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Correlated Coalescing Brownian Flows on $\mathbb{R}$ and the Circle

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Abstract. We consider a stochastic differential equation on the real line which is driven by two correlated Brownian motions B^+ and B^- respectively on the positive half line and the negative half line. We assume $|d\langle B^+, B^-\rangle_t| \leq \rho\, dt$ with $\rho \in [0, 1)$. We prove it has a unique flow solution. Then, we generalize this flow to a flow on the circle, which represents an oriented graph with two edges and two vertices. We prove that both flows are coalescing. Coalescence leads to the study of a correlated reflected Brownian motion on the quadrant. Moreover, we find the distribution of the hitting time to the origin of a reflected Brownian motion. This has implications for the effect of the correlation coefficient ρ on the coalescence time of our flows.

1. Introduction

Stochastic flows on graphs have attracted much interest recently (Hajri and Raimond, 2016, 2014, 2013). As a simple abstraction of change of randomness at a node connecting two edges, flows on $\mathbb{R}$ with the origin having a special role are

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considered, see Le Jan and Raimond (2014). In particular, a stochastic differential equation is used to represent different dynamics at the negative and positive axes for the motion of a single particle. In this paper, we consider the equation

$$dX_t = 1_{\{X_t>0\}} dB_t^+ - 1_{\{X_t\le 0\}} dB_t^-, \qquad X_0 = x \tag{1.1}$$

for the dynamics of the trajectory X in the flow, where B^+ and B^- are correlated standard Brownian motions adapted to the same filtration. Since the usual conditions are not satisfied by the diffusion terms, we first investigate the existence of a strong solution. Then, we show that there exists a stochastic flow on $\mathbb{R}$ based on this differential equation and use it for generalizing to a stochastic flow on the circle.

More precisely, let B^+ and B^- be two Brownian motions, jointly defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ and adapted to a filtration $(\mathcal{F}_t)$ satisfying the usual conditions. By the assumption that B^+ and B^- are Brownian motions, (B^+, B^-) is a martingale with respect to $(\mathcal{F}_t)$ (Revuz and Yor, 1999, pg. 147), with cross variation process $H_t := \langle B^+, B^-\rangle_t$. Since H is of bounded variation (Karatzas and Shreve, 1991), it is almost everywhere differentiable. Let h_t denote the derivative dH_t/dt when it exists. We assume

$$|h_t| \le \rho \tag{1.2}$$

for almost every $t \ge 0$ with $\rho \in [0,1)$ and it follows that

$$|H_t| \le \rho t\,.$$

Examples of martingales such as (B^+, B^-) arise as solutions of stochastic differential equations.

In this paper, we seek for a flow of mapping φ based on SDE (1.1) (Le Jan and Raimond, 2004). Our first result is given as follows.

Theorem 1.1. *There exists a unique coalescing stochastic flow of mappings φ such that for all $x \in \mathbb{R}$ and $s \le t$*

$$\varphi_{s,t}(x) = x + \int_s^t 1_{\{\varphi_{s,u}(x)>0\}} W^+(du) - \int_s^t 1_{\{\varphi_{s,u}(x)\le 0\}} W^-(du)$$

where (W^+, W^-) is a white noise with $|\langle W^+_{s,t}, W^-_{u,v}\rangle| \le \rho|[s,t]\cap[u,v]|$.

For the proof of coalescence, we make use of a reflected Brownian motion (RBM) on the quadrant (Varadhan and Williams, 1985). It is constructed from two particle motion by extracting the parts of the trajectories with opposite signs and assigning each partial trajectory to a coordinate of the RBM. Clearly, two particles have a chance to coalesce only if their signs are different. The coordinates of the RBM are correlated by construction. In the case when the correlation between B^+ and B^- is exactly ρt, we find the distribution of the hitting time T_0 of the RBM to the origin. In this way, we not only extend the results of Le Jan and Raimond (2014) where the finiteness of T_0 is proved for $\rho = 0$, but also find the distribution of T_0, which is of independent interest as well. Its density is given by

$$f_\rho(t) = \frac{x}{2\sqrt{\pi t^3(1+\rho)}}\, e^{-\frac{x^2}{4t(1+\rho)}} \qquad t > 0\,.$$

We show that coalescence occurs faster as ρ gets closer to 1.

Our flows can be considered as interpolations between the flows associated to Tanaka's SDE (Hajri, 2015; Le Jan and Raimond, 2006) and the basic Brownian flow considered in Le Jan and Raimond (2014). When $\langle B^+, B^- \rangle_t = \rho t$ in (1.1), $\rho = 1$ corresponds to Tanaka's SDE and the equation of Le Jan and Raimond (2014) represents the independent case $\rho = 0$. We study the intermediate, but more general cross variation process H_t.

Second, we are interested in a flow on the unit circle $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. When $z \in \mathbb{C}$ is represented as $z = |z|e^{i\theta}$ for $\theta \in \mathbb{R}$, the argument of z, denoted by $\arg(z)$, refers to the angle θ. We embed an oriented graph with two edges and two vertices at 1 and e^{il} in $\mathcal{C}$ where the angle $l \in (0, 2\pi]$ is fixed. In analogy with (1.1), a particle is supposed to follow a different Brownian motion on each edge. This defines a special metric graph and the construction of a flow on this graph follows directly from the theory in Hajri and Raimond (2014). Formally, we require the stochastic flow φ on the circle $\mathcal{C}$ to satisfy the equation

$$\begin{aligned} f(\varphi_{s,t}(z)) \quad &= \quad f(z) + \int_s^t f'(\varphi_{s,u}(z)) 1_{\{\arg(\varphi_{s,u}(z)) \in [0,l)\}} W^+(du) \\ &\quad - \int_s^t f'(\varphi_{s,u}(z)) 1_{\{\arg(\varphi_{s,u}(z)) \in [-2\pi+l,l)\}} W^-(du) \\ &\quad + \frac{1}{2} \int_s^t f''(\varphi_{s,u}(z))\, du \end{aligned} \tag{1.3}$$

for all $f \in C^2(\mathcal{C})$, as a generalization of the stochastic flow solution to (1.1) on $\mathbb{R}$ given in Theorem 1.1. To that end, we refer to some flows of mappings φ^+ and φ^-, which are flow solutions to two forms of SDE (1.1) with consistent Brownian motions. The following is an explicit formula for the flow, but only until the arguments φ^+ and φ^- move a point z on an edge of the circle to the boundary of the other edge.

Theorem 1.2. *There exists a stochastic flow φ on $\mathcal{C}$ satisfying (1.3) and such that*

$$\varphi_{s,t}(z) = \begin{cases} e^{i\varphi^+_{s,t}(arg(z))} & \text{if } \ t < \gamma_s^{z,+} \\ e^{il - i\varphi^-_{s,t}(l - arg(z))} & \text{if } \ t < \gamma_s^{z,-} \end{cases} \tag{1.4}$$

where $arg(z) \in [-2\pi + l, l]$, $z \in \mathcal{C}$, almost surely for all $s < t$ where $\gamma_s^{z,+} = \inf\{r \geq s : \varphi^+_{s,r}(arg(z)) \notin (-2\pi + l, l)\}$ and $\gamma_s^{z,-} = \inf\{r \geq s : \varphi^-_{s,r}(l - arg(z)) \notin (-2\pi + l, l)\}$.

We construct the flow more explicitly and prove that it is coalescing. We first take two particles moving on the plane such that when they are mapped on the circle they move with respect to the flow on the circle. Then, we construct an associated RBM on a bounded domain. We show that this process hits one of the corners in finite time, which in turn implies that coalescence occurs in finite time at one of the vertices. A further remarkable result is on coalescence time. We find its distribution explicitly and see that the closer ρ is to 1, the faster the coalescence occurs.

Our paper is organized as follows. Section 2 gives a series of propositions and lemmas that prove Theorem 1.1 together with the results on the hitting time distribution. In Section 3, we show that the flow on the circle is coalescing, and we also derive the probability law of the hitting time to a vertex.

2. The flow on $\mathbb{R}$

We refer to Le Jan and Raimond (2004) for the definition of stochastic flows with independent increments on $\mathbb{R}$. Starting with the solution of SDE (1.1), we study its flow in this section.

2.1. *Strong solution for one-point motion.* In order to show that the strong solution exists, it is sufficient to show pathwise uniqueness and weak existence. The proof of this fact, as given in Karatzas and Shreve (1991, pg. 309-10) based on Yamada and Watanabe's theorem for SDE's driven by independent Brownian motions, is also valid for our SDE, where the Brownian motions B^+ and B^- are not independent. The crucial point is that (B^+, B^-) with cross variation process H still takes values in a Polish space, namely, $C(\mathbb{R}_+, \mathbb{R}^2)$. In view of this, there exist regular versions of the conditional probability distributions involved and the result follows; see also Revuz and Yor (1999, pg. 368).

For the existence of a weak solution of SDE (1.1), we demonstrate a probability space $(\Omega', \mathcal{H}', \mathbb{P}')$ with filtration $(\mathcal{F}'_t)$ and a process $(X', B^{+'}, B^{-'})$ adapted to $(\mathcal{F}'_t)$ that solves (1.1), as given in the following proposition.

Proposition 2.1. *There exists a weak solution of* (1.1).

Proof: Recall that (B^+, B^-) is a martingale on $(\Omega, \mathcal{H}, \mathbb{P}, (\mathcal{F}_t))$ with cross variation process H. When it exists, a solution X of SDE (1.1) is a Brownian motion by Lévy's characterization theorem, irrespective of the joint law of (B^+, B^-). Although the joint distribution of B^+ and B^- is not specified, it is fixed by H due to Revuz and Yor (1999, Thm.V.3.9) as $dH_t = h_t\, dt$ for almost every $t > 0$. Accordingly, we can find a two-dimensional Brownian motion $B := (B^1, B^2)^T$ such that

$$(B_t^+, B_t^-)^T = \int_0^t \alpha_s\, dB_s \tag{2.1}$$

where α is a matrix-valued process given by

$$\alpha_s = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+h_s} & \sqrt{1-h_s} \\ \sqrt{1+h_s} & -\sqrt{1-h_s} \end{bmatrix}$$

as in Revuz and Yor (1999, Thm.V.3.9). Since α is invertible due to assumption $\rho < 1$ in (1.2), we have $(B_t^1, B_t^2)^T = \int_0^t \alpha_s^{-1}\, d(B_s^+, B_s^-)^T$. Then, B^1 and B^2 are independent Brownian motions adapted to $(\mathcal{F}_t)$ as $\langle B^1, B^2 \rangle_t = 0$, for $t > 0$. Therefore, (2.1) characterizes the joint distribution of (B^+, B^-) uniquely in our original probability space. Now, let

$$\begin{aligned} X'_t &= B_t^+, \\ B_t^{+'} &= \int_0^t 1_{\{X'_s > 0\}} dB_s^+ - \int_0^t 1_{\{X'_s \le 0\}} dB_s^-, \\ B_t^{-'} &= \int_0^t 1_{\{X'_s > 0\}} dB_s^- - \int_0^t 1_{\{X'_s \le 0\}} dB_s^+ . \end{aligned}$$

It follows that $(X', B^{+'}, B^{-'})$ is adapted to $(\mathcal{F}_t)$ and solves (1.1). It is easily checked that $\langle B^{+'}, B^{-'} \rangle_t = H_t$, and $(B^{+'}, B^{-'})$ has the same distribution as (B_t^+, B_t^-) by an analogous relation to (2.1) where B and (B^+, B^-) are replaced by B', and $(B^{+'}, B^{-'})$, respectively. Note that $\rho = 1$ is not problematic for demonstrating a weak solution. In that case, our arguments above can be modified by an enlargement of the probability space as in Revuz and Yor (1999, Thm.V.3.9). □

Pathwise uniqueness will be shown next using Prokaj (2013, Thm.2), equivalently, its generalization, which is given in Fernholz et al. (2013, Thm.6.1), based on the assumption $\rho < 1$. In the Appendix, we also give an alternative proof of Proposition 2.2, which reveals the role of the magnitude of the correlation between B^+ and B^- by following the steps of Hajri and Raimond (2016, Prop.4.5), where $\rho < 1$ is again crucial.

Proposition 2.2. *Pathwise uniqueness holds for (1.1).*

Proof: Let us define $M_t = \frac{B_t^+ + B_t^-}{2}$ and $N_t = \frac{B_t^+ - B_t^-}{2}$. They are continuous local martingales which are strongly orthogonal, i.e., $\langle M, N\rangle_t = 0$. Then (1.1) reduces to

$$dX_t = \mathrm{sgn}(X_t)dM_t + dN_t$$

We look for $c > 0$ such that $d\langle M\rangle_t \le cd\langle N\rangle_t$. This is equivalent to $dH_t \le \frac{c-1}{c+1}\,dt$. Since $h_t \le \rho$ by (1.2), we have $dH_t \le \rho\,dt$ with $\rho < 1$ and the domination relation is satisfied with $c = (1+\rho)/(1-\rho)$. By Prokaj (2013, Thm.2), we may conclude that pathwise uniqueness holds for (1.1). □

2.2. *Coalescence of two particles.* Let X and Y be two solutions of SDE (1.1) with $X_0 = 0, Y_0 = y > 0$, which we also refer as two particles starting at 0 and y. Other starting points can be handled similarly as explained below. For two particles, there is a chance to meet only if X and Y have opposite signs. Otherwise, they move lockstep with either B^+ or B^-. Therefore, we concatenate the trajectories piecewise only for $t > 0$ such that $X_t \le 0$ and $Y_t \ge 0$ in order to prove coalescence. Define the coalescence time

$$T = \inf\{s \ge 0 : X_s = Y_s\}$$

and the quadrant

$$D = \{(x, y) \in \mathbb{R}^2 : x \le 0, y \ge 0\}$$

which will be the domain of an RBM to be defined as follows. Let

$$A_t = \int_0^{t\wedge T} 1_{\{(X_s,Y_s)\in D\}}ds, \qquad \kappa_t = \inf\{s > 0 : A_s > t\}$$

and

$$(X_t^r, Y_t^r) = (X_{\kappa_t}, Y_{\kappa_t}), \qquad t \le A_T.$$

Let $L_t(X)$ and $L_t(Y)$ be the local times of X and Y at 0. Denote by L_t^1 and L_t^2, $\frac{1}{2}L_{\kappa_t}(X)$ and $\frac{1}{2}L_{\kappa_t}(Y)$, respectively. The following lemma identifies (X^r, Y^r) as a reflected Brownian motion in D.

Lemma 2.3. *The process (X^r, Y^r) is a correlated reflected Brownian motion in D, obliquely reflected at the boundary with angle $\pi/4$ and stopped when it hits $(0, 0)$. That is, there exist Brownian motions B^1 and B^2 with $|d\langle B^1, B^2\rangle_t| \le \rho\,dt$ such that for all $t < A_T$*

$$X_t^r = -B_t^1 - L_t^1 + L_t^2\,, \qquad Y_t^r = y + B_t^2 - L_t^1 + L_t^2$$

with $\langle B^1, B^2\rangle_t = \int_0^{\kappa_t} 1_{\{X_s\le 0\}}h_s\,ds$.

Proof: According to SDE (1.1), the solutions X and Y satisfy

$$\begin{aligned} X_t &= \int_0^t 1_{\{X_s>0\}}\, dB_s^+ - \int_0^t 1_{\{X_s\leq 0\}}\, dB_s^-, \qquad (2.2) \\ Y_t &= y + B_t^+. \end{aligned}$$

for Brownian motions B^+ and B^- with $\langle B^+, B^-\rangle_t = H_t$, until

$$\tau_1 := \inf\{t > 0 : Y_t = 0\} .$$

The process (X_t^r, Y_t^r) takes values in $D = \mathbb{R}^- \times \mathbb{R}^+$ for $\kappa_t \leq \tau_1$, equivalently for $t \leq A_{\tau_1}$. To see this, first note that $Y_t \geq 0$ for $t \leq \tau_1 \leq T$ and consider only the coordinate X_t^r. Then, for $t \leq \tau_1$ we have $A_t = \int_0^t 1_{\{X_s<0\}} ds$, which implies that A increases only during the negative excursions of X. As a result, the path $\{X_t^r : 0 \leq t \leq A_{\tau_1}\}$ takes values in $\mathbb{R}^-$ as it is the negative part of X over $[0, \tau_1]$. By definition of the clock A_t, the trajectory of (X^r, Y^r) is formed piecewise from that of (X, Y) as illustrated in dark in Fig.2.1. The process (X_t^r, Y_t^r) is continuous

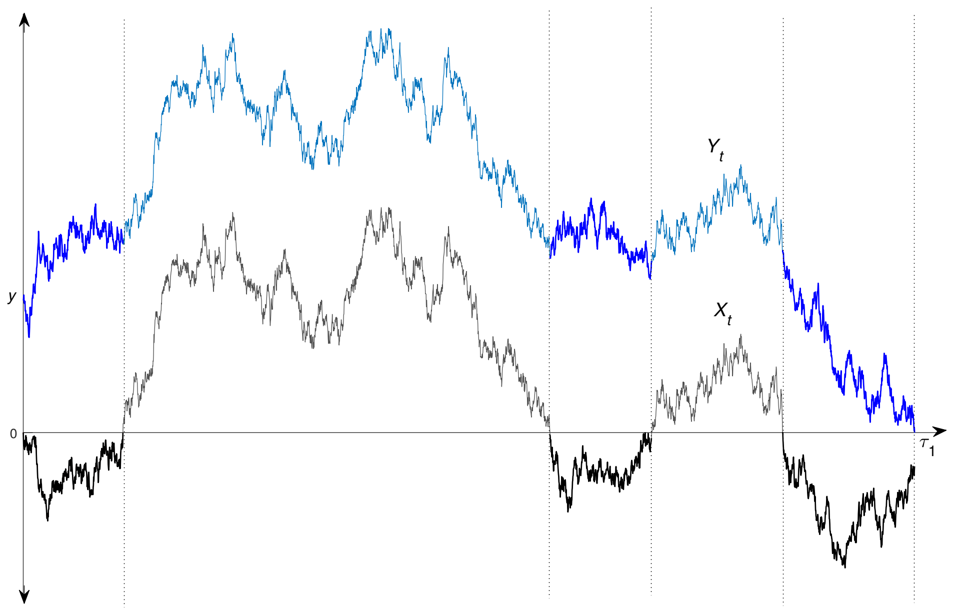

FIGURE 2.1. The darker parts of the trajectories of X_t and Y_t are concatenated piecewise to form (X^r, Y^r).

on $[0, A_{\tau_1}]$ because the value of Y is the same at the start and end points of a positive excursion of X, due to the fact that $X_t - Y_t$ is constant when they both move lockstep with B^+ according to SDE (1.1) during the excursion. Moreover, the evolution of A_t and the role of its right inverse κ_t can be sketched as in Karatzas and Shreve (1991, Rem.6.3.3) to see how they work.

We next derive the Skorohod representation of (X_t^r, Y_t^r) on $[0, A_{\tau_1}]$ to find the angle of reflection and show that is a reflected Brownian motion in $\mathbb{R}^- \times \mathbb{R}$ in this time interval. By Tanaka formula (Karatzas and Shreve, 1991, Prop.3.6.8) for the negative part of X, we have

$$L_t^1 = -X_t^r - B_t^1$$

where $L^1_t = (1/2)L_{\kappa_t}(X)$ and $B^1_t = -\int_0^{\kappa_t} 1_{\{X_s\le 0\}}\,dX_s = \int_0^{\kappa_t} 1_{\{X_s\le 0\}}\,dB^-_s$. Putting $B^2_t := \int_0^{\kappa_t} 1_{\{X_s\le 0\}}\,dB^+_s$, we observe that B^2 is also a Brownian motion and

$$Y^r_t = Y_{\kappa_t} = B^2_t - L^1_t + y$$

holds from (2.2) and the fact that $X_{\kappa_t} + B^1_t = \int_0^{\kappa_t} 1_{\{X_s>0\}}\,dX_s$, which is $-L^1_t$. Now, we have

$$\langle B^1, B^2\rangle_t = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} d\langle B^+, B^-\rangle_s = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} dH_s = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} h_s\,ds\,.$$

It follows that

$$|d\langle B^1, B^2\rangle_t| = |h_{\kappa_t}|\,dt \le \rho\,dt$$

as $|h_t| \le \rho$ for all t, and by definition of κ_t as right continuous inverse of $A_t = \int_0^t 1_{\{X_s\le 0\}}\,ds$. Explicitly, we have

$$(X^r_t, Y^r_t) = (0, y) + (-B^1_t, B^2_t) - (L^1_t, L^1_t) \quad t \le A_{\tau_1}\,.$$

Thus, for $t \le A_{\tau_1}$, (X^r_t, Y^r_t) is a correlated reflected Brownian motion on $\mathbb{R}^- \times \mathbb{R}$ with oblique reflection of angle $\pi/4$ by Harrison and Reiman (1981, Thm.1) for Skorohod representation of multidimensional reflected diffusions, provided that we show L^1_t increases only at those times t such that $X^r_t = 0$. This can be shown easily as in Le Jan and Raimond (2014, Lem.4.3) by approximating the equation $X^r_t = -B^1_t - L^1_t$ in probability with the upcrossings of X when it takes values in $(-\epsilon, 0)$, as $\epsilon \downarrow 0$. Therefore, L^1_t increases only when $X^r_t = 0$ in view of the downcrossing representation of local time (Karatzas and Shreve, 1991, Thm.6.2.23), which is symmetric, to upcrossings.

Now, starting with $(x, 0) := (X_{\tau_1}, Y_{\tau_1})$, with $x < 0$, we replace X, Y, B^+, B^- with $X_{\tau_1+\cdot}, Y_{\tau_1+\cdot}, B^+_{\tau_1+\cdot} - B^+_{\tau_1}, B^-_{\tau_1+\cdot} - B^-_{\tau_1}$. In $(A_{\tau_1}, A_{\tau_2}]$ where $\tau_2 = \inf\{t > 0 : X_t = 0\}$, according to (1.1) we have

$$\begin{aligned} X_t &= x - B^-_t\,, \\ Y_t &= \int_0^t 1_{\{Y_s>0\}}\,dB^+_s - \int_0^t 1_{\{Y_s\le 0\}}\,dB^-_s. \end{aligned}$$

Similar to $[0, A_{\tau_1}]$ above, (X^r_t, Y^r_t) is a correlated reflected Brownian motion on $\mathbb{R} \times \mathbb{R}^+$ with angle of reflection $\pi/4$ for $t \in (A_{\tau_1}, A_{\tau_2}]$. In particular, Y^r is a reflected Brownian motion on $\mathbb{R}^+$ with local time

$$L^2_t = Y^r_t - B^2_t$$

where $B^2_t = \int_0^{\kappa_t} 1_{\{Y_s>0\}}\,dY_s = \int_0^{\kappa_t} 1_{\{Y_s>0\}}\,dB^+_s$ from Tanaka formula. This time, X is on the negative axis and satisfies

$$X^r_t = X_{\kappa_t} = x + L^2_t - B^1_t$$

where we put $B^1_t := -\int_0^{\kappa_t} 1_{\{Y_s\le 0\}}\,dB^-_s$. It follows that

$$(X^r_t, Y^r_t) = (x, 0) + (-B^1_t, B^2_t) + (L^2_t, L^2_t)$$

and $|d\langle B^1, B^2\rangle_t| \le \rho\,dt$ as before.

Alternating as above, the process (X^r, Y^r) constructed in this way is continuous in D by definition and satisfies

$$(X^r_t, Y^r_t) = (0, y) + (-B^1_t, B^2_t) - (L^1_t, L^1_t) + (L^2_t, L^2_t), \qquad \text{for} \quad t < A_T$$

since L^1 and L^2 are not positive at the same time for $t < A_T$. $\square$

Note that the proof of Lemma (2.3) considers a starting point of the form $(0, y)$ or $(x, 0)$. If the process (X, Y) starts from $(x, y) \in D$ with $x < 0 < y$, then with these starting points for (X^r, Y^r) the same proof applies after (X, Y) hits the boundary of D. Clearly, the case $y < 0 < x$ is symmetric with D replaced by $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$. If (x, y) is in the first or third quadrant, with the same sign, then we can define (X^r, Y^r) after the hitting time of one of X and Y to 0. The following lemma indicates coalescence.

Lemma 2.4. $\mathbb{P}\{T < \infty\} = 1$.

Proof: Let L^1, L^2 be as in Lemma 2.3 and define $L^r := L^1 + L^2$, which is the local time of (X^r, Y^r) at the boundary $\{x = 0\} \cup \{y = 0\}$. Let $T_0 = \inf\{t \geq 0 : X^r_t = Y^r_t = 0\}$, the hitting time of the RBM to the origin. Along the very same lines of the proof of Le Jan and Raimond (2014, Lem.4.6), one can show that $\mathbb{P}\{L^r_{T_0} < \infty\} = 1$. The cross variation process of B^1 and B^2 and the upper bound ρ appear in an obvious way in the proof. Then, since $\frac{1}{2}(L_T(X) + L_T(Y)) = L^r_{T_0} < \infty$ and since X is a Brownian motion, it follows that $T < \infty$ a.s. □

Since the solution of (1.1) is strong, when two particles meet they stay together thereafter.

2.3. *Feller property and the flow.* Since there exists a strong solution to (1.1), we may define $P^n_t(x, dy)$, $n \geq 1$, as the law of $(X^1, \ldots, X^n)$ which are n solutions of the same equation with initial conditions $X^i_0 = x_i$, $i = 1, \ldots, n$. We will prove $(P^n_t)_{n\geq 1}$ defines a compatible family of Feller semigroups next.

Lemma 2.5. *The family of Markovian semigroups corresponding to n-point motion corresponding to (1.1) is Feller, for each $n \geq 1$*

Proof: We will check Condition (C) of Le Jan and Raimond (2004, Thm.4.1) as a sufficient condition for Feller property. Let (X, Y) be the two-point motion. Condition (C) is verified if for every $t > 0$ and $\varepsilon > 0$

$$\lim_{|y-x|\to 0} \mathbb{P}^{(2)}_{(x,y)}\{\, |X_t - Y_t| > \varepsilon,\, t < T\} = 0. \tag{2.3}$$

Assume $0 < y - x < \varepsilon$, then

$$\begin{aligned}
\mathbb{P}^{(2)}_{(x,y)}\{Y_t - X_t > \varepsilon, t < T\} &\leq \mathbb{P}^{(2)}_{(x,y)}\{\sup_{t\geq 0}(Y_t - X_t) \geq \varepsilon\} \\
&= \mathbb{P}^{(2)}_{(x,y)}\{Y_0 - X_0 \geq \varepsilon\} + \frac{1}{\varepsilon}\mathbb{E}^{(2)}_{(x,y)}[(y - x)1_{\{Y_0 - X_0 < \varepsilon\}}] \\
&= \frac{y - x}{\varepsilon}
\end{aligned}$$

where the first equality follows from Karatzas and Shreve (1991, problem 1.3.28). Then, (2.3) follows. □

Now, let W^+ and W^- be two given white noises with $|\langle W^+_{s,t}, W^-_{s,t}\rangle| \leq \rho(t - s)$, for $s < t$. By the results of Section 2.1, the equation

$$\varphi_{s,t}(x) = x + \int_s^t 1_{\{\varphi_{s,u}(x)>0\}} W^+(du) - \int_s^t 1_{\{\varphi_{s,u}(x)\leq 0\}} W^-(du) \tag{2.4}$$

has a strong solution φ for each $x \in \mathbb{R}$ and $s \leq t$. As in the proof of Le Jan and Raimond (2014, Thm.1.1), the mappings $\varphi_{s,t} : \mathbb{R} \to \mathbb{R}$, for s, t rational, can be defined on the same probability space. Then, the solutions can be extended to all $s, t \in \mathbb{R}_+$ and they will satisfy (2.4) by continuity in view of Lemma 2.5. Measurability and cocycle property follows by similar arguments as in Hajri (2011, pg. 81) and Kunita (1990, pg. 161). Hence, the proof of Theorem 1.1 is complete.

2.4. *Distribution of* T_0. In this part, we find the distribution of the time it takes the reflected Brownian motion (X^r, Y^r) to hit the origin under the simplified condition that the cross variation process is exactly equal to ρt. Let (X, Y) be a two point motion of our SDE

$$dX_t = 1_{\{X_t>0\}} dB_t^+ - 1_{\{X_t\le 0\}} dB_t^-$$

where $\langle B^+, B^-\rangle_t = \rho t$ with $X_0 = x > 0$ and $Y_0 = 0$. Remember the reflected Brownian motion constructed in subsection 2.2 as

$$X_t^r = x + B_t^1 + L_t^1 - L_t^2\,, \qquad Y_t^r = -B_t^2 + L_t^1 - L_t^2$$

which is on $\mathbb{R}^+\times\mathbb{R}^-$. From the proof of Lemma 2.3, $\langle B^1, B^2\rangle_t = \rho t$ as a consequence of $\langle B^+, B^-\rangle_t = \rho t$. Let $V_t = \frac{1}{\sqrt{2}}(X_t^r - Y_t^r)$. Note that $V_t = 0$ if and only if $X_t^r = Y_t^r = 0$. Now define $M_t := V_t - V_0$ and $T(s) := \inf\{t \ge 0 : \langle M\rangle_t > s\}$. Then, $M_t \in \mathcal{M}_{loc}$, $\langle M\rangle_t = (1+\rho)t$, and $T(s) = \frac{s}{1+\rho}$. By Karatzas and Shreve (1991, Thm. 3.4.6), we may conclude that the time changed process $B_s = M_{\frac{s}{1+\rho}}$ is a standard one dimensional Brownian motion. Now define the following stopping times:

$$T_0 = \inf\{t \ge 0 : V_t = 0\}, \qquad S = \inf\{s \ge 0 : B_s = -\frac{x}{\sqrt{2}}\}.$$

Note that T_0 is also the first hitting time of M_t to $-\frac{x}{\sqrt{2}}$. We know that $\mathbb{P}\{S \in ds\} = \frac{|x|}{2\sqrt{\pi s^3}} e^{-\frac{x^2}{4s}} ds$. Then, we may conclude that

$$\mathbb{P}\{T_0 \in dt\} = (1+\rho)^{-\frac{1}{2}} \frac{x}{2\sqrt{\pi t^3}} e^{-\frac{x^2}{4t(1+\rho)}} dt\,.$$

Now let $\bar{F}_\rho(t) = \mathbb{P}(T_0 > t)$ and define $\bar{F}(t) = \bar{F}_0(t)$. Then, we have

$$\bar{F}_\rho(t) = \int_t^\infty (1+\rho)^{-1/2} \frac{x}{2\sqrt{\pi s^3}} e^{-\frac{x^2}{4s(1+\rho)}} ds\,.$$

By a change of variable s to $s/(1+\rho)$, we get

$$\bar{F}_\rho(t) = \int_{t(1+\rho)}^\infty \frac{x}{2\sqrt{\pi s^3}} e^{-\frac{x^2}{4s}} ds$$

which gives the relation $\bar{F}_\rho(t) = \bar{F}(t(1+\rho))$. That is, the probability of the time it takes the reflected Brownian motion to hit the corner being greater than a fixed $t > 0$ is equal to $\bar{F}(t(1+\rho))$. This probability gets smaller, equivalently the probability of hitting the corner gets larger, as the correlation coefficient ρ increases.

3. Flows on a Circle

In this section, $\mathbb{R}$ is replaced with a circle, where we study an application of our flow (Le Jan and Raimond, 2004).

3.1. *The flow.* We study the flow solution of SDE (1.1) on the unit circle $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$, which will be denoted by φ in this section. An oriented graph with two edges and two vertices at 1 and e^{il} is embedded in $\mathcal{C}$, where $l \in (0, \pi]$ is fixed. We assume that vertex 1 is the tail and e^{il} is the head for both edges. Let $\mathcal{C}^+ = \{z \in \mathcal{C} : \arg(z) \in (0, l)\}$ and $\mathcal{C}^- = \{z \in \mathcal{C} : \arg(z) \in (-2\pi + l, 0)\}$. See Fig. 3.2 for an illustration.

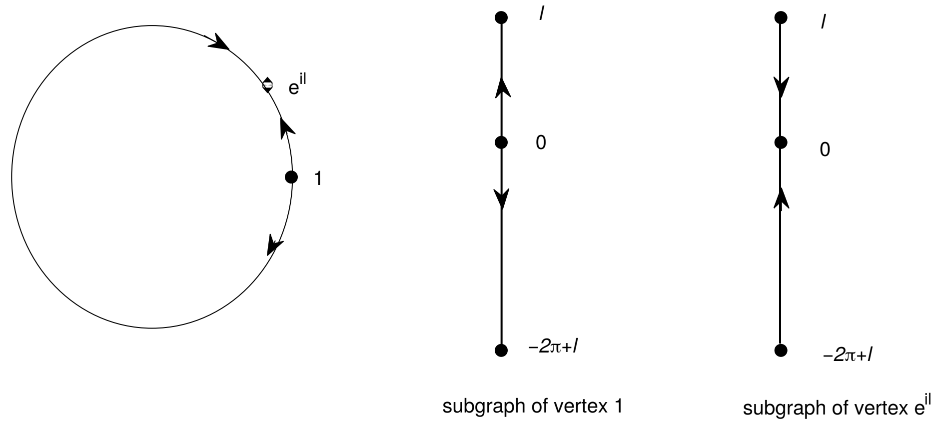

FIGURE 3.2. Illustration of the circle $\mathcal{C}$ and its subgraphs

A function f is said to be differentiable at $z \in \mathcal{C}$ if

$$f'(z) = \lim_{h\to 0} \frac{f(ze^{ih}) - f(z)}{h}$$

exists. For all $f \in C^2(\mathcal{C})$, we require φ to satisfy

$$\begin{aligned} f(\varphi_{s,t}(z)) \quad &= \quad f(z) + \int_s^t f'(\varphi_{s,u}(z))1_{\{\arg(\varphi_{s,u}(z))\in[0,l)\}}W^+(du) \qquad (3.1)\\ &\quad - \int_s^t f'(\varphi_{s,u}(z))1_{\{\arg(\varphi_{s,u}(z))\in[-2\pi+l,0)\}}W^-(du)\\ &\quad + \frac{1}{2}\int_s^t f''(\varphi_{s,u}(z))\,du \end{aligned}$$

by Ito formula, where W^+, W^- are correlated white noises with $\langle W^+_{s,t}, W^-_{u,v}\rangle \le \rho\,(|\,[s,t] \cap [u,v]\,|)$, $\rho \in [0,1)$. Note that $\rho = 1$ gives Tanaka's flow on the circle as studied in Hajri and Raimond (2013), and $\rho = 0$ yields a flow on the circle as a special case of Hajri and Raimond (2014).

The construction of the flow φ on $\mathcal{C}$ follows directly from Hajri and Raimond (2014, Thm.3.2), which theorem is for more general metric graphs. We adapt the steps of the construction there for the special case of our flow on the circle. In particular, φ is given in Theorem 1.2 in terms of the auxiliary flows φ^+ and φ^- on $\mathbb{R}$, which are the respective flow solutions of

$$d\varphi^+_{s,t}(x) = 1_{\{\varphi^+_{s,t}(x)>0\}}W^+(dt) - 1_{\{\varphi^+_{s,t}(x)\le 0\}}W^-(dt)$$

and

$$d\varphi^-_{s,t}(x) = 1_{\{\varphi^-_{s,t}(x)\le 0\}}W^-(dt) - 1_{\{\varphi^-_{s,t}(x)>0\}}W^+(dt)$$

with $x \in \mathbb{R}$. For the flow φ^+, the origin 0 corresponds to 1 on the circle and the first subgraph in Figure 3.2, whereas for φ^-, the second subgraph in Figure 3.2 is valid and the origin 0 corresponds to e^{il} on the circle. The arrows on the edges illustrate the sign, that is orientation, of the white noise W^+ or W^-. The sign is positive if the arrow is in the increasing direction on the real line, and negative otherwise.

For $z \in \mathcal{C}$, let

$$\tau_s^z = \inf\{r \geq s : 1_{\{z \in \mathcal{C}^+\}} e^{i(arg(z) + W^+_{s,r})} + 1_{\{z \in \mathcal{C}^-\}} e^{i(arg(z) - W^-_{s,r})} = 1 \text{ or } e^{il}\}$$

which is the hitting time of the particle to one of the vertices $\{1, e^{il}\}$. We consider the minimum of l and $2\pi - l$, which is l by the assumption $l \in (0, \pi]$, and we let

$$A_{s,t} = \{\sup_{s<u<v<t} \max(|W^+_{u,v}|, |W^-_{u,v}|) < l\}$$

in order to control the support of the flow that will be constructed almost surely. On $A_{s,t}$, let

$$\varphi^0_{s,t}(z) = \begin{cases} 1_{\{z \in \mathcal{C}^+\}} e^{i(x + W^+_{s,t})} + 1_{\{z \in \mathcal{C}^-\}} e^{i(x - W^-_{s,t})} & \text{if} \quad t \leq \tau^z_s \\ e^{i\varphi^+_{s,t}(x)} 1_{\{\varphi^0_{s,\tau^z_s}(z) = 1\}} + e^{i(l - \varphi^-_{s,t}(l-x))} 1_{\{\varphi^0_{s,\tau^z_s}(z) = e^{il}\}} & \text{if} \quad t > \tau^z_s \end{cases} \tag{3.2}$$

with $x = \arg(z)$, and set $\varphi^0_{s,t}(z) = z$ on $A^c_{s,t}$. For $n \in \mathbb{N}$, let $\mathbb{D}_n = \{k2^{-n} : k \in \mathbb{Z}\}$. For $s > 0$, let $s_n = \sup\{u \in \mathbb{D}_n : u \leq s\}$ and $s^+_n = s_n + 2^{-n}$. For every $n \geq 1$ and $s \leq t$, we define

$$\varphi^n_{s,t} = \varphi^0_{t_n,t} \circ \varphi^0_{t_n - 2^{-n}, t_n} \circ \ldots \circ \varphi^0_{s^+_n, s^+_n + 2^{-n}} \circ \varphi^0_{s, s^+_n} . \tag{3.3}$$

Let $\Omega^n_{s,t} = \{\sup_{\{s<u<v<t : |v-u| \leq 2^{-n}\}} \max(|W^+_{u,v}|, |W^-_{u,v}|) < l\}$, and let $\Omega_{s,t} = \cup_n \Omega^n_{s,t}$. Due to continuity of W, we have $\mathbb{P}(\Omega_{s,t}) = 1$. Then, φ on $\mathcal{C}$ is constructed as

$$\varphi_{s,t}(\omega) := \varphi^n_{s,t}(\omega) \tag{3.4}$$

for $\omega \in \Omega_{s,t}$, where $n = n_{s,t} = \inf\{k : \omega \in \Omega^k_{s,t}\}$, and $\varphi(\omega, z) := z$ for $\omega \in \Omega^c_{s,t}$.

3.2. *Flow and Feller property.* We first prove the flow property.

Lemma 3.1. *φ satisfies the flow property: for $s < t < u$, $\varphi_{s,u} = \varphi_{t,u} \circ \varphi_{s,t}$.*

Proof: We first prove the flow property for φ^0. We will show that

$$\varphi^0_{u,v} = \varphi^0_{t,v} \circ \varphi^0_{u,t} \tag{3.5}$$

for all $u < t < v$, almost surely on $A_{u,v}$. Note that $A_{u,v} \subset A_{u,t} \cap A_{t,v}$. Suppose $z \in \mathcal{C}^+$, and assume $\varphi^0_{u,\tau^z_u}(z) = 1$ if $\tau^z_u < v$ for brevity of notation. When $v \leq \tau^z_u$, we have $Z := \phi^0_{u,t}(z) = e^{i(arg(z) + W^+_{u,t})}$. Then, $\tau^Z_t = \tau^z_u$ and (3.5) holds by additivity of the white noise. If $t \leq \tau^z_u < v$, then $\varphi^0_{u,t}(z) =: Z$, $\varphi^0_{u,v}(z) = e^{i\varphi^+_{u,v}(x)}$ with $x = \arg(z)$, and $\tau^Z_t = \tau^z_u < v$. Therefore, we have

$$\varphi^0_{t,v} \circ \varphi^0_{u,t}(z) = \varphi^0_{t,v}(Z) = e^{i\varphi^+_{t,v}(X)} = e^{i(\varphi^+_{t,v} \circ \varphi^+_{u,t}(x))} = e^{i\varphi^+_{u,v}(x)} = \varphi^0_{u,v}(z)$$

where $X = \arg(Z)$, and we used the fact that φ^+ is a flow. If $t > \tau^z_u$, we have

$$\varphi^0_{u,v}(z) = e^{i\varphi^+_{u,v}(x)} = e^{i(\varphi^+_{t,v} \circ \varphi^+_{u,t}(x))}$$

by the flow property of φ^+. On the event $A_{u,v} \cap \{t > \tau_u^z\}$, $\phi_{t,v}^+(Y)$ takes values in $(0, 2\pi)$ for $Y \in (0, 2\pi)$. Since $X := \varphi_{u,t}^+(x) \in (0, 2\pi)$ also in this case, this implies that $e^{i(\varphi_{t,v}^+(\varphi_{u,t}^+(x)))} = e^{i\varphi_{t,v}^+(X)} \in (0, 2\pi)$. Therefore, we can write

$$e^{\varphi_{t,v}^+ \circ \varphi_{u,t}^+(x)} = e^{\varphi_{t,v}^+(X)} = \varphi_{t,v}^0(e^{iX}) = \varphi_{t,v}^0(Z) = \varphi_{t,v}^0(\varphi_{u,t}^0(z))$$

on $A_{u,v} \cap \{t > \tau_u^z\}$. Then, (3.5) follows. The same arguments are valid if $\varphi_{u,\tau_u^z}^0(z) = e^{il}$, but with slightly more involved notation.

Now by definition of $\varphi_{s,t}$ as given in (3.4) almost surely, we have $\varphi_{s,t} = \varphi_{s,t}^n$. Also, for all $s \le u < v \le t$ such that $|v - u| \le 2^{-n}$, we have $\Omega_{s,t}^n \subset A_{u,v}$. Therefore, we can apply the flow property (3.5) on each interval $(u, v) \in \{(s, s_n^+), (s_n^+, s_n^+ + 2^{-n}), \ldots, (t_n, t)\}$. In particular, we consider the finer mesh $s_m^+, s_m^+ + 2^{-m}, \ldots, t_m$ since $s_n^+, s_n^+ + 2^{-n}, \ldots, t_n$ are also in $\mathbb{D}_m$, for $m \ge n_{s,t}$. We get

$$\varphi_{s,t} = \varphi_{s,t}^n = \varphi_{t_m,t}^0 \circ \varphi_{t_m - 2^{-m}, t_m}^0 \circ \ldots \circ \varphi_{s_m^+, s_m^+ + 2^{-m}}^0 \circ \varphi_{s, s_m^+}^0 = \varphi_{s,t}^m$$

for all $m \ge n_{s,t}$ almost surely.

For $m \ge \max(n_{s,u}, n_{s,t}, n_{t,u})$ and $s < t < u$, we have

$$\begin{aligned}
\varphi_{s,u} = \varphi_{s,u}^m &= \varphi_{u_m,u}^m \circ \ldots \circ \varphi_{t_m^+, t_m^+ + 2^{-m}}^m \circ \varphi_{t_m, t_m^+}^m \circ \varphi_{s,t_m}^m \\
&= \varphi_{u_m,u}^m \circ \ldots \circ \varphi_{t_m^+, t_m^+ + 2^{-m}}^m \circ \varphi_{t, t_m^+}^m \circ \varphi_{t_m,t}^m \circ \varphi_{s,t_m}^m \\
&= \varphi_{t,u}^m \circ \varphi_{s,t}^m \;=\; \varphi_{t,u} \circ \varphi_{s,u}
\end{aligned}$$

by the definition of φ^m in (3.3), and the flow property (3.5) for φ^0 on (t_m, t_m^+), equivalently for φ^m on this interval. □

In the following lemma, we prove a sufficient condition for Feller property of φ. By Le Jan and Raimond (2004, Lem. 1.11), an additional condition is

$$\lim_{t \to 0} \mathbb{E}[f(\varphi_{0,t}(z)] = f(z)$$

for all $z \in \mathcal{C}$, but this is satisfied trivially by bounded convergence theorem.

Lemma 3.2. *For all $f \in C(\mathcal{C})$ and $s \le t$*

$$\lim_{d(x,y) \to 0} \mathbb{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] = 0$$

for every $x, y \in \mathcal{C}$.

Proof: First we will show that $\varphi_{s,t}^0$ is Fellerian. For this it is enough to show that $d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y))$ converges to zero in probability for all $t > 0$, and $x \in \mathcal{C}$ as $y \to x$. The proof relies on two facts: φ^+, φ^- are Fellerian and $W_{\tau_0^y}^+$ will converge to $W_{\tau_0^x}^+$ in probability as $y \to x$ (similarly for W^-), so $\mathbb{P}(\varphi_{0,\tau_0^y}^0(y) \ne \varphi_{0,\tau_0^x}^0(x))$ will converge to 0 as $y \to x$. Note that

$$d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y)) = d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y)) 1_{A_{0,t}} + d(x,y) 1_{A_{0,t}^c}\,.$$

Now assume that $x \in \mathcal{C}^+$. Fix $t > 0$. On the event $\{t < \tau_0^x\} \cap \{t < \tau_0^y\}$ we have $d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y)) = |\arg(y) - \arg(x)|$. The probability of the event $\{(t < \tau_0^x) \cap (t \ge \tau_0^y)\}$ will converge to 0 as $y \to x$, and on the event $\{t \ge \tau_0^x\} \cap \{t \ge \tau_0^y\}$, the Feller property of φ^+ and φ^- will give us the desired result. The other cases are similar.

Note that the flow $\varphi^n_{s,t}$ constructed in (6) is Fellerian since it is a composition of Fellerian mappings which are independent from each other. Now let $\epsilon > 0$ and $k \in \mathbb{N}$ such that $\mathbb{P}\{n_{s,t} > k\} < \epsilon$. Then for all $x, y \in \mathcal{C}$, since $\mathbb{P}(\Omega_{s,t}) = 1$, we have

$$\mathbb{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] \leq \sum_{n \leq k} \mathbb{E}[(f \circ \varphi^n_{s,t}(x) - f \circ \varphi^n_{s,t}(y))^2] + 4\epsilon ||f||^2_\infty$$

since $\varphi^n_{s,t}$ is Fellerian for all n, we get

$$\lim_{d(x,y)\to 0} \mathbb{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] \leq 4\epsilon ||f||^2_\infty .$$

As ϵ is arbitrary, the result follows. □

Proof of Theorem 1.2: SDE (1.1) is satisfied on each edge by construction of φ as a stochastic flow and (3.1) follows. We have $\varphi_{s,t} = \varphi^0_{s,t}$ on $A_{s,t}$. It follows that (1.4) holds on $A_{s,t}$ as both pieces of the definition of φ^0 given in (3.2) coincide with φ^+ (and φ^-) after a transformation. We will show this on $\{t < \gamma^{z,+}_s\}$ ($t < \gamma^{z,-}_s$ is similar). For $m \geq n_{s,t}$, we have

$$\varphi_{s,t} = \varphi^0_{t_m,t} \circ \ldots \circ \varphi^0_{s^+_m, s^+_m + 2^{-m}} \circ \varphi^0_{s,s^+_m} \tag{3.6}$$

If $t < \gamma^{z,+}_s$, then each point in the mesh $s^+_m, s^+_m + 2^{-m}, \ldots, t_m$ is also less than $\gamma^{z,+}_s$. Therefore, each $\varphi^0_{u,v}$, where $(u,v) \in \{(s, s^+_m), (s^+_m, s^+_m + 2^{-m}), \ldots, (t_m, t)\}$, will map into $\mathcal{C}^- \cup \{1\} \cup \mathcal{C}^+$, and will coincide with $\varphi^+_{u,v}$ after the isometric transformation from $\mathcal{C}$ to $\mathbb{R}$. That is, we have

$$\begin{aligned} \varphi_{s^+_m, s^+_m+2^{-m}} \circ \varphi_{s,s^+_m}(z) &= \exp[i\varphi^+_{s^+_m, s^+_m+2^{-m}}(\arg(e^{i\varphi^+_{s,s^+_m}(x)}))] \\ &= \exp[i\,\varphi^+_{s^+_m, s^+_m+2^{-m}} \circ \varphi^+_{s,s^+_m}(x)] \;=\; e^{i\varphi^+_{s,s^+_m+2^{-m}}(arg(z))} \end{aligned}$$

with $x = \arg(z)$, and in view of (3.6) we get

$$\varphi_{s,t}(z) = e^{i\varphi^+_{s,t}(arg(z))}$$

almost surely, when $t < \gamma^{z,+}_s$. □

3.3. *Coalescence.* In this section, we will prove that two particles on the unit circle $\mathcal{C}$ with respect to the flow given in Theorem 1.2 meet in finite time almost surely. Consider the two particle motion as represented by (e^{iX_t}, e^{iY_t}) where X and Y are solutions of SDE (1.1), with $(e^{iX_0}, e^{iY_0}) = (e^{ix}, 1)$, $0 < x < l$. This is an embedding of $\mathcal{C}$ into the plane which allows us to focus on the processes X and Y. Let B^+ and B^- be two Brownian motions with $H_t := \langle B^+, B^- \rangle_t$. Each of X and Y has the same increments as B^+ in $\bigcup_{n\in\mathbb{Z}} (2n\pi, 2n\pi + l]$ and as $-B^-$ elsewhere. Furthermore, we identify each interval of the form $[2n\pi, 2n\pi + l]$, $n \neq 0$, with the interval $[0, l]$, and the intervals of the form $[-2n\pi + l, -2(n-1)\pi]$, $n \neq 1$, with $[-2\pi + l, 0]$ since the argument of a point on the circle is mapped to the latter intervals. Then, the coalescence time is given by

$$T = \inf\{s \geq 0 : (X_s, Y_s) = (0,0) \text{ or } (X_s, Y_s) = (l, -2\pi + l)\}$$

and the region

$$D = [0, l] \times [-2\pi + l, 0]$$

is considered. Define, as before, $A_t = \int_0^{t\wedge T} 1_{\{(X_s,Y_s)\in D\}} ds$, its right continuous inverse $\kappa_t = \inf\{s > 0 : A_s > t\}$, and the process $(X_t^r, Y_t^r) = (X_{\kappa_t}, Y_{\kappa_t})$ together with the local times

$$L_t^1 = \frac{1}{2} L_{\kappa_t}(X, 0)\,, \quad L_t^2 = \frac{1}{2} L_{\kappa_t}(X, l)\,, \quad L_t^3 = \frac{1}{2} L_{\kappa_t}(Y, 0)\,, \quad L_t^4 = \frac{1}{2} L_{\kappa_t}(Y, -2\pi+l)$$

where $L_t(X, a)$ and $L_t(Y, b)$ denote the local times of X and Y at a and b, respectively, for $a, b \in \mathbb{R}$.

Lemma 3.3. *Suppose* $(X_0, Y_0) = (x, 0)$ *with* $0 < x < l$. *The process* (X^r, Y^r) *is a correlated reflected Brownian motion in the bounded domain D which is stopped when it hits* $(0, 0)$ *or* $(l, -2\pi + l)$. *That is, there exist two Brownian motions* B^1 *and* B^2 *with* $|\langle B^1, B^2\rangle_t| \le \rho\, dt$ *such that for all* $t < A_T$

$$(X_t^r, Y_t^r) = (x, 0) + (B_t^1, -B_t^2) + (L_t^1, L_t^1) - (L_t^2, L_t^2) - (L_t^3, L_t^3) + (L_t^4, L_t^4)\,.$$

Proof: We consider

$$X_t = x + B_t^+\,, \qquad Y_t = \int_0^t 1_{\{Y_s>0\}} dB_s^+ - \int_0^t 1_{\{Y_s\le 0\}} dB_s^-$$

according to SDE (1.1) until the stopping time $\bar\tau_1 = \tau_1 \wedge \tau_2 \wedge \tau_3$ where

$$\tau_1 = \inf\{s \ge 0 : X_s = 0\},\ \tau_2 = \inf\{s \ge 0 : X_s = l\},\ \tau_3 = \inf\{s \ge 0 : Y_s = -2\pi + l\}.$$

Note that for $t \le \bar\tau_1$, (X_t^r, Y_t^r) takes values in $D = [0, l] \times [-2\pi + l, 0]$. That Y_t^r is a reflected Brownian motion on $[0, 2\pi - l]$ follows similarly as in Lemma 2.3. By Tanaka formula for the negative part of Y, we have

$$L_t^3 = -Y_t^r - B_t^2$$

where $B_t^2 = -\int_0^{\kappa_t} 1_{\{Y_s\le 0\}} dY_s = \int_0^{\kappa_t} 1_{\{Y_s\le 0\}} dB_s^-$ and L^3 turns out to be the local time of Y_t^r at 0. Putting $B_t^1 = \int_0^{\kappa_t} 1_{\{Y_s\le 0\}} dB_s^+$, we observe that it is also a Brownian motion and

$$X_t^r = x + B_t^1 - L_t^3$$

holds by the construction of X and the fact that $\int_0^{\kappa_t} 1_{\{Y_s>0\}} dY_s = -L_t^3$. Note that we have $|d\langle B^1, B^2\rangle_t| \le \rho\, dt$ just as in the proof of Lemma 2.3. Thus, (X_t^r, Y_t^r) is a correlated reflected Brownian motion on $[0, l] \times [-2\pi + l, 0]$. More explicitly, we have

$$(X_t^r, Y_t^r) = (x, 0) + (B_t^1, -B_t^2) - (L_t^3, L_t^3) \qquad t \le A_{\bar\tau_1}\,.$$

We will consider the three possibilities at $\bar\tau_1$ in i)-iii) below depending on the value of $\bar\tau_1$ and continue the construction by considering a new origin and a new subgraph for X or Y to start with.

i) If $\bar\tau_1 = \tau_1$, we have $(X_{\bar\tau_1}, Y_{\bar\tau_1}) = (0, y)$, for some $y \in [-2\pi + l, 0]$. Now, replace X, Y, B^+, B^- with $X_{\bar\tau_1+\cdot}, Y_{\bar\tau_1+\cdot}, B^+_{\bar\tau_1+\cdot}, B^-_{\bar\tau_1+\cdot}$. Define the following stopping times

$$\tau_4 = \inf\{s \ge 0 : X_s = l\} \quad \tau_5 = \inf\{s \ge 0 : Y_s = 0\} \quad \tau_6 = \inf\{s \ge 0 : Y_s = -2\pi + l\}$$

and let $\bar\tau_2 = \tau_4 \wedge \tau_5 \wedge \tau_6$ to confine the process in region D. Construct X and Y in the interval $(\bar\tau_1, \bar\tau_2]$ as

$$Y_t = y - B_t^-\,, \qquad X_t = \int_0^t 1_{\{X_s>0\}} dB_s^+ - \int_0^t 1_{\{X_s\le 0\}} dB_s^-\,.$$

Then, X_t^r is a reflected Brownian motion on $[0, l]$ with local time L_t^1 at 0:

$$L_t^1 = X_t^r - B_t^1$$

by Tanaka formula and again by construction Y satisfies

$$Y_t^r = -B_t^2 + L_t^1$$

where $B_t^1 = \int_0^{\kappa_t} 1_{\{X_s>0\}} dB_s^+$ and $B_t^2 = \int_0^{\kappa_t} 1_{\{X_s>0\}} dB_s^-$ as before. More explicitly, we have

$$(X_t^r, Y_t^r) = (0, y) + (B_t^1, -B_t^2) + (L_t^1, L_t^1) \qquad t \in (A_{\bar{\tau}_1}, A_{\bar{\tau}_2}]$$

ii) If $\bar{\tau}_1 = \tau_2$, then $(X_{\bar{\tau}_1}, Y_{\bar{\tau}_1}) = (l, y)$, $y \in [-2\pi + l, 0]$. We replace X, Y, B^+, B^- with $X_{\bar{\tau}_1+.}$, $Y_{\bar{\tau}_1+.}$, $B^+_{\bar{\tau}_1+.}$, $B^-_{\bar{\tau}_1+.}$. Define the following stopping times

$$\tau_4 = \inf\{s \geq 0 : X_s = 0\} \quad \tau_5 = \inf\{s \geq 0 : Y_s = 0\} \quad \tau_6 = \inf\{s \geq 0 : Y_s = -2\pi + l\}$$

and let $\bar{\tau}_2 = \tau_4 \wedge \tau_5 \wedge \tau_6$. Construct X and Y in the interval $(\bar{\tau}_1, \bar{\tau}_2]$ as

$$Y_t = y - B_t^-, \qquad X_t = l - \int_0^t 1_{\{X_s>l\}} dB_s^- + \int_0^t 1_{\{X_s \leq l\}} dB_s^+ .$$

Then, X_t^r is a reflected Brownian motion on $[0, l]$ with local time L_t^2 at l:

$$L_t^2 = l + B_t^1 - X_t^r$$

by Tanaka formula where $B_t^1 = \int_0^{\kappa_t} 1_{\{X_s \leq l\}} dB_s^+$ is a Brownian motion. Again, by construction Y satisfies

$$Y_t^r = y - B_t^2 - L_t^2$$

where $B_t^2 = \int_0^{\kappa_t} 1_{\{X_s \leq l\}} dB_s^-$ is a Brownian motion. We get

$$(X_t^r, Y_t^r) = (l, y) + (B_t^1, -B_t^2) - (L_t^2, L_t^2), \qquad t \in (A_{\bar{\tau}_1}, A_{\bar{\tau}_2}] .$$

iii) If $\bar{\tau}_1 = \tau_3$, then $(X_{\bar{\tau}_1}, Y_{\bar{\tau}_1}) = (x', -2\pi + l)$, for some $x' \in (0, l)$. Then, we replace X, Y, B^+, B^- with $X_{\bar{\tau}_1+.}$, $Y_{\bar{\tau}_1+.}$, $B^+_{\bar{\tau}_1+.}$, $B^-_{\bar{\tau}_1+}$ and define

$$\tau_4 = \inf\{s \geq 0 : X_s = 0\} \quad \tau_5 = \inf\{s \geq 0 : X_s = l\} \quad \tau_6 = \inf\{s \geq 0 : Y_s = 0\}$$

and let $\bar{\tau}_2 = \tau_4 \wedge \tau_5 \wedge \tau_6$. In $(\bar{\tau}_1, \bar{\tau}_2]$, X and Y satisfy

$$X_t = x' + B_t^+, \qquad Y_t = -2\pi + l + \int_0^t 1_{\{Y_s \leq -2\pi + l\}} dB_s^+ - \int_0^t 1_{\{Y_s > -2\pi + l\}} dB_s^- .$$

Similar to previous cases, Y_t^r is a reflected Brownian motion on $[-2\pi + l, 0]$ with local time L_t^4 at $-2\pi + l$:

$$L_t^4 = Y_t^r + B_t^2 + 2\pi - l$$

where $B_t^2 = \int_0^{\kappa_t} 1_{\{Y_s > -2\pi + l\}} dB_s^-$, and $X_t^r = x' + B_t^1 + L_t^4$ where $B^1 = \int_0^{\kappa_t} 1_{\{Y_s > -2\pi + l\}} dB_s^+$. More explicitly, we have

$$(X_t^r, Y_t^r) = (x, -2\pi + l) + (B_t^1, -B_t^2) + (L_t^4, L_t^4) \quad t \in (A_{\bar{\tau}_1}, A_{\bar{\tau}_2}] .$$

As a result, we get the desired representation. □

Note that other starting points in D can be handled similarly. Now, consider the process $V_t := \frac{X_t^r - Y_t^r}{\sqrt{2}}$. It follows that $V_t = \frac{x}{\sqrt{2}} + \frac{B_t^1 + B_t^2}{\sqrt{2}}$, which is a martingale with $\lim_{t\to\infty} \langle V \rangle_t = \infty$. That is, V_t is a time changed Brownian motion. Define $T_0 = \inf\{t \geq 0 : V_t = 0 \text{ or } V_t = 2\pi\}$. Then, $T_0 < \infty$ almost surely. To see this, note that $0 \leq V_t \leq 2\pi$ for all t, and $V_t = 0$ if and only if $X_t = Y_t = 0$ and $V_t = 2\pi$ if and only if $X_t = l$ and $Y_t = -2\pi + l$. Therefore, the correlated reflected Brownian motion will hit one of the corners $(0, 0)$ or $(l, -2\pi + l)$ in D in finite time, almost surely.

Lemma 3.4. $\mathbb{P}\{T < \infty\} = 1$.

Proof: Let $U_t = \frac{X_t^r + Y_t^r}{\sqrt{2}}$, and note that $\frac{-2\pi + l}{\sqrt{2}} \le U_t \le \frac{l}{\sqrt{2}}$. Putting $L^r = \sqrt{2}(L^1 - L^2 - L^3 + L^4)$, we get

$$U_t = \frac{x}{\sqrt{2}} + \frac{B_t^1 - B_t^2}{\sqrt{2}} + L_t^r$$

and

$$\mathbb{E}[U_{t \wedge T_0}] = \frac{x}{\sqrt{2}} + \mathbb{E}[L_{t \wedge T_0}^r]\,.$$

It follows that $L_{t\wedge T_0}^r$ is finite for each $t > 0$ since U_t is bounded. This yields $L_{T_0}^r < \infty$. In particular, $L_{T_0}^1$, which is equal to $\frac{1}{2} L_T(X, 0)$, is finite almost surely. Since X is a Brownian motion, this implies that $T < \infty$ almost surely. □

3.4. *Distribution of* T_0. In this section, we will use the notation of Lemma 3.3 and the subsequent definitions, and assume that $H_t = \rho t$ with $\rho \in [0, 1)$. Note that $\langle V \rangle_t = (1 + \rho)t$. Define $M_t = V_t - V_0$. Then M is a martingale with $M_0 = 0$. Since $\langle M \rangle_t = (1 + \rho)t$ it is a time changed Brownian motion. Then, by the same arguments as given in Section 2.4, $B_s := M_{s/(1+\rho)}$ is a one dimensional Brownian motion. Recall that $T_0 = \inf\{t \ge 0 : V_t = 0 \text{ or } V_t = 2\pi\}$ and let

$$S = \inf\{s \ge 0 : B_s = -\frac{x}{\sqrt{2}} \ \text{ or } \ B_s = 2\pi - \frac{x}{\sqrt{2}}\}\,.$$

We have $\mathbb{P}\{T_0 \in dt\} = P\{S \in (1 + \rho)dt\}$. Since the distribution of S is known (Borodin and Salminen, 2002) as

$$\mathbb{P}\{S \in dt\} = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi t^3}} e^{-\frac{(2\pi(k+1) - \frac{x}{\sqrt{2}})^2}{2t}} dt\,,$$

the distribution of T_0 is given by

$$\mathbb{P}\{T_0 \in dt\} = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi(1+\rho)t)^3}} e^{-\frac{(2\pi(k+1) - \frac{x}{\sqrt{2}})^2}{2(1+\rho)t}} (1+\rho)dt\,.$$

This leads to the complementary distribution function

$$\bar{F}_\rho(t) = P\{T_0 \ge t\} = \sum_{k=-\infty}^{+\infty} \int_t^\infty (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi(1+\rho)s)^3}} e^{-\frac{(2\pi(k+1) - \frac{x}{\sqrt{2}})^2}{2(1+\rho)s}} (1+\rho)ds$$

where the interchange of integration and summation is justified by uniform convergence of the series. By a change of variable from $(1 + \rho)s$ to s, we get

$$\begin{aligned}
\mathbb{P}\{T_0 \ge t\} &= \sum_{k=-\infty}^{+\infty} \int_{(1+\rho)t}^\infty (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi s^3}} e^{-\frac{(2\pi(k+1) - \frac{x}{\sqrt{2}})^2}{2s}} ds \\
&= \int_{(1+\rho)t}^\infty P\{S \in ds\} \ = \ \mathbb{P}\{S \ge (1 + \rho)t\}\,.
\end{aligned}$$

As a result, it becomes more likely for the reflected process to hit one of the corners before a fixed time t when the correlation coefficient ρ increases.

Appendix

Following the steps of Hajri and Raimond (2016, Prop.4.5), we give an alternative proof of Proposition 2.2, which reveals the role of the correlation between B^+ and B^-. Let (X, B^+, B^-) and (X', B^+, B^-) be two solutions with $X_0 = X'_0 = 0$. Set $\operatorname{sgn}(x) = 1_{\{x>0\}} - 1_{\{x\le 0\}}$. By the occupation times formula

$$\int_{(0,\infty]} L_t^a(X - X')\frac{da}{a} = \int_0^t 1_{\{X_s - X'_s > 0\}} \frac{d\langle X - X'\rangle_s}{X_s - X'_s}\,.$$

Also observe that

$$d\langle X{-}X'\rangle_s = \big[1 - 1_{\{X_s>0,X'_s>0\}} - 1_{\{X_s\le 0,X'_s\le 0\}} + (1_{\{X_s>0,X'_s\le 0\}} + 1_{\{X_s\le 0,X'_s>0\}})h_s\big]2ds.$$

So, we have

$$\begin{aligned} |d\langle X - X'\rangle_s| &\le \big[|1 - 1_{\{X_s>0,X'_s>0\}} - 1_{\{X_s\le 0,X'_s\le 0\}}| \\ &\quad + |1_{\{X_s>0,X'_s\le 0\}} + 1_{\{X_s\le 0,X'_s>0\}}|.|h_s|\big]2ds \\ &\le 2|\operatorname{sgn}(X_s) - \operatorname{sgn}(X'_s)|\,ds \end{aligned}$$

since $|h_s| \le \rho < 1$.

Let $\{f_n\} \subset C^1(\mathbb{R})$ such that $f_n \to \operatorname{sgn}$ pointwise and $\{f_n\}$ is uniformly bounded in total variation. By Fatou's lemma we get

$$\begin{aligned} \int_{(0,\infty]} L_t^a(X - X')\frac{da}{a} &\le 2\liminf_n \int_0^t 1_{\{X_s - X'_s>0\}} \frac{|f_n(X_s) - f_n(X'_s)|}{X_s - X'_s} ds \\ &\le 2\liminf_n \int_0^t 1_{\{X_s - X'_s>0\}} \left| \int_0^1 f'_n(Z_s^u)du \right| ds \end{aligned}$$

where $Z_s^u = (1-u)X_s + uX'_s$. Now observe that

$$\frac{d\langle Z^u\rangle_s}{ds} = \begin{cases} 1 & \text{if } X_s > 0, X'_s > 0 \text{ or } X_s \le 0, X'_s \le 0, \\ 2(1+h_s)u^2 - 2(1+h_s)u + 1 & \text{if } X_s > 0, X'_s \le 0 \text{ or } X_s \le 0, X'_s > 0. \end{cases}$$

This shows that for all $0 \le \rho < 1$ there exists a constant $A > 0$ such that for all $s \ge 0$, and $u \in [0,1]$ $d\langle Z^u\rangle_s \ge \frac{ds}{A}$, since this polynomial in u has its minimum for $u = 1/2$ and it has roots $(1 \pm \sqrt{\frac{(h_s)^2-1}{h_s+1}})\frac{1}{2}$. For $|h_s| \le \rho < 1$ it has no real roots. We have the following inequality:

$$\begin{aligned} \int_{(0,\infty]} L_t^a(X - X')\frac{da}{a} &\le 2A\liminf_n \int_0^1 \int_0^t |f'_n(Z_s^u)|d\langle Z^u\rangle_s du \\ &\le 2A\liminf_n \int_0^1 \int_{\mathbb{R}} |f'_n(a)|L_t^a(Z^u)dadu\,. \end{aligned}$$

Now taking expectations and using Fatou's lemma we get

$$\mathbb{E}\left[\int_{(0,\infty]} L_t^a(X - X')\frac{da}{a}\right] \le 2A\liminf_n \int_{\mathbb{R}} |f'_n(a)|da \sup_{a\in\mathbb{R},u\in[0,1]} \mathbb{E}[L_t^a(Z^u)]\,.$$

By Tanaka's formula, we have

$$\begin{aligned} \mathbb{E}[L_t^a(Z^u)] &= \mathbb{E}[|Z_t^u - a|] - \mathbb{E}[|Z_0^u - a|] - \mathbb{E}\left[\int_0^t \operatorname{sgn}(Z_s^u - a)dZ_s^u\right] \\ &\le \mathbb{E}[|Z_t^u - Z_0^u|]\,. \end{aligned}$$

The right hand side is uniformly bounded with respect to (a, u) which gives us

$$\int_{(0,\infty]} L_t^a(X - X')\frac{da}{a} < \infty\,.$$

Since $\lim_{a\downarrow 0} L^a(X - X') = L^0(X - X')$ this implies that $L_t^0(X - X') = 0$ and thus by Tanaka's formula, $|X - X'|$ is a local martingale which is also a nonnegative supermartingale with $|X_0 - X_0'| = 0$ and finally X and X' are indistinguishable.

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