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Exponential Functionals of Lévy Processes with Jumps

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Abstract. We study the exponential functional $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ of two one-dimensional independent Lévy processes ξ and η, where η is a subordinator. In particular, we derive an integro-differential equation for the density of the exponential functional whenever it exists. Further, we consider the mapping Φ_ξ for a fixed Lévy process ξ, which maps the law of η_1 to the law of the corresponding exponential functional $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$, and study the behaviour of the range of Φ_ξ for varying characteristics of ξ. Moreover, we derive conditions for selfdecomposable distributions and generalized Gamma convolutions to be in the range. On the way we also obtain new characterizations of these classes of distributions.

1. Introduction

Given two independent Lévy processes $(\xi_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ the corresponding *exponential functional* is defined as

$$V := \int_{(0,\infty)} e^{-\xi_{t-}}\, d\eta_t, \tag{1.1}$$

provided that the integral converges a.s. Necessary and sufficient conditions for this convergence in terms of the Lévy characteristics of $(\xi_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ have been given by Erickson and Maller (2005).

Exponential functionals of Lévy processes describe the stationary distributions of generalized Ornstein-Uhlenbeck (GOU) processes. More detailed, if ξ_t tends to $+\infty$ as $t \to \infty$ almost surely, then the law of V defined in (1.1) is the unique stationary

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distribution of the GOU process

$$V_t = e^{-\xi_t}\left(\int_0^t e^{\xi_{s-}}\, d\eta_s + V_0\right), \quad t \geq 0, \tag{1.2}$$

where V_0 is a starting random variable, independent of (ξ, η), on the same probability space (cf. Lindner and Maller (2005, Thm. 2.1)).

Due to their importance in applications and their complexity, exponential functionals have gained a lot of attention from various researchers over the last 25 years. See e.g. the survey Bertoin and Yor (2005) or the more recent research papers Pardo et al. (2012, 2013) for results on exponential functionals of the form $V = \int_0^\infty e^{-\xi_{s-}}\, ds$. Exponential functionals where η is a Brownian motion plus drift have been treated for example in Kuznetsov et al. (2012). The case of general Lévy processes ξ and η has been studied e.g. in our previous papers Behme and Lindner (2013+) and Behme et al. (2014+). Nevertheless, for several of the more concrete results in Behme et al. (2014+), the setting was narrowed down to the case where ξ is a Brownian motion plus drift and η a subordinator.

Still, in general the distribution of exponential functionals is unknown. E.g. Dufresne (cf. Bertoin and Yor (2005, Equation (16))) showed that $V \overset{d}{=} \frac{2}{\sigma^2} G^{-1}_{2a/\sigma^2}$ where G_k is a Gamma$(k, 1)$ random variable, whenever ξ is a Brownian motion with variance σ^2 and drift $a > 0$, and $\eta_t = t$ is deterministic. Here and in the following $\overset{d}{=}$ denotes equality in distribution. A few more concrete distributions of specific exponential functionals have been obtained in Gjessing and Paulsen (1997). Further it has been investigated whether exponential functionals belong to certain classes of distributions. So, as shown in Bertoin et al. (2008), V is selfdecomposable whenever ξ is spectrally negative, i.e. has no positive jumps. In Behme et al. (2012) conditions are derived under which the exponential functional (1.1) is a generalized gamma convolution, where one of the processes is a compound Poisson process.

In this article we focus on the case of exponential functionals as in (1.1) when ξ is a general Lévy process such that $\lim_{t\to\infty} \xi_t = \infty$ and η is a subordinator, independent of ξ. By Behme et al. (2014+, Cor. 1) this means that $V \geq 0$ a.s. and we have the following relationship between the characteristic triplet $(\gamma_\xi, \sigma_\xi^2, \nu_\xi)$ of ξ and the Laplace exponents ψ_η and ψ_μ of η_1 and the distribution μ of V, resp.,

$$\begin{aligned}\psi_\eta(u) =& (\gamma_\xi - \frac{\sigma_\xi^2}{2}) u \psi_\mu'(u) + \frac{\sigma_\xi^2}{2} u^2 \left((\psi_\mu'(u))^2 - \psi_\mu''(u)\right) \\ &+ \int_{\mathbb{R}} \left(e^{\psi_\mu(u) - \psi_\mu(ue^{-y})} - 1 - u\psi_\mu'(u) y \mathbb{1}_{|y|\leq 1}\right) \nu_\xi(dy), \quad u > 0.\end{aligned} \tag{1.3}$$

Starting from this, we will consider several aspects of exponential functionals. In particular, in Section 2, we derive an integro-differential equation for the density of the exponential functional (given its existence) which extends a previous result from Carmona et al. (1997) where η was assumed to be deterministic.

Since selfdecomposable distributions and generalized Gamma convolutions play an important role in the remainder of the paper, we review them and their connection to exponential functionals in Section 3, which also includes some new results on these classes of distributions. Further, Section 4 is concerned with the behaviour of the class of distributions of exponential functionals for varying characteristics of ξ. In Sections 5 and 6 we derive general conditions for selfdecomposable distributions to be given by an exponential functional with predetermined process ξ and also apply these on generalized Gamma convolutions. Finally, Section 7 contains the proof of Proposition 3.7.

Notation. We write $\mu = \mathcal{L}(X)$ if μ is the distribution of the random variable X. The set of all probability distributions on $\mathbb{R}$ ($\mathbb{R}_+$) is denoted by $\mathcal{P}$ ($\mathcal{P}^+$).
For a real-valued Lévy process $(\xi_t)_{t\geq 0}$, the *characteristic exponent* is given by its Lévy-Khintchine formula (e.g. Sato (1999, Thm. 8.1))

$$\begin{aligned} \log \phi_\xi(u) &:= \log \mathbb{E}\left[e^{iu\xi_1}\right] \\ &= i\gamma_\xi u - \frac{1}{2}\sigma_\xi^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu_\xi(dx), \quad u \in \mathbb{R}, \end{aligned} \tag{1.4}$$

where $(\gamma_\xi, \sigma_\xi^2, \nu_\xi)$ is the *characteristic triplet* of the Lévy process ξ. We refer to Sato (1999) for further information on Lévy processes.
In the special case of a subordinator $(\eta_t)_{t\geq 0}$, i.e. of a nondecreasing Lévy process, we will also use its Laplace transform which we denote as $\mathbb{L}_\eta(u) := \mathbb{L}_{\eta_1}(u) = \mathbb{E}[e^{-u\eta_1}] = e^{-\psi_\eta(u)}$, $u \geq 0$, where the *Laplace exponent* ψ_η is a *Bernstein function* (BF), i.e.

$$\psi_\eta(u) = a_\eta u + \int_{(0,\infty)} (1 - e^{-ut})\nu_\eta(dt), \quad u > 0, \tag{1.5}$$

with $a \geq 0$ called the *drift* of η and a Lévy measure ν_η. A thorough introduction to BFs can be found in the monograph Schilling et al. (2012). Remark that general BFs as defined in Schilling et al. (2012) may have an additional constant term, while in this article we restrict on BFs which are Laplace exponents of a probability measure, that is which are zero in zero and hence are of the form (1.5).
Similarly, the Laplace transform of a random variable X on $\mathbb{R}_+$ with $\mu = \mathcal{L}(X)$ is written as $\mathbb{L}_X(u) = \mathbb{L}_\mu(u) = \mathbb{E}[e^{-uX}] = e^{-\psi_X(u)} = e^{-\psi_\mu(u)}$. Please notice, that this notation of Laplace exponents is different from the previous papers Behme and Lindner (2013+); Behme et al. (2014+) but coincides with the notation used in Schilling et al. (2012).

As in Behme and Lindner (2013+); Behme et al. (2014+), given a one-dimensional Lévy process $(\xi_t)_{t\geq 0}$ drifting to $+\infty$, we will consider the mapping

$$\begin{aligned} \Phi_\xi^+ &: D_\xi^+ \to \mathcal{P}^+, \\ \mathcal{L}(\eta_1) &\mapsto \mathcal{L}\left(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s\right), \end{aligned}$$

defined on

$$\begin{aligned} D_\xi^+ := \{\mathcal{L}(\eta_1) : \eta = (\eta_t)_{t\geq 0} &\text{ one-dimensional subordinator independent of } \xi \\ &\text{such that } \int_0^\infty e^{-\xi_{s-}}\, d\eta_s \text{ converges a.s.}\}, \end{aligned}$$

and we denote the range of Φ_ξ^+ by

$$R_\xi^+ := \Phi_\xi^+(D_\xi^+).$$

2. On the density of the exponential functional

As already observed in previous articles, it follows directly from Alsmeyer et al. (2009, Thm. 1.3) that the exponential functional V has a pure-type law, i.e. its distribution is either absolutely continuous, continuous singular or a Dirac measure, where the latter can only be obtained if both processes, ξ and η, are deterministic (c.f. Behme and Lindner (2013+, Prop. 6.1)).

Absolute continuity of exponential functionals has been studied in detail in Bertoin et al. (2008). For the setting of this paper, Bertoin et al. (2008, Thm 3.9) shows in particular, that the exponential functional V as in (1.1) is absolutely continuous, whenever the subordinator η has a strictly positive drift. Further, in Kuznetsov et al. (2012, Cor. 2.5), it is shown that the exponential functional V as in (1.1) is absolutely continuous with continuous density if $\sigma_\xi > 0$.

Nevertheless, if η and ξ both are compound Poisson processes, examples can be constructed in which V is not absolutely continuous (see Lindner and Sato (2009) and Remark 2.2 below).

The following theorem provides an integro-differential equation fulfilled by the density of V whenever it exists. Notice that for the special case of a deterministic process $\eta_t = t$ this result has been obtained in Carmona et al. (1997) using a different technique. In particular, case (2) below is a special case of the results in Carmona et al. (1997) or similarly of Pardo et al. (2013, Thm. 2.3) and is just kept here for completeness.

Theorem 2.1. *Assume that $\xi = (\xi_t)_{t\geq 0}$ is a Lévy process such that $\lim_{t\to\infty} \xi_t = \infty$ and with characteristic triplet $(\gamma_\xi, \sigma_\xi^2, \nu_\xi)$ such that $\int_{[-1,1]} |x| \nu_\xi(dx) < \infty$ and set $\gamma_0 := \gamma_\xi - \int_{[-1,1]} x\nu_\xi(dx)$. Let $\eta = (\eta_t)_{t\geq 0}$ be a subordinator with drift a_η and jump measure ν_η, independent of ξ and such that at least one of the processes ξ and η is non-deterministic.*

(1) *If $\sigma_\xi = 0$, $\gamma_0 > 0$ and $\nu_\xi((0,\infty)) = 0$, then a density $f(t)$, $t \geq 0$, of $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$ exists, which is continuous on $\mathbb{R}_+ \setminus \{\frac{a_\eta}{\gamma_0}\}$, and fulfills*

$$f(t) = 0, \quad t < \frac{a_\eta}{\gamma_0}, \tag{2.1}$$

$$(a_\eta - \gamma_0 t) f(t) = -\int_{\frac{a_\eta}{\gamma_0}}^{t} \Big(\nu_\xi((-\infty, \log\frac{s}{t})) + \nu_\eta((t-s,\infty))\Big) f(s)ds,\ t \geq \frac{a_\eta}{\gamma_0}.$$

(2) *If $\sigma_\xi = 0$, $\gamma_0 > 0$, $\nu_\xi((0,\infty)) > 0$, $\nu_\xi((-\infty,0)) = 0$ and $\nu_\eta \equiv 0$, then a density $f(t)$, $t \geq 0$, of $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$ exists, which is continuous on $\mathbb{R}_+ \setminus \{\frac{a_\eta}{\gamma_0}\}$, and fulfills*

$$f(t) = 0, \quad t > \frac{a_\eta}{\gamma_0}, \tag{2.2}$$

$$(a_\eta - \gamma_0 t) f(t) = \int_t^{\frac{a_\eta}{\gamma_0}} \nu_\xi((\log\frac{s}{t}, \infty)) f(s)ds,\ t \leq \frac{a_\eta}{\gamma_0}.$$

(3) *Otherwise, assume that $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$ is absolutely continuous (with differentiable density $f(t)$, $t \geq 0$, such that $\lim_{t\to 0} t^2 f(t) = 0$ if $\sigma_\xi > 0$), then*

f fulfills λ-a.e. (with λ the Lebesgue measure)

$$a_\eta f(t) - \left(\gamma_0 + \frac{\sigma_\xi^2}{2}\right) t f(t) - \frac{\sigma_\xi^2}{2} t^2 f'(t) \tag{2.3}$$
$$= \int_t^\infty \nu_\xi((\log \frac{s}{t}, \infty)) f(s) ds$$
$$- \int_0^t \left(\nu_\xi((-\infty, \log \frac{s}{t})) + \nu_\eta((t-s, \infty))\right) f(s) ds, \quad t \geq 0.$$

Conversely, if $f(t)$, $t \geq 0$, is a probability density which fulfills (2.1), (2.2) or (2.3) λ-a.e. for some Lévy characteristics $\gamma_0, \sigma_\xi^2, \nu_\xi, a_\eta$ and ν_η, then it is a density of the corresponding exponential functional (1.1).

Proof: Starting from (1.3), multiplying on both sides with $\mathbb{L}_\mu(u) = e^{-\psi_\mu(u)}$ and dividing once by u we obtain for $u > 0$

$$\frac{\psi_\eta(u)}{u}\mathbb{L}_\mu(u) = -(\gamma_0 - \frac{\sigma_\xi^2}{2})\mathbb{L}'_\mu(u) + \frac{\sigma_\xi^2}{2} u \mathbb{L}''_\mu(u) + \int_{\mathbb{R}} \left(\frac{\mathbb{L}_\mu(ue^{-y})}{u} - \frac{\mathbb{L}_\mu(u)}{u}\right) \nu_\xi(dy). \tag{2.4}$$

Now assume that μ has a density, such that $\mathbb{L}_\mu(u) = \int_0^\infty e^{-ut} f(t) dt$. Denote the inverse Laplace transform by $\xrightarrow{\mathbb{L}^{-1}}$, then obviously we have $\mathbb{L}_\mu(u) \xrightarrow{\mathbb{L}^{-1}} f(t)$ λ-a.e. while (assuming $\lim_{t\to 0} t^2 f(t) = 0$ and that f is differentiable) λ-a.e. we get

$$\mathbb{L}'_\mu(u) \xrightarrow{\mathbb{L}^{-1}} -t f(t),$$
$$u\mathbb{L}''_\mu(u) \xrightarrow{\mathbb{L}^{-1}} \frac{d}{dt}(t^2 f(t)) = 2t f(t) + t^2 f'(t),$$
$$\int_{\mathbb{R}} \left(\frac{\mathbb{L}_\mu(ue^{-y})}{u} - \frac{\mathbb{L}_\mu(u)}{u}\right) \nu_\xi(dy) \xrightarrow{\mathbb{L}^{-1}} \int_t^\infty \nu_\xi((\log \frac{s}{t}, \infty)) f(s) ds$$
$$- \int_0^t \nu_\xi((-\infty, \log \frac{s}{t})) f(s) ds,$$

where the last line follows from

$$\int_{\mathbb{R}} \left(\frac{\mathbb{L}_\mu(ue^{-y})}{u} - \frac{\mathbb{L}_\mu(u)}{u}\right) \nu_\xi(dy)$$
$$= \int_{\mathbb{R}} \left(\int_0^\infty e^{-ut} \left(\int_0^{te^y} f(s) ds - \int_0^t f(s) ds\right) dt\right) \nu_\xi(dy)$$
$$= \int_0^\infty e^{-ut} \int_t^\infty f(s) \int_{\log \frac{s}{t}}^\infty \nu_\xi(dy)\, ds\, dt - \int_0^\infty e^{-ut} \int_0^t f(s) \int_{-\infty}^{\log \frac{s}{t}} \nu_\xi(dy)\, ds\, dt.$$

Further for the left hand side of (2.4) with $\psi_\eta(u) = a_\eta u + \int_{(0,\infty)} (1 - e^{-ut}) \nu_\eta(dt)$ we will use that

$$\frac{\int_{(0,\infty)} (1 - e^{-ut}) \nu_\eta(dt)}{u} \mathbb{L}_\mu(u)$$
$$= \int_0^\infty e^{-us} \nu_\eta((s, \infty)) ds\, \mathbb{L}_\mu(u) \xrightarrow{\mathbb{L}^{-1}} \int_0^t \nu_\eta((t-s, \infty)) f(s) ds$$

which is due to the fact that convolutions become multiplications under the Laplace transform. Now, putting all terms together we easily derive (2.3).

Observe that in the setting of case (1), it follows from Behme et al. (2014+, Lemma 1 and Thm. 1) that the measure μ has support $[\frac{a_\eta}{\gamma_0}, \infty)$. Further recall that in this case ξ is a spectrally negative process ξ and hence μ is selfdecomposable and has a continuous density on $(\frac{a_\eta}{\gamma_0}, \infty)$ (cf. Steutel and van Harn (2004, Thm. V.2.16)).
By Behme et al. (2014+, Lemma 1 and Thm. 1) in the setting of case (2) μ has support $[0, \frac{a_\eta}{\gamma_0}]$ and otherwise μ has full support on $[0, \infty)$. Hence we derive the corresponding formulas from (2.3). Existence of a density in case (2) follows from Bertoin et al. (2008, Thm. 3.9), continuity has been proven in Carmona et al. (1997).
For the converse assume that f is a density which fulfills (2.3), then reverting the above we see that its Laplace transform fulfills (1.3) which yields the claim by Behme et al. (2014+, Thm. 3). □

Remark 2.2. In Lindner and Sato (2009) the exponential functional V as in (1.1) has been studied in the case where $(\eta_t)_{t\geq 0}$ is a Poisson process with jump intensity $v > 0$, and $\xi_t = (\log c)N_t$ for $c > 1$ and another (independent) Poisson process $(N_t)_{t\geq 0}$ with jump intensity $u > 0$.
From Theorem 2.1 above, we observe that in this setting, if a density of V exists, then it fulfills λ-a.e.

$$v \int_{(t-1)\vee 0}^{t} f(s)ds = u \int_{t}^{ct} f(s)ds, \quad t \geq 0$$

or in terms of the cumulative distribution function $F(t) = \int_0^t f(s)ds$ and the parameter $q = \frac{v}{u+v} \in (0, 1)$

$$F(t) = (1-q)F(ct) + qF(t-1), \quad t > 0, \qquad \text{where } F(t) = 0, \quad t \leq 0. \tag{2.5}$$

Actually, (2.5) can be shown to hold even if $\mu = \mathcal{L}(V)$ is not absolutely continuous, by a similar proof as for Theorem 2.1. Further, from (2.5) we deduce the self-similarity relation

$$\mu = (1-q)\,\mu \circ T_0^{-1} + q\,\mu \circ T_1^{-1}$$

for μ with weights $\{1-q, q\}$ and

$$T_0 : x \mapsto \frac{x}{c}, \quad T_1 : x \mapsto x + 1.$$

Remark that T_1 is not a contraction and hence μ is not a self-similar measure in the classical and well-studied sense of Hutchinson (1981).
Nevertheless, in Lindner and Sato (2009), the authors proved that μ shares some properties with self-similar measures. In particular, μ is continuous singular if c is a Pisot-Vijayaraghavan number, but for Lebesgue a.a. $c > 1$ there exists $\bar{q} < 1$ such that μ is absolutely continuous for all $q \in (\bar{q}, 1)$.

From the theorem above, we can derive characterizations of densities of selfdecomposable distributions on $\mathbb{R}_+$ as well as of generalized Gamma convolutions. This will be done in Corollaries 3.4 and 3.6 below. For the moment, we end this section with an example of application for Theorem 2.1.

Example 2.3. Assume $L = (L_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma_L, \sigma_L^2, \nu_L)$ and set

$$S_t := [L, L]_t^d = \sum_{0<s\leq t} (\Delta L_s)^2, \quad t \geq 0.$$

Then the COGARCH volatility process with parameters $\beta, \eta, \varphi > 0$ driven by L or S is defined as

$$V_t = e^{-\xi_t} \left(V_0 + \beta \int_{(0,t]} e^{\xi_s} \, ds \right), \quad t \geq 0,$$

where V_0 is a nonnegative random variable, independent of $(L_t)_{t\geq 0}$, and

$$\xi_t = \eta t - \sum_{0<s\leq t} \log(1 + \varphi \Delta S_s), \quad t \geq 0.$$

As originally shown in Klüppelberg et al. (2004, Thm. 3.1), the process defined in (2.3) has a strictly stationary distribution if and only if

$$\int_{\mathbb{R}_+} \log(1 + \varphi y) \, \nu_S(dy) = \int_{\mathbb{R}} \log(1 + \varphi y^2) \, \nu_L(dy) < \eta$$

and in this case, the stationary distribution is given by the distribution of the exponential functional

$$V = \beta \int_{\mathbb{R}_+} e^{-\xi_s} \, ds.$$

Since ξ is spectrally negative by construction, we can apply Theorem 2.1(1) (or Carmona et al. (1997, Prop. 2.1)) to obtain that V has a density $f(t)$, $t \geq 0$, with $f(t) = 0$ for $t < \frac{\beta}{\eta}$, while f is continuous on $(\frac{\beta}{\eta}, \infty)$ fulfilling

$$(\beta - \eta t) f(t) + \int_{\frac{\beta}{\eta}}^{t} \nu_S \left(\left(\frac{t-s}{s\varphi}, \infty \right) \right) f(s) ds = 0, \quad t \geq \frac{\beta}{\eta}. \tag{2.6}$$

Now, if for example $(S_t)_{t\geq 0}$ is chosen to be a Poisson process with intensity $c > 0$, we obtain from (2.6) the following difference-differential equation for the cumulative distribution function $F(t)$ of V

$$\frac{\eta t - \beta}{c} F'(t) = F(t) - F(\frac{\beta}{\eta}), \quad t \geq \frac{\beta}{\eta},$$

with $F(t) = 0$ for $t < \frac{\beta}{\eta}$. Similarly, for the common choice of L having standard normally distributed jumps, one derives the recursive formula

$$f(t) = \frac{2}{\beta - \eta t} \int_{\frac{\beta}{\eta}}^{t} \left(1 - \phi \left(\sqrt{\frac{t-s}{s\varphi}} \right) \right) f(s) ds, \quad t > \frac{\beta}{\eta},$$

where ϕ is the cumulative distribution function of the normal distribution.

3. (Semi-)Selfdecomposability and Generalized Gamma Convolutions

We will use the following notations for the classes of infinitely divisible distributions:

$\mathrm{ID}, \mathrm{ID}^+$	infinitely divisible distributions on $\mathbb{R}, \mathbb{R}_+$ (respectively)
$\mathrm{ID}_{\log}, \mathrm{ID}^+_{\log}$	infinitely divisible distributions on $\mathbb{R}, \mathbb{R}_+$ with finite log-moment

Further the following classes of distributions will be introduced in the next subsections:

L, L^+	selfdecomposable distributions on $\mathbb{R}, \mathbb{R}_+$
$\mathrm{L}(c), \mathrm{L}(c)^+$	c-decomposable, semi-selfdecomposable distributions on $\mathbb{R}, \mathbb{R}_+$
BO	Goldie-Steutel-Bondesson class/ Bondesson's class (on $\mathbb{R}_+$)
T	Thorin's class/ generalized gamma convolutions (on $\mathbb{R}_+$)

3.1. *Selfdecomposability.* A random variable X (or equivalently a probability measure μ) is called *selfdecomposable*, if for all $c \in (0,1)$, there exists a random variable Y_c, independent of X, such that

$$X \overset{d}{=} cX' + Y_c, \tag{3.1}$$

where X' is an independent copy of X. In this case we write $\mu = \mathcal{L}(X) \in \mathrm{L}$. Obviously, for distributions on the positive real line, (3.1) is equivalent to

$$\mathbb{L}_\mu(u) = \mathbb{L}_\mu(cu)\mathbb{L}_{\mu_c}(u), \quad u \geq 0, c \in (0,1),$$

or

$$\psi_\mu(u) - \psi_\mu(cu) = \psi_{\mu_c}(u), \quad u \geq 0, c \in (0,1), \tag{3.2}$$

where $\mu_c = \mathcal{L}(Y_c)$. In particular it is known (cf. Schilling et al. (2012, Prop. 5.17)), that every $\mu \in \mathrm{L}^+$ has a Laplace exponent of the form

$$\psi_\mu(u) = au + \int_0^\infty (1 - e^{-ut}) \frac{k(t)}{t} dt, \quad u \geq 0, \tag{3.3}$$

with $a \geq 0$ called the *drift* of μ and $k : [0,\infty) \to [0,\infty)$ non-increasing.

The following proposition collects characterizations of selfdecomposable distributions in $\mathcal{P}^+$ which we intend to use in this paper. Most of them are well known. We couldn't find characterization (iv) in this form in the literature, so we give a short instructive proof. Alternatively (iv) is easily seen to be equivalent to the characterization of selfdecomposability in Steutel and van Harn (2004, Thm. V.2.9). Further characterizations of selfdecomposable distributions can also be found in Maejima (2015+); Sato (2010); Steutel and van Harn (2004) and for a.s. positive random variables in the recent article Mai et al. (2014+) as well as in Corollary 3.4 below.

Proposition 3.1. *Let $\mu \in \mathcal{P}^+$ be a probability measure with Laplace exponent $\psi_\mu(u)$, $u \geq 0$. Then the following statements are equivalent.*

(i) $\mu \in \mathrm{L}^+$.
(ii) $\psi_{\mu_c}(u) := \psi_\mu(u) - \psi_\mu(cu)$ *is a Bernstein function for all* $c \in (0,1)$.
(iii) $-\psi_{\mu_c}(u) = \psi_\mu(cu) - \psi_\mu(u)$ *is a BF for all* $c > 1$.
(iv) $u \cdot \psi'_\mu(u)$ *is a BF.*
(v) $\mu = \mathcal{L}(\int_{(0,\infty)} e^{-t} dX_t)$ *for some subordinator* $(X_t)_{t\geq 0}$ *with* $\mathbb{E}[\log^+(X_1)] < \infty$.

Proof: Equivalence of (i) and (ii) is well known and follows immediately from the definition of selfdecomposability and the fact that μ_c as in (3.1) is infinitely divisible (see e.g. Sato (1999, Prop. 15.5)). Further by Schilling et al. (2012, Cor. 3.8(iii)) (ii) implies that also $\psi_{\mu_c}(c^{-1}u) = \psi_\mu(c^{-1}u) - \psi_\mu(u)$, $c \in (0,1)$ is a BF, i.e. (iii).

The converse can be seen similarly.
We continue proving that (ii) implies (iv). Assume (ii), then for all $c \in (0,1)$

$$\frac{\psi_\mu(u) - \psi_\mu(u-(1-c)u)}{(1-c)}$$

is a BF in u. Thus

$$u\psi'_\mu(u) = \lim_{c\to 1} \frac{\psi_\mu(u) - \psi_\mu(u-(1-c)u)}{(1-c)}$$

is a BF, too (Schilling et al. (2012, Cor. 3.8(ii))), which shows (iv).
Now assume (iv) and set

$$\psi_X(u) := u\psi'_\mu(u), \quad u \geq 0, \tag{3.4}$$

then ψ_X is a BF with $\psi_X(0) = 0$ and hence there exists a subordinator $(X_t)_{t\geq 0}$ with Laplace exponent ψ_X. Now by Behme et al. (2014+, Thm. 5.1 (ii)) (setting $\sigma = 0$) this implies that

$$\mu = \mathcal{L}\left(\int_{(0,\infty)} e^{-t} dX_t\right). \tag{3.5}$$

Since μ exists by assumption and therefore the integral has to converge, we obtain $\mathbb{E}[\log^+(X_1)] < \infty$ and hence (v).
Finally, $\int_{(0,\infty)} e^{-t} dX_t$ is well known and easily seen to be selfdecomposable (see e.g. Bertoin et al. (2008)) which concludes the proof. □

Remark 3.2. As already observed in Behme et al. (2014+), Equation (3.4) implies in particular, that μ and $\mathcal{L}(X)$ have the same drift and that the Lévy density of μ and the Lévy measure of X are related by

$$k(t) = \nu_X((t,\infty)) \tag{3.6}$$

(see also Barndorff-Nielsen and Shephard (2001, Eq. 4.17)).

Definition 3.3. Differences of BFs as in (ii) and (iii) of the above proposition will appear frequently in the remaining sections of this article. Hence in the following, we refer to the distributions with Laplace exponent ψ_{μ_c} $(c \in (0,1))$ or $-\psi_{\mu_c}$ $(c > 1)$ as *c-factor distributions* of the distribution $\mu \in \mathrm{L}$. Recall that these are always in ID and that they are uniquely determined since $\mu \in \mathrm{ID}$.
In terms of random variables we refer to Y_c as the *c-factor* $(c \in (0,1))$ of X if $X \overset{d}{=} cX' + Y_c$ and we say that Y_c is the *c-factor* $(c > 1)$ for X, if $cX \overset{d}{=} X' + Y_c$.

Further, from Theorem 2.1 above we obtain the following characterization of densities of distributions in L^+. The fact that densities of selfdecomposable distributions fulfill an equality like (3.7) can also be found in Steutel and van Harn (2004, Thm. V.2.16). Here we see that actually all solutions to (3.7) correspond to distributions in L^+.

Corollary 3.4. *Let $f(t)$ be a probability density with support $[a,\infty)$, $a \geq 0$, which is continuous on (a,∞). Then f corresponds to a selfdecomposable distribution, if and only if f fulfills*

$$(a-t)f(t) + \int_a^t \nu((t-s,\infty))f(s)ds = 0, \quad t \geq a, \tag{3.7}$$

for some Lévy measure ν such that $\int_0^\infty \log^+(x)\nu(dx) < \infty$.

Proof: Every distribution $\mu \in \mathrm{L}^+$ which is non-degenerate is absolutely continuous and can be represented as $\mu = \Phi_\xi(\mathcal{L}(L_1))$ for $\xi_t = t$ and some subordinator L with $\mathbb{E}[\log^+(L_1)] < \infty$. Further $\operatorname{supp}(\mu) = [a, \infty)$, $a \geq 0$, implies by Behme et al. (2014+, Thm. 1(ii)) that L has drift a. Hence by Theorem 2.1(1) the density of μ fulfills (3.7).
Conversely, if $f(t)$ is a density with support $[a, \infty)$, $a \geq 0$, which is continuous on (a, ∞) and which fulfills (3.7), then by Theorem 2.1 it is the density of the exponential functional $\int_{(0,\infty)} e^{-t} dL_t$ for some subordinator L with Lévy measure ν and drift $a \geq 0$. Thus we conclude that $\mu \in \mathrm{L}^+$. □

3.2. *Semi-selfdecomposability.* We say a random variable X (or its probability measure μ) is *c-decomposable*, $c \in (0,1)$, or semi-selfdecomposable if (3.1) holds for a $c \in (0,1)$ and a random variable Y_c such that $\mathcal{L}(Y_c) \in \mathrm{ID}$. We write $\mathrm{L}(c), \mathrm{L}^+(c)$ for the class of c-decomposable distributions on $\mathbb{R}$ and $\mathbb{R}_+$, respectively. As in the case of selfdecomposable distributions, we refer to the random variable Y_c in (3.1) as the *c-factor* of X.

By Sato (1999, Prop. 15.5) it holds $\mathrm{L}(c) \subset \mathrm{ID}$.

For probability distributions on $\mathbb{R}_+$ one can characterize c-decomposability in terms of the Laplace exponents. In particular, $\mu \in \mathrm{L}^+(c)$ if and only if $\psi_{\mu_c}(u) = \psi_\mu(u) - \psi_\mu(cu)$, $u > 0$, is a BF. The fact that BFs build a convex cone then implies directly $\mathrm{L}^+(c) \subseteq \mathrm{L}^+_{c^n}$ for all $n \in \mathbb{N}$. More detailed

$$\psi_{\mu_{c^n}}(u) = \sum_{i=0}^{n-1} \psi_{\mu_c}(c^i u) \tag{3.8}$$

is the Laplace exponent of the c^n-factor of $\mu \in \mathrm{L}^+(c)$. Using this one further obtains for any $\mu \in \mathrm{L}^+(c)$

$$\psi_\mu(u) = \lim_{n\to\infty} \psi_{\mu_{c^n}}(u) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \psi_{\mu_c}(c^i u)$$

such that

$$\mathrm{L}^+(c) = \{\mu \in \mathcal{P}^+, \text{ s.t. } \psi_\mu(u) = \sum_{i=0}^{\infty} f(c^i u) \text{ for some BF } f\}.$$

3.3. *Generalized Gamma Convolutions.* The class of generalized Gamma convolutions T is a subclass of the selfdecomposable distributions in $\mathcal{P}^+$. In particular, every $\mu \in \mathrm{T}$ has a Laplace exponent of the form

$$\psi_\mu(u) = au + \int_{(0,\infty)} (1 - e^{-ut}) \frac{k(t)}{t} dt, \quad u \geq 0, \tag{3.9}$$

for some $a \geq 0$ and a completely monotone (CM) function $k : (0, \infty) \to [0, \infty)$.
The class of probability distributions whose Laplace transform is of the form (3.9) for some $a \geq 0$ with $\frac{k(t)}{t}$ CM is called Goldie-Steutel-Bondesson class or simply Bondesson's class (BO). Its Laplace exponents are referred to as complete Bernstein functions (CBF) and they can always be represented as

$$\psi_\mu(u) = au + \int_{(0,\infty)} \frac{u}{u + x} d\rho(x), \quad u \geq 0, \tag{3.10}$$

with $a \geq 0$ and a so-called Stieltjes measure ρ, that is a measure ρ on $(0,\infty)$ for which $\int_{(0,\infty)}(1+x)^{-1}\rho(dx) < \infty$. For further details and an overview of the existing literature we refer to Schilling et al. (2012) and Maejima (2015+).
Recall that BO is the smallest class of distributions which contains all mixtures of exponential distributions and is closed under convolutions and weak limits, while T is the smallest class that contains all gamma distributions and is closed under convolutions and weak limits. Also recall that $\mathrm{T} \subset \mathrm{BO} \subset \mathrm{ID}^+$ and $\mathrm{T} \subset \mathrm{L}^+ \subset \mathrm{ID}^+$, but $\mathrm{L}^+ \not\subset \mathrm{BO}$ and $\mathrm{BO} \not\subset \mathrm{L}^+$.

Generalized Gamma convolutions and distributions in BO are connected via exponential functionals as shown in the following proposition, which has originally been proven in Barndorff-Nielsen et al. (2006, Thm. C(iii)). Nevertheless, we can now give a completely different and shorter proof as we shall do.

Proposition 3.5. *Let $\xi_t = t$. Then*

$$\Phi_\xi(\mathrm{BO} \cap \mathrm{ID}_{\log}) = \mathrm{T}$$

In particular, the distributions in $\mathrm{BO} \cap \mathrm{ID}_{\log}$ with finite Stieltjes measure are mapped surjectively on the generalized Gamma convolutions with $k(0+) < \infty$.

Proof: Assume $\mu \in \mathrm{T} \subset \mathrm{L}^+$, then there exists a Lévy process X with $\mathcal{L}(X_1) \in \mathrm{ID}^+_{\log}$ such that $\Phi_\xi(X_1) = \mu$, i.e. X and μ are related via (3.4) or (3.5). Hence from (3.4) and (3.9)

$$\psi_X(u) = au + u\int_{(0,\infty)} e^{-ut}k(t)dt = au + u\int_{(0,\infty)} e^{-ut}\int_{[0,\infty)} e^{-tx}d\rho(x)dt$$

for some unique measure ρ with $\rho(\{0\}) = \lim_{t\to\infty} k(t) = 0$. Using Tonelli we can proceed

$$\psi_X(u) = au + \int_{(0,\infty)} u\int_{(0,\infty)} e^{-ut}e^{-tx}dt\,d\rho(x) = au + \int_{(0,\infty)} \frac{u}{u+x}d\rho(x).$$

Hence $\psi_X(u)$ is a CBF (see e.g. Schilling et al. (2012, Remark 6.4)) such that $\mathcal{L}(X_1) \in \mathrm{BO}$ by Schilling et al. (2012, Def. 9.1).
Conversely, assume that X is a Lévy process such that $\mathcal{L}(X_1) \in \mathrm{BO} \cap \mathrm{ID}_{\log}$. Then $\Phi_\xi(\mathcal{L}(X_1))$ exists and the same computation backwards proves that $\Phi_\xi(\mathcal{L}(X_1)) \in \mathrm{T}$. The remaining assertion follows directly from an inspection of the above proof. □

From this, we obtain an analogue result to Corollary 3.4 characterizing the densities of distributions in T.

Corollary 3.6. *Let $f(t)$ be a probability density with support $[a,\infty)$, $a \geq 0$, which is continuous on (a,∞). Then f is the density of a generalized gamma convolution, if and only if f fulfills*

$$(a-t)f(t) + \int_a^t f(s)\int_{t-s}^\infty m(x)dx\,ds = 0, \quad t \geq a,$$

for some $m(x) : (0,\infty) \to [0,\infty)$ which is CM and such that $\int_0^\infty \log^+(x)m(x)dx < \infty$.

Proof: The statement follows similarly to Corollary 3.4 with the help of Proposition 3.5. □

As mentioned, the c-factors of selfdecomposable distributions play an important role for our studies. In the following proposition, which is of interest by its own, we will see, that the GGCs are exactly those distributions in L^+ whose c-factors are all in Bondesson's class. Its proof is postponed to the closing section of this article.

Proposition 3.7. *Let $\mu \in \mathrm{T}$, then $\mu_c \in \mathrm{BO}$ for all $c > 0$, $c \neq 1$. Conversely, if $\mu \in \mathrm{L}^+$ with either $\mu_c \in \mathrm{BO}$ for all $c \in (0,1)$, or $\mu_c \in \mathrm{BO}$ for all $c > 1$, then $\mu \in \mathrm{T}$.*

Summarizing, we can state the characterizations of the class T similarly to that of L^+ in Proposition 3.1.

Corollary 3.8. *Let $\mu \in \mathrm{L}^+$ be a probability measure with Laplace exponent $\psi_\mu(u)$, $u > 0$. Then the following statements are equivalent.*

(i) $\mu \in \mathrm{T}$.
(ii) $\psi_{\mu_c}(u) := \psi_\mu(u) - \psi_\mu(cu)$ *is a CBF for all* $c \in (0,1)$.
(iii) $-\psi_{\mu_c}(u) = \psi_\mu(cu) - \psi_\mu(u)$ *is a CBF for all* $c > 1$.
(iv) $u \cdot \psi'_\mu(u)$ *is a CBF.*
(v) $\mu = \mathcal{L}(\int_{(0,\infty)} e^{-t} dX_t)$ *for some subordinator* $(X_t)_{t\geq 0}$ *with* $\mathbb{E}[\log^+(X_1)] < \infty$ *and* $\mathcal{L}(X_1) \in \mathrm{BO}$.

4. Nested ranges

In this section, we will consider what happens with the range R_ξ^+ when we modify the characteristics of ξ. This result has a counterpart in the case when ξ is a Brownian motion (see Behme et al. (2014+, Thm. 5)), although here for some statements we have to restrict on $\mathrm{L} \cap R_\xi^+$. That this restriction is truly necessary will subsequently be shown in Proposition 4.2.

Theorem 4.1. *Let $(\xi_t)_{t\geq 0}$ be a Lévy process with characteristic triplet (γ, σ^2, ν) and write $R^+(\gamma, \sigma^2, \nu) := R_\xi^+$.*
Then if $\sigma^2 \neq 0$

$$R^+(\gamma, \sigma^2, \nu) = R^+(\gamma/\sigma^2, 1, \nu/\sigma^2).$$

Further for $\gamma' \geq \gamma$ it holds

$$\mathrm{L} \cap R^+(\gamma, \sigma^2, \nu) \subseteq \mathrm{L} \cap R^+(\gamma', \sigma^2, \nu), \tag{4.1}$$

while assuming that $\nu((0,\infty)) = 0$ and $\int_{[-1,0)} |x|\nu(dx) < \infty$ we obtain

$$R^+(\gamma, \sigma^2, \nu) \subseteq R^+(\gamma', \sigma^2, \lambda\nu) \tag{4.2}$$

for all $\lambda \in (0,1]$ and γ' such that $\gamma' - \gamma \geq -(1-\lambda)\int_{[-1,0)} x\nu(dx)$.

Proof: By the Lévy-Itô-decomposition we have $\xi_t = \sigma B_t + \tilde{\xi}_t$, where $\sigma = \sqrt{\sigma^2}$ and $(B_t)_{t\geq 0}$ is a standard Brownian motion and independent of $\tilde{\xi}_t$. Hence $(\sigma B_t)_{t\geq 0} \overset{d}{=} (B_{\sigma^2 t})_{t\geq 0}$ and thus $(\sigma B_t + \tilde{\xi}_t)_{t\geq 0} \overset{d}{=} (B_{\sigma^2 t} + \tilde{\tilde{\xi}}_{\sigma^2 t})_{t\geq 0}$ where $\tilde{\tilde{\xi}}$ has characteristic triplet $(\gamma/\sigma^2, 0, \nu/\sigma^2)$.
This implies that for any subordinator $(\eta_t)_{t\geq 0}$, independent of ξ and with $\mathcal{L}(\eta_1) \in D_\xi^+$

$$\int_{(0,\infty)} e^{-\xi_t} d\eta_t = \int_{(0,\infty)} e^{-(\sigma B_t + \tilde{\xi}_t)} d\eta_t \overset{d}{=} \int_{(0,\infty)} e^{-(B_{\sigma^2 t} + \tilde{\tilde{\xi}}_{\sigma^2 t})} d\eta_t$$
$$= \int_{(0,\infty)} e^{-(B_t + \tilde{\tilde{\xi}}_t)} d\eta_{t/\sigma^2}.$$

Thus $\mathcal{L}(\eta_{1/\sigma^2}) \in D^+_{B+\tilde{\tilde{\xi}}}$ and $\Phi^+_{\xi}(\mathcal{L}(\eta_1)) = \Phi^+_{B+\tilde{\tilde{\xi}}}(\mathcal{L}(\eta_{1/\sigma^2}))$ from which we conclude the first assertion.

Now assume $\mu \in R^+(\gamma, \sigma^2, \nu) \cap \mathrm{L}$, then by Behme et al. (2014+, Thm. 3)

$$\begin{aligned} f_\gamma(u) =& (\gamma - \frac{\sigma^2}{2}) u\psi'_\mu(u) + \frac{\sigma^2}{2} u^2 \left((\psi'_\mu(u))^2 - \psi''_\mu(u) \right) \\ &+ \int_{\mathbb{R}} \left(e^{\psi_\mu(u) - \psi_\mu(ue^{-y})} - 1 - u\psi'_\mu(u) y \mathbb{1}_{|y|\leq 1} \right) \nu(dy), \quad u \geq 0, \end{aligned}$$

is the Laplace exponent of some subordinator, i.e. a BF. Observe that for $\gamma' \geq \gamma$

$$f_{\gamma'}(u) = f_\gamma(u) + (\gamma' - \gamma) u\psi'_\mu(u).$$

Since the set of BFs is a convex cone (cf. Schilling et al. (2012, Cor. 3.8(i))) and since by assumption $\mu \in \mathrm{L}^+$ such that $u\psi'_\mu(u)$ is a BF, $f_{\gamma'}(u)$ is again a BF. Hence $\mu \in R^+(\gamma', \sigma^2, \nu)$ by Behme et al. (2014+, Thm. 3).

Finally, assume $\mu \in R^+(\gamma, \sigma^2, \nu)$ where $\nu((0, \infty)) = 0$ and $\int_{[-1,0)} |x| \nu(dx) < \infty$ and set for $\lambda \in (0, 1]$

$$\begin{aligned} g_\lambda(u) =& (\gamma_\lambda - \frac{\sigma^2}{2}) u\psi'_\mu(u) + \frac{\sigma^2}{2} u^2 \left((\psi'_\mu(u))^2 - \psi''_\mu(u) \right) \\ &+ \int_{\mathbb{R}_-} \left(e^{\psi_\mu(u) - \psi_\mu(ue^{-y})} - 1 \right) \lambda\nu(dy), \quad u \geq 0, \end{aligned}$$

where $\gamma_\lambda := \gamma - \lambda \int_{[-1,0)} x\nu(dx)$, then $g_1(u)$ is a BF by assumption. For any $\lambda < 1$ we observe that for $u > 0$

$$\begin{aligned} g_\lambda(u) =& g_1(u) \\ &+ (1-\lambda) \int_{[-1,0)} x\nu(dx) u\psi'_\mu(u) + (1-\lambda) \int_{\mathbb{R}_-} \left(1 - e^{\psi_\mu(u) - \psi_\mu(ue^{-y})} \right) \nu(dy). \end{aligned}$$

Since ξ is spectrally negative, μ is selfdecomposable and thus $\psi_\mu(ue^{-y}) - \psi_\mu(u)$ is a BF for any negative y by Proposition 3.1 (it is the Laplace exponent of the e^{-y}-factor of μ). Hence $e^{\psi_\mu(u) - \psi_\mu(ue^{-y})}$ is CM and we can write

$$e^{\psi_\mu(u) - \psi_\mu(ue^{-y})} = \int_{(0,\infty)} e^{-ut} \mu_{e^{-y}}(dt).$$

Thus for $u > 0$

$$\begin{aligned} g_\lambda(u) = g_1(u) &+ (1-\lambda) \int_{[-1,0)} x\nu(dx) u\psi'_\mu(u) \\ &+ (1-\lambda) \int_{\mathbb{R}_-} \int_{(0,\infty)} \mu_{e^{-y}}(dt) \nu(dy) \left(1 - e^{-ut} \right). \end{aligned}$$

Since $u\psi'_\mu(u)$ is a BF by Proposition 3.1 and since all appearing integrals exist, we conclude that $g_\lambda(u) + (\gamma' - \gamma) u\psi'_\mu(u)$ is again a BF. Hence $\mu \in R^+(\gamma', \sigma^2, \lambda\nu)$ which proves (4.2). □

Proposition 4.2. *Let $(\xi_t)_{t\geq 0}$ be a subordinator with drift $a > 0$ and jump measure ν and set $R^+(a, \nu) := R^+_\xi$. Then for $a' > a$ we have*

$$\mathrm{L} \cap R^+(a, \nu) \subseteq \mathrm{L} \cap R^+(a', \nu),$$

but

$$R^+(a, \nu) \setminus R^+(a', \nu) \neq \emptyset.$$

Proof: The first statement has been shown in Theorem 4.1.
Let $\mu := \Phi_{\xi^{(a)}}(\delta_1)$ be the law of $\int_{(0,\infty)} e^{-\xi_t^{(a)}} dt$, then $\mu \in R^+(a,\nu)$ with $\operatorname{supp}\mu = [0, \frac{1}{a}]$ in case of a non-deterministic ξ and $\operatorname{supp}\mu = \{\frac{1}{a}\}$ if ξ is deterministic (cf. Behme et al. (2014+, Lemma 2.1)).
On the other hand by Behme et al. (2014+, Lemma 2.1 and Thm. 2.2) all distributions in $R^+(a',\nu)$ have support $[0,\infty)$, $[0, \frac{1}{a'}]$ (ξ non-deterministic) or $\{\frac{1}{a'}\}$ (ξ deterministic). Hence $\mu \notin R^+(a',\nu)$. □

In case of varying jump heights, nested ranges cannot be expected. To illustrate this, we consider the case of Poisson processes with varying jump height in which we can fully describe the range as we shall do in the following proposition, which also improves the previous result Behme and Lindner (2013+, Prop. 6.3).

Proposition 4.3. *Assume that $\xi_t = cN_t$ for a Poisson process $N = (N_t)_{t\geq 0}$ with intensity λ and some $c > 0$. Then*

$$\begin{aligned} R_\xi^+ &= \{\mu \in \mathrm{L}_{e^{-c}} \text{ with compound exponentially distributed } e^{-c}\text{-factor}\} \quad (4.3)\\ &= \left\{\mu \in \mathcal{P}^+,\ s.t.\ \psi_\mu(u) = \lim_{n\to\infty} \log\left(\frac{\prod_{k=0}^{n-1}(f(e^{-kc}u)+\lambda)}{\lambda^n}\right) \text{ for some BF } f\right\}. \end{aligned}$$

Proof: In the present case (1.3) reduces to

$$\psi_\eta(u) = \lambda e^{\psi_\mu(u)-\psi_\mu(ue^{-c})} - \lambda, \quad u > 0. \quad (4.4)$$

Set $\tilde{c} = e^{-c}$, then this is equivalent to

$$\psi_{\mu_{\tilde{c}}}(u) = \psi_\mu(u) - \psi_\mu(u\tilde{c}) = \log\left(\frac{\psi_\eta(u)+\lambda}{\lambda}\right),$$

i.e. $\psi_{\mu_{\tilde{c}}}(u)$ is the Laplace exponent of a compound exponential distribution - the distribution of η_T for some exponential random variable T, independent of η - and hence it is the Laplace exponent of an infinitely divisible distribution (cf. Steutel and van Harn (2004, Chapter 3, Thm. 3.6)), i.e. a BF. This proves the first equality in (4.3).
By iterating and taking limits we further obtain

$$\begin{aligned} \psi_\mu(u) &= \lim_{n\to\infty} \psi_{\mu_{\tilde{c}^n}}(u)\\ &= \lim_{n\to\infty}\sum_{k=0}^{n-1}\log\left(\frac{\psi_\eta(\tilde{c}^k u)+\lambda}{\lambda}\right) = \lim_{n\to\infty}\log\left(\frac{\prod_{k=0}^{n-1}(\psi_\eta(\tilde{c}^k u)+\lambda)}{\lambda^n}\right) \end{aligned}$$

which proves the second equality in (4.3). □

Remark 4.4. (1) Although for $n \in \mathbb{N}$ we have $\mathrm{L}^+(e^{-c}) \subseteq \mathrm{L}^+(e^{-nc})$, the ranges $R^+_{\xi^{(n)}}$ for $\xi_t^{(n)} = ncN_t$ with $(N_t)_{t\in\mathbb{N}}$ being a Poisson process are in general not nested. In fact, assume that $\mu \in R^+_{\xi^{(1)}} \subset \mathrm{L}^+(e^{-c}) \subseteq \mathrm{L}^+(e^{-nc})$ is given. Then it can be seen from (3.8) that the e^{-nc}-factor of μ has the same distribution as an independent sum of (scaled) compound exponentially distributed random variables. Such sums are in general not compound exponentially distributed. A counterexample can be constructed using the Gamma(k,θ) distribution with Laplace transform $\mathbb{L}(u) = (\frac{\theta}{\theta+u})^k$, which is a compound exponential distribution if and only if $k \leq 1$ (cf. Steutel and van Harn (2004, Chapter III, Ex. 5.4)). The convolution of a Gamma(k,θ)

distribution and a scaled Gamma(k,θ) distribution with Laplace transform $\mathbb{L}(e^{-c}u) = (\frac{\theta}{\theta+e^{-c}u})^k$ is no compound exponential distribution. This can be seen by applying Steutel and van Harn (2004, Chapter III, Thm. 5.1) and using simple algebra to observe that $\frac{d}{du}(\mathbb{L}(u)\mathbb{L}(e^{-c}u))^{-1}$ is not CM.

(2) Since BFs grow at most linearly (cf. Schilling et al. (2012, Cor. 3.8 (viii))), the above proposition implies that in the given setting $\psi_\mu(u) = o(u^\alpha)$ for any $\alpha > 0$. Hence ψ_μ has zero drift and also no polynomial part (in particular μ can not be stable).

5. Selfdecomposable distributions in the range

In this section, we derive a general criterion for a probability distribution to be in R_ξ^+ for a spectrally negative Lévy process ξ. Recall that in this case $R_\xi^+ \subseteq \mathrm{L}^+$.

Theorem 5.1. *Let $\mu \in \mathrm{L}^+$. Assume that $\xi = (\xi_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma_\xi, \sigma_\xi^2, \nu_\xi)$ such that $\nu_\xi((0,\infty)) = 0$, $\int_{[-1,0)} |x|\nu_\xi(dx) < \infty$ and $\lim_{t\to\infty} \xi_t = \infty$.*
Set $\gamma_0 := \gamma_\xi - \int_{[-1,0)} x\nu_\xi(dx) > 0$, let ν_X be the Lévy measure of the Lévy process X which is related to μ via (3.5) and let μ_c, $c > 1$, be the c-factor distribution of μ as defined in Definition 3.3.

(1) *If $\sigma_\xi^2 = 0$, then $\mu \in R_\xi^+$ if and only if*

$$G_1 : (0,\infty) \to [0,\infty) \qquad (5.1)$$
$$t \mapsto \gamma_0 \nu_X((0,t)) - \int_{\mathbb{R}_-} \mu_{e^{-x}}((0,t))\nu_\xi(dx)$$

is non-decreasing. In this case $\mu = \mathcal{L}(\int_0^\infty e^{-\xi_{t-}} d\eta_t)$, where η is a subordinator, independent of ξ, with Lévy measure $\nu_\eta(dt) = dG(t)$ and drift $a_\eta = \gamma_0 a \geq 0$ where $a \geq 0$ denotes the drift of μ.

(2) *If $\sigma_\xi^2 > 0$, assume that $\nu_\xi(\mathbb{R}_-) < \infty$ and $\nu_X(\mathbb{R}_+) < \infty$. Then $\mu \in R_\xi^+$ if and only if μ has zero drift and ν_X has a density $g(t)$, $t > 0$, such that*

$$\lim_{t\to\infty} tg(t) = \lim_{t\to 0} tg(t) = 0, \qquad (5.2)$$

and such that

$$G_2 : (0,\infty) \to [0,\infty) \qquad (5.3)$$
$$t \mapsto (\gamma_0 + \sigma_\xi^2 \nu_X(\mathbb{R}_+)) \int_0^t g(u)du + \frac{\sigma_\xi^2}{2} tg(t)$$
$$- \frac{\sigma_\xi^2}{2} \int_0^t (g * g)(u)du - \int_{\mathbb{R}_-} \mu_{e^{-y}}((0,t))\nu_\xi(dy)$$

is non-decreasing. In this case $\mu = \mathcal{L}(\int_0^\infty e^{-\xi_{t-}} d\eta_t)$, where η is a subordinator, independent of ξ, with Lévy measure $\nu_\eta(dt) = dG(t)$ and zero drift.

Proof: Observe that $\gamma_0 > 0$, since $\mathbb{E}[\xi_1] > 0$ where

$$\mathbb{E}[\xi_1] = \gamma_\xi + \int_{(-\infty,-1)} x\nu_\xi(dx) = \gamma_0 + \int_{\mathbb{R}_-} x\nu_\xi(dx) = \gamma_0 - \int_{\mathbb{R}_-} |x|\nu_\xi(dx).$$

By Behme et al. (2014+, Thm. 3) a probability distribution $\mu \in \mathcal{P}^+$ is in R_ξ^+ for the given ξ if and only if

$$f(u) := \left(\gamma_\xi - \frac{\sigma_\xi^2}{2}\right) u\psi_\mu'(u) + \frac{\sigma_\xi^2}{2} u^2 \left((\psi_\mu'(u))^2 - \psi_\mu''(u)\right) + \int_{\mathbb{R}_-} \left(e^{-(\psi_\mu(ue^{-y}) - \psi_\mu(u))} - 1 - u\psi_\mu'(u) y \mathbb{1}_{|y|\le 1}\right) \nu_\xi(dy)$$

defines a BF. Since $\mu \in \mathrm{L}^+$, the functions $\psi_X(u) = u\psi_\mu'(u)$ and $-\psi_{\mu_c}(u) = \psi_\mu(cu) - \psi_\mu(u)$, $c > 1$, are again BFs by Proposition 3.1 and

$$f(u) = \gamma_0 \psi_X(u) + \frac{\sigma_\xi^2}{2} \left((\psi_X(u))^2 - u\psi_X'(u)\right) + \int_{\mathbb{R}_-} \left(\exp(\psi_{\mu_{e^{-y}}}(u)) - 1\right) \nu_\xi(dy).$$

As μ_c is the c-factor of μ we have $e^{\psi_{\mu_c}(u)} = \int_{[0,\infty)} e^{-ut} \mu_c(dt)$, and therefore

$$f(u) = \gamma_0 \psi_X(u) + \frac{\sigma_\xi^2}{2} \left((\psi_X(u))^2 - u\psi_X'(u)\right) + \int_{(0,\infty)} (e^{-ut} - 1) \int_{\mathbb{R}_-} \mu_{e^{-y}}(dt) \nu_\xi(dy). \tag{5.4}$$

Now assume that $\sigma_\xi^2 = 0$ and let $a \ge 0$ denote the drift of μ, then it follows via Behme et al. (2014+, Lemma 1 and Thm. 1) that X has drift a such that

$$\psi_X(u) = au + \int_{(0,\infty)} \left(1 - e^{-uy}\right) \nu_X(dy),$$

and inserting this in (5.4) we obtain

$$f(u) = \gamma_0 a u + \int_{(0,\infty)} (1 - e^{-ut}) [\gamma_0 \nu_X(dt) - \int_{\mathbb{R}_-} \mu_{e^{-y}}(dt) \nu_\xi(dy)].$$

For f to be a BF it is now necessary and sufficient that ν_η defined via

$$\nu_\eta(dt) := \gamma_0 \nu_X(dt) - \int_{\mathbb{R}_-} \mu_{e^{-y}}(dt) \nu_\xi(dy)$$

is a Lévy measure, which holds if and only if G_1 is non-decreasing.
In the case that $\sigma_\xi^2 > 0$ first observe that from Behme et al. (2014+, Lemma 1 and Thm. 1) we know that $\mathrm{supp}\mu = [0, \infty)$ which implies that μ has drift 0 and so does X. Further under the assumption that $\nu_X(\mathbb{R}_+) < \infty$ we obtain as in the proof of Behme et al. (2014+, Thm. 7) that

$$(\psi_X(u))^2 = \int_{(0,\infty)} (1 - e^{-ut}) [2\nu_X(\mathbb{R}_+)\nu_X - \nu_X * \nu_X](dt). \tag{5.5}$$

Now suppose $\mu \in R_\xi^+$, then f is a BF, i.e. $f(u) = bu + \int_{(0,\infty)} (1 - e^{-ut}) \nu(dt)$, and we obtain from (5.4)

$$\frac{\sigma_\xi^2}{2} u\psi_X'(u) = -bu + \int_{(0,\infty)} (1 - e^{-ut}) \rho_1(dt) - \int_{(0,\infty)} (1 - e^{-ut}) \rho_2(dt)$$

where

$$\rho_1(dt) := (\gamma_0 + \sigma_\xi^2 \nu_X(\mathbb{R}_+))\nu_X(dt) + \int_{\mathbb{R}_-} \mu_{e^{-y}}(dt)\nu_\xi(dy)$$

$$\rho_2(dt) := \nu(dt) + \frac{\sigma_\xi^2}{2}\nu_X * \nu_X(dt)$$

Proceeding as in the proof of Behme et al. (2014+, Thm. 7(i)) this shows $b = 0$ and that ν_X has the density

$$g(t) = \frac{2}{\sigma_\xi^2 t}(\rho_1(t,\infty) - \rho_2(t,\infty)), \quad t > 0.$$

Since $\nu_\xi(\mathbb{R}_-) < \infty$ and $\nu_X(\mathbb{R}_+) < \infty$, similarly to the argumentation in Behme et al. (2014+, Thm. 7(i)), it follows that (5.2) holds and finally that

$$\nu(dt) = (\gamma_0 + \sigma_\xi^2 \nu_X(\mathbb{R}_+))g(t)dt + \frac{\sigma_\xi^2}{2}d(tg(t)) - \frac{\sigma_\xi^2}{2}(g*g)(t)dt - \int_{\mathbb{R}_-} \mu_{e^{-y}}(dt)\nu_\xi(dy).$$

Thus, if $\mu \in R_\xi^+$, then $\nu(dt)$ has to be a Lévy measure, which proves that G_2 is non-decreasing. Conversely, if G_2 is non-decreasing, define a subordinator η with Lévy measure $\nu(dt) = dG(t)$ and zero drift, then reverting the above, it follows from Behme et al. (2014+, Thm. 3) that $\mu \in R_\xi^+$. □

Example 5.2. Consider the COGARCH volatility process as introduced in Example 2.3. In this case the process ξ has no gaussian part, Lévy measure $\nu_\xi = T(\nu_S)$ for the transformation $T : s \mapsto -\log(1 + \varphi s)$ and $\gamma_0 = \eta > 0$.
Since the integrating process in the case of the COGARCH is deterministic $t \mapsto \beta t$, its Lévy measure is zero and we conclude from Theorem 5.1(i) above that the measure $\mu \in \mathrm{L}^+$, which is the stationary distribution of the COGARCH volatility, has to have drift $a = \frac{\beta}{\eta}$ and that it has to fulfill

$$\eta\nu_X(dt) = \int_{\mathbb{R}_-} \mu_{e^{-x}}(dt)\nu_\xi(dx) = \int_{\mathbb{R}_+} \mu_{1+\varphi s}(dt)\nu_S(ds), \tag{5.6}$$

where X is connected to μ via (3.4).
Observe that it follows directly from this, that

$$k(0+) = \nu_X(\mathbb{R}_+) = \eta^{-1}\nu_S(\mathbb{R}_+),$$

where $k(t), t > 0$, is the factor of the Lévy density of μ as in (3.3).
Assuming e.g. that $(S_t)_{t\geq 0}$ is a Poisson process with intensity $c > 0$, we further obtain from (5.6) that

$$\eta\nu_X(dt) = c\mu_{1+\varphi}(dt),$$

where $\mu_{1+\varphi}$ has the Laplace exponent $\psi_\mu((1+\varphi)u) - \psi_\mu(u)$. Hence in this case, with (3.3) and (3.4) one can deduce the following equation for the Lévy density $m(t) = k(t)/t$, $t > 0$, of μ,

$$\frac{\eta}{c}\int_{(0,\infty)} e^{-ut} t dm(t) + \frac{\eta}{c}\int_{(0,\infty)} e^{-ut} m(t)dt$$
$$= -\exp\left(-\frac{\beta}{\eta}\varphi u - \int_{(0,\infty)} (1 - e^{-\varphi ut})e^{-ut} m(t)dt\right).$$

Example 5.3. Assume μ is positive strictly stable with index $\alpha \in (0,1)$, i.e. $\psi_\mu(u) = cu^\alpha$, for some $c > 0$ and let $(\xi_t)_{t\geq 0}$ be a Lévy process without gaussian part and which fulfills the assumptions of Theorem 5.1. Then $\mu \in R_\xi^+$ if and only if

$$\nu(dt) = \gamma_0 \frac{c\alpha^2}{\Gamma(1-\alpha)} t^{-(1+\alpha)} dt - \int_{(0,\infty)} \mu_{e^{-x}}(dt)\nu_\xi(dx)$$

defines a Lévy measure. In particular observe that $\mu_{e^{-x}}$ has Laplace exponent $cu^\alpha(e^{-\alpha x} - 1)$ and hence $\nu_Y(dt) := \int_{(0,\infty)} \mu_{e^{-x}}(dt)\nu_\xi(dx)$ can be interpreted as the Lévy measure of $(Y_t)_{t\geq 0}$ where $Y_t = S_{\tilde{\xi}_t}$, with $S = (S_t)_{t\geq 0}$ a strictly α-stable subordinator with $\psi_S(u) = u^\alpha$ and $\tilde{\xi}$ a pure-jump subordinator with Lévy measure $\nu_{\tilde{\xi}} = T(\nu_\xi)$ for the transformation $T : x \mapsto c(e^{-\alpha x} - 1)$ (see e.g. Sato (1999, Thm. 30.1)).

6. GGCs in the range

There exist several examples of exponential functionals whose distributions are generalized Gamma convolutions. Just recall Proposition 3.5 or the example mentioned in the introduction, which states that $\int_{(0,\infty)} e^{-(\sigma B_t + at)} dt$ has an inverse Gamma distribution which is a GGC, where $(B_t)_{t\geq 0}$ is a Brownian motion and $\sigma, a > 0$. Further explicit examples of exponential functionals whose distributions are generalized Gamma convolutions can also be found in Behme et al. (2012) and Behme and Bondesson (2015+).

As generalized Gamma convolutions are selfdecomposable, one can also directly transfer the results from the last section to obtain conditions on GGCs to be in the range R_ξ for a given process ξ. Together with the results in Section 3 this then yields the following example.

Example 6.1. Let $\mu \in \mathrm{T}$ have the Laplace exponent (3.9) with $a \geq 0$ and $k(0+) < \infty$, $k'(0+) > -\infty$ and $k(t) \not\equiv 0$. Then by Corollary 3.8 the Lévy measure $\nu_X(dt)$ of the Lévy process X which is related to μ via (3.5) has a density $m(t), t \geq 0$, which is CM, that is $\nu_X((0,t)) = \int_{(0,t)} m(s)ds$.

Assume that $\xi = (\xi_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma_\xi, 0, \nu_\xi)$ such that $\nu_\xi((0,\infty)) = 0$, $\int_{[-1,0)} |x|\nu_\xi(dx) < \infty$, $\nu_\xi \not\equiv 0$ and $\lim_{t\to\infty} \xi_t = \infty$.

Set $\gamma_0 := \gamma_\xi - \int_{[-1,0)} x\nu_\xi(dx) > 0$ and let μ_c, $c > 1$, be the c-factor distribution of μ as defined in Definition 3.3, then by Theorem 5.1 we have $\mu \in R_\xi^+$ if and only if

$$G_1 : (0,\infty) \to [0,\infty)$$
$$t \mapsto \gamma_0 \int_{(0,t)} m(s)ds - \int_{\mathbb{R}_-} \mu_{e^{-x}}((0,t))\nu_\xi(dx)$$

is non-decreasing.

By Proposition 3.7 the c-factor distributions of μ are in BO. Further, for $c > 1$, they have drift $a_c := a(c-1)$ and their CM Lévy densities are given by

$$g_c(t) = \frac{k(c^{-1}t) - k(t)}{t} = t^{-1}\nu_X((c^{-1}t, t]) = t^{-1} \int_{(c^{-1}t,t]} m(s)ds, \quad t > 0,$$

(compare the proof of Proposition 3.7) where the second equality follows from (3.6). Further, by l'Hospital's rule $g_c(0+) < \infty$, since $k(0+) < \infty$ and $|k'(0+)| < \infty$. Therefore the Lévy densities g_c are integrable, which implies that the μ_c are

compound Poisson distributed, as it would have followed similarly from Bondesson (1981, Thm. 6.1). Hence $\mu_c = \mathcal{L}(a_c + \sum_{i=1}^N Y_i^c)$, where $N \sim \text{Poisson}(\lambda_c)$ and where the random variables Y_i^c are i.i.d. with densities $\lambda_c^{-1} g_c(t)$, $t > 0$, with $\lambda_c^{-1} := \int_{(0,\infty)} g_c(t)dt$.
Therefore μ_c has the density

$$e^{-\lambda_c} \sum_{n=1}^{\infty} \frac{\lambda_c^n}{n!} (\lambda_c^{-1} g_c(t-a_c))^{*n} = e^{-\lambda_c} \sum_{n=1}^{\infty} \frac{(g_c(t-a_c))^{*n}}{n!}, \quad t > a_c,$$

and an atom of mass $e^{-\lambda_c}$ in a_c. This yields that $a = 0$ is necessary for μ to be in the range, because otherwise G_1 has negative jumps.
Now for $a = 0$ the term $\int_{\mathbb{R}_-} \mu_{e^{-x}}((0,t))\nu_\xi(dx)$ is differentiable and the function $G_1(t)$, $t > 0$, as above, is non-decreasing if and only if for all $t > 0$

$$\frac{dG_1(t)}{dt} = \gamma_0 m(t) - \int_{\mathbb{R}_-} \exp(-\lambda_{e^{-x}}) \sum_{n=1}^{\infty} \frac{(g_{e^{-x}}(t))^{*n}}{n!} \nu_\xi(dx) \geq 0.$$

For example, assume that μ is a Gamma(k,θ) distribution. Then it has zero drift and its Lévy density is given by $kt^{-1}e^{-\theta t}$ (cf. Sato (1999, Ex. 8.10)) such that it fulfills the above assumptions. Further we deduce $m(t) = k\theta e^{-\theta t}$,

$$g_c(t) = k \cdot \frac{e^{-c^{-1}\theta t} - e^{-\theta t}}{t}, \quad \text{and} \quad \lambda_c = k \log c.$$

Thus

$$\begin{aligned}
\frac{dG_1(t)}{dt} &= \gamma_0 m(t) - \int_{\mathbb{R}_-} e^{kx} \sum_{n=1}^{\infty} \frac{(g_{e^{-x}}(t))^{*n}}{n!} \nu_\xi(dx) \\
&\leq \gamma_0 k\theta e^{-\theta t} - \int_{\mathbb{R}_-} e^{kx} g_{e^{-x}}(t) \nu_\xi(dx) \\
&= ke^{-\theta t} \left(\gamma_0 \theta - \int_{\mathbb{R}_-} e^{kx} \cdot \frac{e^{\theta t(1-e^x)} - 1}{t} \nu_\xi(dx) \right),
\end{aligned}$$

which becomes negative for large t, since $\nu_\xi \not\equiv 0$. Therefore in this case we have shown Gamma$(k,\theta) \notin R_\xi^+$.

Even in the case that ξ has no jumps but a gaussian part, many GGCs can not be in the range as shown in the following.

Proposition 6.2. *Let $\xi_t = \sigma B_t + at$, $a, \sigma > 0$, and let $\mu \in \mathrm{T}$ have the Laplace exponent (3.9) with $k(0+) < \infty$ and $k(t) \not\equiv 0$. Then $\mu \notin R_\xi^+$.*

Proof: Let $\mu \in \mathrm{T}$ with $k(0+) < \infty$ be given and define the subordinator X via (3.4) or (3.5). Then from Proposition 3.5 we know that $\mathcal{L}(X) \in \mathrm{BO}$ with finite Stieltjes measure and as such it has a Laplace exponent of the form

$$\psi_X(u) = bu + \int_0^\infty (1 - e^{-ut}) m(t) dt$$

where $m(t)$ is CM and integrable. From Behme et al. (2014+, Thm. 7) we know that if $\mu \in R_\xi^+$, then necessarily $b = 0$. Further from Behme et al. (2014+, Remark 7(ii)) it follows that if $\mu \in R_\xi^+$, then

$$\left(a + \sigma^2 \int_0^\infty m(t)dt + \frac{\sigma^2}{2} \right) m(t) + \frac{\sigma^2}{2} tm'(t) - \frac{\sigma^2}{2} (m * m)(t) \geq 0, \quad \forall t > 0.$$

Since $m(t)$ is CM, it holds

$$m(t)=\int_{[0,\infty)} e^{-\lambda t}d\rho(\lambda)$$

for some measure ρ with $\rho(\{0\})=\lim_{t\to\infty} m(t)=0$. Hence

$$m'(t)=-\int_{(0,\infty)}\lambda e^{-\lambda t}d\rho(\lambda),\quad \int_{(0,\infty)} m(t)dt=\int_{(0,\infty)}\lambda^{-1}d\rho(\lambda)<\infty,$$

and

$$(m*m)(t)=\int_0^t m(t-s)m(s)ds=\int_{(0,\infty)}\int_{(0,\infty)}\frac{e^{-\zeta t}-e^{-\lambda t}}{\lambda-\zeta}d\rho(\zeta)d\rho(\lambda).$$

So for $\mu\in R_\xi^+$ it is necessary that for all $t>0$

$$\left(a+\sigma^2\int_{(0,\infty)}\lambda^{-1}d\rho(\lambda)+\frac{\sigma^2}{2}\right)\int_{(0,\infty)}e^{-\lambda t}d\rho(\lambda)-\frac{\sigma^2}{2}t\int_{(0,\infty)}\lambda e^{-\lambda t}d\rho(\lambda)$$
$$-\frac{\sigma^2}{2}\int_{(0,\infty)}\int_{(0,\infty)}\frac{e^{-\zeta t}-e^{-\lambda t}}{\lambda-\zeta}d\rho(\zeta)d\rho(\lambda)\geq 0,$$

or equivalently for all $t>0$

$$\frac{1}{t}\int_{(0,\infty)}\left(a+\sigma^2\int_{(0,\infty)}u^{-1}d\rho(u)+\frac{\sigma^2}{2}\right)e^{-\lambda t}d\rho(\lambda)-\int_{(0,\infty)}\frac{\sigma^2}{2}\lambda e^{-\lambda t}d\rho(\lambda)\tag{6.1}$$
$$\geq\frac{1}{t}\int_{(0,\infty)}\int_{(0,\infty)}\frac{\sigma^2}{2}\frac{e^{-\zeta t}-e^{-\lambda t}}{\lambda-\zeta}d\rho(\zeta)d\rho(\lambda).$$

The term on the RHS of (6.1) is non-negative, for the left hand side we observe that by dominated convergence

$$\lim_{t\to\infty}\int_{(0,\infty)}\left(\frac{a+\sigma^2\int_{(0,\infty)}u^{-1}d\rho(u)+\frac{\sigma^2}{2}}{t}-\frac{\sigma^2}{2}\lambda\right)e^{-\lambda t}d\rho(\lambda)$$
$$=\int_{(0,\infty)}\lim_{t\to\infty}\left(\frac{a+\sigma^2\int_{(0,\infty)}u^{-1}d\rho(u)+\frac{\sigma^2}{2}}{t}-\frac{\sigma^2}{2}\lambda\right)e^{-\lambda t}d\rho(\lambda)$$
$$<0$$

in contradiction to (6.1). This proves the proposition. □

7. Proof of Proposition 3.7

For the proof of Proposition 3.7 we need the following two simple lemmata.

Lemma 7.1. *Let $\lambda>0$ be constant, then*

$$f(x)=\frac{1-e^{-\lambda x}}{x},\quad x>0,$$

is completely monotone.

Proof: Obviously f is infinitely often continuously differentiable and it holds $f(x) > 0,\ x > 0$. Further it can be shown by an elementary induction, that the n-th derivative of f is given by

$$f^{(n)}(x) = (-1)^n n! e^{-\lambda x} x^{-(n+1)} \left(e^{\lambda x} - \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} \right). \tag{7.1}$$

It follows from the series representation of the exponential function, that the term in the brackets in (7.1) is positive. Hence $(-1)^n f^{(n)}(x) \geq 0,\ x > 0$, for all n as we had to show. □

Lemma 7.2. *Let $k(x),\ x > 0$, be completely monotone and let $c > 1$ be some constant. Then*

$$f(x) = \frac{k(x) - k(cx)}{x}$$

is completely monotone.

Proof: Assume first that $k(x) = e^{-\lambda x}$ for some $\lambda > 0$. Then

$$f(x) = \frac{e^{-\lambda x} - e^{-\lambda x c}}{x} = e^{-\lambda x} \frac{1 - e^{-\lambda x(c-1)}}{x}$$

is CM since $e^{-\lambda x}$ and $x^{-1}(1 - e^{-\lambda x(c-1)})$ are CM by Lemma 7.1 and since products of CM functions are again CM (cf. Schilling et al. (2012, Cor. 1.6)).
Now let k be an arbitrary CM function, i.e.

$$k(x) = \int_{[0,\infty)} e^{-\lambda x} \rho(d\lambda).$$

Then

$$f(x) = \frac{k(x) - k(cx)}{x} = \int_{[0,\infty)} \frac{e^{-\lambda x} - e^{-\lambda cx}}{x} \rho(d\lambda) = \int_{(0,\infty)} \frac{e^{-\lambda x} - e^{-\lambda cx}}{x} \rho(d\lambda)$$

is an integral mixture of CM functions and hence CM. □

Now we can state the proof of Proposition 3.7.

Proof of Proposition 3.7: Assume $\mu \in \mathrm{T}$, then its Laplace exponent is given by

$$\psi_\mu(u) = au + \int_0^\infty (1 - e^{-ut}) \frac{k(t)}{t} dt, \quad u \geq 0,$$

for some $a \geq 0$ and a CM function k. Hence the Laplace exponent of its c-factor $\mu_c,\ c \in (0,1)$, is by (3.2)

$$\psi_{\mu_c}(u) = \psi_\mu(u) - \psi_\mu(cu) = a(1-c)u + \int_0^\infty (1 - e^{-ut}) \frac{k(t) - k(c^{-1}t)}{t} dt$$

and μ_c is in Bondesson's class if and only if

$$f(t) = \frac{k(t) - k(c^{-1}t)}{t}$$

is CM. This holds by Lemma 7.2.
Analogous calculations show that also $\mu_c,\ c > 1$, is in Bondesson's class.

For the converse assume $\mu \in \mathrm{L}^+$ with $\mu_c \in \mathrm{BO}$ for all $c \in (0,1)$, i.e. $\psi_{\mu_c}(u) = \psi_\mu(u) - \psi_\mu(cu)$ is a CBF for all $c \in (0,1)$. This implies that

$$\begin{aligned}\psi_X(u) &:= u\psi_\mu'(u) = u \lim_{c\to 1} \frac{\psi_\mu(u) - \psi_\mu(u-(1-c)u)}{u(1-c)} \\ &= \lim_{c\to 1} \frac{\psi_\mu(u) - \psi_\mu(u-(1-c)u)}{(1-c)}\end{aligned}$$

is the limit of CBFs and hence a CBF (Schilling et al. (2012, Cor. 7.6)). Similarly, if $\mu_c \in \mathrm{BO}$ for all $c > 1$ one obtains $\psi_X(u)$ as limit of CBFs for $c \searrow 1$.
Now let $(X_t)_{t\geq 0}$ be the subordinator with Laplace exponent ψ_X, then by Behme et al. (2014+, Thm. 4 (ii)) (setting $\sigma = 0$) this is equivalent to $\mu = \Phi_\xi(\mathcal{L}(X_1))$ for $\xi_t = t$. Hence by Proposition 3.5 μ is in T. □

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