_{k \rightarrow \infty} c_k / \sigma_k = 0$, and ϕ symmetric about zero implies $\lim_{k \rightarrow \infty} P(t_k(X) < c_k) = 1/2$.

2.1. *Proof of Theorem 2.2.* Using (2.8) and (2.9)

$$HD_{\mathcal{T}}(a, \mu) \leq \inf_{\alpha \in \ell_0, t_{\alpha}(a) > 0} P(t_{\alpha}(X) \geq t_{\alpha}(a)),$$

and by Markov's inequality

$$HD_{\mathcal{T}}(a, \mu) \leq \inf_{\alpha \in \ell_0, t_{\alpha}(a) > 0} \mathbb{E}(t_{\alpha}^2(X) / t_{\alpha}^2(a)) = \inf_{\alpha \in \ell_0, t_{\alpha}(a) > 0} \frac{\sum_{k \geq 1} \alpha_k^2 \sigma_k^2}{(\sum_{k \geq 1} \alpha_k t_k(a))^2}.$$

Therefore, $HD_{\mathcal{T}}(a, \mu) = 0$ whenever

$$\sup_{\alpha \in \ell_0, t_\alpha(a) > 0} \frac{(\sum_{k \geq 1} \alpha_k t_k(a))^2}{\sum_{k \geq 1} \alpha_k^2 \sigma_k^2} = \infty.$$

Given $a \in B$ such that (2.6) holds, then by setting $\alpha_k = t_k(a)/\sigma_k^2, k = 1, \dots, n$, and zero for $k \geq n$, we therefore have

$$\sup_{\alpha \in \ell_0, t_\alpha(a) > 0} \frac{(\sum_{k \geq 1} \alpha_k t_k(a))^2}{\sum_{k \geq 1} \alpha_k^2 \sigma_k^2} \geq \sup_n \sum_{k=1}^n t_k(a)^2 / \sigma_k^2 = \infty.$$

Thus the theorem is proved as the final assertion that (2.5) follows from (2.7) is now immediate.

2.2. *Proof of Theorem 2.6.* To prove Theorem 2.6 it will be convenient to first prove some lemmas, where we continue to use the parameterization of \mathcal{T} determined in (2.8).

Lemma 2.11. *Let $\{t_k(X) : k \geq 1\}$ be independent with mean zero, $\sigma_k^2 = \mathbb{E}(t_k^2(X)) \in (0, \infty)$, and define $\ell_0^+ = \{\alpha \in \ell_0 : \sum_{k \geq 1} \alpha_k^2 > 0\}$. Furthermore, assume there exists $c \in (0, \infty)$ such that (A-II) holds. Then,*

$$\inf_{\alpha \in \ell_0^+} \frac{[\mathbb{E}((\sum_{k \geq 1} \alpha_k t_k(X))^2)]^2}{\mathbb{E}((\sum_{k \geq 1} \alpha_k t_k(X))^4)} \geq (3c)^{-1}. \tag{2.24}$$

Proof: Expanding the sum to the fourth power we have

$$\mathbb{E}(|\sum_{k \geq 1} \alpha_k t_k(X)|^4) = \sum_{k \geq 1} \alpha_k^4 \mathbb{E}(t_k(X)^4) + 6 \sum_{1 \leq i < j} \alpha_i^2 \alpha_j^2 \mathbb{E}(t_i(X)^2) \mathbb{E}(t_j(X)^2),$$

and hence, since $c \geq 1$,

$$\begin{aligned} \mathbb{E}(|\sum_{k \geq 1} \alpha_k t_k(X)|^4) &\leq 3c [\sum_{k \geq 1} \alpha_k^4 \mathbb{E}^2(t_k(X)^2) + 2 \sum_{1 \leq i < j} \alpha_i^2 \alpha_j^2 \mathbb{E}(t_i(X)^2) \mathbb{E}(t_j(X)^2)] \\ &= 3c [\sum_{k \geq 1} \alpha_k^2 \mathbb{E}(t_k(X)^2)]^2. \end{aligned}$$

□

Lemma 2.12. *If $\{t_k(X) : k \geq 1\}$ are independent and symmetric (about zero), (A-II) holds, and $a \in B$ is such that $\sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} < 1$, then*

$$\inf_{\alpha \in \ell_0} P(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)) > 0.$$

Proof: Take $\delta > 0$ such that $\sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} < (1 - \delta)^2$. Then,

$$|\sum_{k \geq 1} \alpha_k t_k(a)| \leq \|\{\alpha_k \sigma_k\}\|_2 \|\{t_k(a)/\sigma_k\}\|_2 \leq (1 - \delta) \|\{\alpha_k \sigma_k\}\|_2,$$

and hence

$$P(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)) \geq P(\sum_{k \geq 1} \alpha_k t_k(X) \geq (1 - \delta) \|\{\alpha_k \sigma_k\}\|_2).$$

Since $\{t_k(X) : k \geq 1\}$ are independent and symmetric, we thus have

$$P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \geq \frac{1}{2} P\left(\left|\sum_{k \geq 1} \alpha_k t_k(X)\right| \geq (1 - \delta) \|\{\alpha_k \sigma_k\}\|_2\right). \quad (2.25)$$

Now

$$\mathbb{E}\left(\left(\sum_{k \geq 1} \alpha_k t_k(X)\right)^2\right) = \sum_{k \geq 1} \alpha_k^2 \sigma_k^2,$$

and hence (2.25) and the Paley-Zygmund inequality implies

$$\begin{aligned} & P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \\ & \geq \frac{1}{2} P\left(\left|\sum_{k \geq 1} \alpha_k t_k(X)\right|^2 \geq (1 - \delta)^2 \|\{\alpha_k \sigma_k\}\|_2^2\right) \\ & \geq \frac{1}{2} \left\{ [1 - (1 - \delta)^2] \frac{\mathbb{E}\left(\left(\sum_{k \geq 1} \alpha_k t_k(X)\right)^2\right)}{\left(\mathbb{E}\left(\left(\sum_{k \geq 1} \alpha_k t_k(X)\right)^4\right)\right)^{\frac{1}{2}}}\right\}^2. \end{aligned} \quad (2.26)$$

Since (A-II) holds, Lemma 2.11 implies we can combine (2.24) and (2.26) to obtain

$$\inf_{\alpha \in \ell_0} P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \geq \frac{1}{6c} (2\delta - \delta^2)^2 > 0, \quad (2.27)$$

and the lemma is proven. \square

Proof of Theorem 2.6 assuming (A-I) and (A-II). First we observe from Theorem 2.2 that (2.18) is necessary for (2.17). Hence we turn to the converse.

To prove sufficiency we first observe that for each $d \geq 1$ and $\alpha \in \ell_0$

$$\begin{aligned} & P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \quad (2.28) \\ & \geq P\left(\sum_{k=1}^d \alpha_k t_k(X) \geq \sum_{k=1}^d \alpha_k t_k(a), \sum_{k \geq d+1} \alpha_k t_k(X) \geq \sum_{k \geq d+1} \alpha_k t_k(a)\right), \end{aligned}$$

and hence the independence of the $\{t_k(X) : k \geq 1\}$ implies

$$\begin{aligned} & P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \quad (2.29) \\ & \geq P\left(\sum_{k=1}^d \alpha_k t_k(X) \geq \sum_{k=1}^d \alpha_k t_k(a)\right) P\left(\sum_{k \geq d+1} \alpha_k t_k(X) \geq \sum_{k \geq d+1} \alpha_k t_k(a)\right). \end{aligned}$$

Taking d sufficiently large such that

$$\sum_{k \geq d+1} \frac{t_k^2(a)}{\sigma_k^2} < 1, \quad (2.30)$$

we have from Lemma 2.12 that

$$\inf_{\alpha \in \ell_0} P\left(\sum_{k \geq d+1} \alpha_k t_k(X) \geq \sum_{k \geq d+1} \alpha_k t_k(a)\right) > 0. \quad (2.31)$$

Since (2.14) holds we have

$$\inf_{k \in \ell_0} P\left(\sum_{k=1}^d \alpha_k t_k(X) \geq \sum_{k=1}^d \alpha_k t_k(a)\right) > 0, \tag{2.32}$$

and the theorem is proved when (A-I) and (A-II) are assumed

In order to complete the proof of the theorem under the assumptions in (A-III) we need a definition, some additional notation and some lemmas. We start with a definition of an admissible translate for a probability measure. Admissible translates appear in a variety of settings in the literature, sometimes with slightly different meanings, but for probability measures on the Borel subsets of \mathbb{R}^∞ with i.i.d. coordinates our definition below agrees with that used by Shepp in Shepp (1965) for a totally indistinguishable translate. It also agrees with terminology used in the study of centered Gaussian measures μ on a separable Banach space, where the admissible translates are the vectors in the Hilbert space H_μ given in subsection 3.1, and in similar situations for other types of measures.

Definition 2.13. Let μ be a probability measure on the Borel subsets \mathcal{B}_E of an F-space E , and for $x \in E$ and $A \in \mathcal{B}_E$ set $\mu_x(A) = \mu(A - x)$. Then x is an admissible translate of μ if μ_x and μ are mutually absolutely continuous with respect to one another on (E, \mathcal{B}_E) .

Lemma 2.14. *If x is an admissible translate of the probability, μ , then $-x$ is also an admissible translate of μ .*

Proof: Suppose $\mu(A + x) = 0$. Then, x an admissible translate of μ implies $\mu((A + x) - x) = 0$, and hence $\mu(A) = 0$. Conversely, if $\mu(A) = 0$, then $\mu((A + x) - x) = \mu(A) = 0$, and since x is an admissible translate, this implies $\mu(A + x) = 0$. \square

When E is a sequence space, such as \mathbb{R}^∞ or ℓ_p , we denote the typical vector x by writing $x = (x_1, x_2, \dots)$ or $x = \{x_k : k \geq 1\}$.

Lemma 2.15. *Let X take values in the F-space B and assume $\mu = \mathcal{L}(X)$ is defined on (B, \mathcal{B}_B) . Let $\Lambda : B \rightarrow \mathbb{R}^\infty$ be such that*

$$\Lambda(a) = (t_1(a), t_2(a), \dots), a \in B, \tag{2.33}$$

where the maps $t_k(\cdot), k \geq 1$, are \mathcal{B}_B measurable to the reals, and $\mathcal{L}(\Lambda(X)) = \mathcal{L}(-\Lambda(X))$, i.e., $\Lambda(X)$ has a symmetric distribution. If $a \in B$ is such that $(t_1(a), t_2(a), \dots)$ is an admissible translate for the probability measure $\mu^\Lambda(A) = \mu(\Lambda^{-1}(A)) = P(X \in \Lambda^{-1}(A)), A \in \mathcal{B}_{\mathbb{R}^\infty}$, then

$$HD_{\mathcal{T}}(a, \mu) > 0, \tag{2.34}$$

where \mathcal{T} denotes all finite linear combination of the maps $\{t_k : k \geq 1\}$.

Proof: If $HD_{\mathcal{T}}(a, \mu) = 0$, then there exists $f_n \in \ell_0$ such that

$$\lim_{n \rightarrow \infty} P(t_{f_n}(X) \geq t_{f_n}(a)) = 0. \tag{2.35}$$

By taking a subsequence, we may assume that

$$\sum_{n \geq 1} P(t_{f_n}(X) \geq t_{f_n}(a)) < \infty, \tag{2.36}$$

and therefore $P(t_{f_n}(X) \geq t_{f_n}(a) \text{ i.o.}) = 0$.

We now connect this with Λ and the fact that $\Lambda(a)$ is an admissible translate. For this purpose for $n \geq 1$ let $f_n = (\alpha_{1,n}, \alpha_{2,n}, \dots) \in \ell_0$, $x = (x_1, x_2, \dots) \in \mathbb{R}^\infty$, and $\langle f_n, x \rangle = \sum_{k \geq 1} \alpha_{k,n} x_k$. We thus have

$$\begin{aligned} P(t_{f_n}(X) - t_{f_n}(a) \geq 0 \text{ i.o.}) &= P(\langle f_n, \Lambda(X) - \Lambda(a) \rangle \geq 0 \text{ i.o.}) & (2.37) \\ &= \mu(u \in B: \langle f_n, \Lambda(u) - \Lambda(a) \rangle \geq 0 \text{ i.o.}) = \mu^\Lambda(\alpha \in \mathbb{R}^\infty: \langle f_n, \alpha - \Lambda(a) \rangle \geq 0 \text{ i.o.}) \\ &= \mu^\Lambda(\beta + \Lambda(a) \in \mathbb{R}^\infty: \langle f_n, \beta \rangle \geq 0 \text{ i.o.}) = (\mu^\Lambda)_{-\Lambda(a)}(\gamma \in \mathbb{R}^\infty: \langle f_n, \gamma \rangle \geq 0 \text{ i.o.}) = 0. \end{aligned}$$

Further,

$$\begin{aligned} (\mu^\Lambda)_{-\Lambda(a)}(\gamma \in \mathbb{R}^\infty: \langle f_n, \gamma \rangle \geq 0 \text{ i.o.}) &= 0 \\ \text{if and only if } (\mu^\Lambda)_{-\Lambda(a)}(\gamma \in \mathbb{R}^\infty: \langle f_n, \gamma \rangle < 0, \text{ eventually}) &= 1. \end{aligned}$$

But, since $\Lambda(a) = (t_1(a), t_2(a), \dots)$ is an admissible translate for the probability μ^Λ by Lemma 2.14 we also have $-\Lambda(a)$ is an admissible translate and consequently

$$1 = \mu^\Lambda(\gamma \in \mathbb{R}^\infty: \langle f_n, \gamma \rangle < 0, \text{ eventually}) \tag{2.38}$$

By symmetry of the measure μ^Λ we also have

$$1 = \mu^\Lambda(\gamma \in \mathbb{R}^\infty: \langle f_n, \gamma \rangle > 0, \text{ eventually}), \tag{2.39}$$

which yields a contradiction. □

Proof of Theorem 2.6 assuming (A-III). As before, Theorem 2.2 shows (2.18) is necessary for (2.17). Hence we turn to the converse, showing

$$\sum_{k \geq 1} \frac{t_k^2(a)}{\lambda_k^2} < \infty \tag{2.40}$$

implies

$$HD_{\mathcal{T}}(a, \mu) > 0. \tag{2.41}$$

Since (2.16) holds, Lemma 2.15 will show (2.41) for $a \in B$ satisfying (2.40) provided we show $\Lambda(a) = (t_1(a), t_2(a), \dots)$ is an admissible translate of μ^Λ , where $\mu = \mathcal{L}(X)$.

This follows using Kakutani's result on the equivalence of infinite product measures as in Shepp (1965). That is, if $\mu_k = \mathcal{L}(t_k(X))$ and $\nu_k = \mathcal{L}(t_k(X) + t_k(a))$ are mutually absolutely continuous for $k \geq 1$, then $\Lambda(a)$ is an admissible translate for μ^Λ if and only if

$$H(\mu^\Lambda, \mu^{\Lambda+\Lambda(a)}) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k) > 0, \tag{2.42}$$

where $\mu^{\Lambda+\Lambda(a)} = \mathcal{L}(\Lambda(X) + \Lambda(a))$ and

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left(\frac{d\mu_k}{dx} \frac{d\nu_k}{dx} \right)^{\frac{1}{2}} dx. \tag{2.43}$$

Now

$$\frac{d\mu_k}{dx}(s) = \frac{\phi(\frac{s}{\lambda_k})}{\lambda_k}, \tag{2.44}$$

$$\frac{d\nu_k}{dx}(s) = \frac{\phi(\frac{s-t_k(a)}{\lambda_k})}{\lambda_k}, \quad (2.45)$$

and hence

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left(\frac{\phi(\frac{s}{\lambda_k})}{\lambda_k} \frac{\phi(\frac{s-t_k(a)}{\lambda_k})}{\lambda_k} \right)^{\frac{1}{2}} ds = \int_{\mathbb{R}} \left(\phi(t) \phi\left(t - \frac{t_k(a)}{\lambda_k}\right) \right)^{\frac{1}{2}} dt. \quad (2.46)$$

Now (2.40) and ϕ having finite information, since (2.15) holds, combine with part (ii) of Theorem 1 of Shepp (1965) to imply $\Lambda(a)/\lambda \equiv (\frac{t_1(a)}{\lambda_1}, \frac{t_2(a)}{\lambda_2}, \dots)$ is an admissible translate of P^Y , where $Y = \Lambda(X)/\lambda$. Therefore, Kakutani's theorem implies

$$H(P^Y, P^{Y+\Lambda(a)/\lambda}) > 0, \quad (2.47)$$

and since an easy calculation shows

$$H(P^Y, P^{Y+\Lambda(a)/\lambda}) = \prod_{k=1}^{\infty} \int_{\mathbb{R}} \left(\phi(t) \phi\left(t - \frac{t_k(a)}{\lambda_k}\right) \right)^{\frac{1}{2}} dt, \quad (2.48)$$

(2.46), (2.47) and (2.48) combine to imply (2.42). Thus Kakutani's theorem implies $\Lambda(a)$ is an admissible translate of μ^Λ and Lemma 2.15 completes the proof.

2.3. *Proof of Theorem 2.7.* Since $\{t_k : k \geq 1\} \subseteq \mathcal{T}$ we have

$$HD_{\mathcal{T}}(a, \mu_n) \leq \inf_{k \geq 1} \frac{1}{n} \sum_{j=1}^n I(t_k(X_j) \geq t_k(a)), \quad (2.49)$$

and hence it suffices to show for all $n \geq 1$ and $a \in B$ satisfying (2.20)

$$\inf_{k \geq 1} Z_{n,k}(a) = 0, \quad (2.50)$$

with probability one, where

$$Z_{n,k}(a) = \frac{1}{n} \sum_{j=1}^n I(t_k(X_j) \geq t_k(a)). \quad (2.51)$$

Since the random variables $\{t_k(X) : k \geq 1\}$ are independent, the sequences $\{t_k(X_j) : k \geq 1\}$ consist of independent random variables, and as sequences are independent and identically distributed for $j \geq 1$. Hence fix $n \geq 1$ and assume $a \in B$ satisfies (2.20).

Then, the sequence $\{Z_{n,k}(a) : k \geq 1\}$ consists of independent random variables. Furthermore, (2.19) then implies there exists $\delta > 0$ and $\{k_i : i \geq 1\}$ a subsequence of the positive integers such that for all k_i

$$P(t_{k_i}(X) < t_{k_i}(a)) \geq \delta. \quad (2.52)$$

Therefore, for all $n \geq 1$

$$P(Z_{n,k_i}(a) = 0) \geq \delta^n. \quad (2.53)$$

Hence for n fixed, the independence in $k \geq 1$ and the Borel-Cantelli lemma implies

$$P(Z_{n,k_i}(a) = 0 \text{ i.o. in } i) = 1. \quad (2.54)$$

Now (2.54) implies (2.50) with probability one, and the theorem is proved.

3. Examples and explicit forms of half space depth

If $\{t_k(X) : k \geq 1\}$ is a sequence of mean zero Gaussian random variables with variances $\sigma_k^2 \in (0, \infty), k \geq 1$, then Theorems 2.2 and 2.6 readily apply to provide necessary and sufficient conditions for the half space depth in (2.5) to be positive. Theorem 2.7 and Corollary 2.9 also provide information about the empirical depth. However, in this situation we can obtain an explicit formula for this depth. In fact, the formula we obtain holds when the sequence $\{t_k(X) : k \geq 1\}$ consists of i.i.d. scaled symmetric stable random variables, which is interesting since among the non-degenerate stable random variables only the Gaussians have a variance and our conditions are in terms of second and higher moments. We also obtain a formula for the Tukey half space depth for any centered Gaussian measure on a separable Banach space B when the depth is computed using B^* . Finally, we provide some information when the $\{t_k(X) : k \geq 1\}$ are i.i.d. Rademacher random variables, but in this case the results are less explicit.

3.1. *An explicit formula for half region depth for stables.* To obtain an explicit formula for the half space depth for symmetric stable random variables we need the following lemma on the sequence spaces $\ell_p, 0 < p \leq 2$. It is essentially proved in Wheeden and Zygmund (1977), pp. 128-129, and hence we omit further details.

Lemma 3.1. *If $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, and $y = \{y_j : j \geq 1\} \in \mathbb{R}^\infty$, then*

$$\sup_{x \in \ell_0, \|x\|_p=1} \sum_{j \geq 1} |x_j y_j| = \|y\|_q, \tag{3.1}$$

where $\|y\|_q = (\sum_{j \geq 1} |y_j|^q)^{\frac{1}{q}}$ could well be infinity. If $0 < p \leq 1$, then (3.1) holds with $q = \infty$ and $\|y\|_\infty = \sup_{k \geq 1} |y_k|$, which again could be infinite.

Notation 3.2. If $b = \{b_k : k \geq 1\} \in \mathbb{R}^\infty$ and $c = \{c_k : k \geq 1\}$ is a strictly positive sequence, we will write b/c to denote the sequence $\{b_k/c_k : k \geq 1\}$,

Proposition 3.3. *Let S be a non-degenerate symmetric p -stable random variable where $0 < p \leq 2$, and for $c_k \in (0, \infty), k \geq 1$, assume $\{t_k(X) : k \geq 1\}$ are independent with $\mathcal{L}(t_k(X)) = \mathcal{L}(c_k S)$. If \mathcal{T} denotes all finite linear combinations of $\{t_k : k \geq 1\}$ and for $a \in B$ we let $\tau(a) = \{t_k(a) : k \geq 1\}$, then*

$$HD_{\mathcal{T}}(a, \mu) = 1 - P(S \leq \|\tau(a)/c\|_q), \tag{3.2}$$

where $q = \infty$ for $0 < p \leq 1, \frac{1}{p} + \frac{1}{q} = 1$ for $1 < p \leq 2$.

Remark 3.4. If $\|\tau(a)/c\|_q = \infty$ in (3.2), then $HD_{\mathcal{T}}(a, \mu) = 0$ for $0 < p \leq 2$. Moreover, since c is fixed, the depth is continuous as a function of the sequence $\tau(a)/c$ in the q -norm when restricted to the set where $\|\tau(a)/c\|_q < \infty$, but it is highly discontinuous with respect to the product topology on \mathbb{R}^∞ . If $p = 2, S$ has variance one, and $c_k = 1, k \geq 1$, then for any $a \in B$

$$HD_{\mathcal{T}}(a, \mu) = 1 - \Phi(\|\tau(a)\|_2), \tag{3.3}$$

where $\Phi(\cdot)$ is the distribution function of a centered Gaussian random variable with variance one. Also, if $1 < p \leq 2$, then it is easy to see from Remark 2.8 that consistency fails at all $a \in B$ where the depth is strictly positive.

Proof: Since the sequence $\{t_k(X) : k \geq 1\}$ is independent and symmetric, it suffices to show

$$\inf_{\alpha \in \ell_0, t_\alpha(a) > 0} P\left(\frac{t_\alpha(X)}{t_\alpha(a)} \geq 1\right) = 1 - P(S \leq \|\tau(a)/c\|_q) \tag{3.4}$$

Now

$$\frac{t_\alpha(X)}{t_\alpha(a)} = \frac{\sum_{k \geq 1} \alpha_k t_k(X)}{\sum_{k \geq 1} \alpha_k t_k(a)}, \tag{3.5}$$

and using independence and that the random variables $\{t_k(X), k \geq 1\}$ are p -stable, we therefore have

$$\mathcal{L}\left(\frac{t_\alpha(X)}{t_\alpha(a)}\right) = \mathcal{L}\left(\frac{(\sum_{k \geq 1} |\alpha_k c_k|^p)^{\frac{1}{p}}}{\sum_{k \geq 1} \alpha_k t_k(a)} S\right). \tag{3.6}$$

Hence,

$$\inf_{\alpha \in \ell_0, t_\alpha(a) > 0} P\left(\frac{t_\alpha(X)}{t_\alpha(a)} \geq 1\right) = \inf_{\alpha \in \ell_0, t_\alpha(a) > 0} P\left(S \geq \frac{\sum_{k \geq 1} \alpha_k t_k(a)}{(\sum_{k \geq 1} |\alpha_k c_k|^p)^{\frac{1}{p}}}\right). \tag{3.7}$$

Letting $\beta_k = |\alpha_k| \text{sign}(t_k(a)), k \geq 1$, and using the continuity of the distribution of S , we have

$$\inf_{\alpha \in \ell_0, t_\alpha(a) > 0} P\left(\frac{t_\alpha(X)}{t_\alpha(a)} \geq 1\right) = P\left(S \geq \sup_{\beta \in \ell_0, \|\beta\|_\infty > 0} \frac{\sum_{k \geq 1} |\beta_k t_k(a)|}{(\sum_{k \geq 1} |\beta_k c_k|^p)^{\frac{1}{p}}}\right). \tag{3.8}$$

Setting

$$\gamma_k = \frac{\beta_k c_k}{(\sum_{k \geq 1} |\beta_k c_k|^p)^{\frac{1}{p}}}, k \geq 1, \tag{3.9}$$

we have

$$\sup_{\beta \in \ell_0, \|\beta\|_\infty > 0} \frac{\sum_{k \geq 1} |\beta_k t_k(a)|}{(\sum_{k \geq 1} |\beta_k c_k|^p)^{\frac{1}{p}}} = \sup_{\{\gamma \in \ell_0, \|\gamma\|_p = 1\}} \sum_{k \geq 1} |\gamma_k t_k(a)/c_k| = \|\tau(a)/c\|_q, \tag{3.10}$$

where the last equality follows from Lemma 4 provided p and q are related as indicated in the proposition. \square

Our next result obtains the analogue of the $p = q = 2$ case of Proposition 3.3 when X is a centered Gaussian random vector X taking values in a separable Banach space B and the half space depth is given by

$$HD_{B^*}(a, \mu) = \inf_{t \in B^*} P(t(X) \geq t(a)) \tag{3.11}$$

where B^* is the dual of B . As before, we assume X is defined on the probability space (Ω, \mathcal{F}, P) and is measurable from \mathcal{F} to the Borel subsets of B , and let $\mu = \mathcal{L}(X)$.

Let $\|\cdot\|$ and $\|\cdot\|_{B^*}$ denote the norms on B and B^* , respectively. Then, by the Fernique-Landau-Shepp result Fernique (1970); Landau and Shepp (1970) for some $s > 0$

$$\int_B \exp\{s\|x\|^2\} d\mu(x) < \infty, \tag{3.12}$$

and hence the linear map $S : B^* \rightarrow B$ given by the Bochner integral

$$Sf = \int_B xf(x)d\mu(x) \tag{3.13}$$

is continuous from B^* to B . The Hilbert space H_μ generating μ is given by the completion of the range of S with respect to the norm $\|\cdot\|_\mu$ obtained from the inner product

$$\langle Sf, Sg \rangle_\mu = \int_B f(x)g(x)d\mu(x). \tag{3.14}$$

Moreover, H_μ can be viewed as a subset of B , since for $x \in H_\mu$

$$\|x\| \leq \sigma(\mu)\|x\|_\mu \tag{3.15}$$

where

$$\sigma(\mu) \equiv \sup_{\|f\|_{B^*} \leq 1} \left(\int_B f^2(x)d\mu(x) \right)^{1/2} < \infty. \tag{3.16}$$

It is also well known that the support of the Gaussian measure μ is given by the closure of H_μ in B , which we denote by \bar{H}_μ , and that $\mu(H_\mu) = 0$ when H_μ is infinite dimensional. Of course, $H_\mu = \bar{H}_\mu$ when H_μ is finite dimensional, so in that case $\mu(H_\mu) = 1$.

Additional properties relating $H_\mu, \langle \cdot, \cdot \rangle_\mu, B$, and the measure μ can be found in Lemma 2.1 of [Kuelbs and Zinn \(2008\)](#). However, the above suffice to state our proposition on the half space depth in (3.11) for Gaussian measures, and for the following useful lemma used in its proof. The proof of the lemma is in [Kuelbs and Zinn \(2008\)](#), where it appears in a slightly more general form as Lemma 2.2. Once we have this lemma, the remainder of the proof follows as in Proposition 3.3.

Lemma 3.5. *Let μ be a centered Gaussian measure on B , and assume H_μ and $\|\cdot\|_\mu$ are defined as above. If*

$$\theta(x) = \sup_{f \in B^*, \int_B f^2 d\mu \leq 1} f(x), \tag{3.17}$$

then $\theta(x) = \|x\|_\mu$ for $x \in H_\mu$, and $\theta(x) = \infty$ for $x \in B - H_\mu$.

Proposition 3.6. *Let X be a B -valued centered Gaussian random vector as above, and assume $\Phi(\cdot)$ is the distribution function of a mean zero-variance one Gaussian random variable. Then,*

$$HD_{B^*}(a, \mu) = 1 - \Phi(\|a\|_\mu), a \in H_\mu, \tag{3.18}$$

and

$$HD_{B^*}(a, \mu) = 0, a \in B - H_\mu. \tag{3.19}$$

Furthermore,

$$\mu(a \in B : HD_{B^*}(a, \mu) = 0) = 0 \text{ or } 1 \tag{3.20}$$

according as H_μ is a finite dimensional Hilbert space or an infinite dimensional Hilbert space. In addition, if H_μ is infinite dimensional, then for all $a \in H_\mu$ the empirical half space depth given in (2.3) or (2.4) with $\mathcal{T} = B^*$ is zero with P -probability one. Hence the empirical half depth fails to approximate the true half space depth for all $a \in H_\mu$ in this setting, i.e. consistency fails at all such points.

Proof: As mentioned following (3.16), since μ is a Gaussian measure with mean vector zero, $\mu(H_\mu)$ has probability one or zero, according as H_μ is a finite dimensional Hilbert space or an infinite dimensional Hilbert space, so (3.20) follows immediately once we verify (3.18) and (3.19).

Since $f(x)$ is centered Gaussian with variance $\sigma_f^2 \equiv \int_B f^2(x)d\mu(x)$ for $f \in B^*$, it follows that

$$HD_{B^*}(a, \mu) = \inf_{f \in B^*, f(a) > 0} \mu(x : f(x) \geq f(a)) = \inf_{f \in B^*, f(a) > 0} [1 - \Phi(f(a)/\sigma_f)]. \tag{3.21}$$

Using the continuity of Φ and that it is increasing, we thus have

$$HD_{B^*}(a, \mu) = 1 - \Phi\left(\sup_{f \in B^*, f(a) > 0} f(a)/\sigma_f\right). \tag{3.22}$$

Moreover, since

$$\sup_{f \in B^*, f(a) > 0} f(a)/\sigma_f = \theta(a), \tag{3.23}$$

where $\theta(\cdot)$ is as in (3.17), we therefore have (3.18) and (3.19).

If H_μ is infinite dimensional, then there exists a sequence $\{t_k : k \geq 1\} \subseteq B^*$ such that $\{St_k = S(t_k) : k \geq 1\}$ are orthonormal in H_μ , for all $a \in H_\mu$,

$$\lim_{k \rightarrow \infty} t_k(a) = 0,$$

and $\{t_k(X) : k \geq 1\}$ are independent centered Gaussian random variables with variance one. Thus, for all $a \in H_\mu$ and $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{X_j}$, where X, X_1, X_2, \dots are i.i.d. B -valued Gaussian random vectors, the empirical depth

$$HD_{B^*}(a, \mu_n) = \inf_{t \in B^*} \mu_n(x \in B : t(x) \geq t(a)) \leq \inf_{k \geq 1} \frac{1}{n} \sum_{j=1}^n I(t_k(X_j) \geq t_k(a)).$$

Hence, as in the proof of Theorem 2.7 for $n \geq 1$ fixed, the independence in $k \geq 1$ and the Borel-Cantelli lemma easily imply the empirical half space depth is zero with P -probability one. Thus, the proposition is proved. \square

3.2. The Rademacher case. The explicit results obtained in Propositions 3.3 and 3.6 depend on the scaling properties of the symmetric stable laws, and therefore are likely quite special. They also involve continuous distributions, so for contrast we examine the special discrete case where $\{t_k(X) : k \geq 1\}$ are independent Rademacher random variables. Of course, Theorem 2.2 implies that

$$HD_{\mathcal{T}}(a, \mu) = 0 \tag{3.24}$$

whenever $\sum_{k \geq 1} t_k^2(a) = \infty$, but, as can be seen from Lemma 3.8 below, that is not the entire story. Furthermore, although the condition (2.14) in (A-I) of Theorem 2.6 is not applicable, once we prove Lemma 3.8, it is easy to see how a suitable modification of the proof of Theorem 2.6 and the conditions in (A-I) allow us to identify the set where the depth is strictly positive. We also indicate how the ideas in Montgomery-Smith’s paper Montgomery-Smith (1990) provides an alternative approach to obtain the results for Rademacher $\{t_k(X) : k \geq 1\}$. Finally, we point out that Lemma 3.8 can be refined to apply to other bounded random variables, and hence Proposition 3.7 below has comparable analogues. For example, if $\{t_k(X) :$

$k \geq 1$ are independent, symmetric random variables with $\mathbb{E}(t_k^2(X)) \in (0, \infty)$, (A-II) holds, $P(|t_k(X)| \leq b_k) = 1$ where $b_k < \infty$ for all $k \geq 1$, and $\mathcal{L}(t_k(X))$ has support in every neighborhood of b_k and of $-b_k$ for all $k \geq 1$, then

$$\sup_{k \geq 1} \frac{|t_k(a)|}{b_k} > 1 \text{ or } \sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} = \infty \text{ imply } HD_{\mathcal{T}}(a, \mu) = 0,$$

and

$$\sup_{k \geq 1} \frac{|t_k(a)|}{b_k} < 1 \text{ and } \sum_{k \geq 1} \frac{t_k^2(a)}{\sigma_k^2} < \infty \text{ imply } HD_{\mathcal{T}}(a, \mu) > 0.$$

What happens when $\sup_{k \geq 1} \frac{|t_k(a)|}{b_k} = 1$ depends on whether $\mathcal{L}(t_k(X))$ has positive mass at b_k , or not. This can be seen in the following proposition, which summarizes our results for Rademacher variables. Since its proof can easily be modified to obtain the previous results, those details are omitted.

Proposition 3.7. *Let $\{t_k(X) : k \geq 1\}$ be independent Rademacher random variables. Then, (3.24) holds for all $a \in B$ such that $\sum_{k \geq 1} t_k^2(a) = \infty$ or $\sup_{k \geq 1} |t_k(a)| > 1$. In addition,*

$$HD_{\mathcal{T}}(a, \mu) > 0 \tag{3.25}$$

for all $a \in B$ such that $\sum_{k \geq 1} t_k^2(a) < \infty$ and $\sup_{k \geq 1} |t_k(a)| \leq 1$, and consistency fails at all such points $a \in B$.

The proof of the proposition requires the following lemma.

Lemma 3.8. *Let $\{t_k(X) : k \geq 1\}$ be independent Rademacher random variables. If $a \in B$ and $\sup_{k \geq 1} |t_k(a)| > 1$, then*

$$HD_{\mathcal{T}}(a, \mu) = 0. \tag{3.26}$$

Furthermore, if $\sup_{k \geq 1} |t_k(a)| \leq 1$, then for every $d \geq 1$

$$HD_{\mathbb{R}^d}(\Pi_d(a), \mu^{\Pi_d}) \geq 2^{-d}, \tag{3.27}$$

where $\Pi_d(a) = (t_1(a), \dots, t_d(a))$, $a \in B$, and μ^{Π_d} is the image of μ on \mathbb{R}^d via the map $\Pi_d(\cdot) : B \rightarrow \mathbb{R}^d$.

Proof: If $|t_{k_0}(a)| > 1$, and $\hat{\alpha} = \{\delta(k, k_0)\text{sign}(t_k(a)) : k \geq 1\}$ where $\delta(k, k_0) = 1$ when $k = k_0$ and zero otherwise, then

$$HD_{\mathcal{T}}(a, \mu) = \inf_{\alpha \in \ell_0} P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \leq \inf_{\alpha = \hat{\alpha}} P(\alpha(X) \geq \alpha(a)) \tag{3.28}$$

$$= P(\text{sign}(t_{k_0}(a))t_{k_0}(X) \geq |t_{k_0}(a)|) = 0. \tag{3.29}$$

Hence, (3.26) holds.

To verify (3.27) we observe that $\mu^{\Pi_d} = \mathcal{L}(P^{\Pi_d(X)})$, and hence

$$\begin{aligned} HD_{\mathbb{R}^d}(\Pi_d(a), \mu^{\Pi_d}) &= \inf_{\alpha \in \ell_0} P\left(\sum_{k=1}^d \alpha_k t_k(X) \geq \sum_{k=1}^d \alpha_k t_k(a)\right) \\ &\geq \inf_{\alpha \in \ell_0} P\left(\sum_{k=1}^{d-1} \alpha_k t_k(X) \geq \sum_{k=1}^{d-1} \alpha_k t_k(a), \alpha_d t_d(X) \geq \alpha_d t_d(a)\right) \end{aligned}$$

$$\geq \inf_{\alpha \in \ell_0} P\left(\sum_{k=1}^{d-1} \alpha_k t_k(X) \geq \sum_{k=1}^{d-1} \alpha_k t_k(a)\right) P(\alpha_d t_d(X) \geq \alpha_d t_d(a)), \tag{3.30}$$

where the last inequality holds by the independence of the $\{t_k(X) : k \geq 1\}$. Furthermore, for all $k \geq 1$,

$$\inf_{s \in \mathbb{R}} P(st_k(X) \geq st_k(a)) = \min[A_1, A_2], \tag{3.31}$$

where

$$A_1 = \inf_{s \geq 0} P(st_k(X) \geq st_k(a)) = P(t_k(X) \geq t_k(a))$$

and

$$A_2 = \inf_{s < 0} P(st_k(X) \geq st_k(a)) = P(t_k(X) \leq t_k(a)).$$

Hence $|t_k(a)| \leq 1$ implies $A_i \geq 1/2$ for $i = 1, 2$, and (3.31) then implies (3.27) for $d = 1$. Furthermore, (3.31) applied to (3.30) allows us to induct on d proving (3.27) for all $d \geq 1$. \square

Proof of Proposition 3.7. If $\sum_{k \geq 1} t_k^2(a) = \infty$, then (3.24) holds by Theorem 2.2, and when $\sup_{k \geq 1} |t_k(a)| > 1$, we have (3.24) by Lemma 3.8. Hence it remains to show that $\sum_{k \geq 1} t_k^2(a) < \infty$ and $\sup_{k \geq 1} |t_k(a)| \leq 1$ imply (3.25). This follows since (3.27) holds for all $a \in B$ satisfying $\sup_{k \geq 1} |t_k(a)| \leq 1$, and hence, although this is not equivalent (2.14) in (A-I), the proof of Theorem 2.6 shows that if (2.14) is replaced by (3.27) in (A-I), then (3.25) holds provided $\sum_{k \geq 1} t_k^2(a) < \infty$ and $\sup_{k \geq 1} |t_k(a)| \leq 1$ for $a \in B$. Finally, Remark 2.8 following Theorem 2.7 implies that for all such $a \in B$ consistency fails.

Although Proposition 3.7 identifies those $a \in B$ with positive half-space depth for the Rademacher variables, it is unclear what its value might be on such points. Below we obtain some estimates on a lower bound for this depth using two different methods. The first method modifies the estimates in Lemma 2.12 appropriately, and the second applies the delicate results in Montgomery-Smith (1990). However, neither approach yields estimates that apply to all $a \in B$ where the half space depth is positive, and hence they do not identify that collection of points as in Proposition 3.7.

Proposition 3.9. *Let $\{t_k(X) : k \geq 1\}$ be independent Rademacher random variables, and $a \in B$. If $r \in \mathbb{N}$ and δ are such that $\sum_{k > r} t_k^2(a) \leq \frac{1}{4}$ and $\delta\sqrt{r} \leq \frac{1}{4}$, then $\sup_{k \geq 1} |t_k(a)| \leq \delta$ implies*

$$HD_{\mathcal{T}}(a, \mu) = \inf_{\alpha \in \ell_0} P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \geq \frac{3}{32}. \tag{3.32}$$

Proof: If $r, \delta > 0$ and $\alpha \in \ell_0$ are as indicated in the proposition, then

$$\sum_{k \geq 1} \alpha_k t_k(a) = \sum_{k=1}^r \alpha_k t_k(a) + \sum_{k \geq r+1} \alpha_k t_k(a) \leq \delta\sqrt{r}\|\alpha\|_2 + \|\alpha\|_2 \left(\sum_{k > r} t_k^2(a)\right)^{1/2}.$$

Hence, $\sum_{k \geq 1} \alpha_k t_k(a) \leq \frac{1}{2}\|\alpha\|_2$, and

$$P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)\right) \geq P\left(\sum_{k \geq 1} \alpha_k t_k(X) \geq \frac{1}{2}\|\alpha\|_2\right) \tag{3.33}$$

$$= \frac{1}{2}P(|\sum_{k \geq 1} \alpha_k t_k(X)| \geq \frac{1}{2}\|\alpha\|_2) = \frac{1}{2}P(|\sum_{k \geq 1} \alpha_k t_k(X)|^2 \geq \frac{1}{4}\|\alpha\|_2^2). \tag{3.34}$$

Thus the Paley-Zygmund inequality and Lemma 2.11 (with the $t_k(X)$ Rademacher random variables) applied to the last term in (3.34), imply

$$P(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)) \geq \frac{9}{32} \frac{(E[(\sum_{k \geq 1} \alpha_k t_k(X))^2])^2}{E[(\sum_{k \geq 1} \alpha_k t_k(X))^4]} \geq \frac{3}{32}, \tag{3.35}$$

which proves the proposition. □

In order to use the results in Montgomery-Smith (1990) we introduce some norms from the theory of interpolation of Banach spaces. Of course, (3.40) below plays a role analogous to the ℓ_∞ and ℓ_2 assumptions in Proposition 3.7. The notation is from Montgomery-Smith (1990), which defines for $x \in \ell_2$ and $t > 0$

$$K_{1,2}(x, t) \equiv K(x, t : \ell_1, \ell_2) = \inf\{\|x'\|_1 + t\|x''\|_2 : x', x'' \in \ell_2, x' + x'' = x\}, \tag{3.36}$$

and

$$J_{\infty,2}(x, t) \equiv J(x, t : \ell_\infty, \ell_2) = \max\{\|x\|_\infty, t\|x\|_2\}. \tag{3.37}$$

Then, for $t > 0, x \in \ell_2$ we have from Lemma 1 in Montgomery-Smith (1990) that

$$K_{1,2}(x, t) = \sup\{\sum_{k \geq 1} x_k y_k : y \in \ell_2, J_{\infty,2}(y, t^{-1}) \leq 1\}, \tag{3.38}$$

and Theorem 1 of Montgomery-Smith (1990) implies there is a constant $c > 0$ such that for all $x \in \ell_2$ and $t > 0$

$$P(\sum_{k \geq 1} x_k \epsilon_k \geq c^{-1}K_{1,2}(x, t)) \geq c^{-1}e^{-ct^2}, \tag{3.39}$$

where $\{\epsilon_k : k \geq 1\}$ are independent Rademacher random variables.

Proposition 3.10. *Let $\{t_k(X) : k \geq 1\}$ be independent Rademacher random variables, and $a \in B$ is such that $\sum_{k \geq 1} t_k^2(a) < \infty$. If $c > 0$ is as in (3.39) and for some $t_0 > 0$*

$$\max\{c\|\{t_k(a) : k \geq 1\}\|_\infty, t_0^{-1}c\|\{t_k(a) : k \geq 1\}\|_2\} \leq 1, \tag{3.40}$$

then,

$$HD_{\mathcal{T}}(a, \mu) \geq c^{-1}e^{-ct_0^2}. \tag{3.41}$$

Proof: If (3.40) holds, then (3.38) implies for all $\alpha \in \ell_2$ that

$$K_{1,2}(\alpha, t_0) \geq c \sum_{k \geq 1} \alpha_k t_k(a). \tag{3.42}$$

Since symmetry implies

$$HD_{\mathcal{T}}(a, \mu) = \inf_{\alpha \in \ell_0, t_\alpha(a) > 0} P(\sum_{k \geq 1} \alpha_k t_k(X) \geq \sum_{k \geq 1} \alpha_k t_k(a)), \tag{3.43}$$

(3.39), (3.42), and (3.43) combine to imply

$$HD_{\mathcal{T}}(a, \mu) \geq \inf_{\alpha \in \ell_0, t_\alpha(a) > 0} P(\sum_{k \geq 1} \alpha_k t_k(X) \geq c^{-1}K_{1,2}(\alpha, t_0)) \geq c^{-1}e^{-ct_0^2}. \tag{3.44}$$

Therefore, (3.41) holds and the proposition is proved. □

3.3. *Empirical depths for the Mosler-Polyakova version of Liu’s simplicial depth.*

The depths considered to this point have been based on linear combinations of the one dimensional functionals $\{t_k : k \geq 1\}$, but they may as well take values in \mathbb{R}^d . The recent manuscript by Mosler and Polyakova (2012) uses this approach to define depths on B . That is, let $D_d(\cdot, \cdot)$ be a depth on \mathbb{R}^d , and assume Θ is a collection of Borel measurable maps from B to \mathbb{R}^d . Then, for μ a Borel probability measure on B and $a \in B$, define

$$D_\Theta(a, \mu) = \inf_{\theta \in \Theta} D_d(\theta(a), \mu^\theta), \tag{3.45}$$

where $\mu^\theta(A) = \mu(\theta^{-1}(A))$, A a Borel subset of \mathbb{R}^d .

In connection with their application to data clouds in B , the paper Mosler and Polyakova (2012) points out that there may be problems with this sort of depth when Θ is too large. The next proposition provides an explicit example of this problem in connection with the empirical estimation of $D_\Theta(a, \mu)$ when $B = \mathbb{R}^\infty$ and $D_d(\cdot, \cdot)$ is simplicial depth as in Liu (1990). That is, for $x \in \mathbb{R}^d$ and Q a Borel probability measure on \mathbb{R}^d

$$D_d(x, Q) = P(x \in \text{co}(Y_1, \dots, Y_{(d+1)})), \tag{3.46}$$

where $Y_1, \dots, Y_{(d+1)}$ are i.i.d. with law Q and $\text{co}(Y_1, \dots, Y_{(d+1)})$ denotes the open convex hull of $Y_1, \dots, Y_{(d+1)}$. In particular, it is interesting to observe via (3.51)-(3.54) below that empirical estimation is not dependable when enough independence is inherent in the data, even if Θ is only countable.

Some further notation is as follows. Let X, X_1, X_2, \dots be i.i.d. \mathbb{R}^∞ valued random vectors with $X = (\eta_1, \eta_2, \dots)$ where $\eta, \eta_1, \eta_2, \dots$ are i.i.d. real valued random variables, and $X_j = (\eta_{1,j}, \eta_{2,j}, \dots), j \geq 1$. For $x = (x_1, x_2, \dots) \in \mathbb{R}^\infty$, let

$$\theta_k(x) = (x_{i_k+1}, \dots, x_{i_{k+1}}), \tag{3.47}$$

where $i_k = (k - 1)d, k = 1, 2, \dots$, and henceforth assume $\Theta = \{\theta_k : k \geq 1\}$. Then, for $a = (a_1, a_2, \dots, a_d, a_1, a_2, \dots, a_d, \dots) \in \mathbb{R}^\infty$ we have $\theta_k(a) = (a_1, \dots, a_d)$ for all $k \geq 1$. Furthermore, for all $k \geq 1, u \geq 0$, the probability

$$P(\theta_k(a) \in \text{co}(\theta_k(X_{u+1}), \dots, \theta_k(X_{u+(d+1)}))) \tag{3.48}$$

is independent of k and u , and denoted by $\lambda(a)$. For $a \in \mathbb{R}^\infty$, we define

$$Z_{n,k}(a) = \sum_{J_{n,d}} I(\theta_k(a) \in \text{co}(\theta_k(X_{j_1}), \dots, \theta_k(X_{j_{d+1}}))), \tag{3.49}$$

where

$$J_{n,d} = \{(j_1, \dots, j_{d+1}) : 1 \leq j_1 < \dots < j_{d+1} \leq n\}.$$

Then, for $a \in \mathbb{R}^\infty$, and $D_d(x, Q)$ the simplicial depth of (3.46) with $Q = \mathcal{L}(\eta_1, \dots, \eta_d)$, we follow Liu (1988) and Liu (1990), and consider the sample analogue of (3.45) to be

$$D_{\Theta,n}(a) = \inf_{k \geq 1} \frac{Z_{n,k}(a)}{N_{n,d}}, \tag{3.50}$$

where $N_{n,d} = \frac{n!}{(d+1)!(n-d-1)!}$. This is slightly different than what one would have if the empirical simplicial depth were defined in terms of the empirical probability

measure $P_n = \frac{1}{n} \sum_{k=1}^n \delta_{X_k}$, as then $N_{n,d}$ would be replaced by $\frac{n^{(d+1)}}{(d+1)!}$. However, since these quantities differ by a non-random quantity which is $O(\frac{1}{n})$, we lose no generality in using (3.50).

Proposition 3.11. *Let X, X_1, \dots be i.i.d. \mathbb{R}^∞ -valued Borel measurable random vectors on the probability space (Ω, \mathcal{F}, P) as indicated above, and assume η has a probability density on \mathbb{R} . Also, assume $D_\Theta(a, \mu)$ is defined using simplicial depth on \mathbb{R}^d as above, and $\Theta = \{\theta_k : k \geq 1\}$. Then, $\{\theta_k(X) : k \geq 1\}$ are i.i.d. \mathbb{R}^d -valued random vectors with absolutely continuous distribution Q on \mathbb{R}^d , and for $a = (a_1, a_2, \dots, a_d, a_1, a_2, \dots, a_d, \dots) \in \mathbb{R}^\infty$ we have*

$$D_\Theta(a, \mu) = \inf_{k \geq 1} P(\theta_k(a) \in \text{co}(\theta_k(X_1), \dots, \theta_k(X_{d+1}))) = \lambda(a). \tag{3.51}$$

Furthermore, for each $k \geq 1$ we have with P -probability one that

$$\lim_{n \rightarrow \infty} \left| \frac{Z_{n,k}(a)}{N_{n,d}} - \lambda(a) \right| = 0, \tag{3.52}$$

but with probability one

$$|D_{\Theta,n}(a) - D_\Theta(a, \mu)| = \left| \inf_{k \geq 1} \frac{Z_{n,k}(a)}{N_{n,d}} - \lambda(a) \right| = \lambda(a). \tag{3.53}$$

Hence, for $\lambda(a) > 0$, the empirical simplicial depth fails to approximate the true simplicial depth as $n \rightarrow \infty$ in this model.

Remark 3.12. For the proof of the proposition we only need that η does not have an atom of size one, but when using the open convex hull in the definition of simplicial depth something close to absolute continuity is needed to have $\lambda(a) > 0$ for a large collection of points. Furthermore, it is also important in the proofs of various useful properties of the simplicial depth in \mathbb{R}^d for $d \geq 2$. For example, see the results in Liu (1990). Finally, if the integer $d \geq 2$ is fixed, then essentially the same proof provides an analogous result provided we take η, η_1, \dots to be \mathbb{R}^d valued random vectors with absolutely continuous distribution.

Proof. Since $\theta_k(a) = (a_1, \dots, a_d)$ for $k \geq 1$, the independence structure we have assumed in n and k implies that (3.52) follows from the law of large numbers for U -statistics, see, for example, Hoeffding (1961), or Theorem 4.1.4 in de la Peña and Giné (1999). Moreover,

$$\{Z_{n,k}(a) = 0\} = \left\{ \sum_{J_{n,d}} I(\theta_k(a) \notin \text{co}(\theta_k(X_{j_1}), \dots, \theta_k(X_{j_{d+1}}))) = N_{n,d} \right\},$$

where

$$\begin{aligned} & I(\theta_k(a) \notin \text{co}(\theta_k(X_{j_1}), \dots, \theta_k(X_{j_{d+1}}))) \\ &= I(\theta_k(a) \notin \text{co}((\eta_{i_k+1, j_1}, \dots, \eta_{i_k+1, j_{d+1}}), \dots, (\eta_{i_k+1, j_1}, \dots, \eta_{i_k+1, j_{d+1}}))). \end{aligned}$$

Since $\theta_k(a) = (a_1, \dots, a_d)$ we therefore have

$$\{Z_{n,k}(a) = 0\} \supseteq A_{1,n,k} \cup A_{2,n,k},$$

where

$$A_{1,n,k} = \left\{ \sum_{J_{n,d}} I(a_1 < \eta_{i_k+1, j_1}, \dots, a_1 < \eta_{i_k+1, j_{d+1}}) = N_{n,d} \right\},$$

and

$$A_{2,n,k} = \left\{ \sum_{J_{n,d}} I(a_1 > \eta_{i_k+1,j_1}, \dots, a_1 > \eta_{i_k+1,j_{d+1}}) = N_{n,d} \right\}.$$

Now

$$A_{1,n,k} = \{a_1 < \eta_{i_k+1,1}, \dots, a_1 < \eta_{i_k+1,n}\},$$

and

$$A_{2,n,k} = \{a_1 > \eta_{i_k+1,1}, \dots, a_1 > \eta_{i_k+1,n}\},$$

and hence

$$P(A_{1,n,k}) = P(\eta > a_1)^n \text{ and } P(A_{2,n,k}) = P(\eta < a_1)^n.$$

Since η has a continuous distribution function $P(\eta < a_1) + P(\eta > a_1) = 1$ and hence

$$P(Z_{n,k}(a) = 0) \geq P(\eta > a_1)^n + P(\eta < a_1)^n > 0, \quad (3.54)$$

Applying the Borel-Cantelli lemma with $n \geq 1$ fixed, the independence in k and (3.54) implies

$$P(Z_{n,k}(a) = 0 \text{ i.o. in } k) = 1. \quad (3.55)$$

and hence $P(\inf_{k \geq 1} Z_{n,k}(a) = 0) = 1$. Thus (3.55) implies (3.53) with probability one, which proves the proposition.

Acknowledgement: It is a pleasure to thank the referees for a careful reading of the manuscript. Their comments and suggestions led to a number of improvements in the exposition.

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