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On jump-diffusion processes with regime switching: martingale approach

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Abstract. We study jump-diffusion processes with parameters switching at random times. Being motivated by possible applications, we characterise equivalent martingale measures for these processes by means of the relative entropy. The minimal entropy approach is also developed. It is shown that in contrast to the case of Lévy processes, for this model an Esscher transformation does not produce the minimal relative entropy.

1. Introduction

We investigate some basic properties of the jump-diffusion processes

$$X(t) = T^c(t) + N^h(t) + W^\sigma(t), \quad t \geq 0,$$

with time-dependent deterministic driving parameters switching simultaneously at random times. Here W^σ denotes the Wiener part, defined by the stochastic integral (w.r.t. Brownian motion B) of the process which is formed by switching at random times of the deterministic diffusion coefficients $\sigma_i = \sigma_i(t)$, $i \in D := \{1, \ldots, d\}$, $d \geq 2$; by N^h is denoted the jump part, i.e. the stochastic integral (w.r.t. process $N = N(t)$ counting number of the regime switchings) applied to the switching functions $h_i = h_i(t)$, $i \in D$, and T^c is path-by-path integral in time t of switching velocity regimes $c_i = c_i(t)$, $i \in D$ (see the detailed definitions in Section 2). In the case of $d = 2$ and exponentially distributed inter-switching time intervals such

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processes are called telegraph-jump-diffusion processes, Ratanov (2010), or Markov modulated jump-diffusion.

In this paper the random inter-switching time intervals are assumed to be independent and arbitrary distributed. In general, such a process is not Markovian, and it is not a Lévy process as well. We study these processes from the martingale point of view, including Girsanov's measure transform.

Similar models without a diffusion component were considered first in Mel′nikov and Ratanov (2007) and more recently have been analysed in detail by Di Crescenzo and Martinucci (2013); Di Crescenzo et al. (2013), Ratanov (2015b, 2013, 2014b), see also the particular cases in Di Crescenzo (2001); Di Crescenzo and Martinucci (2010). The model with missing jump component is presented in Di Crescenzo et al. (2014); Di Crescenzo and Zacks (2015). The processes with random driving parameters are studied in Ratanov (2013). The recent paper Ratanov (2014a) is related to the model of random switching intensities.

This setting is widely used for applications, see e.g. Weiss (1994). The martingale approach developed in this paper is motivated by financial modelling, see Runggaldier (2003) for jump-diffusion model. See also Ratanov (2007) for jump-telegraph model (and a more detailed presentation in Kolesnik and Ratanov (2013)).

The Markov modulated jump-diffusion model of asset pricing (with additive jumps superimposed on the diffusion) has been studied before, see Guo (2001); Ratanov (2010). This model for a single risky asset possesses infinitely many martingale measures, and thus the market is incomplete. The model can be completed by adding a further asset; for the jump-diffusion model see Runggaldier (2003), and for the telegraph-jump-diffusion model see Ratanov (2010).

In this paper we explore another approach. We describe the set of equivalent martingale measures and determine the Föllmer-Schweizer minimal probability measure (so called the minimal entropy martingale measure (MEMM)), Föllmer and Schweizer (1991). In his seminal paper Frittelli (2000) Fritelli has showed the equivalence between maximisation of expected exponential utility and the minimisation of the relative entropy. Then, by this approach the models based on Lévy processes have been studied in Fujiwara and Miyahara (2003).

Observe that for Lévy processes and for regime switching diffusions the usual technique is based on an Esscher transform, which produces the MEMM, see Fujiwara and Miyahara (2003); Esche and Schweizer (2005) and Elliott et al. (2005, 2007).

In our model this method does not work. The Esscher transform under regime switching does not affect the switching intensities. In the Lévy model Fujiwara and Miyahara (2003) this contradicts the minimal entropy condition (if the process is not already a martingale), see (5.4). In the case of the regime switching model Elliott et al. (2005) some additional entropy given by the jumps embedded into this model can be reduced by more flexible measure transformation, see Section 5.

Moreover, in contrast with Lévy model and with the regime switching diffusions without jumps our model has the following important feature: the entropy minimum as well as calibration of MEMM depend on the time horizon under consideration.

The paper is organised as follows. Section 2 contains the definition and the main properties of the regime switching jump-diffusion processes with arbitrary distributions of inter-switching time intervals. We construct Girsanov's transformation in Section 3. Then, we define the entropy and derive the corresponding Volterra equations. In Section 4 we describe the set of equivalent martingale measures for the regime switching jump-diffusion processes. The minimal entropy equivalent martingale measures are studied in Section 5 for the case of constant parameters.

Applications to financial modelling have been recently published in Ratanov (2015a).

2. Generalised telegraph-jump-diffusion processes. Distributions and expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the given right-continuous filtration $\mathcal{F}_t,\ t \geq 0$, satisfying the usual hypotheses, Jeanblanc et al. (2009). We start with a d-state $\mathcal{F}_t$-adapted semi-Markov (see Jacobsen (2006)) random process $\varepsilon,\ \varepsilon(t) \in D,\ t \geq 0$. The switchings occur at random times $\tau_n,\ n \geq 0,\ \tau_0 = 0$, and process ε is right-continuous with left-hand limits.

Let $N = N(t)$ be the counting process, $N(t) = \max\{n \mid \tau_n \leq t\},\ t \geq 0$.

2.1. *The definition of telegraph-jump-diffusion process.* For the set of deterministic measurable functions $z_i = z_i(t),\ t \geq 0,\ i \in D$, we construct first the piecewise deterministic random process $z^\dagger$ combining $z_i,\ i \in D$, by means of the switching process ε:

$$z^\dagger(t) = \sum_{n=1}^{\infty} z_{\varepsilon(\tau_{n-1})}(t - \tau_{n-1}) \mathbf{1}_{\Delta_n}(t), \qquad t \geq 0. \tag{2.1}$$

Here $\Delta_n := [\tau_{n-1},\ \tau_n),\ n \geq 1$, and $\mathbf{1}_\Delta(\cdot)$ is the indicator function. Process $z^\dagger$ starts from the origin at the switching time $\tau_0 = 0$; at each further switching time $\tau_n,\ n \geq 1$, the process $z^\dagger$ is renewed.

Second, consider the integrals of $z^\dagger$ of the following three types:

(1) the generalised d-state telegraph process (path-by-path integral)

$$T^z(t) = \int_0^t z^\dagger(u)\mathrm{d}u, \qquad t > 0; \tag{2.2}$$

(2) the pure jump process (integral w.r.t. counting process N)

$$N^z(t) = \int_0^t z^\dagger(u)\mathrm{d}N(u), \qquad t > 0, \tag{2.3}$$

(3) the Wiener process (Itô integral)

$$W^z(t) = \int_0^t z^\dagger(u)\mathrm{d}B(u), \qquad t > 0, \tag{2.4}$$

where $B = B(t),\ t \geq 0$, is an $\mathcal{F}_t$-adapted Brownian motion, independent of ε. Note that $W^z = W^z(t),\ t \geq 0$, is a Gaussian $(\mathcal{F}_t, \mathbb{P})$-martingale.

Let $c_i = c_i(t),\ h_i = h_i(t)$ and $\sigma_i = \sigma_i(t),\ t \geq 0,\ i \in D$, be deterministic measurable functions. We assume that functions c_i are locally integrable, and σ_i are locally square integrable,

$$\int_0^t \sigma_i(u)^2 \mathrm{d}u < \infty, \qquad t > 0,\ i \in D.$$

In this paper we analyse the jump-diffusion process with switching regimes $X = X(t),\ t \geq 0$, of the following form

$$X(t) := T^c(t) + N^h(t) + W^\sigma(t), \qquad t \geq 0, \tag{2.5}$$

with the components $T^c,\ N^h$ and W^σ which are defined by (2.2), (2.3) and (2.4) respectively. Process X satisfies the stochastic equation

$$\mathrm{d}X(t) = c_{\varepsilon(\tau_{N(t)})}(t-\tau_{N(t)})\mathrm{d}t + h_{\varepsilon(\tau_{N(t)})}(T_{N(t)})\mathrm{d}N(t) + \sigma_{\varepsilon(\tau_{N(t)})}(t-\tau_{N(t)})\mathrm{d}B(t),\ t > 0.$$

Equivalently, processes $T^c,\ N^h,\ W^\sigma$ can be expressed by summing up the paths between the consequent switching instants τ_n:

$$T^c(t) = \int_0^t c^\dagger(u)\mathrm{d}u = \sum_{n=1}^{N(t)} l_{\varepsilon(\tau_{n-1})}(T_n) + l_{\varepsilon(\tau_{N(t)})}(t - \tau_{N(t)}), \tag{2.6}$$

$$N^h(t) = \int_0^t h^\dagger(u)\mathrm{d}N(u) = \sum_{n=1}^{N(t)} h_{\varepsilon(\tau_{n-1})}(T_n), \tag{2.7}$$

$$W^\sigma(t) = \int_0^t \sigma^\dagger(u)\mathrm{d}B(u) = \sum_{n=1}^{N(t)} w_{\varepsilon(\tau_{n-1})}(T_n) + w_{\varepsilon(\tau_{N(t)})}(t - \tau_{N(t)}), \tag{2.8}$$

where the following notations are used

$$l_i(t) = \int_0^t c_i(u)\mathrm{d}u, \qquad w_i(t) = \int_0^t \sigma_i(u)\mathrm{d}B(u), \quad t \geq 0,\ i \in D. \tag{2.9}$$

Note that $l_i(t),\ i \in D$, are deterministic and the variables $w_i(t),\ i \in D$, are zero-mean normally distributed with the variances $\Sigma_i(t)^2 := \int_0^t \sigma_i(u)^2\mathrm{d}u,\ t > 0,\ i \in D$.

We say that the regime switching jump-diffusion process X defined by (2.5)-(2.9) is characterised by the triplet $\langle c, h, \sigma\rangle$ with distributions of inter-switching time intervals T_n which are determined by the hazard rate functions $\gamma_{ij} = \gamma_{ij}(t)$, $i, j \in D$, (see the definition in (2.11)).

We apply the notations X_i and T_i^c, N_i^h, W_i^σ, if the initial state $i \in D$ of the underlying process ε is given, $\varepsilon(0) = i$.

Further, we will need the following explicit expression for the stochastic exponential $\mathcal{E}_t(X)$ of $X = T^c + N^h + W^\sigma$. It is known, see e.g. Jeanblanc et al. (2009), that

$$\mathcal{E}_t(X) = \exp\left(T^{c-\sigma^2/2}(t) + W^\sigma(t)\right) \prod_{n=1}^{N_t} (1 + h_{\varepsilon(\tau_n)}(T_n)) = \exp\left(Y(t)\right), \tag{2.10}$$

where $Y(t) = T^{c-\sigma^2/2}(t) + N^{\ln(1+h)}(t) + W^\sigma(t),\ t \geq 0$. Here $T^{c-\sigma^2/2}$ is the telegraph process (2.6) with the velocities $c_i - \sigma_i^2/2$ instead of $c_i,\ i \in D$, and $N^{\ln(1+h)}$ is the pure jump process (2.7) with the jump values $\ln(1 + h_i)$, instead of $h_i,\ i \in D$, switching at random times $\tau_n,\ n \geq 1$.

2.2. *Semi-Markov process* $\varepsilon = \varepsilon(t)$. In order to state the distribution of $X(t)$ we introduce conditions on the driving processes ε and $\{\tau_n\}_{n\geq 1}$.

Denote by $\overline{F}_{ij} = \overline{F}_{ij}(t),\ t > 0,\ i, j \in D$, the transition probabilities of the form Jacobsen (2006, (3.17)):

$$\overline{F}_{ij}(t) = \mathbb{P}\{\tau_1 > t,\ \varepsilon(\tau_1) = j \mid \varepsilon(0) = i\}, \quad t > 0,\ i, j \in D.$$

Let $\overline{F}_{ij}(t) > 0,\ t > 0,\ i, j \in D$. Consider the hazard functions Γ_{ij},

$$\Gamma_{ij}(t) = -\ln \overline{F}_{ij}(t), \quad t > 0,\ i, j \in D.$$

Let functions $\overline{F}_{ij}$ be differentiable, $\frac{\mathrm{d}\overline{F}_{ij}}{\mathrm{d}t}(t) = -f_{ij}(t),\ t > 0,\ i, j \in D$. Thus the hazard functions $\Gamma_{ij} = \Gamma_{ij}(t)$ are expressed by

$$\Gamma_{ij}(t) = \int_0^t \gamma_{ij}(u)\mathrm{d}u, \qquad t \geq 0.$$

Here

$$\gamma_{ij}(u) := \frac{f_{ij}(u)}{\overline{F}_{ij}(u)}, \quad u > 0, \qquad i, j \in D, \tag{2.11}$$

are the hazard rate functions, and $f_{ij},\ i, j \in D$, are the density functions of the interarrival times. We assume the non-exploding condition to be hold:

$$\int_0^\infty \gamma_{ij}(u)\mathrm{d}u = +\infty, \qquad i, j \in D. \tag{2.12}$$

Note that

$$\begin{aligned} \overline{F}_{ij}(t) &= \exp\left(-\int_0^t \gamma_{ij}(u)\mathrm{d}u\right), \\ f_{ij}(t) &= \gamma_{ij}(t)\exp\left(-\int_0^t \gamma_{ij}(u)\mathrm{d}u\right), \end{aligned} \qquad t > 0, \quad i, j \in D.$$

The survival function of the first switching time τ_1 is given by

$$\begin{gathered} \overline{F}_i(t) := \mathbb{P}\{\tau_1 > t \mid \varepsilon(0) = i\} = \prod_{j \in D \setminus \{i\}} \overline{F}_{ij}(t) = \exp\left(-\int_0^t \gamma_i(u)\mathrm{d}u\right), \\ t > 0, \qquad i \in D, \end{gathered} \tag{2.13}$$

where

$$\gamma_i = \sum_{j \in D \setminus \{i\}} \gamma_{ij}, \quad i \in D. \tag{2.14}$$

Furthermore, let

$$f_i(t) = -\frac{\mathrm{d}\overline{F}_i}{\mathrm{d}t}(t) = \gamma_i(t)\exp\left(-\int_0^t \gamma_i(u)\mathrm{d}u\right), \quad t > 0, \qquad i \in D. \tag{2.15}$$

Due to (2.13) N is an inhomogeneous Poisson process with switchings at $\tau_n,\ n \geq 1$, and with the instantaneous intensities $\gamma_{ij}(t),\ t > 0,\ i, j \in D$.

If the process ε is observed beginning from the time $s,\ \tau_0 \leq s < \tau_1$, the corresponding conditional distributions can be described by the following conditional survival functions,

$$\overline{F}_{ij}(t \mid s) = \mathbb{P}(\tau_1 > t,\ \varepsilon(\tau_1) = j \mid \tau_1 > s,\ \varepsilon(0) = i) = \frac{\overline{F}_{ij}(t)}{\overline{F}_i(s)}, \tag{2.16}$$

with the density functions

$$f_{ij}(t \mid s) = -\frac{\partial}{\partial t}\overline{F}_{ij}(t \mid s) = \frac{f_{ij}(t)}{\overline{F}_i(s)}, \quad 0 \leq s < t, \quad i, j \in D.$$

Moreover, let

$$\begin{aligned} \overline{F}_i(t \mid s) &:= \mathbb{P}(\tau_1 > t \mid \tau_1 > s,\ \varepsilon(0) = i) = \exp\left(-\int_s^t \gamma_i(u)\mathrm{d}u\right) \\ f_i(t \mid s) &= \gamma_i(t)\exp\left(-\int_s^t \gamma_i(u)\mathrm{d}u\right), \end{aligned} \qquad i \in D, \tag{2.17}$$

see (2.13).

Notice that

$$\begin{aligned} \overline{F}_{ij}(t \mid 0) \equiv& \overline{F}_{ij}(t), \qquad \overline{F}_i(t \mid 0) \equiv \overline{F}_i(t), \\ f_{ij}(t \mid 0) \equiv& f_{ij}(t), \qquad f_i(t \mid 0) \equiv f_i(t), \end{aligned} \qquad t > 0, \qquad i, j \in D.$$

Assume the inter-switching time intervals $T_n = \Delta\tau_n = \tau_n - \tau_{n-1},\ n \geq 1$, to be independent, and process ε to be renewal in the following sense: $n \geq 1$,

$$\mathbb{P}\{T_n > t \mid T_n > s,\ \varepsilon(\tau_{n-1}) = i\} = \overline{F}_i(t \mid s), \quad t > s \geq 0, \qquad i \in D,$$

see (2.16)-(2.17).

2.3. *The distribution and expectation of $X(t)$.* Our further analysis is based on the following observation.

Let $\tau_1 = \tau_1^{(i)}$ be the first switching time, where $i \in D$ is the fixed initial state of the process ε, $\varepsilon(0) = i$. Owing to the renewal character of the counting process N, see Cox (1962), we have the following equalities in distribution:

$$X_i(t) \stackrel{D}{=} (l_i(t) + w_i(t))\mathbf{1}_{\tau_1 > t} + \left[l_i(\tau_1) + w_i(\tau_1) + h_i(\tau_1) + \tilde{X}_{\varepsilon_i(\tau_1)}(t - \tau_1)\right]\mathbf{1}_{\tau_1 < t}, \\ t > 0,\ i \in D, \tag{2.18}$$

where $\tilde{X}$ is the regime switching jump-diffusion process independent of X which starts at time τ_1. Here $\mathbf{1}_A$ is the indicator of event A.

Let

$$\begin{aligned} p_i(x, t) &:= \mathbb{P}(X_i(t) \in \mathrm{d}x)/\mathrm{d}x, \quad t \geq 0, \\ p_i(x, t \mid s) &:= \mathbb{P}(X_i(t) \in \mathrm{d}x \mid N_i(s) = 0)/\mathrm{d}x, \quad t \geq s, \end{aligned} \qquad x \in (-\infty, \infty),\ i \in D,$$

be the density functions of $X_i(t),\ i \in D$. Note that $p_i(x, t \mid 0) = p_i(x, t),\ i \in D$.

In these terms equalities (2.18) take the following form

$$\begin{aligned} p_i(x, t \mid s) =& \psi_i(x - l_i(t), t)\overline{F}_i(t \mid s) \\ &+ \sum_{j \in D \setminus \{i\}} \int_s^t \left(\int_{-\infty}^{\infty} p_j(x - l_i(u) - h_i(u) - y, t - u)\psi_i(y, u)\mathrm{d}y\right) \\ &\times f_{ij}(u \mid s)\mathrm{d}u, \\ & t > s \geq 0,\ x \in (-\infty, \infty),\ i \in D. \end{aligned} \tag{2.19}$$

Here $\psi_i = \psi_i(x, t)$ is the density function of the Gaussian random variable $w_i(t),\ t > 0,\ i \in D$,

$$\psi_i(x, t) = \frac{1}{\Sigma_i(t)\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\Sigma_i(t)^2}\right), \qquad x \in (-\infty, \infty),$$

where $\Sigma_i(t)^2 := \int_0^t \sigma_i(u)^2\mathrm{d}u,\ t > 0,\ i \in D$.

In the Markov case of two-state processes, $d = 2$, with constant parameters $c_i, h_i, \sigma_i,\ i \in \{1, 2\}$, and with exponentially distributed inter-switching times T_n, the distributions of $X_1(t)$ and $X_2(t)$ have been analysed in detail in Ratanov (2010).

Let us study the expectations

$$\mu_i(t) := \mathbb{E}[X_i(t)], \qquad t \geq 0,\ i \in D,$$

and

$$\mu_i(t \mid s) := \mathbb{E}[X_i(t) \mid N_i(s) = 0], \qquad t \geq 0,\ i \in D,$$

of the d-state process $X = X(t),\ t > 0$.

Note that $\mathbb{E}[W^\sigma(t)] = 0,\ \forall t \geq 0$. Moreover, for $t \in [0, s]$, $\mu_i(t \mid s) = l_i(t),\ i \in D$.

To characterise $\mu_i(t),\ \mu_i(t \mid s),\ i \in D$, we use the following notations:

$$\left.\begin{aligned} a_i(t) &:= \int_0^t \left[c_i(u)\overline{F}_i(u) + h_i(u)f_i(u)\right] \mathrm{d}u, && t \geq 0, \\ a_i(t \mid s) &:= l_i(s) + \int_s^t [c_i(u)\overline{F}_i(u \mid s) + h_i(u)f_i(u \mid s)]\mathrm{d}u, && t \geq s \geq 0, \end{aligned}\right\} \quad i \in D.$$

Here $f_i(u),\ \overline{F}_i(u)$ and $f_i(u \mid s),\ \overline{F}_i(u \mid s),\ i \in D$, are defined in (2.13), (2.15) and (2.17).

Proposition 2.1. *The expectations $\mu_i(t),\ t \geq 0,\ i \in D$, satisfy the following Volterra system of integral equations,*

$$\mu_i(t) = a_i(t) + \sum_{j \in D \setminus \{i\}} \int_0^t \mu_j(t-u)f_{ij}(u)\mathrm{d}u, \qquad t \geq 0,\ i \in D. \tag{2.20}$$

If functions $\mu_i(t),\ t \geq 0,\ i \in D$, solve system (2.20), then the conditional expectations $\mu_i(t \mid s)$ are given by

$$\mu_i(t \mid s) = a_i(t \mid s) + \sum_{j \in D \setminus \{i\}} \int_s^t \mu_j(t-u)f_{ij}(u \mid s)\mathrm{d}u, \qquad t \geq s,\ i \in D. \tag{2.21}$$

Proof: By applying (2.19) (with $s = 0$) to

$$\mu_i(t) = \int_{-\infty}^{\infty} xp_i(x, t)\mathrm{d}x$$

one can get

$$\begin{aligned} \mu_i(t) =& l_i(t)\overline{F}_i(t) \\ &+ \sum_{j \in D \setminus \{i\}} \int_0^t \left(\int_{-\infty}^{\infty} \psi_i(y, u)\mathrm{d}y \int_{-\infty}^{\infty} xp_j(x - l_i(u) - h_i(u) - y, t-u)\mathrm{d}x \right) \\ &\qquad\qquad \times f_{ij}(u)\mathrm{d}u \\ =& l_i(t)\overline{F}_i(t) + \sum_{j \in D \setminus \{i\}} \int_0^t \left[\mu_j(t-u) + l_i(u) + h_i(u)\right] f_{ij}(u)\mathrm{d}u. \end{aligned}$$

Integrating by parts we have

$$\begin{aligned} \mu_i(t) = l_i(t)\overline{F}_i(t) + \sum_{j \in D \setminus \{i\}} \Bigg[&\int_0^t \left(\mu_j(t-u) + h_i(u)\right) f_{ij}(u)\mathrm{d}u - l_i(t)\overline{F}_{ij}(t) \\ &+ \int_0^t c_i(u)\overline{F}_{ij}(u)\mathrm{d}u \Bigg], \end{aligned}$$

which gives (2.20). The proof of (2.21) is similar. □

In the case $d = 2$ equations (2.20) are derived e.g. in equations (3.2)-(3.3) of Ratanov (2013).

Corollary 2.2. *The identities*

$$\mu_i(t) \equiv 0,\ t \geq 0,\ \textit{and}\ \mu_i(t \mid s) \equiv l_i(s),\ t \geq s, \qquad i \in D,$$

hold if and only if

$$\gamma_i(t)h_i(t) + c_i(t) \equiv 0, \qquad t \geq 0,\ i \in D, \tag{2.22}$$

where $\gamma_i = \gamma_i(t),\ t \geq 0,\ i \in D$, are the hazard rate functions, which are defined by (2.11), (2.14).

Proof: Notice that systems (2.20) and (2.21) have the trivial solutions $\mu_i(t) \equiv 0$, $t \geq 0,\ i \in D$ and $\mu_i(t \mid s) \equiv l_i(s),\ t \geq s,\ i \in D$, respectively, if and only if $a_i(t) \equiv 0$ and $a_i(t \mid s) \equiv l_0(s),\ i \in D$. These equalities hold, when

$$\left.\begin{aligned} c_i(u)\overline{F}_i(u) + h_i(u)f_i(u) \equiv 0, \\ c_i(u)\overline{F}_i(u \mid s) + h_i(u)f_i(u \mid s) \equiv 0, \end{aligned}\right. \quad u > s > 0,\ i \in D,$$

which is equivalent to (2.22), due to (2.11)-(2.13) and (2.16). □

Remark 2.3. Condition (2.22) has the sense of Doob-Meyer decomposition. These type of conditions for the jump-telegraph processes appears first in Ratanov (2007, Theorem 1) in the case of constant deterministic parameters c, h, γ (see also Kolesnik and Ratanov (2013)). In this case condition (2.22) is intuitively evident. It means that the displacement performed by the telegraph process during a time-period τ equal to the mean-switching-time is identical to the jump's size performed in the opposite direction.

This intuitively explains why this is a martingale condition.

3. Girsanov's transformation

In this section we analyse the problems which are important for applications, e.g. for financial modelling. First, we describe all possible martingales in our setting. Then, we derive a generalisation of Girsanov's Theorem.

3.1. *Martingale's characterisation.* Since the diffusion part $W^\sigma = W^\sigma(t) = \int_0^t \sigma^\dagger(u)\mathrm{d}B(u)$ is already a $\mathbb{P}$-martingale, it is sufficient to investigate the process $X = T^c(t) + N^h(t),\ t \geq 0$.

Theorem 3.1. *Let $X = X(t),\ t \geq 0$, be the process with the parameters $\langle c_i, h_i\rangle,\ i \in D$, switching at random times $\tau_n,\ n \geq 0$. Let the inter-switching times $T_n = \tau_n - \tau_{n-1},\ n \geq 1$, be distributed with hazard rate functions $\gamma_i = \gamma_i(t),\ t \geq 0,\ i \in D$, see (2.11), (2.14).*

Process X is a martingale if and only if the equalities in (2.22) are fulfilled.

Proof: If X is a martingale, then $\mu_0(t) = \mu_1(t) \equiv 0$, which is equivalent to (2.22), see Proposition 2.1.

Conversely, it is known, see Jeanblanc and Rutkowski (2002), Proposition 2.13, that the compensated jump process $N^h - T^{\gamma h}$ is a martingale. Therefore, if identities (2.22) hold, then $-T^c$ is the compensator of N^h and the sum $T^c + N^h$ is a martingale. □

Notice that if jumps vanish, $h_i \equiv 0,\ i \in D$, process X is a martingale only in the trivial case: $c_i \equiv 0,\ i \in D$, and thus $X = 0$, see (2.22).

Corollary 3.2 (Ratanov (2010))**.** *Let $X = X(t),\ t \geq 0$, be the jump-diffusion process with switching* constant *parameters c_i, h_i and $\sigma_i,\ i \in D$. Let $h_i \neq 0,\ i \in D$.*

Process X is a martingale if and only if $c_i/h_i < 0,\ i \in D$, and the distributions of the inter-switching times T_n are exponential, $\mathrm{Exp}(\lambda_i)$, *with parameters $\lambda_i,\ \lambda_i = -c_i/h_i > 0,\ i \in D$. In this case the underlying ε is a Markov process.*

Remark 3.3. In the paper by Di Crescenzo et al Di Crescenzo et al. (2014) the generalised 2-state geometric telegraph-diffusion process $S = S(t)$ with constant parameters c_0, c_1 and σ is studied,

$$S(t) = s_0 \exp\left[T(t) + \sigma B(t)\right],$$

where B is a standard Brownian motion and the inter-switching times are independent and arbitrarily distributed. The authors expected that the process $S = S(t)$ can be transformed in a martingale by superimposing of a jump component. This expectation is not justified.

The process $S(t)/s_0$ is the stochastic exponential of $X = T(t) + \sigma B(t) + \sigma^2 t/2$. After the inclusion of a jump component with *constant* jump amplitudes $h_1, h_2 > -1$, such that $\dfrac{c_i + \sigma^2/2}{h_i} < 0, i \in D = \{1, 2\}$, processes X and S become martingales only in the standard case of exponentially distributed inter-arrival times, $\mathrm{Exp}(\lambda_i)$, with constant intensities

$$\lambda_i = -\frac{c_i + \sigma^2/2}{h_i}, \quad i \in D = \{1, 2\},$$

(see Corollary 3.2).

3.2. *Girsanov's Theorem.* The problem of existence and uniqueness of an equivalent martingale measure is extremely significant for applications, especially in the theory of financial derivatives. It is important to understand how the equivalent martingale measures can be constructed if such a measure exists.

Let $\varepsilon(t) \in D,\ t \geq 0$, be the switching process on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ governed by the hazard rate functions $\gamma_i = \gamma_i^{\mathbb{P}}(t) = f_i(t)/\overline{F}_i(t),\ t > 0, i \in D$, of the inter-switching times, see (2.14), (2.11), satisfying the non-exploding condition (2.12).

Let c_i^* and $h_i^*,\ i \in D$, be measurable functions satisfying the martingale condition (2.22),

$$\gamma_i^{\mathbb{P}}(t) h_i^*(t) + c_i^*(t) \equiv 0 \qquad t \geq 0,\ i \in D. \tag{3.1}$$

We assume $c_i^*,\ i \in D$, to be locally integrable and $h_i^*(t) > -1,\ t \geq 0,\ i \in D$. Thus,

$$c_i^*(t) \leq \gamma_i^{\mathbb{P}}(t), \qquad \forall t > 0,\ i \in D. \tag{3.2}$$

Furthermore, let

$$\int_0^\infty \left(c_i^*(u) - \gamma_i^{\mathbb{P}}(u)\right) \mathrm{d}u = -\infty, \qquad i \in D. \tag{3.3}$$

Consider the jump-diffusion martingale $X^*=T^*(t)+N^*(t)+W^*(t)$ with regime switching, where

$$\begin{aligned}
T^*(t)&=\sum_{n=1}^{N(t)} l^*_{\varepsilon(\tau_{n-1})}(T_n)+l^*_{\varepsilon(\tau_{N(t)})}(t-\tau_{N(t)}),\\
N^*(t)&=\sum_{n=1}^{N(t)} h^*_{\varepsilon(\tau_{n-1})}(T_n),\\
W^*(t)&=\sum_{n=1}^{N(t)} w^*_{\varepsilon(\tau_{n-1})}(T_n)+w^*_{\varepsilon(\tau_{N(t)})}(t-\tau_{N(t)}).
\end{aligned}\tag{3.4}$$

Here, see (2.9),

$$l_i^*(t)=\int_0^t c_i^*(u)\mathrm{d}u,\qquad w_i^*(t)=\int_0^t \sigma_i^*(u)\mathrm{d}B(u),$$

where σ_i^*, $i\in D$, are locally square integrable functions.

Let $Z=Z(t)=\mathcal{E}_t(X^*)$ be the stochastic exponential of X^*. By (2.10)

$$\begin{aligned}
Z(t)=\mathcal{E}_t(X^*)&=\exp\left(T^{c^*-\sigma^{*2}/2}(t)+W^{\sigma^*}(t)\right)\prod_{n=1}^{N_t}(1+h^*_{\varepsilon(\tau_{n-1})}(T_n))\\
&=\exp\left(T^{c^*-\sigma^{*2}/2}(t)+N^{\ln(1+h^*)}+W^{\sigma*}(t)\right).
\end{aligned}\tag{3.5}$$

Here $T^{c^*-\sigma^{*2}/2}$ is the d-state generalised telegraph process (2.6) with the velocity regimes $c_i^*-(\sigma_i^*)^2/2$ instead of c_i, and $N^{\ln(1+h^*)}$ is the pure jump process (2.7) with the jump values $\ln(1+h_i^*)$ instead of h_i, $i\in D$.

Theorem 3.4 (Girsanov's Theorem)**.** *Assume that conditions* (3.1)-(3.3) *hold.*

Let measure $\mathbb{Q}$ be equivalent to $\mathbb{P}$ under the fixed time horizon t, $t\geq 0$, with the density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t}=Z(t).\tag{3.6}$$

Under the measure $\mathbb{Q}$:

(a) *the inter-arrival times $\{T_n=\tau_n-\tau_{n-1}\}_{n\geq 1}$ are independent and distributed with the survival functions*

$$\overline{F}_i^{\mathbb{Q}}(t)=\exp(l_i^*(t))\overline{F}_i^{\mathbb{P}}(t),\qquad t\geq 0,\ i\in D.\tag{3.7}$$

The hazard rate functions $\gamma_i^{\mathbb{Q}}$ of these distributions are given by

$$\gamma_i^{\mathbb{Q}}(t)=\gamma_i^{\mathbb{P}}(t)-c_i^*(t)\equiv(1+h_i^*(t))\gamma_i^{\mathbb{P}}(t),\qquad t\geq 0,\ i\in D,\tag{3.8}$$

and the non-exploding condition

$$\int_0^\infty \gamma_i^{\mathbb{Q}}(u)\mathrm{d}u=+\infty,\qquad i\in D,\tag{3.9}$$

holds;

(b) *the process* $\widetilde{B}(t) = B(t) - L^*(t)$ *is the standard* $\mathbb{Q}$*-Brownian motion, where* $L^*(t)$, $t \geq 0$, *is the generalised telegraph process with switching velocities* σ_i^*, $i \in D$, *i.e.*

$$L^*(t) := T^{\sigma^*}(t) = \sum_{n=1}^{N(t)} \int_0^{T_n} \sigma^*_{\varepsilon(\tau_{n-1})}(u)\mathrm{d}u + \int_0^{t-\tau_{N(t)}} \sigma^*_{\varepsilon(\tau_{N(t)})}(u)\mathrm{d}u;$$

(c) *the* $\mathbb{P}$*-jump-diffusion process* X *with switching regimes, which is defined by* (2.5),

$$X = T^c(t) + N^h(t) + W^\sigma(t),\ t \geq 0,$$

under measure $\mathbb{Q}$ *is still a jump-diffusion process*

$$X = T^{c+\sigma\sigma^*}(t) + N^h(t) + \widetilde{W}^\sigma(t),\ t \geq 0,$$

characterised by $\langle c + \sigma\sigma^*, h, \sigma\rangle$, *with the hazard rate functions* $\gamma_i^{\mathbb{Q}}$ *of the inter-switching times* $\{T_n\}_{n\geq 1}$, *determined by* (3.8). *Here* $\widetilde{W}^\sigma$ *is the stochastic integral* (2.8) *based on the* $\mathbb{Q}$*-Brownian motion* $\widetilde{B}$.

Proof: By definition, see (3.5)-(3.6), we have

$$\begin{aligned}\overline{F}_i^{\mathbb{Q}}(t) =& \mathbb{Q}\{\tau_1 > t \mid \varepsilon(0) = i\} = \mathbb{E}_{\mathbb{P}}\left\{Z(t)\mathbf{1}_{\{\tau_1>t\}} \mid \varepsilon(0) = i\right\} \\ =& \exp\left(\int_0^t [c_i^*(u) - (\sigma_i^*)^2(u)/2]\mathrm{d}u\right) \mathbb{E}\left[\exp(w_i^*(t))\right] \mathbb{P}(\tau_1 > t \mid \varepsilon(0) = i).\end{aligned}$$

Owing to $\mathbb{E}\left[\exp(w_i^*(t))\right] = \exp\left(\frac{1}{2}\int_0^t (\sigma_i^*)^2(u)\mathrm{d}u\right)$ we obtain (3.7).

Note that by (3.7) and (3.3)

$$\exp\left(-\int_0^t \gamma_i^{\mathbb{Q}}(u)\mathrm{d}u\right) = \overline{F}_i^{\mathbb{Q}}(t) = \exp(l_i^*(t))\overline{F}_i^{\mathbb{P}}(t) = \exp\left(\int_0^t \left(c_i^*(u) - \gamma_i^{\mathbb{P}}(u)\right)\mathrm{d}u\right), \tag{3.10}$$

$$t \geq 0,\ i \in D.$$

Hence, $\gamma_i^{\mathbb{Q}} \equiv \gamma_i^{\mathbb{P}} - c_i^*$. Since by (3.1) $c_i^* = -h_i^*\gamma_i^{\mathbb{P}}$, and (3.8) is completely proved. The non-exploding condition (3.9) follows from (3.3).

The part (b) of the theorem follows from the classical Girsanov's Theorem, see e.g. Jeanblanc et al. (2009), Proposition 1.7.3.1.

The part (c) holds by the following observation. The Wiener part W^σ of process X under measure $\mathbb{Q}$ becomes $W^\sigma(t) = \widetilde{W}^\sigma(t) + T^{\sigma\sigma^*}(t)$, see part (b). Here $\widetilde{W}^\sigma$ is the $\mathbb{Q}$-Wiener process, i.e. the Itô integral w.r.t. $\mathbb{Q}$-Brownian motion $\widetilde{B}$, and $T^{\sigma\sigma^*}$ is the $\mathbb{Q}$-telegraph process which is driven by the subsequently switching velocities $\sigma_i(t)\sigma_i^*(t)$, $i \in D$.

Therefore, under measure $\mathbb{Q}$ the process X is still the jump-diffusion process with switching regimes,

$$X(t) = T^{c+\sigma\sigma^*}(t) + N^h(t) + \widetilde{W}^\sigma(t), \qquad t > 0,$$

characterised by $\langle c + \sigma\sigma^*, h, \sigma\rangle$. The theorem is proved. □

3.3. *Relative entropy.* Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent measures. Under the time horizon $t,\ t>0$, the relative entropy of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is defined by the set of functions $H_i(t),\ t>0,\ i\in D:$

$$H_i(t) := \mathbb{E}_{\mathbb{Q}}\left[\ln\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)\ \Big|\ \varepsilon(0)=i\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)\ln\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)\ \Big|\ \varepsilon(0)=i\right], \tag{3.11}$$

see Frittelli (2000). Here the Radon-Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)=\mathcal{E}_t\left(X^*\right)$ is presented by (3.5)-(3.6).

Theorem 3.5. *Let conditions* (3.1)-(3.3) *hold.*

The relative entropy functions H_i are expressed by

$$H_i(t)=\mathbb{E}_{\mathbb{Q}}\left[T_i^{c^*+(\sigma^*)^2/2}(t)+N_i^{\ln(1+h^*)}(t)\right], \qquad t\geq 0,\ i\in D, \tag{3.12}$$

and satisfy the system of the integral equations

$$H_i(t)=a_i(t)+\sum_{j\in D\setminus\{i\}}\int_0^t H_j(t-u)f_{ij}(u)\mathrm{d}u, \qquad t\geq 0,\ i\in D. \tag{3.13}$$

where functions a_i are defined by

$$a_i(t)=\int_0^t b_i(u)\overline{F}_i^{\mathbb{Q}}(u)\mathrm{d}u, \quad t\geq 0,\ i\in D. \tag{3.14}$$

Here

$$\begin{aligned}
\overline{F}_i^{\mathbb{Q}}(u) =&\exp\left(-\int_0^u \gamma_i^{\mathbb{Q}}(u')\mathrm{d}u'\right),\\
b_i(u) =&\gamma_i^{\mathbb{P}}(u)-\gamma_i^{\mathbb{Q}}(u)+\phi_i(u)+\frac{1}{2}\sigma_i^*(u)^2,\\
\phi_i(u) =&\begin{cases}\gamma_i^{\mathbb{Q}}(u)\ln\left[\dfrac{\gamma_i^{\mathbb{Q}}(u)}{\gamma_i^{\mathbb{P}}(u)}\right], & \textit{if } \gamma_i^{\mathbb{P}}(u)\neq 0,\\ 0, & \textit{if } \gamma_i^{\mathbb{P}}(u)=0,\end{cases}\\
& u\geq 0,\ i\in D.
\end{aligned} \tag{3.15}$$

Proof: Owing to (3.5)

$$H_i(t)=\mathbb{E}_{\mathbb{Q}}\left[\ln\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)\ \Big|\ \varepsilon(0)=i\right]=\mathbb{E}_{\mathbb{Q}}\left[T_i^{c^*-(\sigma^*)^2/2}(t)+N_i^{\ln(1+h^*)}(t)+W_i^{\sigma^*}(t)\right], \tag{3.16}$$

where the alternating tendencies $T_i^{c^*-(\sigma^*)^2/2}$, the jump process $N_i^{\ln(1+h*)}$ and the Wiener process $W_i^{\sigma^*}$ are defined by (3.4) (with $c_i^*-(\sigma_i^*)^2/2$ instead of c_i^* and $\ln(1+h_i^*)$ instead of $h_i^*,\ i\in D$).

By Theorem 3.4, part (c), the process $T_i^{c^*-(\sigma^*)^2/2}(t)+N_i^{\ln(1+h^*)}(t)+W_i^{\sigma^*}(t)$ under measure $\mathbb{Q}$ becomes $T_i^{c^*+(\sigma^*)^2/2}(t)+N_i^{\ln(1+h^*)}(t)+\widetilde{W}_i^{\sigma^*}(t)$, where $\widetilde{W}_i^{\sigma^*}(t)$ is the stochastic integral w.r.t. the $\mathbb{Q}$-Brownian motion $\widetilde{B}$. Therefore, $\mathbb{E}_{\mathbb{Q}}\left[\widetilde{W}_i^{\sigma^*}(t)\right]=0,\ t\geq 0$, and

$$\begin{aligned}
H_i(t)=&\mathbb{E}_{\mathbb{Q}}\left[T_i^{c^*+(\sigma^*)^2/2}(t)+N_i^{\ln(1+h^*)}(t)+\widetilde{W}_i^{\sigma^*}(t)\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[T_i^{c^*+(\sigma^*)^2/2}(t)+N_i^{\ln(1+h^*)}(t)\right],
\end{aligned}$$

which gives (3.12).

Equations (3.13)-(3.15) follow from Proposition 2.1 and Theorem 3.4, see (3.8). □

It is easy to see that functions b_i defined by (3.15) are non-negative, $b_i(u) \geq 0,\ u \geq 0,\ i \in D$. Hence, functions a_i defined by (3.14) are also non-negative, $a_i(t) \geq 0,\ t \geq 0,\ i \in D$.

Remark 3.6. By applying the Laplace transform $f \to \hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t$ to (3.13) one can obtain the system:

$$\hat{H}_i(s) = \hat{a}_i(s) + \sum_{j \in D \setminus \{i\}} \hat{f}_{ij}(s)\hat{H}_j(s), \quad s > 0, \qquad i \in D.$$

if the transformations $\hat{a}_i(s),\ i \in D$, exist. The above system yields the unique solution.

For example, if $d = 2$, and $b_1,\ b_2$ (see (3.15)) are constants; if the alternating distributions of inter-arrival times are exponential, $\gamma_i^{\mathbb{Q}}(t) = \lambda_i^* = \text{const},\ i \in \{1, 2\}$, therefore in this simple case

$$\begin{aligned} \hat{H}_1(s) &= \frac{B}{s^2} + \frac{A_1}{s + \lambda_1^* + \lambda_2^*}, \\ \hat{H}_2(s) &= \frac{B}{s^2} + \frac{A_2}{s + \lambda_1^* + \lambda_2^*}, \end{aligned} \quad s > 0,$$

where

$$A_1 = \frac{\lambda_1^*(b_1 - b_2)}{(\lambda_1^* + \lambda_2^*)^2}, \quad A_2 = \frac{\lambda_2^*(b_2 - b_1)}{(\lambda_1^* + \lambda_2^*)^2}, \quad B = \frac{\lambda_2^* b_1 + \lambda_1^* b_2}{\lambda_1^* + \lambda_2^*}. \tag{3.17}$$

This corresponds to the following explicit solution of (3.13): the relative entropy functions $H_1(t), H_2(t),\ t \geq 0$, are expressed by

$$\begin{aligned} H_1(t) = H_1(t; \lambda_1^*, \lambda_2^*) &= Bt + A_1 \left[1 - \mathrm{e}^{-(\lambda_1^* + \lambda_2^*)t}\right], \\ H_2(t) = H_2(t; \lambda_1^*, \lambda_2^*) &= Bt + A_2 \left[1 - \mathrm{e}^{-(\lambda_1^* + \lambda_2^*)t}\right]. \end{aligned} \tag{3.18}$$

4. Equivalent martingale measure

Consider the jump-diffusion process $X = T^c(t) + H^h(t) + W^\sigma(t),\ t \geq 0$, with the switching hazard rate functions $\gamma_i^{\mathbb{P}},\ i \in D$, of inter-arrival times $T_n,\ n \geq 1$, see the definitions in (2.6)-(2.8).

Let the equivalent measure $\mathbb{Q}$ be defined by the Radon-Nikodým density $Z(t) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t},\ t \geq 0$, see (3.5)-(3.6). Let driving parameters $c_i^*,\ h_i^*,\ h_i^* > -1$, and $\sigma_i^*,\ i \in D$, satisfy (3.1)-(3.3). By Theorem 3.4 under measure $\mathbb{Q}$ the hazard rate functions $\gamma_i^{\mathbb{Q}}$ are defined by (3.8).

The family of the equivalent *martingale* measures for X can be disclosed precisely.

Theorem 4.1. *Measure $\mathbb{Q}$ is the martingale measure for process X if and only if*

$$c_i(t) + \sigma_i(t)\sigma_i^*(t) + \gamma_i^{\mathbb{Q}}(t)h_i(t) = 0, \quad t \geq 0,\ i \in D. \tag{4.1}$$

Proof: This result is well known, see e.g. Bellamy and Jeanblanc (2000), Proposition 3.1. The proof follows from Theorem 3.1 and Theorem 3.4.

Let measure $\mathbb{Q}$ be defined by (3.5)-(3.6). Then, by Theorem 3.4, part (c), under measure $\mathbb{Q}$ the process

$$X(t) = T^{c+\sigma\sigma^*}(t) + N^h(t) + \widetilde{W}^\sigma(t), \qquad t > 0,$$

is again the jump-diffusion process with switching regime. The martingale condition (2.22) of Theorem 3.1 becomes (4.1).

The theorem is proved. □

The relative entropy functions $H_i(t),\ t > 0,\ i \in D$, of the *martingale* measure $\mathbb{Q}$ w.r.t. $\mathbb{P}$ solve system (3.13) with functions a_i specified by (3.14) and (3.15). Driving parameters c_i^*, h_i^*, σ_i^* and switching intensities $\gamma_i^{\mathbb{P}}, \gamma_i^{\mathbb{Q}},\ i \in D$, satisfy (3.8) and (4.1).

Consider the following examples when the equivalent martingale measure $\mathbb{Q}$ is unique.

Example 4.2 (**Jump-telegraph process**). Consider process X missing the diffusion component,

$$X(t) = T^c(t) + N^h(t), \qquad t \geq 0,$$

see (2.6)-(2.7).

Assume, that $h_i(t) \neq 0$ almost everywhere,[1] and $h_i(t)$ is of the opposite sign with $c_i(t)$:

$$c_i(t)/h_i(t) < 0, \quad t > 0, \qquad i \in D. \tag{4.2}$$

Moreover, let functions $c_i/h_i,\ i \in D$, be locally integrable and

$$\int_0^\infty \frac{c_i(u)}{h_i(u)} \mathrm{d}u = -\infty, \qquad i \in D. \tag{4.3}$$

Then, by Theorem 4.1 the equivalent martingale measure $\mathbb{Q}$ exists and it is unique with the hazard rate functions of interarrival times defined by

$$\gamma_i^{\mathbb{Q}}(t) = -c_i(t)/h_i(t) > 0, \qquad t > 0, \quad i \in D. \tag{4.4}$$

Here (4.3) is the non-exploding condition for measure $\mathbb{Q}$. The corresponding measure transformation is determined by the functions $c_i^*, h_i^*,\ i \in D$,

$$c_i^*(t) = \gamma_i^{\mathbb{P}}(t) - \gamma_i^{\mathbb{Q}}(t), \qquad h_i^*(t) = -1 + \gamma_i^{\mathbb{Q}}(t)/\gamma_i^{\mathbb{P}}(t), \quad t > 0, \quad i \in D,$$

if $\gamma_i^{\mathbb{P}}(t) > 0$ *a.e.*, see (3.8) and (4.4). [2]

The entropy functions $H_i(t),\ i \in D$, solve system (3.13) with

$$a_i(t) = \int_0^t \left[\gamma_i^{\mathbb{P}}(u) - \gamma_i^{\mathbb{Q}}(u) + \gamma_i^{\mathbb{Q}}(u) \ln \frac{\gamma_i^{\mathbb{Q}}(u)}{\gamma_i^{\mathbb{P}}(u)} \right] \overline{F}_i^{\mathbb{Q}}(u) \mathrm{d}u,$$

where $\gamma_i^{\mathbb{Q}}$ are defined by (4.4). The survival functions $\overline{F}_i^{\mathbb{Q}},\ i \in D$, are defined in (2.13).

---

[1] If, in contrary, $h_i(u) = 0$ on a whole interval, $u \in (a, b) \subset [0, \infty)$, then, due to (4.1) with $\sigma_i \equiv 0$, we have no martingale measures (if $c_i \neq 0$ on the interval (a, b)), or infinitely many ones (if $c_i = 0$ with free fragment of hazard rate function $\gamma_i^{\mathbb{Q}}$).

[2] Measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent. If $\mathbb{P}$-distribution of the interarrival times has a "dead" zone: $\gamma_i^{\mathbb{P}}(u) \equiv 0,\ u \in (a, b)$, for some time interval $(a, b) \subset [0, \infty)$, then due to (3.8) and (4.1) for any martingale measure $\mathbb{Q}$ the hazard rate function $\gamma_i^{\mathbb{Q}}$ also vanishes on (a, b), $\gamma_i^{\mathbb{Q}}(u) \equiv 0$ and $c_i(u) = c_i^*(u) \equiv 0,\ u \in (a, b)$.

If the inequalities (4.2) do not hold, the martingale measure does not exist.

In particular, if the parameters $c_i, h_i,\ h_i \neq 0$, are constant such that $c_i/h_i < 0,\ i \in D$, with exponentially distributed inter-switching times, $\gamma_i^{\mathbb{P}} = \lambda_i,\ i \in D$, then by (4.4) under the martingale measure $\mathbb{Q}$ the inter-switching times are again exponentially distributed with the switching intensities $\lambda_i^{\mathbb{Q}} = -c_i/h_i,\ i \in D$. If $d = 2$, then the closed form of the entropy functions is found, see Remark 3.6, formulae (3.17)-(3.18), and more detailed analysis in Section 5 below. In this case the unique martingale measure $\mathbb{Q}$ is defined by the Radon-Nikodým density (3.5)-(3.6) with constant c_i^* and h_i^*:

$$c_i^* = \lambda_i - \lambda_i^*, \qquad h_i^* = -1 + \frac{\lambda_i^*}{\lambda_i}, \tag{4.5}$$

where $\lambda_i^* = -c_i/h_i > 0,\ i \in D = \{1, 2\}$, are the new alternating intensities of the inter-switching times, see e.g. Kolesnik and Ratanov (2013).

Formulae (3.17)-(3.18) for the entropy functions $H_i(t),\ t \geq 0$, hold with

$$b_i = \lambda_i + \frac{c_i}{h_i} - \frac{c_i}{h_i} \ln\left[-\frac{c_i}{h_i\lambda_i}\right], \qquad i \in D = \{1, 2\}.$$

Example 4.3 (**Diffusion process**). Consider the diffusion process missing the jump component and switching, c is locally integrable and functions $\sigma,\ c/\sigma$ are locally square integrable. Assume that $\sigma(u) \neq 0$ *a.e.*

Let $\mathbb{Q}$ be an equivalent measure. In this case the Radon-Nikodým derivative of $\mathbb{Q}$ is defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t) = \exp\left(\int_0^t \sigma^*(u)\mathrm{d}B(u) - \frac{1}{2}\int_0^t \sigma^*(u)^2\mathrm{d}u\right),$$

where σ^* is the locally square integrable function. By Girsanov's Theorem the process

$$\widetilde{B} = B - \int_0^t \sigma^*(u)\mathrm{d}u$$

is $\mathbb{Q}$-Brownian motion. Hence, process X takes the form

$$X(t) = \int_0^t \left[c(u) + \sigma(u)\sigma^*(u)\right]\mathrm{d}u + \int_0^t \sigma^*(u)\mathrm{d}\widetilde{B}(u), \qquad t \geq 0.$$

This is a martingale if and only if $\sigma\sigma^* \equiv -c$.

By (3.11) the relative entropy $H(t)$ of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is

$$\begin{aligned} H(t) = \mathbb{E}_{\mathbb{Q}}\left[\ln\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \sigma^*(u)\mathrm{d}B(u) - \frac{1}{2}\int_0^t \sigma^*(u)^2\mathrm{d}u\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \sigma^*(u)\mathrm{d}\widetilde{B}(u) + \frac{1}{2}\int_0^t \sigma^*(u)^2\mathrm{d}u\right] \\ &= \frac{1}{2}\int_0^t \sigma^*(u)^2\mathrm{d}u. \end{aligned}$$

Therefore, the relative entropy of the (unique) martingale measure is given by

$$H(t) = \frac{1}{2}\int_0^t \left[\frac{c(u)}{\sigma(u)}\right]^2 \mathrm{d}u, \qquad t > 0.$$

Remark 4.4 (**Diffusion process with switching tendencies and diffusion coefficients**). Consider the case of the *diffusion process,*

$$X(t) = T^c(t) + W^\sigma(t), \qquad t \geq 0,$$

with the switching tendencies $c_i = c_i(t)$ and diffusion coefficients $\sigma_i = \sigma_i(t) \neq 0,\ t > 0,\ i \in D$, where the jump component is missing.

In this case there are infinitely many equivalent martingale measures.

Theorem 4.1 shows that the measure transformation defined by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\ |_{\mathcal{F}_t} = \mathcal{E}_t(X^*)$, see (3.5) with $h_i^* = 0$, $c_i^* = 0$ and $\sigma_i^*(t) = -c_i(t)/\sigma_i(t),\ t \geq 0,\ i \in D$, eliminates the drift component similarly as in Example 4.3. Under measure $\mathbb{Q}$ process X becomes the martingale of the form $X(t) \equiv \widetilde{W}^\sigma(t),\ t \geq 0$, whereas by (4.1) the inter-switching times are *arbitrary distributed.* Here $\widetilde{W}^\sigma$ is the stochastic integral (2.8) based on $\mathbb{Q}$-Brownian motion $\widetilde{B}$.

This model has been analysed in Elliott et al. (2005) by using the Esscher transform under switching regimes. This transformation does not affect the distribution of inter-switching times and the corresponding equivalent martingale measure is of the minimal relative entropy, see Elliott et al. (2005), Proposition 3.1.

In the next section we study in detail the jump-diffusion model with switching regimes based on the Markov underlying process ε. We have discovered that in this case the Esscher transform does not produce the minimal relative entropy.

5. Esscher transform and minimal entropy martingale measure

Typically, the jump-diffusion model with switching regime has no martingale measure or it has infinitely many. The rare examples of the unique martingale measure are presented above (Example 4.2 and Example 4.3). In this section we discuss the case when the infinitely many martingale measures exist and discuss some methods to select one. The first method is based on the so-called the Esscher transform under switching regimes.

Let $X = T^c(t) + N^h(t) + W^\sigma(t),\ t \geq 0$, be the jump-diffusion process with switching regime, see (2.6)-(2.8), defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Let $\sigma_i \neq 0,\ i \in D$, a. s. The case with missed diffusion ($\sigma_i \equiv 0,\ i \in D$) is analysed in Example 4.2.

To choose an equivalent martingale measure by a reasonable way consider the deterministic measurable functions $\theta_i = \theta_i(t),\ t \geq 0,\ i \in D$, which define the regime switching processes $\theta_i^\dagger = \theta_i^\dagger(t),\ t \geq 0,\ i \in D$, similarly as in (2.1). Let measure $\mathbb{Q}_\theta$ (equivalent to $\mathbb{P}$) be defined by the density

$$\frac{\mathrm{d}\mathbb{Q}_\theta}{\mathrm{d}\mathbb{P}}\ |_{\mathcal{F}_t} := \frac{\exp\left(\int_0^t \theta^\dagger(s)\mathrm{d}Y(s)\right)}{\mathbb{E}_\mathbb{P}\left[\exp\left(\int_0^t \theta^\dagger(s)\mathrm{d}Y(s)\right)\ |\ \mathcal{F}_t^\varepsilon\right]}. \tag{5.1}$$

Here $Y(t) = T^{c-\sigma^2/2}(t) + N^{\ln(1+h)}(t) + W^\sigma(t),\ t \geq 0$, see (2.10), and $\mathcal{F}_t^\varepsilon$ is the $\mathbb{P}$-augmentation of the natural filtration generated by ε. This particular choice of the new measure is named a regime switching Esscher transform (or exponential tilting), see Elliott et al Elliott et al. (2005).

It is easy to see that

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t \theta^{\dagger}(s)\mathrm{d}Y(s)\right) \mid \mathcal{F}_t^{\varepsilon}\right] = \exp\left(T^{\theta(c-\sigma^2/2)}(t) + N^{\theta\ln(1+h)}(t)\right) \times \exp\left(T^{\theta^2\sigma^2/2}(t)\right).$$

Therefore the Radon-Nikodým derivative of the Esscher transforms is given by

$$\frac{\mathrm{d}\mathbb{Q}_\theta}{\mathrm{d}\mathbb{P}} |_{\mathcal{F}_t} = \exp\left(W^{\theta\sigma}(t) - T^{\theta^2\sigma^2/2}(t)\right),$$

which corresponds to Radon-Nikodým derivative (3.5) with

$$\sigma_i^* = \theta_i\sigma_i, \quad c_i^* = 0, \quad h_i^* = 0, \qquad i \in D. \tag{5.2}$$

Observe, that by Girsanov's Theorem (see Theorem 3.4, equation (3.8)) due to (5.2) the distribution of inter-switching times are not changed under such defined measure, $\gamma^{\mathbb{Q}_\theta} \equiv \gamma^{\mathbb{P}}$.

Hence, due to (5.2), the martingale condition (see Theorem 4.1, equation (4.1)) can be written as

$$\theta_i(t) = -\frac{c_i(t) + \gamma_i^{\mathbb{P}}(t)h_i(t)}{\sigma_i(t)^2}, \quad t > 0,\ i \in D. \tag{5.3}$$

It is known that the Esscher measure transform defined by (5.1) with parameters $\theta_i,\ i \in D$, determined by (5.3) corresponds to the *minimal relative entropy*, see Elliott et al. (2005), Proposition 3.1. The similar approach with the Esscher measure transform produces the minimal relative entropy in the case of Lévy processes (see Esche and Schweizer (2005); Fujiwara and Miyahara (2003)).

For our model based on Brownian motion *with jumps and with switching regimes* the Esscher transform does not produce the minimal relative entropy.

In the rest of this section for the sake of simplicity, we consider the Markov case with $d = 2$, when the alternating distributions of inter-switching times are exponential both under measure $\mathbb{P}$ and under an equivalent measure $\mathbb{Q}$, i.e.

$$\gamma_i^{\mathbb{P}} = \lambda_i = \text{const} > 0, \qquad \gamma_i^{\mathbb{Q}} = \lambda_i^* = \text{const} > 0, \quad i \in D = \{1, 2\},$$

and the driving parameters $c_i, h_i, \sigma_i,\ i \in D = \{1, 2\}$, are constant. Here measure $\mathbb{Q}$ is defined by (3.4)-(3.6) with constant parameters $c_i^*, h_i^*, \sigma_i^*,\ i \in D = \{1, 2\}$, satisfying (3.1)-(3.3).

To analyse the set of equivalent martingale measures from the viewpoint of the relative entropy we are looking for the solution of the integral equations (3.13). Since this jump-diffusion process X is bounded, by Theorem 2.1 of Frittelli (2000) there exists a unique minimal entropy martingale measure.

As it is shown in Remark 3.6, in this case the relative entropy functions are defined by (3.18):

$$\begin{aligned} H_1(t) = H_1(t; \lambda_1^*, \lambda_2^*) &= Bt + A_1\left[1 - \mathrm{e}^{-(\lambda_1^*+\lambda_2^*)t}\right], \\ H_2(t) = H_2(t; \lambda_1^*, \lambda_2^*) &= Bt + A_2\left[1 - \mathrm{e}^{-(\lambda_1^*+\lambda_2^*)t}\right], \end{aligned} \qquad t \geq 0,$$

where A_1, A_2 and B are defined by (3.17) and (3.15).

Remark 5.1. Note that $B = 0$ (or, equivalently, $b_1 = b_2 = 0$) if and only if process X is already a $\mathbb{P}$-martingale. Indeed, $b_1 = b_2 = 0$ if and only if $\sigma_i^* = 0,\ i \in D = \{1, 2\}$, and $\lambda_i^* = \lambda_i, \ i \in D = \{1, 2\}$, (see (3.15)) , or equivalently, $\mathbb{Q} = \mathbb{P}$. Hence $A_1 = A_2 = 0$ and $H_1(t) = H_2(t) \equiv 0$.

Remark 5.2. Let the jump-diffusion process X be a Lévy process, i.e. the alternation is missing and $c_1 = c_2 = c, \ h_1 = h_2 = h \neq 0, \ \sigma_1 = \sigma_2 = \sigma \neq 0$ are constant. This is the case of a Markov jump-diffusion process.

Let the new measure $\mathbb{Q}$ be defined by (3.4)-(3.6).

Therefore, by (3.18) the relative entropy functions are identical and linear in t,

$$H_1 \equiv H_2 = Bt, \ t \geq 0,$$

where, due to (3.17), $B = b = \lambda - \lambda^* + \lambda^* \ln \lambda^*/\lambda + (\sigma^*)^2/2$, and $A_1 = A_2 = 0$. Here $\lambda = \gamma^{\mathbb{P}}$ and $\lambda^* = \gamma^{\mathbb{Q}}$ are the constant jump intensities under measure $\mathbb{P}$ and measure $\mathbb{Q}$ respectively; the parameter σ^* satisfies martingale condition (4.1):

$$\sigma^* = -\frac{c + \lambda^* h}{\sigma}.$$

In this case the martingale measure with the minimal relative entropy is defined by the jump intensity λ^* which satisfies the algebraic equation:

$$b'(\lambda^*) \equiv \ln \lambda^*/\lambda + \frac{h^2}{\sigma^2}\left(\frac{c}{h} + \lambda^*\right) = 0. \tag{5.4}$$

The latter equation is equivalent to

$$c + \beta^* \sigma^2 + \lambda h \exp(\beta^* h) = 0, \tag{5.5}$$

where the following change of variables $\lambda^* = \lambda \exp(\beta^* h)$ is applied. In this particular case of Lévy process X, equation (5.5) coincides with condition (C) of Fujiwara and Miyahara (2003), which gives the minimal relative entropy $H(t)$ under a measure defined by the Esscher transformation. In this example equation (3.18) becomes

$$H(t) = \left[\lambda(1 - \exp(\beta^* h) + \beta^* h \exp(\beta^* h)) + \frac{h^2}{2\sigma^2}\left(\frac{c}{h} + \lambda \exp(\beta^* h)\right)^2\right] t, \tag{5.6}$$

where β^* is the (unique) solution of (5.5). Equation (5.6) corresponds to equation (3.9) from Fujiwara and Miyahara (2003).

In the case of the jump-diffusion process with alternating parameters, such coincidence is not available. In this case the minimal entropy functions depend on the initial state and they have a bit more complicated behaviour.

Observe that functions $b_1 = b_1(\lambda_1^*, \sigma_1^*), \ b_2 = b_2(\lambda_2^*, \sigma_2^*)$ are expressed by summing up of the two nonnegative and convex functions, $f(x) = a - x + x\ln(x/a), \ x > 0$, (with λ^* for x) and $g(y) = y^2/2$ (with σ^* for y). Hence, b_1 and b_2 are nonnegative and convex. Therefore, function $B = B(\lambda_1^*, \lambda_2^*, \sigma_1^*, \sigma_2^*) = \dfrac{\lambda_2^* b_1 + \lambda_1^* b_2}{\lambda_1^* + \lambda_2^*}$ is also nonnegative.

We analyse the relative entropy functions $H_1 = H_1(t)$ and $H_2 = H_2(t)$ for small and big times t separately. These functions possesses the following time-asymptotics.

Proposition 5.3. *Let the relative entropy functions H_1 and H_2 be defined by* (3.18). *Thus,*

$$H_1(t) \sim b_1 t, \qquad H_2(t) \sim b_2 t, \qquad t \to 0, \tag{5.7}$$

and

$$H_1(t) \sim Bt + A_1, \qquad H_2(t) \sim Bt + A_2, \qquad t \to \infty. \tag{5.8}$$

Proof: As $t \to 0$ formulae (3.17)-(3.18) lead to (5.7):

$$H_i(t) = Bt + A_i \left[1 - \mathrm{e}^{-(\lambda_1^* + \lambda_2^*)t}\right] \sim [B + A_i(\lambda_1^* + \lambda_2^*)]\, t \equiv b_i t, \qquad i \in D = \{1, 2\}.$$

Long-term asymptotic (5.8) is evident. □

We choose the equivalent measure minimising the relative entropy function $\mathbf{H}(t)$ under the martingale condition (4.1), i.e. $\lambda_i^*,\ \sigma_i^*,\ i \in D = \{1, 2\}$, satisfy the following relations:

$$\begin{cases} c_1 + \lambda_1^* \cdot h_1 + \sigma_1^* \cdot \sigma_1 = 0, \\ c_2 + \lambda_2^* \cdot h_2 + \sigma_2^* \cdot \sigma_2 = 0. \end{cases} \tag{5.9}$$

If measure $\mathbb{Q}$ is constructed by way of minimising b_1 and b_2, we say that $\mathbb{Q}$ is the short-term minimal entropy martingale measure (MEMM), see (5.7); measure $\mathbb{Q}$ is the long-term MEMM, if $\mathbb{Q}$ minimises B, see (5.8).

The short-term and the long-term minimal entropy equivalent martingale measures are defined as the solutions of the following optimisation problems subject to martingale condition (5.9):

- the short-term MEMM is defined by solving the problem w.r.t. σ_i^* and $\lambda_i^*,\ i \in D = \{1, 2\}$,

$$\begin{cases} b_1 = \lambda_1 - \lambda_1^* + \lambda_1^* \ln \lambda_1^*/\lambda_1 + (\sigma_1^*)^2/2 \to \min, \\ b_2 = \lambda_2 - \lambda_2^* + \lambda_2^* \ln \lambda_2^*/\lambda_2 + (\sigma_2^*)^2/2 \to \min. \end{cases} \tag{5.10}$$

- the long-term MEMM is defined by solving the problem w.r.t. σ_i^* and $\lambda_i^*,\ i \in D = \{1, 2\}$,

$$B = \frac{\lambda_2^* b_1 + \lambda_1^* b_2}{\lambda_1^* + \lambda_2^*} \to \min. \tag{5.11}$$

Theorem 5.4. *The solutions of problems* (5.10) *and* (5.11) *subject to condition* (5.9) *exist and they are unique.*

Proof: If $\sigma_i \neq 0$, then (5.9) gives

$$\sigma_i^* = -\frac{c_i + \lambda_i^* h_i}{\sigma_i}. \tag{5.12}$$

Hence problem (5.10) is equivalent to minimisation of the function

$$b_i = b_i(\lambda_i^*) := \lambda_i - \lambda_i^* + \lambda_i^* \ln(\lambda_i^*/\lambda_i) + \frac{1}{2\sigma_i^2}\left(c_i + \lambda_i^* h_i\right)^2, \quad i \in D = \{1, 2\}.$$

If, additionally, $h_i = 0$, then problem (5.10) has the unique solution $\lambda_i^* = \lambda_i$. In this case we return to the Markov modulated diffusion process, see Remark 4.4 and Elliott et al. (2005). This confirms again that in this case the minimum of entropy and the Esscher transform (5.1) lead to the same result.

If $\sigma_i = 0,\ h_i \neq 0$, then we have jump-telegraph model (see Example 4.2). In this case the martingale condition (5.9) gives $\lambda_i^* = -c_i/h_i$, which corresponds to the unique equivalent martingale measure.

On the contrary, if $\sigma_i \neq 0,\ h_i \neq 0$, then the minimal entropy and the Esscher transform (5.1) give different results. Observe that $b_i,\ i \in D = \{1,2\}$, with σ_i^* given by (5.12) can be rewritten as

$$b_i = b_i(\lambda_i^*) := \lambda_i - \lambda_i^* + \lambda_i^* \ln(\lambda_i^*/\lambda_i) + \frac{C_i^2}{2}\left(\lambda_i^* - \alpha_i\right)^2, \quad i \in D = \{1,2\}, \tag{5.13}$$

where $C_i^2 = h_i^2/\sigma_i^2 > 0$ and $\alpha_i = -c_i/h_i > 0,\ i \in D = \{1,2\}$.

Differentiating (5.13) we have:

$$\begin{aligned} b_1'(\lambda_1^*) \equiv& \frac{h_1^2}{\sigma_1^2}(c_1/h_1 + \lambda_1^*) + \ln(\lambda_1^*/\lambda_1), \\ b_2'(\lambda_2^*) \equiv& \frac{h_2^2}{\sigma_2^2}(c_2/h_2 + \lambda_2^*) + \ln(\lambda_2^*/\lambda_2). \end{aligned} \tag{5.14}$$

We remark that functions b_1' and b_2' vary monotonically from $-\infty$ to $+\infty$ as λ_1^* and λ_2^* increase from 0 to $+\infty$. Hence, the system $b_1'(\lambda_1^*) = 0,\ b_2'(\lambda_2^*) = 0$ (and minimisation problem (5.10)) has the unique solution.

Moreover, if $\lambda_i^* < -c_i/h_i$, then the solution λ_i^* of the corresponding equation $b_i'(\lambda_i^*) = 0$ satisfies $\lambda_i^*,\ \lambda_i^* > \lambda_i,\ i \in D = \{1,2\}$; if $\lambda_i^* > -c_i/h_i$, then $\lambda_i^* < \lambda_i,\ i \in D = \{1,2\}$. Therefore, the solution λ_i^* of (5.10) is always between λ_i and $-c_i/h_i$, $i \in D = \{1,2\}$, whereas the Esscher transform gives $\lambda_i^* = \lambda_i$, $i \in D = \{1,2\}$.

We solve the problem (5.11) by differentiating again. System

$$\frac{\partial B}{\partial \lambda_1^*} = 0, \qquad \frac{\partial B}{\partial \lambda_2^*} = 0$$

is equivalent to

$$\Phi_1 := (\lambda_1^* + \lambda_2^*)b_1'(\lambda_1^*) + b_2(\lambda_2^*) - b_1(\lambda_1^*) = 0, \tag{5.15}$$

$$\Phi_2 := (\lambda_1^* + \lambda_2^*)b_2'(\lambda_2^*) + b_1(\lambda_1^*) - b_2(\lambda_2^*) = 0. \tag{5.16}$$

By a similar reasoning as before, one can easily see that system (5.15)-(5.16) also has (unique) solution. Indeed, owing to (5.13) and (5.14) we have

$$\Phi_1 = \lambda_2^* \ln(\lambda_1^*/\lambda_1) + b_2(\lambda_2^*) - \lambda_1 + \lambda_1^* + \frac{C_1^2}{2}(\lambda_1^* - \alpha_1)^2 + C_1^2(\lambda_1^* - \alpha_1)(\lambda_2^* + \alpha_1).$$

Thus,

$$\frac{\partial \Phi_1}{\partial \lambda_1^*} = 1 + \frac{\lambda_2^*}{\lambda_1^*} + C_1^2(\lambda_1^* + \lambda_2^*) > 0.$$

Hence, function Φ_1 increases monotonically from $-\infty$ to $+\infty$ as λ_1^* increases from 0 to $+\infty$, which means that equation (5.15) has the unique solution $\lambda_1^* = \phi(\lambda_2^*) > 0$ for any fixed positive λ_2^*. Therefore system (5.15)-(5.16) is equivalent to

$$b_1'(\phi(\lambda_2^*)) + b_2'(\lambda_2^*) = 0.$$

Differentiating this identity and (5.14) one can see that

$$\phi'(\lambda_2^*) = -\frac{C_2^2 + 1/\lambda_2^*}{C_1^2 + 1/\phi(\lambda_2^*)} < 0.$$

Hence, function $b_1'(\phi(\lambda_2^*))$ decreases as λ_2^* goes from 0 to $+\infty$, whereas $b_2'(\lambda_2^*)$ strictly increases from $-\infty$ to $+\infty$. Therefore system (5.15)-(5.16) has the unique solution.

Then, the second derivatives of B are

$$\begin{aligned} B_{11} &= \frac{\lambda_2^*}{\lambda_1^* + \lambda_2^*} b_1''(\lambda_1^*) - \frac{2\lambda_2^*}{(\lambda_1^* + \lambda_2^*)^3} \Phi_1(\lambda_1^*, \lambda_2^*), \\ B_{22} &= \frac{\lambda_1^*}{\lambda_1^* + \lambda_2^*} b_2''(\lambda_2^*) - \frac{2\lambda_1^*}{(\lambda_1^* + \lambda_2^*)^3} \Phi_2(\lambda_1^*, \lambda_2^*), \\ B_{12} = B_{21} &= \frac{\lambda_1^*}{(\lambda_1^* + \lambda_2^*)^3} \Phi_1(\lambda_1^*, \lambda_2^*) + \frac{\lambda_2^*}{(\lambda_1^* + \lambda_2^*)^3} \Phi_2(\lambda_1^*, \lambda_2^*). \end{aligned}$$

Here $B_{ij} = \dfrac{\partial^2 B}{\partial \lambda_i^* \partial \lambda_j^*}$, $i, j \in D = \{1, 2\}$. Hence, point $(\lambda_1^*, \lambda_2^*)$, fitting for system (5.15)-(5.16), gives the minimum of B. □

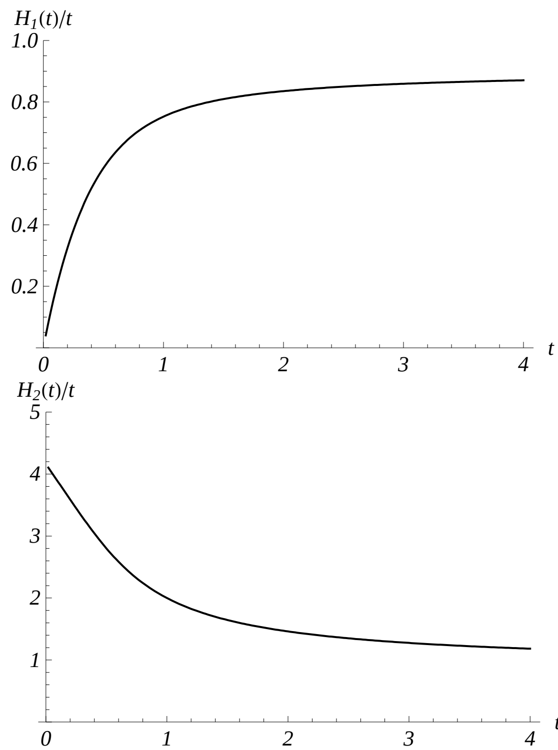

FIGURE 5.1. Plot of $H_i(t)/t$, $t > 0$, where $H_i(t)$, $i \in \{1, 2\}$, is the minimal entropy (for $\lambda_1 = \lambda_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $c_1 = -1$, $c_2 = 3$, $h_1 = 1$, $h_2 = -0.1$).

Remark 5.5. Theorem 5.4 and Proposition 5.3 show that the minimal entropy martingale measure differs from the measure supplied by Esscher transformation. This is confirmed by the plots presented in Fig. 5.1 and Fig. 5.2, where an asymmetric situation is considered. Surprisingly, the minimal entropy is supplied by λ_1^*, λ_2^* which depend on time horizon. Moreover, the entropy functions $H_i(t)$, $t > 0$, $i \in \{1, 2\}$, are not linear (cf. (5.6), Remark 5.2).

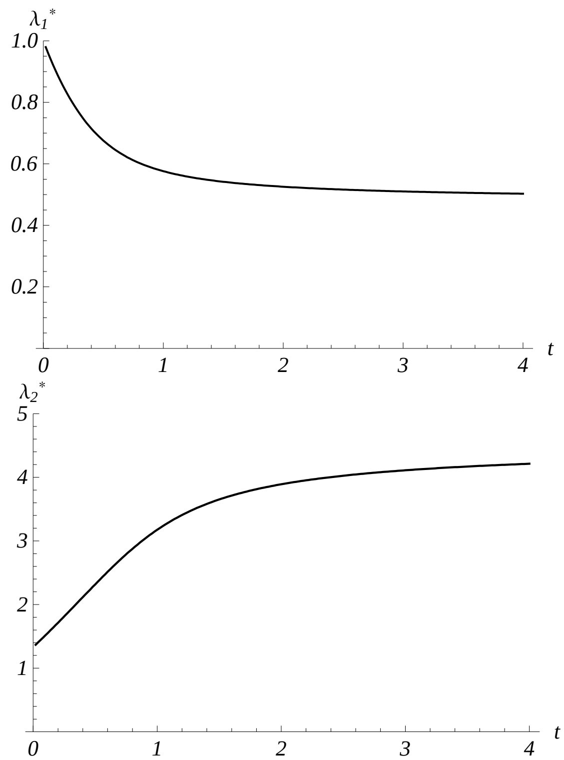

FIGURE 5.2. Plot of *argmin* $H_1(t)$, for the same case of Figure 5.1.

In Elliott et al. (2007) the Esscher transform is applied to option pricing under a Markov-modulated jump-diffusion model.

In the symmetric case,

$$\lambda_1 = \lambda_2,\ \sigma_1 = \sigma_2 \ \text{ and } \ c_1 = -c_2,\ h_1 = -h_2,\ c_1 h_2 = c_2 h_1, \tag{5.17}$$

the solution of the minimal entropy problem is constant.

Proposition 5.6. *In the symmetric case* (5.17) *the problem*

$$\begin{aligned} H_1(t;\ \lambda_1^*, \lambda_2^*) &\to \min, \\ H_2(t;\ \lambda_1^*, \lambda_2^*) &\to \min \end{aligned} \tag{5.18}$$

subject to condition (5.9) *has the unique solution* $\lambda_1^* = \lambda_2^* = \text{const}$, *which does not depend on time* t.

Proof: By Taylor representation we have

$$\begin{aligned} H_1 =& b_1 t + \lambda_1^*(b_1 - b_2)t^2 \cdot \phi\left((\lambda_1^* + \lambda_2^*)t\right), \\ H_2 =& b_2 t + \lambda_2^*(b_2 - b_1)t^2 \cdot \phi\left((\lambda_1^* + \lambda_2^*)t\right), \end{aligned}$$

where

$$\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{(n+2)!}.$$

Notice that in the symmetric case (5.17) function b_1 and b_2 (see (5.13)) are equal: $b_1(\lambda) \equiv b_2(\lambda) = b$, $\lambda > 0$. Hence, the problem (5.18) has the unique constant solution, $\lambda_1^* = \lambda_2^* = \lambda^*$, which corresponds to problem (5.10). The minimal entropy is linear, $H_1(t) = H_2(t) = bt$, where $b = b(\lambda^*)$. □

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