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Survival exponents for fractional Brownian motion with multivariate time

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Abstract. Fractional Brownian motion of index $0 < H < 1$, H-FBM, with d-dimensional time is considered in a spherical domain that contains 0 at its boundary. The main result: the log-asymptotics of the probability that H-FBM does not exceed a fixed positive level is $(H - d + o(1)) \log T$, where $T >> 1$ is the radius of the domain.

1. Introduction

Fractional Brownian motion of index $H \in (0, 1)$, H-FBM, with multivariate time $t \in R^d$ is a centered Gaussian random process $w_H(t)$ with correlation function

$$E w_H(t) w_H(s) = 0.5(|t|^{2H} + |s|^{2H} - |t - s|^{2H}).$$

H-FBM is H-self-similar (H-ss), isotropic, and has stationary increments (si), i.e.,

$$\{w_H(\lambda U t + t_0) - w_H(t_0)\} \doteq \{\lambda^H w_H(t)\}$$

holds in the sense of the equality of finite-dimensional distributions for any fixed t_0, $\lambda > 0$, and orthogonal mapping $U : R^d \to R^d$.

The one-sided exit problem for a random process $\xi(t)$ and its characteristics, the so-called survival exponents:

$$\theta_\xi = \lim_{T \to \infty} - \log P(\xi(t) < 1, t \in \Delta_T)/\psi(T) \tag{1.1}$$

are the subject of intensive analysis in applications. Here Δ_T is an increasing sequence of domains of size T, and $\psi(\cdot)$ is a suitable slowly varying function, typically, $\psi(T) = \log T$ for ss-processes. The greatest progress in this area has been achieved for processes with one-dimensional time. (See surveys by Bray et al., 2013 of the

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physics literature and by Aurzada and Simon, 2015 of the relevant mathematical publications).

H-FBM was one of the first non-trivial examples of non-Markovian processes for which the survival exponents have been found exactly (Molchan, 1999). Namely, for $\psi(T) = \log T$, the survival exponents for H-FBM are

$$\theta_{w_H} = 1 - H, \quad \Delta_T = (0, T) \quad \text{and} \quad \theta_{w_H} = d, \quad \Delta_T = (-T, T)^d. \tag{1.2}$$

Recently, Aurzada et al. (2016) considerably refined the asymptotics of probability

$$p_T = P(w_H(t) < 1,\, t \in \Delta_T), \quad \Delta_T = (0, T) \tag{1.3}$$

and showed that the exponent $\theta = 1 - H$ is universal for a broad class of H-ss processes with stationary increments. The ideas of this work have proved useful in the analysis of the conjecture that $\theta_{w_H} = d - kH$ for $w_H(t)$ in $\Delta_T = [0, T]^k \times [-T, T]^{d-k}$ (Molchan, 2012).

The case $k = 0$ corresponds to the right part of (1.2). The case $k = 1$ is supported by the result which we discuss below: $\theta_{w_H} = d - H$ for fractional Brownian motion in $\Delta_T = T\Delta_1$, where Δ_1 is a unit ball that contains 0 at its boundary.

The main idea of the paper by Aurzada et al. (2016) is to show that for a broad class of si-processes, $\xi(t), \xi(0) = 0$, with discrete time

$$|\Delta_T|\, P(\xi(t) < 1,\, t \in \Delta_T \cap \mathbb{Z}^1) \approx E \max(\xi(t),\, t \in \Delta_T \cap \mathbb{Z}^1), \tag{1.4}$$

where $\Delta_T = [0, T]$, $|\Delta_T| = T$, and $\approx$ means up to a multiplicative term in $T^{o(1)}$. For H-ss processes with continuous time, the right-hand part of (1.4) is proportional to T^H, and therefore the exponent for (1.3) is $1 - H$. However, the result by Aurzada et al. (2016) essentially uses the 1-D nature of time. Considering $|\Delta_T|$ as the volume of Δ_T, relation (1.4) is found to be in formal agreement with the conjecture for $k = 1$, but not for $k > 1$; in addition, (1.4) is very crude for $k = 0$ (see (1.3)). This means that the analysis of the cases $d > 1, k > 1$ needs more ideas.

2. The lower bound

Proposition 2.1. *Let $\xi(t), \xi(0) = 0, t \in R^d$ be a centered isotropic random process with stationary increments. Then*

$$P(\xi(t) < -1, t \in \Delta_T, |t| > 1) \le cT^{-d} E \max(\xi(t), t \in \Delta_T),$$

where $\Delta_T = T\Delta_1$ is a ball of radius T that contains 0 at its boundary.

Consequence 2.2. If $\xi(t)$ is fractional Brownian motion of index $H \in (0, 1)$ in Δ_T, then the survival exponent has the lower bound $\theta^-_{w_H} \ge d - H$.

Remark 2.3. Proposition 2.1 holds for $\Delta_T = [0, T] \times [-T, T]^{d-1}$ as well.

Proof: Let $U_T = \{x_{k,\alpha}, \alpha = 1, 2, ..., n_k; k = 1, 2, ...\}$ be a subset of ball B_T of radius T in R^d; U_T consists of N_T points such that

$$|x_{k,\alpha}| = r_k, \quad |x_{k,\alpha} - x_{m,\beta}| > 1, \quad N_T > CT^d, \quad 1 < r_k < r_{k+1} \le T. \tag{2.1}$$

Consider the following increasing sequence of subsets of U_T:

$$U_{k+1,\alpha} = U_k \cup \bigcup_{\beta=1}^{\alpha} x_{k+1,\beta}, \quad U_k = \{x_{i,\gamma} : |x_{i,\gamma}| \le r_k\}.$$

Fix $\Delta_T = \{t : |t + Te| \le T\}$, where $e = (0, \dots, 0, 1)$. Let $O_{k,\alpha}$ be an orthogonal mapping transforming $x_{k,\alpha}$ in $\tilde{x}_{k,\alpha} = r_k e$. Setting $\tilde{U}_{k,\alpha} = O_{k,\alpha} U_{k,\alpha}$, one has

$$(\tilde{U}_{k,\alpha} - \tilde{x}_{k,\alpha}) \backslash \{0\} \subset \Delta_T \backslash B_1, \quad (k, \alpha) \ne (1, 1). \tag{2.2}$$

Therefore, using the notation $M(A) = \sup(\xi(t),\, t \in A)$, we get

$$p_T(-1) := P(\xi(t) < -1,\, t \in \Delta_T \backslash B_1) \le P(M((\tilde{U}_{k,\alpha} - \tilde{x}_{k,\alpha}) \backslash \{0\}) < -1). \tag{2.3}$$

By the si-property of $\xi(t)$, we can continue

$$= P(M(\tilde{U}_{k,\alpha} \backslash \tilde{x}_{k,\alpha}) - \xi(\tilde{x}_{k,\alpha}) < -1) = P(M(U_{k,\alpha-1}) + 1 < \xi(x_{k,\alpha})). \tag{2.4}$$

The last equality holds because $\xi(t)$ is rotation invariant.

The event $\{M(U_{k,\alpha-1}) + 1 < \xi(x_{k,\alpha})\}$ is measurable relative to the sequence

$$\xi(x_{1,1}), \dots, \xi(x_{1,n_1}); \dots ; \xi(x_{k,1}), \dots, \xi(x_{k,n_k}); \dots \tag{2.5}$$

This event take place when $\xi(x_{k,\alpha})$ is realized as a record in the sequence (2.5) which exceeds the previous one by at least 1. Let ν_T be the number of such records in (2.5). Then, by (2.3, 2.4),

$$(N_T - 1) p_T(-1) \le \sum_{k,\alpha} P(M(U_{k,\alpha}) + 1 < \xi(x_{k,\alpha+1})) = E\nu_T \le E(M(U_T) - \xi(x_{1,1})),$$

where $U_{1,1} = \{x_{1,1}\}, \quad U_{k,n_k} = U_{k+1}, \quad x_{k,n_k+1} = x_{k+1,1}, \quad (k, \alpha) \ne (1, 1)$.

Finally, by (2.1),

$$p_T(-1) \le E(M(U_T))/(N_T - 1) < cT^{-d} E(\sup \xi(t),\, t \in \Delta_T). \tag{2.6}$$

Suppose that $\xi(t)$ is fractional Brownian motion of index $H \in (0, 1)$ in Δ_T. By the standard procedure, we can compare $p_T(-1)$ with

$$p_T(1) = P(w_H(t) < 1,\, t \in (\Delta_T \backslash B_1)). \tag{2.7}$$

For this purpose we can find a continuous function $\varphi_T(t)$ such that

$$\varphi_T(t) = 1,\ |t| > 1,\ \|\varphi_T\|_{H,T} < const, \tag{2.8}$$

where $\|\cdot\|_{H,T}$ is the norm of the Hilbert space $H_H(\Delta_T)$ with the reproducing kernel $Ew_H(t) w_H(s), (t, s) \in \Delta_T \times \Delta_T$ (see for this fact Molchan, 1999 or Appendix). Then

$$p_T(-1) = P(w_H(t) + 2\varphi_T(t) < 1, t \in (\Delta_T \backslash B_1)).$$

According to Aurzada and Dereich (2013),

$$\left| \sqrt{\ln 1/p_T(1)} - \sqrt{\ln 1/p_T(-1)} \right| \le \|2\varphi_T\|_{H,T} / \sqrt{2}. \tag{2.9}$$

From the self-similarity of H-FBM and (2.6) one has

$$p_T(-1) \le cT^{-(d-H)} EM_{w_H}(\Delta_1). \tag{2.10}$$

Combining (2.8-2.10), one has

$$[\ln 1/P(w_H(t) < 1, t \in \Delta_T)]^{1/2} / \sqrt{\ln T} \ge \sqrt{d - H} + O(1/\sqrt{\ln T}), \tag{2.11}$$

i.e., $\theta_{w_H} \ge d - H$. $\square$

3. The upper bound

Below we use notation $M(A) = \sup(w_H(t), t \in A)$ and $|A| = \#\{t : t \in A\}$.

Proposition 3.1. *Let $w_H(t)$ be H-FBM in $\Delta_T = T\Delta_1 \subset R^d$ where Δ_1 is a bounded domain and $0 \in \Delta_1$. Consider a finite 1-net of Δ_T, i.e. a subset $U_T = \{x_k, k = 1, ..., N_T\} \subset \Delta_T$, $\{0\} \notin U_T$ such that*

$$c < N_T/T^d < C \quad \text{and} \quad \Delta_T \subset \bigcup_{r=1}^{N_T} B_1(x_r),$$

where $B_1(x)$ is a unit ball centered at x. Then there is a $0 < q < 1$ such that for all $T > T_0$

$$P(M(\Delta_T) < c_H\sqrt{4d\ln T}) \geq qP(M(U_T) < 0). \tag{3.1}$$

In addition,

$$EM(U_T) = EM(\Delta_T)(1 + o(1)) = T^H EM(\Delta_1)(1 + o(1)),\ T \to \infty. \tag{3.2}$$

Proof: One has

$$P(M(U_T) < 0) \leq P(M(U_T) < 0, A_T) + P(A_T^c), \tag{3.3}$$

where

$$A_T = \{\max_k \max_t (w_H(t) - w_H(x_k), t \in B_1(x_k)) < b_T\}.$$

We can continue the previous inequality

$$\leq P(M(\Delta_T) < b_T) + \sum_k P(\max(w_H(t) - w_H(x_k), t \in B_1(x_k)) > b_T)$$

$$\leq P(M(\Delta_T) < b_T) + N_T P(M(B_1) > b_T) := p_{1,T} + p_{2,T}. \tag{3.4}$$

Applying the Fernique (1975) inequality to $w_H(t)$, we have

$$P(M(B_1) > r_T c_H) \leq c_d \int_{r_T}^{\infty} e^{-u^2/2} du, \quad r_T > (1 + 4d)^{1/2}. \tag{3.5}$$

Hence, setting $b_T = \sqrt{2(2d + \varepsilon)\ln T}c_H, \varepsilon > 0$, one has

$$p_{2,T} < CT^d \cdot T^{-2d-\varepsilon}/\sqrt{\ln T} = CT^{-d-\varepsilon}/\sqrt{\ln T}. \tag{3.6}$$

To show that $p_{2,T} = o(p_{1,T})$, note that $\Delta_T \subset B_{TD}$, where D is the diameter of Δ_1. Therefore

$$p_{1,T} = P(M(\Delta_T) < b_T) \geq P(M(B_{TD}) < b_T) = P(M(B_{T'}) < 1), \tag{3.7}$$

where $T' = TD/b_T^{1/H}$. By Molchan (1999),

$$P(M(B_T) < 1) > cT^{-d-\varepsilon}.$$

Due to (3.6), (3.7), we have

$$p_{2,T}/p_{1,T} < c(\ln T)^{-(1+d/H)/2} = o(1). \tag{3.8}$$

Relations (3.3, 3.4) and (3.8) imply (3.1):

$$P(M(U_T) < 0) \leq (1 + o(1))p_{1,T} \leq (1 + \varepsilon)P(M(\Delta_T) < b_T),$$

where $b_T = \sqrt{4d\ln T}c_H$. To prove relation (3.2), note that

$$M(\Delta_T) \leq M(U_T) + \max_k \max_t (w_H(t) - w_H(x_k), t \in B_1(x_k))$$

$$:= M(U_T) + \delta_T.$$

As above, using the event $A_T = \{\max_k \max_t(w_H(t) - w_H(x_k), t \in B_1(x_k)) < b_T\}$, one has

$$E\delta_T \le b_T + E\delta_T 1_{A_T^c} \le b_T + N_T EM(B_1)[M(B_1) > b_T], \tag{3.9}$$

where $b_T = \sqrt{4d \ln T} c_H$ and $N_T < CT^d$.
Therefore, the second term in (3.9) is $o(1)$, because

$$(EM(B_1)[M(B_1) > b_T])^2 \le EM^2(B_1)P(M(B_1) > b_T) = O(T^{-2d}/\sqrt{\ln T}).$$

Due to (3.9), the relation (3.2) follows from the inequality:

$$\begin{aligned} EM(U_T) &\ge EM(\Delta_T) - E\delta_T \ge EM(\Delta_T) - c\sqrt{\ln T} + o(1) \\ &= T^H EM(\Delta_1) - c\sqrt{\ln T} + o(1). \end{aligned}$$

□

Proposition 3.2. *Let $w_H(t)$, $t \in \Delta_T$ be H-FBM, $\Delta_T = T\Delta_1 \subset R^d$, where Δ_1 is a unit ball and $0 \in \Delta_1$. Then*

$$P(M(\Delta_T) < 1) \ge cT^{-(d-H)}(\sqrt{\ln T})^{-d/H},$$

i.e., the survival exponent for H-FBM in Δ_T has the upper bound $\theta^+_{w_H} \le d - H$.

Corollary 3.3. *Due to Propositions 2.1, 3.2, the survival exponent for H-FBM in Δ_T exists and equals $d - H$.*

Proof: Proceeding as in the proof of Proposition 2.1, we consider again the subset U_T of the ball $B_T \subset R^d : U_T = \{x_{k,\alpha}, \alpha = 1, 2, ..., n_k; k = 1, 2, ...\}$, $\{0\} \notin U_T$. In addition to the properties (2.1), we suppose that the elements of U_T are enumerated in such a way that

$$x_{k,\alpha+1} \in B_2(x_{k,\alpha}) \quad \text{and} \quad x_{k+1,1} \in B_2(x_{k,n(k)}). \tag{3.10}$$

As before,

$$U_{k+1,\alpha} = U_k \cup \bigcup_{\beta=1}^{\alpha} x_{k+1,\beta},\ U_k = \{x_{i,\gamma} : |x_{i,\gamma}| \le r_k\} := U_{k,0};$$

$\Delta_T = \{t : |t + Te| \le T\}$, where $e = (0, ..., 0, 1)$; $O_{k,\alpha}$ is an orthogonal mapping transforming $x_{k,\alpha}$ in $\tilde{x}_{k,\alpha} = r_k e$. Setting $\tilde{U}_{k,\alpha} = O_{k,\alpha}U_{k,\alpha}$, one has

$$(\tilde{U}_{k+1,\alpha} - \tilde{x}_{k+1,\alpha})\backslash\{0\} \subset \Delta_{k+1}\backslash B_1.$$

Due to (3.10), $(\tilde{U}_{k+1,\alpha} - \tilde{x}_{k+1,\alpha})$ is a 2-net in Δ_{k+1}. Therefore, by (3.1), for $k > T_0$

$$\begin{aligned} &P\Big(M(\Delta_k) < c_H\sqrt{4d\ln k}\Big) > qP(M(\tilde{U}_{k,\alpha} - \tilde{x}_{k,\alpha})\backslash\{0\}) < 0) \\ &= qP(M(\tilde{U}_{k,\alpha-1}) - w_H(\tilde{x}_{k,\alpha})) < 0) = qP(M(U_{k,\alpha-1}) < w_H(x_{k,\alpha})). \end{aligned}$$

As a result,

$$\sum_{k=K}^{K'} n_k P\Big(M(\Delta_k) < c_H\sqrt{4d\ln k}\Big) \ge q \sum_{k=K}^{K'} \sum_{\alpha=1}^{n_k} P(M(U_{k,\alpha-1}) < w_H(x_{k,\alpha})) \tag{3.11}$$

where $K = [T]$ and $K' = [T']$.

Similarly to the proof of Proposition 2.1, we conclude that the right-hand part of (3.11) is equal to $qE\nu(T, T')$, where $\nu(T, T')$ is the number of all records in the following sequences:

$$M(U_K), w_H(x_{K+1,1}), \ldots, w_H(x_{K+1,n(K+1)}); \ldots; w_H(x_{K',1}), \ldots, w_H(x_{K',n(K')}).$$

Let $\delta(T,T')$ be the maximum increment between adjacent elements of the sequence $w_H(x_{K,n(K)})$, $w_H(x_{K+1,1})$, $\ldots$, $w_H(x_{K+1,n(K+1)})$; $\ldots$; $w_H(x_{K',1})$, $\ldots$, $w_H(x_{K',n(K')})$. Then

$$M(U_{K'}) - M(U_K) \le (\nu(T,T')+1)\delta(T,T') \le (\nu(T,T')+1)b_T + R_T, \qquad (3.12)$$

where

$$R_T = (|U_{K'}\backslash U_K|+1)\delta(T,T')[\delta(T,T') > b_T].$$

Due to (3.10),

$$ER_T < (|U_{K'}\backslash U_K|+1)^2 \max_{|t|<2} Ew_H(t)[w_H(t) > b_T].$$

Setting $b_T = \sqrt{8d\ln T}c_H$ and $T' - T = \rho T$, we obtain

$$ER_T < cT^{2d}\cdot T^{-2d} = c. \qquad (3.13)$$

By (3.12),

$$b_T E\nu(T,T') > EM(U_{K'}) - EM(U_K) - b_T - ER_T,$$

where, according to (3.2),

$$EM(U_K) = K^H EM(\Delta_1)(1+o(1)).$$

As a result,

$$b_T E\nu(T,T') > c(T^H - \sqrt{\ln T} - 1) = cT^H(1+o(1)). \qquad (3.14)$$

Keeping in mind that the right part of (3.11) is $qE\nu(T,T')$, we have:

$$qE\nu(T',T) \le \sum_{k=K}^{K'} n_k P\Big(M(\Delta_k) < c_H\sqrt{4d\ln k}\Big). \qquad (3.15)$$

Due to the self-similarity of H-FBM,

$$P\Big(M(\Delta_k) < c\sqrt{\ln k}\Big) = P(M(\Delta_{\tilde{k}}) < 1), \quad \tilde{k} = k/\Big(c\sqrt{\ln k}\Big)^{1/H},$$

and therefore the probability term decreases as a function of k. Hence, (3.15) implies

$$\begin{aligned} qE\nu(T',T) &\le |U_{T'}\backslash U_T|\, P\Big(M(\Delta_{T'+1}) < c_H\sqrt{4d\ln(T'+1)}\Big) \\ &\le CT^d P(M(\Delta_{\tilde{T}}) < 1), \end{aligned} \qquad (3.16)$$

where

$$\tilde{T} = T'/\Big(c_H\sqrt{4d\ln T'}\Big)^{1/H} \quad \text{or} \quad T' = \tilde{T}\Big(c_H\sqrt{4d\ln\tilde{T}}\Big)^{1/H}(1+o(1)). \qquad (3.17)$$

Finally, by (3.14, 3.16),

$$b_T^{-1}cT^H(1+o(1)) \le E\nu(T',T) \le q^{-1}CT^d P(M(\Delta_{\tilde{T}}) < 1).$$

Taking into account (3.17) and the relation $T' - T = \rho T$, we get

$$P(M(\Delta_{\tilde{T}}) < 1) \ge c\tilde{T}^{-(d-H)}\Big(\sqrt{\ln\tilde{T}}\Big)^{-d/H}.$$

□

Appendix

Example from Proposition 2.1. Consider H-FBM in domains $\Delta_T = T \cdot \Delta_1$, $0 \in \partial\Delta_1$; then a suitable function $\varphi_T(t)$, $t \in \Delta_T$ can be chosen as follows:

$$\varphi_T(t) = f(|t|/(Tk)) - f(|t|),$$

where $f(x), x \in R^1$ is a finite smooth function such that $f(t) = 1$ for $|x| < 1/2$ and $f(t) = 0$ for $|x| > 1$. Here k is the diameter of Δ_1.

By Molchan (1999), this can be seen as follows. Due to the spectral representation of H-FBM, the Hilbert space $H_H(\Delta_T)$ with the reproducing kernel $Ew_H(t)w_H(s), (t,s) \in \Delta_T \times \Delta_T$ (Lifshits, 2012), is the closure of smooth functions $\varphi(t), \varphi(0) = 0$ relative to the norm

$$\|\varphi\|_{H,T} = \inf_{\tilde{\varphi}} \|\tilde{\varphi}\|_H, \|\psi\|_H = A_H \int \left|\hat{\psi}(\lambda)\right|^2 |\lambda|^{d+2H} d\lambda,$$

Where $\tilde{\varphi}(t)$ is a finite function such that $\tilde{\varphi}(t) = \varphi(t), t \in \Delta_T$; $\hat{\psi}(\lambda), \lambda \in R^d$ is the Fourier transform of $\psi(t)$. Obviously, we have $\varphi_T(0) = 0$, $\varphi_T(1) = 1$ for $t \in \Delta_T \backslash B_1$, and

$$\begin{aligned}\|\varphi\|_{H,T} &< \|f(|t|/Tk) - f(|t|)\|_H < \|f(|t|/Tk)\|_H + \|f\|_H \\ &= ((Tk)^{-H} + 1)\|f\|_H < 2\|f\|_H.\end{aligned}$$

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