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Quenched invariance principle for a long-range random walk with unbounded conductances

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Abstract. We consider a random walk on a random graph (V, E), where V is the set of open sites under i.i.d. Bernoulli site percolation on the d-dimensional integer set $\mathbf{Z}^d$ with $d \geq 2$, and the transition probabilities of the walk are generated by i.i.d. random conductances (positive numbers) assigned to the edges in E. This random walk in random environments has long range jumps and is reversible. We prove the quenched invariance principle for this walk when the random conductances are unbounded from above but uniformly bounded from zero by taking the corrector approach. To this end, we prove a metric comparison between the graph metric and the Euclidean metric on the graph (V, E), an estimation of a first-passage percolation and an almost surely weighted Poincaré inequality on (V, E), which are used to prove the quenched heat kernel estimations for the random walk.

1. Introduction

Let $\{\xi_x,\, x \in \mathbf{Z}^d\}$ denote a collection of i.i.d. Bernoulli random variables with $P(\xi_0 = 1) = p > 0$. The set of open sites of the Bernoulli site percolation is defined as $V = \{x \in \mathbf{Z}^d \mid \xi_x = 1\}$. The set E is defined as consisting of all the line segments which are parallel to the coordinate axes, have the points of V as the end points and contain no other points of V. We refer the set E as the edge set. Thus (V, E) is a random graph. Observe that the edges in E have the form $(x, x + he_i)$, where e_i, $i = 1, \cdots, d$, denote the standard unit vectors of $\mathbf{Z}^d$, $h \geq 1$, $x, x + he_i \in V$ and $x + ke_i \notin V$ for all $1 \leq k < h$ when $h > 1$. We assign to the edge $(x, x + he_i)$ a positive random number $\mu_x^{(i)}$ referred as the conductance of the edge for any $h > 0$.

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We assume that $\{\mu_x^{(i)},\ i=1,\cdots,d,\ x\in\mathbf{Z}^d\}$ is a collection of i.i.d. positive random variables. Thus we get a random weighted graph (V,E,μ) , where μ denotes the set of the conductances of the edges in E.

Suppose that a probability space $(\Omega,\mathcal{A},\mathbb{P})$ carrying the above model, where $\Omega=\{\{0,1\}\times[1,\infty)^d\}^{\mathbf{Z}^d}$, $\mathcal{A}$ denotes the $\sigma-$field generated by the cylinder sets, and $\mathbb{P}$ is the product measure.

In this paper, we assume that

$$d\geq 2,\ \mathbb{P}(\xi_0=1)=p>0,\ \mathbb{P}(\mu(e)\geq 1)=1, \tag{1.1}$$

where $\mu(e)$ denotes the generic conductance of an edge $e\in E$.

Define a subspace $\Omega_0=\{\omega\in\Omega\mid\xi_0(\omega)=1\}$, i.e., $0\in V(\omega)$, $\forall\omega\in\Omega_0$, and the conditional measure on Ω_0 is defined as follows,

$$\mathbb{Q}(\cdot)=\mathbb{P}(\cdot\mid\Omega_0). \tag{1.2}$$

By taking the convention of $\mu(x,y)=0,\ \forall(x,y)\notin E$, we have $\mu(x,y)=\mu(y,x)$ for any $x,y\in\mathbf{Z}^d$. Set $\mu(x)=\sum_y\mu(x,y)$. For fixed $\omega\in\Omega_0$, let $(V(\omega),E(\omega))$ denote the corresponding realization of (V,E). We define a family of transition probabilities on $(V(\omega),E(\omega))$ as follows,

$$P_\omega(x,y)=\frac{\mu(x,y,\omega)}{\mu(x,\omega)},\ \forall x,y\in V(\omega). \tag{1.3}$$

We follow the approach of Barlow and Deuschel (2010) to consider two natural continuous time random walks equipped with the above transition probabilities (1.3), which have the following generators,

$$\mathcal{Q}_{v,\omega}:\mathcal{Q}_{v,\omega}f(x)=\sum_{y:(x,y)\in E(\omega)}\mu(x,y,\omega)(f(y)-f(x)), \tag{1.4}$$

$$\mathcal{Q}_{c,\omega}:\mathcal{Q}_{c,\omega}f(x)=\mu(x,\omega)^{-1}\sum_{y:(x,y)\in E(\omega)}\mu(x,y,\omega)(f(y)-f(x)), \tag{1.5}$$

where $f:V(\omega)\to\mathbf{R}$ is bounded measurable function.

The random walk defined by $\mathcal{Q}_{v,\omega}$ (1.4) will be called the *variable speed random walk* (**VSRW**) which waits at the current position, say x, until a Poisson clock with rate $\mu(x,\omega)$ rings, and then jumps according to the transition probabilities (1.3). The random walk defined by $\mathcal{Q}_{c,\omega}$ (1.5) will be called the *constant speed random walk* (**CSRW**) which waits at each position until a Poisson clock with rate 1 rings, and then jumps according to the same transition probabilities (1.3). In fact, the **VSRW** and the **CSRW** are time-changed processes with respect to each other.

Let $X(\omega)=(X_t(\omega),t\geq 0)$ and $\widetilde{X}(\omega)=(\widetilde{X}_t(\omega),t\geq 0)$ denote the **VSRW** and the **CSRW** on $(V(\omega),E(\omega))$ starting from the origin respectively, i.e., $X_0(\omega)=\widetilde{X}_0(\omega)=0$. Let $P_\omega(\cdot)$ denote the measure of the **VSRW**, and the annealed measure of the **VSRW** is defined as

$$Q(\cdot)=\mathbb{Q}(d\omega)\times P_\omega(\cdot). \tag{1.6}$$

Define the rescaled process of the **VSRW** as follows,

$$X_t^{(\epsilon)}(\omega)=\epsilon X_{t/\epsilon^2}(\omega),\ t\in[0,1],\ \epsilon>0. \tag{1.7}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Under the condition* (1.1), *for almost every* $\omega\in\Omega_0$, *the following hold.*

(1) *For the* ***VSRW*** $X(\omega)$, *under* P_ω, $(X_t^{(\epsilon)}(\omega),\ t \in [0,1])$ *converges weakly to a* $d-$*dimensional Brownian Motion* $(B_t^d,\ t \in [0,1])$ *with the diffusion matrix* $\sigma_v^2 I_d$, *where* I_d *denotes the* $d \times d$ *unit matrix, and* σ_v^2 *is positive and independent of* ω. *Furthermore, the following equation holds,*

$$d\sigma_v^2 = E_Q(|X_1(\omega)|_2^2) - 2\|\nabla\chi\|_Q^2. \tag{1.8}$$

(2) *For the* ***CSRW*** $\widetilde{X}(\omega)$, $(\widetilde{X}_t^{(\epsilon)}(\omega),\ t \in [0,1])$ *converges weakly to a* $d-$*dimensional Brownian Motion* $(\widetilde{B}_t^d,\ t \in [0,1])$ *with the diffusion matrix* $\sigma_c^2 I_d$, *where*

$$\sigma_c^2 = \begin{cases} \sigma_v^2/(2d\mathbb{E}\mu(e)), & \mathbb{E}(\mu(e)) < \infty; \\ 0, & \mathbb{E}(\mu(e)) = \infty. \end{cases} \tag{1.9}$$

In (1.8), $E_Q(|X_1(\omega)|_2^2)$ denotes the annealed quadratic moment of $X_1(\omega)$, where $|\cdot|_2$ denotes the Euclidean norm on $\mathbf{R}^d$, and $2\|\nabla\chi\|_Q^2$ accounts for the effect of the randomness of the environments. $\mathbb{E}(\cdot)$ denotes the expectation with respect to the measure $\mathbb{P}$. The other notations in (1.8) will be explained in Section 3.

To prove Theorem 1.1, we need the quenched heat kernel estimations for the **VSRW** $X(\omega)$ which are provided in the following theorem.

Theorem 1.2. *Write* $P_\omega^{(t)}(x,y) = P_\omega(X_t(\omega) = y \mid X_0(\omega) = x)$. *Under the condition* (1.1), *there exists a family of random variables* $\{U_x,\ x \in \mathbf{Z}^d\}$ *together with a constant* $\alpha \in (0,1)$ *and constants* $c_i > 0$ (*depending on* p, d *and the distribution of* $\mu(e)$) *such that*

$$\mathbb{P}(U_x(\omega) > n) \le c_1 \exp(-c_2 n^\alpha),\ \forall x \in \mathbf{Z}^d, \tag{1.10}$$

and for almost every $\omega \in \Omega$, *the following hold.*

$$P_\omega^{(t)}(x,y) \le c_3 t^{-d/2},\ \forall x,y \in V(\omega),\ t \ge 0. \tag{1.11}$$

For $x, y \in V(\omega)$, *if* $|x-y|_\infty \le t^{1/2}$ *or* $|x-y|_\infty > C_0 U_x(\omega)$, *where the constant* $C_0 = C_0(p,d) \ge 1$, *then*

$$P_\omega^{(t)}(x,y) \le c_4 t^{-d/2} \exp(-c_5|x-y|_\infty^2/t), \qquad t \ge c_6|x-y|_\infty; \tag{1.12}$$

$$P_\omega^{(t)}(x,y) \le c_4 \exp(-c_5|x-y|_\infty(1 \vee \log|x-y|_\infty/t)), \qquad t \le c_6|x-y|_\infty. \tag{1.13}$$

And if $U_x(\omega) \vee d_\omega(x,y) \le c_7 t^{1/2}$, *then*

$$P_\omega^{(t)}(x,y) \ge c_8 t^{-d/2}. \tag{1.14}$$

The motivation of this paper is to address the problem of the quenched invariance principle for a long range reversible random walk in random environment. By "long range" we mean that the step size of the random walk has no uniform finite upper bound. Indeed, the length of an edge of E obeys the geometric distribution with parameter p under the measure $\mathbb{P}$, so the step size of the **VSRW** is unbounded when $p < 1$. We take the corrector approach developed in recent years under the setting of the *random conductance model* (**RCM**) on the integer lattice, e.g., Sidoravicius and Sznitman (2004); Berger and Biskup (2007); Mathieu and Piatnitski (2007); Mathieu (2008); Biskup and Prescott (2007); Barlow and Deuschel (2010); Andres et al. (2013) and the recent survey paper Biskup (2011). To this end, we need the heat kernel estimations for the random walk under the Euclidean metric. Since the heat kernel estimations are naturally expressed under the graph metric, we also need a comparison between the Euclidean metric and the graph metric. Usually only the heat kernel upper bounds are needed to prove the functional CLT, but due to the unboundedness (from above) of the conductances we also need the heat kernel lower bounds. Generally, the existence of long edges leads to complicated graph structure which makes the heat kernel estimations hard to get even under the graph metric, and makes the comparison between the Euclidean metric and the graph metric hard either. Due to the specific structure of (V, E), we will show that the suitable heat kernel estimations for the **VSRW** can be proved.

The model of this paper is closely related to the **RCM** of Barlow and Deuschel (2010) in that the graph $(V(\omega), E(\omega))$ and the integer lattice share a feather which renders the Nash inequality to hold on both of them on one hand and the edge weights are uniformly bounded from zero, i.e., no dilution of the edges, on the other hand. These features are essential for the methods used in this paper. The difficulties of our model are mainly due to the existence of long edges in the graph $(V(\omega), E(\omega))$ and the unboundedness of the conductances. Naturally one may consider the cases where the conductances are not bounded from zero. Under the setting of the **RCM** on the integer lattice the quenched invariance principle was already established with the conductances unbounded from zero, for example, see Andres et al. (2013) for the conductances unbounded both from zero and from above which extends the results of Barlow and Deuschel (2010), see Biskup and Prescott (2007) and Mathieu (2008) for the conductances unbounded from zero but bounded from above, see Sidoravicius and Sznitman (2004), Berger and Biskup (2007) and Mathieu and Piatnitski (2007) on the supercritical Bernoulli bond percolation cluster as a special **RCM**, and all the proofs of them rely in some way on the Gaussian heat kernel upper bounds for a walk on the supercritical Bernoulli bond percolation cluster as that in Barlow (2004) and Andres et al. (2013). In this view, to extend the result of this paper to the setting of the conductances unbounded from zero, at least a weak Poincaré inequality as that of Lemma 2.8 and a metric comparison result as that of Lemma 2.2 need to hold on the " *infinite Bernoulli bond percolation cluster*" on $(V(\omega), E(\omega))$, but we are not able to establish these at the present time. Note that our methods to prove Lemma 2.8 and Lemma 2.2 are not valid in this setting since they rely on the independence structure of (V, E), but the result of Lemma 2.5 may be of help.

Now we sketch the proofs of this paper. The key observations for the proof of the metric comparison between the Euclidean metric and the graph metric on the graph $(V(\omega), E(\omega))$ in Lemma 2.1 and Lemma 2.2 are that under the measure $\mathbb{P}$ the lengths of the edges of E without intersection are i.i.d geometric random variables and any self-avoiding path (without intersection) on (V, E) consists of i.i.d. Bernoulli random variables indexed by the sites in the path. We get a quenched locally isoperimetric inequality on $(V(\omega), E(\omega))$ in Lemma 2.7 by exploiting the structure of the random graph (V, E) which leads to a weak Poincaré inequality of Lemma 2.8, and then to a weighted Poincaré inequality of Proposition 2.11 by a result of Barlow (2004). We use the Loomis-Whitney inequality to get an (globally) isoperimetric inequality on $(V(\omega), E(\omega))$ exactly as the case of the integer lattice which leads to the Nash inequality on $(V(\omega), E(\omega))$. To treat the unboundedness of the conductance, we also give a first-passage percolation estimation on (V, E) in Proposition 2.12 by applying the theory of the first-passage percolation developed under the setting of the integer lattice (e.g., Kesten (1986)). With the above estimations and inequalities we prove the heat kernel estimations of Theorem 1.2 by invoking the general results developed in Barlow and Deuschel (2010) and Barlow (2004). With the heat kernel bounds of Theorem 1.2 the proof of the functional CLT for the **VSRW** and the **CSRW** is carried out by taking the corrector approach as that of Barlow and Deuschel (2010) where the corrector is constructed based on a time discretization of the **VSRW**, see (3.1).

The structure of this paper is arranged as follows: In Section 2, we give the proof of Theorem 1.2, and this is the main part of this paper. Explicitly, in Subsection 2.1, the metric comparison between the Euclidean metric and the graph metric on $(V(\omega), E(\omega))$ is proved; In Subsection 2.2, we prove a weak Poincaré on $(V(\omega), E(\omega))$ which leads to a weighted Poincaré inequality on $(V(\omega), E(\omega))$; In Subsection 2.3, we give a first passage percolation estimation on (V, E) which is effectively a metric comparison between the graph metric and the metric derived from the first passage percolation; In Subsection 2.4, the proof of Theorem 1.2 is carried out by combining the previous results which we have proved. In Section 3, using the result of Theorem 1.2 we construct the corrector based on $\widehat{X}(\omega)$, a time discretization of the **VSRW** $X(\omega)$, and prove Theorem 1.1.

2. Heat Kernel Estimations

2.1. *Metric Comparison.* Let $|\cdot|_q$, $q \in [1, \infty]$, denote the $q-$norm on $\mathbf{R}^d$, e.g., for $x = (x_1, \cdots, x_d) \in \mathbf{Z}^d$, $|x|_1 = \sum_{i=1}^d |x_i|$ and $|x|_\infty = \max\{|x_1|, \cdots, |x_d|\}$. Let l^q, $q \in [1, \infty]$, denote the metrics derived from the norm $|\cdot|_q$ on $\mathbf{R}^d$ respectively and d_ω denote the graph metric on the graph $(V(\omega), E(\omega))$. For a set A (of sites or edges), $|A|$ denotes the cardinality of it, i.e., the number of elements contained in A.

For fixed $\omega \in \Omega$, let $B_\infty(x, n)$, $B_{d_\omega}(x, n)$, $x \in V(\omega)$ denote the balls centered at x with radius n under the metric l^∞ and the metric d_ω respectively, i.e.,

$$B_\infty(x, n) = \{y \in \mathbf{Z}^d \mid |y - x|_\infty \le n\}, \quad B_{d_\omega}(x, n) = \{y \in V(\omega) \mid d_\omega(x, y) \le n\}.$$

We will show that the two metrics, l^∞ and d_ω, are comparable at large scale. By this we mean that for any $x \in V(\omega)$ there exist constants C_0 and C_1 such that when n is large enough, the following relations hold,

$$B_{d_\omega}(x, n) \subset B_\infty(x, C_0 n) \text{ and } B_\infty(x, n) \cap V(\omega) \subset B_{d_\omega}(x, C_1 n). \tag{2.1}$$

Note that the metrics l^q, $q \in [1, \infty]$, are mutually comparable.

We consider the minimum radius n for the relation (2.1) to hold and give the bounds of the tail probabilities of this minimum radius under the measure $\mathbb{P}$. These are contained in Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *There exist constants $C_0 = C_0(p, d) \ge 1$ and $c_i = c_i(p, d) > 0$, such that for any $x \in V(\omega)$,*

$$u_x(\omega) = \min\{n > 0 \mid \forall m \ge n,\ B_{d_\omega}(x, m) \subset B_\infty(x, C_0 m)\},$$

we have

$$\mathbb{P}(u_x(\omega) > n) \le c_1 \exp(-c_2 n).$$

Proof: Without loss of generality, we consider the case with the center of the ball at the origin 0, and let $\xi_0(\omega) = 1$, otherwise there is nothing to prove.

By the definition of (V,E), all edges in E are line segments in $\mathbf{Z}^d$ which are parallel to the axes. It is easy to see that the lengths of the edges under the metric l^∞ are identically distributed geometric random variables with parameter p under the measure $\mathbb{P}$. Let $G(e)$ denote the length of the edge $e \in E$. Observe that for any collection of edges $A \subset E$, if the edges of A do not intersect except at the end vertices of them, then $\{G(e),\ e \in A\}$ is a collection of i.i.d. geometric variables with parameter p under the measure $\mathbb{P}$.

Let R_n denote a self avoiding path starting from the origin with n edges of E. Define

$$R_n^{(i)} = \{e \in R_n \mid \ e \text{ is parallel to the } i-th \text{ axis.}\},\ i = 1, \cdots, d.$$

Observe that the edges in any $R_n^{(i)}$ do not intersect except at the end vertices of them, so by the above discussions, $G(e),\ e \in R_n^{(i)}$, are i.i.d. geometric variables with parameter p under $\mathbb{P}$. Let r_n denote the other end point of the path R_n. Then we have

$$\begin{aligned}
\mathbb{P}(|r_n|_\infty \geq C_0 n) &\leq \mathbb{P}(\max_{i\in\{1,\cdots,d\}} \sum_{e\in R_n^{(i)}} G(e) \geq C_0 n) \\
&\leq \sum_{i=1}^{d} \mathbb{P}(\sum_{e\in R_n^{(i)}} G(e) \geq C_0 n) \\
&\leq \sum_{i=1}^{d} \exp(-aC_0 n)\mathbb{E}[\exp(a\sum_{e\in R_n^i} G(e))] \\
&\leq d\exp(-aC_0 n)[\mathbb{E}\exp(aG(e))]^n, \qquad (2.2)
\end{aligned}$$

where the values of the constants a and C_0 are to be determined.

Since, by definition, the degree of each vertex in the graph (V,E) equals $2d$, the total number of paths staring from the origin with n distinct edges is less than $2d(2d-1)^{n-1}$. And observe that $B_{d_\omega}(0,n) \subset B_\infty(0,C_0 n)$ is equivalent to $\max_{x\in B_{d_\omega}(0,n)} |x|_\infty \leq C_0 n$. It is easy to show that the value $\max_{x\in B_{d_\omega}(0,n)} |x|_\infty$ is achieved at the boundary of $B_{d_\omega}(0,n)$ which have graph distance n from the origin.

By the inequality (2.2), to choose $a > 0$ such that $\mathbb{E}\exp(aG(e)) < \infty$ and C_0 large enough, there exist positive constants c_1, c_2, such that we have

$$\begin{aligned}
\mathbb{P}[B_{d_\omega}(0,n) \notin B_\infty(0,C_0 n)] &\leq 2d(2d-1)^{n-1} d\exp(-aC_0 n)[\mathbb{E}\exp(aG(e))]^n \\
&\leq c_1\exp(-c_2 n).
\end{aligned}$$

Using the above inequality we get that

$$\begin{aligned}
\mathbb{P}(u_0(\omega) > n) &\leq \mathbb{P}(\exists m \geq n,\ B_{d_\omega}(0,m) \subsetneq B_\infty(0,C_0 m)) \\
&\leq \sum_{m=n}^{\infty} c_1\exp(-c_2 m) \leq c\exp(-c_2 n).
\end{aligned}$$

By the definition of (V,E), there exists $x \in B_{d_\omega}(0,n)$ such that $|x|_\infty \geq n$ almost surely, so the constant $C_0 \geq 1$. Thus the proof is completed. □

Lemma 2.2. *There exist finite constants $C_1 = C_1(p,d) > 0$ and $c_i = c_i(p,d) > 0$, such that for any $x \in V(\omega)$,*

$$v_x(\omega) = \min\{n > 0 \mid \forall m \geq n,\ B_\infty(x,m) \cap V(\omega) \subset B_{d_\omega}(x,C_1 m)\},$$

we have

$$\mathbb{P}(v_x(\omega) > n) \le c_1 \exp(-c_2 n).$$

Proof: Without loss of generality, we consider the case with the center of the ball at the origin 0. Let $\xi_0(\omega) = 1$, otherwise there is nothing to prove.

Define the $(d-1)-$dimensional hyperplanes of $\mathbf{Z}^d$ as follows,

$$H_i(0) = \{x = (x_1, \cdots, x_d) \in \mathbf{Z}^d \mid x_i = 0\},\ i = 1, \cdots, d.$$

Let $\mathcal{P}_i$ denote the projection operator onto $H_i(0)$, i.e., $\mathcal{P}_i(x)_i = 0$ for any $x \in \mathbf{Z}^d$.

Write $V_n = B_\infty(0,n) \cap V$. Let (V_n, E_n) denote the induced subgraph of (V,E) on V_n. For a fixed $x \in V_n$, we seek a path in (V_n, E_n) connecting x and 0 .

Define $I(x) = \{i \mid x_i = 0, i \le d.\}$. The cardinal number $|I(x)|$ equals the number of the zero coordinate elements of x, in particular $|I(0)| = d$. We perform the following greedy procedure to find a path connecting x and 0.

Step 0 : If $|I(x)| < d$, set $\pi^{(0)}(x) = \{x\}$, then go to Step 1; otherwise $x = 0$.

Step 1 : If there exists $j \notin I(x)$ with $\mathcal{P}_j(x) \in V$, then set $x^{(1)} = \mathcal{P}_j(x)$.

While if $\forall j \notin I(x)$, $\mathcal{P}_j(x) \notin V$, then pick the $i-$th coordinate axis with $x_i \ne 0$, and let x moving along the $i-$th axis towards 0, until find a site x' with $x' \in V$ such that there exists $j \notin I(x')$ with $\mathcal{P}_j(x') \in V$ (Note that $I(x) = I(x')$), and set $x^{(1)} = \mathcal{P}_j(x')$.

Denote the path connecting x and $x^{(1)}$ by $[x, x^{(1)}]$ and set $\pi^{(1)}(x) = \pi^{(0)}(x) \cup [x, x^{(1)}]$. It is easy to see that $I(x^{(1)}) = I(x) \cup \{j\}$ and $|I(x^{(1)})| = |I(x)| + 1$.

To justify the above procedure, we will show that the length of the segment $[x, x']$ is finite almost surely.

Let e_i denote the $i-$th standard unit vector of $\mathbf{Z}^d$. Then by the definition of V, we have

$$\mathbb{P}(x + e_i \in V) = p,\ \mathbb{P}\{\exists j \notin I(x), \mathcal{P}_j(x + e_i) \in V\} = 1 - (1-p)^{d-|I(x)|-1}.$$

Thus $\mathbb{P}(x' = x + e_i) = p(1-(1-p)^{d-|I(x)|-1})$ by independence. Furthermore,

$$\mathbb{P}(x' = x + he_i) = p(1-(1-p)^{d-|I(x)|-1}),\ \forall h \ne 0,\ h \in \mathbf{Z}$$

and all the events $\{x' = x + he_i\}$ are independent by the definition of V. Thus the length of the segment $[x, x']$ obeys the geometric distribution with parameter $p(1-(1-p)^{d-|I(x)|-1})$ under $\mathbb{P}$, and thus is finite almost surely.

We say that $[x, x']$ is a crossing segment if it crosses the boundary of a hyperplane, say H_i, which is equivalent to that $x_i x'_i < 0$.

Step 2 : To repeat Step 1 starting from the site $x^{(1)}$, denote the resulting site by $x^{(2)}$ and set $\pi^{(2)}(x) = \pi^{(1)}(x) \cup [x^{(1)}, x^{(2)}]$. We have $|I(x^{(2)})| = |I(x)| + 2$. Then to repeat Step 1 starting from $x^{(2)}$, and so on. After $d - |I(x)|$ repetitions, we get the site $x^{(d-|I(x)|)}$ and the path $\pi^{(d-|I(x)|)}(x)$. Since $|I(x^{(d-|I(x)|)})| = |I(x)| + d - |I(x)| = d$, $x^{(d-|I(x)|)} = 0$ by the definition, the resulting path $\pi^{(d-|I(x)|)}(x)$ is a path connecting x and 0.

Write $\pi(x) = \pi^{(d-|I(x)|)}(x)$. Let $|\pi(x)|$ denote the number of sites in $\pi(x)$. The following properties about the path $\pi(x)$ are immediate.

(1) The path $\pi(x)$ passes each point of $\mathbf{Z}^d$ at most once;

(2) If $\pi(x)$ does not contain a crossing segment, then $|\pi(x)| = |x|_1$;

(3) The path $\pi(x)$ contains at most $|I(x)|-1$ distinct crossing segments of which the lengths are independent geometric variables under $\mathbb{P}$ with parameters $p(1-(1-p)^k)$, $k \in \{1, \cdots, |I(x)|-1\}$, respectively.

Define

$$\eta(x) = \sum_{k=1}^{|I(x)|-1} \eta_k(x), \tag{2.3}$$

where $\eta_k(x)$ denotes a geometric variable with parameter $p(1-(1-p)^k)$ and the variables $\eta_k(x),\ k \in \{1, \cdots, |I(x)|-1\}$, are independent.

By the above properties of $\pi(x)$ and the definition of $\eta(x)$ (2.3), we have

$$|\pi(x)| \le |x|_1 + 2\eta(x). \tag{2.4}$$

Observe that $\{\xi_z,\ z \in \pi(x)\}$ is a collection of i.i.d. Bernoulli variables, and $\{z \in \pi(x), \xi_z = 1\}$ are the end vertexes of the edges in $\pi(x)$. Thus the number of edges in $\pi(x)$ equals $|\{z \in \pi(x),\ \xi_z = 1\}| - 1$ and $|\{z \in \pi(x),\ \xi_z = 1\}| - 2$ is a Binomial variable $B(|\pi(x)| - 1, p)$ under the measure $\mathbb{P}$. And to have $\pi(x) \subset (V_n, E_n)$, it is sufficient that $\eta(x) \le n$, see (2.3).

Combining the above discussions with (2.4), (2.3) and $|x|_1 < dn$, $\forall x \in V_n$, there exist a constant $C_1 = C_1(p,d) > 0$ and constants $c_i = c_i(p,d) > 0$ such that

$$\begin{aligned}
& \mathbb{P}(V_n(\omega) \subsetneq B_{d_\omega}(0, C_1 n)) \\
= \;& \mathbb{P}(\exists x \in V_n(\omega),\ d_\omega(0,x) > C_1 n) \\
\le \;& \sum_{|x|_\infty \le n} \mathbb{P}(x \in V(\omega),\ d_\omega(0,x) > C_1 n) \\
\le \;& \sum_{|x|_\infty \le n} [\mathbb{P}(|\{z \in \pi(x),\ \xi_z = 1\}| - 1 > C_1 n) + \mathbb{P}(\eta(x) > n)] \\
\le \;& \sum_{|x|_\infty \le n} [\mathbb{P}(B((d+2)n, p) > C_1 n) + \mathbb{P}(\eta(x) > n)] \\
\le \;& (2n+1)^d [c \exp(-c_2 n) + \exp(-c_1 n) \mathbb{E}(\exp c_1 \eta(x))] \\
\le \;& c \exp(-c_3 n),
\end{aligned} \tag{2.5}$$

where the constant $c_1 > 0$ is small enough such that $E(\exp c_1 \eta(x)) < \infty$, the constant C_1 is large enough such that we can use the large deviation estimations of the Binomial variable (see e.g., Theorem 2.2.3 of Dembo and Zeitouni, 1998) and the constant $c > 0$ assumes different values at different places.

Using the above inequality (2.5), we have

$$\begin{aligned}
\mathbb{P}(v_0(\omega) > n) &= \mathbb{P}(\exists m \ge n,\ V_m(\omega) \subsetneq B_{d_\omega}(x, C_1 m)) \\
&\le \sum_{m \ge n} \mathbb{P}(V_m(\omega) \subsetneq B_{d_\omega}(x, C_1 m)) \\
&\le \sum_{m \ge n} c \exp(-c_3 m) \le c' \exp(-c_3 n).
\end{aligned}$$

Thus the proof is completed. □

Remark 2.3. The metric comparison result of Lemma 2.2 has a similar form as that of the metric comparison result on the supercritical site percolation cluster on $\mathbf{Z}^d$ (under the l^1 metric), see e.g., Theorem 1.3 of Drewitz et al. (2012). But the result of Lemma 2.2 for supercritical p cannot be directly obtained from that on the

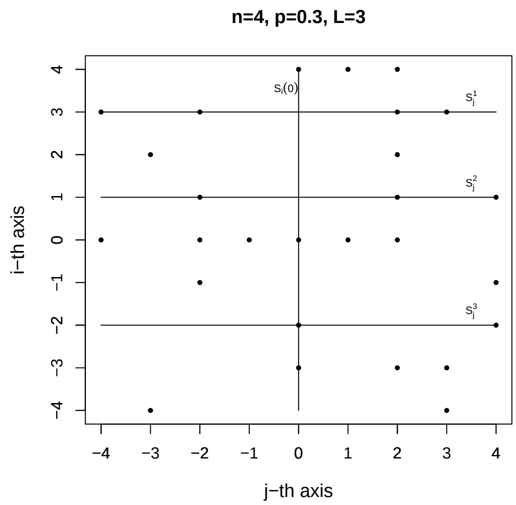

FIGURE 2.1.

supercritical site percolation cluster, because the site percolation cluster contains no long edges and a significant portion of V are not contained in it.

2.2. *A Weighted Poincaré Inequality on* $(V(\omega), E(\omega))$. Recalling that (V_n, E_n) denotes the induced subgraph of (V, E) on $B_\infty(0, n)$ (see Lemma 2.2), i.e.,

$$V_n = B_\infty(0, n) \cap V, \quad E_n = \{(x, y) \in E \mid x, y \in V_n\}. \tag{2.6}$$

With abuse of notation, let $\mathcal{P}_i$ denote the projection operator along the $i-$th axis, i.e., its action on $x \in \mathbf{Z}^d$ drops the $i-$th coordinate element of x (c.f. the $\mathcal{P}_i$ in Lemma 2.2).

For any $x \in \mathbf{Z}^d$, let $S_i(x)$ denote the line which contains the site z and is parallel to the $i-$th axis. For a finite number L, let $S_j^1, \cdots, S_j^L$, denote a collection of L parallel lines in the $i - j$ plane of $\mathbf{Z}^d$, which are parallel to the $j-$th axis. To illustrate these definitions, Figure 2.1 depicts a sample of the intersection of V_4 and the $i-j$ plane, where the dotted sites represent the open sites of the site percolation with parameter $p = 0.3$, and three randomly chosen parallel lines, S_j^1, S_j^2, S_j^3, and the line $S_i(0)$ are included. We consider the sets in the forms,

$$S_i(x) \cap V_n, \ |x|_\infty \le n, \ 1 \le i \le d, \tag{2.7}$$

$$\bigcup_{1 \le m \le L} \mathcal{P}_i(S_j^m \cap V_n), \ 1 \le i \ne j \le d, \ S_j^m \cap V_n \ne \emptyset. \tag{2.8}$$

We provide the uniform bound of the size of the sets in the forms (2.7) and (2.8) in Lemma 2.4.

Lemma 2.4. *Under the condition* (1.1)*, there exist constants* $c_i = c_i(p,d) > 0$ *such that*

$$\begin{aligned} & \mathbb{P}\{|S_i(x)\cap V_n|/(2n+1)\notin[p/2,2p],\ |x|_\infty\le n,\ 1\le i\le d.\} \\ \le\ & c_1\exp(-c_2 n), \qquad (2.9) \\ & \mathbb{P}\{\max_{\substack{1\le i\ne j\le d \\ \forall S_j^m,\ S_j^m\cap V_n\ne\emptyset,\ 1\le m\le L.}}[1-|\bigcup_{1\le m\le L}\mathcal{P}_i(S_j^m\cap V_n)|/(2n+1)]\ge 2(1-p)^L\} \\ \le\ & c_3\exp(-c_4 n). \qquad (2.10) \end{aligned}$$

Proof: Observe that for any $1\le i\le d$ and x with $|x|_\infty\le n$, $|S_i(x)\cap V_n|$ equals the number of open sites contained in a line segment having $2n+1$ sites on it . Then $|S_i(x)\cap V_n|$ obeys the Binomial distribution with parameters $2n+1$ and p under the measure $\mathbb{P}$.

Note that the number of distinct sets of the form (2.7) equals $d(2n+1)^{d-1}$. Then using the large deviation estimation of the Binomial variable, we have

$$\text{LHS of (2.9)}\ \le d(2n+1)^{d-1}\cdot c_1\exp(-cn)\le c_1\exp(-c_2 n).$$

To prove (2.10), we consider a set of the form (2.8), $\bigcup_{1\le m\le L}\mathcal{P}_i(S_j^m\cap V_n)\}$. For an element $z\in\mathcal{P}_i(\cup_{m=1}^L S_j^m\cap B_\infty(0,n))$, we say that z is unoccupied with respect to the parallel lines, $S_j^1,\cdots,S_j^L$, under the action of $\mathcal{P}_i$, if the L sites, $z(m)\in S_j^m$, $1\le m\le L$, which are mapped to z by $\mathcal{P}_i$, are closed sites, i.e., $z(m)\notin V$, otherwise we say that z is occupied. Then the set $\bigcup_{1\le m\le L}\mathcal{P}_i(S_j^m\cap V_n)$ consists of the occupied elements. By the definition of V, the probability of an element being unoccupied equals $(1-p)^L$ and the events of being unoccupied across the different elements of $\mathcal{P}_i(\cup_{m=1}^L S_j^m\cap B_\infty(0,n))$ are independent. Observe that $|\mathcal{P}_i(\cup_{m=1}^L S_j^m\cap B_\infty(0,n))| = 2n+1$, so the number of unoccupied elements obeys the Binomial distribution with parameters $2n+1$ and $(1-p)^L$ under the measure $\mathbb{P}$. Note that the sum of the number of unoccupied elements and the number of occupied elements equals $2n+1$. By combining the above observations, $2n+1-|\bigcup_{1\le m\le L}\mathcal{P}_i(S_j^m\cap V_n)\}|$ obeys the Binomial distribution with parameters $2n+1$ and $(1-p)^L$, and the density of the unoccupied elements equals $1-|\bigcup_{1\le m\le L}\mathcal{P}_i(S_j^m\cap V_n)|/(2n+1)$.

Note that the number of the sets of the form (2.8) equals $2\binom{d}{2}\binom{2n+1}{L}(2n+1)^{d-2}$. Then using the large deviation estimation of the Binomial variable, we have

$$\text{LHS of (2.10)}\ \le 2\binom{d}{2}\binom{2n+1}{L}(2n+1)^{d-2}\cdot c_3\exp(-cn)\le c_3\exp(-c_4 n).$$

□

Let $V_{n,k}$ denote the intersection of V_n and a $k-$dimensional subspace of $\mathbf{Z}^d$, for example, the subspace generated by the first k standard unit vectors, $e_1,\cdots,e_k$. In particular, $V_{n,d}=V_n$ and $V_{n,1}$ is the intersection of V_n and an axis of $\mathbf{Z}^d$, c.f. (2.7). Let $V_n^{(x)}=B_\infty(x,n)\cap V$, and $V_{n,k}^{(x)}$ denote the intersection of $V_n^{(x)}$ with a $k-$dimensional subspace containing x.

For any fixed $V_{n,k}$, $1\le k\le d$, and let e_i denote a unit vector not used in the definition of $V_{n,k}$ when $k<d$. By the Borel-Cantelli lemma, the result of Lemma 2.4 implies that for all large n,

$$p(2n+1)^k/2\le|V_{n,k}^{(je_i)}(\omega)|\le 2p(2n+1)^k,\ 1\le k\le d,\ |j|\le n,\ \mathbb{P}-a.s., \qquad (2.11)$$

$$|\cup_{|j_m|\le n,\ 1\le m\le L}\mathcal{P}_i(V_{n,k}^{(j_m e_i)}(\omega))|\ge(1-2(1-p)^L)(2n+1)^k,\ \mathbb{P}-a.s., \qquad (2.12)$$

where L denotes a finite number. Define

$$w_0(\omega) = \min\{n' > 0 \mid \forall n \geq n', \text{ the inequalities (2.11) and (2.12) hold.}\}, \quad (2.13)$$

and for any $x \in \mathbf{Z}^d$, $w_x(\omega)$ is defined similarly as $w_0(\omega)$ with $V_n^{(x)}$ taking the place of V_n.

Let $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$ denote the induced graph on $V_n^{(x)}(\omega)$ for fixed $\omega \in \Omega$. For a finite subset $A \subset V_n^{(x)}(\omega)$, define the edge-boundary of A as follows,

$$\partial_{E_n^{(x)}}(A) = \{(y, y') \in E_n^{(x)}(\omega) \mid y, y' \in V_n^{(x)}(\omega),\ y \in A,\ y' \notin A\}. \quad (2.14)$$

Let $m(\cdot)$ denote the counting measure on $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$, i.e., $m(y, y') = 1$ for any $(y, y') \in E_n^{(x)}(\omega)$ and $m(y) = \sum_{(y,y') \in E_n^{(x)}(\omega)} m(y, y')$ for $y \in V_n^{(x)}(\omega)$. Thus $m(y)$ equals the degree of the vertex y in the graph $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$. For any $A \subset V_n^{(x)}(\omega)$, set $m(A) = \sum_{y \in A} m(y)$.

The isoperimetric constant of the graph $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$ is defined as

$$I_n^{(x)}(\omega) = \min_{\substack{A \in V_n^{(x)}(\omega) \\ |A| \leq |V_n^{(x)}(\omega)|/2}} \frac{|\partial_{E_n^{(x)}}(A)|}{m(A)}. \quad (2.15)$$

We will prove a lower bound for $I_n^{(x)}(\omega)$, i.e., an isoperimetric inequality, in Lemma 2.7. To this end we prove Lemma 2.5 at first, which is the analogue of A.3 of Deuschel and Pisztora (1996) under the setting of this paper.

Lemma 2.5. *For any $A \subset V_n^{(x)}(\omega)$, $x \in \mathbf{Z}^d$, and an arbitrary $\epsilon > 0$, there exist a finite $w_x(\omega)$ and a constant $\delta = \delta(\epsilon, p, d) > 0$ such that if*

$$\frac{|\partial_{E_n^{(x)}}(A)|}{\sum_{i=1}^d |\mathcal{P}_i(A)|} \leq \delta,\ n \geq w_x(\omega), \quad (2.16)$$

then

$$|A| \geq (1 - \epsilon)|V_n^{(x)}(\omega)|,\ \mathbb{P} - a.s. \quad (2.17)$$

And there exist constants $c_i = c_i(p, d) > 0$ such that

$$\mathbb{P}(w_x(\omega) > n) \leq c_1 \exp(-c_2 n). \quad (2.18)$$

Proof: We consider the case of the graph $(V_n(\omega), E_n(\omega))$ with $\omega \in \Omega$, and the general case can be proved similarly.

We will take an induction on the dimension. To do this, we consider the analogue problems on the induced $k-$dimensional subgraphs $(V_{n,k}^{(x)}(\omega), E_{n,k}^{(x)}(\omega))$ of $(V_n(\omega), E_n(\omega))$ for $2 \leq k \leq d$. For the convenience of presentation, we concentrate on the special subgraph $(V_{n,k}(\omega), E_{n,k}(\omega))$, where $V_{n,k}(\omega)$ is the intersection of $V_n(\omega)$ with the $k-$dimensional subspace of $\mathbf{Z}^d$ generated by the first k standard unit vectors of $\mathbf{Z}^d$. The other cases are similar.

We consider the problem: For any $(V_{n,k}(\omega), E_{n,k}(\omega))$ and arbitrary $\epsilon_k > 0$, $2 \leq k \leq d$, there exists a constant $\delta_k > 0$ such that for any subset $A_k \subset V_{n,k}(\omega)$ and large n,

$$\begin{aligned} &|\partial_{E_{n,k}}(A_k)| \leq \delta_k \sum_{i=1}^k |\mathcal{P}_i(A_k)| \\ \Rightarrow &|A_k| \geq (1 - \epsilon_k)|V_{n,k}(\omega)|,\ \mathbb{P}{-}a.s. \end{aligned} \quad (2.19)$$

Note that all the estimates about the size of the various subsets of $(V_n(\omega), E_n(\omega))$, which will be used in the proof, are contained in (2.11) and (2.12), and when $n \geq w_0(\omega)$ these estimates are valid to use, see (2.13).

Write $\partial(A_k) = \partial_{E_{n,k}}(A_k)$. Since the edges of $\partial(A_k)$ are line segments which are parallel to the axes, we define

$$\partial_i(A_k) = \{e \in \partial(A_k) \mid e \text{ is parallel to the } i\text{−th axis.}\},\ i \leq k \leq d.$$

Then

$$\partial(A_k) = \cup_{i=1}^k \partial_i(A_k),\ |\partial(A_k)| = \sum_{i=1}^k |\partial_i(A_k)|.$$

By the above relations, the conditions in (2.19) are equivalent to

$$\sum_{i=1}^k |\partial_i(A_k)| \leq \delta_k \sum_{i=1}^k |\mathcal{P}_i(A_k)|,\ 2 \leq k \leq d. \tag{2.20}$$

Now we consider the case of $(V_{n,2}(\omega), E_{n,2}(\omega))$. Suppose that there exist a constant δ_2 with its value to be determined later and a subset $A_2 \subset V_{n,2}(\omega)$ such that

$$\sum_{i=1}^2 |\partial_i(A_2)| \leq \delta_2 \sum_{i=1}^2 |\mathcal{P}_i(A_2)|. \tag{2.21}$$

Without loss of generality, we assume that

$$|\partial_2(A_2)|/|\mathcal{P}_2(A_2)| \geq |\partial_1(A_2)|/|\mathcal{P}_1(A_2)|, \tag{2.22}$$

since the other case will be similar. From (2.22) and (2.21), we have

$$|\partial_1(A_2)| \leq \delta_2 |\mathcal{P}_1(A_2)|. \tag{2.23}$$

Observe that, for any $x \in A_2$, if $S_1(x) \cap V_{n,2}(\omega) \subsetneq A_2$, then there exists at least one boundary edge in $\partial_1(A_2)$ which is contained in $S_1(x)$ (viewing the edge as a line segment). Thus by (2.23), there must exists a site $x \in A_2$ such that $S_1(x) \cap V_{n,2}(\omega) \subset A_2$ when δ_2 is small, say $\delta_2 < 1$, and $|S_1(x) \cap V_{n,2}(\omega)| \geq (2n+1)p/2$ by (2.11). Note that $\mathcal{P}_i(A_2) \leq 2n+1$, $i = 1, 2$. By (2.21) we have

$$\begin{aligned} \frac{|\partial_2(A_2)|}{|\mathcal{P}_2(A_2)|} &= \frac{|\partial_2(A_2)|}{\sum_{i=1}^2 |\mathcal{P}_i(A_2)|} \cdot \frac{\sum_{i=1}^2 |\mathcal{P}_i(A_2)|}{|\mathcal{P}_2(A_2)|} \\ \leq \quad & \delta_2 \cdot \frac{\sum_{i=1}^2 |\mathcal{P}_i(A_2)|}{|S_1(z) \cap V_{n,2}(\omega)|} \leq \delta_2 \cdot \frac{4(2n+1)}{p(2n+1)} = 4\delta_2/p. \end{aligned} \tag{2.24}$$

We claim that for any finite number L, there exist $x(1), \cdots, x(L) \in S_1(x) \cap V_{n,2}(\omega)$ such that the following hold for small δ_2,

$$S_2(x(l)) \cap V_{n,2}(\omega) \subset A_2,\ \forall l \in \{1, \cdots, L\}. \tag{2.25}$$

If not, we have $|\partial_2(A_2)| \geq (2n+1)p/2 - L$, which contradicts (2.21) with small δ_2 since $\sum_{i=1}^2 |\mathcal{P}_i(A)| \leq 2(2n+1)$.

Set

$$\epsilon_0 = 2(1-p)^L. \tag{2.26}$$

By (2.25) and (2.12) we conclude that

$$|\mathcal{P}_2(A_2)| \geq (1-\epsilon_0)(2n+1),\ n \geq w_0(\omega). \tag{2.27}$$

Substituting (2.27) into (2.24) we get a better bounds,

$$|\partial_2(A_2)|/|\mathcal{P}_2(A_2)| \leq 2\delta_2/(1-\epsilon_0). \tag{2.28}$$

By viewing the edge of $\partial_2(A_2)$ as a set of sites, we observe that

$$\begin{aligned}&\mathcal{P}_2(x) \notin \mathcal{P}_2(\partial_2(A_2)) \Rightarrow S_2(x) \cap V_{n,2}(\omega) \subset A_2,\ \forall x \in A_2,\\ &\text{and } |\mathcal{P}_2(\partial_2(A_2))| \leq |\partial_2(A_2)|. \end{aligned} \tag{2.29}$$

By (2.27), (2.28) and (2.29), the number of the distinct sets of the form $S_2(x) \cap V_{n,2}(\omega) \in A_2$, $x \in A_2$, is at least

$$(1 - \epsilon_0 - 2\delta_2/(1-\epsilon_0))(2n+1). \tag{2.30}$$

With (2.30) and (2.11), we get that

$$\begin{aligned}|V_{n,2}(\omega)| - |A_2| &\leq (\epsilon_0 + 2\delta_2/(1-\epsilon_0))(2n+1)\cdot 2p(2n+1)\\ &\leq 4[\epsilon_0 + 2\delta_2/(1-\epsilon_0)]|V_{n,2}(\omega)|. \end{aligned} \tag{2.31}$$

Thus for arbitrary $\epsilon_2 > 0$, we choose δ_2 and ϵ_0 such that $\epsilon_2 \geq 4[\epsilon_0 + 2\delta_2/(1-\epsilon_0)]$, and solving for δ_2 we get that

$$\delta_2 \leq 1/2[\epsilon_2/4 - \epsilon_0(1 + \epsilon_2/4 - \epsilon_0)]. \tag{2.32}$$

Since the constant ϵ_0 in (2.32) can be made arbitrarily small by taking large L, see (2.26), we can find a $\delta_2 > 0$ for any $\epsilon_2 > 0$ such that (2.19) holds.

Thus we complete the proof of the two dimensional case.

Now suppose that on any $k-$dimensional subgraph of $(V_n(\omega), E_n(\omega))$, the conclusion of (2.19) holds with parameters ϵ_k and δ_k for $2 \leq k$. We start to derive the $(k+1)-$dimensional case of (2.19).

Again we concentrate on the case of $(V_{n,k+1}(\omega), E_{n,k+1}(\omega))$, and the other cases can be treated similarly. By (2.20), we assume that

$$\sum_{i=1}^{k+1} |\partial_i(A_{k+1})| \leq \delta_{k+1} \sum_{i=1}^{k+1} |\mathcal{P}_i(A_{k+1})|, \tag{2.33}$$

where $A_{k+1} \subset V_{n,k+1}(\omega)$ and $\delta_{k+1} > 0$ is a constant with its value to be determined later. Without loss of generality, we assume that

$$|\partial_{k+1}(A_{k+1})|/|\mathcal{P}_{k+1}(A_{k+1})| \geq \max_{1\leq i\leq k} \{|\partial_i(A_k+1)|/|\mathcal{P}_i(A_{k+1})|\}, \tag{2.34}$$

and the other cases are similar. By (2.34) and (2.33), we have

$$\sum_{i=1}^{k} |\partial_i(A_{k+1})| \leq \delta_{k+1} \sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1})|. \tag{2.35}$$

Write $V_{n,k,j}(\omega) = V_{n,k}^{(je_{k+1})}(\omega)$, where e_{k+1} denotes the $(k+1)-$th unit vector of $\mathbf{Z}^d$ and $-n \leq j \leq n$. We consider the following induced $k-$dimensional subgraphs,

$$(V_{n,k,j}(\omega), E_{n,k,j}(\omega)),\ -n \leq j \leq n.$$

By writing $A_{k+1,j} = A_{k+1} \cap V_{n,k,j}(\omega)$, $j \in [-n,n]$, we have

$$A_{k+1} = \cup_{j=-n}^{n} A_{k+1,j},\ |A_{k+1}| = \sum_{j=-n}^{n} |A_{k+1,j}|. \tag{2.36}$$

By writing $\partial(A_{k+1,j}) = \partial_{E_{n,k,j}(\omega)}(A_{k+1,j})$, $j \in [-n,n]$ we have

$$\sum_{j=-n}^{n} |\partial(A_{k+1,j})| = \sum_{i=1}^{k} |\partial_i(A_{k+1})|,\ \sum_{j=-n}^{n}\sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j})| = \sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1})|. \tag{2.37}$$

Substituting the above relation (2.37) into (2.35), we get that

$$\sum_{j=-n}^{n} |\partial(A_{k+1,j})| \le \delta_{k+1} \sum_{j=-n}^{n} \sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j})|. \tag{2.38}$$

By the above inequality (2.38), there must exist $j_1 \in [-n, n]$ such that

$$|\partial(A_{k+1,j_1})| \le \delta_{k+1} \sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j_1})|. \tag{2.39}$$

By (2.39) and the induction assumption, using (2.11) we conclude that

$$|A_{k+1,j_1}| \ge (1-\epsilon_k)|V_{n,k,j_1}(\omega)| \ge p(1-\epsilon_k)(2n+1)^k/2, \tag{2.40}$$

if

$$\delta_{k+1} \le \delta_k. \tag{2.41}$$

Using (2.40) and (2.33), we derive an upper bound for $\partial_{k+1}(A_k+1)|/|\mathcal{P}_{k+1}(A_{k+1})|$,

$$\begin{aligned} \frac{|\partial_{k+1}(A_{k+1})|}{|\mathcal{P}_{k+1}(A_{k+1})|} &= \frac{|\partial_{k+1}(A_{k+1})|}{\sum_{i=1}^{k+1} |\mathcal{P}_i(A_{k+1})|} \cdot \frac{\sum_{i=1}^{k+1} |\mathcal{P}_i(A_{k+1})|}{|\mathcal{P}_{k+1}(A_{k+1})|} \\ \le \quad & \delta_{k+1} \cdot \frac{\sum_{i=1}^{k+1} |\mathcal{P}_i(A_{k+1})|}{|A_{k+1,j}|} \le \delta_{k+1} \cdot \frac{2(k+1)(2n+1)^k}{p(1-\epsilon_k)(2n+1)^k} \\ \le \quad & 5k\delta_{k+1}/p \quad (\epsilon_k \le 1/2), \end{aligned} \tag{2.42}$$

where the first inequality follows from (2.33) and the second inequality follows from that $|\mathcal{P}_i(A_{k+1})| \le (2n+1)^k$ and (2.40).

By the observation (2.29), using (2.40), (2.42) and (2.11) we get that

$$\begin{aligned} &|A_{k+1}| \ge [|A_{k+1,j_1}| - 5k\delta_{k+1}|\mathcal{P}_{k+1}(A_{k+1})|/p] \cdot p(2n+1)/2 \\ \ge &[(1-\epsilon_k)p/2 - 5k\delta_{k+1}/p](2n+1)^k \cdot (2n+1)p/2 \ge cn^{k+1}, \end{aligned} \tag{2.43}$$

where $c > 0$ and δ_{k+1} is chosen such that

$$\delta_{k+1} < (1-\epsilon_k)p^2/(10k). \tag{2.44}$$

We claim that, for any finite number L, there exist L distinct number, $j_1, \cdots, j_L \in [-n, n]$, such that

$$|\partial(A_{k+1,j_l})| \le \delta_{k+1} \sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j_l})|, \ \forall l \in \{1, \cdots, L\}. \tag{2.45}$$

We have shown that there exists one such set, i.e., A_{k+1,j_1} in (2.39). We will show that there exists another such set, say A_{k+1,j_2}, by deriving a contradiction.

To this end, we assume that

$$|\partial(A_{k+1,j})| > \delta_{k+1} \sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j})|, \ \forall j \in [-n, n]/\{j_1\}. \tag{2.46}$$

We apply the Loomis-Whitney inequality (see e.g., Lemma 6.31 of Lyons and Peres, 2011) to each subgraph of $(V_{n,k,j}(\omega), E_{n,k,j}(\omega))$, $j \in [-n, n]/\{j_1\}$, and get that, for any $\in [-n, n]/\{j_1\}$,

$$\sum_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j})| \ge k(\prod_{i=1}^{k} |\mathcal{P}_i(A_{k+1,j})|)^{1/k} \ge k|A_{k+1,j}|^{1-1/k}. \tag{2.47}$$

Using (2.36), (2.43) and (2.47), we have

$$\begin{aligned}&\sum_{j\in[-n,n]/\{j_1\}}\sum_{i=1}^{k}|\mathcal{P}_i(A_{k+1,j})| \geq k\sum_{j\in[-n,n]/\{j_1\}}|A_{k+1,j}|^{1-1/k}\\ \geq\ & k(\sum_{j\in[-n,n]/\{j_1\}}|A_{k+1,j}|)^{1-1/k} = k(|A_{k+1}|-|A_{k+1,j_1}|)^{1-1/k}\\ \geq\ & ckn^{k-1/k}, \end{aligned} \tag{2.48}$$

where the second inequality follows by Jensen's inequality, and the last inequality follows from that $|A_{k+1,j_1}| \leq 2p(2n+1)^k$ by (2.11) and $|A_{k+1}| \geq cn^{k+1}$ by (2.43).

Note that $\sum_{i=1}^{k}|\mathcal{P}_i(A_{j_1})| \leq k(2n+1)^{k-1}$. Using (2.46) and (2.48) we have

$$\begin{aligned}&\frac{\sum_{j\in[-n,n]}|\partial(A_{k+1,j})|}{\sum_{i=1}^{k}|\mathcal{P}_i(A_{k+1})|}\\ \geq\ & \frac{\sum_{j\in[-n,n]}|\partial(A_{k+1,j})| - \sum_{i=1}^{k}|\mathcal{P}_i(A_{k+1,j_1})|}{\sum_{j\in[-n,n]/\{j_1\}}\sum_{i=1}^{k}|\mathcal{P}_i(A_{k+1,j})|}\\ \geq\ & \frac{\sum_{j\in[-n,n]/\{j_1\}}|\partial(A_{k+1,j})| - k(2n+1)^{k-1}}{\sum_{j\in[-n,n]/\{j_1\}}\sum_{i=1}^{k}|\mathcal{P}_i(A_{k+1,j})|}\\ >\ & \delta_{k+1} - \frac{k(2n+1)^{k-1}}{ckn^{k-1/k}}. \end{aligned}$$

The above inequality contradicts (2.38) for all large n. So there must exist a set A_{k+1,j_2} which satisfies (2.45). To repeat the above arguments L times establishes the claim (2.45).

From (2.45), if $\delta_{k+1} \leq \delta_k$, i.e., (2.41) holds, then by the induction assumption we have

$$|A_{k+1,j_l}| \geq (1-\epsilon_k)|V_{n,k,j_l}|,\ \forall l \in \{1,\cdots,L\}.$$

Using (2.12) and the above inequalities we conclude that

$$|\mathcal{P}_{k+1}(A_{k+1})| \geq (1 - L\epsilon_k - \epsilon_0)(2n+1)^k, \tag{2.49}$$

where $\epsilon_0 = 2(1-p)^L$, see (2.26).

To bring (2.49) back into (2.42) we get a better bound as follows,

$$|\partial_{k+1}(A_k+1)|/|\mathcal{P}_{k+1}(A_{k+1})| \leq (k+1)\delta_{k+1}/(1-L\epsilon_k-\epsilon_0).$$

By the observation (2.29), using the above inequality with (2.49) and (2.11) we get that

$$\begin{aligned}&|V_{n,k+1}(\omega)| - |A_{k+1}|\\ \leq\ & [L\epsilon_k + \epsilon_0 + (k+1)\delta_{k+1}/(1-L\epsilon_k-\epsilon_0)]\cdot(2n+1)^{d+1}\cdot 2p\\ \leq\ & 4[L\epsilon_k + \epsilon_0 + (k+1)\delta_{k+1}/(1-L\epsilon_k-\epsilon_0)]\cdot|V_{n,k+1}(\omega)|. \end{aligned} \tag{2.50}$$

For any $\epsilon_{k+1} > 0$, we choose $\delta_{k+1} > 0$ such that

$$\epsilon_{k+1} \geq 4[L\epsilon_k + \epsilon_0 + (k+1)\delta_{k+1}/(1-L\epsilon_k-\epsilon_0)].$$

Solving the above inequality for δ_{k+1}, we get that

$$\delta_{k+1} \leq (k+1)^{-1}[\epsilon_{k+1}/4 - (L\epsilon_k+\epsilon_0)(1+\epsilon_{k+1}/4 - L\epsilon_k - \epsilon_0)]. \tag{2.51}$$

In (2.51), the constant ϵ_0 can be made arbitrarily small by taking large L, see (2.26), and ϵ_k can be arbitrarily small with $\delta_k > 0$ by the induction assumption, so δ_{k+1} is well chosen by (2.51) and (2.41), i.e., for any fixed $\epsilon_{k+1} > 0$, $\delta_{k+1} > 0$ can be achieved. Thus the proof for the $(k+1)-$dimensinal subgraph of $(V_n(\omega), E_n(\omega))$ is completed.

By induction, for any $\epsilon = \epsilon_d > 0$, there exists a constant $\delta = \delta_d > 0$ such that if a subset $A \in V_n(\omega)$ satisfies (2.16), then it also satisfies (2.17).

By the definition of $w_0(\omega)$, see (2.13), using Lemma 2.4 we have

$$\mathbb{P}(w_0(\omega) > n) \le \sum_{m \ge n} (c_1 \exp(-c_2 m) + c_3 \exp(-c_4 m)) \le c \exp(-c' n),$$

where the constants $c, c' > 0$. Thus the proof is completed. □

Remark 2.6. Note that the basic procedures in the proof of Lemma 2.5 for the two dimensional case and the higher dimensional case, i.e., the induction step, are the same, and this is partially reflected in the resemblance between the forms of the inequalities (2.32) and (2.51).

In the proof of Lemma 2.5, the subscript k in A_k, ϵ_k, δ_k, et al., is only intended to indicate that the corresponding objects are dependent on the dimension k. While ϵ_k and δ_k generally can take arbitrary value, we implicitly assumed that the function $\delta_k(\epsilon_k)$ at the critical value depends on the dimension k. Indeed, there do exist restrictions on the values of ϵ_k and δ_k expressed in (2.32),(2.41) and (2.51). In practice, we start with a set $A = A_d$ and $\delta = \delta_d > 0$ in relation (2.16) to find the feasible δ_k and ϵ_k, $2 \le k < d$, with given $\epsilon = \epsilon_d$. In light of (2.41), we may take fixed $\delta = \delta_k$ for all $2 \le k \le d$. In viewing of (2.41) and (2.51), the corresponding sequence $\{\epsilon_k\}$ is strictly monotone, i.e., $\epsilon_d > \cdots > \epsilon_2$, and ϵ_d/ϵ_2 can be quit large for small p or large d. Intuitively, it is said that to have a subset, say A, to occupy a small portion of a high dimensional cube, say $V_n(\omega)$, there must exist some lower dimensional sub-cubes, say $V_{n,2}$, of which a large portion are contained in A.

Using Lemma 2.5, we derive a lower bound for the isoperimetric constant $I_n^{(x)}(\omega)$ defined in (2.15).

Lemma 2.7. *For almost every $\omega \in \Omega$ and $x \in \mathbf{Z}^d$, when $n \ge w_x(\omega)$ and the graph $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$ is connected, there exists a constant $c = c(p, d) > 0$ such that*

$$I_n^{(x)}(\omega) \ge c n^{-1}. \tag{2.52}$$

In particular, for $x \in V(\omega)$, the inequality (2.52) holds when $n \ge v_x(\omega) \vee w_x(\omega)$.

Proof: At first, we check the connectivity of the graph $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$, otherwise $I_n^{(x)}(\omega) = 0$ by the definition (2.15).

From the proof of Lemma 2.2, c.f. (2.5), if $x \in V(\omega)$ and $n \ge v_x(\omega)$, the graph is connected. While if $x \notin V(\omega)$, we choose a site $x' \in V(\omega)$ which is a nearest site of x under the metric l^∞. With this choice, if $n \ge v_{x'}(\omega) + |x - x'|_\infty$, then the graph $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$ is connected either by the same reason.

Using Lemma 2.5 with $\epsilon < 1/2$, for any $A \in V_n^{(x)}(\omega)$, if $n \ge w_x(\omega)$ and $|A| \le |V_n^{(x)}(\omega)|/2$, then there exists a constant $\delta > 0$ such that

$$|\partial_{E_n^{(x)}}(A)| \ge \delta \sum_{i=1}^{d} |\mathcal{P}_i(A)|. \tag{2.53}$$

Using the Loomis-whitney inequality, see e.g., Lemma 6.31 of Lyons and Peres (2011), we have

$$\sum_{i=1}^{d}|\mathcal{P}_i(A)| \geq d(\prod_{i=1}^{d}|\mathcal{P}_i(A)|)^{1/d} \geq d|A|^{1-1/d}. \tag{2.54}$$

Recall that $m(\cdot)$ denotes the counting measure on $(V_n^{(x)}(\omega), E_n^{(x)}(\omega))$. Observe that for any $y \in V_n^{(x)}(\omega)$, $m(y) \leq 2d$. Thus for $A \in V_n^{(x)}(\omega)$, $m(A) \leq 2d|A|$. Then by the definition of $I_n^{(x)}(\omega)$ (2.15), using (2.53) and (2.54) we have

$$\begin{aligned} I_n^{(x)}(\omega) &= \min_{\substack{A\in V_n^{(x)}(\omega)\\ |A|\leq |V_n^{(x)}(\omega)|/2}} \frac{|\partial_{E_n^{(x)}}(A)|}{m(A)} \geq \min_{\substack{A\in V_n^{(x)}(\omega)\\ |A|\leq |V_n^{(x)}(\omega)|/2}} \frac{|\partial_{E_n^{(x)}}(A)|}{2d|A|} \\ &\geq \min_{\substack{A\in V_n^{(x)}(\omega)\\ |A|\leq |V_n^{(x)}(\omega)|/2}} \delta/2\cdot|A|^{-1/d} \geq \delta/2^{1+1/d}\cdot|V_n^{(x)}(\omega)|^{-1/d} > \delta n^{-1}/8, \end{aligned}$$

where the rough bound $|V_n^{(x)}(\omega)| \leq (2n+1)^d$ is used in the last inequality. □

Using Lemma 2.7, we prove a weak Poincaré inequality on $B_{d_\omega}(x,n)$.

Lemma 2.8. *Set $C_W = C_0C_1$. For almost every $\omega\in\Omega$, if $n \geq u_x(\omega)\vee v_x(\omega)\vee w_x(\omega)$ for $x \in V(\omega)$, there exists a constant $C_P > 0$ such that*

$$\inf_{a\in\mathbf{R}}\sum_{y\in B_{d_\omega}(x,n)}(f(y)-a)^2m(y) \leq C_Pn^2\sum_{(y,y')\in E(B_{d_\omega}(x,C_Wn))}(f(y)-f(y'))^2m(y,y'),$$

for any $f : B_{d_\omega}(x,C_Wn) \to \mathbf{R}$, where $E(B_{d_\omega}(x,C_Wn))$ denotes the edge set on $B_{d_\omega}(x,C_Wn)$ and $m(\cdot)$ denotes the counting measure on the graph.

Proof: By Lemma 2.1 and Lemma 2.2, if $n \geq u_x(\omega)\vee v_x(\omega)$, we have

$$B_{d_\omega}(x,n) \subset B_\infty(x,C_0n)\cap V(\omega) \subset B_{d_\omega}(x,C_0C_1n) = B_{d_\omega}(x,C_Wn),\ \mathbb{P}-a.s. \tag{2.55}$$

Recall that $B_\infty(x,C_0n)\cap V(\omega) = V_{C_0n}^{(x)}(\omega)$. Note that $x\in V(\omega)$ and $n \geq v_x(\omega)\vee w_x(\omega)$. Then, by Lemma 2.7, the isoperimetric inequality (2.52) implies that

$$I_{C_0n}^{(x)}(\omega) \geq cn^{-1},$$

since $C_0 \geq 1$, see Lemma 2.1, where the constant $c > 0$.

By invoking Lemma 3.3.7 of Saloff-Coste (1997), the above isoperimetric inequality implies the following Poincaré inequality,

$$\inf_{a\in\mathbf{R}}\sum_{y\in V_{C_0n}^{(x)}(\omega)}(f(y)-a)^2m(y) \leq cn^2\sum_{(y,y')\in E_{C_0n}^{(x)}(\omega)}(f(y)-f(y'))^2m(y,y'), \tag{2.56}$$

for any function $f : V_{C_0n}^{(x)}(\omega) \to \mathbf{R}$, where $m(\cdot)$ denotes the counting measure.

For any function $f : B_{d_\omega}(x, C_W n) \to \mathbf{R}$, using (2.56) and (2.55) we have

$$\begin{aligned}
&\inf_{a\in\mathbf{R}} \sum_{y\in B_{d_\omega}(x,n)} (f(y)-a)^2 m(y) \\
\le\ & \inf_{a\in\mathbf{R}} \sum_{y\in V^{(x)}_{C_0 n}(\omega)} (f(y)-a)^2 m(y) \\
\le\ & cn^2 \sum_{(y,y')\in E^{(x)}_{C_0 n}(\omega)} (f(y)-f(y'))^2 m(y,y') \\
\le\ & cn^2 \sum_{(y,y')\in E(B_{d_\omega}(x,C_W n))} (f(y)-f(y'))^2 m(y,y'),
\end{aligned}$$

where in the second inequality (2.56) is used and in the first and last inequality the relation in (2.55) is used. Thus the proof is completed. □

The following definition adapts the Definition 1.7 of Barlow (2004) to the setting of this paper.

Definition 2.9. For $x \in V(\omega)$, we say that $B_{d_\omega}(x, n)$ is *good* if $n \ge u_x(\omega) \vee v_x(\omega) \vee w_x(\omega)$, and say that $B_{d_\omega}(x, N)$ is *very good* if any $B_{d_\omega}(y, n)$ with $B_{d_\omega}(y, n) \subset B_{d_\omega}(x, N)$, $N^{1/(d+2)} \le n \le N$, is *good*.

Lemma 2.10. *Define $z_x(\omega) = \min\{n > 0 \mid \forall m \ge n,\ B_{d_\omega}(x, m)$ is very good.$\}$, for $x \in V(\omega)$. There exist constants $c_i = c_i(p, d) > 0$ such that*

$$\mathbb{P}(z_x(\omega) > n) \le c_1 \exp(-c_2 n^{1/(d+2)}).$$

Proof: Using Lemma 2.1, Lemma 2.2 and Lemma 2.5, by the Definition 2.9 we have

$$\begin{aligned}
&\mathbb{P}(z_x(\omega) > n) = \mathbb{P}(\exists m \ge n, B_{d_\omega}(x, m) \text{ is not } \textit{very good.}) \\
\le\ & \sum_{m=n}^{\infty} \mathbb{P}(B_{d_\omega}(x, m) \text{ is not } \textit{very good.}) \\
\le\ & \sum_{m=n}^{\infty} [\mathbb{P}(\exists m^{1/(d+2)} \le k \le m,\ y \in B_{d_\omega}(x, m),\ B_{d_\omega}(y, k) \text{ is not } \textit{good.})] \\
\le\ & \sum_{m=n}^{\infty} [\sum_{k=m^{1/(d+2)}}^{m} cm^d \mathbb{P}(u_y \vee v_y \vee w_y > k) + \mathbb{P}(u_x > m)] \\
\le\ & \sum_{m=n}^{\infty} [\sum_{k=m^{1/(d+2)}}^{m} cm^d \exp(-c_1 k) + c\exp(-c_2 m)] \\
\le\ & c\exp(-c_3 n^{1/(d+2)}),
\end{aligned}$$

where the constants $c_i > 0$ and $c > 0$ takes different values at different places. □

Let $B^c_{d_\omega}(x, n)$ denote the complement of $B_{d_\omega}(x, n)$ in $(V(\omega), E(\omega))$ and define

$$\varphi(y) = \left(\frac{n \wedge d_\omega(y, B^c_{d_\omega}(x, n))}{n}\right)^2,\ y \in B_{d_\omega}(x, n),$$

where $d_\omega(y, B^c_{d_\omega}(x, n)) = \min\{d_\omega(y, y') \mid y' \in B^c_{d_\omega}(x, n)\}$.

A weighted Poincaré inequality on $B_{d_\omega}(x, n)$ is contained in Proposition 2.11.

Proposition 2.11. *Suppose that $B_{d_\omega}(x,n)$ is very good. Then for almost every $\omega \in \Omega$, there exists a constant $c = c(p,d) > 0$ such that the following holds,*

$$\begin{aligned} &\inf_{a\in\mathbf{R}} \sum_{y\in B_{d_\omega}(x,n)} (f(y)-a)^2\varphi(y)m(y) \\ \leq\ & cn^2 \sum_{(y,y')\in E(B_{d_\omega}(x,n))} (f(y)-f(y'))^2\varphi(y)\wedge\varphi(y')m(x,y), \end{aligned}$$

for any function $f : B_{d_\omega}(x,n) \to \mathbf{R}$.

With Lemma 2.8 and Lemma 2.10, Proposition 2.11 can be proved by following exactly the lines of Theorem 4.8 of Barlow (2004) except only that the lower bound for the volume of ball was used there, but here, the upper bound for the volume of ball will be used instead (under the graph metric). So we omit the details of the proof.

2.3. *First-Passage Percolation on (V,E).* Define

$$t(x,y) = \mu^{-1/2}(x,y),\ \forall (x,y)\in E.$$

Since for any $e \in E$, $\mathbb{P}(\mu(e) \in [1,\infty)) = 1$, $\mathbb{P}(t(e) \in (0,1]) = 1$. We consider the first-passage percolation on the random graph (V,E) associated with the positive random variables $\{t(e),\ e \in E\}$, where $t(e)$ is referred as the traversing time of the edge $e \in E$.

For any $x, y \in V$, let $\pi(x,y)$ denote a self-avoiding path connecting x and y, and $\Pi(x,y)$ denote the collection of all the self-avoiding paths connecting x and y. Set $S(\pi(x,y)) = \sum_{e\in\pi(x,y)} t(e)$. Then $S(\pi(x,y))$ denotes the time needed to traverse the path $\pi(x,y)$. The percolation time between the sites of V are defined as follows,

$$d^f(x,y) = \inf_{\pi(x,y)\in\Pi(x,y)} S(\pi(x,y)),\ x,y\in V.$$

It is easy to see that d^f is a metric on V. For $x \in V$, let $B_{d^f}(x,n)$ denote the ball centered at x with radius n under the metric d^f. It is trivial that $B_{d_\omega}(x,n) \subset B_{d^f}(x,n)$ almost surely, since for any $e \in E$, $t(e) \leq 1$. The following result suggests that the two metrics d_ω and d^f are actually comparable under certain conditions.

Proposition 2.12. *For any $x \in V$, there exist positive constants C_2 and c_i (depending on p, d, and the distribution of $\mu(e)$) such that the following holds,*

$$\mathbb{P}(B_{d^f}(x,C_2 n) \subsetneq B_{d_\omega}(x,n)) \leq c_1 \exp(-c_2 n). \tag{2.57}$$

Proof: Without loss of generality, we consider a self avoiding path with length n starting from the origin, $(0, e_1, \cdots, e_n)$. Let $v_0 = 0, v_1, \cdots, v_n$ denote the consecutive end vertices of the edges of this path. We define a sequence of subindex numbers as follows,

$$s_0 = 0,\ s_{i+1} = \min\{j > s_i \mid j \leq n,\ |v_j - v_{s_i}|_\infty \geq M\}, \tag{2.58}$$

where M is a constant with its value to be determined. Let s_K denote the largest index number, i.e., $\max_{s_K\leq m\leq n} |v_{s_K} - v_m|_\infty < M$. Then the following inequality holds,

$$K+1 \geq n/(2M+1)^d, \tag{2.59}$$

because a self avoiding path on (V,E) passes no more than $(2M+1)^d$ sites before it leaves the ball centered at any site with radius M under the metric l^∞, c.f. (2.58).

To prove (2.57), it is sufficient to prove that

$$\begin{aligned} & \mathbb{P}(\exists \text{ a self avoiding path } \pi \text{ with at least } n \text{ edges, s.t. } S(\pi) \leq C_2 n. \ ) \\ \leq \ & c_1 \exp(-c_2 n), \end{aligned} \tag{2.60}$$

where $c_i > 0$ are constants, and $C_2 > 0$ with its value to be determined.

From (2.58), we observe that any sub-path $(v_{s_{i-1}}, \cdots, v_{s_i})$,$1 \leq i \leq K$, passes only one boundary site x of the ball $B_\infty(v_{s_{i-1}}, M)$ (x may not belong to V), and all the sub-paths are identically distributed under the measure $\mathbb{P}$. Since the collection of random variables indexed by the edges of a self avoiding path are i.i.d. variables, the B-K inequality can be applied. With these observations, by the similar arguments as that of Proposition 5.8 Kesten (1986) we have

$$\text{LHS of (2.60)} \leq \sum_{K+1 \geq n/(2M+1)^d} \exp(\gamma C_2 n) \left[\sum_{|x|_\infty = M} \mathbb{E} \exp(-\gamma S(0,x)) \right]^K,$$

where $S(0,x)$ denotes the time needed to traverse a self avoiding path starting from 0 and ending at the first time of passing a boundary site, which is x, of the ball $B_\infty(0, M)$ ($x \notin V$ is allowed), and $\gamma > 0$ is a constant with its value to be determined.

We choose M and γ such that the following holds,

$$\sum_{|x|_\infty = M} \mathbb{E} \exp(-\gamma S(0,x)) < 1.$$

The above inequality is possible since $\mathbb{P}(t(e) > 0) = 1$ and we can choose an arbitrary large γ. By the inequality (2.59), to choose $C_2 > 0$ small enough, we get (2.60). Thus the proof is completed. □

The following definition adapts the Definition 2.9 of Barlow and Deuschel (2010).

Definition 2.13. For $x \in V(\omega)$, $\lambda > 1$, $\kappa > 1$, $\beta \in (0,1)$, we say that (x,n) is $(\lambda,\kappa) -$ *good* if $B_{d^f}(x, m/\lambda) \subset B_{d_\omega}(x,m) \subset B_\infty(x, \kappa m)$, $\forall m \geq n$, and say that (x,n) is $(\lambda,\kappa) -$ *very good* if for any $y \in B_{d_\omega}(x,n)$, $m \geq n^\beta$, (y,m) is $(\lambda,\kappa) -$ *good*.

Lemma 2.14. *Define* $s_x(\omega) = \min\{n > 0 \mid \forall m \geq n, \ (x,m) \text{ is } (\lambda,\kappa)- \text{very good.}\}$, *for* $x \in V(\omega)$. *For any* $x \in V$, *there exist constants* $c_i > 0$ (*depending on* p, d *and the distribution of* $\mu(e)$) *such that*

$$\mathbb{P}(s_x(\omega) > n) \leq c_1 \exp(-c_2 n^\beta),$$

where the constant $\beta \in (0,1)$.

Proof: From Lemma 2.1 and Proposition 2.12, we have

$$\begin{aligned} & \mathbb{P}(s_x(\omega) > n) = \mathbb{P}(\exists m \geq n, \exists y \in B_{d_\omega}(x,m), \ (y, m^\beta) \text{ is not } (\lambda,\kappa) - \textit{good.}) \\ \leq \ & \sum_{m=n}^{\infty} \sum_{|y-x|_\infty \leq C_0 m} \sum_{k=m^\beta}^{\infty} [\mathbb{P}\{B_{d^f}(y, C_2 k) \subsetneq B_{d_\omega}(y,k)\} + \mathbb{P}(u_y(\omega) > k)] \\ & + \mathbb{P}(u_x(\omega) > n) \\ \leq \ & \sum_{m=n}^{\infty} \sum_{|y-x|_\infty \leq C_0 m} \sum_{k=m^\beta}^{\infty} (c\exp(-c_1 k) + c\exp(-c_2 k)) + c\exp(-c_3 n) \\ \leq \ & c\exp(-c_4 n^\beta), \end{aligned}$$

where the constants $c_i > 0$ and $c > 0$ takes different values at different places. □

Remark 2.15. The condition of $(\lambda, \kappa) - good$ is about the metric comparisons between the metrics d^f, d_ω and d_∞, c.f. Lemma 2.10 of Barlow and Deuschel (2010), which will be needed in the proof of the quenched heat kernel upper bounds of Theorem 1.2, c.f. Theorem 2.19 of Barlow and Deuschel (2010). The metric comparison between d^f and l^∞ can also be used to obtain that the **VSRW** $X(\omega)$ is conservative almost surely by the arguments of Lemma 2.11 of Barlow and Deuschel (2010).

2.4. *The Proof of Theorem 1.2.*

Proof: For fixed $\omega \in \Omega$, let $(V(\omega), E(\omega))$ denote the corresponding graph. For any finite $A \subset V(\omega)$, the following isoperimetric inequality on $(V(\omega), E(\omega))$ holds,

$$|\partial(A)| \geq 2\sum_{i=1}^{d} |\mathcal{P}_i(A)| \geq 2d(\prod_{i=1}^{d} |\mathcal{P}_i(A)|)^{1/d} \geq 2d|A|^{1-1/d}, \tag{2.61}$$

where $\partial(A)$ denotes the edge boundary of A, the first inequality is due to the fact that each element of $\mathcal{P}_i(A)$ corresponds at least two boundary edges in $\partial(A)$ by the definition of $(V(\omega), E(\omega))$, and the last inequality follows by applying the Loomis-Whitney inequality to A (e.g., Lemma 6.31 of Lyons and Peres, 2011).

Let $m(\cdot)$ denote the counting measure on $(V(\omega), E(\omega))$, i.e., $m(x) = 2d$ for any $x \in V(\omega)$ and $m(x, y) = 1$ for any $(x, y) \in E(\omega)$. It is known that the isoperimetric inequality (2.61) implies the Nash inequality on $(V(\omega), E(\omega))$ (see e.g., Proposition 14.1 of Woess, 2000), that is

$$1/2 \sum_{(x,y)\in E(\omega)} (f(x) - f(y))^2 m(x, y) \geq C_N \|f\|_2^{2+4/d} \|f\|_1^{-4/d} \tag{2.62}$$

for any $f \in L^1(V(\omega), m) \cap L^2(V(\omega), m)$.

Since $\mu(x, y, \omega) \geq 1$, $\forall (x, y) \in E(\omega)$, the left-hand side of (2.62) is increasing when $\{m(x, y), (x, y) \in E(\omega)\}$ is substituted by $\{\mu(x, y, \omega), (x, y) \in E(\omega)\}$, so the Nash inequality holds also on the weighted graph $(V(\omega), E(\omega), \mu)$ which is known to imply the uniform heat kernel upper bounds (1.11), see e.g., Corollary 14.5 of Woess (2000).

Also, the graph $(V(\omega), E(\omega))$ is connected by Lemma 2.2, the counting measure is the reversible measure of the **VSRW** and the weight of any edge $\mu(x, y, \omega) \geq 1$. These properties together with the Nash inequality on $(V(\omega), E(\omega), \mu)$ constitute the assumptions of Theorem 2.19 of Barlow and Deuschel (2010). Then by invoking Theorem 2.19 of Barlow and Deuschel (2010), for any $x, y \in V(\omega)$, when $d_\omega(x, y) \leq t^{1/2}$ or $d_\omega(x, y) \geq s_x(\omega) \vee c_0$ (c.f. Lemma 2.14), we have

$$P_\omega^{(t)}(x, y) \leq c_1 t^{-d/2} \exp(-c_2 d_\omega(x, y)^2/t), \qquad t \geq c_3 d_\omega(x, y), \tag{2.63}$$

$$P_\omega^{(t)}(x, y) \leq c_1 \exp(-c_2|x - y|_\infty(1 \vee \log d_\omega(x, y)/t)), \qquad t \leq c_3 d_\omega(x, y), \tag{2.64}$$

where the constants $c_i > 0$.

Through the metric comparison between l^∞ and d_ω derived from Lemma 2.1, that is,

$$|x - y|_\infty \leq C_0 d_\omega(x, y), \text{ when } |x - y|_\infty > C_0 u_x(\omega), \; x, y \in V(\omega),$$

(2.63) and (2.64) are transferred to the heat kernel upper bounds (1.12) and (1.13) under the metric l^∞.

The weighted Poincaré inequality on $(V(\omega), E(\omega), \mu)$ follows from Proposition 2.11 similarly as the situation of the above Nash inequality. Thus, when $z_x(\omega) \vee d_\omega(x,y) \le t^{1/2}$ (c.f. Lemma 2.10), the lower bounds (1.14) is proved by following the arguments of Proposition 5.1 of Barlow (2004) (see also Proposition 3.2 of Barlow and Deuschel, 2010).

Set $U_x(\omega) = s_x(\omega) \vee z_x(\omega)$ (note that $z_x(\omega) \ge u_x(\omega)$) and $\alpha = \beta \wedge 1/(d+2)$, the tail probability of $U_x(\omega)$ (1.10) follows by using Lemma 2.10 and Lemma 2.14.

Thus we complete the proof of Theorem 1.2. □

Remark 2.16. Suppose that, instead of assigning independent conductances to the edges of (V, E), we let the conductance of each edge equals its edge length or a monotone function (bounded from zero) of its edge length, then we still have the quenched weighted Poincaré inequality (c.f. Proposition 2.11)and the Nash inequality as (2.62) under this setting, and thus the uniform upper bounds (1.11) and the lower bounds (1.14) of Theorem 1.2 hold in this setting . But for a gaussian heat kernel bounds to hold, we need an analogue result of Proposition 2.12 in this setting.

3. Quenched Invariance Principle

For fixed $\omega \in \Omega_0$, we define a discrete time random walk $\widehat{X}(\omega) = (\widehat{X}_n(\omega))$ from the **VSRW** $X(\omega)$ as follows,

$$\widehat{X}_n(\omega) = X_n(\omega),\ n \in \mathbf{N}_0, \tag{3.1}$$

where $\mathbf{N}_0$ denotes the non-negative integer set.

Let $\tau_x : \Omega \to \Omega$, $x \in \mathbf{Z}^d$, denote the natural translation on the environment space Ω derived from that on $\mathbf{Z}^d$. For any unit vector e_i of $\mathbf{Z}^d$, the derived translation $T(e_i)$ from that on V is defined as follows,

$$l_{e_i}(\omega) = min\{n > 0 \mid 0 \in V(\omega),\ ne_i \in V(\omega)\},\ T(e_i)\omega = \tau_{l_{e_i}(\omega)e_i}\omega, \tag{3.2}$$

where $l_{e_i}(\omega)$ is well defined almost surely due to the fact that l_{e_i} is a geometric random variable by the definition of V, or that τ_{e_i} is ergodic with respect to the product measure $\mathbb{P}$ and $\mathbb{P}(0 \in V) = p > 0$ and using the individual ergodic theorem. Since $\mathbb{P}$ is a product measure, τ_{e_i} is invertible and ergodic with respect to $\mathbb{P}$. Note that $T(e_i)$ is the induced translation on Ω_0. Thus $T(e_i)$ is invertible and ergodic with respect to $\mathbb{Q}$, see (1.2), by Lemma 3.3 of Berger and Biskup (2007).

The environments viewed by the particle $\widehat{X}(\omega)$, defined as $(\tau_{\widehat{X}_n(\omega)}\omega)$, is a reversible Markov chain on Ω_0, since $\widehat{X}(\omega)$ is a reversible Markov chain. By the similar arguments as that of Lemma 3.4 of Berger and Biskup (2007) or Lemma 4.3 of De Masi et al. (1989), the Markov chain $(\tau_{\widehat{X}_n(\omega)}\omega)$ with initial measure $\mathbb{Q}$ is ergodic and thus the continuous time Markov chain $(\tau_{X_t(\omega)}\omega, t \ge 0)$ with initial measure $\mathbb{Q}$ is ergodic by the definition of $\widehat{X}(\omega)$, see (3.1).

The corrector can be defined from many perspectives, for example, Kozlov (1985); Sidoravicius and Sznitman (2004); Berger and Biskup (2007); Biskup and Prescott (2007); Mathieu and Piatnitski (2007); Barlow and Deuschel (2010); Biskup (2011), etc. We will use the electrical network theory to construct the corrector for $\widehat{X}(\omega) = (\widehat{X}_n(\omega),\ n \in \mathbf{N}_0)$.

At first we use the environmental Markov chain $(\tau_{\widehat{X}_n(\omega)}\omega,\ n \in \mathbf{N}_0)$, $\omega \in \Omega_0$, to introduce a (weighted) graph structure on the environmental space Ω_0. It is known that a weighted graph underlies a reversible Markov chain. Let $(\Omega_0, \mathcal{E}, C)$ denote the weighted graph underlying the reversible Markov chain $(\tau_{\widehat{X}_n(\omega)}\omega,\ n \in \mathbf{N}_0)$ for $\omega \in \Omega_0$, where $\mathcal{E}$ denotes the edge set and C denotes the set of the weights of the edges in $\mathcal{E}$. In definition, for any $\omega \in \Omega_0$ and $x \in V(\omega)$, $(\omega, \tau_x(\omega)) \in \mathcal{E}$ and the edge weight $C(\omega, \tau_x\omega) = P_\omega^{(1)}(0,x)$, because $P_\omega^{(1)}(0,x) > 0$ for any $x \in V(\omega)$ and $\sum_x P_\omega^{(1)}(0,x) = 1$. By the symmetry of the transition probability we also have $C(\omega, \tau_x\omega) = C(\tau_x\omega, \omega)$. We take the convention that $C(\omega, \tau_x\omega) = 0$ for $x \notin V(\omega)$, i.e., $(\omega, \tau_x\omega) \notin \mathcal{E}$. Since the measure $\mathbb{Q}$ is the invariant measure for $(\tau_{\widehat{X}_n(\omega)}\omega)$, we will apply the electrical network theory to the weighted graph $(\Omega_0, \mathcal{E}, C)$ equipped with the vertex measure $\mathbb{Q}$, see e.g., Section 2 of Chapter 1, p.14, of Woess (2000).

By the electrical network theory, there are two Hilbert spaces defined naturally on the weighted graph $(\Omega_0, \mathcal{E}, C)$ equipped with the vertex measure $\mathbb{Q}$, denoted by $\mathbf{L}^2(\Omega_0, \mathbb{Q})$ and $\mathbf{L}^2(\mathcal{E}, Q)$, which are equipped with the following inner products respectively,

$$\langle f_1, f_2\rangle_{\mathbb{Q}} = \int (f_1(\omega), f_2(\omega))\mathbb{Q}(d\omega), \qquad f_1, f_2 : \Omega_0 \to \mathbf{R}^d, \tag{3.3}$$

$$\langle g_1, g_2\rangle_Q = 1/2 \int \sum_x \frac{(g_1(\omega, \tau_x\omega), g_2(\omega, \tau_x\omega))}{C(\omega, \tau_x\omega)}\mathbb{Q}(d\omega), \qquad g_1, g_2 : \mathcal{E} \to \mathbf{R}^d, \tag{3.4}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbf{R}^d$ and the factor $1/2$ in (3.4) comes from the fact that the integral counts each edge of $\mathcal{E}$ twice. Let $\|\cdot\|_{\mathbb{Q}}$ and $\|\cdot\|_Q$ denote the norms induced by the inner products (3.3) and (3.4) respectively. Throughout the paper the symbol Q stands for the annealed measure, see (1.6), and we use Q in (3.4) to indicate the fact that the transition probabilities, i.e., $C(\omega, \tau_x\omega) = P_\omega^{(1)}(0,x)$, are incorporated in the integral, see also the equation (3.8) where the annealed measure $\mathbb{Q}(d\omega) \times P_\omega^{(1)}(0,x)$ is explicit.

For any function $f(\omega) : \Omega_0 \to \mathbf{R}^d$, write $\nabla_x f(\omega) = f(\tau_x\omega) - f(\omega)$, $x \in \mathbf{Z}^d$. We define two difference operators as follows,

$$\nabla : (\nabla f)(\omega, \tau_x\omega) = C(\omega, \tau_x\omega)\nabla_{-x} f(\tau_x\omega) \qquad \forall f(\omega) : \Omega_0 \longrightarrow \mathbf{R}^d, \tag{3.5}$$

$$\nabla^* : (\nabla^* g)(\omega) = \sum_x g(\omega, \tau_x\omega) \qquad \forall g(\omega, \omega') : \mathcal{E} \longrightarrow \mathbf{R}^d. \tag{3.6}$$

The Laplace operator $\mathcal{L}$ on the graph $(\Omega, \mathcal{E}, C)$ is defined as

$$\mathcal{L} f(\omega) = \sum_x \nabla_x f(\omega) P_\omega^{(1)}(0,x).$$

Combining the above with (3.5) and (3.6), we get the following operator equation,

$$\mathcal{L} = -\nabla^* \cdot \nabla. \tag{3.7}$$

By (3.5) and (3.4), for any function $f : \Omega_0 \to \mathbf{R}^d$, we have

$$\|\nabla f\|_Q^2 = 1/2 \int \sum_x (\nabla_x f(\omega), \nabla_x f(\omega)) P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega), \tag{3.8}$$

i.e., $\|\nabla f\|_Q^2$ is the Dirichlet integral of the function f. We introduce another function space $\mathbf{D}_Q$ as follows,

$$\mathbf{D}_Q = \{f : \Omega_0 \to \mathbf{R}^d \mid \|\nabla f\|_Q < \infty\}. \tag{3.9}$$

We consider a function $\phi(\omega):\Omega_0\to\mathbf{Z}^d$, which is a solution of the following equations,

$$\phi(\omega,x):=\nabla_x\phi(\omega)=x,\ x\in V(\omega),\ \mathbb{Q}-a.s. \tag{3.10}$$

To see that such a solution exists, observe that

$$\phi(\omega,y)-\phi(\omega,x)=\phi(\tau_x\omega,y-x),\ x,y\in V(\omega),\ \mathbb{Q}-a.s., \tag{3.11}$$

since both sides of (3.11) equal $y-x$ by (3.10). The cocycle property (or shift covariance) (3.11) implies that the vector field $\{\phi(\omega,x),\ x\in V(\omega),\ \omega\in\Omega_0\}$ is a potential field generated by some function on Ω_0 which is the solution of (3.10) and we denote by $\phi(\omega)$. In the sequel, we use $\{\phi(\omega,x),\ x\in V(\omega)\}$ to record the displacements of $\widehat{X}(\omega)$.

The existence of the corrector and some basic properties of it are contained in Lemma 3.1.

Lemma 3.1. *There exists a function $\chi:\Omega_0\to\mathbf{R}^d$ such that $\|\nabla\chi\|_Q<\infty$. Write $\chi(\omega,x)=\nabla_x\chi(\omega),\ x\in\mathbf{Z}^d$. For almost every $\omega\in\Omega_0$, we have*

(1) $\chi(\omega,y)-\chi(\omega,x)=\chi(\tau_x\omega,y-x)$, *for any* $x,y\in V(\omega)$,
(2) $\mathcal{L}(\phi(\omega)+\chi(\omega))=0$,
(3) $\|\nabla(\phi+\chi)\|_Q^2=\|\nabla\phi\|_Q^2-\|\nabla\chi\|_Q^2$.

Proof: For any function $f\in\mathbf{L}^2(\Omega_0,\mathbb{Q})$, by (3.3) and (3.8), we have

$$\begin{aligned}\|\nabla f\|_Q^2 &= 1/2\int\sum_x(\nabla_x f(\omega),\nabla_x f(\omega))P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega)\\ &\le \int\sum_x(f(\tau_x\omega),f(\tau_x\omega))P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega)+\int\sum_x(f(\omega),f(\omega))P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega)\\ &= \int\sum_{-x}(f(\omega),f(\omega))P_\omega^{(1)}(0,-x)\mathbb{Q}(d\omega)+\int(f(\omega),f(\omega))\mathbb{Q}(d\omega)\\ &= 2\|f\|_{\mathbb{Q}}^2,\end{aligned}$$

where the inequality follows by the Cauchy-Schwartz inequality and in the second equality we used the shift invariance of the measure $\mathbb{Q}$ and the symmetry of the transition probabilities, i.e., $P_\omega(0,x)=P_\omega(x,0)$.

By the above inequality, we have

$$\|\nabla\|=\sup_{f\in\mathbf{L}^2(\Omega_0,\mathbb{Q})}\frac{\|\nabla f\|_Q}{\|f\|_{\mathbb{Q}}}\le\sqrt{2}.$$

This is to say that ∇ is a bounded linear operator from the Hilbert space $\mathbf{L}^2(\Omega_0,\mathbb{Q})$ to the Hilbert space $\mathbf{L}^2(\mathcal{E},Q)$. From the knowledge of functional analysis there exists a unique dual operator of ∇ which is ∇^* by the definition (3.6). Further, $\|\nabla^*\|=\|\nabla\|\le\sqrt{2}$.

By the equation (3.7), we get an integration by parts formula,

$$\langle\nabla f,\nabla g\rangle_Q=\langle f,\nabla^*\cdot\nabla g\rangle_{\mathbb{Q}}=-\langle f,\mathcal{L}g\rangle_{\mathbb{Q}},\ \forall f\in\mathbf{L}^2(\Omega_0,\mathbb{Q}),\ \forall g\in\mathbf{D}_Q. \tag{3.12}$$

To make the formula (3.12) meaningful, it is left to show that $\mathcal{L}g \in \mathbf{L}^2(\Omega_0, \mathbb{Q})$ for any function $g \in \mathbf{D}_Q$ as follows,

$$\begin{aligned}\|\mathcal{L}g\|_{\mathbb{Q}}^2 &= \int (\mathcal{L}g(\omega), \mathcal{L}g(\omega))\mathbb{Q}(d\omega) \\ &= \int (\sum_x \nabla_x g(\omega) P_\omega^{(1)}(0,x), \sum_x \nabla_x g(\omega) P_\omega^{(1)}(0,x))\mathbb{Q}(d\omega) \\ &\leq \int \sum_x (\nabla_x g(\omega), \nabla_x g(\omega)) P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega) \\ &= 2\|\nabla g\|_Q^2 < \infty,\end{aligned}$$

where the inequality follows by the Jensen's inequality and the last inequality is due to $g \in \mathbf{D}_Q$, see (3.9).

Since ∇^* is a bounded linear operator from the Hilbert space $\mathbf{L}^2(\mathcal{E}, Q)$ to the Hilbert space $\mathbf{L}^2(\Omega_0, \mathbb{Q})$, the null space of ∇^*, denoted by $\mathbf{L}^{2,*}(\mathcal{E}, Q)$, is a complete subspace of $\mathbf{L}^2(\mathcal{E}, Q)$. We get the following orthogonal decomposition of $\mathbf{L}^2(\mathcal{E}, Q)$,

$$\mathbf{L}^2(\mathcal{E}, Q) = \mathbf{L}^{2,*}(\mathcal{E}, Q) \oplus \mathbf{L}^{2,\perp}(\mathcal{E}, Q), \tag{3.13}$$

where $\mathbf{L}^{2,\perp}(\mathcal{E}, Q)$ denotes the completion of $\mathbf{L}^{2,*}(\mathcal{E}, Q)$.

Using (1.13) of Theorem 1.2, there exist constants $c_i > 0$ and $\alpha > 0$ such that

$$P_\omega^{(1)}(0,x) \leq c_4 \exp(-c_5|x|_\infty), \text{ when } |x|_\infty > C_0 U_0(\omega), \tag{3.14}$$

$$\mathbb{P}(U_0(\omega) > n) \leq c_1 \exp(-c_2 n^\alpha). \tag{3.15}$$

Recall the function $\phi(\omega)$ defined by (3.10). Using (3.8), (3.14) and (3.15) we have

$$\begin{aligned}\|\nabla\phi\|_Q^2 &= 1/2 \int \sum_x |x|_2^2 P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega) \\ \leq \quad & \mathbb{E}(C_0^2 U_0^2(\omega)) + \sum_{|x|_\infty \geq C_0 U_0(\omega)} |x|_2^2 \cdot c_4 \exp(-c_5|x|_\infty) < \infty.\end{aligned}$$

Then $\phi \in \mathbf{D}_Q$, see (3.9).

Let $\mathcal{P}$ denote the projection operator onto $\mathbf{L}^{2,\perp}(\mathcal{E}, Q)$. Then $\mathcal{P}$ is a linear operator because $\mathbf{L}^{2,\perp}(\mathcal{E}, Q)$ is a complete subspace of a Hilbert space. Since $\phi \in \mathbf{D}_Q$, for any $(\omega, \tau_x\omega) \in \mathcal{E}$, using the projection theorem of the Hilbert space we define $\chi(\omega, x)$ as the unique solution of the following equation,

$$\chi(\omega, x) = \mathcal{P}(-\phi(\omega, x)), \ \forall (\omega, \tau_x\omega) \in \mathcal{E}. \tag{3.16}$$

Since $(V(\omega), E(\omega))$ is connected almost surely by Lemma 2.2, using the linearity of $\mathcal{P}$, (3.16) and (3.11), by extending, we have

$$\begin{aligned}\chi(\omega, y) - \chi(\omega, x) &= \mathcal{P}(-\phi(\omega, y)) - \mathcal{P}(-\phi(x, \omega)) = -\mathcal{P}[\phi(\omega, y) - \phi(\omega, x)] \\ &= \mathcal{P}(-\phi(\tau_x\omega, y - x)) = \chi(\tau_x\omega, y - x), \ \forall x, y \in V(\omega), \ \mathbb{Q} - a.s.\end{aligned} \tag{3.17}$$

Using the cocycle property (3.17), by the same argument as that used in the definition of $\phi(\omega)$, see (3.10), there exists a function $\chi \in \mathbf{D}_Q$ such that

$$\chi(\omega, x) = \nabla_x \chi(\omega), \ x \in V(\omega), \ \mathbb{Q}{-}a.s. \tag{3.18}$$

The vector field $\{\chi(\omega, x), \ x \in V(\omega)\}$ is the corrector for the walk $\widehat{X}(\omega)$.

By (3.16), (3.18) and the orthogonal decomposition (3.13), we get that

$$\nabla(\phi + \chi) \in \mathbf{L}^{2,*}(\mathcal{E}, Q) \text{ and } \nabla\chi \in \mathbf{L}^{2,\perp}(\mathcal{E}, Q). \tag{3.19}$$

By (3.12), (3.19) and the definition of $\mathbf{L}^{2,*}(\mathcal{E},Q)$, we have

$$\langle f,\mathcal{L}(\phi+\chi)\rangle_{\mathbb{Q}}=-\langle\nabla f,\nabla(\phi+\chi)\rangle_Q=-\langle f,\nabla^*\cdot\nabla(\phi+\chi)\rangle_{\mathbb{Q}}=0,\ \forall f\in\mathbf{L}^2(\Omega_0,\mathbb{Q}).$$

Since $\mathbf{L}^2(\Omega_0,\mathbb{Q})$ is complete, the above equation implies that

$$\mathcal{L}(\phi(\omega)+\chi(\omega))=0,\ \mathbb{Q}-a.s. \tag{3.20}$$

By (3.19) and (3.13), we have

$$\begin{aligned}\|\nabla(\phi+\chi)\|_Q^2&=\|\nabla\phi\|_Q^2+2\langle\nabla\phi,\nabla\chi\rangle_Q+\|\nabla\chi\|_Q^2\\&=\|\nabla\phi\|_Q^2-\|\nabla\chi\|_Q^2.\end{aligned} \tag{3.21}$$

By (3.18), (3.17), (3.20) and (3.21), the proof is completed. □

Let $x_n(\omega),\ n\in\mathbf{Z}$, denote the sequence of sites which are the intersection of $V(\omega)$ with an axis of $\mathbf{Z}^d$, c.f. (3.2). We have the following one-dimensional sublinearity of the corrector $\{\chi(\omega,x_n(\omega)),n\in\mathbf{Z}\}$.

Lemma 3.2.

$$\lim_{|n|\to\infty}|n|^{-1}\chi(\omega,x_n(\omega))=0,\ \mathbb{Q}-a.s.$$

Proof: For fixed $\omega\in\Omega_0$, write $x_n:=x_n(\omega),\ n\in\mathbf{Z}$. By (1.14) of Theorem 1.2, there exists an integer valued variable $S_0(\omega)$ such that

$$P_\omega^{(n)}(0,x_1)\geq cn^{-d/2},\ \forall n\geq S_0(\omega),$$

where $c>0$ and $\mathbb{P}(S_0(\omega)>n)\leq c_1\exp(-c_2n^{\alpha/2})$ with $c_1,c_2>0$ and $\alpha>0$.

Then for $1\leq\gamma<2$, by writing $q_n:=P_\omega^{(n)}(0,x_1)$, using the above inequalities we have

$$\begin{aligned}&\mathbb{E}(|\chi(\omega,x_1)|_2^\gamma 1_{\{S_0(\omega)=n\}})\\=\ &\mathbb{E}[|\chi(\omega,x_1)|_2^\gamma q_n^{\gamma/2}q_n^{-\gamma/2}1_{\{S_0(\omega)=n\}}]\\\leq\ &(\mathbb{E}[|\chi(\omega,x_1)|_2^2q_n])^{\gamma/2}\cdot(cn^{\gamma d/(4-2\gamma)}\mathbb{P}(S_0=n))^{1-\gamma/2}\\\leq\ &[\mathbb{E}(\sum_x|\chi(\omega,x)|_2^2P_\omega^{(n)}(0,x))]^{\gamma/2}\cdot cn^{\gamma d/4}\cdot\exp(-c_2(1-\gamma/2)n^{\alpha/2})\\\leq\ &(2n\|\nabla\chi\|_Q)^\gamma\cdot cn^{\gamma d/4}\cdot\exp(-c'n^{\alpha/2}),\end{aligned} \tag{3.22}$$

where the constants $c,c'>0$ is due to $\gamma<2$, the first inequality follows by Hölder's inequality and the last inequality is due to the shift invariance of $\mathbb{Q}$ and the markov property of $\widehat{X}(\omega)$, c.f. Lemma 5.8 of Barlow and Deuschel (2010).

Since $|\chi(\omega,x_1)|_2^\gamma=\sum_{n=1}^\infty|\chi(\omega,x_1)|_2^\gamma 1_{\{S_0(\omega)=n\}}$, using (3.22), $\|\nabla\chi\|_Q<\infty$ by Lemma 3.1, and the monotone convergence theorem of the integral we have

$$\mathbb{E}(|\chi(\omega,x_1)|_2^\gamma)\leq\sum_{n=1}^\infty c\|\nabla\chi\|_Q^\gamma n^{\gamma+\gamma d/4}\exp(-c'n^{\alpha/2})<\infty. \tag{3.23}$$

Since $\|\nabla\chi\|_Q<\infty$, by approximation there exist a sequence of bounded functions $f_n(\omega),\ n\geq1$, such that

$$\nabla_{x_1}f_n(\omega)\xrightarrow{\text{In norm }\|\cdot\|_Q}\chi(\omega,x_1).$$

Since $\mathbb{Q}(P_\omega^{(1)}(0,x_1)>0)=1$, the above convergence implies that

$$\nabla_{x_1}f_n(\omega)\xrightarrow{\text{In }\mathbb{Q}-\text{probability}}\chi(\omega,x_1). \tag{3.24}$$

Note that $1 \le \gamma < 2$. Thus the inequality (3.23) implies that $\{\nabla_{x_1} f_n(\omega),\ n \ge 1,\ \chi(\omega, x_1)\}$ is uniformly integrable under the measure $\mathbb{Q}$, so the convergence of (3.24) also holds in $\mathbf{L}^1(\mathbb{Q})$. Then we get that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\chi(\omega, x_1)) &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}(\nabla_{x_1} f_n(\omega)) \\ &= \lim_{n\to\infty} [\mathbb{E}_{\mathbb{Q}}(f_n(\tau_{x_1}\omega)) - \mathbb{E}_{\mathbb{Q}}(f_n(\omega))] = 0, \end{aligned} \tag{3.25}$$

where the shift invariance of $\mathbb{Q}$ is used in the last equality.

We have shown that the induced translation on Ω_0 is ergodic with respect to $\mathbb{Q}$, see (3.2). Using the cocycle property of the corrector, see Lemma 3.1, and (3.25) we have

$$\lim_{n\to\infty} n^{-1}\chi(\omega, x_n) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \chi(\tau_{x_{i-1}}\omega, x_i - x_{i-1}) = \mathbb{E}_{\mathbb{Q}}(\chi(\omega, x_1)) = 0 \tag{3.26}$$

Similarly, the above relation (3.26) also holds when $n \to -\infty$. Thus the proof is completed. □

With Lemma 3.2, by the same arguments as that of Theorem 5.4 of Berger and Biskup (2007) we have the following multi-dimensional averaged sublinearity of the corrector.

Lemma 3.3. *For any $\epsilon > 0$, we have*

$$\limsup_{n\to\infty} (2n+1)^{-d} \sum_{\substack{|x|_\infty \le n \\ x \in V(\omega)}} 1_{\{|\chi(\omega,x)|_\infty \ge \epsilon n\}} = 0,\ \mathbb{Q} - a.s.$$

Now we prove the tightness of the laws of the rescaled **VSRW** $(X_t^{(\epsilon)}(\omega),\ t \in [0,1])$, $\epsilon > 0$, see (1.7). This is the analogue of Theorem 4.11 of Barlow and Deuschel (2010).

Lemma 3.4. *The family of the laws of $(X_t^{(\epsilon)}(\omega),\ t \in [0,1])_{\epsilon>0}$ is tight almost surely.*

Proof: For fixed ω, define

$$\rho(x, N) = \inf\{t > 0 \mid d_\omega(X_{t,x}(\omega), x) > N\},$$

i.e., $\rho(x, N)$ is the first time when the **VSRW** starting from x exits the ball $B_{d_\omega}(x, N)$.

Under the setting of this paper, by invoking Proposition 2.18 of Barlow and Deuschel (2010), there exist constants $c_i > 0$ such that if

$$N \ge c_1,\ t \ge c_1 N,\ (y, c_2 t/N) \text{ is } (\lambda, \kappa) - good,\ \forall y \in B_{d_\omega}(x, N), \tag{3.27}$$

then we have

$$P_\omega(\rho(x, N) < t) \le c_3 \exp(-c_4 N^2/t). \tag{3.28}$$

By Lemma 2.14, when $n \ge s_x(\omega)$, (x, n) is $(\lambda, \kappa) - very\ good$, and there exist constants $c, c' > 0$ and $\beta \in (0, 1)$ such that

$$\mathbb{P}(s_x(\omega) > n) \le c \exp(-c' n^{\beta/2}). \tag{3.29}$$

When N is large enough and ϵ small enough such that the condition (3.27) is satisfied with $1/\epsilon^2$ substituting for t and N/ϵ for N, using (3.28) we have

$$P_\omega(\rho(0, N/\epsilon) < 1/\epsilon^2) \le c_3 \exp(-c_4 N^2) \xrightarrow{N\to\infty} 0. \tag{3.30}$$

We will show that for arbitrary small $\eta > 0$, there exist a constant $\delta > 0$ such that

$$P_\omega\{\sup_{\substack{|s_1-s_2|\le\delta\\ s_i\in[0,1]}} \epsilon|X_{s_1/\epsilon^2}(\omega) - X_{s_2/\epsilon^2}(\omega)|_\infty \ge \eta\} \xrightarrow{\epsilon,\delta\to 0} 0. \tag{3.31}$$

By the triangle inequality, we have

$$\begin{aligned} &\sup_{\substack{|s_1-s_2|\le\delta\\ s_i\in[0,1]}} \epsilon|X_{s_1/\epsilon^2}(\omega) - X_{s_2/\epsilon^2}(\omega)|_\infty \ge \eta\} \\ &\le 2\max_{k\le 1/\delta}\sup_{s\in[k\delta,(k+1)\delta]} |X_{s/\epsilon^2}(\omega) - X_{k\delta/\epsilon^2}(\omega)|_\infty. \end{aligned}$$

By the above inequality, for small ϵ we have

$$\begin{aligned} & P_\omega\{\sup_{\substack{|s_1-s_2|\le\delta\\ s_i\in[0,1]}} \epsilon|X_{s_1/\epsilon^2}(\omega) - X_{s_2/\epsilon^2}(\omega)| \ge \eta\} \le P_\omega(\rho(0,N/\epsilon) < 1/\epsilon^2) \\ & +P_\omega\{\rho(0,N/\epsilon) \ge 1/\epsilon^2,\ 2\max_{k\le 1/\delta}\sup_{s\in[k\delta,(k+1)\delta]} |X_{s/\epsilon^2}(\omega) - X_{k\delta/\epsilon^2}(\omega)|_\infty \ge \eta/\epsilon\} \\ \le\ & c_3\exp(-c_4N^2) + 1/\delta P_\omega(\rho(y,\eta/(2\kappa\epsilon)) \le \delta/\epsilon^2,\ |y|_\infty \le \kappa N/\epsilon) \\ \le\ & c_3\exp(-c_4N^2) + 1/\delta c_3\exp(-c_4\eta^2/(4\kappa^2\delta)) \\ \le\ & c\exp(-c'\eta^2/\delta) \xrightarrow{\delta\to 0} 0. \end{aligned} \tag{3.32}$$

Now we check the conditions implicitly assumed in the derivation of (3.32).

Using (3.29) and Borel-Cantelli lemma, the following condition is satisfied when ϵ is small enough,

$$\eta/(2\kappa\epsilon) \ge \max_{|y|_\infty\le\kappa N/\epsilon} s_y(\omega),\ \mathbb{Q}-a.s., \tag{3.33}$$

where we used that $(0, N/\epsilon)$ is $(\lambda,\kappa)-$*good* for small ϵ which implies $B_{d_\omega}(0,N/\epsilon) \subset B_\infty(0,\kappa N/\epsilon)$, see Definition 2.13.

With (3.33), the condition (3.27) is satisfied with ϵ small enough for the application of (3.28) in the second and third inequality of (3.32). In the last inequality of (3.32), we chose N such that $N^2 \ge \eta^2/(4\kappa^2\delta)$.

With (3.30) and (3.31), invoking Theorem 7.2 of Ethier and Kurtz (2005) establishes the tightness of the laws of $(X_t^{(\epsilon)}(\omega),\ t\in[0,1])_{\epsilon>0}$. □

We are ready to prove Theorem 1.1.

Proof: Write the rescaled **VSRW** as $X^{(\epsilon)}(\omega) = (X_t^{(\epsilon)},\ t\in[0,1]))$, see (1.7).

We have proved the tightness of the laws of $(X_t^{(\epsilon)},\ t\in[0,1]))_{\epsilon>0}$ in Lemma 3.4. Thus to prove the functional CLT for the rescaled **VSRW** $X^{(\epsilon)}(\omega)$, it remains to prove the finite dimensional convergence of $X^{(\epsilon)}(\omega)$. By the markov property of the **VSRW**, it is sufficient to prove that

$$n^{-1/2}\widehat{X}_n(\omega)\ \text{ satisfies CLT},\ \mathbb{Q}-a.s., \tag{3.34}$$

since $\widehat{X}_n(\omega) = X_n(\omega)$ for $n\in\mathbf{N}_0$ by the definition (3.1).

Define

$$\widehat{M}_n(\omega) = \widehat{X}_n(\omega) + \chi(\omega,\widehat{X}_n(\omega)), n\in\mathbf{N}_0. \tag{3.35}$$

Let $(\mathcal{F}_n(\omega), n\in\mathbf{N}_0)$ denote the $\sigma-$fields generated by the random walk $(\widehat{X}_n(\omega),\ n\in\mathbf{N}_0)$, and let $E_\omega(\cdot)$ denote the expectation with respect to the random walk measure

P_ω. Using the markov property of $\widehat{X}(\omega)$ and the cocycle property of the corrector (see Lemma 3.1), we have

$$\begin{aligned} & E_\omega[\widehat{M}_{n+1}(\omega) \mid \mathcal{F}_n(\omega)] \\ = \; & E_\omega[\widehat{M}_n(\omega) \mid \mathcal{F}_n(\omega)] + \sum_x [x + \chi(\tau_{\widehat{X}_n(\omega)}\omega, x)] P_{\tau_{\widehat{X}_n(\omega)}\omega}(0,x) \\ = \; & \widehat{M}_n(\omega) + \mathcal{L}[\phi(\tau_{\widehat{X}_n(\omega)}\omega) + \chi(\tau_{\widehat{X}_n(\omega)}\omega)] = \widehat{M}_n(\omega), \; \mathbb{Q} - a.s., \end{aligned}$$

where the last equality follows by Lemma 3.1. Then $(\widehat{M}_n(\omega))$ is a martingale with respect to $(\mathcal{F}_n(\omega),\ n \in \mathbf{N}_0)$ almost surely.

To prove (3.34), we prove the following martingale CLT first,

$$n^{-1/2}\widehat{M}_n(\omega) \;\text{ satisfies CLT, }\; \mathbb{Q} - a.s. \tag{3.36}$$

Let $a \in \mathbf{R}^d$ be a fixed vector with the Euclidean norm $|a|_2 = 1$. Then $((a, M_n(\omega)), n \in \mathbf{N}_0)$ is a martingale almost surely. For a positive number K, we define

$$\mathcal{U}_n(K,\omega) = n^{-1} \sum_{i=1}^n E_\omega[(a, \widehat{M}_i(\omega) - \widehat{M}_{i-1}(\omega))^2 1_{|(a,\widehat{M}_i(\omega)-\widehat{M}_{i-1}(\omega))| \geq K} \mid \mathcal{F}_{i-1}(\omega)].$$

Since $(\tau_{\widehat{X}_n(\omega)}\omega)$ is ergodic with respect to $\mathbb{Q}$, using (3.8) and $|a|_2 = 1$ we have

$$\begin{aligned} & \lim_{n\to\infty} n^{-1} \sum_{i=1}^n E_\omega[(a, \widehat{M}_i(\omega) - \widehat{M}_{i-1}(\omega))^2 \mid \mathcal{F}_{i-1}(\omega)] \\ = \; & \int \sum_x (a, x + \chi(\omega, x))^2 P_\omega^{(1)}(0,x) \mathbb{Q}(d\omega) = 2\|(a, \nabla(\phi+\chi))\|_Q^2 \\ \leq \; & |a|_2^2 \int \sum_x |\nabla_x(\phi(\omega) + \chi(\omega))|_2^2 P_\omega^{(1)}(0,x)\mathbb{Q}(d\omega) \\ = \; & 2\|\nabla(\phi+\chi)\|_Q^2 = 2\|\nabla\phi\|_Q^2 - 2\|\nabla\chi\|_Q^2 < \infty, \end{aligned} \tag{3.37}$$

where the inequality follows by the Cauchy-Schwartz inequality and in the last equality and the last inequality we used Lemma 3.1.

By (3.37), we have

$$\lim_{n\to\infty} \mathcal{U}_n(0,\omega) = 2\|(a, \nabla(\phi+\chi))\|_Q^2 < \infty,\ P_\omega - a.s. \text{ and } \mathbb{Q} - a.s.$$

Since for arbitrary large K and arbitrary small $\delta > 0$, $\delta\sqrt{n} > K$ when n is large enough, using the monotonicity of $\mathcal{U}_n(K,\omega)$ in K and the finiteness of (3.37) we have

$$\lim_{n\to\infty} \mathcal{U}_n(\delta\sqrt{n},\omega) \leq \lim_{n\to\infty} \mathcal{U}_n(K,\omega) \xrightarrow{K\to\infty} 0,\ P_\omega - a.s. \text{ and } \mathbb{Q} - a.s.$$

The above two relations constitute the Lindeberg-Feller conditions for the martingale CLT, e.g., Theorem 7.7.3 of Durrett (2005). Then $(a, \widehat{M}_n(\omega))$ converges weakly to a Gaussian random variable almost surely. Since a is arbitrary, by the Cramer-Wald device the martingale CLT (3.36) is proved.

It remains to show that the rescaled corrector, see (3.35) and (3.36), is negligible in P_ω-probability when n goes to infinity. By Theorem 1.2, for arbitrary small constant $\epsilon > 0$, there exist a constant $M = M(\epsilon) > 0$ and $U_0(\omega) < \infty$ such that

$$P_\omega(|\widehat{X}_n(\omega)|_\infty \leq M\sqrt{n}) \geq 1 - \epsilon,\ \forall n \geq U_0^2(\omega),\ \mathbb{Q} - a.s. \tag{3.38}$$

For arbitrary $\delta > 0$, using (3.38), the uniform heat kernel upper bounds (1.11) and the averaged sublinearity of the corrector in Lemma 3.3, we have

$$\begin{aligned} & P_\omega(|\chi(\omega, \widehat{X}_n(\omega))|_\infty \geq \delta\sqrt{n}) \\ \leq \;& \epsilon + P_\omega(|\chi(\omega, \widehat{X}_n(\omega))|_\infty \geq \delta\sqrt{n},\, |\widehat{X}_n(\omega)|_\infty \leq M\sqrt{n}) \\ \leq \;& \epsilon + cn^{-d/2} \sum_{\substack{|x|_\infty \leq M\sqrt{n} \\ x\in V(\omega)}} 1_{\{|\chi(\omega,x)|_\infty \geq \delta\sqrt{n}\}} \xrightarrow{n\to\infty} \epsilon. \end{aligned}$$

Since ϵ and δ in the above inequality are arbitrary, we have proved that

$$n^{-1/2}\chi(\omega, \widehat{X}_n(\omega)) \xrightarrow{\text{In } P_\omega-\text{probability}} 0,\; \mathbb{Q}-a.s. \tag{3.39}$$

Combining (3.39), (3.35) and the martingale CLT (3.36), we get the CLT for $\widehat{X}(\omega)$ (3.34) by an application of the Slutsky's theorem. Then the functional CLT for $X^{(\epsilon)}(\omega)$ is established.

Because the environments are rotation invariant and coordinate-wise independent, the resulted diffusion matrix has the form $\sigma_v^2 I_d$, here I_d denotes the $d\times d$ unit matrix. Using the last equality of (3.37), (3.10) and (3.8), we have

$$d\sigma_v^2 = 2\|\nabla\phi\|_Q^2 - 2\|\nabla\chi\|_Q^2 = E_Q(|X_1(\omega)|_2^2) - 2\|\nabla\chi\|_Q^2, \tag{3.40}$$

where the factor d of the left side is due to the fact that the right side equals the sum of the diagonal elements of the diffusion matrix. The positivity of σ_v^2 can be proved by using the heat kernel upper bounds (1.11) or by the arguments of Remark 1.2(2) of Sidoravicius and Sznitman (2004). By (3.40), the equation (1.8) holds. Thus the proof of Theorem 1.1 for the **VSRW** $X(\omega)$ is completed.

Set $A(t) = \int_0^t \mu(X_s(\omega), \omega)ds$. By the ergodicity of the **VSRW** $X(\omega)$, we have

$$\lim_{t\to\infty} A(t)/t = \mathbb{E}(\mu(0,\omega)) = 2d\mathbb{E}(\mu(e)),\; \mathbb{Q}-a.s. \tag{3.41}$$

Define $\vartheta(t) = \inf\{s > 0 \mid A(s) \geq t\}$. We have

$$\widetilde{X}_t(\omega) = X_{\vartheta(t)}(\omega),\; t \geq 0,\; \mathbb{Q}-a.s., \tag{3.42}$$

$$\lim_{t\to\infty} \vartheta(t)/t = (2d\mathbb{E}\mu(e))^{-1} := a,\; \mathbb{Q}-a.s., \tag{3.43}$$

where (3.43) follows from (3.41).

Let $\widetilde{X}^{(\epsilon)}(\omega) = (\widetilde{X}_t^{(\epsilon)}(\omega),\; t \in [0,1])$ denote the rescaled **CSRW**, where $\widetilde{X}_t^{(\epsilon)}(\omega) = \epsilon\widetilde{X}_{t/\epsilon^2}(\omega)$.

When $\mathbb{E}(\mu(e)) = \infty$, we have $\lim_{t\to\infty}\vartheta(t)/t = 0$ by (3.43). Then using (3.42) and the weak convergence of $X^{(\epsilon)}(\omega)$, we get that $\widetilde{X}^{(\epsilon)}$ weakly converges to a degenerate Brownian Motion by the Slutsky's theorem and the equation (1.9) holds in this case.

When $\mathbb{E}(\mu(e)) < \infty$, we have $a > 0$ by (3.43). By the weak convergence of $X^{(\epsilon)}(\omega)$ and the scaling of the Brownian Motion, we have

$$X_{at}^{(\epsilon)}(\omega) \Longrightarrow \sqrt{a}B_t^d,\; t \in [0,1], \tag{3.44}$$

where the symbol "$\Longrightarrow$" stands for weak convergence and $(B_t^d,\, t \in [0,1])$ denotes the limiting $d-$dimensional Brownian Motion for $X^{(\epsilon)}(\omega)$. By the functional CLT (3.44) and the continuity of the Brownian Motion, for arbitrary constants $\varepsilon > 0$ and $\eta > 0$, there exists a constant $\delta > 0$ such that, when ϵ is small enough, we have

$$P_\omega\{\sup_{\substack{|s-t|\leq\delta \\ s,t\in[0,1]}} |X_{at}^{(\epsilon)}(\omega) - X_{as}^{(\epsilon)}(\omega)|_\infty > \varepsilon\} \leq \eta. \tag{3.45}$$

By (3.43), for arbitrary $\delta > 0$, when ϵ is small enough, we have

$$\sup_{t\in[0,1]} |\vartheta(t/\epsilon^2) - at/\epsilon^2| \le \delta/\epsilon^2, \ \mathbb{Q}{-}a.s. \tag{3.46}$$

To combine (3.45) and (3.46), when ϵ is small enough, we have,

$$P_\omega\{\sup_{t\in[0,1]} |\widetilde{X}_t^{(\epsilon)}(\omega) - X_{at}^{(\epsilon)}(\omega)|_\infty > \varepsilon\} \le \eta.$$

Since η and ε in the above inequality are arbitrary, we get that

$$\lim_{\epsilon\downarrow 0}\left(\widetilde{X}_t^{(\epsilon)}(\omega) - X_{at}^{(\epsilon)}(\omega)\right) = 0, \text{ in } P_\omega\text{−probability}, \ \mathbb{Q}{-}a.s.$$

By invoking the Slutsky's theorem, the above relation and the weak convergence (3.44) imply that

$$\widetilde{X}_t^{(\epsilon)}(\omega) \Longrightarrow \sqrt{a}B_t^d, \ t \in [0,1].$$

Then the diffusion constant $\sigma_c^2 = \sigma_v^2/(2d\mathbb{E}\mu(e))$ by (3.43), i.e., the equation (1.9) holds. The proof of Theorem 1.1 is completed. □

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