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# Optimal Convergence Rates and One-Term Edgeworth Expansions for Multidimensional Functionals of Gaussian Fields

**Simon Campese**

Université du Luxembourg
Faculté des Sciences, de la Technologie et de la Communication
Unité de Recherche en Mathémathiques
6, rue Coudenhove Kalergi, 1359 Luxembourg
*E-mail address*: simon.campese@uni.lu

**Abstract.** We develop techniques for determining the exact asymptotic speed of convergence in the multidimensional normal approximation of smooth functions of Gaussian fields. As a by-product, our findings yield exact limits and often give rise to one-term generalized Edgeworth expansions increasing the speed of convergence. Our main mathematical tools are Malliavin calculus, Stein's method and the Fourth Moment Theorem. This work can be seen as an extension of the results of Nourdin and Peccati (2009a) to the multi-dimensional case, with the notable difference that in our framework covariances are allowed to fluctuate. We apply our findings to exploding functionals of Brownian sheets, vectors of Toeplitz quadratic functionals and the Breuer-Major Theorem.

## 1. Introduction

Let $X$ be an isonormal Gaussian process on some real, separable Hilbert space $\mathcal{H}$ and $(F_n)$ be a sequence of centered, real-valued functionals of $X$ with converging covariances. Moreover, assume that $F_n \xrightarrow{\mathcal{L}} Z$, where $Z$ is a centered Gaussian random variable and $\xrightarrow{\mathcal{L}}$ denotes convergence in law. In Nourdin and Peccati (2009b), Nourdin and Peccati used a combination of Stein's method (see Nourdin and Peccati (2012); Chen et al. (2011); Chen and Shao (2005); Reinert (2005); Stein (1986, 1972)) and Malliavin calculus (see Nourdin and Peccati (2012); Nualart (2006);

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Janson (1997)) to derive the bound

$$|\mathrm{E}[g(F_n)] - \mathrm{E}[g(Z)]| \leq M(g)\, \varphi(F_n) \tag{1.1}$$

and used it to prove estimates for several probabilistic distances $d(F_n, Z)$ (among them the Fortet-Mourier, Kolmogorov and Wasserstein distances). The quantity $\varphi(F_n)$ in the bound (1.1) is defined by

$$\varphi(F_n) = \sqrt{\operatorname{Var} \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}} + \left| \mathrm{E}\left[F_n^2\right] - \mathrm{E}\left[Z^2\right] \right|,$$

where $g$ is a sufficiently smooth function, $M(g)$ is a constant depending on $g$ and the random variable $\left\langle DF_n, -DL^{-1}F_n \right\rangle_{\mathfrak{H}}$ involves the Malliavin derivative operator $D$ and the pseudo-inverse $L^{-1}$ of the Ornstein Uhlenbeck generator $L$ (see for example Nourdin and Peccati (2012) or Nualart (2006) for definitions).

This approach was pushed further by the same two authors in Nourdin and Peccati (2009a). Disregarding technicalities, they showed that if

$$\left( F_n, \frac{\left\langle DF_n, -DL^{-1}F_n \right\rangle_{\mathfrak{H}} - \mathrm{E}\left\langle DF_n, -DL^{-1}F_n \right\rangle_{\mathfrak{H}}}{\sqrt{\operatorname{Var} \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}}} \right)$$

jointly converges in law to a Gaussian random vector $(Z_1, Z_2)$, it holds that

$$\frac{P(F_n \leq z) - \Phi(z)}{\varphi(F_n)} = \frac{\mathrm{E}\left[1_{(-\infty, z]}(F_n)\right] - \Phi(z)}{\varphi(F_n)} \to \frac{\rho}{3} \Phi^{(3)}(z), \tag{1.2}$$

where $\rho = \mathrm{E}[Z_1 Z_2]$ and $\Phi$ is the cumulative distribution function of the Gaussian random variable $Z$ and $\Phi^{(3)}$ denotes its third derivative, thus providing exact asymptotics for the difference $P(F_n \leq z) - \Phi(z)$.

Recently, Nourdin, Peccati and Réveillac showed in Nourdin et al. (2010b) that the bound (1.1) also has a multidimensional version. It can still be written as

$$|\mathrm{E}[g(F_n)] - \mathrm{E}[g(Z)]| \leq M(g)\, \varphi(F_n) \tag{1.3}$$

but now the functionals $F_n$ and the normal $Z$ are $\mathbb{R}^d$-valued and the function $g$ has to be in $\mathcal{C}^2(\mathbb{R}^d)$ with bounded first and second derivatives. Moreover, the quantities $\varphi(F_n)$ are now given by $\varphi(F_n) = \Delta_\Gamma(F_n) + \Delta_C(F_n)$, where

$$\Delta_\Gamma(F_n) = \sqrt{\sum_{i,j=1}^{d} \operatorname{Var} \Gamma_{ij}(F_n)},$$

$$\Delta_C(F_n) = \sqrt{\sum_{i,j=1}^{d} \left(\mathrm{E}[F_{i,n} F_{j,n}] - \mathrm{E}[Z_i Z_j]\right)^2}$$

and $\Gamma_{ij}(F_n) = \left\langle DF_{i,n}, -DL^{-1}F_{j,n} \right\rangle_{\mathfrak{H}}$. As every Lipschitz function can be approximated by $\mathcal{C}^2$ functions with bounded derivatives up to order two, (1.3) yields an upper bound for the Wasserstein-distance (see Nourdin et al. (2010b)), which, to the knowledge of the author, is the strongest distance that has been achieved via an approach based on Stein's method (see the discussion before Theorem 4 in Chatterjee and Meckes (2008)). One should note that, using methods of Malliavin calculus, it is however possible to prove that, in several cases, the central limit theorems implied by the bound (1.3) take place in the total variation distance (see Nourdin and Poly (2013, Theorem 5.2)). One should also note that another bound for the difference on the left hand side of (1.1) is given by the maximum of the third and fourth cumulants of $F_n$ (see Biermé et al. (2012)) and that this bound is in fact optimal in total variation distance, if the sequence $(F_n)$ lives in a fixed Wiener chaos (see Nourdin and Peccati (2013)).

The main result of this paper is Theorem 3.2, which provides exact asymptotics for the difference

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z_n)\right]}{\varphi(F_n)}, \tag{1.4}$$

where $(Z_n)$ is a sequence of Gaussian random vectors that has the same covariance structure as $(F_n)$. Analogously to the one-dimensional case, the random sequences $\left(F_n, \widetilde{\Gamma}_{ij}(F_n)\right)_{n\geq 1}$, where $\widetilde{\Gamma}_{ij}(F_n)$ is a normalized version of $\Gamma_{ij}(F_n)$, will play a crucial role.

Assuming converging covariances, we are able to obtain an exact and explicit limit for the quantity (1.4), where the Gaussian sequence $(Z_n)$ is replaced by a single Gaussian vector $Z$. This is Theorem 3.4, which can be seen as a multi-dimensional analogue to (1.2). As a by-product, we obtain the optimality of $\varphi(F_n)$ for the Wasserstein distance $d_W$, by which we mean the existence of positive constants $c_1$ and $c_2$ such that

$$c_1 \leq \frac{d_W(F_n, Z)}{\varphi(F_n)} \leq c_2$$

for $n \geq n_0$. Note that the mere existence of these constants is not hard to prove. Indeed, a suitable upper bound $c_2$ can always obtained from (1.3) and by choosing $g$ in (1.3) to depend only on one coordinate, the problem of finding lower bounds can essentially be reduced to the one-dimensional findings of Nourdin and Peccati (2009a).

Taking these results a step further, we provide one-term generalized Edgeworth expansions that speed up the convergence of $(\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z_n)\right])$ (or the respective sequence with $Z_n$ replaced by $Z$ in the converging variances case) towards zero.

As an important special case, we apply Theorems 3.2 and 3.4 and their implications to random sequences $(F_n)$, whose components are elements of some Wiener chaos (that can vary by component). In this case, the sufficient conditions for our results simplify substantially and can exclusively be expressed in terms of contractions of the respective kernels (or even cumulants in the case of the second chaos). In many cases, the only contractions one has to look at are those where all kernels are taken from the same component of $F_n$, in the spirit of part (B) of the Fourth Moment Theorem 2.8.

The remainder of the paper is organized as follows. In the preliminary Section 2, we introduce the necessary mathematical theory and gather some results from the existing literature. Our main results in a general framemork are presented in Section 3. In the following Section 4, these results are then specialized to the case where all components of $(F_n)$ are multiple integrals. We conclude by applying our methods to several examples, namely step functions, exploding integrals of Brownian sheets, continuous time Toeplitz quadratic functionals and the Breuer-Major Theorem.

## 2. Preliminaries

2.1. *Metrics for probability measures and asymptotic normality.* We fix a positive integer $d$ and denote by $\mathcal{P}(\mathbb{R}^d)$ the set of all probability measures on $\mathbb{R}^d$. If $X$ is a $\mathbb{R}^d$-valued random vector, we denote its law by $P_X$. If $(P_n) \subset \mathcal{P}(\mathbb{R}^d)$ is a sequence of probability measures, weakly converging to some limit $P$, we can always find an almost surely converging sequence $(X_n)$ of $\mathbb{R}^d$-valued random vectors, such that $X_n$ has law $P_n$. This is the well-known Skorokhod representation theorem, which we will state here for convenience.

**Theorem 2.1** (Skorokhod representation theorem, Skorohod (1956))**.** *Let $(P_n)_{n\geq 0} \subset \mathcal{P}(\mathbb{R}^d)$ be a a sequence of probability measures such that $P_n \xrightarrow{\mathcal{L}} P_0$. Then there exists a sequence $(X_n)_{n\geq 0}$ of $\mathbb{R}^d$-valued random vectors, defined on some common probability space $(\Omega^*, \mathcal{F}^*, P^*)$, such that $P_{X_n} = P_n$ and $X_n \to X$ $P$-almost surely.*

Given a metric $\gamma$ on $\mathcal{P}(\mathbb{R}^d)$, we say that $\gamma$ metrizes the weak convergence on $\mathcal{P}(\mathbb{R}^d)$, if for all $P \in \mathcal{P}(\mathbb{R}^d)$ and sequences $(P_n) \subseteq \mathcal{P}(\mathbb{R}^d)$ the following equivalence holds:

$$\gamma(P_n, P) \to 0 \quad \Leftrightarrow \quad P_n \xrightarrow{\mathcal{L}} P.$$

Two prominent examples are the Prokhorov metric $\rho$ and the Fortet-Mourier metric $\beta$, defined by

$$\rho(P,Q) = \inf\left\{\varepsilon > 0\colon P(A) \leq Q(A^\varepsilon) + \varepsilon \quad \text{for every Borel set } A \subseteq \mathbb{R}^d\right\},$$

and

$$\beta(P,Q) = \sup\left\{ \left| \int f \,\mathrm{d}(P-Q) \right| : \|f\|_\infty + \|f\|_L \leq 1 \right\}.$$

Here, $A^\varepsilon = \{x\colon \|x-y\| \leq \varepsilon \text{ for some } y \in A\}$, $\|\cdot\|$ is the $\varepsilon$-hull with respect to the Euclidean norm and $\|\cdot\|_L$ denotes the Lipschitz seminorm. For double sequences of probability measures whose elements are asymptotically close with respect to one of these two metrics, a result similar to the Skorokhod Representation Theorem 2.1 holds.

**Theorem 2.2** (Dudley (2002), Th. 11.7.1)**.** *Let $(P_n)_{n\geq 1}$, $(Q_n)_{n\geq 1} \subseteq \mathcal{P}(\mathbb{R}^d)$ be two sequences of probability measures. Then the following three conditions are equivalent.*

*a)* $\beta(P_n, Q_n) \to 0$

*b)* $\rho(P_n, Q_n) \to 0$

*c) There exist two sequences $(X_n)$ and $(Y_n)$ of $\mathbb{R}^d$-valued random vectors, defined on some common probabilty space $(\Omega^*, \mathcal{F}^*, P^*)$, such that $P_{X_n} = P_n$ and $P_{Y_n} = Q_n$ for $n \geq 1$ and $X_n - Y_n \to 0$ $P$-almost surely.*

Note that the Skorokhod Representation Theorem 2.1 is not a simple corollary of Theorem 2.2. Also, the distances $\beta$ and $\rho$ can not easily be replaced by other metrics (see Dudley (2002, p.418) for details and counterexamples). Theorem 2.2 is the motivation for our following definition of asymptotically close normality.

**Definition 2.3.** Let $(X_n)$ be a sequence of $\mathbb{R}^d$-valued random vectors with finite first and second moments. We say that $(X_n)$ is *asymptotically close to normal* (in short: *ACN*), if

$$\beta(P_{X_n}, P_{Z_n}) \to 0,$$

where the probabilty measures $P_{Z_n}$ are laws of $d$-dimensional Gaussian random variables $Z_n$, whose first and second moments coincide with those of $X_n$.

Note that we consider (almost surely) constant random vectors as being "degenerated" Gaussians. Thus, by the above definition, all sequences of random vectors whose second moments eventually vanish are ACN. By Theorem 2.2, we could of course replace the Fortet-Mourier metric $\beta$ with the Prokhorov metric $\rho$. It is clear that if $(X_n)$ is ACN, the same is true for any of its components $(X_{i,n})$. Furthermore, if all first and second moments of $(X_n)$ converge (or, as a special case, are equal), being ACN is equivalent to converging in law to a Gaussian random variable $Z$ (with the limiting moments as parameters). Indeed, the triangle inequality gives

$$\rho(P_{X_n}, P_Z) \leq \rho(P_{X_n}, P_{Z_n}) + \rho(P_{Z_n}, P_Z).$$

We will use the following asymptotic notation for two positive sequences $(a_n)$ and $(b_n)$ throughout the text. We write$(a_n) \preccurlyeq (b_n)$, if there exists a positive constant $c$ such that $a_n \leq c\, b_n$ for $n \geq n_0$ and $(a_n) \asymp (b_n)$, if $(a_n) \preccurlyeq (b_n)$ and $(b_n) \preccurlyeq (a_n)$ holds. For brevity, we often drop the braces and just write $a_n \preccurlyeq b_n$, $a_n \asymp b_n$, etc.

2.2. *Hermite polynomials, integration by parts and the transformation $U_{g,C}$.* Fix a positive integer $d$. Elements of the set $\mathbb{N}_0^d$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, will be called ($d$-dimensional) multi-indices. We define $|\alpha| = \sum_{i=1}^d \alpha_i$ and call this sum the order of $\alpha$. For a $d$-dimensional vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define $x^\alpha := \prod_{i=1}^d x_i^{\alpha_i}$. Multi-indices of order one will sometimes be denoted by $e_i$, where the index $i$ marks the position of the non-zero entry. Thus, for example, $x^{e_i} = x_i$. It is clear that every multi-index $\alpha$ can be written as a sum of $|\alpha|$ multi-indices $l_1, \dots, l_{|\alpha|}$ of order one, and that this sum is unique up to the order of the summands. We will call the set $\{l_1, \dots, l_{|\alpha|}\}$ of these multi-indices the *elementary decomposition* of $\alpha$. For example, the elementary decomposition for the multi-index $(2, 0, 1)$ is $\{(1, 0, 0), (1, 0, 0), (0, 0, 1)\}$.

For any multi-index $\alpha$, the multidimensional *Hermite polynomials* $H_\alpha(x, \mu, C)$ are defined by

$$H_\alpha(x, \mu, C) = \frac{(-1)^{|\alpha|} \partial_\alpha \phi_d(x, \mu, C)}{\phi_d(x, \mu, C)}, \tag{2.1}$$

where $\phi_d(x, \mu, C)$ denotes the density of a $d$-dimensional Gaussian random variable with mean vector $\mu$ and positive definite covariance matrix $C$ (see for example McCullagh (1987, Section 5.4)). Note that in the case $\mu = 0$ and $d = C = 1$, this definition yields the well known one-dimensional Hermite polynomials. The first few multidimensional Hermite polynomials are given by $H_0(x, \mu, C) = 1$,

$$H_{e_i}(x, \mu, C) = \sum_{k=1}^d c_{ik}\, (x_k - \mu_k)$$

and

$$H_{e_i+e_j}(x, \mu, C) = H_{e_i}(x, \mu, C)\, H_{e_j}(x, \mu, C) - c_{ij},$$

where $1 \leq i, j \leq d$ and $C^{-1} = (c_{ij})_{1 \leq i, j \leq d}$ denotes the inverse of $C$.

The polynomial $H_\alpha(x, \mu, C)$ is of order $|\alpha|$ and one can show that for fixed $\mu$ and $C$, the family $\{H_\alpha(x, \mu, C) \colon \alpha \in \mathbb{N}_0^d\}$ is orthogonal in $L^2(\mathbb{R}^d, \phi_d(x, \mu, C))$.

Furthermore, by integration by parts, we obtain the identity

$$\mathrm{E}\left[\partial_i f(Z)\, H_\alpha(Z,\mu,C)\right] = \mathrm{E}\left[f(Z)\, H_{\alpha+e_i}(Z,\mu,C)\right], \tag{2.2}$$

where $1 \leq i \leq d$, $\alpha \in \mathbb{N}_0^d$, $f \in \mathrm{Lip}(\mathbb{R}^d)$ with at most polynomial growth and $Z$ is a Gaussian random variable with mean $\mu$ and covariance matrix $C$. Note that the left hand side of (2.2) is well-defined by Rademacher's theorem, which guarantees the differentiability of the Lipschitz continuous function $f$ almost everywhere. We will also make use of another integration by parts formula, which can be verified by direct calculation, namely

$$\mathrm{E}\left[f(Z)\, Z_i\right] = \sum_{j=1}^{d} \mathrm{E}\left[Z_i Z_j\right] \mathrm{E}\left[\partial_j f(Z)\right], \tag{2.3}$$

where $1 \leq i \leq d$, $f$ as above and $Z$ a $d$-dimensional Gaussian random variable (with possibly singular covariance matrix).

For a given Lipschitz function $g\colon \mathbb{R}^d \to \mathbb{R}$ and a positive semi-definite and symmetric matrix $C$ of dimension $d \times d$, we define $U_{g,C}\colon \mathbb{R}^d \to \mathbb{R}$ by

$$U_{g,C}(x) = \int_a^b \frac{\upsilon'(t)}{\upsilon(t)} \left( \mathrm{E}\left[g(N)\right] - \mathrm{E}\left[ g\left( \upsilon(t)x + \sqrt{1-\upsilon^2(t)}N \right)\right]\right) \mathrm{d}t, \tag{2.4}$$

where $N$ is a $d$-dimensional centered Gaussian random variable with covariance $C$, $-\infty \leq a < b \leq \infty$ and $\upsilon\colon (a,b) \to (0,1)$ is a diffeomorphism with $\lim_{t\to a+} \upsilon(t) = 0$ (and therefore $\lim_{t \to b-} \upsilon(t) = 1$). From the change of variables $\upsilon(t) = s$, we see that $U_{g,C}$ does not depend on the particular choice of $\upsilon$ and by choosing $\upsilon(t) = \mathrm{e}^{-t}$ on the interval $(0,\infty)$, we can write

$$U_{g,C}(x) = \int_0^\infty \left(P_t g(x) - P_\infty g(x)\right) \mathrm{d}t,$$

where $P_t g(x) = \mathrm{E}\left[g\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}N\right)\right]$, $P_\infty g(x) := \lim_{t\to\infty} P_t g(x) = \mathrm{E}\left[g(N)\right]$ and $N$ is defined as above. The operators $P_t$ form the well-known Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$ (see Nourdin and Peccati (2012, Chapter 1) for details).

Before stating some properties of $U_{g,C}$, let us introduce some more notation. If $f \in C^k(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$ denotes a multi-index with elementary decomposition $\{l_1, l_2, \dots, l_{|\alpha|}\}$, we write $\partial_\alpha f$ or $\partial_{l_1 l_2 \cdots l_{|\alpha|}} f$ instead of the more cumbersome $\frac{\partial^{|\alpha|} f}{\partial x_{l_1} \partial x_{l_2} \cdots \partial x_{l_{|\alpha|}}}$.

**Lemma 2.4.** *Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz-function with at most polynomial growth. Furthermore, let $Z$ be a centered, $d$-dimensional Gaussian random variable with covariance matrix $C$ and define $U_{g,C}$ via (2.4). Then the following is true.*

*a) The function $U_{g,C}$ satisfies the multidimensional Stein equation*

$$\left\langle C, \operatorname{Hess} U_{g,C}(x) \right\rangle_{H.S.} - \left\langle x, \nabla U_{g,C}(x)\right\rangle_{\mathbb{R}^d} = g(x) - \mathrm{E}\left[g(Z)\right].$$

*b) If $g$ is $k$-times differentiable with bounded derivatives up to order $k$, the same is true for $U_{g,C}$. In this case, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, the derivatives are given by*

$$\partial_\alpha U_{g,C}(x) = \int_a^b \upsilon'(t)\, \upsilon^{|\alpha|-1}(t)\ \mathrm{E}\left[\partial_\alpha g\left(\upsilon(t)x + \sqrt{1-\upsilon^2(t)}N\right)\right] \mathrm{d}t \tag{2.5}$$

*and it holds that*

$$|\partial_\alpha U_{g,C}(x)| \leq \frac{\|\partial_\alpha g\|_\infty}{|\alpha|} \tag{2.6}$$

*and*

$$\mathrm{E}\left[\partial_\alpha U_{g,C}(Z)\right] = \frac{1}{|\alpha|}\,\mathrm{E}\left[\partial_\alpha g(Z)\right]. \tag{2.7}$$

*Proof*: For a proof of part a) see Nourdin et al. (2010b). Repeated differentiation under the integral sign (the first one being justified by the Lipschitz property of $g$) shows formula (2.5), of which the bound (2.6) is an immediate consequence. To show (2.7), we again use formula (2.5) and the fact that $\upsilon(t)Z + \sqrt{1-\upsilon^2(t)}N$ has the same law as $Z$. This gives

$$\begin{aligned}
\mathrm{E}\left[\partial_\alpha U_{g,C}(Z)\right] &= \int_a^b \upsilon'(t)\upsilon^{|\alpha|-1}(t)\,\mathrm{E}\left[\partial_\alpha g\left(\upsilon(t)Z + \sqrt{1-\upsilon^2(t)}N\right)\right]\mathrm{d}t \\
&= \int_a^b \upsilon'(t)\upsilon^{|\alpha|-1}(t)\,\mathrm{d}t\,\mathrm{E}\left[\partial_\alpha g\left(Z\right)\right] \\
&= \frac{1}{|\alpha|}\,\mathrm{E}\left[\partial_\alpha g\left(Z\right)\right].
\end{aligned}$$

□

2.3. *Isonormal Gaussian processes and Wiener chaos.* For a detailed discussion of the notions introduced in this section, we refer to Nourdin and Peccati (2012) or Nualart (2006).

Fix a real separable Hilbert space $\mathfrak{H}$ and a family $X = \{X(h) \colon h \in \mathfrak{H}\}$ of centered Gaussian random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$, such that the isometry property $\mathrm{E}\left[X(g)X(h)\right] = \langle g, h\rangle_{\mathfrak{H}}$ holds for $g, h \in \mathfrak{H}$. Such a family $X$ is called an *isonormal Gaussian process* over $\mathfrak{H}$. Without loss of generality, we assume that the $\sigma$-field $\mathcal{F}$ is generated by $X$. For $q \geq 1$, we denote the $q$th tensor product of $\mathfrak{H}$ by $\mathfrak{H}^{\otimes q}$ and the $q$th symmetric tensor product of $\mathfrak{H}$ by $\mathfrak{H}^{\odot q}$. Furthermore, we define $\mathcal{H}_q$, the *Wiener chaos of order* $q$ (with respect to $X$), to be the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the set $\{H_q(X(h)) \colon h \in \mathfrak{H},\ \|h\|_{\mathfrak{H}} = 1\}$, where $H_q(x) = H_q(x, 1)$ denotes the $q$th Hermite polynomial, defined by (2.1). The mapping $I_q(h^{\otimes q}) = q!H_q(X(h))$, where $\|h\|_{\mathfrak{H}} = 1$, can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$, equipped with the modified norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$, and the $q$th Wiener chaos $\mathcal{H}_q$. Wiener chaoses of different orders are orthogonal. More precisely, if $f_i \in \mathfrak{H}^{\odot q_i}$ and $f_j \in \mathfrak{H}^{\odot q_j}$ for $q_i, q_j \geq 1$ it holds that

$$\mathrm{E}\left[I_{q_i}(f_i)\,I_{q_j}(f_j)\right] = \begin{cases} q_i!\,\langle f_i, f_j\rangle_{\mathfrak{H}} & \text{if } q_i = q_j \\ 0 & \text{if } q_i \neq q_j. \end{cases} \tag{2.8}$$

Furthermore, the *Wiener chaos decomposition* tells us that the space $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the $\mathcal{H}_q$. As a consequence, any square-integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$F = \mathrm{E}\left[F\right] + \sum_{q=1}^{\infty} I_q(f_q). \tag{2.9}$$

where the kernels $f_q \in \mathfrak{H}^{\odot q}$ are uniquely defined. This identity is called the *chaos expansion* of $F$.

If $\{\psi_k\colon k \geq 1\}$ is a complete orthonormal system in $\mathfrak{H}$, $f_i \in \mathfrak{H}^{\odot q_i}$, $f_j \in \mathfrak{H}^{\odot q_j}$ and $r \in \{0, \dots, q_i \wedge q_j\}$, the *contraction* $f_i \otimes_r f_j$ of $f_i$ and $f_j$ of order $r$ is the element of $\mathfrak{H}^{\otimes(q_i+q_j-2r)}$ defined by

$$f_i \otimes_r f_j = \sum_{l_1,\dots,l_r=1}^{\infty} \langle f_i, \psi_{l_1} \otimes \dots \otimes \psi_{l_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle f_j, \psi_{l_1} \otimes \dots \otimes \psi_{l_r} \rangle_{\mathfrak{H}^{\otimes r}} . \tag{2.10}$$

The contraction $f_i \otimes_r f_j$ is not necessarily symmetric. We denote its canonical symmetrization by $f_i \widetilde{\otimes}_r f_j \in \mathfrak{H}^{\odot q_i+q_j-2r}$. Note that $f_i \otimes_0 f_j$ is equal to the usual tensor product $f_i \otimes f_j$ of $f_i$ and $f_j$. Furthermore, if $q_i = q_j$, we have that $f_i \otimes_{q_i} f_j = \langle f_i, f_j \rangle_{\mathfrak{H}^{\otimes q_i}}$. The well known multiplication formula

$$I_{q_i}(f_i)\, I_{q_j}(f_j) = \sum_{r=0}^{q_i \wedge q_j} \beta_{q_i,q_j}(r)\, I_{q_i+q_j-2r}(f_i \widetilde{\otimes}_r f_j), \tag{2.11}$$

where

$$\beta_{a,b}(r) = r! \binom{a}{r} \binom{b}{r}, \tag{2.12}$$

gives us the chaos expansion of the product of two multiple integrals.

When $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, where $(A, \mathcal{A})$ is a Polish space, $\mathcal{A}$ is the associated Borel $\sigma$-field and the measure $\mu$ is positive, $\sigma$-finite and non-atomic, one can identify the symmetric tensor product $\mathfrak{H}^{\odot q}$ with the Hilbert space $L^2_s(A^q, \mathcal{A}^q, \nu^{\otimes q})$, which is defined as the collection of all $\nu^{\otimes q}$-almost everywhere symmetric functions an $A^q$, that are square-integrable with respect to the product measure $\nu^{\otimes q}$. In this case, the random variable $I_q(h)$, $h \in \mathfrak{H}^{\odot q}$, coinicides with the multiple Wiener-Itô integral of order $q$ of $h$ with respect to the Gaussian measure $B \mapsto X(1_B)$, where $B \in \mathcal{A}$ and $\nu(A) < \infty$. Furthermore, the contraction (2.10) can be written as

$$\begin{aligned}(f_i \otimes_r f_j)&(t_1, \dots, t_{q_i+q_j-2r}) \\ &= \int_{A^r} f_i(t_1, \dots, t_{q_i-r}, s_1, \dots, s_r) f_j(t_{q_i-r+1}, \dots, t_{q_i+q_j-2r}, s_1, \dots, s_r) \\ &\qquad\qquad \mathrm{d}\nu(s_1) \dots \mathrm{d}\nu(s_r).\end{aligned} \tag{2.13}$$

2.4. *Operators from Malliavin calculus.* In this section, we introduce the operators $D$, $L$ and $L^{-1}$ from Malliavin calculus, which will appear in the statements of our main results. This exposition is by no means complete, most notably, we do not introduce the divergence operator. Again, we refer to Nourdin and Peccati (2012) or Nualart (2006) for a full discussion.

If $\mathcal{S}$ is the set of all cylindrical random variables of the type

$$F = g(X(h_1), \dots, X(h_k)),$$

where $k \geq 1$, $h_i \in \mathfrak{H}$ for $1 \leq i \leq k$ and $g\colon \mathbb{R}^k \to \mathbb{R}$ is an infinitely differentiable function with compact support, the Malliavin derivative $DF$ with respect to $X$ is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$DF = \sum_{i=1}^{k} \partial_i(X(h_1), \dots, X(h_k)) h_i.$$

Iterating this procedure, we obtain higher derivatives $D^m F$ for any $m \geq 2$, which are elements of $L^2(\Omega, \mathfrak{H}^{\odot m})$. For $m, p \geq 1$, $\mathbb{D}^{m,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{m,p}$, which is defined by

$$\|F\|_{m,p}^p = \mathrm{E}\left[|F|^p\right] + \sum_{i=1}^m \mathrm{E}\left[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right].$$

If $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, with $\nu$ non-atomic, the Malliavin derivative of a random variable $F$ having the chaos expansion (2.9) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_t F = \sum_{q=1}^\infty q I_{q-1}(f_q(\cdot, t)), \quad t \in A. \tag{2.14}$$

The *Ornstein-Uhlenbeck generator* $L$ is defined by $L = \sum_{q=0}^\infty -qJ_q$. Here, $J_q$ denotes the orthogonal projection onto the $q$th Wiener chaos. The domain of $L$ is $\mathbb{D}^{2,2}$. Similarly, we define its pseudo-inverse $L^{-1}$ by $L^{-1} = \sum_{q=1}^\infty -\frac{1}{q} J_q$. This pseudo-inverse is defined on $L^2(\Omega)$ and for any $F \in L^2(\Omega)$ it holds that $L^{-1}F$ lies in the domain of $L$. The name pseudo-inverse is justified by the relation

$$LL^{-1}F = F - \mathrm{E}[F],$$

valid for any $F \in L^2(\Omega)$.

2.5. *Cumulants.* Recall the multi-index notation introduced in the first paragraph of Section 2.2.

Let $F = (F_1, \dots, F_d)$ be a $\mathbb{R}^d$-valued random vector. For a multi-index $\alpha$, we set $F^\alpha = \prod_{k=1}^d F_k^{\alpha_k}$ and, with slight abuse of notation, $|F|^\alpha = \prod_{k=1}^d |F_k|^{\alpha_k}$. The moments $\mu_\alpha(F)$ of $F$ of order $|\alpha|$ are then defined by $\mu_\alpha(F) = \mathrm{E}[F^\alpha]$, provided that the expectation on the right hand side is finite. Analogously, one defines the absolute moments $\mu_\alpha(|F|)$. We denote by $\phi_F(t) = \mathrm{E}[\exp(\mathrm{i}\langle t, F\rangle_{\mathbb{R}^d})]$ the characteristic function of $F$. If $\mu_\alpha(|F|) < \infty$, the *joint cumulant* $\kappa_\alpha(F)$ of order $|\alpha|$ of $F$ is defined by

$$\kappa_\alpha(F) = (-\mathrm{i})^{|\alpha|} \partial_\alpha \log \phi_F(t)|_{t=0}.$$

Given all joint cumulants $\kappa_\alpha(F)$ up to some order exist, we can compute the moments up to the same order by Leonov and Shiryaev's formula (see Peccati and Taqqu (2011, Proposition 3.2.1))

$$\mu_\alpha(F) = \sum_\pi \kappa_{b_1}(F)\kappa_{b_2}(F)\cdots\kappa_{b_m}(F), \tag{2.15}$$

where the sum is taken over all partitions $\pi = \{B_1, \dots, B_m\}$ of the elementary decomposition of $\alpha$ and the multi-indices $b_k$ are defined by $b_k = \sum_{l_j \in B_k} l_j$. For example, if $1 \le i, j, k \le d$, we get $\mu_{e_i}(F) = \kappa_{e_i}(F)$, $\mu_{e_i+e_j}(F) = \kappa_{e_i+e_j}(F) + \kappa_{e_i}(F)\kappa_{e_j}(F)$ and

$$\begin{aligned}\mu_{e_i+e_j+e_k}(F) &= \kappa_{e_i+e_j+e_k}(F) \\ &\quad + \kappa_{e_i}(F)\kappa_{e_j+e_k}(F) + \kappa_{e_j}(F)\kappa_{e_i+e_k}(F) + \kappa_{e_k}(F)\kappa_{e_i+e_j}(F) \\ &\quad + \kappa_{e_i}(F)\kappa_{e_j}(F)\kappa_{e_k}(F)\end{aligned}$$

for the moments of order one, two and three, respectively. Note that if $F$ is centered, all moments of order less than four coincide with the respective cumulants.

2.6. *Generalized Edgeworth expansions.* Let $F_1$ and $F_2$ be two $\mathbb{R}^d$-valued random vectors with finite absolute moments up to some order $m \in \mathbb{N}_0 \cup \{\infty\}$ and consider the problem of approximating $F_1$ in terms of $F_2$. The classical Edgeworth expansion provides such an approximation in terms of formal "moments", which we will now describe.

For every multi-index $\alpha$ of order at most $m$, we define formal "cumulants" $\widetilde{\kappa}_\alpha(F_1, F_2)$ by $\widetilde{\kappa}_\alpha(F_1, F_2) = \kappa_\alpha(F_1) - \kappa_\alpha(F_2)$ and use Shiryaev's formula (2.15) to define corresponding formal "moments" $\widetilde{\mu}_\alpha(F_1, F_2)$. Two things are important to note at this point. In general, $\widetilde{\mu}_\alpha(F_1, F_2) \neq \mu_\alpha(F_1) - \mu_\alpha(F_2)$ and the collection $\{\widetilde{\kappa}_\alpha(F_1, F_2) \colon |\alpha| \leq m\}$ can not be represented as cumulants associated with some random variable. If we now assume that $F_1$ and $F_2$ both have densities, say $f_1$ and $f_2$, the classical Edgeworth expansion of order $m$ for the density $f_1$ then reads

$$f_1(x) \sim f_2(x) + \sum_{1 \leq |\alpha| \leq m} \frac{(-1)^{|\alpha|}}{|\alpha|!} \, \widetilde{\mu}_\alpha(F_1, F_2) \partial_\alpha f_2(x). \tag{2.16}$$

In the most prominent example where this is the case, $F_1$ is a normalized sum of iid random variables and $F_2$ is Gaussian. In this case, the Edgeworth expansion can be used to improve the speed of convergence in the classical central limit theorem. For details, we refer to Hall (1992), McCullagh (1987, Chapter5) and Bhattacharya and Ranga Rao (1986).

For our framework, however, the classical Edgeworth expansion is too rigid, as we cannot assume the existence of (smooth) densities. Therefore, instead of expanding the density $f_1$ in terms of $f_2$ and its derivatives, we pass to the distributional operators $g \mapsto \mathrm{E}\left[g(F_1)\right]$ and $g \mapsto \mathrm{E}\left[g(F_2)\right]$, defined on the space of infinitely differentiable functions with compact support. The expansion (2.16) becomes

$$\mathrm{E}\left[g(F_1)\right] \sim \mathrm{E}\left[g(F_2)\right] + \sum_{1 \leq |\alpha| \leq m} \frac{\widetilde{\mu}_\alpha(F_1, F_2)}{|\alpha|!} \, \mathrm{E}\left[\partial_\alpha g(F_2)\right]. \tag{2.17}$$

In the case of existing smooth densities it holds that $\mathrm{E}\left[g(F_1)\right] = \int_{-\infty}^{\infty} g(x) f_1(x) \, \mathrm{d}x$ and, by integration by parts,

$$\mathrm{E}\left[\partial_\alpha g(F_2)\right] = \int_{-\infty}^{\infty} \partial_\alpha g(x) f_2(x) \, \mathrm{d}x = (-1)^{|\alpha|} \int_{-\infty}^{\infty} g(x) \partial_\alpha f_2(x) \, \mathrm{d}x,$$

so that (2.17) is obtained in a natural way from (2.16), by multiplying with the test function $g$ and integrating on both sides.

This leads us to the following definition of a generalized Edgeworth expansion.

**Definition 2.5** (Generalized Edgeworth expansion)**.** If $g$ is $m$-times differentiable and has bounded derivatives up to order $m$, we define the *generalized* $m$*th order Edgeworth expansion* $\mathcal{E}_m(F_1, F_2, g)$ of $\mathrm{E}\left[g(F_1)\right]$ around $\mathrm{E}\left[g(F_2)\right]$ by

$$\mathcal{E}_m(F_1, F_2, g) = \mathrm{E}\left[g(F_2)\right] + \sum_{1 \leq |\alpha| \leq m} \frac{\widetilde{\mu}_\alpha(F_1, F_2)}{|\alpha|!} \, \mathrm{E}\left[\partial_\alpha g(F_2)\right]. \tag{2.18}$$

If $Z$ is a $d$-dimensional centered normal with covariance matrix $C$ (the case that we will exclusively consider in the sequel), formula (2.2) yields

$$\mathcal{E}_m(F_1, Z, g) = \mathrm{E}\left[g(Z)\right] + \sum_{1 \leq |\alpha| \leq m} \frac{\widetilde{\mu}_\alpha(F_1, F_2)}{|\alpha|!} \, \mathrm{E}\left[g(Z) H_\alpha(Z, C)\right],$$

where the Hermite polynomials $H_\alpha(x, C)$ are defined by (2.1) (recall our convention that we drop the mean as an argument if it is zero).

2.7. *Cumulant formulas for chaotic random vectors.* When dealing with functionals of an isonormal Gaussian process, their cumulants can be generalized in terms of Malliavin operators. This (among other things) is the content of Nourdin and Peccati (2010) and Noreddine and Nourdin (2011) (see also Nourdin and Peccati (2012, Chapter8)), which we will summarize here.

Let $F = (F_1, \dots, F_d)$ be a $\mathbb{R}^d$-valued random vector whose components are functionals of some isonormal Gaussian process $X$ and let $l_1, l_2, \dots$ be a sequence of $d$-dimensional multi-indices of order one. If $F^{l_1} \in \mathbb{D}^{1,2}$, we set $\Gamma_{l_1}(F) = F_i$. Inductively, if $\Gamma_{l_1,l_2,\dots,l_k}(F)$ is a well-defined element of $L^2(\Omega)$ for some $k \geq 1$, we define

$$\Gamma_{l_1,\dots,l_{k+1}}(F) = \left\langle DF_{l_{k+1}}, -DL^{-1}\Gamma_{l_1,\dots,l_k}(F)\right\rangle_{\mathfrak{H}}. \tag{2.19}$$

The question of existence is answered by the following lemma.

**Lemma 2.6** (Noreddine, Nourdin Noreddine and Nourdin (2011))**.** *With the notation as above, fix an integer $j \geq 1$ and assume that $F_i \in \mathbb{D}^{j,2^j}$ for $1 \leq i \leq d$. Then it holds that $\Gamma_{l_1,\dots,l_k}(F)$ is a well-defined element of $\mathbb{D}^{j-k+1,2^{j-k+1}}$ for all $k = 1, \dots, j$. In particular, $\Gamma_{l_1,\dots,l_j}(F) \in \mathbb{D}^{1,2} \subset L^2(\Omega)$ and the quantity $\mathrm{E}\left[\Gamma_{l_1,\dots,l_j}(F)\right]$ is well-defined and finite.*

Using these random elements, we can now state a formula for the cumulants of $F$.

**Theorem 2.7** (Noreddine, Nourdin Noreddine and Nourdin (2011))**.** *Let $\alpha$ be a $d$-dimensional multi-index with elementary decomposition $\{l_1, \dots, l_{|\alpha|}\}$. If $F_i \in \mathbb{D}^{|m|,2^{|m|}}$ for $1 \leq i \leq d$, then*

$$\kappa_\alpha(F) = \sum_\sigma \mathrm{E}\left[\Gamma_{l_1,l_{\sigma(2)},l_{\sigma(3)},\dots,l_{\sigma(\alpha)}}(F)\right], \tag{2.20}$$

*where the sum is taken over all permutations $\sigma$ of the set $\{2, 3, \dots, |\alpha|\}$.*

We again stress that – as the labeling of the elementary decomposition is arbitrary – we can freely choose the fixed first element $l_1$. For the case $d = 1$, this formula has been proven in Nourdin and Peccati (2010).

To simplify notation, we will frequently write $\Gamma_{i_1 i_2 \cdots i_k}(F)$ instead of the more cumbersome $\Gamma_{e_{i_1},e_{i_2},\dots,e_{i_k}}(F)$. The random variable $\Gamma_{e_1,e_2}(F) = \left\langle DF_1, -DL^{-1}F_2\right\rangle_{\mathfrak{H}}$ will for example also be denoted by $\Gamma_{12}(F)$.

If all components of $F$ are elements of (possibly different) Wiener chaoses, formula (2.20) can be stated in terms of contractions. We state two special cases here and refer to Noreddine and Nourdin (2011) for a general formula. As a first special case, assume that $F_i = I_{q_i}(f_i)$ where $q_i \geq 1$ and $f_i \in \mathfrak{H}^{\odot q_i}$ for $1 \leq i \leq d$. In this case, for $1 \leq i, j, k \leq d$, the third-order cumulants are given by

$$\kappa_{e_i+e_j+e_k} = \begin{cases} c\left\langle f_i \widetilde{\otimes}_r f_j, f_k\right\rangle_{\mathfrak{H}^{\otimes q_k}} & \text{if } r := \frac{q_i+q_j-q_k}{2} \in \{1, 2, \dots, q_i \wedge q_j\}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.21}$$

where $c$ is some positive constant depending on the chaotic orders $q_i$, $q_j$ and $q_k$. As a second special case, assume that the components $F_i$ are all of the form $F_i = I_2(f_i)$, with $f_i \in \mathfrak{H}^{\odot 2}$ for $1 \leq i \leq d$. In this case, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \geq 2$ it holds that

$$\kappa_\alpha(F) = 2^{|\alpha|-1} \sum_\sigma \left\langle (\cdots (f_{i_1} \widetilde{\otimes}_1 f_{i_{\sigma(1)}}) \widetilde{\otimes}_1 f_{i_{\sigma(2)}}) \ldots) \widetilde{\otimes}_1 f_{i_{\sigma(|\alpha|-1)}}, f_{i_{\sigma(|\alpha|)}} \right\rangle_{\mathfrak{H}^{\otimes 2}}, \quad (2.22)$$

where the sum is taken over all permutations $\sigma$ of the set $\{2, 3, \ldots, |\alpha|\}$ and the indices $i_1, \ldots, i_{|\alpha|}$ are defined as follows: If $\{l_1, \ldots, l_{|\alpha|}\}$ is the elementary decomposition of $\alpha$, we set $i_j = k$ if $l_j = e_k$, $j = 1, \ldots, |\alpha|$. To illustrate this formula, we have for example

$$\kappa_{(2,1)}(F) = 8 \left\langle f_1 \widetilde{\otimes}_1 f_1, f_2 \right\rangle_{\mathfrak{H}^{\otimes 2}}, \quad (2.23)$$

or, with a different labelling of the elementary decomposition,

$$\kappa_{(2,1)}(F) = 8 \left( \left\langle f_1 \widetilde{\otimes}_1 f_2, f_1 \right\rangle_{\mathfrak{H}^{\otimes 2}} + \left\langle f_2 \widetilde{\otimes}_1 f_1, f_1 \right\rangle_{\mathfrak{H}^{\otimes 2}} \right). \quad (2.24)$$

One can verify by direct computations that the right hand sides of (2.23) and (2.24) are indeed equal.

2.8. *Limit theorems for vectors of multiple integrals.* In this section, we gather two results from the literature which we will use extensively in the sequel. The first is a version of the so called *Fourth Moment Theorem* (see Peccati (2007, Theorem 3)) for fluctuating covariances, that is based on the findings in Nualart and Peccati (2005); Nualart and Ortiz-Latorre (2008) and Peccati and Tudor (2005) for the converging covariance case.

**Theorem 2.8** (Fourth Moment Theorem, Peccati (2007))**.**

*(A)* *Let $q \geq 1$ and $(F_n)_{n \geq 1} = (I_q(f_n))_{n \geq 1}$ be a sequence of multiple integrals and assume that there exists a constant $M$ such that $\mathrm{E}\left[F_n^2\right] \leq M$ for $n \geq 1$. Then the following conditions are equivalent.*
- *(i)* *$(F_n)_{n \geq 1}$ is ACN*
- *(ii)* $\mathrm{E}\left[F_n^4\right] - 3\,\mathrm{E}\left[F_n^2\right]^2 \to 0$
- *(iii)* *For $1 \leq r \leq q-1$ it holds that* $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \to 0$
- *(iv)* $\mathrm{Var}\left(\Gamma_{11}(F_n)\right) \to 0$

*If the variance of $F_n$ converges to some limit $c$, conditions (i)-(iv) are equivalent to*
- *(i') $F_n \xrightarrow{d} Z$, where $Z$ is a centered normal with variance $c$.*

*(B)* *Let $(F_n)_{n \geq 1} = (F_{1,n}, \ldots, F_{d,n})_{n \geq 1}$ be a random sequence such that $F_{i,n} = I_{q_i}(f_{i,n})$, $q_i \geq 1$, for $1 \leq i \leq d$ and the covariances of $(F_n)$ are uniformly bounded. Then $(F_n)$ is ACN if and only if $(F_{i,n})$ is ACN for $1 \leq i \leq d$.*

Secondly, we will make use of the following central limit theorem for the case where one component of the random vectors $F_n$ has a finite chaos expansion. As this result is an immediate consequence of the findings in Peccati (2007), we omit the proof.

**Lemma 2.9.** *Let $(F_n)_{n \geq 1} = (F_{1,n}, \ldots, F_{d,n})_{n \geq 1}$ be a sequence of random vectors such that $F_{i,n} = I_{q_i}(F_{i,n})$ for $n \geq 1$ and $1 \leq i \leq d$. Furthermore, let $G_n = \sum_{k=1}^M I_k(g_{k,n})$ for $n \geq 1$. If*

$$\sum_{i=1}^d \sum_{r=1}^{q_i - 1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i - r)}} \to 0 \quad (2.25)$$

*and*

$$\sum_{k=2}^{M}\sum_{s=1}^{k-1}\|g_{k,n}\otimes_s g_{k,n}\|_{\mathfrak{H}^{\otimes 2(q_k-s)}}\to 0, \tag{2.26}$$

*then* $(F_n, G_n)_{n\geq 1}$ *is ACN.*

## 3. Main results

In this section, for some fixed positive integer $d$, we denote by $(F_n)_{n\geq 1}=(F_{1,n},F_{2,n},\dots,F_{d,n})_{n\geq 1}$ a sequence of centered, $\mathbb{R}^d$-valued random vectors such that $F_{i,n}\in\mathbb{D}^{1,4}$ for $1\leq i\leq d$. We also introduce a normalized sequence $(\widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ for $1\leq i,j\leq d$, which is defined by

$$\widetilde{\Gamma}_{ij}(F_n)=\frac{\Gamma_{ij}(F_n)-\mathrm{E}\left[\Gamma_{ij}(F_n)\right]}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}=\frac{\Gamma_{ij}(F_n)-\mathrm{E}\left[F_{i,n}F_{j,n}\right]}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}},$$

where $\Gamma_{ij}(F_n)$ is given by (2.19). Furthermore, for $1\leq i,j\leq d$, let $(Z_n)_{n\geq 1}=(Z_{1,n},\dots,Z_{d,n})_{n\geq 1}$ be a centered sequence of Gaussian random variables such that $Z_n$ has the same covariance as $F_n$ for $n\geq 1$. The following crucial identity is the starting point of our investigations.

**Theorem 3.1** (Nourdin et al. (2010a))**.** *Let $g\in\mathcal{C}^2(\mathbb{R}^d)$ and $Z$ be a $d$-dimensional normal vector with covariance matrix $C$. Then, for every $n\geq 1$, it holds that*

$$\mathrm{E}\left[g(F_n)\right]-\mathrm{E}\left[g(Z)\right]=\sum_{i,j=1}^{d}\mathrm{E}\left[\partial_{ij}U_{g,C}(F_n)\ (\Gamma_{ij}(F_n)-C_{ij})\right], \tag{3.1}$$

*where $U_{g,C}$ is defined by* (2.4)*.*

Identity (3.1) has been derived in Nourdin et al. (2010a) by the so called "smart path method" and Malliavin calculus. If the covariance matrix $C$ is positive definite, one can give an alternative proof by using Stein's method (see Nourdin et al. (2010b, proof of Theorem 3.5)).

A straightforward application of the Cauchy-Schwarz inequality to identity (3.1) yields the bound

$$\left|\mathrm{E}\left[g(F_n)\right]-\mathrm{E}\left[g(Z)\right]\right|\leq\frac{\sqrt{d}}{2}\left(\sup_{|\alpha|=2}\|\partial_\alpha g\|_\infty\right)\varphi_C(F_n), \tag{3.2}$$

where $\varphi_C(F_n)=\Delta_\Gamma(F_n)+\Delta_C(F_n)$ and the quantities $\Delta_\Gamma(F_n)$ and $\Delta_C(F_n)$, that already appeared in the Introduction, are defined by

$$\Delta_\Gamma(F_n)=\|\Gamma(F_n)-\operatorname{Cov}(F_n)\|_{H.S.}=\sqrt{\sum_{i,j=1}^{d}\operatorname{Var}\,\Gamma_{ij}(F_n)} \tag{3.3}$$

and

$$\Delta_C(F_n)=\|\operatorname{Cov}(F_n)-\operatorname{Cov}(Z)\|_{H.S.}=\sqrt{\sum_{i,j=1}^{d}\left(\mathrm{E}\left[F_iF_j\right]-C_{ij}\right)^2}. \tag{3.4}$$

Here, $\|\cdot\|_{H.S.}$ denotes the Hilbert-Schmidt matrix norm. Note that $\Delta_C(F_n)$ is equal to zero if and only if $F_n$ has covariance matrix $C$ and that $\Delta_\Gamma(F_n)$ is equal to zero if $F_n$ is Gaussian. The latter follows from the fact that $|\Gamma_{ij}(F_n)|$ is constant if $F_{i,n}$

and $F_{j,n}$ are Gaussian, which can be seen by applying the bound (3.2) to the vector $(F_{i,n}, F_{j,n})$ and a centered Gaussian vector $(Z_1, Z_2)$ with the same covariance.

Assume now that $\varphi_C(F_n)$ converges to zero. For the one-dimensional case $d = 1$, an adaptation of the arguments in Nourdin and Peccati (2009a) provides conditions under which the ratio

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]}{\varphi_C(F_n)}$$

converges to some real number. If this number is non-zero, this implies in particular that the rate $\varphi_C(F_n)$ is optimal, in the sense that there exist positive constants $c_1$, $c_2$ and $n_0$ such that

$$c_1 \leq \frac{\left|\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]\right|}{\varphi_C(F_n)} \leq c_2$$

for $n \geq n_0$. As already mentioned in the introduction, by approximating a Lipschitz function by functions with bounded derivatives, this implies that $\varphi_C(F_n)$ is optimal for the one-dimensional Wasserstein distance (see Nourdin and Peccati (2009a) for details). By considering coordinate projections, optimality in multiple dimensions case can immediately be reduced to the one-dimensional case. However, obtaining exact asymptotics is a much more involved task, as the next two theorems show.

**Theorem 3.2** (Exact asymptotics for the fluctuating variance case)**.** *Assume that $\Delta_\Gamma(F_n) \to 0$ and let $g\colon \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with bounded derivatives up to order three. If, for $1 \leq i, j \leq d$, the random sequences $\big(F_n, \widetilde{\Gamma}_{ij}(F_n)\big)_{n\geq 1}$ are ACN whenever $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ it holds that*

$$\frac{1}{\Delta_\Gamma(F_n)}\Big( \mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z_n)\right] \\ - \frac{1}{3}\sum_{i,j,k=1}^{d} \sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}\rho_{ijk,n}\, \mathrm{E}\left[\partial_{ijk} g(Z_n)\right] \Big) \to 0. \quad (3.5)$$

*Here, the constants $\rho_{ijk,n}$ are defined by*

$$\rho_{ijk,n} = \mathrm{E}\left[\widetilde{Z}_{ij,n} Z_{k,n}\right]$$

*whenever $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ holds and $(F_N, \widetilde{\Gamma}_{ij,n})_{n\geq 1}$ is ACN with corresponding Gaussian sequence $(Z_n, \widetilde{Z}_{ij,n})$, and $\rho_{ijk,n} = 0$ otherwise.*

*Remark* 3.3. Clearly, the condition $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ in the above Theorem expresses the fact that we can neglect those summands of $\Delta_\Gamma(F_n)$ that vanish “too fast” and therefore do not contribute to the overall speed of convergence of $\Delta_\Gamma(F_n)$.

*Proof of Theorem 3.2:* By applying Theorem 3.1, we get

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z_n)\right]}{\Delta_\Gamma(F_n)} \\ = \sum_{i,j=1}^{d} \frac{\sqrt{\operatorname{Var}\,\Gamma_{ij}(F_n)}}{\Delta_\Gamma(F_n)}\, \mathrm{E}\left[\partial_{ij} U_{g,C_n}(F_n)\, \widetilde{\Gamma}_{ij}(F_n)\right]. \quad (3.6)$$

The bound (2.6) for the derivatives of $U_{g,C_n}$ and the fact that $\widetilde{\Gamma}_{ij}(F_n)$ has unit variance immediately implies that the expectations occuring in the sum on the right hand side of (3.6) are bounded. Therefore, we only have to examine those summands in the same sum, for which $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ is true (as all others vanish in the limit). Now choose $1 \leq i,j \leq d$ such that this asymptotic condition holds. Due to our assumption, Theorem 2.2 implies the existence of random vectors $(F_n^*, \widetilde{\Gamma}_{ij}(F_n)^*)$ and Gaussian random variables $(Z_n^*, \widetilde{Z}_{ij,n}^*)$, defined on some common probability space, such that $(F_n^*, \widetilde{\Gamma}_{ij}(F_n)^*)$ has the same law as $(F_n, \widetilde{\Gamma}_{ij}(F_n))$, $(Z^*, \widetilde{Z}_{ij,n}^*)$ has the same law as $(Z, \widetilde{Z}_{ij,n})$ and $(F_n^* - Z_n^*, \widetilde{\Gamma}_{ij}(F_n)^* - \widetilde{Z}_{ij,n}^*) \to 0$ almost surely. Thus we can write

$$\mathrm{E}\left[\partial_{ij} U_{g,C_n}(F_n)\widetilde{\Gamma}_{ij}(F_n)\right] = \eta_{ij,n}^1 + \eta_{ij,n}^2 + \eta_{ij,n}^3, \tag{3.7}$$

where

$$\begin{aligned}
\eta_{ij,n}^1 &= \mathrm{E}\left[\left(\partial_{ij} U_{g,C_n}(F_n^*) - \partial_{ij} U_{g,C_n}(Z_n^*)\right) \widetilde{\Gamma}_{ij}(F_n)^*\right], \\
\eta_{ij,n}^2 &= \mathrm{E}\left[\partial_{ij} U_{g,C_n}(Z_n^*) \left(\widetilde{\Gamma}_{ij}(F_n)^* - \widetilde{Z}_{ij,n}^*\right)\right]
\end{aligned}$$

and

$$\eta_{ij,n}^3 = \mathrm{E}\left[\partial_{ij} U_{g,C_n}(Z_n^*)\widetilde{Z}_{ij,n}^*\right].$$

The integration by parts formula (2.3) and Lemma 2.4b) yield

$$\eta_{ij,n}^3 = \frac{1}{3}\sum_{k=1}^d \mathrm{E}\left[Z_{k,n}^* \widetilde{Z}_{ij,n}^*\right] \mathrm{E}\left[\partial_{ijk} U_{g,C_n}(Z_n^*)\right],$$

so that the proof is finished as soon as we have established that

$$\eta_{ij,n}^1 \to 0 \quad \text{and} \quad \eta_{ij,n}^2 \to 0.$$

But this is an immediate consequence of the Lipschitz continuity of $\partial_{ij} U_{g,C_n}$, the fact that $\widetilde{\Gamma}_{ij}(F_n)^*$ has unit variance (implying uniform integrability of the sequences $(\widetilde{\Gamma}_{ij}(F_n)^*)_{n\geq 0}$ and $(\widetilde{\Gamma}_{ij}(F_n)^* - \widetilde{Z}_{ij,n})_{n\geq 0}$) and the bound (2.6). □

**Theorem 3.4** (Exact asymptotics for the converging variance case)**.** *Assume that $\Delta_\Gamma(F_n) \to 0$ and let $g\colon \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with bounded derivatives up to order three. If there exists a covariance matrix $C$ such that $\Delta_C(F_n) \to 0$ and, for $1 \leq i,j \leq d$, the random sequences $\left(F_n, \widetilde{\Gamma}_{ij}(F_n)\right)_{n\geq 1}$ converge in law to a centered Gaussian random vector $(Z, \widetilde{Z}_{ij})$ whenever*

$$\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)} + \left|\mathrm{E}\left[F_{i,n}F_{j,n}\right] - C_{ij}\right| \asymp \varphi_C(F_n) \tag{3.8}$$

*it holds that*

$$\frac{1}{\varphi_C(F_n)}\left(\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right] - \frac{1}{2}\sum_{i,j=1}^d \left(\mathrm{E}\left[F_{i,n}F_{j,n}\right] - C_{ij}\right)\mathrm{E}\left[\partial_{ij} g(Z)\right] \right.$$
$$\left. -\frac{1}{3}\sum_{i,j,k=1}^d \sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}\rho_{ijk}\,\mathrm{E}\left[\partial_{ijk} g(Z)\right]\right) \to 0. \tag{3.9}$$

*Here, the constants $\rho_{ijk}$ are defined by $\rho_{ijk} = \mathrm{E}\left[\widetilde{Z}_{ij} Z_k\right]$ whenever (3.8) is true and $\rho_{ijk} = 0$ otherwise.*

*Proof*: Theorem 3.1 implies

$$\frac{\mathrm{E}\left[g(F_n)\right]-\mathrm{E}\left[g(Z)\right]}{\varphi_C(F_n)} = \sum_{i,j=1}^{d}\left(\frac{\mu_{ij}(F_n)-C_{ij}}{\varphi_C(F_n)}\,\mathrm{E}\left[\partial_{ij}U_{g,C}(F_n)\right] + \frac{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}{\varphi_C(F_n)}\,\mathrm{E}\left[\partial_{ij}U_{g,C}(F_n)\,\widetilde{\Gamma}_{ij}(F_n)\right]\right). \tag{3.10}$$

Arguing as in the proof of Theorem 3.2, we see that all expectations occuring in the sum on the right hand side of (3.10) are bounded. Therefore, we can choose $1 \leq i,j \leq d$ and assume that (3.8) is true (as otherwise the corresponding summand would vanish in the limit). By the boundedness of the second derivatives of $U_{g,C}$ (see (2.6)) and our assumption of convergence in law, we get

$$\mathrm{E}\left[\partial_{ij}U_{g,C}(F_n)\right] \to \mathrm{E}\left[\partial_{ij}U_{g,C}(Z)\right]$$

and

$$\mathrm{E}\left[\partial_{ij}U_{g,C}(F_n)\widetilde{\Gamma}_{ij}(F_n)\right] \to \mathrm{E}\left[\partial_{ij}U_{g,C}(Z)\widetilde{Z}_{ij}\right].$$

The integration by parts formula (2.3) and Lemma 2.4b) now yield

$$\mathrm{E}\left[\partial_{ij}U_{g,C}(Z)\right] = \frac{1}{2}\,\mathrm{E}\left[\partial_{ij}g(Z)\right]$$

and

$$\mathrm{E}\left[\partial_{ij}U_{g,C}(Z)\widetilde{Z}_{ij}\right] = \frac{1}{3}\sum_{k=1}^{d}\mathrm{E}\left[\widetilde{Z}_{ij}Z_k\right]\mathrm{E}\left[\partial_{ijk}Z\right],$$

finishing the proof. □

*Remark* 3.5. If the covariance $C$ of the Gaussian random variable $Z$ is positive definite, the Hermite polynomials $H_\alpha(x,C)$ form an orthonormal basis for the space $L^2(\mathbb{R}^d,\gamma_C)$, where $\gamma_C$ is the density of $Z$, so that an expansion of the form $g(x) = \sum_\alpha H_\alpha(x,C)$ exists for all $x \in \mathbb{R}^d$. Thus, the integration by parts formula

$$\mathrm{E}\left[\partial_\alpha g(Z)\right] = \mathrm{E}\left[g(Z)H_\alpha(Z,C)\right],$$

valid for any multi-index $\alpha$ up to order three, yields a neccessary condition for the limit to be non-zero: $g$ must not be orthogonal (in $L^2(\mathbb{R}^d,\gamma_C)$) to all second- and third-order Hermite polynomials.

An immediate consequence of the Theorems 3.2 and 3.4 is the following corollary.

**Corollary 3.6** (Sharp bounds and exact limits)**.**

*a) In the setting of Theorem 3.2, the* $\liminf$ *and* $\limsup$ *of the sequence*

$$\left(\frac{\left|\mathrm{E}\left[g(F_n)\right]-\mathrm{E}\left[g(Z_n)\right]\right|}{\Delta_\Gamma(F_n)}\right)_{n\geq 1}$$

*coincide with those of the sequence*

$$\left(\frac{1}{3\Delta_\Gamma(F_n)}\sum_{i,j,k=1}^{d}\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}\rho_{ik,n}\,\mathrm{E}\left[\partial_{ijk}g(Z_n)\right]\right)_{n\geq 1}.$$

*b) In the setting of Theorem 3.4, the* lim inf *and* lim sup *of the sequence*

$$\left(\frac{\mathrm{E}\,[g(F_n)] - \mathrm{E}\,[g(Z)]}{\varphi_C(F_n)}\right)_{n\geq 1} \tag{3.11}$$

*coincide with those of*

$$\left(\frac{1}{2}\sum_{i,j=1}^{d}\frac{\mathrm{E}\,[F_{i,n}F_{j,n}] - C_{ij}}{\varphi_C(F_n)}\,\mathrm{E}\,[\partial_{ij}g(Z)] + \frac{1}{3}\sum_{i,j,k=1}^{d}\frac{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}{\varphi_C(F_n)}\rho_{ijk}\,\mathrm{E}\,[\partial_{ijk}g(Z)]\right)_{n\geq 1}. \tag{3.12}$$

*In particular, if the sequence (3.12) converges, it provides the exact limit of the sequence (3.11).*

If $d = 1$ and the $F_n$ all have identical variances, the assumptions of Corollary 3.6b) are always satisfied and one obtains an analogue of Theorem 3.1 in Nourdin and Peccati (2009a).

If the third-order moments $\mu_\alpha(F_n)$ of the random vectors $F_n$ exist, the third order Edgeworth expansion $\mathcal{E}_3(F_n, Z, g)$, introduced in section 2.5, is well-defined for any Gaussian random vector $Z$ and any three-times differentiable function $g$ with bounded derivatives up to order three. The next theorem shows how these expansions can be used to increase the speed of convergence.

**Theorem 3.7** (One-term Edgeworth expansions)**.** *Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with bounded derivatives up to order three and assume that $F_n$ has finite moments up to order three for $n \geq 1$, and moreover*

$$\left(\frac{\sum_{k=1}^d \mu_{ijk}(F_n)}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}\right)_{n\geq 1} \tag{3.13}$$

*is bounded whenever $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}/\Delta_\Gamma(F_n) \to 0$.*

*a) If all assumptions of Theorem 3.2a) are satisfied, it holds that*

$$\frac{\mathrm{E}\,[g(F_n)] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \to 0. \tag{3.14}$$

*b) If all assumptions of Theorem 3.2b) are satisfied, it holds that*

$$\frac{\mathrm{E}\,[g(F_n)] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \to 0. \tag{3.15}$$

*Remark* 3.8. The third order Edgeworth expansion $\mathcal{E}_3(F_n, Z_n, g)$ in (3.14) takes the explicit form

$$\mathcal{E}_3(F_n, Z_n, g) = \mathrm{E}\,[g(Z_n)] + \sum_{i,j,k=1}^{d}\frac{\mu_{ijk}(F_n)}{3!}\,\mathrm{E}\,[\partial_{ijk}g(Z_n)] \tag{3.16}$$

whereas $\mathcal{E}_3(F_n, Z, g)$ in (3.15) is given by

$$\mathrm{E}\,[g(Z)] + \sum_{i,j=1}^{d}\frac{C_{ij} - \mu_{ij}(F_n)}{2}\,\mathrm{E}\,[\partial_{ij}g(Z)] + \sum_{i,j,k=1}^{d}\frac{\mu_{ijk}(F_n)}{3!}\,\mathrm{E}\,[\partial_{ijk}g(Z)] \tag{3.17}$$

*Proof of Theorem 3.7:* By Theorem 3.2a), it is sufficient to show that

$$\frac{1}{3}\sum_{i,j,k=1}^{d}\frac{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}{\Delta_\Gamma(F_n)}\left(\rho_{ijk,n}-\frac{\mu_{ijk}(F_n)}{2\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}\right)\operatorname{E}\left[\partial_{ijk}g(Z_n)\right]=0, \tag{3.18}$$

Fix $1\leq i,j\leq d$. If

$$\frac{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}{\Delta_\Gamma(F_n)}\to 0,$$

the quantity

$$\frac{\mu_{ijk}(F_n)}{2\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}$$

is bounded by assumption, so that the corresponding summand in the sum (3.18) vanishes in the limit. If $\limsup\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}/\Delta_\Gamma(F_n)$ is positive, the sequence $(F_n,\widetilde{\Gamma}_{ij,n})_{n\geq 1}$ is ACN. Thus, for $n\geq 1$, there exists Gaussian random variables $(Z_n,\widetilde{Z}_{ij,n})$ with the same covariance as $(F_n,\widetilde{\Gamma}_{ij,n})$. By definition, we get

$$\rho_{ijk,n}=\operatorname{E}\left[Z_{k,n}\widetilde{Z}_{ij,n}\right]=\frac{\operatorname{E}\left[F_{k,n}\Gamma_{ij}(F_n)\right]}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}=\frac{\operatorname{E}\left[\Gamma_{ijk}(F_n)\right]}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}.$$

The cumulant formula (2.20) and the fact that $\operatorname{Var}\Gamma_{ij}(F_n)=\operatorname{Var}\Gamma_{ji}(F_n)$ now yields

$$\rho_{ijk,n}+\rho_{jik,n}=\frac{\kappa_{ijk}(F_n)}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}=\frac{\mu_{ijk}(F_n)}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}$$

so that (3.18) follows.

Likewise, by Theorem 3.4, it is sufficient for the proof of assertion b) to show that

$$\frac{1}{3}\sum_{i,j,k=1}^{d}\frac{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}{\varphi_C(F_n)}\left(\rho_{ijk}-\frac{\mu_{ijk}(F_n)}{2\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}\right)\operatorname{E}\left[\partial_{ijk}G(Z)\right]\to 0, \tag{3.19}$$

Again, by assumption, if $1\leq i,j\leq d$ is such that $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}/\varphi_C(F_n)\to 0$, the corresponding summand in (3.19) vanishes in the limit. If, on the other hand, $\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}\asymp\varphi_C(F_n)$, the sequence $(F_n,\widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ converges in law to $(Z,\widetilde{Z}_{ij})$. Therefore,

$$\operatorname{E}\left[\widetilde{\Gamma}_{ij,n}F_{k,n}\right]\to\operatorname{E}\left[\widetilde{Z}_{ij}Z_k\right]=\rho_{ijk} \tag{3.20}$$

for $1\leq k\leq d$. The cumulant formula (2.20) gives

$$\frac{\mu_{ijk}(F_n)}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}=\frac{\kappa_{ijk}(F_n)}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}=\operatorname{E}\left[\widetilde{\Gamma}_{ij,n}F_{k,n}\right]+\operatorname{E}\left[\widetilde{\Gamma}_{ji,n}F_{k,n}\right],$$

which together with (3.20) implies that

$$\frac{\mu_{ijk}(F_n)}{\sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}}\to\rho_{ijk}+\rho_{jik}.$$

This immediatiely yields (3.19), finishing the proof. □

## 4. The case of multiple integrals

In this section, we specialize our results to the case where the components of the sequence $(F_n)_{n\geq 1}$ are vectors of multiple integrals. As in the previous section, we fix an integer $d \geq 1$ and study a sequence $(F_n)_{n\geq 1} = (F_{1,n}, \dots, F_{d,n})_{n\geq 1}$ of $\mathbb{R}^d$-valued random vectors, but now each component $F_{i,n}$ is a multiple integral of the form $F_{i,n} = I_{q_i}(f_{i,n})$ where $g_i \geq 1$ and $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Recall the definitions for the random variables $\Gamma_{ij}(F_n)$, $\widetilde{\Gamma}_{ij}(F_n)$ and $Z_n$, which were given in the first paragraph of the previous section.

Let us begin by deducing explicit representations of some of the crucial quantities of the last section. Using the product formula (2.11) and the orthogonality property (2.8) of multiple integrals, we see that

$$\Gamma_{ij}(F_n) = \frac{1}{q_j} \langle DF_{i,n}, DF_{j,n} \rangle_{\mathfrak{H}} \\ = \sum_{r=1}^{q_i \wedge q_j} q_i \, \beta_{q_i-1,q_j-1}(r-1) \, I_{q_i+q_j-2r}(f_{i,n} \widetilde{\otimes}_r f_{j,n}) \quad (4.1)$$

and

$$\operatorname{Var} \Gamma_{ij}(F_n) \\ = \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} (q_i + q_j - 2r)! \, q_i^2 \, \beta^2_{q_i-1,q_j-1}(r-1) \, \| f_{i,n} \widetilde{\otimes}_r f_{j,n} \|^2_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}, \quad (4.2)$$

where the positive constants $\beta_{a,b}(r)$ are defined by (2.12).

As all constants in the sum on the right hand side of (4.2) are positive, this implies that

$$\operatorname{Var} \Gamma_{ij}(F_n) \asymp \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \| f_{i,n} \widetilde{\otimes}_r f_{j,n} \|^2_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}} \quad (4.3)$$

and therefore

$$\Delta_\Gamma(F_n) \asymp \left( \sum_{i,j=1}^{d} \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \| f_{i,n} \widetilde{\otimes}_r f_{j,n} \|^2_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}} \right)^{1/2}. \quad (4.4)$$

If, for some integers $i, j, k$ with $1 \leq i, j, k \leq d$, it holds that $r := \frac{q_i+q_j-q_k}{2} \in \{1, 2, \dots, q_i \wedge q_j\}$, formula (2.21) and the Cauchy-Schwarz inequality yield

$$\mu_{ijk}(F_n) = \kappa_{e_i+e_j+e_k} F_n \leq c \| f_i \widetilde{\otimes}_r f_j \|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}} \, \| f_k \|_{\mathfrak{H}^{\otimes q_k}}.$$

Combining this with (4.3), we see that the quantity (3.13) from Theorem 3.7 is bounded and therefore one-term Edgeworth expansions are always possible whenever the corresponding ACN-conditions from Theorems 3.2 and 3.4 are verified. Thus we have proven the following proposition.

**Proposition 4.1** (Exact asymptotics and Edgeworth expansions for multiple integrals)**.** *In the above framework, let $g \colon \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with bounded derivatives up to order three.*

*a) If, for $1 \leq i,j \leq d$, the random sequence $(F_n, \widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ is ACN whenever*

$$\limsup_n \sqrt{\operatorname{Var}\Gamma_{ij}(F_n)}/\Delta_\Gamma(F_n) > 0,$$

*it holds that*

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \to 0.$$

*b) If there exists a covariance matrix $C$ such that $\Delta_C(F_n) \to 0$ and, for $1 \leq i,j \leq d$, the random sequence $(F_n, \widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ is ACN whenever*

$$\limsup_n \sqrt{\operatorname{Var}\Gamma_{ij}(F_n)} + \left|\mathrm{E}\left[F_{i,n}F_{j,n}\right] - C_{ij}\right| / \varphi_C(F_n) > 0,$$

*it holds that*

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \to 0.$$

Sufficient conditions for the sequences $(F_n, \widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ to be ACN are given by the following proposition.

**Proposition 4.2.** *The sequence $(F_n, \widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ is ACN if*

$$\sum_{i=1}^{d}\sum_{r=1}^{q_i-1} \left\| f_{i,n} \otimes_r f_{i,n} \right\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \to 0 \tag{4.5}$$

*and*

$$\frac{\sum_{r=1}^{q_i\wedge q_j - \delta_{q_iq_j}} \sum_{s=1}^{q_i+q_j-2r-1} \left\| (f_{i,n}\widetilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n}\widetilde{\otimes}_r f_{j,n}) \right\|_{\mathfrak{H}^{\otimes 2(q_i+q_j-2r-1-s)}}}{\sum_{r=1}^{q_i\wedge q_j - \delta_{q_iq_j}} \left\| f_{i,n}\widetilde{\otimes}_r f_{j,n} \right\|^2_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}} \to 0. \tag{4.6}$$

*Proof*: This is a direct consequence of (4.1), (4.2) and Lemma 2.9. □

If we assume that $F_i = I_2(f_i)$ for $f_i \in \mathfrak{H}^{\odot 2}$, $1 \leq i \leq d$, we can state conditions for the ACN property of $(F_n, \widetilde{\Gamma}_{ij}(F_n))_{n\geq 1}$ which only involve cumulants. This is due to the well known formula (see Fox and Taqqu (1987))

$$\begin{aligned}\kappa_{k\times e_i}(F) &= 2^{k-1}(k-1)!\ \operatorname{Tr}\left(H_{f_i}^k\right) \\ &= 2^{k-1}(k-1)! \left\langle f_i \otimes_1^{(k-1)} f_i, f_i \right\rangle_{\mathfrak{H}} = 2^{k-1}(k-1)! \sum_{n=1}^{\infty} \lambda_{f_i,n}^k,\end{aligned} \tag{4.7}$$

where $H_f \colon \mathfrak{H} \to \mathfrak{H}$ is the Hilbert-Schmidt operator defined by $H_f(g) = f \otimes_1 g$ and $\{\lambda_{f,n} \colon n \geq 1\}$ are its eigenvalues.

In particular, we have

$$\begin{aligned}\kappa_{4e_i}(F) &= 2^3\, 3! \left\langle f_i \otimes_1^{(3)} f_i, f_i \right\rangle_{\mathfrak{H}} \\ &= 2^3\, 3!\ \langle f_i \otimes_1 f_i, f_i \otimes_1 f_i\rangle_{\mathfrak{H}} = 2^3\, 3!\ \|f_i \otimes_1 f_i\|^2_{\mathfrak{H}}\end{aligned} \tag{4.8}$$

and

$$\kappa_{8e_i}(F) = 2^7\, 7! \left\langle f_i \otimes_1^{(7)} f_i, f_i \right\rangle_{\mathfrak{H}} = 2^7\, 7!\ \|(f_i \otimes_1 f_i) \otimes_1 (f_i \otimes_1 f_i)\|^2_{\mathfrak{H}}. \tag{4.9}$$

Using Propositions 4.1, 4.2 and the Cauchy-Schwarz inequality, we can now deduce the following Proposition. We will however omit these elementary calculations, as the Proposition will also follow as a special case from Theorem 4.6 of the forthcoming section.

**Proposition 4.3.** *Let* $(F_n) = (F_{1,n}, \dots, F_{d,n})$ *be a sequence of random vectors whose components are elements of the second chaos,* $g \colon \mathbb{R}^d \to \mathbb{R}$ *be three times differentiable with bounded derivatives up to order three and assume that*

$$\sum_{i=1}^{d} \kappa_{4e_i}(F_n) \to 0. \tag{4.10}$$

*a) If, for* $1 \le i \le d$, *it holds that* $\kappa_{4e_i}(F_n) \asymp \sum_{i=1}^{d} \kappa_{4e_i}(F_n)$ *implies*

$$\frac{\kappa_{8e_i}(F_n)}{\kappa_{4e_i}(F_n)^2} \to 0, \tag{4.11}$$

*then*

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \to 0.$$

*b) If there exists a covariance matrix* $C$ *such that* $\Delta_C(F_n) \to 0$ *and, for* $1 \le i \le d$, *the convergence* (4.11) *is implied by*

$$\kappa_{4e_i}(F_n) + \left|\kappa_{e_i+e_j}(F_n) - C_{ij}\right| \asymp \sum_{i=1}^{d} \kappa_{4e_i}(F_n) + \sqrt{\sum_{i,j=1}^{d} \left(\kappa_{e_i+e_j}(F_n) - C_{ij}\right)^2},$$

*then*

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \to 0.$$

Note that in the case $d = 1$, part b) becomes a (weaker) version of Proposition 3.8 from Nourdin et al. (2010b). Thus, in the second chaos, the joint speed of convergence can be compeletely characterized by the coordinate sequences.

4.1. *Majorizing integrals and the role of mixed contractions.* We now turn to the question whether the mixed contractions (i.e. those for which $i \neq j$) in the numerator and denominator of condition (4.6) are neccessary to ensure that $(F_n, \widetilde{\Gamma}_{ij}(F_n))_{n \ge 1}$ is ACN. It will turn out that in some cases, most notably the one where all kernels are non-negative, we can replace condition (4.6) by a similar fraction containing only non-mixed contractions. In these cases, the "interplay" of the different kernels thus has no influence on the speed of convergence.

To be able to develop our theory, we assume that $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, where $(A, \mathcal{A})$ is a Polish space, $\mathcal{A}$ is the associated Borel $\sigma$-field and the measure $\nu$ is positive, $\sigma$-finite and non-atomic. This can be done without loss of generality (see Nualart and Peccati (2005, section2.2)). The components of the random vectors $F_n$ under examination are still multiple integrals.

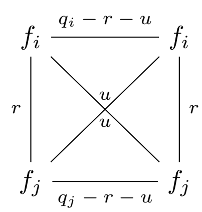
If $f_i \in \mathfrak{H}^{\odot q_i}$ and $f_j \in \mathfrak{H}^{\odot q_j}$ are two symmetric kernels and $1 \le r \le q_i \wedge q_j$, then according to formula (2.13) we can write

$$\|f_i \widetilde{\otimes}_r f_j\|^2_{\mathfrak{H}^{\otimes q_i+q_j-2r}} = \sum_{u=0}^{(q_i \wedge q_j)-r} c_u \, G_r(f_i, f_j, u), \tag{4.12}$$

where the $c_u$ are some positive universal constants not depending on $f_i$ and $f_j$ and each $G_r(f_i, f_j, u)$ is an integral of the form

$$\int_{A^u}\int_{A^u}\int_{A^r}\int_{A^r}\int_{A^n}\int_{A^m} f_i(v,x,y) f_i(v,\widetilde{x},\widetilde{y}) f_j(\widetilde{x},y,w) f_j(x,\widetilde{y},w) \\ \mathrm{d}\mu^{\otimes m}(v)\,\mathrm{d}\mu^{\otimes n}(w)\,\mathrm{d}\mu^{\otimes r}(x)\,\mathrm{d}\mu^{\otimes r}(\widetilde{x})\,\mathrm{d}\mu^{\otimes u}(y)\,\mathrm{d}\mu^{\otimes u}(\widetilde{y}), \quad (4.13)$$

where $m = q_i - r - u$ and $n = q_j - r - u$. We can visualize each of these integrals by an integer weighted, undirected graph



, (4.14)

by identifying each kernel occuring in the integral with a vertex and drawing an edge with weight $l$ between two functions, if $l$ variables of these two functions coincide. For example, the edge with label $q_i - r - u$ in the above graph corresponds to the variable $v$ in the integral (4.13). Due to the symmetry of the kernels involved, we can freely translate back and forth between the explicit notation (4.13) and the visual notation (4.14) without losing any information. To avoid cumbersome treatment of degenerate cases, we adopt the convention that edges with weight zero are non-existent.

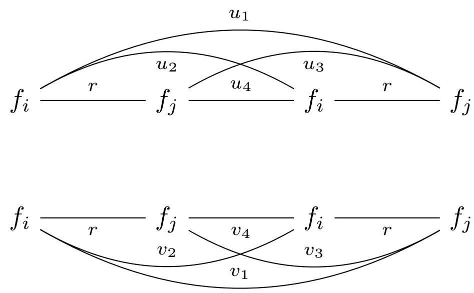
Analogously, we can write

$$\|(f_i \widetilde{\otimes}_r f_j) \otimes_s (f_i \widetilde{\otimes}_r f_j)\|^2_{\mathfrak{H}^{\otimes 2(q_i+q_j-2r-s)}} = \sum_{\lambda \in \Lambda} c_\lambda\, G_{r,s}(f_i, f_j, \lambda), \quad (4.15)$$

where $\Lambda$ is some finite index set, the $c_\lambda$ are positive constants and the $G_{r,s}(f_i, f_j, \lambda)$ are integrals of the form

$$\int f_i(\cdot) f_i(\cdot) f_i(\cdot) f_i(\cdot) f_j(\cdot) f_j(\cdot) f_j(\cdot) f_j(\cdot)$$

involving four copies of the kernels $f_i$ and $f_j$, respectively, which are obtained by first choosing $2(q_i + q_j)$ pairs of variables, then identifying variables that have been paired and finally integrating with respect to the $2(q_i + q_j)$ resulting variables. The only constraint one has to obey is that two variables that stem from the same kernel must not be paired. One could write this with a lot more rigour (using, for example, diagrams and partitions, see Peccati and Taqqu (2011), or a visual method similar to ours, see Marinucci (2008)) but for our purposes it is enough to know that each of this integrals can be visualized as a graph with eight vertices (four of them labeled with $f_i$ and $f_j$, respectively) that contains

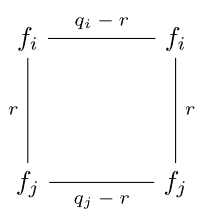



as a subgraph, where $\sum_i u_i = \sum_i v_i = s$ and some of the $u_i$ and $v_i$ can be zero (recall our convention that an edge with weight zero is non-existent). Note that in the original graph there are always edges (to be precise, exactly $2(q_i + q_j - r) - s$ of them) connecting the "upper" and "lower" groups of four edges.

If we are given an integral $G$ occuring in the representations (4.12) or (4.15), we can arbitrarily divide the four or eight kernels appearing in the integrands into two sets $A$ and $B$ and use the Cauchy-Schwarz inequality to obtain a bound of the type $|G| \leq G_1^{1/2} G_2^{1/2}$, where the integrals $G_1$ and $G_2$ only involve kernels in the set $A$ and $B$, respectively. Of course, $G_1$ and $G_2$ can also be visualized by graphs. In fact, these two graphs can be obtained without analytical detour by the following, purely visual "cut-mirror-merge"-operation on the graph of $G$:

1) Divide the vertices of $G$ into two groups $A$ and $B$.
2) Erase all edges that connect vertices of different groups, thus obtaining two subgraphs. We refer to a vertex adjacent to an edge that has been erased as a *bordering vertex*.
3) For each of these two subgraphs, take a copy of this subgraph and connect each bordering vertex of the subgraph with the corresponding vertex in the copy by an edge. The weight of this edge is equal to the sum of the weights of all erased edges that were adjacent to this vertex and have been erased in step two.

For example, starting from the integral given by the graph (4.14), if we choose two identical sets consisting of one $f_i$ and one $f_j$, respectively, the resulting graphs for $G_1$ and $G_2$ are identical as well and given by



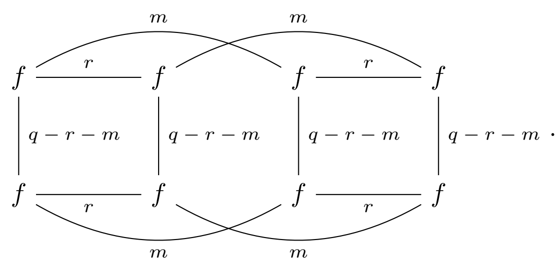

Translated back into the language of integrals, this is just the well-known fact that

$$\|f_i \widetilde{\otimes}_r f_j\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}} \leq \|f_i \otimes_r f_j\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}},$$

which can of course be proven much more concisely by a direct calculation. However, the advantage of working with graphs reveals itself when dealing with the integrals on the right hand side of (4.15). We will see this when proving the forthcoming Majorizing Lemma, that plays a key role in this section.

If $f_i = f_j$, some integrals appearing in the sum on the right hand side of (4.12) are of special interest, as they dominate all others (in a sense that will be made clear in the sequel). We title them *majorizing integrals*. They are defined as follows.

**Definition 4.4** (Majorizing integrals)**.** For $f \in \mathfrak{H}^{\odot q}$, $1 \le r \le q-1$ and $0 \le m \le q-r$, the *majorizing integrals* $M_r(f,m)$ are given by the graph




(4.16)

Observe that the integrals $M_r(f,m)$ are non-negative and appear in the expansion of the type (4.15) for the norm $\|(f \widetilde{\otimes}_r f) \otimes_s (f \widetilde{\otimes}_r f)\|_{\mathfrak{H}^{\otimes 4(q-r)-2s}}$. Also, by grouping the inner four vertices in the graph (4.16) and applying "cut-mirror-merge", we see that

$$M_r(f,m) \le \|f \otimes_r f\|^4_{\mathfrak{H}^{\otimes 2(q-r)}},$$

with equality if $m \in \{0, q-r\}$.

**Lemma 4.5** (Majorizing Lemma)**.** *Let $q_i$ and $q_j$ be two positive integers and $f_i \in \mathfrak{H}^{\odot q_i}$, $f_j \in \mathfrak{H}^{\odot q_j}$ be two symmetric kernels. For given integers $r$ and $s$ with $1 \le r \le q_i \wedge q_j - \delta_{q_i,q_j}$ and $1 \le s \le q_i + q_j - 2r - 1$, let $G_{r,s}(f_i, f_j, \lambda)$ be one of the summands in the representation (4.15). Then, for $1 \le k \le 4$, there exists integers $n_k \in \{0, 1, \dots, q_i - r\}$ and $m_k \in \{0, 1, \dots, q_j - r\}$ with $m_k + n_k = s$, such that*

$$G_{r,s}(f_i, f_j, \lambda)^8 \le \prod_{k=1}^{4} \big( M_r(f_i, m_k)\, M_r(f_j, n_k) \big). \tag{4.17}$$

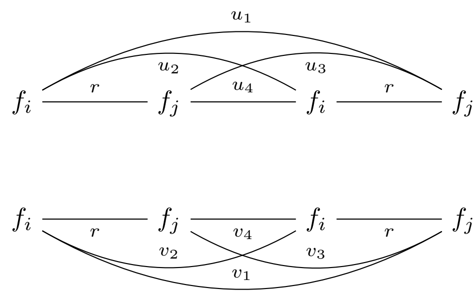
*Proof*: We will iteratively apply Cauchy-Schwarz (using the visual method developed above) to obtain the chain

$$G_{r,s}(f_i, f_j, \lambda)^8 \le (G'_1\, G''_1)^4 \le \big(G'_{2,1}\, G'_{2,2}\, G''_{2,1}\, G''_{2,2}\big)^2 \le \prod_{k=1}^{4} \big( M_r(f_i, m_k)\, M_r(f_j, n_k) \big), \tag{4.18}$$

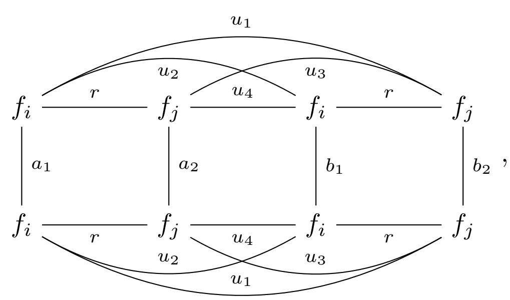
where all primed and double-primed quantities are integrals which will be described by their corresponding graphs.

As already mentioned, a graph associated with an integral $G_{r,s}(f_i, f_j, \lambda)$ in the sum (4.15) has eight vertices (four of them labeled with $f_i$, the other four with $f_j$)

and contains

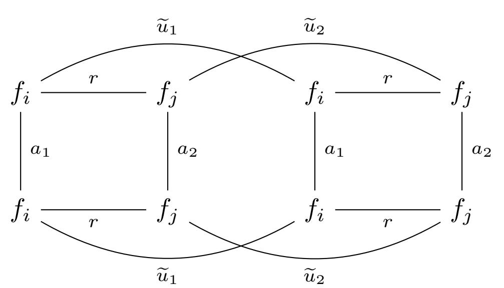



as a subgraph, where $\sum_i u_i = \sum_i v_i = s$ and some of the $u_i$ and $v_i$ can be zero (recall our convention that an edge with weight zero is non-existent). We now apply the Cauchy-Schwarz inequality for the first time, grouping the four vertices connected by the $u_i$- and $v_i$-edges. The resulting bounding integrals $G_1'$ and $G_1''$ are given by the graph




where $a_1 + a_2 + b_1 + b_2 = 2(q_i + q_j - r) - s > 0$ and the same graph with $u_i$ replaced by $v_i$. We now continue to apply Cauchy-Schwarz to $G_1'$. The exact same operations then have to be performed with $G_1''$ to obtain the final result.

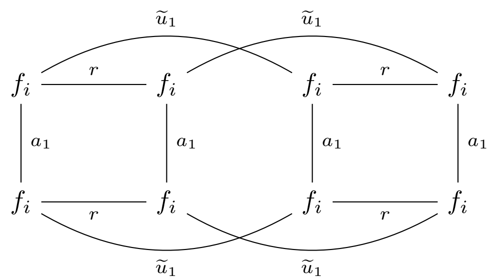
In the graph of $G_1'$, we group the four vertices connected by $a_i$- and $b_i$-edges respectively and then apply Cauchy-Schwarz. This yields bounding integrals $G_{2,1}'$, given by a "cube" of the form




(4.19)

where $0 \le a_1 + a_2 \le 2(q_i + q_j - r) - s$, $0 \le \widetilde{u}_1, \widetilde{u_2} \le s$ and $\widetilde{u}_1 + \widetilde{u}_2 = s$, and $G_{2,2}'$, given by the same graph with the $a_i$ replaced by $b_i$.

From the graphs for $G_{2,1}'$ and $G_{2,2}'$, by grouping the four $f_i$- and $f_j$-vertices together and then applying Cauchy-Schwarz another time, we now obtain graphs that represent majorizing integrals. For example, starting from the graph (4.19) for $G'_{2,1}$, we obtain




with $0 \leq \widetilde{u}_1 \leq s$ and $a_1 = q_i - r - \widetilde{u}_1$ and the same graph with $f_i$, $\widetilde{u}_1$ and $a_1$ replaced by $f_j$, $\widetilde{u}_2$ and $a_2$, respectively.

Starting from $G''_1$, the integrals $G''_{2,1}$ and $G''_{2,1}$ as well as the corresponding majorizing integrals are obtained analogously. Finally, a careful inspection of the single steps indicated above yields that the majorizing integrals obtained after the final application of the Cauchy-Schwarz inequality are indeed of the form stated in (4.17). □

We are now ready to prove the main theorem of this section.

**Theorem 4.6.** *Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with bounded derivatives up to order three and assume that the following conditions are true.*

*(i)*

$$\sum_{i=1}^{d} \sum_{r=1}^{q_i-1} \| f_{i,n} \otimes_r f_{i,n} \|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \to 0.$$

*(ii) For those $i, j \in \{1, 2, \ldots, d\}$for which $\sqrt{\operatorname{Var} \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ it holds that*

$$\sum_{r=1}^{q_i-1} \| f_{i,n} \widetilde{\otimes}_r f_{i,n} \|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \asymp \sum_{r=1}^{q_i-1} \| f_{i,n} \otimes_r f_{i,n} \|_{\mathfrak{H}^{\otimes 2(q_i-r)}}. \tag{4.20}$$

*and*

$$\frac{\sum_{r=1}^{q_i-1} \sum_{s=1}^{q_i-r-1} M_r(f_{i,n}, s)}{\sum_{r=1}^{q_i-1} \| f_{i,n} \otimes_r f_{i,n} \|^4_{\mathfrak{H}^{\otimes 2(q_i-r)}}} \to 0. \tag{4.21}$$

*Then it holds that*

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \to 0. \tag{4.22}$$

*If, in addition, there exists a covariance matrix $C$ such that $\Delta_C(F_n) \preccurlyeq \Delta_\Gamma(F_n)$, then*

$$\frac{\mathrm{E}\left[g(F_n)\right] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \to 0. \tag{4.23}$$

*Proof*: Let $i, j \in \{1, 2, \ldots, d\}$ such that $\sqrt{\operatorname{Var} \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ and assume, without loss of generality, that $q_i \leq q_j$.

We will show that

$$\operatorname{Var} \Gamma_{ij}(F_n) \succcurlyeq \sqrt{\operatorname{Var} \Gamma_{ii}(F_n) \operatorname{Var} \Gamma_{jj}(F_n)} \tag{4.24}$$

and

$$\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \sum_{s=1}^{q_i+q_j-2r-1} \|(f_{i,n} \widetilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \widetilde{\otimes}_r f_{j,n})\|^2_{\mathfrak{H}^{\otimes 2(q_i+q_j-2r-s)}} \preccurlyeq \sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}}\,, \quad (4.25)$$

where

$$A_{1,n} = \left( \sum_{r=1}^{q_i-1} \sum_{s=1}^{2(q_i-r)-1} M_r(f_{i,n}, s) \right) \left( \sum_{r=1}^{q_j-1} \sum_{s=1}^{2(q_j-r)-1} M_r(f_{j,n}, s) \right)$$

$$A_{2,n} = \left( \sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|^4_{\mathfrak{H}^{\otimes 2(q_i-r)}} \right) \left( \sum_{r=1}^{q_j-1} \sum_{s=1}^{2(q_j-r)-1} M_r(f_{j,n}, s) \right)$$

and

$$A_{3,n} = \left( \sum_{r=1}^{q_j-1} \|f_{j,n} \otimes_r f_{j,n}\|^4_{\mathfrak{H}^{\otimes 2(q_j-r)}} \right) \left( \sum_{r=1}^{q_i-1} \sum_{s=1}^{2(q_i-r)-1} M_r(f_{i,n}, s) \right)$$

As a consequence, we get

$$\frac{\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \sum_{s=1}^{q_i+q_j-2r-1} \|(f_{i,n} \widetilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \widetilde{\otimes}_r f_{j,n})\|_{\mathfrak{H}^{\otimes 2(q_i+q_j-2r-1-s)}}}{\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{\mathfrak{H}^{\otimes (q_i+q_j-2r)}}} \preccurlyeq \left( \frac{\sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}}}{\operatorname{Var} \Gamma_{ii}(F_n) \operatorname{Var} \Gamma_{jj}(F_n)} \right)^{1/2} \to 0,$$

where the convergence to zero is implied by assumptions (4.21) and (4.20). In view of Proposition 4.2, this proves (4.22). By the same argument we obtain (4.23), as $\Delta_C(F_n) \preccurlyeq \Delta_\Gamma(F_n)$ implies that

$$\sqrt{\operatorname{Var} \Gamma_{ij}(F_n)} + |\operatorname{E}[F_{i,n} F_{j,n}] - C_{ij}| \asymp \varphi_C(F_n)$$

is equivalent to $\sqrt{\operatorname{Var} \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$.

To prove (4.24), note that the Cauchy-Schwarz inequality implies for $1 \leq r < q_i$ that

$$\begin{aligned} \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{\mathfrak{H}^{\otimes (q_i+q_j-2r)}} &\leq \|f_{i,n} \otimes_r f_{j,n}\|^2_{\mathfrak{H}^{\otimes (q_i+q_j-2r)}} \\ &\leq \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j-r)}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{r=1}^{q_i \wedge q_j - 1} \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{\mathfrak{H}^{\otimes (q_i+q_j-2r)}} \\ &\quad \leq \sum_{r=1}^{q_i \wedge q_j - 1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j-r)}} \\ &\quad \leq \left( \sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \right) \left( \sum_{r=1}^{q_j-1} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j-r)}} \right) \end{aligned}$$

Together with assumption (4.20), this gives

$$\begin{aligned}
\mathrm{Var}_{ij}(F_n) &= \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}} \\
&\preccurlyeq (1-\delta_{q_i q_j}) \|f_{i,n} \widetilde{\otimes}_{q_i} f_{j,n}\|^2_{\mathfrak{H}^{\otimes(q_j-q_i)}} + \sqrt{\mathrm{Var}_{ii}(F_n)\,\mathrm{Var}_{jj}(F_n)} \\
&\preccurlyeq (1-\delta_{q_i q_j}) \|f_{i,n} \widetilde{\otimes}_{q_i} f_{j,n}\|^2_{\mathfrak{H}^{\otimes(q_j-q_i)}} + \mathrm{Var}_{ii}(F_n) + \mathrm{Var}_{jj}(F_n) \\
&\preccurlyeq \Delta_\Gamma(F_n)^2,
\end{aligned}$$

where we set $\mathfrak{H}^{\otimes 0} = \mathbb{R}$. As $\sqrt{\mathrm{Var}_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$, this shows

$$\mathrm{Var}_{ij}(F_n) \asymp (1-\delta_{q_i q_j}) \|f_{i,n} \widetilde{\otimes}_{q_i} f_{j,n}\|^2_{\mathfrak{H}^{\otimes(q_j-q_i)}} + \sqrt{\mathrm{Var}_{ii}(F_n)\,\mathrm{Var}_{jj}(F_n)}$$

and therefore (4.24).

Now let $1 \le r \le q_i \wedge q_j - \delta_{q_i q_j}$, $1 \le s \le q_i + q_j - 2r - 1$ and $G_{r,s}(f_{i,n}, f_{j,n}, \lambda)$ be an integral from the right hand side of representation (4.15). By the Majorizing Lemma 4.5, we can find integers $n_k \in \{0, 1, \dots, q_i - r\}$ and $m_k \in \{0, 1, \dots, q_j - r\}$ such that $m_k + n_k = s$ for $1 \le k \le 4$ and

$$G_{r,s}(f_{i,n}, f_{j,n}, \lambda) \le \prod_{k=1}^{4} \big(M_r(f_{i,n}, m_k) M_r(f_{j,n}, n_k)\big)^{1/8}.$$

As clearly

$$M_r(f_{i,n}, m_k)\, M_r(f_{j,n}, n_k) \le A_{1,n} + A_{2,n} + A_{3,n},$$

this gives

$$G_{r,s}(f_{i,n}, f_{j,n}, \lambda) \le \prod_{k=1}^{4} (A_{1,n} + A_{2,n} + A_{3,n})^{1/8} \le \sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}}.$$

The representation (4.15) thus yields

$$\|(f_{i,n} \widetilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \widetilde{\otimes}_r f_{j,n})\|^2_{\mathfrak{H}^{\otimes 2(q_i+q_j-2r-s)}} \preccurlyeq \sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}}\,,$$

wich immediately implies (4.25). □

*Remark* 4.7. a) Note that (4.20) is satisfied, if the kernels $f_{i,n}$ are either all non-negative or all non-positive for $n \ge n_0$.

b) In the case where $q_i = 2$ for $1 \le i \le d$, we obtain Proposition 4.3 as a special case of Theorem 4.6. Indeed, (4.20) is trivially satisfied, and, by (4.8) and (4.9), conditions (i) and (ii) are equivalent to (4.10) and (4.11), respectively.

## 5. Examples

In this section, we provide several examples that illustrate our techniques.

5.1. *Step functions and matrix representations.* We start with a counterexample, that in a way shows that the kernels involved can not be too "simple" in order for our techniques to work. Let $\mathfrak{H} = L^2([0,1), \mu)$, where $\mu$ is the Lebesgue measure and partition $[0,1)$ into $N$ equidistant intervals $\alpha_1, \alpha_2, \dots, \alpha_N$ where $\alpha_k = \left[\frac{k-1}{N}, \frac{k}{N}\right)$

for $k = 1, \dots, N$. Using this partition, we endow $[0,1)^2$ with a grid and define a symmetric kernel $f \in \mathfrak{H}^{\odot 2}$ that is constant on each sector by

$$f(x,y) = \sum_{i,j=1}^{N} a_{ij}\, 1_{\alpha_i}(x) 1_{\alpha_j}(y), \tag{5.1}$$

where the $a_{ij}$ are real constants and $a_{ij} = a_{ji}$. Of course, $f$ is uniquely determined by the symmetric matrix $A = (a_{ij})_{1 \le i,j \le N}$. If $g$ is another kernel of the type (5.1), given by a matrix $B = (b_{ij})_{1 \le i,j \le N}$, we have

$$\begin{aligned}(f \otimes_1 g)(x,y) &= \int_0^1 \left( \sum_{i,j=1}^{N} a_{ij}\, 1_{\alpha_i}(t) 1_{\alpha_j}(x) \right) \left( \sum_{k,l=1}^{N} b_{kl}\, 1_{\alpha_k}(t) 1_{\alpha_l}(y) \right) \mathrm{d}\mu(t) \\ &= \sum_{i,j,l=1}^{N} a_{ij}\, b_{jl}\, \mu(\alpha_j)\, 1_{\alpha_i}(x) 1_{\alpha_l}(y) \\ &= \frac{1}{N} \sum_{i,j,l=1}^{N} a_{ij}\, b_{jl} 1_{\alpha_i}(x) 1_{\alpha_l}(y)\end{aligned}$$

and

$$(f \widetilde{\otimes}_1 g)(x,y) = \frac{1}{2N} \sum_{i,j,l=1}^{N} (a_{ij}\, b_{jl} + a_{lj}\, b_{ji})\, 1_{\alpha_i}(x) 1_{\alpha_l}(y).$$

Therefore, $f \otimes_1 g$ can be identified with the matrix $C = \frac{1}{N} AB$ and $f \widetilde{\otimes}_1 g$ by $\frac{1}{2}\left(C + C^T\right)$, where $C^T$ denotes the transpose of $C$. Analogously, one can show that

$$\langle f, g \rangle_{\mathfrak{H}^{\otimes 2}} = \frac{1}{N^2} \langle A, B \rangle_{H.S.} = \frac{\operatorname{tr}(AB^T)}{N^2}.$$

By formula (4.7), it is now easy to see that for any $m \ge 2$ we get

$$\kappa_m(I_2(f)) = 2^{m-1}(m-1)! \frac{\operatorname{tr} A^m}{N^m} \tag{5.2}$$

and therefore, by the Cauchy-Schwarz inequality,

$$\frac{3!^2}{2 \times 7!} \frac{\kappa_8(I_2(f))}{\kappa_4(I_2(f))^2} = \frac{\operatorname{tr}\left(A^8\right)}{\operatorname{tr}\left(A^4\right)^2} = \left(1 + \frac{\sum_{1 \le k,l \le N, k \ne l} \lambda_k^4 \lambda_l^4}{\sum_{1 \le k \le N} \lambda_k^8}\right)^{-1} \ge \frac{1}{2}, \tag{5.3}$$

where $(\lambda_k)_{1 \le k \le N}$ denotes the eigenvalue sequence of $A$. Now fix $d \ge 1$ and choose a sequence $(N_n)$ of positive integers greater than two. For $n \ge 1$, define random vectors $F_n = (I_2(f_{1,n}), \dots, I_2(f_{d,n}))$, where the kernels $f_{i,n}$ are given by symmetric $(N_n \times N_n)$-matrices $A_{i,n}$, $1 \le i \le d$, such that

$$\frac{\operatorname{tr} A_{i,n}^4}{N_n^4} \to 0 \qquad (n \to \infty)$$

for $1 \le i \le d$. Then, by the Fourth Moment Theorem 2.8, $F_n$ converges in distribution to a $d$-dimensional, centered Gaussian random vector. However, by (5.3), condition (4.11) is never satisfied and thus Proposition 4.6 fails to provide optimal rates of convergence.

*Remark* 5.1. The following explicit example illustrates how symmetrization can drastically increase the speed of convergence (see Nourdin and Rosinski (2011), Remark 3.2(3) for another example): If $N = 2$ and the kernels $f$ and $g$ are represented by the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively, it holds that $AB + BA = 0$ and $\operatorname{tr}(ABBA) = 2$. Therefore, $\|f \widetilde{\otimes}_1 g\|^2_{\mathfrak{H}^{\otimes 2}} = 0$ while $\|f \otimes_1 g\|^2_{\mathfrak{H}^{\otimes 2}} = \frac{1}{8}$.

5.2. *Exploding integrals of Brownian sheets.* Let $W = \{W(t_1, \dots, t_l) \colon 0 \leq t_1, \dots, t_l \leq 1\}$ be a standard Brownian sheet on $[0,1]^l$, i.e. a centered Gaussian process such that

$$\mathrm{E}\left[W(s_1, \dots, s_l) W(t_1, \dots, t_l)\right] = \prod_{i=1}^{l} (s_i \wedge t_i)$$

for all $(s_1, \dots, s_l), (t_1, \dots, t_l) \in [0,1]^l$. We can identify the Gaussian space generated by $W$ with an isonormal process $X$ on $\mathfrak{H} = L^2([0,1]^l)$ via

$$W(t_1, \dots, t_l) \sim X\left( \prod_{i=1}^{l} 1_{[0,t_i]} \right).$$

For positive $\varepsilon$, we now define

$$F_\varepsilon = \int_{[0,1]^l} \frac{W(t_1, \dots, t_l)^2}{(t_1\, t_2 \cdots t_l)^{2-\varepsilon}} \, \mathrm{d}(t_1, \dots, t_l). \tag{5.4}$$

An application of Jeulin's lemma (see Jeulin (1980, Lemma 1, p. 44)) shows that $F_\varepsilon$ "explodes" in the limit, i.e. that $P$-almost surely

$$F_\varepsilon \to \infty \qquad (\varepsilon \to 0).$$

However, for the normalized sequence $\widetilde{F}_\varepsilon$, defined by

$$\widetilde{F}_\varepsilon = \frac{F_\varepsilon - \mathrm{E}\left[F_\varepsilon\right]}{\sqrt{\mathrm{E}\left[F_\varepsilon^2\right]}}, \tag{5.5}$$

the central limit theorem

$$\widetilde{F}_\varepsilon \xrightarrow{\mathcal{L}} Z \sim \mathcal{N}(0,1) \qquad (\varepsilon \to 0). \tag{5.6}$$

holds. This is a consequence of the Fourth Moment Theorem 2.8 and the forthcoming formula (5.12) that provides asymptotics for the cumulants of $\widetilde{F}_\varepsilon$. For slightly different exploding functionals of the above type, an analogous central limit theorem was established in Peccati and Yor (2004) for the case $l = 1$, Deheuvels et al. (2006) for the case $l = 2$ and Nualart and Peccati (2005) for the case $l > 2$. Exact asymptotics in the Kolmogorov distance were provided in Nourdin and Peccati (2009a). Here, we are interested in vectors of such functionals.

Routine calculations show that

$$\widetilde{F}_\varepsilon = \frac{F_\varepsilon - \mu_\varepsilon}{\sigma_\varepsilon} = \frac{1}{\sigma_\varepsilon} I_2(f_\varepsilon),$$

where

$$f_\varepsilon(x_1, \dots, x_l, y_1, \dots, y_l) = \prod_{k=1}^{l} \int_0^1 \frac{1_{[0,t_k]}(x_k) 1_{[0,t_k]}(y_k)}{t_k^{2-\varepsilon}} \, \mathrm{d}t_k, \tag{5.7}$$

$\mu_\varepsilon = \mathrm{E}\left[\widetilde{F}_\varepsilon\right]$, and $\sigma_\varepsilon^2 = \mathrm{E}\left[\widetilde{F}_\varepsilon^2\right]$.

From (5.7) we conclude that if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ are $k$ positive numbers, then

$$\left\langle (\cdots((f_{\varepsilon_1} \widetilde{\otimes}_1 f_{\varepsilon_2}) \widetilde{\otimes}_1 f_{\varepsilon_3}) \widetilde{\otimes}_1 \cdots) \widetilde{\otimes}_1 f_{\varepsilon_{k-1}}, f_{\varepsilon_k} \right\rangle_{\mathfrak{H}} = C(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)^l, \tag{5.8}$$

where

$$C(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) = \int_{[0,1]^k} \frac{(s_1 \wedge s_2)(s_2 \wedge s_3) \cdots (s_{k-1} \wedge s_k)(s_k \wedge s_1)}{s_1^{2-\varepsilon_1} s_2^{2-\varepsilon_2} \cdots s_k^{2-\varepsilon_k}} \, \mathrm{d}(s_1, \ldots, s_k). \tag{5.9}$$

For convenience, we will write $C_k(\varepsilon) = C(\varepsilon, \ldots, \varepsilon)$, if all $k$ arguments are equal. With this notation, we have $\mu_\varepsilon = C(\varepsilon)^l$ and $\sigma_\varepsilon^2 = C_2(\varepsilon)^l$. By partitioning the $k$-dimensional unit interval into simplexes, the integral (5.9) can be computed explicitly. These calculations yield that

$$C(\varepsilon_1, \ldots, \varepsilon_k) = \widetilde{c}(\varepsilon_1, \ldots, \varepsilon_k) \times \frac{k!}{\varepsilon_1 + \ldots + \varepsilon_k}. \tag{5.10}$$

Here, $\widetilde{c}(\varepsilon_1, \ldots, \varepsilon_k)$ is the canonical symmetrization of

$$c(\varepsilon_1, \ldots, \varepsilon_k) = \big((1+\varepsilon_1)(1+\varepsilon_1+\varepsilon_2) \cdots (1+\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_{k-1})\big)^{-1}.$$

Observe that $0 < \widetilde{c}(\varepsilon_1, \ldots, \varepsilon_k) < 1$ and $\widetilde{c}(\varepsilon_1, \ldots, \varepsilon_k) \to 1$ if $\varepsilon_1, \ldots, \varepsilon_k \to 0$.

Now fix $d \geq 1$ and, for $\varepsilon_1, \ldots, \varepsilon_d > 0$, define $\widetilde{F}_{(\varepsilon_1, \ldots, \varepsilon_d)} = (\widetilde{F}_{\varepsilon_1}, \ldots, \ldots, \widetilde{F}_{\varepsilon_d})$. As the inner product on the left hand side of (5.8) does not depend on the order in which the $k$ kernels $f_{\varepsilon_1}, f_{\varepsilon_2}, \ldots, f_{\varepsilon_k}$ are contracted, the cumulant formula (2.22) yields that

$$\kappa_\alpha(\widetilde{F}_{(\varepsilon_1, \ldots, \varepsilon_d)}) = 2^{|\alpha|-1} (|\alpha|-1)! \, C(\varepsilon_{l_1}, \varepsilon_{l_2}, \ldots, \varepsilon_{l_{|\alpha|}})^l \prod_{k=1}^{|\alpha|} C_2(\varepsilon_{l_k})^{-l/2}, \tag{5.11}$$

where $\alpha \in \mathbb{N}_0^d$ with elementary decomposition $\{l_1, \ldots, l_{|\alpha|}\}$. Identity (5.10) shows that

$$\kappa_\alpha(\widetilde{F}_n)^{-l} \asymp \frac{\left(\varepsilon_{l_1} \cdots \varepsilon_{l_{|\alpha|}}\right)^{1/2}}{\varepsilon_{l_1} + \cdots + \varepsilon_{l_{|\alpha|}}}. \tag{5.12}$$

This allows us to apply Proposition 4.6 to the vector $\widetilde{F}_{(\varepsilon_1, \ldots, \varepsilon_d)}$. We obtain the following explicit result.

**Proposition 5.2.** *For $\varepsilon_1, \ldots, \varepsilon_d > 0$, let $\widetilde{F}_{\varepsilon_1, \ldots, \varepsilon_d} = (\widetilde{F}_{\varepsilon_1}, \ldots, \widetilde{F}_{\varepsilon_d})$ be defined by (5.5) and let $g \colon \mathbb{R}^2 \to \mathbb{R}$ be a three times differentiable function with bounded derivatives up to order three. Then it holds that*

$$\frac{\mathrm{E}\left[g(\widetilde{F}_{\varepsilon_1, \ldots, \varepsilon_d})\right] - \mathcal{E}_3(\widetilde{F}_{\varepsilon_1, \ldots, \varepsilon_d}, Z_{\varepsilon_1, \ldots, \varepsilon_d}, g)}{\sqrt{\sum_{i=1}^d \varepsilon_i^l}} \to 0, \qquad (\varepsilon_1, \ldots, \varepsilon_d \to 0), \tag{5.13}$$

*where $Z_{\varepsilon_1, \ldots, \varepsilon_d}$ is a centered Gaussian random variable with the same covariance matrix as $\widetilde{F}_{(\varepsilon_1, \ldots, \varepsilon_d)}$. If, in addition, for $1 \leq i, j \leq d$ it holds that*

$$\frac{2}{\sqrt{\frac{\varepsilon_i}{\varepsilon_j}} + \sqrt{\frac{\varepsilon_j}{\varepsilon_i}}} \to C_{ij}^{1/l}, \qquad (\varepsilon_1, \ldots, \varepsilon_d \to 0), \tag{5.14}$$

*and*

$$\frac{\sum_{i,j=1}^{d}\left(\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j}}+\sqrt{\frac{\varepsilon_j}{\varepsilon_i}}\right)^{-l}-\frac{C_{ij}}{2^l}\right)^2}{\sum_{i=1}^{d}\varepsilon_i^l}\to 0, \qquad (\varepsilon_1,\dots,\varepsilon_d\to 0), \tag{5.15}$$

*where* $C_{ij}\geq 0$*, then*

$$\frac{\mathrm{E}\left[g(\widetilde{F}_{(\varepsilon_1,\dots,\varepsilon_d)})\right]-\mathcal{E}_3(\widetilde{F}_{(\varepsilon_1,\dots,\varepsilon_d)},Z,g)}{\sqrt{\sum_{i=1}^{d}\varepsilon_i^l}}\to 0 \qquad (\varepsilon_1,\dots,\varepsilon_d\to 0)$$

*and*

$$\lim_{\varepsilon_1,\dots,\varepsilon_d\to 0}\frac{\mathrm{E}\left[g(\widetilde{F}_{(\varepsilon_1,\dots,\varepsilon_d)})\right]-\mathrm{E}\left[g(Z)\right]}{\varphi_C(\widetilde{F}_{(\varepsilon_1,\dots,\varepsilon_d)})}$$
$$=4\sqrt{6}\lim_{\varepsilon_1,\dots,\varepsilon_d\to 0}\frac{\sum_{i,j,k=1}^{d}\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j\varepsilon_k}}+\sqrt{\frac{\varepsilon_j}{\varepsilon_i\varepsilon_k}}+\sqrt{\frac{\varepsilon_k}{\varepsilon_i\varepsilon_j}}\right)^{-l}\mathrm{E}\left[\partial_{ijk}g(Z)\right]}{\sqrt{\sum_{i,j=1}^{d}\left(\frac{1}{\varepsilon_i}+\frac{1}{\varepsilon_j}\right)^{-l}}}.$$

*Here,* $\varphi_C(\widetilde{F}_{(\varepsilon_1,\dots,\varepsilon_d)})^2\asymp\sum_{i=1}^{d}\varepsilon_i^l$ *and* $Z$ *is a* $d$*-dimensional, centered Gaussian random variable with covariance* $C=(C_{ij})_{i,j=1}^{d}$*.*

Note that by (5.11) and (5.10), the Edgeworth expansions $\mathcal{E}_3(\widetilde{F}_{\varepsilon_1,\dots,\varepsilon_d},Z_{\varepsilon_1,\dots,\varepsilon_d},g)$ and $\mathcal{E}_3(\widetilde{F}_{\varepsilon_1,\dots,\varepsilon_d},Z,g)$ can be calculated explicitly.

To illustrate this, we choose a positive sequence $(a_n)_{n\geq 1}$ converging to zero and two positive numbers $\xi$ and $\zeta$. Then it holds that $\widetilde{F}_{(\xi\cdot a_n,\zeta\cdot a_n)}$ has covariance

$$C=\begin{pmatrix}1 & \rho^l\\ \rho^l & 1\end{pmatrix}$$

for all $n\geq 1$, where

$$\rho=\frac{2}{\sqrt{\frac{\xi}{\zeta}}+\sqrt{\frac{\zeta}{\xi}}}.$$

Thus, the conditions (5.14) and (5.15) are trivially satisfied and all conclusions of Proposition 5.2 hold.

5.3. *Continuous time Toeplitz quadratic functionals.* Let $(X_t)_{t\geq 0}$ be a centered, real valued Gaussian process with a covariance function $r$ of the form $r(t)=\mathrm{E}\left[X_uX_{u+t}\right]=\widehat{f}(t)$, where $f\colon\mathbb{R}\to\mathbb{R}$ is an integrable, even function, customarily called the spectral density of the process $(X_t)$ and $\widehat{f}$ denotes its Fourier transform $t\mapsto\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xt}f(x)\,\mathrm{d}x$. If $h\colon\mathbb{R}\to\mathbb{R}$ is another integrable even function with Fourier transform $\widehat{h}$ and $T>0$, we define the Toeplitz functional $Q_{h,T}$ associated with $h$ and $T$ by

$$Q_{h,T}:=\int_{[0,T]^2}\widehat{h}(t-s)X_tX_s\,\mathrm{d}(s,t).$$

and denote a normalized version by

$$\widetilde{Q}_T = \frac{Q_{h,T} - \mathrm{E}\left[Q_{h,T}\right]}{\sqrt{T}}.$$

In the following, we want to apply our results to sequences of random vectors whose components are (normalized) Toeplitz functionals, analogous to the treatment in Nourdin and Peccati (2009a) for the one-dimensional case.

For $T > 0$ and $\psi \in L^1(\mathbb{R})$, the *truncated Toeplitz operator* $B_T(\psi)$, defined on $L^2(\mathbb{R})$, is given by

$$B_T(\psi)(u)(t) = \int_0^T u(x)\widehat{\psi}(t-x)\,\mathrm{d}x.$$

As usual, if $\psi_1, \dots, \psi_m \in L^1(R)$ and $j \geq 1$, we write

$$B_T(\psi_m)B_T(\psi_{m-1})\cdots B_T(\psi_1) = B_T(\psi_m) \circ B_T(\psi_{m-1}) \circ \cdots \circ B_T(\psi_1)$$

and

$$\left(B_T(\psi_2)B_T(\psi_1)\right)^j = \underbrace{B_T(\psi_2)B_T(\psi_1)\dots B_T(\psi_2)B_T(\psi_1)}_{j \text{ times}}$$

for its operator products and powers, respectively. Explicitly, the above operator product takes the form

$$\begin{aligned}&\big(B_T(\psi_m)B_T(\psi_{m-1})\cdots B_T(\psi_1)(u)\big)(t)\\ &= \int_0^T \cdots \int_0^T \int_0^T u(x_1)\widehat{\psi}_1(x_2-x_1)\widehat{\psi}_2(x_3-x_2)\cdots\widehat{\psi}_m(t-x_m)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_m.\end{aligned}$$

To adapt the setting to our framework, we introduce the Hilbert space of complex-valued, square integrable and even functions $h\colon \mathbb{R} \to \mathbb{C}$, equipped with the inner product $\langle h_1, h_2 \rangle_{\mathfrak{H}} = \int_{-\infty}^{\infty} h_1(x)\overline{h_2(x)}f(x)\,\mathrm{d}x$. The process $(X_t)_{t\geq 0}$ can then be identified with an isonormal Gaussian process on the real subspace $\mathfrak{H}$ generated by the family $\{x \mapsto \mathrm{e}^{\mathrm{i}xt}\colon t > 0\}$. This allows us to represent the normalized Toeplitz functional $\widetilde{Q}_{h,T}$ as a multiple integral of second order with kernel $\varphi_{h,T}$, which is given by

$$\varphi_{h,T}(x,y) = \int_{[0,T]^2} \widehat{h}(t-s)\mathrm{e}^{\mathrm{i}(sx+ty)}\,\mathrm{d}s\,\mathrm{d}t. \tag{5.16}$$

We now choose even functions $h_1, \dots, h_d \in L^1(\mathbb{R})$ be even functions and define a random vector $F_T = (F_{1,T}, \dots, F_{d,T})$ by setting $F_{i,T} = \widetilde{Q}_{h_i,T}$ for $1 \leq i \leq d$ and $T0$. The following Theorem collects some results from the literature.

**Theorem 5.3.** *Let $\alpha \in \mathbb{N}_0^d$ be a multi-index with $|\alpha| \geq 2$ and elementary decomposition $\{\, l_1, \dots, l_{|\alpha|}\,\}$. Then the following is true.*

*a) The cumulant $\kappa_\alpha(F_T)$ is given by*

$$\kappa_\alpha(F_T) = T^{-|\alpha|/2}\, 2^{|\alpha|-1}\,(|\alpha|-1)!\ \mathrm{Tr}\left[B_T(f)^{|\alpha|}\prod_{i=1}^{|\alpha|} B_T(h_{l_i})\right]. \tag{5.17}$$

*b) If $f \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$ and $h_i \in L^1(\mathbb{R}) \cap L^{q_i}(\mathbb{R})$ such that $1/q_0 + 1/q_i \leq 1/\,|\alpha|$ for $1 \leq i \leq d$, then*

$$\lim_{T\to\infty} T^{|\alpha|/2-1}\kappa_\alpha(F_T) = 2^{|\alpha|-1}(|\alpha|-1)! \int_{-\infty}^{\infty} f^{|\alpha|}(x)\prod_{i=1}^{|\alpha|} h_{l_i}(x)\,\mathrm{d}x. \tag{5.18}$$

*c) If $f \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$ and $h_i \in L^1(\mathbb{R}) \cap L^{q_i}(\mathbb{R})$ such that $1/q_0 + 1/q_i \leq 1/2$ for $1 \leq i \leq d$, then*

$$F_T \xrightarrow{d} \mathcal{N}(0, C), \qquad (T \to \infty), \tag{5.19}$$

*where the covariance matrix $C = (C_{ij})_{1 \leq i,j \leq d}$ is given by*

$$C_{ij} = 2 \int_0^\infty f^2(x) h_i(x) h_j(x) \, \mathrm{d}x.$$

*Proof*: Part a) follows from a straightforward adaptation of the arguments in Grenander and Szegő (1984, Chapter 11) to multiple dimensions. Part b) follows from a) and Ginovian (1994, Theorem 1a)). Finally, part c) is a consequence of part b) and the Fourth Moment Theorem 2.8. For the one-dimensional case ($d = 1$), part c) was first proven in Ginovian (1994). Weaker conditions for the convergence (5.19) to take place can be found in Ginovyan and Sahakyan (2007). □

We are now able to prove the following Edgeworth expansion.

**Proposition 5.4.** *In the above framework, assume that $f \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$, $h_i \in L^1(\mathbb{R}) \cap L^{q_i}(\mathbb{R})$ such that $1/q_0 + 1/q_i \leq 1/8$ for $1 \leq i \leq d$ and let $g \colon \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with bounded derivatives up to order three. Then it holds that*

$$\frac{\mathrm{E}\left[g(F_T)\right] - \mathcal{E}_3(F_T, Z, g)}{\varphi_C(F_T)} \to 0 \quad (T \to \infty), \tag{5.20}$$

*where $Z$ is a $d$-dimensional Gaussian random variable with covariance matrix $C = (C_{ij})_{1 \leq i,j \leq d}$ given by $C_{ij} = 2 \int_0^\infty f^2(x) h_i(x) h_j(x) \, \mathrm{d}x$.*

*Proof*: By Theorem 5.3b), we immediately verify the conditions of Proposition 4.6b) and obtain the result. □

5.4. *Edgeworth expansions for the Breuer-Major Theorem.* Define $B = \{B_x \colon x \geq 0\}$ to be a fractional Brownian motion with Hurst index $H \in (0, \frac{1}{2})$, i.e. a centered Gaussian process with covariance

$$\mathrm{E}\left[B_x B_y\right] = \frac{1}{2} \left( x^{2H} + y^{2H} - |x - y|^{2H} \right), \qquad x, y \geq 0.$$

For fixed $H \in (0, \frac{1}{2})$, the Gaussian space generated by $B$ can be identified with an isonormal Gaussian process $X = \{X(h) \colon h \in \mathfrak{H}\}$, where the real and separable Hilbert space $\mathfrak{H}$ is the closure of the set of all $\mathbb{R}$-valued step functions on $\mathbb{R}_+$ with respect to the inner product

$$\left\langle 1_{[0,x]}, 1_{[0,y]} \right\rangle_{\mathfrak{H}} := \mathrm{E}\left[B_x B_y\right].$$

In particular, we have $B_x = X(1_{[0,x]})$ for $x > 0$. For more details on fractional Brownian motion see for example Nourdin and Peccati (2012); Nualart (2006) or Nourdin (2012). We denote the covariance function of the stationary increment process $(B_{x+1} - B_x)_{x \geq 0}$ by

$$\rho(t) := \mathrm{E}\left[B_{x+1-x}\left(B_{x+1+t} - B_{x+t}\right)\right] = \frac{1}{2} \left( |t+1|^{2H} + |t-1|^{2H} - 2\,|t|^{2H} \right).$$

Now choose $d$ integers $q_i \geq 2$ and, for $T > 0$, define the centered random vectors $F_T = (F_{1,T}, \ldots, F_{d,T})$ by

$$F_{i,T} = \frac{1}{\sqrt{T}} \int_0^T H_{q_i}(B_{u+1} - B_u) \, \mathrm{d}u. \tag{5.21}$$

We immediately see that the covariance matrix $C_T = (C_{ij,T})_{1\le i,j\le d}$ of $F_T$ is given by

$$C_{ij,T} = \mathrm{E}\left[F_{i,T}F_{j,T}\right] = \delta_{q_i q_j}\frac{q_i!}{T}\int_{[0,T]^2}\rho^{q_i}(u-v)\,\mathrm{d}(u,v)$$

and converges to $C = (C_{ij})_{1\le i,j\le d}$ for $T\to\infty$, where

$$C_{ij} = \delta_{q_i q_j}\, q_i! \int_{-\infty}^{\infty}\rho^{q_i}(x)\,\mathrm{d}x.$$

It is well known (see for example Breuer and Major (1983) or Giraitis and Surgailis (1985)) that for each component $F_{i,T}$ the central limit theorem

$$F_{i,T}\xrightarrow{\mathcal{L}}\mathcal{N}(0,C_{ii}), \qquad (T\to\infty)$$

holds and the Fourth Moment Theorem 2.8A therefore implies the joint convergence of $F_T$ towards a centered $d$-dimensional Gaussian random vector $Z$ with covariance $C$. By applying our methods, we are able to derive a fluctuating and non-fluctuating Edgeworth expansion for $F_T$, which in many cases yield exact asymptotics.

**Theorem 5.5.** *If, in the above setting, $g$ is three times differentiable with bounded derivatives up to order three it holds that*

$$\frac{\mathrm{E}\left[g(F_T)\right]-\mathcal{E}_3(F_T,Z,g)}{1/\sqrt{T}}\to 0, \qquad (T\to\infty). \tag{5.22}$$

*Remark* 5.6. For a non-trivial application of Theorem 5.5 at least one of the integers $q_i$ should be even. Indeed, otherwise the Edgeworth expansion $\mathcal{E}_3(F_T,Z_T,g)$ (or $\mathcal{E}_3(F_T,Z,g)$, respectively) would merely reduce to the expectation $\mathrm{E}\left[g(Z_T)\right]$ (or $\mathrm{E}\left[g(Z)\right]$).

*Proof of Theorem 5.5:* . We want to apply Theorem 4.6. Observe that by the well-known relation $H_{q_i}(B_{u+1}-B_u) = I_q(1_{[u,u+1]}^{\otimes q_i})$ we can represent each component $F_{i,T}$ by a multiple integral $I_{q_i}(f_{i,T})$, where the kernels $f_{i,T}$ are given by

$$f_{i,T} = \frac{1}{\sqrt{T}}\int_0^T 1_{[u,u+1]}^{\otimes q}\,\mathrm{d}u.$$

Straightforward calculations now yield

$$\begin{aligned}
&\|f_{i,T}\otimes_r f_{i,T}\|^2_{\mathfrak{H}^{\otimes 2(q_i-r)}}\\
&\quad = \frac{1}{T^2}\int_{[0,T]^4}\rho(v_1-u_1)^r\rho(v_2-u_2)^r\\
&\qquad\qquad \rho(v_2-v_1)^{q_i-r}\rho(u_2-u_1)^{q_i-r}\,\mathrm{d}(u_1,u_2,v_1,v_2)\\
&\quad \asymp \frac{1}{T}\int_{[-T,T]^3}\rho(x_1)^r\rho(x_2)^r\rho(x_3)^{q_i-r}\rho(x_1-x_2+x_3)^{q_i-r}\,\mathrm{d}(x_1,x_2,x_3),
\end{aligned}$$

where the integral in the last line, which has been obtained by a change of variables, converges for $T\to\infty$ (due to the restriction $H\in(0,1/2)$). As symmetrizing only changes the exponents of the factors of the integrand, we see that

$$\|f_{i,T}\otimes_r f_{i,T}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \asymp \|f_{i,T}\widetilde{\otimes}_r f_{i,T}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \asymp \frac{1}{\sqrt{T}}. \tag{5.23}$$

Moreover, for $1 \le r \le q_i - 1$ and $1 \le m \le q_i - r - 1$, we see that the majorizing integrals $M_r(f_{i,T}, s)$ are given by

$$\begin{aligned} M_r(f_{i,T}, s) = \frac{1}{T^4} \int_{[0,T]^8} & \big(\rho(v_1-u_1)\rho(v_2-u_2)\rho(v_3-u_3)\rho(v_4-u_4)\big)^r \\ & \big(\rho(u_2-u_1)\rho(v_2-v_1)\rho(u_4-u_3)\rho(v_4-v_3)\big)^s \\ & \big(\rho(u_3-u_1)\rho(v_3-v_1)\rho(u_4-u_2)\rho(v_4-v_2)\big)^{q_i-r-s} \\ & \mathrm{d}(u_1,\dots,u_4,v_1,\dots,v_4) \\ \asymp \frac{1}{T^3} \int_{[-T,T]^7} & \big(\rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4)\big)^r \\ & \big(\rho(x_5)\rho(x_2+x_5-x_1)\rho(x_6)\rho(x_4+x_6-x_3)\big)^s \\ & \big(\rho(x_7)\rho(x_3+x_7-x_1)\rho(x_6+x_7-x_5) \\ & \qquad \rho(x_4+x_6+x_7-x_5-x_2)\big)^{q_i-r-s} \\ & \mathrm{d}(x_1,\dots,x_7). \end{aligned}$$

By the same argument as before, the integral converges for $T \to \infty$ and we get

$$M_r(f_{i,T}, s) \asymp \frac{1}{T^3}. \tag{5.24}$$

Together with the asymptotic relation (5.23), this shows that conditions conditions (i) and (ii) of Theorem 4.6 are satisfied. It remains to show that $\Delta_C(F_T) \preccurlyeq \Delta_\Gamma(F_T)$, so that (5.22) follows. by Theorem 4.6. A linear change of variables allows us to write

$$C_{ij,T} - C_{ij} = \delta_{q_i q_j} \frac{q_i!}{T} \int_0^T \int_{(-\infty,-v)\cup(T-v,\infty)} \rho^{q_i}(w)\,\mathrm{d}w\,\mathrm{d}v.$$

Using the well known asymptotic relation $\rho(t) \asymp t^{2(H-1)}$, we get

$$C_{ij,T} - C_{ij} \asymp \delta_{q_i q_j} T^{2(H-1)q_i+1}$$

and thus

$$\Delta_C(F_T) \asymp T^{1+2(H-1)q_{\min}}.$$

As, by (5.23), $\Delta_\Gamma(F_T) \asymp 1/\sqrt{T}$ and $1 + 2(H-1)q_{\min} < -1/2$, the proof is finished. □

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