ALEA, Lat. Am. J. Probab. Math. Stat. 16, 141–177 (2019)
DOI: 10.30757/ALEA.v16-06

Limit theorems for random simplices in high dimensions

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Abstract. Let $r = r(n)$ be a sequence of integers such that $r \leq n$ and let $X_1, \dots, X_{r+1}$ be independent random points distributed according to the Gaussian, the Beta or the spherical distribution on $\mathbb{R}^n$. Limit theorems for the log-volume and the volume of the random convex hull of $X_1, \dots, X_{r+1}$ are established in high dimensions, that is, as r and n tend to infinity simultaneously. This includes Berry-Esseen-type central limit theorems, log-normal limit theorems, and moderate and large deviations. Also different types of mod-ϕ convergence are derived. The results heavily depend on the asymptotic growth of r relative to n. For example, we prove that the fluctuations of the volume of the simplex are normal (respectively, log-normal) if $r = o(n)$ (respectively, $r \sim \alpha n$ for some $0 < \alpha < 1$).

Received by the editors November 3rd, 2017; accepted November 15th, 2018.
2010 *Mathematics Subject Classification.* 52A22, 52A23, 60D05, 60F05, 60F10.
Key words and phrases. Central limit theorem, large deviations, moderate deviations, mod-ϕ convergence, random simplex, volume.

1. Introduction

In the last decades, random polytopes have become one of the most central models studied in stochastic geometry. In particular, they have seen numerous applications to other branches of mathematics such as asymptotic geometric analysis, compressed sensing, computational geometry, optimization or multivariate statistics; see, for example, the survey articles of Bárány (2007), Hug (2013) and Reitzner (2010) for further details and references. The focus in most works has been on models of the following type. First, we fix a space dimension $n \in \mathbb{N}$ and a probability measure μ on $\mathbb{R}^n$. Then, we let $X_1, \dots, X_r$, where $r \geq n+1$, be independent random points in $\mathbb{R}^n$ that are distributed according to μ. A random polytope P_r now arises by taking the convex hull of the point set $X_1, \dots, X_r$. Starting with the seminal paper of Rényi and Sulanke (1963), the asymptotic behaviour of the expectation and the variance of the volume or the number of faces of P_r has been studied intensively, as $r \to \infty$, while keeping n fixed. Moreover, it has been investigated whether these quantities satisfy a 'typical' and 'atypical' behaviour, i.e., fulfil a central limit theorem, large or moderate deviation principles and concentration inequalities, respectively, to name just a few topics of current research.

However, up to a few exceptions it has not been investigated what happens if the space dimension n is not fixed, but tends to infinity. The only such exceptions we were able to localize in the literature are the papers of Ruben (1977), Mathai (1982), Anderson (1986) and Maehara (1980). It is shown in the first two of these works that for any *fixed* $r \in \mathbb{N}$ the r-volume of the convex hull of $r+1 \leq n+1$ independent and uniform random points of which some are in the interior of the n-dimensional unit ball and the others on its boundary, is asymptotically normal, as $n \to \infty$. The third one establishes the same result in the situation where the r points are distributed according to the so-called Beta-type distribution in the n-dimensional unit ball. The fourth mentioned text generalizes the set-up to an arbitrary underlying n-fold product distribution on $\mathbb{R}^n$.

On the other hand, the regime in which r and n tend to infinity *simultaneously* is not treated in these papers. The purpose of the present text is to close this gap and to prove a collection of probabilistic limit theorems for the r-volume of the convex hull of $r+1 \leq n+1$ random points that are distributed according to certain classes of probability distributions that allow for explicit computations, especially focusing on different regimes of growths of the parameter r relative to n. More precisely, we distinguish between the following three regimes. The first one is the case where r grows like $o(n)$ with the dimension n, which means that r/n converges to zero, as $n \to \infty$. This of course includes the situations where r is fixed – covering thereby the case considered in the four papers mentioned above – or behaves like n^α wir $\alpha \in (0,1)$, to give just two examples (let us emphasize at this point that we interpret expressions like $\sqrt{n}$ or $n/2$ as $\lfloor \sqrt{n} \rfloor$ and $\lfloor n/2 \rfloor$, respectively, in what follows). Secondly, the underlying situation might be the one where r is asymptotically equivalent to αn for some $\alpha \in (0,1)$. Lastly, we analyse the setting where $n - r = o(n)$, as $n \to \infty$. In particular, for $r = n$ we arrive in the situation where we choose $n+1$ random points and thus their convex hull is nothing but a full-dimensional simplex in $\mathbb{R}^n$.

Our paper and the results we are going to present (and which represent a 'complete' description of the high-dimensional probabilistic behaviour of the underlying random simplices) are organized as follows. In Section 2 we introduce the different random point models we consider and state formulas for the moments of the volume of the random simplices induced by the convex hulls of these point sets. By using these moments, we are then able to derive the precise distributions of the previously mentioned volumes. In Section 3 we start with the first limit theorems. By using the method of cumulants we give 'optimal' Berry-Esseen bounds and moderate deviation principles for the logarithmic volumes of our random simplices. Then, we transfer the limit theorem from the log-volume to the volume itself and obtain thereby a phase transition in the limiting behaviour depending on the choice of the parameter r. Section 4 establishes results concerning so-called mod-ϕ convergence and is also the starting point to prove the results presented in Section 5, where we add large deviation principles to the moderate ones obtained earlier in Section 3.

2. Models, volumes and probabilistic representations

2.1. *The four models.* In this paper we consider convex hulls of random points $X_1, X_2, \ldots$ We only consider the following four models which allow for explicit computations. These models were identified in Miles (1971) and Ruben and Miles (1980), respectively, see also Kabluchko et al. (2018, Section 3.4).

(a) The *Gaussian model*: $X_1, X_2, \ldots$ are i.i.d. with standard normal density

$$f(x) = (2\pi)^{-n/2} \cdot \mathrm{e}^{-\frac{1}{2}|x|^2}, \quad x \in \mathbb{R}^n.$$

(b) The *Beta model* with parameter $\nu > 0$: $X_1, X_2, \ldots$ are i.i.d. points in the ball of radius 1 with density

$$f(x) = \frac{1}{\pi^{n/2}} \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1 - |x|^2\right)^{(\nu-2)/2}, \quad x \in \mathbb{R}^n, \quad |x| < 1.$$

(c) The *Beta prime model* with parameter $\nu > 0$: $X_1, X_2, \ldots$ are i.i.d. points with density

$$f(x) = \frac{1}{\pi^{n/2}} \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1 + |x|^2\right)^{-(n+\nu)/2}, \quad x \in \mathbb{R}^n.$$

(d) The *spherical model*: $X_1, X_2, \ldots$ are uniformly distributed on the sphere of radius 1 centered at the origin of $\mathbb{R}^n$.

Remark 2.1. Observe that in the Beta prime model the power is $(n + \nu)/2$ (which depends on n) rather than just $\nu/2$.

2.2. *Moments for the volumes of random simplices and parallelotopes.* Let $1 \le r \le n$ be an integer and $X_1, \ldots, X_{r+1}$ be independent random points in $\mathbb{R}^n$ that are distributed according to one of the distributions introduced in Section 2.1. By $\mathcal{V}_{n,r}$ we denote the r-dimensional volume of the simplex with vertices $X_1, \ldots, X_{r+1}$. Moreover, we use the symbol $\mathcal{W}_{n,r}$ to indicate the r-dimensional volume of the parallelotope spanned by the vectors $X_1, \ldots, X_r$. Note that up to a factor $r!$, $\mathcal{W}_{n,r}$ is the same as the r-dimensional volume of the simplex with vertices $0, X_1, \ldots, X_r$. We start by recalling formulas for the moments of $\mathcal{W}_{n,r}$. Moments of integer orders can directly be computed using the well-known linear Blaschke-Petkantschin formula from integral geometry together with an induction argument.

Theorem 2.2 (Moments for parallelotopes)**.** *Let $\mathcal{W}_{n,r}$ be the volume of the r-dimensional parallelotope spanned by the vectors $X_1, \ldots, X_r$ chosen according to one of the above four models.*

(a) *In the Gaussian model we have, for all real $k \geq 0$,*

$$\mathbb{E}[\mathcal{W}_{n,r}^{2k}] = \prod_{j=1}^{r} \left[2^k \frac{\Gamma\Big(\frac{n-r+j}{2} + k\Big)}{\Gamma\Big(\frac{n-r+j}{2}\Big)} \right].$$

(b) *In the Beta model with parameter $\nu > 0$ we have, for all real $k \geq 0$,*

$$\mathbb{E}[\mathcal{W}_{n,r}^{2k}] = \prod_{j=1}^{r} \left[\frac{\Gamma\Big(\frac{n-r+j}{2} + k\Big)\Gamma\Big(\frac{n+\nu}{2}\Big)}{\Gamma\Big(\frac{n-r+j}{2}\Big)\Gamma\Big(\frac{n+\nu}{2} + k\Big)} \right].$$

(c) *In the Beta prime model with parameter $\nu > 0$ we have, for all real $k \in (0, \frac{\nu}{2}]$,*

$$\mathbb{E}[\mathcal{W}_{n,r}^{2k}] = \prod_{j=1}^{r} \left[\frac{\Gamma\Big(\frac{n-r+j}{2} + k\Big)\Gamma\Big(\frac{\nu}{2} - k\Big)}{\Gamma\Big(\frac{n-r+j}{2}\Big)\Gamma\Big(\frac{\nu}{2}\Big)} \right].$$

(d) *In the spherical model we have, for all real $k \geq 0$,*

$$\mathbb{E}[\mathcal{W}_{n,r}^{2k}] = \prod_{j=1}^{r} \left[\frac{\Gamma\Big(\frac{n-r+j}{2} + k\Big)\Gamma\Big(\frac{n}{2}\Big)}{\Gamma\Big(\frac{n-r+j}{2}\Big)\Gamma\Big(\frac{n}{2} + k\Big)} \right].$$

Proof: The formula in (a) can be concluded from Mathai (1999) or Ruben (1979). Formula (b) is Theorem 19.2.5 from Mathai (2001), Formula (c) is Theorem 19.2.6 from Mathai (2001). Formula (d) is the limiting case of (c) for $\nu \downarrow 0$ but is actually also contained both in Theorems 19.2.5 and 19.2.6 from Mathai (2001) because these deal with a slightly more general model which allows for some points to be chosen uniformly on the unit sphere. □

For simplices, the moments are very similar. The products appearing in the formulas for simplices are the same as for parallelotopes, but certain additional factors involving the Γ-function appear. Again, for moments of integer order, a direct proof for these formulas can be carried out using the affine Blaschke-Petkantschin formula and an induction argument (compare, for example, with the proof of Schneider and Weil (2008, Theorem 8.2.3) for the special case of the Beta model with $\nu = 2$ and the spherical model.)

Theorem 2.3 (Moments for simplices)**.** *Let $\mathcal{V}_{n,r}$ be the volume of the r-dimensional simplex with vertices $X_1, \ldots, X_{r+1}$ chosen according to one of the above four models.*

(a) *In the Gaussian model we have, for all real $k \geq 0$,*

$$\mathbb{E}[(r!\mathcal{V}_{n,r})^{2k}] = (r+1)^k \prod_{j=1}^{r} \left[2^k \frac{\Gamma\big(\frac{n-r+j}{2} + k\big)}{\Gamma\big(\frac{n-r+j}{2}\big)} \right].$$

(b) *In the Beta model with parameter $\nu > 0$ we have, for all real $k \geq 0$,*

$$\mathbb{E}[(r!\mathcal{V}_{n,r})^{2k}] = \prod_{j=1}^{r} \left[\frac{\Gamma\left(\frac{n-r+j}{2} + k\right)}{\Gamma\left(\frac{n-r+j}{2}\right)} \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2} + k\right)} \right] \times \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2} + k\right)} \frac{\Gamma\left(\frac{r(n+\nu-2)+(n+\nu)}{2} + (r+1)k\right)}{\Gamma\left(\frac{r(n+\nu-2)+(n+\nu)}{2} + rk\right)}.$$

(c) *In the Beta prime model with parameter $\nu > 0$ we have, for all real $0 \leq k < \frac{\nu}{2}$,*

$$\mathbb{E}[(r!\mathcal{V}_{n,r})^{2k}] = \prod_{j=1}^{r} \left[\frac{\Gamma\left(\frac{n-r+j}{2} + k\right)}{\Gamma\left(\frac{n-r+j}{2}\right)} \frac{\Gamma\left(\frac{\nu}{2} - k\right)}{\Gamma\left(\frac{\nu}{2}\right)} \right] \times \frac{\Gamma\left(\frac{\nu}{2} - k\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{(r+1)\nu}{2} - rk\right)}{\Gamma\left(\frac{(r+1)\nu}{2} - (r+1)k\right)}.$$

(d) *In the spherical model we have, for all real $k \geq 0$,*

$$\mathbb{E}[(r!\mathcal{V}_{n,r})^{2k}] = \prod_{j=1}^{r} \left[\frac{\Gamma\left(\frac{n-r+j}{2} + k\right)}{\Gamma\left(\frac{n-r+j}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} + k\right)} \right] \times \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} + k\right)} \frac{\Gamma\left(\frac{r(n-2)+n}{2} + (r+1)k\right)}{\Gamma\left(\frac{r(n-2)+n}{2} + rk\right)}.$$

Proof: Formula (a) is Equation (70) in Miles (1971). Formula (b) is Equation (74) in Miles (1971). Formula (c) is Equation (72) in Miles (1971). Finally, Formula (d) is obtained from (b) by letting $\nu \to 0$. Note that the formula in Miles (1971) contains a typo, which is corrected, for example, in Chu (1993). Also Miles (1971) considers only integer moments. Extension to non-integer moments can be found in Kabluchko et al. (2018+). □

Observe that the moments in the spherical case can be obtained from the moments in the Beta model by taking $\nu = 0$ there. In fact, the uniform distribution on the sphere is the weak limit of the Beta distribution as $\nu \downarrow 0$; see the proof of Theorem 2.7, below. Since the proofs of our limit theorems are based on the above formulas for the moments, we may and will consider the spherical and the Beta models together, the former being the special case of the latter with $\nu = 0$. We refrain from stating the limit theorems in the Beta prime case because they are very similar to the Beta case.

2.3. *Distributions for the volumes of random simplices and parallelotopes.* The purpose of this section is to derive probabilistic representations for the random variables $\mathcal{W}_{n,r}^2$ and $\mathcal{V}_{n,r}^2$ for the four models introduced in Section 2.1. For this, we need to recall certain standard distributions. A random variable has a Gamma distribution with shape $\alpha > 0$ and scale $\lambda > 0$ if its density is given by

$$g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}, \quad t \geq 0.$$

Especially if $\alpha = d/2$ for some $d \in \mathbb{N}$ and $\lambda = 1/2$, we speak about a χ^2 distribution with d degrees of freedom. A random variable has a Beta distribution with parameters $\alpha_1 > 0, \alpha_2 > 0$ if its density is

$$g(t) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}, \quad t \in (0,1).$$

Finally, a random variable has a Beta prime distribution with parameters $\alpha_1 > 0$, $\alpha_2 > 0$ if its density is

$$g(t) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} t^{\alpha_1 - 1}(1+t)^{-\alpha_1 - \alpha_2}, \quad t > 0.$$

Note that the Beta prime distribution coincides, up to rescaling, with the Fisher F distribution. We agree to denote by χ_d^2, respectively $\Gamma_{\alpha,\lambda}, \beta_{\alpha_1,\alpha_2}, \beta'_{\alpha_1,\alpha_2}$, a random variable with χ^2-distribution with $d \in \mathbb{N}$ degrees of freedom and the Gamma, Beta or Beta prime distribution with corresponding parameters, respectively. We shall also use the notation $X \sim \text{Beta}(\alpha_1, \alpha_2)$ or $X \sim \text{Beta}'(\alpha_1, \alpha_2)$ to indicate that a random variable X has a Beta or a Beta prime distribution with parameters α_1 and α_2, respectively. Also, we agree that all such variables appearing below are assumed to be independent. We recall that the moments (of real order $k \geq 0$, as long as they exist) of these distributions are given by:

$$\mathbb{E}[\chi_d^{2k}] = 2^k \frac{\Gamma\left(\frac{d}{2} + k\right)}{\Gamma\left(\frac{d}{2}\right)},$$

$$\mathbb{E}[\beta_{\alpha_1,\alpha_2}^k] = \frac{\Gamma(\alpha_1 + \alpha_2)\Gamma(\alpha_1 + k)}{\Gamma(\alpha_1)\Gamma(\alpha_1 + \alpha_2 + k)},$$

$$\mathbb{E}[(\beta'_{\alpha_1,\alpha_2})^k] = \frac{\Gamma(\alpha_1 + k)\Gamma(\alpha_2 - k)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Using Theorem 2.2 we first obtain probabilistic representations for the volume of random parallelotopes spanned by vectors whose distributions belong to one of the classes introduced in Section 2.1.

Theorem 2.4 (Distributions for parallelotopes)**.** *Let $\mathcal{W}_{n,r}$ be the volume of the r-dimensional parallelotope spanned by the vectors $X_1, \ldots, X_r$ chosen according to one of the above four models.*

(a) *In the Gaussian model we have* $\mathcal{W}_{n,r}^2 \overset{d}{=} \prod_{j=1}^{r} \chi_{n-r+j}^2$.

(b) *In the Beta model we have* $\mathcal{W}_{n,r}^2 \overset{d}{=} \prod_{j=1}^{r} \beta_{\frac{n-r+j}{2}, \frac{\nu+r-j}{2}}$.

(c) *In the Beta prime model we have* $\mathcal{W}_{n,r}^2 \overset{d}{=} \prod_{j=1}^{r} \beta'_{\frac{n-r+j}{2}, \frac{\nu}{2}}$.

(d) *In the spherical model we have* $\mathcal{W}_{n,r}^2 \overset{d}{=} \prod_{j=1}^{r} \beta_{\frac{n-r+j}{2},\frac{r-j}{2}}$.

The random variables involved in the products are assumed to be independent.

The distribution of the volume of a random simplex generated by one of the four models is more involved and can be derived from Theorem 2.3.

Theorem 2.5 (Distributions for simplices). *Let $\mathcal{V}_{n,r}$ be the volume of the r-dimensional simplex with vertices $X_1,\dots,X_{r+1}$ chosen according to the one of the above four models.*

(a) *In the Gaussian model we have*

$$(r!\mathcal{V}_{n,r})^2 \overset{d}{=} (r+1)\prod_{j=1}^{r}\chi^2_{n-r+j}.$$

(b) *In the Beta model we have*

$$\xi(1-\xi)^r(r!\mathcal{V}_{n,r})^2 \overset{d}{=} (1-\eta)^r\prod_{j=1}^{r}\beta_{\frac{n-r+j}{2},\frac{\nu+r-j}{2}},$$

where $\xi,\eta\sim Beta(\frac{n+\nu}{2},\frac{r(n+\nu-2)}{2})$ are random variables such that ξ is independent of $\mathcal{V}_{n,r}$, while η is independent of $\beta_{\frac{n-r+j}{2},\frac{\nu+r-j}{2}}$, $j=1,\dots,r$.

(c) *In the Beta prime model we have*

$$(1+\eta)^r(r!\mathcal{V}_{n,r})^2 \overset{d}{=} \xi^{-1}(1+\xi)^{r+1}\prod_{j=1}^{r}\beta'_{\frac{n-r+j}{2},\frac{\nu}{2}},$$

where $\xi,\eta\sim Beta'(\frac{\nu}{2},\frac{r\nu}{2})$ are random variables such that η is independent of $\mathcal{V}_{n,r}$, while ξ is independent of $\beta'_{\frac{n-r+j}{2},\frac{\nu}{2}}$, $j=1,\dots,r$.

(d) *In the spherical model we have*

$$\xi(1-\xi)^r(r!\mathcal{V}_{n,r})^2 \overset{d}{=} (1-\eta)^r\prod_{j=1}^{r}\beta_{\frac{n-r+j}{2},\frac{r-j}{2}},$$

where $\xi,\eta\sim Beta(\frac{n}{2},\frac{r(n-2)}{2})$ are random variables such that ξ is independent of $\mathcal{V}_{n,r}$, while η is independent of $\beta_{\frac{n-r+j}{2},\frac{r-j}{2}}$, $j=1,\dots,r$.

Proof: The assertion in (a) follows directly from Theorem 2.3 (a) combined with the fact that the kth moment of a χ^2_{n-r+j} random variable is given by

$$2^k\frac{\Gamma(\frac{n-r+j}{2}+k)}{\Gamma(\frac{n-r+j}{2})}.$$

To prove (b) we define $\alpha_1 := \frac{n+\nu}{2}$ and $\alpha_2 := \frac{r(n+\nu-2)}{2}$. Denoting by $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x,y>0$, the Beta function, we observe that, since $\xi,\eta\sim \mathrm{Beta}(\alpha_1,\alpha_2)$,

$$\begin{aligned}\mathbb{E}[(1-\eta)^{rk}] &= \frac{1}{B(\alpha_1,\alpha_2)}\int_0^1 x^{\alpha_1-1}(1-x)^{\alpha_2+rk-1}\,\mathrm{d}x\\ &= \frac{B(\alpha_1,\alpha_2+rk)}{B(\alpha_1,\alpha_2)}\end{aligned}$$

and

$$\mathbb{E}[\xi^k(1-\xi)^{rk}] = \frac{1}{B(\alpha_1,\alpha_2)}\int_0^1 x^{\alpha_1+k-1}(1-x)^{\alpha_2+rk-1}\,\mathrm{d}x = \frac{B(\alpha_1+k,\alpha_2+rk)}{B(\alpha_1,\alpha_2)}.$$

This implies that

$$\frac{\mathbb{E}[(1-\eta)^{rk}]}{\mathbb{E}[\xi^k(1-\xi)^{rk}]} = \frac{B(\alpha_1,\alpha_2+rk)}{B(\alpha_1+k,\alpha_2+rk)} = \frac{\Gamma(\alpha_1+\alpha_2+(r+1)k)\Gamma(\alpha_1)}{\Gamma(\alpha_1+k)\Gamma(\alpha_1+\alpha_2+rk)} = \frac{\Gamma\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}+(r+1)k\Big)\Gamma\Big(\frac{n+\nu}{2}\Big)}{\Gamma\Big(\frac{n+\nu}{2}+k\Big)\Gamma\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}+rk\Big)}$$

and this is precisely the last factor in the formula for the moments, see Theorem 2.3 (b).

Next, we consider (c). Since $\xi,\eta \sim \mathrm{Beta}'(\alpha_1,\alpha_2)$ with $\alpha_1 = \frac{\nu}{2}$ and $\alpha_2 = \frac{r\nu}{2}$, we apply the formula $\int_0^\infty x^{\alpha_1-1}(1+x)^{-\alpha_1-\alpha_2}\mathrm{d}x = B(\alpha_1,\alpha_2)$ to obtain

$$\mathbb{E}[(1+\eta)^{rk}] = \frac{1}{B(\alpha_1,\alpha_2)}\int_0^\infty x^{\alpha_1-1}(1+x)^{-\alpha_1-(\alpha_2-rk)}\mathrm{d}x = \frac{B(\alpha_1,\alpha_2-rk)}{B(\alpha_1,\alpha_2)}$$

and

$$\mathbb{E}\Big[\xi^{-k}(1+\xi)^{(r+1)k}\Big] = \frac{1}{B(\alpha_1,\alpha_2)}\int_0^\infty x^{\alpha_1-k-1}(1+x)^{-\alpha_1-\alpha_2-(r+1)k}\mathrm{d}x = \frac{B(\alpha_1-k,\alpha_2-rk)}{B(\alpha_1,\alpha_2)}.$$

It follows that

$$\frac{\mathbb{E}\Big[\xi^{-k}(1+\xi)^{(r+1)k}\Big]}{\mathbb{E}[(1+\eta)^{rk}]} = \frac{B(\alpha_1-k,\alpha_2-rk)}{B(\alpha_1,\alpha_2-rk)} = \frac{\Gamma(\alpha_1-k)\Gamma(\alpha_1+\alpha_2-rk)}{\Gamma(\alpha_1+\alpha_2-(r+1)k)\Gamma(\alpha_1)} = \frac{\Gamma\Big(\frac{\nu}{2}-k\Big)}{\Gamma\Big(\frac{\nu}{2}\Big)}\frac{\Gamma\Big(\frac{(r+1)\nu}{2}-rk\Big)}{\Gamma\Big(\frac{(r+1)\nu}{2}-(r+1)k\Big)},$$

which is exactly the last factor in the formula for the moments given by Theorem 2.3 (c).

The assertion in (d) follows as a limit case from that in (b), as $\nu \downarrow 0$. □

Remark 2.6. The distributional equality in Theorem 2.5 (a) has already been noted by Miles (1971, Section 13). The other probabilistic representations in (b)–(d) seem to be new.

2.4. *Distance distributions.* As in the previous sections let $X_1,\dots,X_{r+1}$ be independent random points that are distributed according to one of the four models from Section 2.1. Our interest now lies in the distance from the origin to the r-dimensional affine subspace spanned by $X_1,\dots,X_{r+1}$.

Theorem 2.7 (Distance distributions)**.** *Let $X_1, \ldots, X_{r+1}$ be chosen according to one of the above four models and denote by $\mathcal{D}_{n,r}$ the distance from the origin to the r-dimensional affine subspace spanned by $X_1, \ldots, X_{r+1}$.*

(a) *In the Gaussian model we have $\mathcal{D}_{n,r}^2 \overset{d}{=} (r+1)^{-1}\chi^2_{n-r}$.*

(b) *In the Beta model we have $\mathcal{D}_{n,r}^2 \overset{d}{=} \beta_{\frac{n-r}{2}, \frac{\nu(r+1)+r(n-1)}{2}}$.*

(c) *In the Beta prime model we have $\mathcal{D}_{n,r}^2 \overset{d}{=} \beta'_{\frac{n-r}{2}, \frac{\nu(r+1)}{2}}$.*

(d) *In the spherical model we have $\mathcal{D}_{n,r}^2 \overset{d}{=} \beta_{\frac{n-r}{2}, \frac{r(n-1)}{2}}$.*

Proof: The density of $\mathcal{D}_{n,r}$ in the cases (a)–(c) can be computed from a formula on page 16 in Ruben and Miles (1980). In fact, for the Gaussian model we obtain that $\mathcal{D}_{n,r}$ has density

$$h \mapsto c_{n,r}\, h^{n-r-1}\, e^{-\frac{h^2(r+1)}{2}}, \qquad h > 0,$$

which implies (a). For the Beta model we obtain the density

$$h \mapsto c_{n,r,\nu}\, h^{n-r-1}(1-h^2)^{\frac{r(n+1)}{2}+\frac{(r+1)(\nu-2)}{2}}, \qquad 0 < h < 1,$$

for $\mathcal{D}_{n,r}$ and (b) follows. Next, for the Beta prime model the density of $\mathcal{D}_{n,r}$ is given by

$$h \mapsto c_{n,r,\nu}\, h^{n-r-1}(1+h^2)^{\frac{r(n+1)}{2}-\frac{(r+1)(n+\nu)}{2}}, \qquad h > 0,$$

whence (c) follows. Finally, the spherical model follows from the Beta model in the limit, as $\nu \downarrow 0$. In fact, since the centred ball of radius 1 can be regarded as a compact metric space, the family of probability measures $(\mathbb{P}_\nu)_{\nu>0}$ with densities $f_\nu(|x|) := \mathrm{const}(1-|x|^2)^{(\nu-2)/2}$, $\nu > 0$, is tight for each $n \in \mathbb{N}$. Thus, $(\mathbb{P}_\nu)_{\nu>0}$ is weakly sequentially compact, i.e., there exist weakly convergent subsequences $(\mathbb{P}_{\nu_n})_{n\in\mathbb{N}}$ with $\nu_n \downarrow 0$. For each such sequence ν_n the limiting probability measure is easily seen to have the following two properties: (i) it is rotation invariant and (ii) it is concentrated on the boundary of the centred ball of radius 1, that is, the radius 1 sphere. In other words, the limit must coincide with the normalized spherical Lebesgue measure on that sphere. Now, as $\nu \downarrow 0$ and since $(x_1, \ldots, x_{r+1}) \mapsto \mathrm{dist}(0, \mathrm{aff}(x_1, \ldots, x_{r+1}))$ is a bounded continuous function on the $(r+1)$-st cartesian power of the unit ball, the density in (d) is the limit of the density in (b). □

3. Cumulants, Berry-Esseen bounds and moderate deviations

In this section we shall concentrate on the Gaussian, the Beta and the spherical model, for which the random variables $\mathcal{V}_{n,r}$ have finite moments of all orders for any $n \in \mathbb{N}$ and $r \le n$.

3.1. *Cumulants for logarithmic volumes.* For a random variable X with $\mathbb{E}[|X|^m] < \infty$ for some $m \in \mathbb{N}$, we write $c^m[X]$ for the mth order cumulant of X, that is,

$$c^m[X] = (-\mathrm{i})^m \frac{\mathrm{d}^m}{\mathrm{d}t^m} \log \mathbb{E}[\exp(\mathrm{i}tX)]\Big|_{t=0}, \tag{3.1}$$

where $\mathfrak{i}$ stands for the imaginary unit. It is well known that sharp bounds for cumulants lead to fine probabilistic estimates for the involved random variables. For the volume of a random simplex with Gaussian or Beta distributed vertices we shall establish the following cumulant bound. In what follows we shall write $a_n \sim b_n$ for two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ if $a_n/b_n \to 1$, as $n \to \infty$. Let us define the random variable $\mathcal{L}_{n,r} := \log(r!\mathcal{V}_{n,r})$.

Theorem 3.1. *Let $X_1, \ldots, X_{r+1}$ be chosen according to one of the four models presented in the previous section, and let $\alpha \in (0,1)$.*

(a) *For the Gaussian model we have*

$$\mathbb{E}\mathcal{L}_{n,r} \sim \frac{r}{2}\log n,$$

$$\operatorname{Var}\mathcal{L}_{n,r} \sim \begin{cases} \frac{r}{2n} & : r = o(n) \\ \frac{1}{2}\log\frac{1}{1-\alpha} & : r \sim \alpha n \\ \frac{1}{2}\log\frac{n}{n-r+1} & : n-r = o(n) \end{cases}$$

and, for $m \geq 3$,

$$|c^m[\mathcal{L}_{n,r}]| \leq \begin{cases} C^m(m-1)!rn^{1-m} & : r = o(n) \text{ or } r \sim \alpha n \\ 2\,(m-1)! & : \text{for arbitrary } r(n)\,, \end{cases}$$

where $C \in (0,\infty)$ is a constant not depending on n and m.

(b) *For the Beta model and the spherical model we have*

$$\operatorname{Var}\mathcal{L}_{n,r} \sim \begin{cases} \frac{r}{2(r+1)n} & : r = \text{const} \\ \frac{r^2}{4n^2} + \frac{1}{2n} & : r \to \infty \text{ such that } r = o(n) \\ \frac{1}{2}\log\frac{1}{1-\alpha} - \frac{\alpha}{2} & : r \sim \alpha n \\ \frac{1}{2}\log\frac{n}{n-r+1} & : n-r = o(n) \end{cases}$$

and, for all $m \geq 3$ and $n \geq 3$,

$$|c^m[\mathcal{L}_{n,r}]| \leq \begin{cases} C^m m!rn^{1-m} & : r = o(n) \text{ or } r \sim \alpha n \\ 2\cdot 6^m m! & : \text{for arbitrary } r(n)\,, \end{cases}$$

where $C \in (0,\infty)$ is a constant not depending on n and m.

The proof of Theorem 3.1 is to some extent canonical and roughly follows Döring and Eichelsbacher (2013a). In particular, it is based on an asymptotic analysis, as $|z| \to \infty$, of the digamma function $\psi(z) = \psi^{(0)}(z) := \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z)$, and the polygamma functions

$$\psi^{(m)}(z) := \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\log\Gamma(z), \qquad m \in \mathbb{N}.$$

We start with the following lemma.

Lemma 3.2. *Let $m \in \mathbb{N}$. Then, as $|z| \to \infty$ in $|\arg z| < \pi - \varepsilon$,*

$$\psi(z) = \log z + O(1/z) \quad \text{and} \quad \psi^{(m)}(z) = (-1)^{m-1}\frac{(m-1)!}{z^m} + O(1/z^{m+1})\,. \tag{3.2}$$

Moreover, for all $z > 0$,

$$|\psi^{(m)}(z)| \leq \frac{(m-1)!}{z^m} + \frac{m!}{z^{m+1}}\,. \tag{3.3}$$

Proof: The asymptotic relations can be found in Abramowitz and Stegun (1992, pp. 259–260). To prove the inequality, note that

$$|\psi^{(m)}(z)| = \sum_{k=0}^{\infty} \frac{m!}{(z+k)^{m+1}} \le \frac{m!}{z^{m+1}} + m! \int_z^{\infty} \frac{\mathrm{d}x}{x^{m+1}} = \frac{m!}{z^{m+1}} + \frac{(m-1)!}{z^m},$$

where we estimated the sum by the integral because the function $x \mapsto x^{-(m+1)}$, $x > 0$, is decreasing. □

Lemma 3.3. *As $n \to \infty$, one has*

$$\frac{1}{2}\sum_{j=1}^{n} \psi\left(\frac{j}{2}\right) \sim \frac{n}{2}\log n, \quad \frac{1}{4}\sum_{j=1}^{n} \psi^{(1)}\left(\frac{j}{2}\right) = \frac{1}{2}\log n + c_1 + o(1), \tag{3.4}$$

where $c_1 = \frac{1}{2}(\gamma + 1 + \frac{\pi^2}{8})$ with the Euler-Mascheroni constant γ. Moreover, for all $m \ge 3$,

$$\frac{1}{2^m}\left|\sum_{j=1}^{n} \psi^{(m-1)}\left(\frac{j}{2}\right)\right| \le 2(m-1)!\,. \tag{3.5}$$

Proof: The asymptotic relations (3.4) can essentially be found in Döring and Eichelsbacher (2013a) (where the constant c_1 has been computed explicitly). The first one follows from $\psi(z) = \log z + O(1/z)$ as $z \to \infty$ together with $\sum_{j=1}^{n} \log\frac{j}{2} \sim n\log n$ as $n \to \infty$. To prove the second one, write

$$\frac{1}{4}\sum_{j=1}^{n} \psi^{(1)}\left(\frac{j}{2}\right) - \frac{1}{2}\log n = \frac{1}{4}\sum_{j=1}^{n}\left(\psi^{(1)}\left(\frac{j}{2}\right) - \frac{2}{j}\right) + \frac{1}{2}\left(\sum_{j=1}^{n}\frac{1}{j} - \log n\right)$$

and observe that the series $\sum_{j=1}^{\infty}(\psi^{(1)}(\frac{j}{2}) - \frac{2}{j})$ converges because $\psi^{(1)}(z) - \frac{1}{z} = O(\frac{1}{z^2})$ as $z \to \infty$. The claim follows since $\sum_{j=1}^{n}\frac{1}{j} - \log n$ converges to the Euler constant γ.

To prove inequality (3.5), use Lemma 3.2 to get

$$\begin{aligned}\frac{1}{2^m}\left|\sum_{j=1}^{n} \psi^{(m-1)}\left(\frac{j}{2}\right)\right| &\le \frac{1}{2^m}\sum_{j=1}^{\infty}\left(\frac{(m-2)!}{(j/2)^{m-1}} + \frac{(m-1)!}{(j/2)^m}\right) \\ &\le (m-1)!\left(\frac{1}{4}\zeta(2) + \zeta(3)\right)\end{aligned}$$

for all $m \ge 3$, where we used the inequality $(m-2)! \le \frac{1}{2}(m-1)!$. The constant in the brackets is smaller than 2. □

Since the moments of $\mathcal{V}_{n,r}$ both, for the Gaussian and the Beta model, involve the same product of fractions of Gamma functions, we prepare the proof of Theorem 3.1 with the following lemma. We define

$$S_{n,r}(z) := \sum_{j=1}^{r}\left[\log\Gamma\left(\frac{n-r+j+z}{2}\right) - \log\Gamma\left(\frac{n-r+j}{2}\right)\right], \qquad z > 0.$$

Lemma 3.4. (a) *If $r = o(n)$ then, as $n \to \infty$,*

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} \sim \begin{cases} \frac{r}{2}\log n & : m = 1 \\ \frac{(-1)^m}{2}(m-2)!\, r\, n^{-(m-1)} & : m \ge 2. \end{cases}$$

(b) *If $r \sim \alpha n$ for some $\alpha \in (0,1)$ then, as $n \to \infty$,*

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} \sim \begin{cases} \frac{\alpha n}{2} \log n & : m = 1 \\ \frac{1}{2} \log \frac{1}{1-\alpha} & : m = 2 \\ \frac{(-1)^m (m-3)!}{2 \cdot n^{m-2}} \left(\frac{1}{(1-\alpha)^{m-2}} - 1 \right) & : m \geq 3. \end{cases}$$

(c) *If $n - r = o(n)$ then, as $n \to \infty$,*

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} \sim \begin{cases} \frac{n}{2} \log n & : m = 1 \\ \frac{1}{2} \log \frac{n}{n-r+1} & : m = 2. \end{cases}$$

(d) *For $m \geq 2$ and if $r = o(n)$ or $r \sim \alpha n$, $\alpha \in (0,1)$, then there is a constant C which may depend on α (but does not depend on m, n) such that*

$$\left| \frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} \right| \leq C^m (m-1)! r n^{1-m}.$$

(e) *Finally, for $m \geq 3$ and without any conditions on r, we have*

$$\left| \frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} \right| \leq 2(m-1)!\,.$$

Proof: Let us prove (a), (b), (c) for $m = 1$. We have

$$\frac{\mathrm{d}}{\mathrm{d}z} S_{n,r}(z)\Big|_{z=0} = \frac{1}{2} \sum_{j=1}^{r} \psi\left(\frac{n-r+j}{2}\right) = \frac{1}{2} \sum_{j=1}^{n} \psi\left(\frac{j}{2}\right) - \frac{1}{2} \sum_{j=1}^{n-r} \psi\left(\frac{j}{2}\right),$$

and all three statements follow easily from the relation $\frac{1}{2} \sum_{j=1}^{n} \psi\left(\frac{j}{2}\right) \sim \frac{n}{2} \log n$; see Lemma 3.3.

Next we prove (a), (b), (c) for $m \geq 2$. We have

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} = \frac{1}{2^m} \sum_{j=1}^{r} \psi^{(m-1)}\left(\frac{n-r+j}{2}\right)$$

and again we can conclude (a) by using Equation (3.2) of Lemma 3.2. To prove (b) for $m = 2$, apply the second asymptotics in (3.4) of Lemma 3.3 to get

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{n,r}(z)\Big|_{z=0} &= \frac{1}{4} \sum_{j=1}^{r} \psi^{(1)}\left(\frac{n-r+j}{2}\right) \\ &= \frac{1}{2} \log n + c_1 - \frac{1}{2} \log(n-r) - c_1 + o(1) \\ &= \frac{1}{2} \log \frac{n}{n-r} + o(1) = \frac{1}{2} \log \frac{1}{1-\alpha} + o(1). \end{aligned}$$

To prove (b) for $m \geq 3$, note that for $r \sim \alpha n$,

$$\begin{aligned} \frac{1}{2^m} \sum_{j=1}^{r} \psi^{(m-1)}\left(\frac{n-r+j}{2}\right) &\sim \frac{1}{2^m} \sum_{k=n-r+1}^{n} \frac{(-1)^{m-2}(m-2)!}{(k/2)^{m-1}} \\ &= \frac{(-1)^m (m-2)!}{2} \left[\sum_{k=1}^{n} \frac{1}{k^{m-1}} - \sum_{k=1}^{n-r} \frac{1}{k^{m-1}} \right] \\ &\sim \frac{(-1)^m (m-3)!}{2 \cdot n^{m-2}} \left(\frac{1}{(1-\alpha)^{m-2}} - 1 \right), \end{aligned}$$

using the asymptotics for the tail of the Riemann zeta series. Finally, to prove (c) for $m = 2$ use the formula $\frac{1}{4}\sum_{j=1}^{r}\psi^{(1)}\left(\frac{n-r+j}{2}\right) = \frac{1}{2}\log n + O(1)$ following from (3.4) to get

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}S_{n,r}(z)\Big|_{z=0} = \frac{1}{4}\sum_{j=1}^{r}\psi^{(1)}\left(\frac{n-r+j}{2}\right)$$
$$= \frac{1}{2}\log n + O(1) - \frac{1}{2}\log(n-r+1) - O(1)$$
$$= \frac{1}{2}\log\frac{n}{n-r+1} + O(1) \sim \frac{1}{2}\log\frac{n}{n-r+1}$$

because $\frac{n}{n-r+1} \to \infty$. We added the term $+1$ to make the formula work in the case $r = n$.

Let us prove (d). Since the function $|\psi^{(m-1)}(z)| = \sum_{k=0}^{\infty}\frac{(m-2)!}{(z+k)^m}$ is decreasing, we can write

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}z^m}S_{n,r}(z)\Big|_{z=0}\right| = \frac{1}{2^m}\sum_{j=1}^{r}\left|\psi^{(m-1)}\left(\frac{n-r+j}{2}\right)\right| \le \frac{r}{2^m}\left|\psi^{(m-1)}\left(\frac{n-r+1}{2}\right)\right|,$$

and the claim follows from the estimates $|\psi^{(m-1)}(z)| \le 2\cdot(m-1)!z^{1-m}$, $z \ge 1$, which is a consequence of Lemma 3.2, and $n-r+1 > n/C$ for sufficiently large C.

Let us prove (e). If $m \ge 3$ and r is arbitrary, we observe that the function $\psi^{(m-1)}(z)$, $z > 0$, has the same sign as $(-1)^m$ and hence

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}z^m}S_{n,r}(z)\Big|_{z=0}\right| = \frac{1}{2^m}\sum_{j=1}^{r}\left|\psi^{(m-1)}\left(\frac{n-r+j}{2}\right)\right| \le \frac{1}{2^m}\sum_{j=1}^{n}\left|\psi^{(m-1)}\left(\frac{j}{2}\right)\right|.$$

Then, the result follows in view of inequality (3.5) of Lemma 3.3. Thus, the proof is complete. □

Proof of Theorem 3.1: Denote the moment generating function of $\mathcal{L}_{n,r} = \log(r!\mathcal{V}_{n,r})$ by

$$M_{n,r}(z) := \mathbb{E}[\exp(z\mathcal{L}_{n,r})] = \mathbb{E}[(r!\mathcal{V}_{n,r})^z].$$

We start with the Gaussian model. Recalling the moment formula from Theorem 2.3 (a), we see that

$$\log M_{n,r}(z) = S_{n,r}(z) + \frac{z}{2}\log(r+1) + \frac{zr}{2}\log 2$$

and hence

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m}\log M_{n,r}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}S_{n,r}(z) + \mathbf{1}_{\{m=1\}}\frac{1}{2}\log(r+1) + \mathbf{1}_{\{m=1\}}\frac{r}{2}\log 2$$

for all $m \in \mathbb{N}$. By taking $z = 0$ it follows that

$$c^m[\mathcal{L}_{n,r}] = \frac{\mathrm{d}^m}{\mathrm{d}z^m}S_{n,r}(z)\Big|_{z=0} + \mathbf{1}_{\{m=1\}}\frac{1}{2}\log(r+1) + \mathbf{1}_{\{m=1\}}\frac{r}{2}\log 2.$$

Using Lemma 3.4 we immediately get the required asymptotic formulae for $\mathbb{E}\mathcal{L}_{n,r} = c^1[\mathcal{L}_{n,r}]$ and $\operatorname{Var}\mathcal{L}_{n,r} = c^2[\mathcal{L}_{n,r}]$. The estimates for the cumulants $c^m[\mathcal{L}_{n,r}]$, $m \ge 3$, follow from Lemma 3.4 (d), (e).

Next, we consider the Beta model and prove part (b) of the theorem. Recalling the moment formula from Theorem 2.3 (b) and denoting by $M_{n,r}(z)$ again the moment generating function of $\mathcal{L}_{n,r}$, we see that

$$\log M_{n,r}(z) = S_{n,r}(z) + \log\Gamma\Big(\frac{r(n+\nu-2)+(n+\nu)}{2} + \frac{(r+1)z}{2}\Big) + (r+1)\log\Gamma\left(\frac{n+\nu}{2}\right) - \log\Gamma\Big(\frac{r(n+\nu-2)+(n+\nu)}{2} + \frac{rz}{2}\Big) - (r+1)\log\Gamma\left(\frac{n+\nu}{2}+\frac{z}{2}\right).$$

It follows that, for $m \in \mathbb{N}$, $\frac{\mathrm{d}^m}{\mathrm{d}z^m}\log M_{n,r}(z)$ equals

$$\begin{aligned} &\frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z) + \Big(\frac{r+1}{2}\Big)^m \psi^{(m-1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2} + \frac{(r+1)z}{2}\Big) \\ &\quad - \Big(\frac{r}{2}\Big)^m \psi^{(m-1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2} + \frac{rz}{2}\Big) \\ &\quad - \frac{r+1}{2^m}\psi^{(m-1)}\left(\frac{n+\nu}{2}+\frac{z}{2}\right). \end{aligned} \tag{3.6}$$

Taking $z=0$, we obtain

$$\begin{aligned} c^m[\mathcal{L}_{n,r}] &= \frac{\mathrm{d}^m}{\mathrm{d}z^m} S_{n,r}(z)\Big|_{z=0} + \Big(\frac{r+1}{2}\Big)^m \psi^{(m-1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}\Big) \\ &\quad - \Big(\frac{r}{2}\Big)^m \psi^{(m-1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}\Big) - \frac{r+1}{2^m}\psi^{(m-1)}\left(\frac{n+\nu}{2}\right). \end{aligned} \tag{3.7}$$

Let us compute the asymptotics of $\operatorname{Var}\mathcal{L}_{n,r} = c^2[\mathcal{L}_{n,r}]$. First we do this under the assumption $r = o(n)$ or $r \sim \alpha n$, where $\alpha \in (0,1)$. The estimate $n - r > \varepsilon n$, which is valid for some $\varepsilon > 0$ and all sufficiently large n, allows us to replace all terms of the form $O(\frac{1}{n-r})$ by $O(\frac{1}{n})$ in the following asymptotic computations. First of all, using the formula $\psi^{(1)}(z) = 1/z + O(1/z^2)$ as $z \to \infty$, we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{n,r}(z)\Big|_{z=0} &= \frac{1}{4}\sum_{j=1}^{r}\psi^{(1)}\left(\frac{n-r+j}{2}\right) \\ &= \frac{1}{4}\sum_{j=1}^{r}\left(\frac{2}{n-r+j} + O\left(\frac{1}{(n-r+j)^2}\right)\right) \\ &= \frac{H_n - H_{n-r}}{2} + O\left(\frac{r}{n^2}\right), \end{aligned}$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number. Using the formula $H_n = \log n + \gamma + 1/(2n) + O(1/n^2)$, as $n \to \infty$, and the similar relation for H_{n-r}, we arrive at

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{n,r}(z)\Big|_{z=0} &= \frac{1}{2}\log\frac{n}{n-r} + \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n-r}\right) + O\left(\frac{1}{n^2}\right) + O\left(\frac{r}{n^2}\right) \\ &= \frac{1}{2}\log\frac{n}{n-r} + O\left(\frac{r}{n^2}\right). \end{aligned}$$

Again using the formula $\psi^{(1)}(z) = 1/z + O(1/z^2)$ as $z \to \infty$, we obtain

$$\psi^{(1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}\Big) = \frac{2}{n(r+1)+O(r)} + O\left(\frac{1}{n^2r^2}\right) = \frac{2}{n(r+1)} + O\left(\frac{1}{n^2 r}\right)$$

and

$$\psi^{(1)}\left(\frac{n+\nu}{2}\right) = \frac{2}{n} + O\left(\frac{1}{n^2}\right).$$

Recalling (3.7) and taking everything together, we obtain

$$\begin{aligned}\operatorname{Var}\mathcal{L}_{n,r} = c^2[\mathcal{L}_{n,r}] &= \frac{1}{2}\log\frac{n}{n-r} + \frac{(r+1)^2-r^2}{4}\frac{2}{n(r+1)} - \frac{r+1}{4}\frac{2}{n} + O\Big(\frac{r}{n^2}\Big)\\ &= \frac{1}{2}\log\frac{n}{n-r} - \frac{r^2}{2n(r+1)} + O\Big(\frac{r}{n^2}\Big)\\ &= -\frac{1}{2}\log\Big(1-\frac{r}{n}\Big) - \frac{r}{2n}\frac{1}{1+\frac{1}{r}} + O\Big(\frac{r}{n^2}\Big)\end{aligned} \tag{3.8}$$

provided that $r = o(n)$ or $r \sim \alpha n$ with $\alpha \in (0,1)$. Let us now consider some special cases. If $r \geq 1$ is constant, the Taylor expansion of the logarithm yields

$$\operatorname{Var}\mathcal{L}_{n,r} = \frac{r}{2n} - \frac{r^2}{2n(r+1)} + O\left(\frac{1}{n^2}\right) = \frac{r}{2n(r+1)} + O\left(\frac{1}{n^2}\right) \sim \frac{r}{2n(r+1)}.$$

In the case when $r \to \infty$ but $r = o(n)$, we use the expansions

$$-\log\Big(1-\frac{r}{n}\Big) = \frac{r}{n} + \frac{r^2}{2n^2} + o\left(\frac{r^2}{n^2}\right) \quad\text{and}\quad \frac{1}{1+\frac{1}{r}} = 1 - \frac{1}{r} + o\left(\frac{1}{r}\right),$$

as $n \to \infty$, to obtain

$$\operatorname{Var}\mathcal{L}_{n,r} = \frac{r^2}{4n^2} + o\left(\frac{r^2}{n^2}\right) + \frac{1}{2n} + o\left(\frac{1}{n}\right) + O\Big(\frac{r}{n^2}\Big) = \frac{r^2}{4n^2} + \frac{1}{2n} + o\left(\frac{r^2}{4n^2} + \frac{1}{2n}\right).$$

Finally, in the case when $r \sim \alpha n$ with $\alpha \in (0,1)$, (3.8) evidently implies that

$$\lim_{n\to\infty}\operatorname{Var}\mathcal{L}_{n,r} = \frac{1}{2}\log\frac{1}{1-\alpha} - \frac{\alpha}{2}.$$

Let us now compute the asymptotics of $\operatorname{Var}\mathcal{L}_{n,r} = c^2[\mathcal{L}_{n,r}]$ in the case $n - r = o(n)$. Using the formula $\psi^{(1)}(z) = 1/z + O(1/z^2)$ as $z \to \infty$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}S_{n,r}(z)\Big|_{z=0} = \frac{1}{4}\sum_{j=1}^{r}\psi^{(1)}\left(\frac{n-r+j}{2}\right) = \frac{H_n - H_{n-r}}{2} + O\left(\frac{1}{n}\right),$$

Using the formulas $H_n = \log n + O(1)$ and $H_{n-r} = \log(n-r+1) + O(1)$ (where $+1$ is needed to make the expression well-defined in the case $r = n$), we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}S_{n,r}(z)\Big|_{z=0} = \frac{1}{2}\log\frac{n}{n-r+1} + O(1).$$

By the formula $\psi^{(1)}(z) = O(1/z)$ as $z \to \infty$, we have

$$\psi^{(1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}\Big) = O\left(\frac{1}{n^2}\right), \quad \psi^{(1)}\left(\frac{n+\nu}{2}\right) = O\left(\frac{1}{n}\right).$$

Plugging everything into (3.7) yields

$$\operatorname{Var}\mathcal{L}_{n,r}=c^2[\mathcal{L}_{n,r}]=\frac{1}{2}\log\frac{n}{n-r+1}+O(1)\sim\frac{1}{2}\log\frac{n}{n-r+1}$$

because $\frac{n}{n-r+1}\to\infty$, thus proving the required asymptotics of the variance.

Next we prove the bounds on the cumulants assuming that $r=o(n)$ or $r\sim\alpha n$. Recall from Lemma 3.4(d) the estimate

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}z^m}S_{n,r}(z)\Big|_{z=0}\right|\le C^m(m-1)!rn^{1-m}.$$

Further, since $\nu\ge 0$, we have

$$\frac{r(n+\nu-2)+(n+\nu)}{2}\ge\frac{r(n-2)}{2}.$$

Since the function $|\psi^{(m-1)}(z)|$ is non-increasing, we have, using also the estimate $|\psi^{(m-1)}(z)|\le 2\cdot(m-1)!z^{1-m}$,

$$\begin{aligned}\left|\psi^{(m-1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}\Big)\right|&\le\left|\psi^{(m-1)}\Big(\frac{r(n-2)}{2}\Big)\right|\\&\le 2^m(m-1)!r^{1-m}(n-2)^{1-m}.\end{aligned}$$

By the mean value theorem, we also have $(r+1)^m-r^m\le m(r+1)^{m-1}$, hence

$$\begin{aligned}&\frac{(r+1)^m-r^m}{2^m}\left|\psi^{(m-1)}\Big(\frac{r(n+\nu-2)+(n+\nu)}{2}\Big)\right|\\&\le m!\left(\frac{r+1}{r}\right)^{m-1}(n-2)^{1-m}\\&\le 6^m m!n^{1-m}\end{aligned}$$

because $n-2\ge n/3$ for $n\ge 3$. Similarly, by the non-increasing property of $|\psi^{(m-1)}(z)|$ and the estimate $|\psi^{(m-1)}(z)|\le 2\cdot(m-1)!z^{1-m}$, we have

$$\frac{r+1}{2^m}\left|\psi^{(m-1)}\left(\frac{n+\nu}{2}\right)\right|\le\frac{r+1}{2^m}\left|\psi^{(m-1)}\left(\frac{n}{2}\right)\right|\le 2r(m-1)!n^{1-m}.$$

Recalling (3.7) and taking the above estimates together, we arrive at the required estimate

$$|c^m[\mathcal{L}_{n,r}]|\le C^m m!rn^{1-m}$$

for a sufficiently large constant $C>0$ not depending on n and m. To prove the bound $|c^m[\mathcal{L}_{n,r}]|\le 2\cdot 6^m m!$ without restrictions on $r(n)$, we argue as above except for using Lemma 3.4(e) to bound the derivative of $S_{n,r}$:

$$|c^m[\mathcal{L}_{n,r}]|\le 2(m-1)!+6^m m!n^{1-m}+2r(m-1)!n^{1-m}\le 2\cdot 6^m m!.$$

Finally, we consider the spherical model. Since the results for the Beta model are independent of the parameter ν, they carry over to the spherical model which appears as a limiting case, as $\nu\downarrow 0$. □

Remark 3.5. Let us argue that in the case when $r=o(n)$,

$$\operatorname{Var}\mathcal{L}_{n,r}\ge\frac{r}{2n(r+1)}(1+o(1)).\tag{3.9}$$

Recall (3.8). We have $\log\frac{n}{n-r}\geq\frac{r}{n}$, so that $\frac{1}{2}\log\frac{n}{n-r}-\frac{r^2}{2n(r+1)}\geq\frac{r}{2n(r+1)}$. Also, $\frac{r}{n^2}=o(\frac{r}{2n(r+1)})$ and we can conclude that (3.9) holds.

3.2. *Berry-Esseen bounds and moderate deviations for the log-volume.* We introduce some terminology. One says that a sequence $(\nu_n)_{n\in\mathbb{N}}$ of probability measures on a topological space E fulfils a large deviation principle (LDP) with speed a_n and (good) rate function $I:E\to[0,\infty]$, if I is lower semi-continuous, has compact level sets and if for every Borel set $B\subseteq E$,

$$-\inf_{x\in\mathrm{int}(B)}I(x)\leq\liminf_{n\to\infty}a_n^{-1}\log\nu_n(B)\leq\limsup_{n\to\infty}a_n^{-1}\log\nu_n(B)\leq-\inf_{x\in\mathrm{cl}(B)}I(x)\,,$$

where int(B) and cl(B) stand for the interior and the closure of B, respectively. A sequence $(X_n)_{n\in\mathbb{N}}$ of random elements in E satisfies a LDP with speed a_n and rate function $I:E\to[0,\infty]$, if the family of their distributions does. Moreover, if the rescaling a_n lies between that of a law of large numbers and that of a central limit theorem, one usually speaks about a moderate deviation principle (MDP) instead of a LDP with speed a_n and rate function I, see Dembo and Zeitouni (2010).

We shall say that a sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ satisfying $\mathbb{E}|X_n|^2<\infty$ for all $n\in\mathbb{N}$ fulfils a Berry-Esseen bound (BEB) with speed $(\varepsilon_n)_{n\in\mathbb{N}}$ if

$$\sup_{t\in\mathbb{R}}\Big|\mathbb{P}\Big(\frac{X_n-\mathbb{E}[X_n]}{\sqrt{\operatorname{Var}X_n}}\leq t\Big)-\Phi(t)\Big|\leq c\,\varepsilon_n\,,$$

where $c>0$ is a constant not depending on n and $\Phi(\,\cdot\,)$ denotes the distribution function of a standard Gaussian random variable.

Theorem 3.6 (BEB and MDP for the log-volume)**.** *Let $X_1,\ldots,X_{r+1}$ be chosen according to the Gaussian, the Beta or the spherical model.*

(a) *For the Gaussian model define*

$$\varepsilon_n:=\frac{1}{\sqrt{rn}}\ \textit{if}\ r=o(n)\ \textit{or}\ r\sim\alpha n$$

and

$$\varepsilon_n:=\frac{1}{\sqrt{\log\frac{n}{n-r+1}}}\ \textit{if}\ n-r=o(n),$$

where $\alpha\in(0,1)$. Then $\mathcal{L}_{n,r}$ satisfies a BEB with speed ε_n. Further, let $(a_n)_{n\in\mathbb{N}}$ be such that $a_n\to\infty$ and $a_n\varepsilon_n\to0$, as $n\to\infty$. Then $\mathcal{L}_{n,r}$ satisfies a MDP with speed a_n rate function $I(x)=\frac{x^2}{2}$.

(b) *For the Beta model and the spherical model define*

$$\varepsilon_n:=\frac{1}{\sqrt{n}}\ \textit{if}\ r=c,\quad\varepsilon_n:=\frac{1}{n}\ \textit{if}\ r\sim\alpha n$$

and

$$\varepsilon_n:=\frac{1}{\sqrt{\log\frac{n}{n-r+1}}}\ \textit{if}\ n-r=o(n)$$

with $\alpha\in(0,1)$ and $c\in\mathbb{N}$. Then $\mathcal{L}_{n,r}$ satisfies a BEB with speed ε_n. Further, let $(a_n)_{n\in\mathbb{N}}$ be such that $a_n\to\infty$ and $a_n\varepsilon_n\to0$, as $n\to\infty$. Then $\mathcal{L}_{n,r}$ satisfies a MDP with speed a_n rate function $I(x)=\frac{x^2}{2}$.

Proof: Let us estimate the cumulants of the normalized random variables

$$\widetilde{\mathcal{L}}_{n,r} := (\mathcal{L}_{n,r} - \mathbb{E}[\mathcal{L}_{n,r}])/\sqrt{\operatorname{Var}\mathcal{L}_{n,r}}.$$

In the Gaussian case, from Theorem 3.1 we conclude that, for $m \geq 3$,

$$|c^m[\widetilde{\mathcal{L}}_{n,r}]| = \frac{|c^m[\mathcal{L}_{n,r}]|}{(\operatorname{Var}\mathcal{L}_{n,r})^{m/2}} \leq \begin{cases} \frac{c_1^m (m-1)!}{(\sqrt{rn})^{m-2}} & : r = o(n) \text{ or } r \sim \alpha n \\ \frac{c_2^m (m-1)!}{\left(\sqrt{\log \frac{n}{n-r+1}}\right)^m} & : n - r = o(n) \end{cases}$$

with constants $c_1, c_2 > 0$ not depending on m and n. For the Beta and the spherical model we have the bounds

$$|c^m[\widetilde{\mathcal{L}}_{n,r}]| \leq \begin{cases} \frac{|c^m[\mathcal{L}_{n,r}]|}{(r/(2(r+1)n))^{m/2}} & : r = o(n) \\ \frac{|c^m[\mathcal{L}_{n,r}]|}{(\frac{1}{2}\log(\frac{1}{1-\alpha}) - \frac{\alpha}{2})^{m/2}} & : r \sim \alpha n \\ \frac{|c^m[\mathcal{L}_{n,r}]|}{(\frac{1}{2}\log\frac{n}{n-r+1})^{m/2}} & : n - r = o(n) \end{cases}$$

$$\leq \begin{cases} c_4^m\, m!\, r\left(\frac{1}{\sqrt{n}}\right)^{m-2} & : r = o(n) \\ c_5^m\, m!\,\left(\frac{1}{n}\right)^{m-2} & : r \sim \alpha n \\ c_6^m\, m!\left(\frac{1}{\sqrt{\log\frac{n}{n-r+1}}}\right)^{m-2} & : n - r = o(n) \end{cases}$$

with constants $c_1, \ldots, c_6 > 0$ not depending on m and n (here, we used Remark 3.5 if $r = o(n)$). The result follows now from Döring and Eichelsbacher (2013b, Theorem 1.1) and Saulis and Statulevičius (1991, Corollary 2.1) (notice, however, that in the regime $r = o(n)$ this argument only applies if $r = c$ is constant). □

Remark 3.7. Starting with the cumulant bounds presented in Theorem 3.1 one can also derive

(i) concentration inequalities,
(ii) bound for moments of all orders,
(iii) Cramér-Petrov type results concerning the relative error in the central limit theorem,
(iv) strong laws of large numbers

for the random variables $\widetilde{\mathcal{L}}_{n,r}$ from the results presented in Saulis and Statulevičius (1991, Chapter 2) (see also Grote and Thäle, 2018a,b).

Remark 3.8. Note that if in the Beta model $r = o(n)$ we still have that $\widetilde{\mathcal{L}}_{n,r}$ satisfies a central limit theorem. Indeed, the proof of Theorem 3.6 shows that in this case we have the cumulant bound $|c^m[\widetilde{\mathcal{L}}_{n,r}]| \leq c^m m! n^{\frac{1}{2}(2-m)+1}$ for all $m \geq 3$ and a constant $c > 0$. This implies that $|c^m[\widetilde{\mathcal{L}}_{n,r}]| \to 0$, as $n \to \infty$, for all $m \geq 4$ and so, the central limit theorem follows.

While we were able in Theorem 3.6 to derive precise Berry-Esseen bounds by using cumulant bounds, we can state a 'pure' central limit theorem for the log-volume in an even more general setup. The following result can directly be concluded by extracting subsequences and then by applying the result of Theorem 3.6 and Remark 3.8.

Corollary 3.9 (Central limit theorem for the log-volume)**.** *Let $r = r(n)$ be an arbitrary sequence of integers such that $r(n) \leq n$ for any $n \in \mathbb{N}$. Further, let for* *each* $n \in \mathbb{N}$, $X_1, \dots, X_{r+1}$ *be independent random points chosen according to the Gaussian, the Beta or the spherical model, and put* $\mathcal{L}_{n,r} := \log(r!\mathcal{V}_{n,r})$. *Then,*

$$\frac{\mathcal{L}_{n,r} - \mathbb{E}[\mathcal{L}_{n,r}]}{\sqrt{\operatorname{Var} \mathcal{L}_{n,r}}} \xrightarrow[n\to\infty]{d} Z,$$

where $Z \sim \mathcal{N}(0,1)$ *is a standard Gaussian random variable.*

3.3. *Central and non-central limit theorem for the volume.* After having investigated asymptotic normality for the log-volume of a random simplex, we turn now to its actual volume, that is, the random variable $\mathcal{V}_{n,r}$.

Theorem 3.10 (Distributional limit theorem for the volume)**.** *Let* $X_1, \dots, X_{r+1}$ *be chosen according to the Gaussian model, the Beta model or the spherical model, and let* $\alpha \in (0,1)$. *Let* $Z \sim \mathcal{N}(0,1)$ *be a standard Gaussian random variable.*

(1) *If* $r = o(n)$, *then for suitable normalizing sequences* $a_{n,r}$ *and* $b_{n,r}$ *the following convergence in distribution holds, as* $n \to \infty$*:*

$$\frac{\mathcal{V}_{n,r} - a_{n,r}}{b_{n,r}} \xrightarrow[n\to\infty]{d} Z.$$

(2) *If* $r \sim \alpha n$ *for some* $\alpha \in (0,1)$, *then for a suitable normalizing sequence* $b_{n,r}$ *we have*

$$\frac{\mathcal{V}_{n,r}}{b_{n,r}} \xrightarrow[n\to\infty]{d} \begin{cases} e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}\,Z} & : \text{in the Gaussian model} \\ e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}-\frac{\alpha}{2}}\,Z} & : \text{in the Beta or spherical model.} \end{cases}$$

Remark 3.11. In the third case, i.e., if $n - r = o(n)$, there is no non-trivial distributional limit theorem for the random variable $\mathcal{V}_{n,r}$ under affine re-scaling. The reason is that the variance of $\log \mathcal{V}_{n,r}$ tends to $+\infty$ in this situation.

The main ingredient in the proof of Theorem 3.10 in the case where $r = o(n)$ is the so-called 'Delta-Method', which is well known and commonly used in statistics, cf. Bickel and Doksum (2015, Lemma 5.3.3).

Proof of Theorem 3.10: From Corollary 3.9 we know that with the sequences $c_{n,r} = \mathbb{E}\log\mathcal{V}_{n,r}$ and $d_{n,r} = \sqrt{\operatorname{Var}\log\mathcal{V}_{n,r}}$ it holds that

$$\frac{\log\mathcal{V}_{n,r} - c_{n,r}}{d_{n,r}} \xrightarrow[n\to\infty]{d} Z.$$

By the Skorokhod–Dudley lemma (Kallenberg, 2002, Theorem 4.30), we can construct random variables $\mathcal{V}^*_{n,r}$ and Z^* on a different probability space such that $\mathcal{V}^*_{n,r} \overset{d}{=} \mathcal{V}_{n,r}$, $Z^* \overset{d}{=} Z$, and

$$Z^*_n := \frac{\log\mathcal{V}^*_{n,r} - c_{n,r}}{d_{n,r}} \xrightarrow[n\to\infty]{a.s.} Z^*.$$

So, we have $\mathcal{V}^*_{n,r} = e^{d_{n,r}Z^*_n + c_{n,r}}$, where $Z^*_n \to Z^*$ a.s., as $n \to \infty$.

Consider first the Gaussian model in the case $r \sim \alpha n$. Then, by Theorem 3.1(a) we have

$$d_{n,r} = \sqrt{\operatorname{Var}\log\mathcal{V}_{n,r}} \sim \sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}.$$

With the aid of Slutsky's lemma it follows that

$$\frac{\mathcal{V}^*_{n,r}}{e^{c_{n,r}}} = e^{d_{n,r} Z_n^*} \xrightarrow[n\to\infty]{a.s.} e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}\, Z^*}.$$

Passing back to the original probability space, we obtain the distributional convergence

$$\frac{\mathcal{V}_{n,r}}{e^{c_{n,r}}} \xrightarrow[n\to\infty]{d} e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}\, Z}.$$

The proof for the Beta or spherical model in the case $r \sim \alpha n$ is similar, only the expression for the asymptotic variance being different.

Consider now the Gaussian model in the case $r = o(n)$. Then, by Theorem 3.1(a),

$$d_{n,r} = \sqrt{\operatorname{Var}\log \mathcal{V}_{n,r}} \underset{n\to\infty}{\longrightarrow} 0.$$

Using the formula $\lim_{x\to 0}(e^x - 1)/x = 1$ and the Slutsky lemma, we obtain

$$\frac{\frac{\mathcal{V}^*_{n,r}}{e^{c_{n,r}}} - 1}{d_{n,r}} = \frac{e^{d_{n,r} Z_n^*} - 1}{d_{n,r} Z_n^*} \cdot Z_n^* \xrightarrow[n\to\infty]{a.s.} Z^*.$$

Passing back to the original probability space and taking $b_{n,r} = e^{c_{n,r}} d_{n,r}$ and $a_{n,r} = e^{c_{n,r}}$, we obtain the required distributional convergence. The proof for the Beta or spherical model in the case $r = o(n)$ is similar, one only needs to take into account that also in this case we have that $\sqrt{\operatorname{Var}\log \mathcal{V}_{n,r}} \underset{n\to\infty}{\longrightarrow} 0$ by Theorem 3.1 (b). □

4. Mod-ϕ convergence

4.1. *Definition.* Mod-ϕ convergence is a powerful notion that was introduced and studied in Delbaen et al. (2015); Féray et al. (2016); Jacod et al. (2011); Kowalski et al. (2015); Kowalski and Nikeghbali (2010), to mention only some references. Once an appropriate version of mod-ϕ convergence has been established, one gets for free a whole collection of limit theorems including the central limit theorem, the local limit theorem, moderate and large deviations, and a Cramér–Petrov asymptotic expansion Féray et al. (2016).

The aim of the present Section 4 is to establish mod-ϕ convergence for the log-volumes of the random simplices. Note that the mod-ϕ convergence we establish in the present section together with the general results from Féray et al. (2016) also imply some of the results we proved in the previous section by means of the cumulant method. On the other hand, we would like to emphasize that this is not the case if $r \sim \alpha n$, for example.

There are many definitions of mod-ϕ convergence. Here, we use one of the strongest ones, c.f. Féray et al. (2016, Definition 1.1). Consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ with moment generating functions $\varphi_n(t) = \mathbb{E}[e^{tX_n}]$ defined on some strip $S = \{z \in \mathbb{C} : c_- < \operatorname{Re} t < c_+\}$. The sequence $(X_n)_{n\in\mathbb{N}}$ *converges in the mod-ϕ sense*, where ϕ is an infinite-divisible distribution with moment generating function $\int_{-\infty}^{\infty} e^{tx}\phi(dx) = e^{\eta(t)}$, if

$$\lim_{n\to\infty} \frac{\mathbb{E}[e^{tX_n}]}{e^{w_n \eta(t)}} = \psi(t)$$

locally uniformly on S, where $(w_n)_{n\in\mathbb{N}}$ is some sequence converging to $+\infty$, and $\psi(t)$ is an analytic function on S. As explained in references cited above, mod-ϕ convergence roughly means that X_n has approximately the same distribution as the

w_n-th convolution power of the infinitely divisible distribution ϕ. The "difference" between these distributions is measured by the "*limit function*" ψ that plays a crucial role in the theory.

4.2. *The Barnes G-function.* The Barnes function is an entire function of the complex argument z defined by the Weierstrass product

$$G(z) = (2\pi)^{z/2} \mathrm{e}^{-\frac{1}{2}(z+(1+\gamma)z^2)} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^k \mathrm{e}^{\frac{z^2}{2k}-z},$$

where γ is the Euler constant. The Barnes G-function satisfies the functional equation

$$G(z+1) = \Gamma(z)G(z).$$

By induction, one deduces that for all $n \in \mathbb{N}_0$,

$$\prod_{k=1}^{n} \Gamma(k+z) = \frac{G(z+n+1)}{G(z+1)}. \tag{4.1}$$

We shall need the Stirling-type formula for G, see Barnes (1900, p. 285),

$$\log G(z+1) = \frac{1}{2}z^2 \log z - \frac{3}{4}z^2 + \frac{z}{2}\log(2\pi) - \frac{1}{12}\log z + \zeta'(-1) + O(1/z), \tag{4.2}$$

uniformly as $|z| \to +\infty$ such that $|\arg z| < \pi - \varepsilon$, where $\zeta'(-1)$ is the derivative of the Riemann ζ-function. The value of $\zeta'(-1)$ can be expressed through the Glaisher–Kinkelin constant, but it cancels in all our calculations because we use (4.2) only via the following lemma.

Lemma 4.1. *Let $|z| \to \infty$ such that $|\arg z| < \pi - \varepsilon$. Let also $a = a(z) \in \mathbb{C}$ be such that $a/z \to 0$. Then, we have*

$$\log G(z+a+1) - \log G(z+1) = a\left(z \log z - z + \log\sqrt{2\pi}\right) + \frac{1}{2}a^2 \log z + O\left(\frac{|a|^3+1}{z}\right).$$

Proof: Applying (4.2) we obtain that

$$\log G(z+a+1) - \log G(z+1) = \frac{1}{2}A_n + B_n + C_n + D_n + O(1/z),$$

where

$$\begin{aligned}
A_n &= (z+a)^2 \log(z+a) - z^2 \log z \\
&= (z^2 + a^2 + 2za)\left(\log z + \frac{a}{z} - \frac{a^2}{2z^2} + O\left(\frac{a^3}{z^3}\right)\right) - z^2 \log z \\
&= za - \frac{1}{2}a^2 + a \log z + 2za \log z + 2a^2 + O\left(\frac{a^3}{z}\right), \\
B_n &= -\frac{3}{4}\left((z+a)^2 - z^2\right) = -\frac{3}{4}a^2 - \frac{3}{2}za, \\
C_n &= \frac{1}{2}a\log(2\pi), \\
D_n &= -\frac{1}{12}(\log(z+a) - \log z) = O\left(\frac{a}{z}\right).
\end{aligned}$$

Taking everything together we get

$$\log G(z+a+1) - \log G(z+1) = a\left(z\log z - z + \log\sqrt{2\pi}\right) + \frac{1}{2}a^2\log z + O\left(\frac{|a|^3+1}{z}\right)$$

and complete the proof of the lemma. □

4.3. *Mod-ϕ convergence for fixed $r\in\mathbb{N}$.* Recall that $\mathcal{V}_{n,r}$ denotes the volume of an r-dimensional random simplex in $\mathbb{R}^n$ whose $r+1$ vertices are distributed according to one of the four models presented in Section 1. We define, as usual, $\mathcal{L}_{n,r} := \log(r!\mathcal{V}_{n,r})$. The next two propositions show that if $r\in\mathbb{N}$ is fixed, we have mod-ϕ convergence.

Proposition 4.2. *Fix some $r\in\mathbb{N}$ and consider the Gaussian model. Then, as $n\to\infty$, the sequence $n(\mathcal{L}_{n,r} - \frac{r}{2}\log n - \frac{1}{2}\log(r+1))$ converges in the mod-ϕ sense with $\eta(t) = \frac{1}{2}((t+1)\log(t+1) - t)$ and parameter $w_n = rn$, namely*

$$\lim_{n\to\infty}\frac{\mathbb{E}\mathrm{e}^{tn(\mathcal{L}_{n,r}-\frac{r}{2}\log n - \frac{1}{2}\log(r+1))}}{\mathrm{e}^{rn\eta(t)}} = (t+1)^{-\frac{r(r+1)}{4}}$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\setminus(-\infty,-1)$.

Proof: An important formula we will often use describes the asymptotic behaviour of the Gamma function; it can be found in Abramowitz and Stegun (1992, Eq. 6.1.39 on p. 257) or derived from the Stirling formula, and reads as follows. For fixed $a>0$, $b\in\mathbb{R}$ it holds that

$$\Gamma(az+b)\sim(2\pi)^{1/2}\exp(-az)\,(az)^{az+b-1/2}, \tag{4.3}$$

as $|z|\to\infty$and $|\arg z|<\pi-\varepsilon$. From the moment formula in Theorem 2.3(a) we obtain

$$\mathbb{E}e^{tn\mathcal{L}_{n,r}} = (r+1)^{\frac{tn}{2}}2^{\frac{tnr}{2}}\prod_{j=1}^{r}\frac{\Gamma\left(\frac{(t+1)n-r+j}{2}\right)}{\Gamma\left(\frac{n-r+j}{2}\right)}.$$

Using (4.3) we deduce that

$$\prod_{j=1}^{r}\frac{\Gamma\left(\frac{(t+1)n-r+j}{2}\right)}{\Gamma\left(\frac{n-r+j}{2}\right)} \sim \prod_{j=1}^{r} e^{-\frac{tn}{2}}\left(\frac{n}{2}\right)^{\frac{tn}{2}}(t+1)^{\frac{(t+1)n}{2}+\frac{j-r-1}{2}}$$
$$= e^{-\frac{tnr}{2}}\left(\frac{n}{2}\right)^{\frac{rtn}{2}}(t+1)^{\left(\frac{(t+1)n}{2}-\frac{1}{2}\right)r-\frac{r(r-1)}{4}}. \tag{4.4}$$

Thus,

$$\mathbb{E}e^{tn\mathcal{L}_{n,r}}\sim(r+1)^{\frac{tn}{2}}e^{-\frac{tnr}{2}}n^{\frac{tnr}{2}}(t+1)^{\left(\frac{(t+1)n}{2}-\frac{1}{2}\right)r-\frac{r(r-1)}{4}}.$$

Taking the logarithm and subtracting $\frac{r}{2}\log n$ and $\frac{1}{2}\log(r+1)$, we conclude that

$$\log\mathbb{E}\mathrm{e}^{tn(\mathcal{L}_{n,r}-\frac{r}{2}\log n-\frac{1}{2}\log(r+1))} = \frac{nr}{2}\Big((t+1)\log(t+1)-t\Big) - \frac{r(r+1)}{4}\log(t+1)+o(1) \tag{4.5}$$

and the result follows. □

Remark 4.3. From the previous proof it easily follows that the asymptotic relation (4.5) is still valid if r growths with n in such a way that $r = o(n)$. This observation will be used below in the context of large deviation principles.

Proposition 4.4. *Fix some $r \in \mathbb{N}$ and consider the Beta or the spherical model. Then, $n\mathcal{L}_{n,r}$ converges in the mod-ϕ sense with*

$$\eta(t) = \frac{(r+1)(t+1)}{2}\log((r+1)(t+1)) - \frac{r(t+1)+1}{2}\log(r(t+1)+1) - \frac{t+1}{2}\log(t+1)$$

and parameter $w_n = n$, namely

$$\lim_{n\to\infty}\frac{\mathbb{E}e^{tn\mathcal{L}_{n,r}}}{e^{n\eta(t)}} = (1+t)^{\frac{1-\nu(r+1)}{2}-\frac{r(r-1)}{4}}\left(\frac{(r+1)(t+1)}{r(t+1)+1}\right)^{\frac{\nu(r+1)-2r-1}{2}}$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\setminus(-\infty,-1)$.

Proof: From the moment formula in Theorem 2.3(b) we have

$$\mathbb{E}e^{tn\mathcal{L}_{n,r}} = \prod_{j=1}^{r}\left[\frac{\Gamma\left(\frac{n-r+j}{2}+\frac{tn}{2}\right)}{\Gamma\left(\frac{n-r+j}{2}\right)}\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2}+\frac{tn}{2}\right)}\right] \times \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2}+\frac{tn}{2}\right)}\frac{\Gamma\left(\frac{r(n+\nu-2)+(n+\nu)}{2}+\frac{(r+1)tn}{2}\right)}{\Gamma\left(\frac{r(n+\nu-2)+(n+\nu)}{2}+\frac{rtn}{2}\right)}.$$

First of all, by (4.3),

$$\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2}+\frac{tn}{2}\right)} \sim (1+t)^{\frac{1}{2}-\frac{\nu}{2}-\frac{(1+t)n}{2}}\left(\frac{n}{2}\right)^{-\frac{tn}{2}}e^{\frac{tn}{2}}.$$

It follows from (4.4) that the first product in the moment formula asymptotically behaves like

$$\prod_{j=1}^{r}\left[\frac{\Gamma\left(\frac{n-r+j}{2}+\frac{tn}{2}\right)}{\Gamma\left(\frac{n-r+j}{2}\right)}\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2}+\frac{tn}{2}\right)}\right] \sim (1+t)^{-\frac{r\nu}{2}-\frac{r(r-1)}{4}}.$$

Again using (4.3), we obtain

$$\frac{\Gamma\left(\frac{r(n+\nu-2)+(n+\nu)}{2}+\frac{(r+1)tn}{2}\right)}{\Gamma\left(\frac{r(n+\nu-2)+(n+\nu)}{2}+\frac{rtn}{2}\right)} \sim ((r+1)(t+1))^{\frac{n(r+1)(t+1)}{2}+\frac{\nu(r+1)-2r-1}{2}}\left(\frac{n}{2}\right)^{\frac{tn}{2}}e^{-\frac{tn}{2}} \times (r(t+1)+1)^{-\frac{n(r(t+1)+1)}{2}-\frac{\nu(r+1)-2r-1}{2}}.$$

Thus, as $n \to \infty$, we get

$$\begin{aligned}\log \mathbb{E}e^{tn\mathcal{L}_{n,r}} &= \left(\frac{1-\nu(r+1)}{2} - \frac{r(r-1)}{4} - \frac{(1+t)n}{2}\right)\log(1+t) \\ &\quad + \left(\frac{n(r+1)(t+1)}{2} + \frac{\nu(r+1)-2r-1}{2}\right)\log((r+1)(t+1)) \\ &\quad - \left(\frac{n(r(t+1)+1)}{2} + \frac{\nu(r+1)-2r-1}{2}\right)\log(r(t+1)+1) + o(1)\end{aligned}$$

and the result follows. □

4.4. *Mod-ϕ convergence for the ExpGamma distribution.* Many examples of mod-ϕ convergence are known in probability, number theory, statistical mechanics and random matrix theory. The most basic cases are probably the mod-Gaussian and mod-Poisson convergence, which can be found in Féray et al. (2016); Jacod et al. (2011); Kowalski and Nikeghbali (2010), but there are also examples of mod-Cauchy (Delbaen et al., 2015; Kowalski et al., 2015) and even mod-uniform (Féray et al., 2016, §7.4) convergence. The aim of the present section is to add one more item to this list by proving a convergence modulo a tilted 1-stable totally skewed distribution.

Let X_n be a random variable having a Gamma distribution with shape n and rate 1, that is the probability density of X_n is $\frac{1}{\Gamma(n)}x^{n-1}\mathrm{e}^{-x}$, $x > 0$. The distribution of $\log X_n$ is called the ExpGamma distribution. The probability density of $-\log X_n$ is given by

$$\frac{1}{\Gamma(n)}\mathrm{e}^{-\mathrm{e}^{-x}}\mathrm{e}^{-xn}, \quad x \in \mathbb{R},$$

and is the limiting probability density of the n-th order upper order statistic in an i.i.d. sample of size $N \to \infty$ from the max-domain of attraction of the Gumbel distribution, or, equivalently, the density of the n-th upper order statistic in the Poisson point process with intensity $\mathrm{e}^{-x}\mathrm{d}x$, $x \in \mathbb{R}$; see Leadbetter et al. (1983, Theorem 2.2.2). It is easy to check that $\mathbb{E}\log X_n = \Gamma'(n)/\Gamma(n) = \psi(n)$ is the digamma function.

Theorem 4.5. *The sequence of random variables $n(\log X_n - \psi(n))$ converges in the mod-ϕ sense with $\eta(t) = (t+1)\log(t+1) - t$ and parameter $w_n = n$, namely*

$$\lim_{n\to\infty}\frac{\mathbb{E}\mathrm{e}^{tn(\log X_n - \psi(n))}}{\mathrm{e}^{n((t+1)\log(t+1)-t)}} = \frac{\mathrm{e}^{t/2}}{\sqrt{t+1}}$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\backslash(-\infty,-1)$.

Proof: By the properties of the Gamma distribution, we have

$$\mathbb{E}\mathrm{e}^{tn(\log X_n - \psi(n))} = \mathrm{e}^{-tn\psi(n)}\mathbb{E}X_n^{tn} = \mathrm{e}^{-tn\psi(n)}\frac{\Gamma(tn+n)}{\Gamma(n)}.$$

The Stirling formula states that $\Gamma(z) \sim \sqrt{2\pi/z}\,(z/\mathrm{e})^z$ uniformly as $|z| \to \infty$ in such a way that $|\arg z| < \pi - \varepsilon$. Using the Stirling formula together with the asymptotics

$\psi(n) = \log n - \frac{1}{2n} + o(\frac{1}{n})$, we obtain

$$\mathrm{e}^{-tn\psi(n)} \frac{\Gamma(tn+n)}{\Gamma(n)} \sim \mathrm{e}^{-tn(\log n - \frac{1}{2n})} \frac{\sqrt{\frac{2\pi}{tn+n}} \left(\frac{tn+n}{\mathrm{e}}\right)^n}{\sqrt{\frac{2\pi}{n}} \left(\frac{n}{\mathrm{e}}\right)^n} = \frac{\mathrm{e}^{t/2}}{\sqrt{t+1}} \mathrm{e}^{n((t+1)\log(t+1)-t)},$$

which proves the claim. □

Remark 4.6. Consider an α-stable random variable $Z_1 \sim S_1(\pi/2, -1, 0)$ with $\alpha = 1$, skewness $\beta = -1$, and scale $\sigma = \pi/2$, where we adopt the parametrization used in the book of Samorodnitsky and Taqqu (1994). It is known Samorodnitsky and Taqqu (1994, Proposition 1.2.12) that the cumulant generating function of this random variable is given by

$$\log \mathbb{E}\mathrm{e}^{tZ_1} = t \log t, \quad \mathrm{Re}\, t \geq 0.$$

Note that $\mathbb{E}\mathrm{e}_1^Z = 1$ and consider an exponential tilt of Z_1, denoted Z_2, whose probability density is

$$\mathbb{P}[Z_2 \in \mathrm{d}x] = \mathrm{e}^x \mathbb{P}[Z_1 \in \mathrm{d}x], \quad x \in \mathbb{R}.$$

Finally, observe that $\mathbb{E}Z_2 = \mathbb{E}[\mathrm{e}^{Z_1} Z_1] = (t^t)'|_{t=1} = 1$ and consider the centered version $Z := Z_2 - 1$. The cumulant generating function of Z is given by

$$\log \mathbb{E}\mathrm{e}^{tZ} = (t+1)\log(t+1) - t, \quad \mathrm{Re}\, t \geq -1.$$

As an exponential tilt of an infinitely divisible distribution, Z is itself infinitely divisible. Thus, in Theorem 4.5 and Proposition 4.2 we have a mod-ϕ convergence modulo a tilted totally skewed 1-stable distribution.

4.5. *Mod-ϕ convergence in the full dimensional case.* In this section we consider the full-dimensional case $r = n$, i.e., we are interested in the random variable $\mathcal{L}_{n,n}$.

Proposition 4.7. *Consider the Gaussian model and let $m_n = \frac{1}{2}(n \log n - n + \frac{1}{2}\log n + \log(2^{3/2}\pi))$. Then, $\mathcal{L}_{n,n} - m_n$ converges in the mod-Gaussian sense (meaning that $\eta(t) = \frac{1}{2}t^2$) with parameter $w_n = \frac{1}{2}\log\frac{n}{2}$, namely*

$$\lim_{n\to\infty} \frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,n}-m_n)}}{\mathrm{e}^{\frac{1}{4}t^2 \log \frac{n}{2}}} = \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right) G\left(1+\frac{t}{2}\right)},$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\backslash\{-1,-2,\ldots\}$.

Proof: In view of Theorem 2.3 (a) and (4.1), we can express the moment generating function of $\mathcal{L}_{n,n}$ in terms of the Barnes G-function as

$$\begin{aligned}\mathbb{E}\mathrm{e}^{t\mathcal{L}_{n,n}} &= \mathbb{E}[(n!\mathcal{V}_{n,n})^t] \\ &= (n+1)^{\frac{t}{2}}\, 2^{\frac{tn}{2}} \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{n+1}{2}\right)} \cdot \frac{G(1)}{G\left(\frac{n+2}{2}\right)} \cdot \frac{G\left(\frac{n+1}{2}+\frac{t}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)} \cdot \frac{G\left(\frac{n+2}{2}+\frac{t}{2}\right)}{G\left(1+\frac{t}{2}\right)},\end{aligned} \tag{4.6}$$

where $G(1) = 1$. For the function

$$\psi(t) := \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right) G\left(1+\frac{t}{2}\right)} \tag{4.7}$$

we have

$$\log \mathbb{E}e^{t\mathcal{L}_{n,n}} = \frac{t}{2}\log(n+1) + \frac{tn}{2}\log 2 + \log\psi(t) + \log G\left(\frac{n-1}{2}+\frac{t}{2}+1\right) - \log G\left(\frac{n-1}{2}+1\right) + \log G\left(\frac{n}{2}+\frac{t}{2}+1\right) - \log G\left(\frac{n}{2}+1\right).$$

Applying Lemma 4.1 two times and using the formula

$$((n+b)\log(n+b) - (n+b)) - (n\log n - n) = b\log n + o(1),$$

where b is any constant, we obtain

$$\log \mathbb{E}e^{t\mathcal{L}_{n,n}} = \log\psi(t) + \frac{t}{2}\left(n\log n - n + \frac{1}{2}\log n + \log(2^{3/2}\pi)\right) + \frac{1}{4}t^2\log\frac{n}{2} + o(1). \tag{4.8}$$

This completes the argument. □

Remark 4.8. We notice that in the full dimensional, Gaussian case $r = n$ our random variables are equivalent to those considered in Dal Borgo et al. (2017) and one can follow our result also from their Theorem 5.1. Nevertheless, we decided to include our independent and much shorter proof.
Their paper deals with the determinant of certain random matrix models and has a completely different focus. On the other hand, let us emphasize that even in this special case the distributions appearing in Dal Borgo et al. (2017) are in fact different from (but very close to) those we obtain.

Proposition 4.9. *Consider the Beta model with parameter $\nu > 0$ or the spherical model (in which case $\nu = 0$) and let $\widetilde{m}_n = \frac{1}{2}(\frac{1}{2}\log n - n + 1 - \nu + \log(2^{3/2}\pi))$. Then, $\mathcal{L}_{n,n} - \widetilde{m}_n$ converges in the mod-Gaussian sense (meaning that $\eta(t) = \frac{1}{2}t^2$) with parameter $w_n = \frac{1}{2}\log\frac{n}{2} - \frac{1}{2}$, namely*

$$\lim_{n\to\infty} \frac{\mathbb{E}e^{t(\mathcal{L}_{n,n}-\widetilde{m}_n)}}{e^{\frac{1}{4}t^2(\log\frac{n}{2}-1)}} = \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)G\left(1+\frac{t}{2}\right)},$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\backslash\{-1,-2,\ldots\}$.

Proof: For the purposes of this proof let $\mathcal{L}^{\mathrm{G}}_{n,n}$ denote the Gaussian analogue of $\mathcal{L}_{n,n}$. In view of the connection between the Gaussian and the Beta model, see Theorem 2.3(a),(b), the moment generating function of $\mathcal{L}_{n,n}$ is given by

$$\log \mathbb{E}e^{t\mathcal{L}_{n,n}} = \log \mathbb{E}e^{t\mathcal{L}^{\mathrm{G}}_{n,n}} - \frac{t}{2}\log(n+1) - \frac{tn}{2}\log 2 + (n+1)\log\left(\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{\nu+t}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{n(n+\nu-1)+nt+t+\nu}{2}\right)}{\Gamma\left(\frac{n(n+\nu-1)+nt+\nu}{2}\right)}\right).$$

Using a second-order Stirling approximation for the logarithms of the Gamma functions, we obtain

$$(n+1)\log\left(\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{\nu+t}{2}\right)}\right) = \frac{(n+1)t}{2}\log\frac{2}{n} - \frac{t}{4}(t-2+2\nu) + o(1)$$

and similarly

$$\log\left(\frac{\Gamma\left(\frac{n(n+\nu-1)+nt+t+\nu}{2}\right)}{\Gamma\left(\frac{n(n+\nu-1)+nt+\nu}{2}\right)}\right) = t\log n - \frac{t}{2}\log 2 + o(1).$$

Denoting by $\psi(t)$ the function defined at (4.7) and using (4.8) we conclude that, after simplification of the resulting terms,

$$\log \mathbb{E}e^{t\mathcal{L}_{n,n}} = \log\psi(t) + t\widetilde{m}_n + \frac{t^2}{4}\Big(\log\frac{n}{2} - 1\Big) + o(1)$$

from which the result follows. □

4.6. *Case of fixed codimension.* Consider the case in which the codimension of the simplex $n-r$ stays fixed, while $n\to\infty$. Of course, if $n-r=0$, we recover the full-dimensional case.

Proposition 4.10. *Consider the Gaussian model and let m_n be the same as in Proposition 4.7. Let $d\in\mathbb{N}$ be fixed and take $r=n-d$, where $n\to\infty$. Then, $\mathcal{L}_{n,r}-m_n$ converges in the mod-Gaussian sense (meaning that $\eta(t)=\frac{1}{2}t^2$) with parameter $w_n=\frac{1}{2}\log\frac{n}{2}$, namely*

$$\lim_{n\to\infty}\frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,r}-m_n)}}{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{2}}} = \frac{G\left(\frac{d+1}{2}\right)\cdot G\left(\frac{d+2}{2}\right)}{2^{\frac{td}{2}}\,G\left(\frac{d+1}{2}+\frac{t}{2}\right)\cdot G\left(\frac{d+2}{2}+\frac{t}{2}\right)}.$$

The convergence is uniform as long as t stays in any compact subset of $\mathbb{C}\backslash\{-d-1,-d-2,\ldots\}$.

Proof: First, we observe that Theorem 2.5 implies the distributional representation

$$\begin{aligned}\mathcal{L}_{n,n}-\frac{1}{2}\log(n+1) \overset{d}{=} \Big(\mathcal{L}_{n-r,n-r}-\frac{1}{2}\log(n-r+1)\Big)\\ +\Big(\mathcal{L}'_{n,r}-\frac{1}{2}\log(r+1)\Big),\end{aligned}\tag{4.9}$$

where $\mathcal{L}'_{n,r}$ is a copy of $\mathcal{L}_{n,r}$ independent of $\mathcal{L}_{n-r,n-r}$. Since $n-r=d$, this implies that

$$\mathbb{E}e^{t(\mathcal{L}_{n,r}-m_n)} = \frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,n}-m_n)}}{\mathbb{E}\mathrm{e}^{t\mathcal{L}_{d,d}}}\,e^{\frac{t}{2}\log(d+1)}\,e^{\frac{t}{2}\log\left(\frac{n-d+1}{n+1}\right)}.$$

Applying Proposition 4.7 to the numerator and (4.6) to the denominator, we conclude that

$$\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,r}-m_n)} \sim \frac{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{2}}\frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)G\left(1+\frac{t}{2}\right)}}{(d+1)^{\frac{t}{2}}\,2^{\frac{td}{2}}\,\frac{G\left(\frac{1}{2}\right)}{G\left(\frac{d+1}{2}\right)}\cdot\frac{G(1)}{G\left(\frac{d+2}{2}\right)}\cdot\frac{G\left(\frac{d+1}{2}+\frac{t}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)}\cdot\frac{G\left(\frac{d+2}{2}+\frac{t}{2}\right)}{G\left(1+\frac{t}{2}\right)}}\cdot(d+1)^{\frac{t}{2}},$$

which implies the claim. □

Proposition 4.11. *Consider the Beta model with parameter $\nu>0$ or the spherical model (in which case $\nu=0$) and let $\tilde{m}_n$ be the same as in Proposition 4.9. Let $d\in\mathbb{N}$ be fixed and take $r=n-d$, where $n\to\infty$. Then, $\mathcal{L}_{n,r}-\tilde{m}_n-\frac{d-1}{2}\log\frac{n}{2}$*

converges in the mod-Gaussian sense (meaning that $\eta(t) = \frac{1}{2}t^2$*) with parameter* $w_n = \frac{1}{2}\log\frac{n}{2} - \frac{1}{2}$*, namely*

$$\lim_{n\to\infty} \frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,r} - \widetilde{m}_n - \frac{d-1}{2}\log\frac{n}{2})}}{\mathrm{e}^{\frac{1}{4}t^2(\log\frac{n}{2}-1)}} = \frac{G\left(\frac{d+1}{2}\right)\cdot G\left(\frac{d+2}{2}\right)}{2^{\frac{td}{2}}\, G\left(\frac{d+1}{2}+\frac{t}{2}\right)\cdot G\left(\frac{d+2}{2}+\frac{t}{2}\right)}.$$

The convergence is uniform as long as t *stays in any compact subset of* $\mathbb{C}\backslash\{-d-1, -d-2, \dots\}$.

Proof: The computations are similar to those in the proof of Proposition 4.9, but slightly more involved. Again we let $\mathcal{L}_{n,r}^{\mathrm{G}}$ to be the Gaussian analogue of $\mathcal{L}_{n,r}$. By Theorem 2.3 (a), (b), the moment generating function of $\mathcal{L}_{n,r}$ is given by

$$\log \mathbb{E}\mathrm{e}^{t\mathcal{L}_{n,r}} = \log \mathbb{E}\mathrm{e}^{t\mathcal{L}_{n,r}^{\mathrm{G}}} - \frac{t}{2}\log(r+1) - \frac{tr}{2}\log 2 + (r+1)\log\left(\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{\nu+t}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{(r+1)t}{2}\right)}{\Gamma\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{rt}{2}\right)}\right). \tag{4.10}$$

Using the Stirling series for the logarithm of the Gamma function, we obtain

$$(r+1)\log\left(\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{\nu+t}{2}\right)}\right) = \frac{(n+1)t}{2}\log\frac{2}{n} - \frac{t}{4}(t-2+2\nu) + \frac{d-1}{2}t\log\frac{n}{2} + o(1)$$

and

$$\log\left(\frac{\Gamma\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{(r+1)t}{2}\right)}{\Gamma\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{rt}{2}\right)}\right) = t\log n - \frac{t}{2}\log 2 + o(1).$$

Using the behavior of $\mathcal{L}_{n,r}^{\mathrm{G}}$ stated in Proposition 4.10, we obtain, after some transformations,

$$\log \mathbb{E}\mathrm{e}^{t\mathcal{L}_{n,r}} = \log\left(\frac{G\left(\frac{d+1}{2}\right)\cdot G\left(\frac{d+2}{2}\right)}{2^{\frac{td}{2}}\, G\left(\frac{d+1}{2}+\frac{t}{2}\right)\cdot G\left(\frac{d+2}{2}+\frac{t}{2}\right)}\right) + t\widetilde{m}_n + \frac{t^2}{4}\left(\log\frac{n}{2}-1\right) + \frac{d-1}{2}t\log\frac{n}{2} + o(1),$$

which yields the claim. □

4.7. *Case of diverging codimension.* In this section we consider the case when the codimension of the simplex goes to $+\infty$.

Proposition 4.12. *Consider the Gaussian model and let* m_n *be the same as in Proposition 4.7. If* $r = r(n)$ *is such that* $n - r \to \infty$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\mathrm{e}^{t\left(\mathcal{L}_{n,r} - (m_n - m_{n-r}) - \frac{1}{2}\log\left(\frac{(r+1)(n-r)}{n}\right)\right)}}{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{n-r}}} = 1.$$

If, additionally, $n - r = o(n)$*, then we have mod-Gaussian convergence (meaning that* $\eta(t) = \frac{1}{2}t^2$*) with parameter* $w_n = \frac{1}{2}\log\frac{n}{n-r} \to \infty$ *and limiting function identically equal to* 1.

Proof: From Proposition 4.7 we know that

$$\lim_{n\to\infty}\frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,n}-m_n)}}{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{2}}}=\lim_{n\to\infty}\frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n-r,n-r}-m_{n-r})}}{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n-r}{2}}}=\frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)G\left(1+\frac{t}{2}\right)}.$$

Using the distributional identity (4.9) it follows that

$$\begin{aligned}\mathbb{E}\mathrm{e}^{t\left(\mathcal{L}_{n,r}-(m_n-m_{n-r})-\frac{1}{2}\log\left(\frac{(r+1)(n-r+1)}{n+1}\right)\right)}&=\frac{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n,n}-m_n)}}{\mathbb{E}\mathrm{e}^{t(\mathcal{L}_{n-r,n-r}-m_{n-r})}}\\&\sim\frac{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{2}}}{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n-r}{2}}}\\&\sim\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{n-r}},\end{aligned}$$

which implies the claim after observing that $\log(n+1)=\log n+o(1)$ and $\log(n-r+1)=\log(n-r)+o(1)$. Observe also that if $n-r=o(n)$, then $w_n\to\infty$, as $n\to\infty$, which is otherwise not the case. □

Proposition 4.13. *Consider the Beta model with parameter $\nu>0$ or the spherical model (in which case $\nu=0$) and let m_n be the same as in Proposition 4.7. If $r=r(n)$ is such that $n-r=o(n)$ as $n\to\infty$, then,*

$$\lim_{n\to\infty}\frac{\mathbb{E}\mathrm{e}^{t\left(\mathcal{L}_{n,r}-(m_n-m_{n-r}-\frac{r+1}{4n}(t-2+2\nu))-\frac{1}{2}\log\frac{(n-r)(1+r)}{n^{1+r}}\right)}}{\mathrm{e}^{\frac{1}{4}t^2\log\frac{n}{n-r}}}=1.$$

That is, we have mod-Gaussian convergence (meaning that $\eta(t)=\frac{1}{2}t^2$) with parameter $w_n=\frac{1}{2}\log\frac{n}{n-r}$ and limiting function identically equal to 1.

Proof: Denote by $\mathcal{L}^{\mathrm{G}}_{n,r}$ the Gaussian analogue of $\mathcal{L}_{n,r}$. Observe that relation (4.10) still holds. Regarding the first term in this relation, we know from Proposition 4.12 that

$$\log\mathbb{E}\mathrm{e}^{t\mathcal{L}^{\mathrm{G}}_{n,r}}=t(m_n-m_{n-r})+\frac{t}{2}\log\left(\frac{(r+1)(n-r)}{n}\right)+\frac{1}{4}t^2\log\frac{n}{n-r}+o(1).$$

Again, a second-order Stirling expansion yields

$$(r+1)\log\left(\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{\nu+t}{2}\right)}\right)=(r+1)\frac{t}{2}\log\frac{2}{n}-\frac{r+1}{n}\frac{t}{4}(t-2+2\nu)+o(r/n^2)$$

and

$$\begin{aligned}\log\left(\frac{\Gamma\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{(r+1)t}{2}\right)}{\Gamma\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{rt}{2}\right)}\right)&=\frac{t}{2}\log\left(\frac{r(n+\nu-2)+n+\nu}{2}+\frac{rt}{2}\right)+o(1)\\&=\frac{t}{2}\log\left(\frac{(r+1)n}{2}\right)+o(1).\end{aligned}$$

Taking everything together, we obtain

$$\begin{aligned}\log\mathbb{E}\mathrm{e}^{t\mathcal{L}_{n,r}}=t\Big(m_n-m_{n-r}-\frac{r+1}{4n}(t-2+2\nu)\Big)+\frac{t}{2}\log\frac{(n-r)(1+r)}{n^{1+r}}\\+\frac{1}{4}t^2\log\left(\frac{n}{n-r}\right)+o(1).\end{aligned}$$

This yields the claim, since $w_n = \frac{1}{2}\log\frac{n}{n-r} \to \infty$, as $n \to \infty$ by the assumption that $n - r = o(n)$. □

Remark 4.14. In Eichelsbacher and Knichel (2017) a wide class of random variables with gamma-type moments is studied, which especially comprises the random variables $\mathcal{V}_{n,r}$ considered in this paper. For this general class of random variables mod-ϕ convergence is proved using different methods on the technical side. Hence, the limit theorems presented in this section also follow from these general results as special cases, see Eichelsbacher and Knichel (2017, Section 7).

5. Large deviations

The purpose of this section is to derive large deviation principles, recall the definition at the beginning of Section 3.2. Again, we restrict to the Gaussian, the Beta and the spherical model, which admit finite moments of all orders.

5.1. *The Gaussian model.*

We start with the Gaussian model and recall the notation $\mathcal{L}_{n,r} := \log(r!\mathcal{V}_{n,r})$. Using the Gärtner–Ellis theorem we derive large deviation principles from the following Proposition.

Proposition 5.1. (a) *Let $r = o(n)$, as $n \to \infty$. Then, we have*

$$j_1(t) := \lim_{n\to\infty} \frac{1}{rn} \log \mathbb{E}e^{tn(\mathcal{L}_{n,r} - \frac{r}{2}\log n - \frac{1}{2}\log(r+1))} = \begin{cases} \frac{1}{2}((t+1)\log(t+1) - t) & : t \geq -1 \\ +\infty & : \text{otherwise.} \end{cases}$$

(b) *If $r \sim \alpha n$, $\alpha \in (0,1)$, we have*

$$j_2(t) := \lim_{n\to\infty} \frac{1}{\alpha n^2} \log \mathbb{E}e^{tn(\mathcal{L}_{n,r} - \frac{\alpha n}{2}(\log n + \log(1-\alpha)))} = \begin{cases} \frac{2+2t-\alpha}{4}\log\left(\frac{1+t-\alpha}{1-\alpha}\right) - \frac{t}{2} & : t \geq \alpha - 1 \\ +\infty & : \text{otherwise.} \end{cases}$$

(c) *Let $d \in \mathbb{N}$ and assume that $d = n - r$, as $n \to \infty$, and $m_n = \frac{1}{2}(n\log n - n + \frac{1}{2}\log n + \log(2^{3/2}\pi))$ as in Proposition 4.10. Then, we have*

$$\lim_{n\to\infty} \frac{1}{\frac{1}{2}\log\frac{n}{2}} \log \mathbb{E}e^{t(\mathcal{L}_{n,r} - m_n)} = \frac{1}{2}t^2.$$

(d) *Let $r = r(n)$ be such that $n - r = o(n)$, as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty} \frac{1}{\frac{1}{2}\log\frac{n}{n-r}} \log \mathbb{E}e^{t\left(\mathcal{L}_{n,r} - (m_n - m_{n-r}) - \frac{1}{4}\log\left(\frac{(r+1)(n-r+1)}{n+1}\right)\right)} = \frac{1}{2}t^2.$$

Proof: Part (a) is a consequence of Remark 4.3. The proofs of the (c) and (d) directly follow from the proofs of Propositions 4.10 and 4.12 in the previous section, respectively.

We turn now to the case that $r \sim \alpha n$. Due to the asymptotic formula (4.3) we obtain for all $\alpha \in (0,1)$, $t \geq 0$ and $j \in \mathbb{N}$ that

$$\begin{aligned}
&\log\left(\frac{\Gamma\left(\frac{(1+t-\alpha)n+j}{2}\right)}{\Gamma\left(\frac{(1-\alpha)n+j}{2}\right)}\right)\\
&\quad\sim \log\left(\frac{\exp(-\frac{(1+t-\alpha)n}{2})\left(\frac{(1+t-\alpha)n}{2}\right)^{\frac{(1+t-\alpha)n}{2}+\frac{j-1}{2}}}{\exp(-\frac{(1-\alpha)n}{2})\left(\frac{(1-\alpha)n}{2}\right)^{\frac{(1-\alpha)n}{2}+\frac{j-1}{2}}}\right)\\
&\quad= \log\left(\exp\left(-\frac{tn}{2}\right)\left(\frac{n}{2}\right)^{\frac{tn}{2}}\frac{(1+t-\alpha)^{\frac{(1+t-\alpha)n}{2}}}{(1-\alpha)^{\frac{(1-\alpha)n}{2}}}\left(\frac{1+t-\alpha}{1-\alpha}\right)^{\frac{j-1}{2}}\right)\\
&\quad= -\frac{tn}{2}+\frac{tn}{2}\log\left(\frac{n}{2}\right)+\frac{(1+t-\alpha)n}{2}\log(1+t-\alpha)\\
&\qquad\qquad -\frac{(1-\alpha)n}{2}\log(1-\alpha)+\frac{j-1}{2}\log\left(\frac{1+t-\alpha}{1-\alpha}\right),
\end{aligned}$$

as $n \to \infty$, and thus

$$\begin{aligned}
&\frac{1}{\alpha n^2}\log \mathbb{E}e^{tn\mathcal{L}_{n,r}}\\
&\quad= \frac{1}{\alpha n^2}\left[\frac{tn}{2}\log(\alpha n+1)+\frac{t\alpha n^2}{2}\log 2+\sum_{j=1}^{\alpha n}\log\left(\frac{\Gamma\left(\frac{(1+t-\alpha)n+j}{2}\right)}{\Gamma\left(\frac{(1-\alpha)n+j}{2}\right)}\right)\right]\\
&\quad\sim -\frac{t}{2}+\frac{t}{2}\log(n)+\frac{1+t-\alpha}{2}\log(1+t-\alpha)\\
&\qquad\quad -\frac{1-\alpha}{2}\log(1-\alpha)+\frac{\alpha}{4}\log\left(\frac{1+t-\alpha}{1-\alpha}\right)\\
&\quad= -\frac{t}{2}+\frac{t}{2}\log(n)+\frac{2+2t-\alpha}{4}\log(1+t-\alpha)-\frac{2-\alpha}{4}\log(1-\alpha)\,.
\end{aligned}$$

This directly yields the result in the case $r \sim \alpha n$ in view of the moment formula for Gaussian simplices stated in Section 2.1. □

We turn now to the large deviation principles for the log-volume of Gaussian simplices.

Theorem 5.2 (LDP for Gaussian simplices). (a) *Let $r = o(n)$, as $n \to \infty$. Then, $\frac{1}{r}(\mathcal{L}_{n,r} - \frac{r}{2}\log n - \frac{1}{2}\log(r+1))$ satisfies a LDP with speed rn and rate function*

$$I(x) = \frac{1}{2}(e^{2x}-1) - x\,, \qquad x \in \mathbb{R}\,.$$

(b) *If $r \sim \alpha n$, $\alpha \in (0,1)$, then, $\frac{1}{\alpha n}(\mathcal{L}_{n,r} - \frac{\alpha n}{2}(\log n + \log(1-\alpha)))$ satisfies a LDP with speed αn^2 and rate function*

$$I(x) = \sup_{t \geq \alpha - 1}\{tx - j_2(t)\}\,, \qquad x \in \mathbb{R}\,,$$

where $j_2(t)$ is the function from Proposition 5.1 (b).

(c) *Let $d \in \mathbb{N}$ and assume that $d = n-r$, as $n \to \infty$, and $m_n = \frac{1}{2}(n \log n - n + \frac{1}{2}\log n + \log(2^{3/2}\pi))$. Then, $\frac{1}{\frac{1}{2}\log\frac{n}{2}}(\mathcal{L}_{n,r} - m_n)$ satisfies a LDP with speed $\frac{1}{2}\log\frac{n}{2}$ and rate function*

$$I(x) = \frac{1}{2}x^2, \qquad x \in \mathbb{R}.$$

(d) *Let $r = r(n)$ be such that $n-r \to \infty$, as $n \to \infty$. If $n-r = o(n)$, as $n \to \infty$, then, $\frac{1}{\frac{1}{2}\log\frac{n}{n-r}}\big(\mathcal{L}_{n,r} - (m_n - m_{n-r}) - \frac{1}{2}\log\big(\frac{(r+1)(n-r+1)}{n+1}\big)\big)$ satisfies a LDP with speed $\frac{1}{2}\log\frac{n}{n-r}$ and rate function*

$$I(x) = \frac{1}{2}x^2, \qquad x \in \mathbb{R}.$$

Proof of Theorem 5.2: Let $r = o(n)$, as $n \to \infty$. Then, by the Gärtner–Ellis theorem (cf. Section 2.3 in Dembo and Zeitouni, 2010) and Proposition 5.1, the random variables $\frac{1}{r}(\mathcal{L}_{n,r} - \frac{r}{2}\log n - \frac{1}{2}\log(r+1))$ satisfy a LDP with speed rn and rate function

$$I(x) = \sup_{t \geq -1}\Big[tx - \frac{1}{2}((t+1)\log(t+1) - t)\Big],$$

i.e., the Legendre-Fenchel transform of the function $\frac{1}{2}((t+1)\log(t+1) - t)$. For each $x \in \mathbb{R}$ the supremum is attained at $t = e^{2x} - 1$, which yields the result of (a). The same argument implies the LDP for the other regimes of r as well. □

5.2. *The Beta and the spherical model.* Now, we turn to the Beta model with parameter $\nu > 0$ and the spherical model, i.e., $\nu = 0$, and recall that $\mathcal{L}_{n,r} := \log(r!\mathcal{V}_{n,r})$, where $\mathcal{V}_{n,r}$ is the volume of the r-dimensional simplex with vertices $X_1, \ldots, X_{r+1}$ chosen according to the Beta or the spherical distribution, respectively. Similar to the Gaussian case, we start with the following proposition that will imply the large deviation principles.

Proposition 5.3. (a) *Let $r \in \mathbb{N}$ be fixed. Then, we have*

$$j_3(t) := \lim_{n\to\infty}\frac{1}{n}\log \mathbb{E}e^{tn\mathcal{L}_{n,r}} = \begin{cases} \eta(t) & : t \geq -1 \\ +\infty & \text{otherwise}, \end{cases}$$

where η is the function from Proposition 4.4.

(b) *If $r \sim \alpha n$, $\alpha \in (0,1)$, we have*

$$j_4(t) := \lim_{n\to\infty}\frac{1}{\alpha n^2}\log \mathbb{E}e^{tn\mathcal{L}_{n,r}} = \begin{cases} \eta(t) & : t \geq -1 \\ +\infty & : \text{otherwise}, \end{cases}$$

where $\eta(t)$ is the function given by

$$\eta(t) := \frac{2+2t-\alpha}{4}\log(1+t-\alpha) - \frac{2-\alpha}{4}\log(1-\alpha) - \frac{1+t}{2}\log(1+t).$$

(c) *Let $d \in \mathbb{N}$ and assume that $d = n - r$, as $n \to \infty$, and let $\widetilde{m}_n = \frac{1}{2}(\frac{1}{2}\log n - n + 1 - \nu + \log(2^{3/2}\pi))$ as in Proposition 4.9. Then, we have*

$$\lim_{n\to\infty} \frac{1}{\frac{1}{2}(\log\frac{n}{2} - 1)} \log \mathbb{E}e^{t(\mathcal{L}_{n,r} - \widetilde{m}_n - \frac{d-1}{2}\log\frac{n}{2})} = \frac{1}{2}t^2.$$

(d) *Let $r = r(n)$ be such that $n - r = o(n)$, and let $m_n = \frac{1}{2}(n\log n - n + \frac{1}{2}\log n + \log(2^{3/2}\pi))$ be as in Proposition 4.10. Then*

$$\lim_{n\to\infty} \frac{1}{\frac{1}{2}\log\frac{n}{n-r}} \times \log \mathbb{E}e^{t\left(\mathcal{L}_{n,r} - (m_n - m_{n-r} - \frac{r+1}{4n}(t-2+2\nu)) - \frac{1}{2}\log\frac{(n-r)(1+r)}{n^{1+r}}\right)} = \frac{1}{2}t^2.$$

Proof of Proposition 5.1: For $t \geq -1$ the assertions in (a) follows from Proposition 4.4. Recall from Theorem 2.5 that the distribution of $\mathcal{V}_{n,r}$ involves Beta random variables $Z := \beta_{\frac{\nu+r-j}{2}, \frac{n-r+j}{2}}$ with $j \leq r$. Writing

$$\mathbb{E}e^{\frac{tn}{2}\log Z} = \mathbb{E}Z^{\frac{tn}{2}} = c\int_0^1 z^{\frac{n-r+j}{2} + \frac{tn}{2} - 1}(1-z)^{\frac{\nu+r-j}{2} - 1}\,\mathrm{d}z$$

we see that the exponent at z is less than -1 for sufficiently large n if $t < -1$. This implies that $\mathbb{E}e^{\frac{tn}{2}\log Z} \to +\infty$ and completes the proof of (a).

Now, let us turn towards the case $r \sim \alpha n$, $\alpha \in (0,1)$ in (b). Similar to what has been done in the Gaussian setting, we obtain by using the asymptotic formula (4.3) for all $\nu > 0$,

$$\begin{aligned}
&(\alpha n + 1)\log\left(\frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{(1+t)n+\nu}{2}\right)}\right) \\
&\quad\sim (\alpha n + 1)\left(\frac{tn}{2} - \frac{tn}{2}\log\left(\frac{n}{2}\right) - \frac{(1+t)n + \nu - 1}{2}\log(1+t)\right) \\
&\quad\sim \frac{t\alpha n^2}{2} - \frac{t\alpha n^2}{2}\log\left(\frac{n}{2}\right) - \frac{(1+t)\alpha n^2 + \alpha n(\nu - 1)}{2}\log(1+t),
\end{aligned}$$

as $n \to \infty$, and for all $t \geq 0$,

$$\begin{aligned}
\log&\left(\frac{\Gamma\left(\frac{\alpha n(n+\nu-2)+n+tn(\alpha n+1)+\nu}{2}\right)}{\Gamma\left(\frac{\alpha n(n+\nu-2)+n+tn\alpha n+\nu}{2}\right)}\right) \sim -\frac{tn}{2} + \frac{tn}{2}\log\left(\frac{n}{2}\right) \\
&+ \frac{\alpha n(n+\nu-2) + n + tn(\alpha n + 1) + \nu}{2}\log\left(\alpha(n+\nu-2) + 1 + t(\alpha n + 1)\right) \\
&- \frac{\alpha n(n+\nu-2) + n + tn\alpha n + \nu}{2}\log\left(\alpha(n+\nu-2) + 1 + t\alpha n\right).
\end{aligned}$$

Thus, by using the calculations made in the Gaussian case above, we conclude that

$$\begin{aligned}
&\frac{1}{\alpha n^2}\log \mathbb{E}e^{tn\mathcal{L}_{n,r}}\\
&= \frac{1}{\alpha n^2}\Bigg[(\alpha n+1)\log\left(\frac{\Gamma\big(\frac{n+\nu}{2}\big)}{\Gamma\big(\frac{(1+t)n+\nu}{2}\big)}\right)\\
&\quad +\log\left(\frac{\Gamma\big(\frac{\alpha n(n+\nu-2)+n+tn(\alpha n+1)+\nu}{2}\big)}{\Gamma\big(\frac{\alpha n(n+\nu-2)+n+tn\alpha n+\nu}{2}\big)}\right)+\sum_{j=1}^{\alpha n}\log\left(\frac{\Gamma\big(\frac{(1+t-\alpha)n+j}{2}\big)}{\Gamma\big(\frac{(1-\alpha)n+j}{2}\big)}\right)\Bigg]\\
&\sim \frac{t}{2}-\frac{t}{2}\log\left(\frac{n}{2}\right)-\frac{1+t}{2}\log(1+t)+\frac{1+t}{2}\log\left(\alpha(n+\nu-2)+1+t(\alpha n+1)\right)\\
&\quad -\frac{1+t}{2}\log\left(\alpha(n+\nu-2)+1+t\alpha n\right)-\frac{t}{2}+\frac{t}{2}\log\left(\frac{n}{2}\right)\\
&\quad +\frac{2+2t-\alpha}{4}\log\left(1+t-\alpha\right)-\frac{2-\alpha}{4}\log\left(1-\alpha\right)\\
&\sim -\frac{1+t}{2}\log(1+t)+\frac{2+2t-\alpha}{4}\log\left(1+t-\alpha\right)-\frac{2-\alpha}{4}\log\left(1-\alpha\right),
\end{aligned}$$

as $n\to\infty$. This directly yields the result in the case where $r\sim\alpha n$, again taking into account the moment representation in the Beta model stated in Section 2.1. Since there is no dependence on the parameter ν in the result concerning the Beta model, the one regarding the spherical model is implied by considering the limiting case $\nu\downarrow 0$ as seen several times before.
The proofs of the (c) and (d) directly follow from the proofs of Propositions 4.11 and 4.13 in the previous section, respectively. □

Now, we are able to state the large deviation principles for the Beta and the spherical model. Their proofs follow the same lines as the ones in the Gaussian case presented above by using the Gärtner–Ellis theorem. For this reason we have decided to skip them.

Theorem 5.4 (LDP for Beta-type and spherical simplices)**.** (a) *Let $r\in\mathbb{N}$ be fixed. Then, $\mathcal{L}_{n,r}$ satisfies a LDP with speed n and rate function*

$$I(x)=\sup_{t\in\mathbb{R}}\big\{tx-j_3(t)\big\},$$

where $j_3(t)$ is the function from Proposition 5.3 (a).

(b) *If $r\sim\alpha n$, $\alpha\in(0,1)$, then, $\frac{1}{\alpha n}\mathcal{L}_{n,r}$ satisfies a LDP with speed αn^2 and rate function*

$$I(x)=\sup_{t\in\mathbb{R}}\left\{tx-j_4(t)\right\},$$

where $j_4(t)$ is the function from Proposition 5.3 (b).

(c) *Let $d\in\mathbb{N}$ and assume that $d=n-r$, as $n\to\infty$, and $\widetilde{m}_n=\frac{1}{2}(\frac{1}{2}\log n-n+1-\nu+\log(2^{3/2}\pi))$. Then, $\frac{1}{\frac{1}{2}(\log\frac{n}{2}-1)}(\mathcal{L}_{n,r}-\widetilde{m}_n-\frac{d-1}{2}\log\frac{n}{2})$ satisfies a LDP with speed $\frac{1}{2}(\log\frac{n}{2}-1)$ and rate function*

$$I(x)=\frac{1}{2}x^2\,,\qquad x\in\mathbb{R}\,.$$

(d) *Let* $r = r(n)$ *be such that* $n-r = o(n)$, *and let* $m_n = \frac{1}{2}(n\log n - n + \frac{1}{2}\log n + \log(2^{3/2}\pi))$ *be defined as in Proposition 4.10. Then,* $\frac{1}{\frac{1}{2}\log\frac{n}{n-r}}\big(\mathcal{L}_{n,r} - (m_n - m_{n-r} - \frac{r+1}{4n}(t-2+2\nu)) - \frac{1}{2}\log\frac{(n-r)(1+r)}{n^{1+r}}\big)$ *satisfies a LDP with speed* $\frac{1}{2}\log\frac{n}{n-r}$ *and rate function*

$$I(x) = \frac{1}{2}x^2\,, \qquad x \in \mathbb{R}\,.$$

Remark 5.5. One can combine Theorem 5.4 with the contraction principle from large deviation theory to obtain a LDP for $\mathcal{V}_{n,r}$, that is, for the volume of the random simplex itself in the cases that $r = o(n)$ and $r \sim \alpha n$ for some $\alpha \in (0,1)$.

Acknowledgements

JG has been supported by the German Research Foundation (DFG) via Research Training Group RTG 2131 *High dimensional Phenomena in Probability – Fluctuations and Discontinuity*. ZK and CT were supported by the DFG Scientific Network *Cumulants, Concentration and Superconcentration*.

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