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An Example of non-quenched Convergence in the Conditional CLT for Discrete Fourier Transforms

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Abstract. A recent result by Barrera and Peligrad shows that the quenched central limit theorem holds for the components of the discrete Fourier transforms of a stationary process in L^2 orthogonal to the subspace of functions that are measurable with respect to the initial sigma-field. In this paper we address the question of whether the quenched CLT remains true for the Fourier transforms without taking orthogonal projections, as could be expected in view of previous, related results about the annealed convergence of the process under consideration.

We give a negative answer to this question by exhibiting an example of a process satisfying the hypothesis of Barrera and Peligrad's result for which the Fourier transforms, without substracting the respective components, do not satisfy a quenched limit theorem. The proof combines ideas from a construction due to Volný and Woodroofe with an interpretation of the results by Barrera and Peligrad for the case of linear processes, and with applications of some previous results related to discrete Fourier transforms.

1. Introduction and Background

1.1. *General Setting.* Let $(X_n)_{n\in\mathbb{Z}}$ be a strictly stationary centered, ergodic sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This is: $X_n = X_0 \circ T^n$, where $T : \Omega \to \Omega$ is an ergodic, bimeasurable, invertible transformation and $EX_0 = 0$. Assume that $X_0 \in L^2_{\mathbb{P}}$ is $\mathcal{F}_0-$measurable, where $\mathcal{F}_0 \subset \mathcal{F}$ is a sigma algebra satisfying $\mathcal{F}_0 \subset T^{-1}\mathcal{F}_0$. Define $\mathcal{F}_n := T^{-n}\mathcal{F}_0$ for all $n \in \mathbb{Z}$ and $\mathcal{F}_{-\infty} := \cap_{n\in\mathbb{Z}}\mathcal{F}_n$, and denote by E_n the conditional expectation with respect to $\mathcal{F}_n$. So $E_nZ := E[Z|\mathcal{F}_n]$ for every integrable random variable Z.

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Define, for every $\theta \in [0, 2\pi)$ the *$n-th$ discrete Fourier transform of $(X_k(\omega))_k$ at* θ by

$$S_n(\theta, \omega) := \sum_{k=0}^{n-1} e^{ik\theta} X_k(\omega). \tag{1.1}$$

When $\theta \in (0, 2\pi)$ is fixed, we will denote by $S_n(\theta)$ the random variable $S_n(\theta, \cdot)$. In the special case $\theta = 0$ we denote by S_n the random variable $S_n(0, \cdot)$. So $S_n(\omega) := \sum_{k=0}^{n-1} X_n(\omega)$.

Assume also that E_0 is regular. This is, that there exists a family of measures $\{\mathbb{P}_\omega\}_{\omega\in\Omega}$ such that for every integrable function X,

$$\omega \mapsto \int_\Omega X(z) d\mathbb{P}_\omega(z) \tag{1.2}$$

defines a version of $E_0 X$.

Finally, denote by λ the Lebesgue measure on $[0, 2\pi)$ (or any other Borelian in $\mathbb{R}$) with the borelian sigma algegra $\mathcal{B}$.

1.2. *Quenched Convergence.* In the context of section 1.1, and given a distribution function F_Y (associated to some random variable Y defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$), we say that the process $(Y_n)_n$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ *converges to Y in the quenched sense*, denoted here by $Y_n \Rightarrow_Q Y$, if for almost every ω, and every continuous and bounded function f, $E^\omega f(Y_n) \to_{n\to\infty} Ef(Y)$, where E^ω denotes integration with respect to $\mathbb{P}_\omega$ and

$$Ef(Y) := \int_{\mathbb{R}} f(y)\, dF_Y(y) = \int_{\Omega'} f \circ Y(\omega')\, d\mathbb{P}'(\omega').$$

In other words, we require that $E_0 f(Y_n) \to Ef(Y)$ $\mathbb{P}$-a.s. (over a set not depending of f). By Portmanteau's theorem, this amounts to say that for almost every ω,

$$\mathbb{P}_\omega(Y_n \le y) \to F_Y(y)$$

at every continuity point y of F_Y.

Barrera and Peligrad, in Barrera and Peligrad (2016), proved the following quenched limit theorem:

Theorem 1.1. *There exists a set $I \subset (0, 2\pi)$ with $\lambda(I) = 2\pi$ such that, for all $\theta \in I$, the complex-valued random variables*

$$Y_n(\theta) := \frac{1}{\sqrt{n}}(S_n(\theta) - E_0 S_n(\theta)) \tag{1.3}$$

converge to a complex Gaussian random variable under $\mathbb{P}_\omega$ for all ω in a set Ω_θ with $\mathbb{P}(\Omega_\theta) = 1$. The asymptotic distribution of the real and imaginary parts corresponds to a bivariate Gaussian random variable with independent entries, each with mean zero and variance

$$\sigma^2(\theta) = \lim_{n\to\infty} \frac{E_0|S_n(\theta) - E_0 S_n(\theta)|^2}{2n}.$$

(the limit exists with probability one, and it is nonrandom).

It is worth to remark that, without additional assumptions, the methods of Barrera and Peligrad (2016) do not give a description of the elements in the set I. The martingale version of the theorem, used to approximate the general case, works provided that $e^{-2i\theta}$ is not an eigenvalue of the Koopman operator associated to T (namely $f \mapsto f \circ T$ for all $f \in L^2_{\mathbb{P}}(\Omega, \mathbb{C})$), and therefore we consider these as exceptional values. A consideration of the classical case $\theta = 0$ shows that more hypotheses may be needed to guarantee the convergence in distribution of $Y_n(\theta)$ outside of I.

Observation 1. Note that quenched convergence implies convergence in distribution ("annealed" convergence) by the dominated convergence theorem: for every uniformly continuous and bounded function f,

$$\int_\Omega f(Y_n)\, d\mathbb{P}(\omega) = \int_\Omega E_0 f(Y_n)(\omega)\, d\mathbb{P}(\omega) \to_{n\to\infty} \int_\Omega Ef(Y)\, d\mathbb{P}(\omega) = Ef(Y).$$

In particular, Theorem 1.1 relates to some previous results about annealed convergence, see for instance Peligrad and Wu (2010) and the references therein.

1.2.1. *Possible limit Laws for a given initial point.* Suppose that we know of an integrable process $(Y_n)_n$ that $Y_n - E_0 Y_n \Rightarrow_Q Y$, say $Y_n - E_0 Y_n \Rightarrow Y$ under $\mathbb{P}_\omega$ for $\omega \in \Omega_0$, where $\mathbb{P}(\Omega_0) = 1$. What are the possible limit laws for $(Y_n)_n$ under $\mathbb{P}_\omega$, for a fixed ω?

To answer this question we depart from the following result (see the proof of Lemma 18 in Barrera and Peligrad (2016)):

Lemma 1.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\mathcal{F}_0 \subset \mathcal{F}$ is a $\sigma-$algebra for which $E_0 := E[\,\cdot\,|\mathcal{F}_0]$ admits a regular version in the sense explained above (see (1.2)), $X : \Omega \to \mathbb{C}$ is Borel measurable and $Z : \Omega \to \mathbb{C}$ is $\mathcal{F}_0-$measurable, then there exists $\Omega_Z \subset \Omega$ with $\mathbb{P}(\Omega_Z) = 1$ such that, for every $g : \mathbb{C}^2 \to \mathbb{R}$ continuous and bounded*

$$E^\omega[g(X, Z(\omega))] = E^\omega[g(X, Z)] \tag{1.4}$$

for all $\omega \in \Omega_Z$.

This lemma, in combination with Proposition 4.6 and Proposition 4.8 in the appendix, gives the following result.

Proposition 1.3 (Possible Limit Laws for a Fixed Starting Point.)**.** *With the notation of Lemma 1.2, assume that $(Y_n)_n$ is an integrable (complex-valued) process such that $Y_n - E_0 Y_n \Rightarrow Y$ under $\mathbb{P}_\omega$ for all $\omega \in \Omega_0$ ($\Omega_0 \subset \Omega$ is any given set, not even assumed measurable), and let $\Omega_1 := \cap_n \Omega_{E_0 Y_n}$ ($\Omega_{E_0 Y_n}$ is the set specified in the conclusion of Lemma 1.2). Then given $\omega \in \Omega_0 \cap \Omega_1$, $Y_n \Rightarrow Z_\omega$ under $\mathbb{P}_\omega$ if and only if $L(\omega) = \lim_{n\to\infty} E_0 Y_n(\omega)$ exists, in which case*

$$Z_\omega = Y + L(\omega)$$

(in distribution).

Proof: Given $\omega \in \Omega_0 \cap \Omega_1$ and any bounded and continuous function $g : \mathbb{C} \to \mathbb{R}$

$$E^\omega g(Y_n - E_0 Y_n(\omega)) = E^\omega g(Y_n - E_0 Y_n) \to_n Eg(Y),$$

so that $Y_n - E_0 Y_n(\omega) \Rightarrow Y$ under $\mathbb{P}_\omega$. From $Y_n = Y_n - E_0 Y_n(\omega) + E_0 Y_n(\omega)$ the conclusion follows via Proposition 4.6 if Y is not constant and via Proposition 4.8 if Y is constant (see also Remark 4.9 after those propositions). □

2. The Question

With the notation introduced in section 1.1 let us define, for every $\theta \in [0, 2\pi)$,

$$Z_n(\theta) := \frac{1}{\sqrt{n}} S_n(\theta). \tag{2.1}$$

In view of Theorem 2.1 in Peligrad and Wu (2010) and Observation 1, it is natural to ask whether the random centering "$-E_0 S_n(\theta)$" in (1.3) is necessary to obtain the quenched convergence stated in Theorem 1.1. We point out that Theorem 2.1 in Peligrad and Wu (2010) assumes, in addition to the hypotheses in Theorem 1.1, that $(X_n)_n$ is a *regular process*, namely

$$E[X_0|\mathcal{F}_{-\infty}] = 0. \tag{2.2}$$

The first observation in the direction of an answer is given by the following lemma.

Lemma 2.1. *With the notation of Theorem 1.1, fix $\theta \in (0, 2\pi)$ (θ may or may not be in I), assume that $Y_n(\theta) \Rightarrow_Q Y_\theta$ for some Y_θ, and let $Z_n(\theta)$ be given by (2.1). Then $Z_n(\theta) \Rightarrow_Q Y_\theta$ if and only if $E_0 S_n(\theta) = o(\sqrt{n})$ almost surely.*

Proof: *Sufficiency.* First note that for any set $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \int_A \mathbb{P}_\omega(A) d\,\mathbb{P}(\omega).$$

Applying this observation to the complement of $[E_0 S_n(\theta) = o(\sqrt{n})]$ we see that if $E_0 S_n(\theta) = o(\sqrt{n})$ $\mathbb{P}-$a.s. then $E_0 S_n(\theta) = o(\sqrt{n})$ $\mathbb{P}_\omega-$a.s. for $\mathbb{P}-$a.e. ω, and therefore the asymptotic distributions of $Y_n(\theta)$ and $Z_n(\theta)$ (if any) must be the same under $\mathbb{P}_\omega$ for $\mathbb{P}-$ a.e ω. This proves sufficiency.

Necessity. We appeal to Proposition 1.3 by taking, in place of Ω_0,

$$\Omega_\theta := \{\omega \in \Omega : Y_n(\theta) \Rightarrow Y_\theta \quad \textit{under } \mathbb{P}_\omega\}$$

and by replacing Y_n by $Y_n(\theta)$. This gives that $Y_\theta = Y_\theta + \lim_n (E_0 S_n(\theta, \omega))/\sqrt{n}$ (in distribution) for $\omega \in \Omega_{1,\theta} := \Omega_1 \cap \Omega_\theta$ where $\Omega_1 := \cap_n \Omega_{E_0 Y_n(\theta)}$ (see the notation in Proposition 1.3). It follows that

$$\lim_n \frac{E_0 S_n(\theta, \omega)}{\sqrt{n}} = 0 \quad \textit{for all } \omega \in \Omega_{1,\theta}$$

□

Thus if we can provide an example of a regular process $(X_n)_n$ satisfying the hypothesis of Theorem 1.1 for which

$$\mathbb{P}\left(\limsup_{n\to\infty} \left|\frac{1}{\sqrt{n}} E_0 S_n(\theta)\right| > 0\right) > 0 \quad \textit{for } \theta \textit{ in a set } I' \textit{ with } \lambda(I') > 0 \tag{2.3}$$

we will prove, in particular, the necessity of random centering for a nonnegligible subset of I (namely $I \cap I'$).

In their paper Volný and Woodroofe (2010), Volny and Woodroofe provide an example of a sequence $(X_n)_n$ for which a quenched CLT holds for $(Y_n(0))_n$ but not for $(Z_n(0))_n$. In this paper, we adapt their construction to give an example satisfying (2.3) with $I' = [0, 2\pi)$. The random centering "$-E_0 S_n(\theta)$" is therefore a necessary condition for the conclusion of Theorem 1.1 to hold.

The main novelty adapting the example in Volný and Woodroofe (2010), which arises from a careful construction of a sequence $(a_n)_n$ of nonnegative coefficients of a linear process is that, in order to guarantee the validity of the "inductive step" defining a_{n+1} from $a_1, \ldots, a_n$, one needs to prove that a certain type of convergence is uniform in θ (see Lemma 3.2 below). While it would be sufficient to prove this uniform convergence for θ in (for instance) an open subinterval I' of $[0, 2\pi)$ in order to construct a valid example, a compactness argument allows us to do it for $I' = [0, 2\pi)$.

Concretely, we will give an example of a regular process $(X_n)_n$ for which

$$\mathbb{P}\left(\limsup_{n\to\infty}\left|\frac{1}{\sqrt{n}}E_0 S_n(\theta)\right| = \infty\right) = 1 \quad \textit{for all } \theta \in [0, 2\pi). \tag{2.4}$$

For this process, Proposition 1.3 shows that the "random centering" not only is necessary to obtain the asymptotic limit specified in Theorem 1.1, but that $S_n(\theta)/\sqrt{n}$ cannot admit an asymptotic limit whatsoever under $\mathbb{P}_\omega$ for $\mathbb{P}$-a.e ω.

The rest of the paper is presented as follows: in Section 3 we specialize our study to the case in which $(X_k)_{k\in\mathbb{Z}}$ is a linear process generated by convolution of a sequence of i.i.d random variables and a sequence in $l^2(\mathbb{N})$. For this family of processes, we give an interpretation of the results above in terms of convergence of Fourier series of perodic functions (Proposition 3.1), and introduce a result necessary to construct the example mentioned above (Lemma 3.2). In Section 4 we present the construction itself.

3. The Fourier Transforms of a Linear Process

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $l^2(\mathbb{N})$ (namely $\sum_n a_n^2 < \infty$), and let $(\xi_k)_{k\in\mathbb{Z}}$ be an i.i.d. sequence of centered, square integrable random variables defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. The *linear process* $(X_k)_{k\in\mathbb{Z}}$ generated by (a_k) and (ξ_k) is defined by

$$X_k := \sum_{j\in\mathbb{N}} a_j\, \xi_{k-j}. \tag{3.1}$$

The orthogonality of $(\xi_k)_k$ shows that X_k is well defined as en element of $L^2_{\mathbb{P}'}$, and that $EX_k^2 = ||\xi_0||_2^2 \sum_j a_j^2$.

We can regard $\Omega = \mathbb{R}^{\mathbb{Z}}$ as a probability space whose sigma algebra is the product sigma algebra and whose probability measure is $\mathbb{P} = \mathbb{P}'\xi^{-1}$, where $\xi : \Omega' \to \Omega$ is given by $\xi(\omega') := (\xi_j(\omega'))_{j\in\mathbb{Z}}$. It is well known that, with this structure (because $(\xi_k)_k$ is i.i.d.), the left shift $T : \Omega \to \Omega$ characterized by $x_k \circ T = x_{k+1}$, where $x_j : S \to \mathbb{R}$ is the projection on the $j-$th coordinate, is weakly mixing and therefore also ergodic.

Note that the sequence of coordinate functions $(x_j)_j$ is a "copy" of the sequence $(\xi_j)_j$: both are sequences of i.i.d functions with the same common distribution. In particular, X_k can be regarded as the function $X_k : \Omega \to \mathbb{R}$ given by $X_k(\omega) = X_k((x_j(\omega))_j) := \sum_j a_j x_{k-j}(\omega)$. Via this identification we can think of ξ_j as defined on $(\Omega, \mathcal{F}, \mathbb{P})$, which we will do for simplicity.

Clearly, $X_k = X_0 \circ T^k$. In this way $(X_k)_{k\in\mathbb{Z}}$ can be interpreted as a strictly stationary, centered, and ergodic sequence in $L^2_{\mathbb{P}}$.

Define $\mathcal{F}_n := \sigma((x_k)_{k\le n})$ for all $n \in \mathbb{Z}$ and let $\mathbb{P}_\omega$ be the measure corresponding to "partial integration with respect to the future". This is: given $\omega_0 \in \Omega$, $\mathbb{P}_{\omega_0} = \mathbb{P}\pi_{\omega_0}^{-1}$ where $\pi_{\omega_0} : S \to S$ is given by

$$x_k(\pi_{\omega_0}(\omega)) = \begin{cases} x_k(\omega_0) & \text{if } k \le 0 \\ x_k(\omega) & \text{if } k > 0 \end{cases}$$

This brings us to the setting introduced in Section 1.1 ($X_0 \in L^2_{\mathbb{P}}$ is $\mathcal{F}_0-$measurable, $\mathcal{F}_n = T^{-n}\mathcal{F}_0$ and so on). By Kolmogorov's zero-one law, $(X_n)_n$ is a regular process. This is: (2.2) is satisfied.

Note also that, since T is weakly mixing, the only eigenvalue of the Koopman operator associated to T is $\lambda_0 = 1$ (see for instance Petersen (1989), p.65).

Coefficients of the Fourier Transforms. Since $(a_n)_n \in l^2(\mathbb{N})$, Carleson's theorem (Carleson (1966)) guarantees the convergence a.s., with respect to the Lebesgue measure λ, of the (equivalence class of $\mathcal{B}$-measurable) function(s) $[0, 2\pi) \to \mathbb{C}$ defined by the series

$$\theta \mapsto f(\theta) = \sum_{j\ge 0} a_j e^{ij\theta} \tag{3.2}$$

and $f(\theta)$, thus defined, is a $2\pi-$periodic function, square integrable over $[0, 2\pi)$, and satisfying $\hat{f}(n) = a_n$, where $\hat{f}$ denotes the Fourier transform

$$\hat{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ixy} f(y)\, d\lambda(y).$$

Denote by $f_k(\theta) := \sum_{j=0}^{k-1} a_j e^{ij\theta}$ ($f_k = 0$ if $k \le 0$). Then, for the process $(X_n)_n$ given by (3.1), we have the following two expressions for $S_n(\theta)$:

$$S_n(\theta) = \sum_{j=-\infty}^{n-1} (f_{-j+n} - f_{-j})(\theta)\xi_j e^{ij\theta}, \tag{3.3}$$

$$S_n(\theta) = \sum_{k=0}^{\infty} a_k \sum_{j=0}^{n-1} e^{ij\theta}\xi_{j-k}. \tag{3.4}$$

Now let us denote, for all $k \in \mathbb{Z}$,

$$\zeta_{-k}(\theta) := \sum_{j=0}^{k} e^{-ij\theta}\xi_{-j} \tag{3.5}$$

(note that $\zeta_{-k} = 0$ if $k < 0$). Then from (3.3) and (3.4) the following two equalities follow respectively:

$$E_0 S_n(\theta) = \sum_{j\le 0} \xi_j (f_{-j+n} - f_{-j})(\theta) e^{ij\theta}, \tag{3.6}$$

$$E_0 S_n(\theta) = \sum_{j\ge 0} a_j (\zeta_{-j} - \zeta_{-j+n})(\theta) e^{ij\theta}. \tag{3.7}$$

In particular

$$E_0|S_n(\theta) - E_0 S_n(\theta)|^2 = E_0|\sum_{j=1}^{n-1} e^{ij\theta}\xi_j f_{-j+n}(\theta)|^2 =$$

$$||\xi_0||_2^2 \sum_{j=1}^{n-1} |f_{n-j}(\theta)|^2 = ||\xi_0||_2^2 \sum_{j=1}^{n-1} |f_j(\theta)|^2,$$

so that

$$\lim_{n\to\infty} \frac{E_0|S_n(\theta) - E_0 S_n(\theta)|^2}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} ||\xi_0||_2^2 |\, f_j(\theta)|^2 = |\,||\xi_0||_2 f(\theta)|^2.$$

Using this, we get the following version of Theorem 1.1.

Proposition 3.1. *For a linear process (3.1) and almost every $\theta \in (0, 2\pi)$, (1.3) is asymptotically normally distributed under $\mathbb{P}_\omega$, for $\mathbb{P}$-almost every ω, with independent real and imaginary parts, each with mean zero and variance*

$$\sigma_\theta^2 = \frac{|\,||\xi_0||_2 f(\theta)|^2}{2},$$

where f is given by (3.2).

By Cuny and Peligrad (2013), p.4075 (Section 4.1) applied to the sequence $(\delta_{1j})_{j\in\mathbb{Z}}$ (δ_{ij} denotes the Kronecker $\delta-$function) and the fact that T is weakly mixing, the following law of the iterated logarithm holds[1]: for every $t \in (0, 2\pi) \setminus \{\pi\}$

$$\limsup_{n\to\infty} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n \log\log n}} = ||\xi_0||_2. \tag{3.8}$$

$\mathbb{P}-$almost surely.

If $\theta = 0$ or $\theta = \pi$, the L.I.L. as stated above holds with $||\xi_0||_2$ replaced by $\sqrt{2}\,||\xi_0||_2$ (note also that the processes $(\zeta_n(0))_n$ and $(\zeta_n(\pi))_n$ are real-valued).

The equality (3.8) clearly implies that $\limsup_n \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} = \infty$ $\mathbb{P}$-a.s. The following lemma states that the divergence occurs "at comparable speeds" for every θ.

Lemma 3.2. *Consider ζ_{-k} as defined by (3.5). Then for every $\lambda \in \mathbb{R}$ and every $0 < \eta \le 1$ there exists $N \in \mathbb{N}$ satisfying*

$$\mathbb{P}\left(\max_{1\le n\le N} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} > \lambda\right) \ge 1 - \eta$$

for all $\theta \in [0, 2\pi)$. In particular

$$\mathbb{P}\left(\max_{1\le n\le m} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} \ge \lambda\right) \ge 1 - \eta$$

for all $m \ge N$.

[1]Note that, for the linear process $(\xi_n)_{n\in\mathbb{Z}}$, corresponding to convolution with $(\delta_{1j})_{j\in\mathbb{Z}}$, the spectral density with respect to Lebesgue measure is the constant function $||\xi_0||_2^2/2\pi$.

Proof: Fix $\lambda \in \mathbb{R}$ and $0 < \eta \leq 1$. Let[2] $\theta \in [0, 2\pi]$ and $\epsilon > 0$ be given and define

$$E_{\epsilon,m}(\theta) := \left[\inf_{|\delta|<\epsilon} \left\{ \max_{1\leq n\leq m} \frac{|\zeta_{-n}(\theta+\delta)|}{\sqrt{n}} \right\} > \lambda \right]$$

and

$$E_m(\theta) := \left[\max_{1\leq n\leq m} \frac{|\zeta_{-n}(\theta)|}{\sqrt{n}} > \lambda \right].$$

Note that, for fixed m, the sequence of sets $E_{\epsilon,m}(\theta)$ is decreasing with respect to ϵ ($\epsilon_1 < \epsilon_2$ implies that $E_{\epsilon_2,m}(\theta) \subset E_{\epsilon_1,m}(\theta)$), and that the (random) function $\theta \mapsto \max_{1\leq n\leq m} |\zeta_{-n}(\theta)|/\sqrt{n}$ is continuous for all m. In particular

$$\bigcup_{\epsilon>0} E_{\epsilon,m}(\theta) = E_m(\theta), \tag{3.9}$$

where the union is increasing over ϵ decreasing to 0.

Now, there exists a minimal $N(\theta)$ such that $\mathbb{P}(E_{N(\theta)}(\theta)) > 1 - \eta$. To see this note that the family $\{E_k(\theta)\}_{k\geq 0}$ is increasing with k, and its union contains the set

$$[\limsup_n |\zeta_{-n}(\theta)|/\sqrt{n} > \lambda]$$

which has $\mathbb{P}-$measure 1 by (3.8).

It follows from (3.9) that there exists an ϵ_θ such that

$$\mathbb{P}(E_{\epsilon_\theta,N(\theta)}(\theta)) > 1 - \eta\,. \tag{3.10}$$

Now, the family of sets $\{(\theta - \epsilon_\theta, \theta + \epsilon_\theta)\}_{\theta\in[0,2\pi]}$ is an open cover of $[0, 2\pi]$, and therefore it admits an open subcover $\{(\theta_j - \epsilon_j, \theta_j + \epsilon_j)\}_{j=1}^r$ where $\epsilon_j := \epsilon_{\theta_j}$.

Let $N = \max\{N(\theta_1), \ldots, N(\theta_r)\}$. We claim that, for every $\theta \in [0, 2\pi]$

$$\mathbb{P}(E_N(\theta)) > 1 - \eta\,.$$

Indeed, given $\theta \in [0, 2\pi]$, with $\theta_j - \epsilon_j < \theta < \theta_j + \epsilon_j$,

$$E_N(\theta) \supset E_{N(\theta_j)}(\theta) = \left[\max_{1\leq n\leq N(\theta_j)} \frac{|\zeta_{-n}(\theta_j + \theta - \theta_j)|}{\sqrt{n}} > \lambda \right] \supset E_{\epsilon_j,N(\theta_j)}(\theta_j)\,,$$

and the conclusion follows from (3.10) and the definition of $E_N(\theta)$. ☐

4. The Example

We now proceed to prove, by an explicit construction, the following theorem:

Theorem 4.1. *There exists a square summable sequence $(a_n)_n$ such that, if $(X_n)_n$ is defined by (3.1) and $S_n(\theta)$ is defined by (1.1), then*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} = \infty \right) = 1$$

for all $\theta \in [0, 2\pi)$.

[2] We work over the invetrval $[0, 2\pi]$ (instead of $[0, 2\pi)$). This has no effect for the validity of the conclusion and is assumed in order to take advantage of compactness, as will be clear along the proof.

Before giving the proof we depart from the following observation: if $(n_k)_{k\geq 0}$ is a strictly increasing sequence of natural numbers and if $(a_j)_j$ is square summable and satisfies $a_j = 0$ if $j \notin \{n_k\}_k$ then, using (3.7) we have, for every given $k \in \mathbb{N}$,

$$E_0 S_n(\theta) = \sum_{j=0}^{k} e^{in_j\theta} a_{n_j} (\zeta_{-n_j} - \zeta_{-n_j+n})(\theta) + \sum_{j=k+1}^{\infty} e^{in_j\theta} a_{n_j} (\zeta_{-n_j} - \zeta_{-n_j+n})(\theta) =: A_k(n,\theta) + B_k(n,\theta) \tag{4.1}$$

so that

$$\begin{aligned} &\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k\right) \geq \\ &\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} \frac{|A_k(n,\theta)|}{\sqrt{n}} \geq 2^{k+1}\right) - \mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} \frac{|B_k(n,\theta)|}{\sqrt{n}} \geq 2^k\right) \geq \\ &\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} \frac{|A_k(n,\theta)|}{\sqrt{n}} \geq 2^{k+1}\right) - \mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} |B_k(n,\theta)| \geq 2^k\right). \end{aligned} \tag{4.2}$$

Now, if $n_{k-1} < n \leq n_k$ then, actually

$$A_k(n,\theta) = \sum_{j=0}^{k-1} e^{in_j\theta} a_{n_j} \zeta_{-n_j}(\theta) + e^{in_k} a_{n_k} (\zeta_{-n_k} - \zeta_{-n_k+n})(\theta).$$

The first summand at the right hand side in this expression is bounded by

$$\sum_{j=1}^{k-1} \sum_{r=0}^{n_j} |a_{n_j}||\xi_{-r}|$$

and therefore there exists $\lambda_k > 0$ such that

$$\mathbb{P}\left(\left|\sum_{j=0}^{k-1} e^{in_j\theta} a_{n_j} \zeta_{-n_j}(\theta)\right| > \lambda_k\right) \leq \left(\frac{1}{2}\right)^{k+2} \tag{4.3}$$

for all $\theta \in [0, 2\pi]$.

All together (4.1), (4.2) and (4.3) give the following result.

Lemma 4.2. *Let $(n_k)_k$ be a strictly increasing sequence of natural numbers and let $(a_j)_j$ be a square summable sequence with $a_j = 0$ for $j \notin \{n_k\}_k$. Then for every sequence of real numbers $(\lambda_k)_k$ satisfying (4.3) the following inequality holds*

$$\begin{aligned} \mathbb{P}&\left(\max_{n_{k-1}<n\leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k\right) \\ &\geq \mathbb{P}\left(a_{n_k} \max_{n_{k-1}<n\leq n_k} \frac{|(\zeta_{-n_k} - \zeta_{-n_k+n})(\theta)|}{\sqrt{n}} \geq \lambda_k + 2^{k+1}\right) \\ &\quad - \mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} |B_k(n,\theta)| \geq 2^k\right) - \left(\frac{1}{2}\right)^{k+2} \end{aligned} \tag{4.4}$$

for all $\theta \in [0, 2\pi]$.

This completes the set of pieces to construct the example stated by Theorem 4.1.

Proof of Theorem 4.1: Following Volný and Woodroofe (2010), assume that $||\xi_0||_2 = 1$ and let $(n_j)_{j\geq 0}$, $(a_j)_{j\geq 0}$, and $(\lambda_j)_{j\geq 0}$ be defined inductively as follows: $n_0 = 1$, $\lambda_0 = 0$, $a_0 = 0$, $a_1 = \frac{1}{2}$, and given $n_0, \cdots, n_{k-1}$, $a_0, \dots, a_{n_{k-1}}$ and $\lambda_0, \dots, \lambda_{k-1}$, let λ_k be such that

$$\mathbb{P}\left(\left|\sum_{j=1}^{k-1} a_{n_j} e^{in_j\theta}\zeta_{-n_j}(\theta)\right| > \lambda_k\right) \leq \left(\frac{1}{2}\right)^{k+2},$$

(see(4.3)) and let $n_k > n_{k-1}$ be such that

$$\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} \frac{|(\zeta_{-n_{k-1}} - \zeta_{-n_{k-1}+n})(\theta)|}{\sqrt{n}} \geq \frac{\lambda_k + 2^{k+1}}{a_{n_{k-1}}}\right) \geq 1 - \left(\frac{1}{2}\right)^{k+1} \tag{4.5}$$

for all $\theta \in [0, 2\pi]$. The choice of n_k is possible according to Lemma 3.2 ($|(\zeta_{-n_{k-1}} - \zeta_{-n_{k-1}+n})(\theta)|$ and $|\zeta_{-n}(\theta)|$ have the same distribution). Then define $a_{n_k} = \frac{1}{2^k\sqrt{n_{k-1}}}$ and $a_j = 0$ for $n_{k-1} < j < n_k - 1$.

The sequences $(a_j)_{j\geq 0}$ and $(\lambda_k)_k$, thus defined, satisfy the hypotheses of Lemma 4.2 and therefore, by the estimates (4.4) and (4.5),

$$\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} \frac{|E_0 S_n(\theta)|}{\sqrt{n}} \geq 2^k\right) \geq 1 - \left(\frac{1}{2}\right)^{k+2} - \mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} |B_k(n,\theta)| \geq 2^k\right)$$

for all $\theta \in [0, 2\pi]$.

We claim that, under the present conditions,

$$\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k} |B_k(n,\theta)| \geq 2^k\right) \leq \left(\frac{1}{2}\right)^{k+2} \tag{4.6}$$

for $k \geq 3$.

Fix $k \geq 3$. First we recall the following (Doob's) maximal inequality for martingales (see for instance Revuz and Yor (1999), p.53): if $(M_n)_n$ is a L^p submartingale (namely $||M_n||_p := (E|M_n|^p)^{1/p} < \infty$ for all n) for some $p > 1$ then

$$||\sup_{k\leq n} M_k||_p \leq \frac{p}{p-1}||M_n||_p\,. \tag{4.7}$$

Now, for fixed θ, $(|\zeta_{-n}(\theta)|)_{n\geq 0}$ is an L^2 submartingale (with respect to $(\mathcal{G}_n)_n$, where $\mathcal{G}_k = \sigma((\xi_{-j})_{j\leq k})$) and therefore, by Doob's maximal inequality (4.7):

$$E\left(\max_{k\leq n}|\zeta_{-k}(\theta)|\right) \leq \left|\left|\max_{k\leq n}|\zeta_{-k}(\theta)|\right|\right|_2 \leq 2\,||\zeta_{-n}(\theta)||_2 = 2\sqrt{n+1}.$$

This gives

$$E\left(\max_{n_{k-1}<n\leq n_k} |B_k(n,\theta)|\right) \leq \sum_{j=k+1}^{\infty} a_{n_j} E\left(\max_{k\leq n_k - n_{k-1}} |\zeta_{-k}(\theta)|\right) \leq$$

$$\sum_{j=k+1}^{\infty} \frac{1}{2^{j-1}}\sqrt{\frac{n_k - n_{k-1} + 1}{n_{j-1}}} \leq \frac{1}{2^{k-1}},$$

and therefore, by Markov's inequality

$$\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k}|B_k(n,\theta)|\geq 2^k\right)\leq \frac{1}{2^{2^{k-1}}}\leq\left(\frac{1}{2}\right)^{k+2}$$

as claimed.

To finish the proof we observe that a combination of (4.4), (4.5) and (4.6) gives, under the present choices of $(a_k)_k$ and $(n_k)_k$, that

$$\mathbb{P}\left(\max_{n_{k-1}<n\leq n_k}\frac{|E_0S_n(\theta)|}{\sqrt{n}}<2^k\right)\leq\left(\frac{1}{2}\right)^{k+1}$$

so that, by the first Borel-Cantelli Lemma

$$\max_{n_{k-1}<n\leq n_k}\frac{|E_0S_n(\theta)|}{\sqrt{n}}\geq 2^k \textit{ except for finitely many } k\textit{'s},$$

$\mathbb{P}-$a.s. This clearly implies that $\limsup_n |E_0S_n(\theta)|/\sqrt{n}=\infty$ $\mathbb{P}-$a.s. □

Appendix: convergence of Types

In this section we present some facts about Convergence of Types in a form that is convenient for the proofs given along this paper.

A distribution function F is *non-degenerate* if it is not the indicator function of some interval $[a,+\infty)$. This is, if it does not correspond to a constant random variable. We recall the following *Convergence of Types* theorem (Billingsley (1995), Theorem 14.2).

Lemma 4.3. *Let F_n, F and G be distribution functions, and a_n, u_n, b_n, v_n be constants with $a_n>0$, $u_n>0$. If F, G are nondegenerate, $F_n(a_nx+b_n)\Rightarrow F(x)$, and $F_n(u_nx+v_n)\Rightarrow G(x)$ then there exist $a=\lim_n a_n/u_n$, $b=\lim_n(b_n-v_n)/u_n$ and $G(x)=F(ax+b)$.*

Note that, necessarily, $a>0$ (as otherwise G would be constant).

We will translate this statement to a statement about convergence of stochastic processes (with a restricted choice of u_n, v_n, see Proposition 4.6 below). To begin with, we remind the following elementary facts, here $\to_P$ denotes convergence in probability.

(1) If a is constant then $U_n\Rightarrow a$ if and only if $U_n\to_P a$.
(2) If $U_n\Rightarrow W$ and $V_n\to_P 0$ then $U_n+V_n\Rightarrow W$.
(3) If $(a_n)_n$ is a sequence of constant functions then $a_n\Rightarrow A$ if and only if $a=\lim_n a_n$ exists (and therefore $A=a$ a.s.).

Lemma 4.4. *If $Y_k\Rightarrow Y$ and $\{c_k\}_k\subset\mathbb{R}$ are such that $Y_k+c_k\Rightarrow 0$, then $Y=-\lim_k c_k$. In particular, Y is a constant function.*

Proof: Note that $c_k=-Y_k+(Y_k+c_k)\Rightarrow -Y$ because $Y_k+c_k\Rightarrow 0$. Now use 3. above. □

Corollary 4.5. *If X is not constant, $X_n\Rightarrow X$, and a_n, b_n are such that $a_nX_n+b_n\Rightarrow 0$, then $a_n\to 0$ and $b_n\to 0$.*

Proof: If $0 < a := \limsup_n a_n \le \infty$ and $a_{n_k} \to_{k\to\infty} a$ with $a_{n_k} > 0$, then applying Lemma 4.4 with $Y_k = X_{n_k}$ and $c_k = b_{n_k}/a_{n_k}$ we conclude that X is constant. This proves that, necessarily, $\limsup_n a_n \le 0$. A similar argument shows that $\liminf_n a_n \ge 0$, and therefore $\lim_n a_n = 0$.

The fact that $b_n \to 0$ follows from here applying Lemma 4.4 again, because $a_n X_n \Rightarrow 0$. □

These results give rise to the following proposition

Proposition 4.6. *If X is not constant, $X_n \Rightarrow X$ and $a_n > 0$, b_n are such that $a_n X_n + b_n \Rightarrow Y$, then there exists $a = \lim_n a_n$, $b = \lim_n b_n$ and, therefore, $Y = aX + b$ (in distribution).*

Proof: If Y is constant then, from $a_n X_n + b_n - Y \Rightarrow 0$ (see 1. above) it follows, via Corollary 4.5, that $\lim_n a_n = 0$ and $\lim_n b_n = Y$.

If Y is not constant we apply Lemma 4.3 with F_n, F, and G the distribution functions of X_n, X and Y respectively, and with $u_n = 1$, $v_n = 0$. □

Remark 4.7. Taking $X_n = 1$ (the constant function), $a_n = n$, and $b_n = -n$, we see that the given restriction on X (to be non-constant) is necessary.

We finish this appendix with a lemma covering the asymptotically degenerate case.

Proposition 4.8. *If X is constant, $X_k \Rightarrow X$, and $X_k + c_k \Rightarrow Z$ then $c = \lim_k c_k$ exists and $Z = X + c$.*

Proof: Use $X + c_k = (X - X_k) + X_k + c_k \Rightarrow Z$ by 2. above. The conclusion follows from 3. □

Remark 4.9. Propositions 4.6 and 4.8 remain valid if the processes involved are complex-valued, provided that the constants $(a_n)_n$ in Proposition 4.6 are still real and positive (all the other constants can be assumed complex). To see this for Proposition 4.6 note that if X_n, X are complex valued, $X_n \Rightarrow X$, and $\mathbf{u} \in \mathbb{R}^2$ is any vector then, by the mapping theorem

$$\mathbf{u} \cdot (a_n X_n + b_n) = a_n(\mathbf{u} \cdot X_n) + \mathbf{u} \cdot b_n \Rightarrow \mathbf{u} \cdot Y$$

so that, by the real valued case just proved, there exists $a = \lim_n a_n$ and $b_{\mathbf{u}} = \lim_n \mathbf{u} \cdot b_n$. Since $\mathbf{u}$ is arbitrary, there actually exists $b = \lim_n b_n$. The second conclusion ($Y = aX + b$ in distribution) follows at once from the Cramer-Wold theorem. The argument for Proposition 4.8 is similar.

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