ALEA, Lat. Am. J. Probab. Math. Stat. 12, 1001–1029 (2015)

On level and collision sets of some Feller processes

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Abstract. This paper is about lower and upper bounds for the Hausdorff dimension of the level and collision sets for a class of Feller processes. Our approach is motivated by analogous results for Lévy processes by Hawkes (1974) (for level sets) and Taylor (1966) and Jain and Pruitt (1969) (for collision sets). Since Feller processes lack independent or stationary increments, the methods developed for Lévy processes cannot be used in a straightforward manner. Under the assumption that the Feller process possesses a transition probability density, which admits lower and upper bounds of a certain type, we derive sufficient conditions for regularity and non-polarity of points; together with suitable time changes this allows us to get upper and lower bounds for the Hausdorff dimension.

1. Introduction

In this paper, we study the Hausdorff dimension of the level and collision sets for a certain class of strong Feller processes; concrete examples were constructed in Knopova and Kulik (2015) and Knopova and Kulik (2016) under rather general assumptions, see Assumption A below. This assumption guarantees, in particular, that the process is a strong Feller process admitting a transition probability density which enjoys upper and lower estimates of "compound kernel" type, see (2.7) and (2.8).

Let us briefly describe the problems which are discussed in this paper. Let X be a (strong) Feller process with values in $\mathbb{R}^n$. Then

$$\{s : X_s(\omega) \in D\} \quad \text{for any Borel set } D \subset \mathbb{R}^n \tag{1.1}$$

Received by the editors January 12, 2015; accepted September 1st, 2015.

2010 *Mathematics Subject Classification.* Primary 60G17. Secondary: 60J75; 60J25; 28A78; 35S05.

Key words and phrases. Feller process, pseudo-differential operator, symbol, Hausdorff dimension, image sets, level sets, regular points, polar sets, Kato class.

Financial support through the Scholarship for Young Scientists 2012–2014, Ukraine (for Victoria Knopova) and DFG grant Schi 419/8-1 (for René L. Schilling) is gratefully acknowledged.

denotes a level set of X, i.e. the (random) set of times when X visits the set D.

We adapt the techniques from Hawkes (1974), see also Hawkes (1970) and Hawkes (1971), to obtain bounds on the Hausdorff dimension of such level sets. The idea used in Hawkes (1974) is based on the notion of subordination (in the sense of Bochner, i.e. a random time change by an independent increasing Lévy process), and on knowledge of the Hausdorff dimension of the range of a γ-stable subordinator T_t^γ (cf. Lemma 4.2 below).

The proof presented in Hawkes (1974) heavily relies on the fact that X is a Lévy process; a key ingredient is a criterion for the polarity of points in terms of the characteristic exponent of the Lévy process X. For general Markov processes such a result is not available, and so we need an essentially different approach. The first problem which we encounter in the investigation of the level set (1.1), is how to check that the process X a.s. enters D; in other words: when is the starting point x *regular* for D. We can overcome this problem using some abstract potential theory and the Kato class; this requires, however, upper and lower estimates for the transition density $p_t(x, y)$ of X which allows us to characterize the notion of a Kato class (with respect to $p_t(x, y)$; see Definition 3.1) and regular points for D. For d-sets this problem simplifies and, at least for certain values of d, any point in the topological boundary ∂D is regular for D. Using the structure of the estimates for $p_t(x, y)$, we can establish similar assertions on the polarity of sets and regularity of points for the subordinate (i.e. time-changed) process $X_{T_t^\gamma}$.

In Theorem 2.1 we use the indices $\gamma_{\inf}$ and $\gamma_{\sup}$—these characterize the set D "in the eyes" of the time-changed process $X_{T_t^\gamma}$—to obtain uniform upper and lower bounds on the random set $\dim\{s : X_s(\omega) \in D\}$; here D is a d-set and the process starts from a point x which belongs to the topological closure $\overline{D}$ of D. In the one-dimensional case we obtain (Proposition 2.2) the exact value of the Hausdorff dimension of the zero-level set $\{s : X_s(\omega) = 0\}$. This result can be pushed a bit further: in dimension one we show (Proposition 2.3) that this value is also the Hausdorff dimension of the set of times, at which two independent copies of X meet.

The second half of the paper is on collision sets. Motivated by our findings in Proposition 2.3 and the results from Taylor (1966) and Jain and Pruitt (1969), we investigate the Hausdorff dimension of the collision set

$$A(\omega) := \big\{x \in \mathbb{R} : X_t^1(\omega) = X_t^2(\omega) = x \quad \text{for some } t > 0\big\}$$

of two independent copies X^1 and X^2 of X; from now on we assume that X is one-dimensional and recurrent. Since recurrence reflects the behaviour of the process as time tends to infinity, it cannot be deduced from Assumption A (which is essentially a condition on short times). Some examples of recurrent processes which fit our setting are given in Section 6. In order to get bounds on the Hausdorff dimension of $A(\omega)$, we compare the polar sets of the process (X^1, X^2) with the polar sets of symmetric stable processes with parameters α and β. The idea to use the range of a stable process as a "gauge" in order to express the Hausdorff dimension of a Borel set in $\mathbb{R}^n$ is due to Taylor (1966); in its original version it heavily relies on the fact that the process X is a Lévy process. In the present paper, we use the symmetric stable ("gauge") processes in a different way, especially when establishing the lower bound for the Hausdorff dimension.

Let us briefly mention some known results. We refer to Xiao (2004) for an extensive survey on sample path properties of Lévy processes, in particular, for various dimension results on level, intersection and image sets. Most results essentially depend on the independence and stationarity of increments of Lévy processes, while for general Markov processes much less is known. For Lévy-type processes the behaviour of the symbol of the corresponding generator allows us to get the results on the Hausdorff dimension of the image sets, see e.g. Schilling (1998), Knopova et al. (2015), and the monograph Böttcher et al. (2013); in Shieh (1995) conditions are given, such that Markov processes collide with positive probability, and Shieh and Xiao (2010) studies the Hausdorff and packing dimensions of the image sets of self-similar processes.

Our paper is organized as follows. In Section 2 we explain the notation and state our main results. Section 3 is devoted to some facts and auxiliary statements from probabilistic potential theory; these are interesting in their own right. The proofs of the main results are given in Sections 4 and 5. Examples of recurrent processes, which satisfy Assumption A can be found in Section 6. Finally, the (rather technical) proofs of some auxiliary statements are given in the appendix.

2. Setting and main results

We begin with the description of the class of stochastic processes which we are going to consider. Denote by $C_\infty^k(\mathbb{R}^n)$ and $C_c^k(\mathbb{R}^n)$ the spaces of k times continuously differentiable functions which vanish at infinity (with all derivatives) and which are compactly supported, respectively. For $f \in C_\infty^2(\mathbb{R}^n)$ we consider the following Lévy-type operator

$$\mathcal{L}f(x) := a(x)\cdot\nabla f(x)+\int_{\mathbb{R}^n\setminus\{0\}}\big(f(x+h)-f(x)-h\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|h|)\big)m(x,h)\,\mu(dh), \tag{2.1}$$

where $a:\mathbb{R}^n\to\mathbb{R}^n$, $m:\mathbb{R}^n\times\mathbb{R}^n\to(0,\infty)$ are measurable functions and μ is a Lévy measure, i.e. a measure on $\mathbb{R}^n\setminus\{0\}$ such that $\int_{\mathbb{R}^n\setminus\{0\}}\big(1\wedge|h|^2\big)\mu(dh)<\infty$.

Denote by $\hat f(x) := (2\pi)^{-n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$ the Fourier transform. It is not hard to see that we can rewrite $\mathcal{L}$ as a pseudo-differential operator

$$\mathcal{L}f(x) := -\int_{\mathbb{R}^n} e^{i\xi\cdot x}q(x,\xi)\hat f(\xi)\,d\xi, \qquad f\in C_c^\infty(\mathbb{R}^n),$$

with *symbol* $q:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$. The symbol is given by the Lévy–Khintchine representation

$$q(x,\xi) = -ia(x)\cdot\xi+\int_{\mathbb{R}^n\setminus\{0\}}\big(1-e^{ih\cdot\xi}+ih\cdot\xi\mathbb{1}_{(0,1)}(|h|)\big)m(x,h)\,\mu(dh). \tag{2.2}$$

We will frequently compare the variable-coefficient operator $\mathcal{L}$ with an operator $\mathcal{L}_0$ (with bounded coefficients), defined by

$$\mathcal{L}_0 f(x) = -\int_{\mathbb{R}^n} e^{ix\cdot\xi}q(\xi)\hat f(\xi)\,d\xi,$$

with the real-valued symbol

$$q(\xi) = \int_{\mathbb{R}^n\setminus\{0\}}\big(1-\cos(\xi\cdot h)\big)\,\mu(dh). \tag{2.3}$$

The symbol $q(\xi)$ is the characteristic exponent of a symmetric Lévy process Z_t in $\mathbb{R}^n$, i.e. $\mathbb{E}e^{i\xi\cdot Z_t} = e^{-tq(\xi)}$. Define

$$q^U(\xi) := \int_{\mathbb{R}^n\setminus\{0\}} \big((\xi\cdot h)^2 \wedge 1\big)\,\mu(dh) \quad \text{and} \quad q^L(\xi) := \int_{0<|\xi\cdot h|\le 1} (\xi\cdot h)^2\,\mu(dh)$$

and

$$q^*(r) := \sup_{\ell\in\mathbb{S}^n} q^U(r\ell),$$

where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^n$. The functions q^U and q^L are, up to multiplicative constants, upper and lower bounds for $q(\xi)$ (cf. Knopova and Kulik (2013); Knopova (2013)):

$$(1-\cos 1)q^L(\xi) \le q(\xi) \le 2q^U(\xi).$$

The key regularity assumption in Knopova and Kulik (2015, 2016) is the following comparison result:

$$\exists\kappa\ge 1 \quad \forall r\ge 1\ :\ q^*(r) \le \kappa \inf_{\ell\in\mathbb{S}^n} q^L(r\ell). \tag{2.4}$$

This condition means that the function $q(\xi)$ does not oscillate "too much". For example, if $q(\xi) = |\xi|^\alpha$ one can check that (2.4) holds true with $\kappa = 2/\alpha$. Motivated by this example, we use the notation

$$\alpha := 2/\kappa \tag{2.5}$$

with $\kappa \ge 1$ from (2.4). Moreover, (2.4) implies, see Knopova and Kulik (2013); Knopova (2013), that

$$q(\xi) \ge c|\xi|^\alpha, \quad |\xi|\ge 1. \tag{2.6}$$

We refer to Knopova and Kulik (2013) for examples which illustrate this condition.
In Knopova and Kulik (2016) it was shown that, under the following assumptions

Assumption A.

1) *The Lévy measure μ is such that (2.4) holds;*
2) *There exist constants $c_1, c_2, c_3 > 0$, such that*
$$|a(x)| \le c_1 \quad \text{and} \quad c_2 \le m(x,u) \le c_3;$$
3) *The functions $a(x)$ and $m(x,u)$ are locally Hölder continuous in x with some index $\lambda\in(0,1]$;*
4) *Either $\alpha > 1$, with α as in (2.4), (2.5),*
or $a(x)\equiv 0$ and $m(x,h) = m(x,-h)$, $\mu(dh) = \mu(-dh)$,

the operator $\mathcal{L}$ extends to the generator of a (strong) Feller process X, which has a transition probability density $p_t(x,y)$. This density is continuous as a function of $(t,x,y)\in[t_0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, $t_0>0$, and satisfies the following upper and lower bounds:

$$p_t(x,y) \ge \rho_t^n f_{\mathrm{low}}((y-x)\rho_t), \quad t\in(0,1],\ x,y\in\mathbb{R}^n, \tag{2.7}$$

and

$$p_t(x,y) \le \rho_t^n \big(f_{\mathrm{up}}(\rho_t\,\cdot) * Q_t\big)(y-x), \quad t\in(0,1],\ x,y\in\mathbb{R}^n, \tag{2.8}$$

where $(Q_t)_{t\ge 0}$ is a family of sub-probability measures,

$$\rho_t := \inf\{r>0:\ q^*(r)\ge 1/t\},$$

$$f_{\mathrm{low}}(z) := a_1(1-a_2|z|)_+ \quad \text{and} \quad f_{\mathrm{up}}(z) := a_3 e^{-a_4|z|}, \quad z\in\mathbb{R},$$

where $a_i > 0$, $i = 1, \dots, 4$, are constants and $x_+ := \max(x, 0)$. The family of sub-probability measures $(Q_t)_{t\geq 0}$ has been explicitly constructed in Knopova and Kulik (2016); for our purposes the exact form of the Q_t is not important.

Unless otherwise specified, $X = (X_t)_{t\geq 0}$ will always denote an $\mathbb{R}^n$-valued Feller process as above, with law $\mathbb{P}^x(X_t \in dy) = p_t(x, y)\, dy$, $t > 0$.

There are many Feller and Lévy-type processes satisfying the conditions required in Assumption A. Note that the integro-differential structure of the generator—as in (2.2), but with a jump kernel (compensator of the jumping measure) $N(x, dh)$ instead of $m(x, h)\, \mu(dy)$ and with a second-order term—is, in fact, necessary for Feller processes and more general semimartingales, at least if the test functions $C_c^\infty(\mathbb{R}^n)$ are in the domain of the generator, see Böttcher et al. (2013). This means that the main restriction is the fact that $N(x, dy)$ is absolutely continuous w.r.t. some Lévy measure and the absence of a second-order diffusion part; just as in the Lévy case, the latter would dominate the short-time path behaviour. Below we give a few typical examples of Feller processes satisfying our assumptions.

- Any rotationally symmetric Lévy process whose Lévy measure has a (rotationally symmetric) density $g(|u|)$ satisfying[1]

$$\int_0^a r^2 g(r)\, dr \asymp a^2 \int_a^\infty g(r)\, dr. \tag{2.9}$$

A concrete example when such condition is satisfied is given in Example 6.1 below.

- Any Lévy process whose Lévy measure is radially symmetric, i.e.

$$\mu(dh) = \int_0^\infty \int_{\mathbb{S}^n} \delta_{r\zeta}(dh)\, m(dr)\, \mu_0(d\zeta),$$

where μ_0 is a finite measure on $\mathbb{S}^n$; we assume, in addition, that $r \mapsto m(\mathbb{R}\backslash(-r^2, r^2))$ is regularly varying at 0. Then $q^L(\xi) \asymp f(|\xi|)$, where $f(|\xi|) = \int_{r|\xi|\leq 1} r^2|\xi|^2\, m(dr)$ is regularly varying at infinity as we have the representation

$$f(|\xi|) = |\xi|^2 \int_0^{1/|\xi|^2} m\{r : r^2 > s\}\, ds,$$

see Bingham et al. (1987) [Proposition 1.5.8]. Fix some $\ell \in \mathbb{S}^n$, and rewrite q^L as

$$\begin{aligned} q^L(|\xi|\ell) &= \int_{|\xi||\ell\cdot h|\leq 1} |\xi|^2 (\ell\cdot h)^2\, \mu(dh) \\ &= \int_{h\neq 0} \mathbb{1}_{\{|\xi||\ell\cdot h|\leq 1\}} |\xi|^2 (\ell\cdot h)^2\, \mu(dh) \\ &= \int_0^\infty \int_{\mathbb{S}^n} \mathbb{1}_{\{|\xi| r|\ell\cdot\zeta|\leq 1\}} |\xi|^2 r^2 (\ell\cdot\zeta)^2\, m(dr)\, \mu_0(d\zeta). \end{aligned} \tag{2.10}$$

Since $|\ell\cdot\zeta| \leq 1$, we get $\mathbb{1}_{\{|\xi| r|\ell\cdot\zeta|\leq 1\}} \geq \mathbb{1}_{\{|\xi| r\leq 1\}}$, and so

$$\begin{aligned} q^L(|\xi|\ell) &\geq \int_0^\infty \mathbb{1}_{\{|\xi| r\leq 1\}} |\xi|^2 r^2\, m(dr) \int_{\mathbb{S}^n} (\ell\cdot\zeta)^2\, \mu_0(d\zeta) \\ &= f(|\xi|) \int_{\mathbb{S}^n} (\ell\cdot\zeta)^2\, \mu_0(d\zeta). \end{aligned}$$

[1] We write $f(t) \asymp g(t)$ or $f \asymp g$ if there is an absolute constant $0 < c < \infty$ such that $c^{-1} f(t) \leq g(t) \leq c f(t)$ for all t (in the specified domain)

On the other hand, the last line of (2.10) reads

$$q^L(|\xi|\ell) = \int_{\mathbb{S}^n} f(|\xi||\ell\cdot\zeta|)\,\mu_0(d\zeta).$$

Since the function $f(r)$ is regularly varying at infinity, there exists some $C>0$, such that $f(cr)\le Cf(r)$ for any $c\in(0,1]$ and sufficiently large values of $r\gg 1$, see Bingham et al. (1987) [Theorem 1.5.6]. Therefore, we get

$$q^L(|\xi|\ell) \le C\mu_0(\mathbb{S}^n) f(|\xi|), \quad |\xi|\gg 1.$$

Observe also, that the function q^U is differentiable almost everywhere, and the derivative with respect to the radial component equals

$$\frac{\partial}{\partial r} q^U(r\ell) = \frac{2}{r} q^L(r\ell).$$

for any $\ell\in\mathbb{S}^n$ and $r>0$. Therefore, we deduce from our previous calculations that

$$1 \asymp \lim_{r\to\infty}\frac{q^L(\lambda r\ell)}{q^L(r\ell)} \asymp \lim_{r\to\infty}\frac{q^U(\lambda r\ell)}{q^U(r\ell)};$$

for the second equivalence relation we use l'Hospital's rule. Thus, condition (2.4) holds true.

- Any Lévy process from the previous example, which is perturbed by a non-constant drift $a(x)$ and such that $q(\xi)\ge c\,|\xi|^{1+\epsilon}$ for some $\epsilon>0$;

- (Weak) solutions to SDEs driven by symmetric α-stable Lévy noise ($1<\alpha<2$) and Hölder continuous coefficients, see Knopova and Kulik (2014) for the existence of such weak solutions, as well as for a simplified version of the parametrix method.

- Stable-type processes (in the sense of Z.-Q. Chen and T. Kumagai) where $m(x,h)$ is jointly continuous, bounded and bounded away from 0 and with an α-stable Lévy measure $\mu(dh)=|h|^{-\alpha-d}\,dh$.

In general, the main problem is to show that (2.4) holds true, which is a condition on the Lévy measure. To wit, this condition holds true for the "discretized version" of an α-stable Lévy measure in $\mathbb{R}^n$:

$$\mu(dh) = \sum_{k=-\infty}^{\infty} 2^{k\gamma} m_{k,\upsilon}(dh), \quad 0<\gamma<2\upsilon,$$

where $m_{k,\upsilon}(dh)$ is the uniform distribution on the sphere $\mathbb{S}_{k,\upsilon}$ centered at 0 with radius $2^{-k\upsilon}$, $\upsilon>0$, $k\in\mathbb{Z}$, $\upsilon>0$, $0<\gamma<2\upsilon$, see Knopova (2013). In this example $q^U(\xi)\asymp q(\xi)\asymp q^L(\xi)\asymp|\xi|^\alpha$, where $\alpha=\gamma/\upsilon\in(0,2)$; see Knopova and Kulik (2013) for further examples in this direction. On the other hand, Lévy measures of the form $\sum_{k=0}^\infty a_k\delta_{h_k}$, $a_k,h_k>0$ for rapidly growing weights $a_k\to\infty$ and $h_k\to 0$ are exactly those measures which create oscillations in the symbol q, making (2.4) impossible, see Farkas et al. (2001) [Example 1.1.15].

In Section 6 we consider further examples of processes which satisfy Assumption A and are recurrent (which is needed in the second main result of our paper).

In order to state our result on the bound for the Hausdorff dimension of level sets we need to define two auxiliary indices. Recall that a set D is called a d-set, if there exists a measure $\varpi\in\mathcal{M}_b^+(\overline{D})$, $\operatorname{supp}\varpi=\overline{D}$, such that

$$c_1 r^d \le \varpi\big(B(x,r)\cap\overline{D}\big)\le c_2 r^d, \quad x\in\overline{D},\ r>0; \tag{2.11}$$

the corresponding measure ϖ is called a d-measure. Denote by $\mathcal{M}_b^+(\overline{D})$ the family of all finite Borel measures with support in $\overline{D} \subset \mathbb{R}^n$. For a d-set D we define

$$\begin{aligned}\gamma_{\inf} &:= \inf \Bigg\{ \gamma \in [0,1] \,:\, \int_0^1 \frac{\varpi\big(B(x,r)\big)}{(q^*)^\gamma(1/r)} \frac{dr}{r^{n+1}} < \infty, \qquad (2.12)\\ &\qquad\qquad \text{for a } d\text{-measure } \varpi \text{ on } D \Bigg\}\\ &= \inf \left\{ \gamma \in [0,1] \,:\, \int_0^1 \frac{r^d}{(q^*)^\gamma(1/r)} \frac{dr}{r^{n+1}} < \infty \right\},\\ \gamma_{\sup} &:= \sup \Bigg\{ \gamma \in [0,1] \,:\, x \mapsto \int_0^1 \frac{\varpi\big(B(x,r)\cap\overline{D}\big)}{(q^*)^\gamma(1/r)} \frac{dr}{r^{n+1}} \qquad (2.13)\\ &\qquad\qquad \text{is unbounded } \forall \varpi \in \mathcal{M}_b^+(\overline{D}) \Bigg\}.\end{aligned}$$

Let us give an intuitive explanation of the meaning of the indices $\gamma_{\inf}$ and $\gamma_{\sup}$. Denote by $T^\gamma = (T_t^\gamma)_{t\geq 0}$, $\gamma \in (0,1)$, a γ-stable subordinator, i.e. a real-valued Lévy process with increasing sample paths such that $t^{-1/\gamma} T_t^\gamma = T_1^\gamma$ in distribution for all $t > 0$. Assume that T^γ is independent of X. Intuitively, $\gamma_{\inf}$ is the smallest γ for which the time-changed process $X_{T_t^\gamma}$ still can see the set $\overline{D}$, and $\gamma_{\sup}$ is the largest γ, for which $\overline{D}$ is polar for $X_{T_t^\gamma}$.

We can now state our first main result.

Theorem 2.1. *Suppose that the Feller process X with generator $\mathcal{L}$ satisfies Assumption A, and $D = \overline{D} \subset \mathbb{R}^n$ is a closed d-set with $d > n - \alpha$. If $x \in D$, then*[2]

$$1 - \gamma_{\inf} \leq \dim\{s \,:\, X_s^x \in D\} \leq 1 - \gamma_{\sup}, \quad \mathbb{P}^x\text{-a.s.} \tag{2.14}$$

where $\gamma_{\inf}$ and $\gamma_{\sup}$ are given by (2.12) *and* (2.13)*, respectively.*

In the one-dimensional case we can get a result which closely resembles those in Hawkes (1974) for Lévy processes. Denote by

$$X^{-1}(\{0\},\omega) := \{s > 0 \,:\, X_s(\omega) = 0\}, \quad \text{where } X_0(\omega) = 0,$$

the zero-level set of X and set

$$\gamma^* := \inf\left\{ \gamma \in [0,1] \,:\, \int_0^1 \frac{1}{(q^*(1/s))^\gamma} \frac{ds}{s^2} < \infty \right\}.$$

The corollary below follows from Theorem 2.1 if we take $D = \{0\}$, $d = 0$ and $\alpha > 1$; in this case points are non-polar for X.

Corollary 2.2. *Let X be a Feller process with generator $\mathcal{L}$ and suppose that Assumption A is satisfied. Let $n = 1$ and $\alpha > 1$. Then*

$$\dim X^{-1}(\{0\},\omega) = 1 - \gamma^* \quad \mathbb{P}^0\text{-a.s.}$$

In particular, if $q^*(\xi) \asymp |\xi|^\alpha$ $(|\xi| \geq 1)$, then $\gamma^* = 1/\alpha$.

[2]Here, as well as in the rest of the paper, "dim" stands for the Hausdorff dimension.

Corollary 2.2 can also be used to calculate the Hausdorff dimension of the set of collision times of independent copies X^1, X^2 of X:

$$\Theta(\omega) := \left\{t \geq 0 \,:\, X_t^1 = X_t^2 = x \quad \text{for some } x \in \mathbb{R}^n\right\}.$$

Proposition 2.3. *Suppose that X is a one-dimensional $(n = 1)$ Feller process with generator $\mathcal{L}$ and that Assumption A is satisfied. Let $\alpha > 1$, and denote by X^1 and X^2 two independent copies of X. Then*

$$\dim \Theta(\omega) = 1 - \gamma^* \quad \mathbb{P}\text{-}a.s.$$

Our second main result concerns the Hausdorff dimension of the collision set

$$A(\omega) := \left\{x \in \mathbb{R} \,:\, X_t^1(\omega) = X_t^2(\omega) = x \quad \text{for some } t > 0\right\} \tag{2.15}$$

$(X^1, X^2$ are two independent copies of $X)$.

Theorem 2.4. *Let X be a one-dimensional $(n = 1)$ Feller process with generator $\mathcal{L}$ and suppose that Assumption A is holds. If X is recurrent and if the function $q(\xi)$ from (2.3) satisfies*

$$c_1|\xi|^\alpha \leq q(\xi) \leq c_2|\xi|^\beta \quad \textit{for all } |\xi| \geq 1, \tag{2.16}$$

for some constants $c_1, c_2 > 0$ and $1 < \alpha \leq \beta < 2$, then the Hausdorff dimension of the collision set $A(\omega)$ is estimated from above and below as

$$\alpha - 1 \leq \dim A(\omega) \leq \beta - 1 \quad \mathbb{P}^x\text{-}a.s. \textit{ for all } x \in \mathbb{R}.$$

3. Some auxiliary results from potential theory

A central problem is which points can be hit by the process X. For this we need a few tools from potential theory. The following definition is taken from Kuwae and Takahashi (2007).

Definition 3.1. Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^n$-valued Markov process admitting a transition density $p_t(x, y)$ and ϖ a Borel measure on $\mathbb{R}^n$. The measure ϖ belongs to the *Kato class* $\mathcal{S}_K$ with respect to $p_t(x, y)$, if

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} p_s(x, y)\, \varpi(dy)\, ds = 0. \tag{3.1}$$

Let $r_\lambda(x, y)$, $\lambda > 0$, be the λ-potential density of X, i.e.

$$r_\lambda(x, y) := \int_0^\infty e^{-\lambda s} p_s(x, y)\, ds.$$

We can extend the resolvent operator from functions $f \in L_1(\mathbb{R}^n)$ to (finite) measures: For $\lambda > 0$ and any finite measure ϖ we can define the operator

$$R_\lambda \varpi(x) := \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda s} p_s(x, y)\, \varpi(dy)\, ds = \int_{\mathbb{R}^n} r_\lambda(x, y)\, \varpi(dy).$$

A Borel set $D \subset \mathbb{R}^n$ is *polar* for $X = (X_t)_{t\geq 0}$, if $\mathbb{P}^x(\tau_D < \infty) = 0$ for all $x \in \mathbb{R}^n$, where

$$\tau_D := \inf\{t > 0 \,:\, X_t \in D\}$$

is the first hitting time of the set D.

Remark 3.2. It is shown in Kuwae and Takahashi (2006) that the condition (3.1) is equivalent to "$\lim_{\lambda\to\infty}\sup_x R_\lambda\varpi(x)=0$". The set D is polar if and only if $R_0\varpi(x)$ is unbounded for any finite non-zero measure ϖ with compact support contained in $\overline{D}$, see Blumenthal and Getoor (1968) [p. 285].

In order to make sure that the process X enters the set D, we need to take the starting point x from the *fine closure* (i.e. the closure in the fine topology) of D. Recall from Blumenthal and Getoor (1968) [p. 87, Exercise 4.9] that the fine closure $\widetilde{D}$ of a set D is $D\cup D^r$, where D^r denotes the set of *regular points* of D, i.e.

$$D^r := \{x\in\mathbb{R}^n \,:\, \mathbb{P}^x(\tau_D=0)=1\}.$$

We need to characterize the regular points for D. The following elementary result *should* be known, but we could not find a reference and so we include the short proof.

Lemma 3.3. *Let $D\subset\mathbb{R}^n$ and assume that there exists a finite measure $\varpi\in\mathcal{S}_K$ (w.r.t. $p_t(x,y)$) with $\operatorname{supp}\varpi=\overline{D}$. If a point $x\in\mathbb{R}^n$ satisfies*

$$\liminf_{\lambda\to\infty}\frac{R_\lambda\varpi(x)}{\sup_{y\in\overline{D}}R_\lambda\varpi(y)}=c(x)>0, \tag{3.2}$$

then x is regular for D. In particular, if a point x is not regular for D, then the constant $c(x)$ in (3.2) is necessarily equal to 0.

Proof: Let ϖ be a finite measure such that $\operatorname{supp}\varpi=\overline{D}$ and $\varpi\in\mathcal{S}_K$. By Dynkin (1965) [Vol. 1, p. 194, Theorem 6.6], there exists a continuous additive functional[3] A_t satisfying

$$\mathbb{E}^x A_t=\int_0^t\int_{\mathbb{R}^n}p_s(x,y)\,\varpi(dy)\,ds.$$

Using standard arguments, we find for any $\lambda>0$ and $x\in\mathbb{R}^n$

$$\mathbb{E}^x\int_0^m e^{-\lambda t}\,dA_t=\int_0^m e^{-\lambda t}\,d\mathbb{E}^x A_t=\int_0^m e^{-\lambda t}p_t(x,y)\,\varpi(dy)\,dt. \tag{3.3}$$

Passing to the limit as $m\to\infty$, we get

$$\mathbb{E}^x\int_0^\infty e^{-\lambda t}\,dA_t=R_\lambda\varpi(x). \tag{3.4}$$

[3]that is, $A_{t+s}=A_s+A_t\circ\theta_s$ for any $t,s>0$ where θ_s is the shift operator.

Let $\tau := \tau_D$ be the hitting time of the set D. By construction, the additive functional A_t satisfies $A_t = 0$ for $t < \tau$. Thus,

$$\begin{aligned}
&\lambda \mathbb{E}^x \int_0^m e^{-\lambda t} A_t\, dt \\
&\quad = \underbrace{\lambda\, \mathbb{E}^x \int_0^m e^{-\lambda t} A_t \mathbb{1}_{\{\tau > m\}}\, dt}_{=0} + \lambda \mathbb{E}^x \int_\tau^m e^{-\lambda t} A_t \mathbb{1}_{\{\tau \le m\}}\, dt \\
&\quad = \lambda \mathbb{E}^x \int_0^{m-\tau} e^{-\lambda(t+\tau)} A_{t+\tau} \mathbb{1}_{\{\tau \le m\}}\, dt \\
&\quad = \mathbb{E}^x \left[e^{-\lambda\tau} \mathbb{1}_{\{\tau \le m\}} \mathbb{E}^{X_\tau} \left(\lambda \int_0^{m-\tau} e^{-\lambda t} A_t\, dt \right) \right] \\
&\quad = \mathbb{E}^x \left[e^{-\lambda\tau} \mathbb{1}_{\{\tau \le m\}} \mathbb{E}^{X_\tau} \left(\int_0^{m-\tau} e^{-\lambda t}\, dA_t \right) \right] - e^{-\lambda m}\, \mathbb{E}^x \left[\mathbb{1}_{\{\tau \le m\}} \mathbb{E}^{X_\tau} A_{m-\tau} \right] \\
&\quad = \mathbb{E}^x \left[e^{-\lambda\tau} \mathbb{1}_{\{\tau \le m\}} \mathbb{E}^{X_\tau} \left(\int_0^{m-\tau} e^{-\lambda t}\, dA_t \right) \right] - e^{-\lambda m} \mathbb{E}^x A_m.
\end{aligned}$$

For the last step we used the continuity of A_t to get $A_\tau = 0$ and, by the additive property,

$$\begin{aligned}
\mathbb{E}^x A_m = \mathbb{E}^x A_\tau + \mathbb{E}^x \left[A_{m-\tau} \circ \theta_\tau \mathbb{1}_{\{\tau \le m\}} \right] &= \mathbb{E}^x \left[\mathbb{E}^x \left(A_{m-\tau} \circ \theta_\tau \mid \mathcal{F}_\tau \right) \right] \\
&= \mathbb{E}^x \left[\mathbb{1}_{\{\tau \le m\}} \mathbb{E}^{X_\tau} A_{m-\tau} \right].
\end{aligned}$$

These calculations, when combined with (3.3) and integration by parts, yield

$$\mathbb{E}^x \int_0^m e^{-\lambda t}\, dA_t = \mathbb{E}^x \left[e^{-\lambda\tau} \mathbb{1}_{\{\tau \le m\}} \mathbb{E}^{X_\tau} \left(\int_0^{m-\tau} e^{-\lambda t}\, dA_t \right) \right],$$

and passing to the limit as $m \to \infty$ we finally arrive at

$$R_\lambda \varpi(x) = \mathbb{E}^x \left[e^{-\lambda\tau} R_\lambda \varpi(X_\tau) \right].$$

Since $X_\tau \in \overline{D}$, the last equality implies

$$\frac{R_\lambda \varpi(x)}{\sup_{y \in \overline{D}} R_\lambda \varpi(y)} \le \mathbb{E}^x e^{-\lambda\tau}. \tag{3.5}$$

Note that $\{\tau > 0\}$ is a "tail event", i.e. it has probability 0 or 1. Taking the lower limit $\liminf_{\lambda \to \infty}$ on both sides, we get a contradiction to (3.2), unless $\tau \equiv 0$. Thus, $\mathbb{P}^x(\tau > 0) = 0$. □

Remark 3.4. For a symmetric Markov process X, the relation (3.4) is known for all measures which have *finite energy integrals*, see Fukushima et al. (2011) [pp. 223–226, Theorem 5.1.1, Lemma 5.1.3].

It is possible to give a more explicit sufficient condition for a point x to be regular for D; this requires further knowledge of the structure of D, for instance that D is a d-set.

Lemma 3.5. *Let $D \subset \mathbb{R}^n$ be a d-set and assume that the corresponding d-measure ϖ belongs to $\mathcal{S}_K$ w.r.t. $p_t(x, y)$. Then any point of $\overline{D}$ is regular for D, i.e. $\overline{D} = D \cup D^r = \widetilde{D}$.*

In order to keep the presentation transparent, we defer the rather technical proof of this lemma to the appendix.

Here is a criterion for the non-polarity of a set D based on the inequality (3.5).

Corollary 3.6. *Assume that there exists some* $\varpi \in \mathcal{S}_K$ *w.r.t.* $p_t(x,y)$ *such that* $\operatorname{supp}\varpi = \overline{D}$. *Then the set* D *is non-polar for* X, *i.e.*

$$\mathbb{P}^x(\tau_D < \infty) > 0. \tag{3.6}$$

Proof: We know from Kuwae and Takahashi (2006), see also Remark 3.2, that $\varpi \in \mathcal{S}_K$ satisfies $\sup_x R_\lambda \varpi(x) < \infty$ for some $\lambda > 0$. From (3.5) we derive

$$\frac{R_\lambda \varpi(x)}{\sup_{y\in\overline{D}} R_\lambda \varpi(y)} \le \mathbb{P}^x(\tau_D < \infty).$$

Let us show that $R_\lambda \varpi(x) > 0$. For this we show that

$$p_t(x,y) > 0 \quad \text{for all } t > 0,\ \ x, y \in \mathbb{R}^n.$$

There is a minimal N, such that the distance from x to y can be covered by N balls of radius less than $(2a_2\rho_{t/N})^{-1}$ (where $a_2 > 0$ is the constant appearing in the representation of f_{low}), i.e. the smallest N, for which the inequality

$$\frac{|x-y|}{N} \le \frac{1}{a_2\rho_{t/N}} \tag{3.7}$$

holds true. Observe that $q^*(r) \le c_1 r^2$, $r \ge 1$, implying $c_2 t^{-1/2} \le \rho_t$, for all t small enough. Hence, (3.7) is valid for all $N \ge (a_2c_2|x-y|)^2/t$. Therefore, putting $y_0 = x$, $y_N = y$, we get

$$\begin{aligned} p_t(x,y) &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\prod_{i=1}^N p_{t/N}(y_{i-1}, y_i) \right) dy_1 \dots dy_N \\ &\ge \int_{B(y_0,(2a_2\rho_{t/N})^{-1})} \cdots \int_{B(y_{N-1},(2a_2\rho_{t/N})^{-1})} \prod_{i=1}^N p_{t/N}(y_{i-1}, y_i)\, dy_i \\ &\ge c_0 \rho_{t/N}^{Nn}. \end{aligned}$$

In the last line we use (2.7) which gives

$$p_{t/N}(y_{i-1}, y_i) \ge 2^{-1} a_1 \rho_{t/N}^n \quad \forall y_i \in B(y_{i-1}, (2a_2\rho_{t/N})^{-1}).$$

Thus, the transition probability density $p_t(x,y)$ is strictly positive, which implies

$$R_\lambda \varpi(x) \ge e^{-\lambda} \int_0^1 \int_D p_t(x,y)\, \varpi(dy)\, dt > 0.$$

Hence, we get (3.6). □

Remark 3.7. *a*) Under the assumptions of Corollary 3.6 one has $\mathbb{P}^x(\tau_D < \infty) > c_K$ uniformly for all $x \in K$ where $K \subset \mathbb{R}^n$ is a compact set.

b) If, in addition, the process X is recurrent, then $\mathbb{P}^x(\tau_D < \infty) = 1$, see Sharpe (1988) [p. 60].

c) Suppose that X is one-dimensional ($n = 1$) and $\int_1^\infty q^*(s)^{-1}\, ds < \infty$. Then there exists a local time for any point $x \in \mathbb{R}$, see Knopova and Kulik (2016). Let $D = \{x\}$, where x is the starting point of X_t. Then $R_\lambda \varpi(x) = \sup_{y\in\mathbb{R}} R_\lambda \varpi(y)$, i.e. the left-hand side of (3.2) is equal to 1, implying that every point is regular for itself.

On the other hand, if $n \geq 2$, we always have $\int_{|\xi|\geq 1} q^*(\xi)^{-1}\,d\xi = \infty$, i.e. for $n \geq 2$ points are polar.

4. Proof of Theorem 2.1 and Proposition 2.3

Throughout this section $X = (X_t)_{t\geq 0}$ is a Feller process as in Section 2. Let $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ be a further probability space and define on this space a γ-stable subordinator $T^\gamma = (T_t^\gamma)_{t\geq 0}$, $\gamma \in (0,1)$. T_t^γ has a transition probability density $\sigma_t^{(\gamma)}(s)$, and

$$\int_0^\infty e^{-\lambda s}\sigma_t^{(\gamma)}(s)\,ds = e^{-t\lambda^\gamma}, \quad \lambda > 0,\ t > 0.$$

From this we immediately get the following scaling property

$$\sigma_t^{(\gamma)}(s) = t^{-1/\gamma}\sigma_1^{(\gamma)}(st^{-1/\gamma}). \tag{4.1}$$

Let $X_t^\gamma := X_{T_t^\gamma}$ be the subordinate process. Its transition probability density $p_t^{(\gamma)}(x,y)$ is given by

$$p_t^{(\gamma)}(x,y) = \int_0^\infty p_s(x,y)\sigma_t^{(\gamma)}(s)\,ds, \tag{4.2}$$

see, for example, Jacob (2001) [Theorem 4.3.1].

The technical proof of the following lemma is deferred to the appendix. Recall that $\mathcal{S}_K$ denotes the Kato class of measures, cf. Definition 3.1. If $\gamma = 1$, $T_t^{(\gamma)} \equiv t$, and the 'subordinate' kernel $p_t^{(1)}(x,y)$ is just $p_t(x,y)$.

Lemma 4.1. a) *Suppose that ϖ satisfies*

$$\int_0^1 \sup_x \frac{\varpi(B(x,r))}{(q^*)^\gamma(1/r)}\,\frac{dr}{r^{n+1}} < \infty, \quad \textit{for some} \quad \gamma \in (0,1]. \tag{4.3}$$

Then $\varpi \in \mathcal{S}_K$ with respect to $p_t^{(\gamma)}(x,y)$.

b) *Suppose that $\varpi \in \mathcal{S}_K$ with respect to $p_t^{(\gamma)}(x,y)$, where $\gamma \in (0,1]$. Then*

$$\lim_{t\to 0}\sup_x \int_0^t \frac{\varpi(B(x,r))}{(q^*)^\gamma(1/r)}\,\frac{dr}{r^{n+1}} = 0. \tag{4.4}$$

The next lemma is due to Hawkes (1974) [Lemma 2.1], cf. also Hawkes (1971) [Proof of Theorem 1]; it plays the key role in the proof of Theorem 2.1.

Lemma 4.2. *Let T^γ be a stable subordinator of index $\gamma \in (0,1)$, and let $B \subset [0,\infty)$ be a Borel set. Then*

$$\mathbb{P}\left(T_t^\gamma \in B \quad \textit{for some } t > 0\right) = 0 \quad \textit{implies} \quad \dim B \leq 1-\gamma,$$

while

$$\mathbb{P}\left(T_t^\gamma \in B \quad \textit{for some } t > 0\right) > 0 \quad \textit{implies} \quad \dim B \geq 1-\gamma.$$

We are now ready for the

Proof of Theorem 2.1: By assumption, D is a closed d-set; pick a corresponding d-measure ϖ on D. For $d > n - \alpha$ we have

$$\int_0^1 \sup_x \frac{\varpi(B(x,r))}{q^*(1/r)} \frac{dr}{r^{n+1}} \le c_1 \int_0^1 \frac{r^d}{q^*(1/r)} \frac{dr}{r^{n+1}} \le c_2 \int_0^1 r^{d+\alpha-n-1}\, dr < \infty,$$

where we used that $q^*(r) \ge cr^\alpha$, cf. (2.6). By Lemma 4.1 (used for $\gamma = 1$) we have $\varpi \in \mathcal{S}_K$ w.r.t. $p_t(x,y)$, and by Lemma 3.5 all points of D are regular for $D = \overline{D}$.

As $X_0 = x \in D$, the set $\{s : X_s(\omega) \in D\}$ is a.s. non-empty, and therefore the random set

$$\begin{aligned} W &:= \big\{(\omega,\omega^*) : X_{T_t^\gamma(\omega^*)}(\omega) \in D \quad \text{for some } t > 0\big\} \\ &= \big\{(\omega,\omega^*) : T_t^\gamma(\omega^*) \in \{s : X_s(\omega) \in D\} \quad \text{for some } t > 0\big\} \end{aligned} \tag{4.5}$$

is well-defined and non-void.

First we calculate the lower bound of the Hausdorff dimension of the random set $\{s : X_s(\omega) \in D\}$. Assume that $\gamma \in (\gamma_{\inf}, 1)$. Recall that the transition probability density of the subordinate process $X_{T_t^\gamma(\omega^*)}(\omega)$ is given by (4.2). By Lemma 4.1, $\varpi \in \mathcal{S}_K$ with respect to $p_t^{(\gamma)}(x,y)$ for any $\gamma \in (\gamma_{\inf}, 1)$. Using Lemma 3.5 we see that the points, which are regular for D "in the eyes" of the original process X, are still regular for D and the subordinate process $X_{T_t^\gamma}$—whenever $\gamma \in (\gamma_{\inf}, 1)$. This implies that the set W has full $\mathbb{P}^x \otimes \mathbb{P}^*$-measure. Thus, (4.5) yields

$$\begin{aligned} 1 &= (\mathbb{P}^x \otimes \mathbb{P}^*)(W) \\ &= \int_\Omega \mathbb{P}^*\Big(\omega^* : T_t^\gamma(\omega^*) \in \{s : X_s(\omega) \in D\} \quad \text{for some } t > 0\Big)\, \mathbb{P}^x(d\omega), \end{aligned}$$

which in turn gives

$$\mathbb{P}^x\Big(\omega : \mathbb{P}^*\left[\omega^* : T_t^\gamma(\omega^*) \in \{s : X_s(\omega) \in D\} \quad \text{for some } t > 0\right] > 0\Big) = 1.$$

Now Lemma 4.2 shows $\dim\{s : X_s(\omega) \in D\} \ge 1 - \gamma$ with $\mathbb{P}^x$-probability 1; letting $\gamma \downarrow \gamma_{\inf}$ along a countable sequence we arrive at

$$\dim\{s : X_s(\omega) \in D\} \ge 1 - \gamma_{\inf} \qquad \mathbb{P}^x\text{-a.s.}$$

To show the upper bound in (2.14), we take $\gamma \in (0, \gamma_{\sup})$. By the definition of $\gamma_{\sup}$,

$$x \mapsto \int_0^\delta \frac{\varpi(B(x,r))}{(q^*)^\gamma(1/r)} \frac{dr}{r^{n+1}}$$

is unbounded for any finite measure ϖ supported in D. There exist, see (7.5) below, constants a, $b, \delta(T) > 0$ such that

$$\int_0^T \int_D p_t^{(\gamma)}(x,y)\, \varpi(dy)\, dt \ge a \int_0^{\delta(T)} \frac{\varpi(B(x,r))}{(q^*)^\gamma(1/r)} \frac{dr}{r^{n+1}}.$$

Thus, $R_0\varpi(x)$ is unbounded and, by Remark 3.2, the set D is polar for X_t^γ. Therefore, $(\mathbb{P}^x \otimes \mathbb{P}^*)(W) = 0$ and, consequently,

$$\mathbb{P}^x\Big(\omega : \mathbb{P}^*\left[\omega^* : T_t^\gamma(\omega^*) \in \{s : X_s^x(\omega) \in D\} \quad \text{for some } t > 0\right] = 0\Big) = 1.$$

This means that $\{s : X_s^x(\omega) \in D\}$ is polar for T_t^γ with $\mathbb{P}^x$-probability 1. Applying Lemma 4.2 we get $\dim\{s : X_s^x(\omega) \in D\} \le 1 - \gamma$ with $\mathbb{P}^x$-probability 1. Letting $\gamma \uparrow \gamma_{\sup}$ along a countable sequence, the upper bound in (2.14) follows. □

Proof of Proposition 2.3: Since the processes X^1 and X^2 are, up to different starting points, i.i.d. copies, the transition probability density of $\tilde{X}_t := X_t^1 - X_t^2$ is given by

$$\tilde{p}_t(x,y) = \int_{\mathbb{R}} p_t(x+x_0, z+y) p_t(x_0, z)\, dz;$$

here $x_0 \in \mathbb{R}$ is the starting point of X_t^2. Let us estimate $\tilde{p}_t(x,y)$ using the upper bounds (2.8) for $p_t(x,y)$. By the triangle inequality we have for any $\epsilon > 0$ and $w_1, w_2 \in \mathbb{R}$

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_t^2 e^{-a_4\rho_t\,|z+y-x-x_0-w_1|} e^{-a_4\rho_t\,|x_0-z+w_2|}\, dz \\
&\qquad \le \rho_t\, e^{-a_4\epsilon\rho_t\,|y-x-w_1+w_2|} \int_{\mathbb{R}} \rho_t\, e^{-a_4(1-\epsilon)\rho_t\cdot(|z+y-x-x_0-w_1|+|x_0-z+w_2|)}\, dz \\
&\qquad \le c\rho_t\, e^{-a_4\epsilon\rho_t\,|y-x-w_1+w_2|}.
\end{aligned}$$

This yields the following upper bound for $\tilde{p}_t(x,y)$:

$$\begin{aligned}
\tilde{p}_t(x,y) &\le a_3^2 \iiint_{\mathbb{R}^3} \rho_t^2 e^{-a_4\rho_t\,|z-x-x_0-w_1|} e^{-a_4\rho_t\,|z-x_0-w_2|}\, dz\, Q_t(dw_1)\, Q_t(dw_2) \\
&\le C\rho_t \left(f_{\text{up}}^{\epsilon}(\rho_t\cdot\,) * \tilde{Q}_t \right)(y-x),
\end{aligned}$$

where $\tilde{Q}_t(dw) := \int_{\mathbb{R}} Q_t(dw+v)\, Q_t(dv)$ is again a sub-probability measure. In other words, the transition probability density of $\tilde{X}$ has an upper bound of the same form as $p_t(x,y)$.

To show the lower bound, take x, y such that $\rho_t\,|y-x| \le a_2^{-1}(1-a_2\epsilon)$, where $\epsilon > 0$ is small. Then

$$\begin{aligned}
\tilde{p}_t(x,y) &\ge a_1^2\rho_t^2 \int_{\mathbb{R}} f_{\text{low}}((y+z-x-x_0)\rho_t) f_{\text{low}}((x_0-z)\rho_t)\, dz \\
&\ge a_1\rho_t \int_{|v|\le\epsilon} f_{\text{low}}(\rho_t(y-x-v/\rho_t)) f_{\text{low}}(v)\, dv.
\end{aligned}$$

Since for $|v| \le \epsilon$

$$1 - a_2\rho_t\,|y-x-v/\rho_t| \ge 1 - a_2\epsilon - a_2\rho_t\,|y-x| = (1-\epsilon a_2)\left(1 - \frac{a_2}{1-a_2\epsilon}\rho_t\,|y-x|\right),$$

we get for all x, y such that $\rho_t\,|y-x| \le a_2^{-1}(1-a_2\epsilon)$ the estimate

$$\tilde{p}_t(x,y) \ge c_1\rho_t\,(1 - c_2\rho_t\,|y-x|)$$

with $c_1 = a_1(1-a_2\epsilon)\int_{|v|\le\epsilon} f_{\text{low}}(v)\, dv$ and $c_2 = a_2(1-a_2\epsilon)^{-1}$. Thus, the lower bound for $\tilde{p}_t(x,y)$ is also of the same form as the one for $p_t(x,y)$.

We have shown that the symmetrized process $\tilde{X}$ satisfies the estimates (2.7) and (2.8), and these estimates are the essential ingredient in the proof of Corollary 2.2.[4] Thus, we can apply Corollary 2.2, and the proof is finished. □

[4]Notice that Assumption A in Corollary 2.2 is just used to ensure that we have (2.7) and (2.8).

5. Proof of Theorem 2.4

Throughout this section we work under the assumptions of Theorem 2.4: Assumption A holds and the process X is recurrent. We denote the transition probability density by $p_t(x,y)$, $t>0$. Recall that the process X is called

1) (*neighbourhood*) *recurrent* if

$$\forall x \in \mathbb{R} \quad \forall \text{open sets } G \subset \mathbb{R} \,:\, \mathbb{P}^x \left(X_t \in G \quad \text{for some } t>0 \right) = 1.$$

2) *point recurrent*, if

$$\forall x, y \in \mathbb{R} \,:\, \mathbb{P}^x \left(X_t = y \quad \text{for some } t>0 \right) = 1.$$

Using the arguments of Jain and Pruitt (1969) [Lemma 4.1] we can show that, in the setting of Theorem 2.4, the recurrence of X already implies point recurrence.

Lemma 5.1. *The process X is point recurrent.*

Proof: Write $\tau_y := \inf\{t>0 \,:\, X_t = y\}$ for the hitting time of $\{y\}$ and set

$$\Phi(x,y) := \mathbb{P}^x \left(X_t = y \quad \text{for some } t>0 \right) = \mathbb{E}^x \mathbb{1}_{\{\tau_y<\infty\}}.$$

Let us show that for X any singleton $\{x\}$ is regular for itself. By (2.8) and the inequality $\rho_t \le ct^{-1/\alpha}$, $t \in (0,1]$—this follows from (2.6)—we have for $\alpha>1$

$$\sup_{x,y\in\mathbb{R}} \int_0^t p_s(x,y)\, ds \le c_1 \int_0^t \rho_s \, ds \le c_2 \int_0^t s^{-1/\alpha}\, ds \le c_3 t^{1-1/\alpha}, \quad t \in (0,1].$$

Thus, any measure of the form $\varpi = c\delta_y$ for $c \ge 0$ and some y belongs to the Kato class $\mathcal{S}_K$ w.r.t. $p_t(x,y)$. By Lemma 3.5, any point $y \in \mathbb{R}$ is regular for itself for X. Then

$$\Phi(y,y) = 1,$$

because $\{\tau_x = 0\} = \bigcap_{\epsilon>0}\{X_t = x \ \text{ for some } \ t \in (0,\epsilon)\}$, and because of the regularity $\mathbb{P}^x(\tau_x = 0) = 1$.

Let us show that the function $\Phi(\cdot,y)$ is excessive. Denote by $(P_t)_{t\ge 0}$ the semigroup given by the kernel $p_t(x,y)$. Since

$$\begin{aligned} \Phi(X_t(\omega),y) &= \mathbb{P}^{X_t(\omega)} \left(X_s = y \quad \text{for some } s>0 \right) \qquad \text{for } \mathbb{P}^x\text{-a.a. } \omega \\ &= \mathbb{P}^x \left(X_{t+s} = y \quad \text{for some } s>0 \right), \end{aligned}$$

we have

$$P_t\Phi(\cdot,y)(x) = \mathbb{E}^x \Phi(X_t,y) = \mathbb{P}^x \left(X_{t+s} = y \quad \text{for some } s>0 \right) \le \Phi(x,y),$$

and by the dominated convergence theorem $P_t\Phi(x,y) \uparrow \Phi(x,y)$ as $t \to 0$. Since X is recurrent, all excessive functions are constant, see Sharpe (1988) [Exercise 10.39]; hence, we get $\Phi(x,y) \equiv 1$ for all $x,y \in \mathbb{R}$. □

Remark 5.2. Let X^1 and X^2 be two independent copies of X. Then the symmetrized process $\tilde{X} = X^1 - X^2$ is point recurrent.

Let β be the exponent appearing in the upper bound in (2.16).

Lemma 5.3. *Let X^1 and X^2 be independent copies of X, and denote by Z^β a symmetric β-stable Lévy process in $\mathbb{R}^2$. Let D be a subset of the diagonal in $\mathbb{R}^2$. If D is polar for Z^β, then it is polar for (X^1,X^2).*

Proof: Denote by $\mathfrak{p}_t(x,y)$, $x=(x_1,x_2)$, $y=(y_1,y_2)$, the transition probability density of the bivariate process (X^1,X^2). Suppose that $|x-y|\le\epsilon$ for some sufficiently small $\epsilon>0$. Using the lower estimates (2.7) for $p_t(x_i,y_i)$, $i=1,2$, we get

$$\begin{aligned}\int_0^1 \mathfrak{p}_t(x,y)\,dt &\ge a_1^2\int_0^1 (1-a_2\rho_t\,|x_1-y_1|)_+(1-a_2\rho_t\,|x_2-y_2|)_+\rho_t^2\,dt\\ &\ge a_1^2(1-a_2)^2\int_{c\phi(|x-y|)}^1 \rho_t^2\,dt,\end{aligned}$$

where $\phi(r):=1/q^U(1/r)$; with this choice of $\phi(r)$ we have $\rho_t\,|x-y|<1$. Changing variables gives

$$\int_0^1 \mathfrak{p}_t(x,y)\,dt \ge c_1\int_{|x-y|}^{1/\rho_1}\frac{1}{r^3q^U(1/r)}\,dr\ge c_2|x-y|^{\beta-2},$$

where we use that ρ_t is the inverse of q^* and (2.4), as well as $\left(q^U(r)\right)'=2q^L(r)/r$ a.e. and (2.16). The expression in the last line is (up to a constant) the potential of the process Z^β. Thus, for $|x-y|<\epsilon$ the potential of (X^1,X^2) is bounded from below by the potential $U(x):=|x|^{\beta-2}$ of Z^β. Now

$$\int_{|x-y|>\epsilon}\frac{1}{|x-y|^{2-\beta}}\,\varpi(dy)\le\epsilon^{\beta-2}\varpi(D)\quad\text{for all finite measures }\varpi. \tag{5.1}$$

By Remark 3.2 the set D is polar for Z^β if and only if the potential of Z^β is unbounded for any finite measure $\varpi\neq 0$ with $\operatorname{supp}\varpi\subset\overline{D}$, i.e.

$$\sup_x U\varpi(x)=\sup_x\int\frac{1}{|x-y|^{2-\beta}}\,\varpi(dy)=\infty.$$

Because of (5.1) this happens if and only if

$$\sup_x\int_{|x-y|\le\epsilon}\frac{1}{|x-y|^{2-\beta}}\,\varpi(dy)=\infty. \tag{5.2}$$

Thus, if (5.2) holds true, then $\sup_x\mathfrak{R}_0\varpi(x)=\infty$, where $\mathfrak{R}_0$ is the 0-resolvent for (X^1,X^2); by Remark 3.2 the set D is polar for (X^1,X^2). □

The next lemma is from Taylor (1966) [Theorem 4], see also Jain and Pruitt (1969), and it plays the key role in the proof of Theorem 2.4.

Lemma 5.4. *Suppose that A is an analytic subset of $\mathbb{R}^n (n=1,2)$, and $Z_t^{\zeta,n}$ is any symmetric ζ-stable Lévy process in $\mathbb{R}^n$. Then*

$$\dim A=n-\inf\left\{\zeta\,:\,A\textit{ is non-polar for }Z^{\zeta,n}\right\}.$$

Proof of Theorem 2.4: Let $A(\omega)$ be the collision set defined in (2.15). Since the one-dimensional process X^1-X^2 is point recurrent, cf. Remark 5.2, the set $A(\omega)$ is a.s. non-empty. Instead of $A(\omega)$ we consider the following set on the diagonal of $\mathbb{R}^2$:

$$\begin{aligned}\hat{A}(\omega)&:=\left\{(x,x)\in\mathbb{R}^2\,:\,(X_t^1(\omega),X_t^2(\omega))=(x,x)\quad\text{for some }t>0\right\}\\ &\equiv\left\{(x,x)\in\mathbb{R}^2\,:\,\tau^x(\omega)<\infty\right\},\end{aligned}$$

where $\tau^x := \inf\{t > 0 : (X_t^1, X_t^2) = (x, x)\}$. There is a one-to-one correspondence between $\hat{A}(\omega)$ and $A(\omega)$, and their Hausdorff dimensions coincide. For our needs it is more convenient to work with the set $\hat{A}(\omega)$.

Define on a further probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ a symmetric θ-stable Lévy process $Z_t^{\theta,1}(\omega')$, $t \geq 0$, taking values on the diagonal of $\mathbb{R}^2$ and with $\theta < 2 - \beta$. We are going to show that

$$\mathbb{P}'\Big(\omega' : Z_t^{\theta,1}(\omega') \in \hat{A}(\omega) \quad \text{for some } t > 0\Big) = 0 \tag{5.3}$$

for almost all ω; this means that $\hat{A}(\omega)$ is a.s. polar for $Z^{\theta,1}(\omega')$.

Let

$$\Gamma := \Big\{(\omega, \omega') : Z_t^{\theta,1}(\omega') \in \hat{A}(\omega) \quad \text{for some } t > 0\Big\}.$$

Then, by the definition of $\hat{A}(\omega)$,

$$\Gamma = \Big\{(\omega, \omega') : (X_t^1(\omega), X_t^2(\omega)) = (x, x) \in \hat{B}(\omega') \quad \text{for some } t > 0\Big\},$$

where $\hat{B}(\omega') := \operatorname{Range} Z_t^{\theta,1}(\omega')$. In Blumenthal and Getoor (1960) it is shown that $\dim \hat{B}(\omega') = \theta$; by Lemma 5.4 we get

$$2 - \inf\Big\{\zeta > 0 : \hat{B}(\omega') \text{ is non-polar for } Z^{\zeta,2}\Big\} = \dim \hat{B}(\omega') = \theta < 2 - \beta,$$

and so

$$\beta < \inf\Big\{\zeta > 0 : \hat{B}(\omega') \text{ is non-polar for } Z^{\zeta,2}\Big\}.$$

Thus, the set $\hat{B}(\omega')$ is for almost all ω' polar for the process $Z_t^{\beta,2}$. By Lemma 5.3 the set $\hat{B}(\omega')$ is polar for $(X_t^1(\omega), X_t^2(\omega))$ for almost all ω'. By Fubini's theorem we have $\mathbb{P} \otimes \mathbb{P}'(\Gamma) = 0$; therefore, (5.3) holds true, showing that $\hat{A}(\omega)$ is polar for $Z^{\theta,1}$ for all $\theta < 2 - \beta$. Thus, by Lemma 5.4

$$\dim \hat{A}(\omega) = 1 - \inf\Big\{\theta > 0 : \hat{A}(\omega) \text{ is non-polar for } Z^{\theta,1}\Big\} \leq 1 - (2 - \beta) = \beta - 1.$$

Next, we are going to show that $\dim \hat{A}(\omega) \geq \alpha - 1$. Choose $\theta \in (2 - \alpha, 2)$, and let $Z^{\theta,1}$ be a symmetric θ-stable Lévy process on the diagonal in $\mathbb{R}^2$. Denote by $\hat{B}(\omega')$ its range; by Blumenthal and Getoor (1960), $\dim \hat{B}(\omega') = \theta$. By Frostman's lemma, cf. e.g. Schilling and Partzsch (2014) [p. 387, Theorem A.44], there exists a measure ϖ on $\hat{B}(\omega') \cap K$ (K is a compact subset of the diagonal in $\mathbb{R}^2$) such that

$$\varpi\big(B(z, r)\big) \leq C r^{\theta - \epsilon}, \quad z \in \hat{B}(\omega'),\ r > 0. \tag{5.4}$$

Denote by $\mathfrak{p}_t(x, y)$ the transition probability density of (X_t^1, X_t^2). A direct calculation shows (cf. (7.1) in the appendix for details of the first estimate) that (5.4) implies

$$\int_0^1 \int_{\hat{B}(\omega')\cap K} \mathfrak{p}_t(x,y)\,\varpi(dy)\,dt$$
$$\le c_1 \int_0^1 \int_0^\infty \rho_t^2 \sup_{x\in\mathbb{R}^2} \varpi\{y : |x-y| \le c_2 r/\rho_t\}\, e^{-r}\,dr\,dt$$
$$\le c_3 \int_0^1 \rho_t^{2-\theta+\epsilon}\,dt$$
$$\le c_4 \int_0^1 t^{-(\theta-2+\epsilon)/\alpha}\,dt < \infty,$$

which shows that $\varpi \in \mathcal{S}_K$ w.r.t. $\mathfrak{p}_t(x,y)$. Hence, by Corollary 3.6 the set $\hat{B}(\omega')$ is non-polar for (X^1, X^2).

By $\mathbb{P}^{(z,z)}$ we indicate that the starting point of the process (X^1, X^2) is (z,z). For all $(z,z) \in \mathbb{R}^2$

$$\mathbb{P}^{(z,z)} \otimes \mathbb{P}'\Big((\omega,\omega') : (X_t^1(\omega), X_t^2(\omega)) = (x,x) \in \hat{B}(\omega') \quad \text{for some } t>0\Big) > 0.$$

By Fubini's theorem, there is a set $F \in \mathcal{F}$ with $\mathbb{P}^{(z,z)}(F) > 0$ such that

$$\forall \omega \in F : \mathbb{P}'\Big(\omega' : Z_t^{\theta,1}(\omega') \in \hat{A}(\omega) \quad \text{for some } t>0\Big) > 0.$$

Let us show that there exists an open subset $\mathcal{O}$ of the diagonal in $\mathbb{R}^2$ such that

$$\inf_{(z,z)\in\mathcal{O}} \mathbb{P}^{(z,z)}(F) \ge \delta > 0. \tag{5.5}$$

Indeed, for any $s > 0$ we have

$$\mathbb{E}^{(z,z)}\mathbb{E}'\mathbb{1}_{\{\exists t>0\,:\,(X_t^1,X_t^2)\in\hat{B}(\omega')\}}$$
$$\ge \mathbb{E}^{(z,z)}\mathbb{E}'\mathbb{1}_{\{\exists t>s\,:\,(X_t^1,X_t^2)\in\hat{B}(\omega')\}}$$
$$= \mathbb{E}^{(z,z)}\Big(\mathbb{E}^{(X_s^1,X_s^2)}\mathbb{E}'\mathbb{1}_{\{\exists t>s\,:\,(X_{t-s}^1,X_{t-s}^2)\in\hat{B}(\omega')\}}\Big)$$
$$= \mathbb{E}^{(z,z)}\Big(\mathbb{E}^{(X_0^1,X_0^2)}\mathbb{E}'\mathbb{1}_{\{\exists t>s\,:\,(X_{t-s}^1\circ\theta_s,X_{t-s}^2\circ\theta_s)\in\hat{B}(\omega')\}}\Big)$$
$$= \mathbb{E}^{(z,z)}\Big(\mathbb{E}'\mathbb{1}_{\{\exists r>0\,:\,(X_r^1\circ\theta_s,X_r^2\circ\theta_s)\in\hat{B}(\omega')\}}\Big).$$

Denote by $\theta_s^{-1}F = \{\omega : \theta_s\omega \in F\}$ the shift of the set F. Clearly, $\theta_s^{-1}\{Y \in \Gamma\} = \{Y \circ \theta_s \in \Gamma\}$ for any random variable $Y(\omega)$ and a set Γ in the state space of Y, and so

$$\mathbb{P}^{(z,z)}(F) \ge \mathbb{P}^{(z,z)}(\theta_s^{-1}F).$$

This inequality implies that $\mathbb{P}^{(z,z)}(F)$ is excessive:

$$P_t\mathbb{P}^{(z,z)}(F) = \mathbb{E}^{(z,z)}\mathbb{P}^{(X_t,X_t)}(F) = \mathbb{E}^{(z,z)}\Big(\mathbb{E}^{(z,z)}\Big[\mathbb{1}_{\theta_t^{-1}F}\,\Big|\,\mathcal{F}_t\Big]\Big)$$
$$= \mathbb{E}^{(z,z)}\mathbb{1}_{\theta_t^{-1}F} = \mathbb{P}^{(z,z)}(\theta_t^{-1}F) \le \mathbb{P}^{(z,z)}(F),$$

and, by the dominated convergence theorem, $P_t\mathbb{P}^{(z,z)}(F) \uparrow \mathbb{P}^{(z,z)}(F)$ as $t \to 0$. Assumption A implies that the process X is a strong Feller process, cf. Section 2, which means that any excessive function is lower semicontinuous, see Blumenthal and Getoor (1968) [p. 77, Exercise 2.16]. Since $z \mapsto \mathbb{P}^{(z,z)}(F)$ is lower semi-continuous we get (5.5).

Fix $\epsilon > 0$, and define

$$\tau_1 := \inf\{t > \epsilon \,:\, X_t^1 = X_t^2\}, \quad \tau_1^x := \inf\{t > \epsilon : \quad X_t^1 = X_t^2 = x\}, \quad x \in \mathbb{R},$$

and

$$\hat{A}_1(\omega) := \left\{(x,x) \in \mathbb{R}^2 \,:\, \tau_1^x(\omega) < \infty\right\}.$$

We have

$$\mathbb{P}'\Big(\omega' \,:\, Z_t^{\theta,1}(\omega') \in \hat{A}_1(\omega) \quad \text{for some } t \in (0,\tau_1]\Big) > 0, \quad \forall \omega \in F.$$

Thus,

$$\inf\Big\{\zeta \,:\, \hat{A}_1(\omega) \quad \text{is non-polar for } Z^{\zeta,1}\Big\} \leq 2-\alpha,$$

which implies by Lemma 5.4 that for all $\omega \in F$

$$\begin{aligned}\dim \hat{A}_1(\omega) &= 1 - \inf\Big\{\zeta \,:\, \hat{A}_1(\omega) \quad \text{is non-polar for } Z^{\zeta,1}\Big\} \\ &\geq 1-(2-\alpha) = \alpha - 1.\end{aligned}$$

For $(z,z) \in \mathcal{O}$ we get

$$\mathbb{P}^{(z,z)}\Big(\dim \hat{A}_1(\cdot) \geq \alpha - 1\Big) \geq \mathbb{P}^{(z,z)}(F) \geq \delta > 0. \tag{5.6}$$

Let us now show that

$$\dim \hat{A}(\omega) \geq \alpha - 1 \quad \mathbb{P}^{(z,z)}\text{-a.e. for all } z \in \mathbb{R}.$$

Let $\tau_0(\omega) = 0$ and define

$$\tau_n(\omega) := \inf\left\{t > \tau_{n-1}(\omega) + \epsilon \,:\, (X_t^1(\omega), X_t^2(\omega)) = (x,x) \in K\right\},$$

where K is as above. Since the process $X^1 - X^2$ is point recurrent, the stopping times τ_n are almost surely finite. Define $G_1(\omega) := \dim \hat{A}_1(\omega)$, and for $n \geq 2$

$$G_n(\omega) := \dim\left\{(x,x) \in \mathbb{R}^2 \,:\, X_t^1 = X_t^2 = x \quad \text{for some } t \in (\tau_{n-1}(\omega), \tau_n(\omega)]\right\}.$$

Note that $\dim \hat{A}(\omega) \geq \sup_n G_n(\omega)$. Using the Markov property and (5.6) we get

$$\begin{aligned}
&\mathbb{P}^{(z,z)}\Big(\dim \hat{A} < 1-\theta\Big) \\
&\quad \leq \mathbb{P}^{(z,z)}\Big(\sup_n G_n < 1-\theta\Big) \\
&\quad \leq \mathbb{P}^{(z,z)}\Big(\max_{i\leq n} G_i < 1-\theta\Big) \\
&\quad = \mathbb{E}^{(z,z)}\Big(\mathbb{1}_{\{\max_{1\leq i\leq n-1} G_i < 1-\theta\}} \mathbb{E}^{(z,z)}\left[\mathbb{1}_{\{G_n<1-\theta\}} \,\big|\, \mathcal{F}_{\tau_{n-1}}\right]\Big) \\
&\quad = \mathbb{E}^{(z,z)}\Big(\mathbb{1}_{\{\max_{1\leq i\leq n-1} G_i < 1-\theta\}} \mathbb{E}^{(X^1_{\tau_{n-1}}, X^2_{\tau_{n-1}})}\left[\mathbb{1}_{\{G_1<1-\theta\}}\right]\Big) \\
&\quad \leq (1-\delta)\mathbb{E}^{(z,z)}\left(\mathbb{1}_{\{\max_{1\leq i\leq n-1} G_i < 1-\theta\}}\right) \\
&\quad \leq (1-\delta)^n
\end{aligned}$$

for all $n \geq 1$. Therefore, $\dim A(\omega) \geq 1-\theta$ $\mathbb{P}$-a.s. Letting $\theta \downarrow 2-\alpha$ along a countable sequence, the claim follows. □

6. Examples: Recurrent processes satisfying Assumption A

In this section we give examples of processes X which satisfy Assumption A and which are recurrent. For simplicity, we will assume that the space dimension $n = 1$.

Example 6.1. Let

$$j(x,u) := \big(n(x,u) + n(u,x)\big)\mathfrak{g}(x-u),$$

where the function $n(x,u)$ is strictly positive, uniformly bounded and Hölder continuous in both variables, and $\mathfrak{g}$ satisfies (2.9). In particular, (2.9) holds true for $\mu(du) = \mathfrak{g}(u)du$, if

a) $\mathfrak{g}$ is even and $\mathfrak{g}(h) \leq C|h|^{-1-\eta}$ for some $\eta > 1$ and all $|u| \geq 1$;
b) There exists some $\epsilon \in (0,1)$ such that $h^{2+\epsilon}\mathfrak{g}(h)$ is increasing on $(0,1]$;
c) There exits some $\delta > 1$ such that the function $h^{\delta}\mathfrak{g}(h)$ is decreasing on $(0,1]$.

Let us check (2.9). As in Section 2 we write

$$q(\xi) = \int_{\mathbb{R}\setminus\{0\}} \big(1 - \cos(\xi h)\big)\mathfrak{g}(h)\,dh$$

and we define the corresponding upper and lower symbols $q^U(\xi)$ and $q^L(\xi)$. The conditions a)–c) imply (2.4). Indeed, by b) we have for $|\xi| \geq 1$

$$q^L(\xi) = \frac{2}{\xi}\int_0^1 v^2\mathfrak{g}\left(\tfrac{v}{\xi}\right)\,dv = \frac{2}{\xi}\int_0^1 v^{-\epsilon}v^{2+\epsilon}\mathfrak{g}\left(\tfrac{v}{\xi}\right)\,dv \leq \frac{c_1}{\xi}\mathfrak{g}\left(\tfrac{1}{\xi}\right),$$

and by c) we get

$$q^L(\xi) = \frac{2}{\xi}\int_0^1 v^2\mathfrak{g}\left(\tfrac{v}{\xi}\right)\,dv \geq \frac{2}{\xi}\int_0^1 v^{2-\delta}v^{\delta}\mathfrak{g}\left(\tfrac{v}{\xi}\right)\,dv \geq \frac{c_2}{\xi}\mathfrak{g}\left(\tfrac{1}{\xi}\right),$$

$$\int_{1/\xi}^1 \mathfrak{g}(h)\,dh = \int_{1/\xi}^1 h^{-\delta}h^{\delta}\mathfrak{g}(h)\,dh \leq \frac{c_3}{\xi}\mathfrak{g}\left(\tfrac{1}{\xi}\right),$$

which implies (2.4). Since $(1-\cos 1)q^L(\xi) \leq q(\xi) \leq 2q^U(\xi)$, we have for large $|\xi|$ that $q(\xi) \asymp |\xi|^{-1}\mathfrak{g}\left(|\xi|^{-1}\right)$.

Note that the estimate in a) gives

$$\begin{aligned} q(\xi) &= \int_{|h|\leq 1}(1-\cos\xi h)\,\mathfrak{g}(h)\,dh + \int_{|h|\geq 1}(1-\cos\xi h)\mathfrak{g}(h)\,dh \\ &\leq c_1\xi^2 + c_2|\xi|^{\eta}\int_{\xi}^{\infty}(1-\cos v)\frac{dv}{v^{1+\eta}} \\ &\leq c_3|\xi|^{2\wedge\eta}, \qquad |\xi| \leq 1. \end{aligned}$$

Assume also that

$$\int_{|h|\leq 1} |h|\big|j(x,x+h) - j(x,x-h)\big|\,dh < \infty.$$

Then the function

$$k(x) := \frac{1}{2}\int_{|h|\leq 1}\big(j(x,x+h) - j(x,x-h)\big)h\,dh$$

is well-defined.

Consider the operator $\mathcal{L}$ defined by (2.1) with $a(x) = k(x)$, and

$$m(x,h)\,\mu(dh) = j(x,x+h)\,dh = \big(n(x,x+h) + n(x+h,x)\big)\mathfrak{g}(h)\,dh.$$

Then

$$\mathcal{L}f(x) = \int_{\mathbb{R}\setminus\{0\}} \big(f(x+h) - f(x)\big) j(x, x+h)\, dh,$$

which is a symmetric operator, and generates a (symmetric) Dirichlet form

$$\mathcal{E}(\phi,\phi) = \frac{1}{2}\int_{\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}} (\phi(x+h) - \phi(x))^2\, j(x, x+h)\, dx\, dh.$$

The form $\mathcal{E}(\cdot,\cdot)$ is comparable with the Dirichlet form $\mathcal{E}^q(\cdot,\cdot)$ corresponding to the Lévy process Z with characteristic exponent q:

$$\mathcal{E}(\phi,\phi) \asymp \mathcal{E}^q(\phi,\phi) := \int_{\mathbb{R}} q(\xi)|\hat{\phi}(\xi)|^2 d\xi = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}\setminus\{0\}} \big(\phi(x+h) - \phi(x)\big)^2 \mathfrak{g}(h)\, dh\, dx.$$

The Dirichlet form $\mathcal{E}^q$ is recurrent, because the related Lévy process Z is recurrent by the Chung-Fuchs criterion, i.e. $\int_{|\xi|\le 1} q(\xi)^{-1}\, d\xi = \infty$. By Oshima's criterion, cf. Ōshima (1992), the form $\mathcal{E}$ is also recurrent implying the recurrence of the related process X.

Example 6.2. Let $\mathcal{L}$ be the generator defined by (2.1). In order to construct an example of a non-symmetric recurrent Markov process satisfying Assumption A, we use the approach from Wang (2008). Note that our Assumption A implies the condition (H) needed in Wang (2008). According to Wang (2008) [Theorem 1.4] the following condition is sufficient for the recurrence of the process X:

$$B(x)x + D(x)|x| \le C \quad \text{for sufficiently large } |x|, \tag{6.1}$$

where $B(x) := b(x) + \int_{1<|z|\le|x|} z m(x,z)\, \mu(dz)$ and $D(x) := \int_{|z|\ge|x|} |z| m(x,z)\, \mu(dz)$, with $b(x)$ and $m(x,u)$ from the representation of $\mathcal{L}$ in (2.1). Thus, by Wang (2008) [Theorem 1.4], the process X which corresponds to (2.1) is recurrent, if (6.1) holds true.

7. Appendix

Proof of Lemma 3.5. Without loss of generality we may assume that D is a closed set. We begin with the upper bound for

$$R_\lambda \varpi(x) = \int_0^1 \int_D e^{-\lambda t} p_t(x,y)\, \varpi(dy)\, dt + \int_1^\infty \int_D e^{-\lambda t} p_t(x,y)\, \varpi(dy)\, dt.$$

The upper estimate for the second term can be proved in the same way as Kuwae and Takahashi (2006) [(3.3)]: for any $x \in D$ and $\lambda > 0$ one finds that

$$\int_1^\infty \int_D e^{-\lambda t} p_t(x,y)\, \varpi(dy)\, dt \le \frac{e^{-\lambda}}{1-e^{-\lambda}} \sup_{x\in D} \int_0^1 \int_D p_s(x,y)\, \varpi(dy)\, ds,$$

where we used in the last line that the integral on the right-hand side is finite since $\varpi \in \mathcal{S}_K$. Therefore,

$$\begin{aligned}\int_0^1 \int_D e^{-\lambda t} p_t(x,y)\, \varpi(dy)\, dt &\le R_\lambda \varpi(x) \\ &\le \left(1 + \frac{e^{-\lambda}}{1-e^{-\lambda}}\right) \sup_{x\in D} \int_0^1 \int_D e^{-\lambda t} p_t(x,y)\, \varpi(dy)\, dt.\end{aligned}$$

Using the upper and lower bounds (2.7), (2.8) for the heat kernel, we obtain by a change of variables

$$\begin{aligned}
&\int_0^1 \int_D e^{-\lambda t} p_t(x,y)\,\varpi(dy)\,dt \\
&\qquad \geq a_1 \int_0^1 \int_D e^{-\lambda t} \rho_t^n (1 - a_2|x-y|\rho_t)_+\,\varpi(dy)\,dt \\
&\qquad = a_1 \int_0^1 \int_0^1 e^{-\lambda t} \rho_t^n\,\varpi\left\{y \in D \,:\, (1 - a_2|x-y|\rho_t)_+ \geq 1 - r\right\}\,dr\,dt \\
&\qquad = a_1 \int_0^1 \int_0^1 e^{-\lambda t} \rho_t^n\,\varpi\left\{y \in D \,:\, a_2|x-y|\rho_t \leq r\right\}\,dr\,dt.
\end{aligned}$$

Using the lower bound in (2.11) for the d-measure ϖ, we get for $x \in D$

$$\begin{aligned}
\int_0^1 \int_D e^{-\lambda t} p_t(x,y)\,\varpi(dy)\,dt &\geq c_1 a_2^{-d} \int_0^1 \int_0^1 \rho_t^{n-d} e^{-\lambda t} r^d\,dt\,dr \\
&= c\lambda^{-1} \int_0^\lambda e^{-u} \rho_{u/\lambda}^{n-d}\,du.
\end{aligned}$$

Similarly, using (2.8) we get

$$\begin{aligned}
&\int_0^1 \int_D e^{-\lambda t} p_t(x,y)\,\varpi(dy)\,dt \\
&\leq a_3 \int_0^1 \int_{\mathbb{R}^n} \int_D e^{-\lambda t} \rho_t^n e^{-a_4|x-y-z|\rho_t}\,\varpi(dy)\,Q_t(dz)\,dt \\
&\leq a_3 \int_0^1 \int_{\mathbb{R}^n} \int_0^\infty e^{-\lambda t} \rho_t^n\,\varpi\left\{y \in D \,:\, a_4|x-y-z|\rho_t \leq r\right\} e^{-r}\,dr\,Q_t(dz)\,dt \\
&\leq c_4 \int_0^1 \int_0^\infty e^{-t\lambda} \rho_t^n \sup_{w \in \mathbb{R}^n} \varpi\left\{y \in D \,:\, a_4|w-y|\rho_t \leq r\right\} e^{-r}\,dr\,dt \\
&\leq c_5 \int_0^1 e^{-t\lambda} \rho_t^{n-d}\,dt \cdot \int_0^\infty r^d e^{-r}\,dr \\
&= c_6 \lambda^{-1} \int_0^\lambda e^{-u} \rho_{u/\lambda}^{n-d}\,du.
\end{aligned} \tag{7.1}$$

Here we used the upper bound (2.11) for small r, and the fact that $\operatorname{supp}\varpi = D$, which implies $\sup_x \varpi(B(x,r)) \leq C$ for large r. This proves that $\sup_y R_\lambda\varpi(y) < \infty$.

Therefore, we see

$$\liminf_{\lambda\to\infty} \frac{R_\lambda\varpi(x)}{\sup_{y\in D} R_\lambda\varpi(y)} \geq \liminf_{\lambda\to\infty} \frac{R_\lambda\varpi(x)}{\sup_{y\in\mathbb{R}^n} R_\lambda\varpi(y)} \geq c > 0,$$

and by Lemma 3.3 all points of D are regular. □

Proof of Lemma 4.1. The case $\gamma = 1$ is already contained in Knopova and Kulik (2016). Therefore, we consider only $\gamma \in (0,1)$. Without loss of generality we may assume that D is closed.

a) Under our assumptions the transition density $p_t(x,y)$ of the process X satisfies (2.7) and (2.8) for $t \in (0,1]$. Using (2.8) and the scaling property of the subordinator (4.1), we get for any $T \in (0,1]$

$$\begin{aligned}
&\int_0^T \int_D p_t^{(\gamma)}(x,y)\,\varpi(dy)\,dt\\
&\qquad\le C\int_0^T\int_D\int_0^1 \rho_s^n\big(f_{\mathrm{up}}(\rho_s\cdot)*Q_s\big)(x-y)t^{-1/\gamma}\sigma_1^{(\gamma)}(t^{-1/\gamma}s)\,ds\,\varpi(dy)\,dt\\
&\qquad\quad+\int_0^T\int_D\int_1^\infty p_s(x,y)t^{-1/\gamma}\sigma_1^{(\gamma)}(t^{-1/\gamma}s)\,ds\,\varpi(dy)\,dt\\
&\qquad=: CI_1(x,T)+I_2(x,T).
\end{aligned}$$

We estimate $I_1(x,T)$ and $I_2(x,T)$ separately. For $I_2(x,T)$ we have

$$\begin{aligned}
I_2(x,T) &= \int_0^T\int_{t^{-1/\gamma}}^\infty\int_D p_{\tau t^{1/\gamma}}(x,y)\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,d\tau\,dt\\
&= \int_{T^{-1/\gamma}}^\infty\int_{\tau^{-\gamma}}^T\int_D p_{\tau t^{1/\gamma}}(x,y)\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dt\,d\tau\\
&= \int_{T^{-1/\gamma}}^\infty\int_1^{\tau^\gamma T}\int_D p_{v^{1/\gamma}}(x,y)\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dv\,d\tau.
\end{aligned}$$

Note that $\sup_{x,y\in\mathbb{R}^n} p_t(x,y)\le c$ for all $t\ge 1$. Indeed, since for $0<\epsilon<1$ we have $p_\epsilon(x,y)\le C_\epsilon$ for all $x,y\in\mathbb{R}^n$, the Chapman–Kolmogorov relation implies

$$p_t(x,y)=\int_{\mathbb{R}^n} p_{t-\epsilon}(x,z)p_\epsilon(z,y)\,dz\le C_\epsilon.$$

Therefore,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} I_2(x,T) &\le c_1\int_{T^{-1/\gamma}}^\infty(\tau^\gamma T-1)\varpi(D)\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,d\tau\\
&\le c_1\varpi(D)T\int_{T^{-1/\gamma}}^\infty\sigma_1^{(\gamma)}(\tau)\,d\tau\le c_1\varpi(D)T\xrightarrow[T\to0]{}0,
\end{aligned}$$

where we used that $\int_0^\infty\sigma_1^{(\gamma)}(\tau)\,d\tau=1$.

For the first integral expression $I_1(x,T)$ we have

$$\begin{aligned}
&I_1(x,T)\\
&=\int_0^T\int_0^{t^{-1/\gamma}}\int_D\rho^n_{\tau t^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{\tau t^{1/\gamma}}\cdot)*Q_{\tau t^{1/\gamma}}\big)(x-y)\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,d\tau\,dt\\
&=\int_0^{T^{-1/\gamma}}\int_0^T\int_D\rho^n_{\tau t^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{\tau t^{1/\gamma}}\cdot)*Q_{\tau t^{1/\gamma}}\big)(x-y)\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dt\,d\tau\\
&\quad+\int_{T^{-1/\gamma}}^\infty\int_0^{\tau^{-\gamma}}\int_D\rho^n_{\tau t^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{\tau t^{1/\gamma}}\cdot)*Q_{\tau t^{1/\gamma}}\big)(x-y)\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dt\,d\tau\\
&=\int_0^{T^{-1/\gamma}}\int_0^{\tau^\gamma T}\int_D\rho^n_{v^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{v^{1/\gamma}}\cdot)*Q_{v^{1/\gamma}}\big)(x-y)\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dv\,d\tau\\
&\quad+\int_{T^{-1/\gamma}}^\infty\int_0^1\int_D\rho^n_{v^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{v^{1/\gamma}}\cdot)*Q_{v^{1/\gamma}}\big)(x-y)\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dv\,d\tau\\
&=: I_{11}(x,T)+I_{12}(x,T).
\end{aligned}$$

For $I_{12}(x,D)$ we have

$$I_{12}(x,T)=\left[\int_{T^{-1/\gamma}}^{\infty}\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,d\tau\right]\int_0^1\!\!\int_D\rho^n_{v^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{v^{1/\gamma}}\cdot)*Q_{v^{1/\gamma}}\big)(x-y)\,\varpi(dy)\,dv.$$

Since $\lim_{T\to 0}\int_{T^{-1/\gamma}}^{\infty}\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,d\tau=0$, we get $\lim_{T\to0}I_{12}(x,T)=0$, if we can show that

$$\sup_{x\in\mathbb{R}^n}\int_0^1\int_D\rho^n_{v^{1/\gamma}}\big(f_{\mathrm{up}}(\rho_{v^{1/\gamma}}\cdot)*Q_{v^{1/\gamma}}\big)(x-y)\,\varpi(dy)\,dv<\infty\quad\text{for some }\gamma\in(\gamma_{\inf},1).\tag{7.2}$$

Set $\ell:=e_1=(1,0,\dots0)^\top$ and $\theta_t:=\inf\{r\,:\,q^U(r\ell)\geq 1/t\}$. Because of (2.4) we have $\theta_t\asymp\rho_t$ for all $t\in(0,1]$. Moreover, the mapping $r\mapsto q^U(r\ell)$ is absolutely continuous, and we have, cf. Knopova (2013),

$$q^U(r_2\ell)-q^U(r_1\ell)=2\int_{r_1}^{r_2}\frac{q^L(v\ell)}{v}\,dv,\qquad 0<r_1<r_2.\tag{7.3}$$

Thus, the above calculations give

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n}\int_0^1\int_D\rho^n_{v^{1/\gamma}}\cdot\big(f_{\mathrm{up}}(\rho_{v^{1/\gamma}}\cdot)*Q_{v^{1/\gamma}}\big)(x-y)\,\varpi(dy)\,dv\\
&\leq c_1\sup_{x\in\mathbb{R}^n}\int_0^1\int_D\theta^n_{v^{1/\gamma}}\cdot\big(f_{\mathrm{up}}(c_2\theta_{v^{1/\gamma}}\cdot)*Q_{v^{1/\gamma}}\big)(x-y)\,\varpi(dy)\,dv\\
&=c_1a_3\sup_{x\in\mathbb{R}^n}\int_0^1\int_{\mathbb{R}^n}\int_0^\infty\theta^n_{v^{1/\gamma}}\cdot\varpi\left\{y\in D\,:\,e^{-c_2a_4|x-y-z|\theta_{v^{1/\gamma}}}\geq s\right\}\otimes\\
&\qquad\qquad\otimes ds\,Q_{v^{1/\gamma}}(dz)\,dv\\
&=c_1c_2a_3a_4\sup_{x\in\mathbb{R}^n}\int_0^1\int_{\mathbb{R}^n}\int_0^\infty\theta^n_{v^{1/\gamma}}\cdot\varpi\left\{y\in D:|x-y-z|\theta_{v^{1/\gamma}}\leq r\right\}e^{-c_2a_4r}\otimes\\
&\qquad\qquad\otimes dr\,Q_{v^{1/\gamma}}(dz)\,dv\\
&=C_1\int_0^\infty\int_0^1\int_{\mathbb{R}^n}\sup_{x\in\mathbb{R}^n}\frac{\varpi\left\{y\in D\,:\,|x-y-z|\leq ur\right\}}{(q^U)^\gamma(\ell u^{-1})}e^{-C_2r}\,\tilde{Q}_u(dz)\,\frac{du}{u^{n+1}}\,dr\\
&\leq\kappa C_1\int_0^\infty\int_0^1\sup_{\xi\in\mathbb{R}^n}\frac{\varpi\left\{y\in D\,:\,|\xi-y|\leq ur\right\}}{(q^*)^\gamma(1/u)}e^{-C_2r}\,\frac{du}{u^{n+1}}\,dr,
\end{aligned}\tag{7.4}$$

where $\tilde{Q}_u(dz)=Q_{v^{1/\gamma}}(dz)$ under the change of variables $v=(q^U)^{-\gamma}(\ell u^{-1})$, which was done in the second line from below. Note that $\frac{1}{t}=q^U(\theta_t\ell)$, and that by (7.3) and $q^U\asymp q^L$, one has

$$\frac{dv}{du}\asymp(q^U)^{-\gamma}(\ell u^{-1})u^{-1}.$$

The constant κ is from (2.4).

Let us estimate the integrals in the last line of (7.4). Without loss of generality we assume that $C_2=1$. Put $h(r):=\sup_{\xi\in\mathbb{R}^n}\varpi\left\{y\in D\,:\,|\xi-y|\leq r\right\}$. Split

$$J:=\left[\int_0^1\int_0^1+\int_1^\infty\int_0^1\right]\frac{h(ur)}{(q^*)^\gamma(1/u)}\,\frac{du}{u^{n+1}}\,e^{-r}\,dr=:J_1+J_2.$$

From the monotonicity of $h(r)$ and the assumption (4.3)

$$J_1 \le \int_0^1 \frac{h(u)}{(q^*)^\gamma(1/u)}\,\frac{du}{u^{n+1}} \cdot \int_0^1 e^{-r}\,dr < \infty.$$

Using the monotonicity of q^*, we get

$$\begin{aligned} J_2 &\le c_1 \int_1^\infty \left[\int_0^r \frac{h(v)}{(q^*)^\gamma(1/v)}\,\frac{dv}{v^{n+1}}\right] r^n e^{-r}\,dr \\ &= \left[\int_1^\infty \int_0^1 + \int_1^\infty \int_1^r\right] \frac{h(v)}{(q^*)^\gamma(1/v)}\, r^n e^{-r}\,\frac{dv}{v^{n+1}}\,dr =: J_{21} + J_{22}. \end{aligned}$$

Clearly, $J_{21} < \infty$. For J_{22} we have

$$\begin{aligned} J_{22} &\le \int_0^\infty \left[\int_v^\infty r^n e^{-r}\,dr\right] \frac{h(v)}{(q^*)^\gamma(1/v)}\,\frac{dv}{v^{n+1}} \\ &\le c_2 \int_0^\infty e^{-\epsilon v}\left[\int_v^\infty r^n e^{-(1-\epsilon)r}\,dr\right] \frac{h(v)}{(q^*)^\gamma(1/v)}\,\frac{dv}{v^{n+1}} \\ &\le c_3 \int_0^\infty e^{-\epsilon v} \frac{h(v)}{(q^*)^\gamma(1/v)}\,\frac{dv}{v^{n+1}}. \end{aligned}$$

By (4.3) and the fact that the integrand is bounded by $Ce^{\epsilon v}$ for $v > 1$ we derive that the integral in the last line is finite. Thus, (7.2) holds true, implying that $\sup_{x\in\mathbb{R}^n} I_{12}(x,T) \to 0$ as $T \to 0$.

Let us estimate $I_{11}(x,T)$. Define $\phi(u) := 1/\theta_u$,

$$I(v,\tau,T) := \mathbb{1}_{\{\tau \le T^{-1/\gamma}\}} e^{-\epsilon v/(2\phi(T^{1/\gamma}\tau))},$$

and recall that $h(r) := \sup_{\xi\in\mathbb{R}^n} \varpi\{y \in D : |\xi - y| \le r\}$. By the same arguments as those which we have used in (7.4), we derive

$$\begin{aligned} &\sup_{x\in\mathbb{R}^n} I_{11}(x,T) \\ &\le c_1 \sup_{x\in\mathbb{R}^n} \int_0^{T^{-1/\gamma}} \int_0^{T\tau^\gamma} \int_D \theta^n_{v^{1/\gamma}} \cdot \big(f_{\mathrm{up}}(c_2\theta_{v^{1/\gamma}}\cdot) * Q_{v^{1/\gamma}}\big)(x-y)\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\varpi(dy)\,dv\,d\tau \\ &\le c_2 \int_0^{T^{-1/\gamma}} \int_0^\infty \int_0^{\phi(T^{1/\gamma}\tau)} \frac{h(ur)}{(q^*)^\gamma(1/u)}\, e^{-c_3 r}\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\frac{du}{u^{n+1}}\,dr\,d\tau \\ &\le c_2 \int_0^{T^{-1/\gamma}} \int_0^1 \int_0^{\phi(T^{1/\gamma}\tau)} \frac{h(u)}{(q^*)^\gamma(1/u)}\, e^{-c_3 r}\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\frac{du}{u^{n+1}}\,dr\,d\tau \\ &\qquad + c_2 \int_0^{T^{-1/\gamma}} \int_1^\infty \int_0^{\phi(T^{1/\gamma}\tau)} \frac{h(ur)}{(q^*)^\gamma(1/u)}\, e^{-c_3 r}\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\frac{du}{u^{n+1}}\,dr\,d\tau. \end{aligned}$$

The first term is estimated from above by

$$\int_0^{T^{-1/\gamma}} \int_0^{\phi(T^{1/\gamma}\tau)} \frac{h(u)}{(q^*)^\gamma(1/u)}\,\tau^{-\gamma}\sigma_1^{(\gamma)}(\tau)\,\frac{du}{u^{n+1}}\,d\tau,$$

which tends to zero as $T \to 0$ by the dominated convergence theorem.

For the second term we have for some $\epsilon > 0$

$$\begin{aligned}
&\int_0^{T^{-1/\gamma}} \int_1^\infty \int_0^{r\phi(T^{1/\gamma}\tau)} \frac{h(v)}{(q^*)^\gamma(1/v)}\, r^n e^{-c_3 r} \tau^{-\gamma} \sigma_1^{(\gamma)}(\tau)\, \frac{dv}{v^{n+1}}\, dr\, d\tau \\
&\qquad \le \int_0^{T^{-1/\gamma}} \int_0^\infty \int_{v/\phi(T^{1/\gamma}\tau)}^\infty \frac{h(v)}{(q^*)^\gamma(1/v)}\, r^n e^{-c_3 r} \tau^{-\gamma} \sigma_1^{(\gamma)}(\tau) dr\, \frac{dv}{v^{n+1}}\, d\tau \\
&\qquad \le c_4 \int_0^{T^{-1/\gamma}} \int_0^\infty e^{-\epsilon v/\phi(T^{1/\gamma}\tau)} \frac{h(v)}{(q^*)^\gamma(1/v)} \tau^{-\gamma} \sigma_1^{(\gamma)}(\tau)\, \frac{dv}{v^{n+1}}\, d\tau \\
&\qquad \le c_4 \int_0^\infty \int_0^\infty \frac{e^{-\epsilon v/2\phi(1)} h(v)}{(q^*)^\gamma(1/v)}\, \tau^{-\gamma} \sigma_1^{(\gamma)}(\tau) I(v,\tau,T)\, \frac{dv}{v^{n+1}}\, d\tau.
\end{aligned}$$

Note that $I(v,\tau,T) \le 1$, and $\lim_{T\to 0} I(v,\tau,T) = 0$ a.e. From Euler's Gamma-integral $s^{-a} = \Gamma(a)^{-1} \int_0^\infty e^{-sx} x^{a-1}\, dx$, $a > 0$, we derive

$$\begin{aligned}
\int_0^\infty s^{-a} \sigma_1^{(\gamma)}(s)\, ds &= \int_0^\infty \int_0^\infty \frac{e^{-sx} x^{a-1}}{\Gamma(a)}\, \sigma_1^{(\gamma)}(s)\, ds\, dx \\
&= \int_0^\infty \frac{e^{-x^\gamma} x^{a-1}}{\Gamma(a)}\, dx = \frac{\Gamma(a/\gamma)}{\gamma \Gamma(a)}.
\end{aligned}$$

By the dominated convergence theorem, $\lim_{T\to 0} \sup_{x\in\mathbb{R}^n} I_{11}(x,T) = 0$. This finishes the proof of a).

b) Without loss of generality we may assume that $T \in (0, 1/2]$. Using (2.7), we have

$$\begin{aligned}
&\int_0^T \int_D p^{(\gamma)}(t,x,y)\, \varpi(dy)\, dt \\
&\qquad \ge \int_0^T \int_D \int_0^\infty \rho_s^n f_{\mathrm{low}}(\rho_s(x-y)) t^{-1/\gamma} \sigma_1^{(\gamma)}(t^{-1/\gamma} s)\, ds\, \varpi(dy)\, dt \\
&\qquad \ge \int_0^T \int_D \int_0^1 \rho_s^n f_{\mathrm{low}}(\rho_s(x-y)) t^{-1/\gamma} \sigma_1^{(\gamma)}(t^{-1/\gamma} s)\, ds\, \varpi(dy)\, dt \\
&\qquad = \int_0^T \int_0^{t^{-1/\gamma}} \int_D \rho_{\tau t^{1/\gamma}}^n f_{\mathrm{low}}(\rho_{\tau t^{1/\gamma}}(x-y)) \sigma_1^{(\gamma)}(\tau)\, \varpi(dy)\, d\tau\, dt \\
&\qquad = \int_0^{T^{-1/\gamma}} \int_0^T \int_D \rho_{\tau t^{1/\gamma}}^n f_{\mathrm{low}}(\rho_{\tau t^{1/\gamma}}(x-y)) \sigma_1^{(\gamma)}(\tau)\, \varpi(dy)\, dt\, d\tau \\
&\qquad\quad + \int_{T^{-1/\gamma}}^\infty \int_0^{\tau^{-\gamma}} \int_D \rho_{\tau t^{1/\gamma}}^n f_{\mathrm{low}}(\rho_{\tau t^{1/\gamma}}(x-y)) \sigma_1^{(\gamma)}(\tau)\, \varpi(dy)\, dt\, d\tau \\
&\qquad \ge \int_0^{T^{-1/\gamma}} \int_0^T \int_D \rho_{\tau t^{1/\gamma}}^n f_{\mathrm{low}}(\rho_{\tau t^{1/\gamma}}(x-y)) \sigma_1^{(\gamma)}(\tau)\, \varpi(dy)\, dt\, d\tau \\
&\qquad = \int_0^{T^{-1/\gamma}} \int_0^{\tau^\gamma T} \int_D \rho_{v^{1/\gamma}}^n f_{\mathrm{low}}(\rho_{v^{1/\gamma}}(x-y)) \tau^{-\gamma} \sigma_1^{(\gamma)}(\tau)\, \varpi(dy)\, dv\, d\tau \\
&\qquad \ge \int_1^{2^{1/\gamma}} \tau^{-\gamma} \sigma_1^{(\gamma)}(\tau)\, d\tau \int_0^T \int_D \rho_{v^{1/\gamma}}^n f_{\mathrm{low}}(\rho_{v^{1/\gamma}}(x-y))\, \varpi(dy)\, dv.
\end{aligned}$$

Using the form of f_{low} and the fact that $\rho_t \asymp \theta_t$, we see that the double integral is bounded from below by

$$\int_0^T \int_D \theta^n_{v^{1/\gamma}} f_{\text{low}}(c\theta_{v^{1/\gamma}}(x-y))\,\varpi(dy)\,dv,$$

where $c > 0$ is some constant. Proceeding as in the estimate for $I_{11}(x,T)$, we get for this expression

$$\begin{aligned}
&\int_0^T \int_D \theta^n_{v^{1/\gamma}} f_{\text{low}}(c_2\theta_{v^{1/\gamma}}(x-y))\,\varpi(dy)\,dv \\
&\qquad = \int_0^T \int_0^1 \theta^n_{v^{1/\gamma}} \varpi\left\{y \in D \,:\, c_2 d_2 \theta_{v^{1/\gamma}}|x-y| \le r\right\}\,dr\,dv \\
&\qquad \ge c_3 \int_0^{\phi(T^{1/\gamma})} \int_0^1 \frac{\varpi\left\{y\in D \,:\, |x-y| \le ru\right\}}{(q^*)^\gamma(1/u)}\,dr\,\frac{du}{u^{n+1}} \\
&\qquad \ge \frac{c_3}{2} \int_0^{\phi(T^{1/\gamma})} \frac{\varpi\left\{y \in D \,:\, |x-y| \le 2^{-1}u\right\}}{(q^*)^\gamma(1/u)}\,\frac{du}{u^{n+1}}.
\end{aligned}$$

Combining everything, we have shown for some constant $C > 0$

$$\int_0^T \int_D p_t^{(\gamma)}(x,y)\,\varpi(dy)\,dt \ge C\int_0^{\phi(T^{1/\gamma})} \frac{\varpi\left\{y\in D \,:\, |x-y|\le 2^{-1}u\right\}}{(q^*)^\gamma(1/u)}\,\frac{du}{u^{n+1}}. \tag{7.5}$$

Therefore, whenever $\varpi \in \mathcal{S}_K$ with respect to $p_t^{(\gamma)}(x,y)$, then (4.4) holds true. $\square$

Acknowledgements

The authors would like to thank an anonymous referee for his careful reading of the paper, helpful criticism and comments. This helped to make the article more accessible.

References

N.H. Bingham, C.M. Goldie and J.L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (1987). ISBN 0-521-30787-2. MR898871.

R.M. Blumenthal and R.K. Getoor. A dimension theorem for sample functions of stable processes. *Illinois J. Math.* **4**, 370–375 (1960). MR0121881.

R.M. Blumenthal and R.K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York-London (1968). MR0264757.

B. Böttcher, R. Schilling and J. Wang. *Lévy matters. III: Lévy-type processes: construction, approximation and sample path properties*, volume 2099 of *Lecture Notes in Mathematics*. Springer, Cham (2013). ISBN 978-3-319-02683-1; 978-3-319-02684-8. MR3156646.

E.B. Dynkin. *Markov processes. Vols. I, II*, volume 121 & 122 of *Die Grundlehren der Mathematischen Wissenschaften*. Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg (1965). MR0193671.

W Farkas, N Jacob and R.L. Schilling. Function spaces related to continuous negative definite functions: ψ-Bessel potential spaces. *Dissertationes Math. (Rozprawy Mat.)* **393**, 62 (2001). MR1840499.

M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition (2011). ISBN 978-3-11-021808-4. MR2778606.

J. Hawkes. Polar sets, regular points and recurrent sets for the symmetric and increasing stable processes. *Bull. London Math. Soc.* **2**, 53–59 (1970). MR0261693.

J. Hawkes. On the Hausdorff dimension of the intersection of the range of a stable process with a Borel set. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **19**, 90–102 (1971). MR0292165.

J. Hawkes. Local times and zero sets for processes with infinitely divisible distributions. *J. London Math. Soc. (2)* **8**, 517–525 (1974). MR0362509.

N. Jacob. *Pseudo differential operators and Markov processes. Vol. I: Fourier analysis and semigroups*. Imperial College Press, London (2001). ISBN 1-86094-293-8. MR1873235.

N. Jain and W.E. Pruitt. Collisions of stable processes. *Illinois J. Math.* **13**, 241–248 (1969). MR0240873.

V. Knopova. Compound kernel estimates for the transition probability density of a Lévy process in $\mathbb{R}^n$. *Teor. Ĭmovīr. Mat. Stat.* (89), 51–63 (2013). MR3235175.

V. Knopova and A. Kulik. Intrinsic small time estimates for distribution densities of Lévy processes. *Random Oper. Stoch. Equ.* **21** (4), 321–344 (2013). MR3139314.

V. Knopova and A. Kulik. Parametrix construction of the transition probability density of the solution to an SDE driven by α-stable noise. *ArXiv Mathematics e-prints* (2014). arXiv: 1412.8732.

V. Knopova and A. Kulik. Parametrix construction for certain Lévy-type processes. *Random Oper. Stoch. Equ.* **23** (2), 111–136 (2015). MR3353627.

V. Knopova and A. Kulik. Intrinsic compound kernel estimates for the transition probability density of a Lévy type process and their applications. *To appear in Probab. Math. Stat.* (2016). arXiv: 1308.0310.

V. Knopova, R.L. Schilling and J. Wang. Lower bounds of the Hausdorff dimension for the images of Feller processes. *Statist. Probab. Lett.* **97**, 222–228 (2015). MR3299774.

K. Kuwae and M. Takahashi. Kato class functions of Markov processes under ultracontractivity. In *Potential theory in Matsue*, volume 44 of *Adv. Stud. Pure Math.*, pages 193–202. Math. Soc. Japan, Tokyo (2006). MR2277833.

K. Kuwae and M. Takahashi. Kato class measures of symmetric Markov processes under heat kernel estimates. *J. Funct. Anal.* **250** (1), 86–113 (2007). MR2345907.

Y. Ōshima. On conservativeness and recurrence criteria for Markov processes. *Potential Anal.* **1** (2), 115–131 (1992). MR1245880.

R.L. Schilling. Feller processes generated by pseudo-differential operators: on the Hausdorff dimension of their sample paths. *J. Theoret. Probab.* **11** (2), 303–330 (1998). MR1622573.

R.L. Schilling and L. Partzsch. *Brownian motion. An introduction to stochastic processes*. De Gruyter Graduate. De Gruyter, Berlin, second edition (2014). ISBN 978-3-11-030729-0; 978-3-11-030730-6. MR3234570.

M. Sharpe. *General theory of Markov processes*, volume 133 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA (1988). ISBN 0-12-639060-6.

MR958914.

N.-R. Shieh and Y. Xiao. Hausdorff and packing dimensions of the images of random fields. *Bernoulli* **16** (4), 926–952 (2010). MR2759163.

N.R. Shieh. Collisions of Markov processes. *Tokyo J. Math.* **18** (1), 111–121 (1995). MR1334709.

S.J. Taylor. Multiple points for the sample paths of the symmetric stable process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **5**, 247–264 (1966). MR0202193.

J. Wang. Criteria for ergodicity of Lévy type operators in dimension one. *Stochastic Process. Appl.* **118** (10), 1909–1928 (2008). MR2454470.

Y. Xiao. Random fractals and Markov processes. In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, volume 72 of *Proc. Sympos. Pure Math.*, pages 261–338. Amer. Math. Soc., Providence, RI (2004). MR2112126.