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On 3-dimensional Berry's model

Federico Dalmao, Anne Estrade and José R. León

DMEL, Universidad de la República
Salto, Uruguay.
E-mail address: fdalmao@unorte.edu.uy

MAP5, UMR CNRS 8145, Université de Paris
Paris, France.
E-mail address: anne.estrade@u-paris.fr

IMERL, Universidad de la República
Montevideo, Uruguay.
E-mail address: rlramos@fing.edu.uy

Abstract. This work aims to study the dislocation or nodal lines of 3D Berry's random waves model. Their expected length is computed both in the isotropic and anisotropic cases, being them compared. Afterwards, in the isotropic case the asymptotic variance and distribution of the length are studied as the domain grows to the whole space. We find different orders of magnitude for the variance and different limit distributions for different submodels. The study includes the Berry's monochromatic random waves, the Bargmann-Fock model and the Black-Body radiation.

1. Introduction

In the last few years, nodal sets, also called as dislocation sets or zero sets, of several classes of random waves have received a lot of attention from Number Theory, Topological Analysis, Differential Geometry, Probability Theory, etc. While studying the random billiards, Berry (2002) argued that in the microscopic scale several models as arithmetic random waves on the torus or spherical harmonics, although they verify some boundary conditions, converge towards an universal Gaussian model, which is called Berry's random waves model. Canzani and Hanin (2020) studied the universality phenomenon in general Riemannian manifolds. The reader can find results on arithmetic random waves defined on the flat torus (Cammarota,

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2019; Dalmao et al., 2019) and on random spherical harmonics in Cammarota and Marinucci (2019); Fantaye et al. (2019); Marinucci and Rossi (2021) and references therein, see also Rossi (2019) for a survey on both subjects. The nodal sets of Berry's planar random waves, *i.e.* the random eigenfunctions of the 2D Euclidean Laplacian operator, have been studied in Nourdin et al. (2019) where Central Limit Theorems are obtained for the nodal length in the real case and for the number of phase singularities in the complex case. Whereas all the previous references are concerned with 2-dimensional isotropic random fields, one can also find studies in more general frameworks. In Estrade and Fournier (2020) anisotropic random waves are considered in any dimension. In Kratz and Vadlamani (2018); Müller (2017), similar central limit results are obtained for any Minkowski functional of excursion sets in the general framework of stationary Gaussian fields whose covariance function is fast enough decreasing at infinity.

Our motivation mostly comes from the seminal paper Berry and Dennis (2000) and from Dennis (2007) where the authors show how the expectation and the second moment of certain functionals of the nodal sets can be computed. Moreover, in Dennis (2007) (where a more formal approach from the mathematical point of view is presented) a variety of problems which are in close relation with the computation of the measure of the zero set of random waves in 2D and 3D are exhibited. Also, the two point correlation is introduced defining it as a second order Rice's function. The main tools are the different forms of the Kac-Rice formulas (see Azaïs and Wschebor, 2009 and the references therein) and Hermite/Wiener expansions, see Peccati and Taqqu (2011) and references therein.

In the present paper we study complex-valued 3-dimensional Berry's random waves models with a focus on the length of the dislocation or nodal lines. We obtain the expected length in a very general framework which includes anisotropy. In order to study the asymptotic variance and the limit distribution we restrict our attention to the isotropic case and we let the domain increase to the whole space. It can be shown that this is equivalent to consider a fixed domain and taking the high energy limit, see Canzani and Hanin (2020); Nourdin et al. (2019) and Remark 3.6 below. We find different orders of magnitude for the variance and different limit distributions for different models. More precisely, we establish the order of the limit variance and the asymptotic normality in a framework including Berry's monochromatic random waves, Black-Body radiation and Bargmann-Fock waves and we include a power law model which has an asymptotic variance of different order and which presents a non-Gaussian limit distribution yielding a non-central limit theorem.

The paper is organized as follows. Section 2 presents the model as well as some particular cases. Our main results, namely Theorems 3.1-3.3 and Propositions 3.4-3.5, as well as some remarks are presented in Section 3. Section 4 is devoted to the study of the first moment of the dislocation length; in particular, it contains the proof of Theorem 3.1. A special section (Section 5) is devoted to the Itô-Wiener's chaotic decomposition given by Hermite expansion. Section 6 is dedicated to the proofs of the main results. In Subsection 6.1 we prove Theorem 3.3 and we show which particular cases shown in Section 2 are covered by this theorem. The two specific cases of Berry's monochromatic random waves and the power law model are studied in Subsections 6.2 and 6.3 respectively.

2. The model

Consider a 3-dimensional Berry's random waves model $\psi : \mathbb{R}^3 \to \mathbb{C}$ given by

$$\psi(x) = \int_{\mathbb{R}^3} \exp(i\langle \mathbf{k}, x\rangle)\, \frac{dW_\Pi(\mathbf{k})}{|\mathbf{k}|}, \quad x \in \mathbb{R}^3, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the usual inner product and 2-norm in $\mathbb{R}^3$ respectively. Besides, W_Π is a (complete) complex-valued Gaussian random measure on $\mathbb{R}^3$ with (real) control measure Π, *i.e.* Π is a positive non-atomic measure on $\mathbb{R}^3$ satisfying

$$\mathbb{E}\left(\int_A \frac{dW_\Pi(\mathbf{k})}{|\mathbf{k}|}\, \overline{\int_B \frac{dW_\Pi(\mathbf{k})}{|\mathbf{k}|}}\right) = 2\int_{A\cap B} \frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2}, \tag{2.2}$$

for any Borel sets A, B in $\mathbb{R}^3$. We further assume that

$$\int_{\mathbb{R}^3} \frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2} = 1,$$

that $\Pi(\mathbb{R}^3) < \infty$ and that $\Pi(-A) = \Pi(A)$ for any Borel set $A \subset \mathbb{R}^3$.

Actually, if $W_\Pi = W_\Pi^1 + iW_\Pi^2$ with real independent W_Π^j $(j = 1, 2)$ then (2.2) holds for W_Π^j $(j = 1, 2)$ without 2 in the right-hand side factor.

As a consequence, the random field ψ is Gaussian, stationary, centered but not necessarily isotropic. We denote by ξ and η the real and imaginary parts of ψ, that is $\psi = \xi + i\eta$. The random fields ξ and η are independent and identically distributed with common covariance function prescribed by

$$\begin{aligned} r(x) &:= \mathbb{E}(\xi(0)\xi(x)) = \mathbb{E}(\eta(0)\eta(x)), \quad x \in \mathbb{R}^3 \\ &= \int_{\mathbb{R}^3} \exp(i\langle \mathbf{k}, x\rangle)\, \frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2}. \end{aligned} \tag{2.3}$$

Note that the normalization $\int_{\mathbb{R}^3} \frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2} = 1$ yields $r(0) = 1$ and that

$$\mathbb{E}(\psi(0)\overline{\psi}(x)) = 2r(x).$$

Furthermore, the condition $\Pi(\mathbb{R}^3) < \infty$ implies that ψ, ξ, η are almost surely C^2.

Using the vocabulary introduced in Estrade and Fournier (2020), ξ and η are random waves whose associated random wavevector admits $\frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2}$ as distribution. In what follows, we will call Π the *power spectrum* although this word is usually reserved to the isotropic framework. Indeed, the random wave ψ can be isotropic or not according to the fact that the covariance function $r(x)$ only depends on $|x|$ or not, which only depends on the choice of Π.

Let us look at the model in the isotropic case. We write $\mathbf{k} = \rho\mathbf{u}$ with $\rho > 0$ and $\mathbf{u} \in \mathbb{S}^2$, being $\mathbb{S}^2$ the unitary sphere in $\mathbb{R}^3$. We consider the case where the image of measure $\frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2}$ through the change of variables $\mathbf{k} \mapsto (\rho, u) \in \mathbb{R}^+ \times \mathbb{S}^2$ writes out

$$\Pi^{rad}(d\rho) \otimes d\sigma(\mathbf{u}), \tag{2.4}$$

for some measure Π^{rad} defined on $\mathbb{R}^+$ and where $d\sigma$ stands for the surface measure on $\mathbb{S}^2$. The normalization on the power spectrum imposes that $\Pi^{rad}(\mathbb{R}^+) = \frac{1}{4\pi}$.

The covariance function is then given by

$$\begin{aligned} r(x) &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}^2} \exp(i\rho|x|\langle \mathbf{u}, \mathbf{e}\rangle)\, d\sigma(\mathbf{u}) \right) \Pi^{rad}(d\rho) \\ &= 4\pi \int_{\mathbb{R}^+} \frac{\sin(\rho|x|)}{\rho|x|}\, \Pi^{rad}(d\rho), \end{aligned} \tag{2.5}$$

being $\mathbf{e}$ a fixed point in $\mathbb{S}^2$.

In view of (2.5), we recognize the covariance function involved in Berry and Dennis model (Berry and Dennis, 2000). Note that our normalization on Π^{rad} differs from (3.11) in Berry and Dennis (2000).

Some particular isotropic cases. In the following examples named as Examples 2.1, 2.2, 2.3 and 2.4, we assume that the power spectrum admits a density with respect to Lebesgue measure and since we focus on isotropic examples we write it as $f(|.|)$. Hence the two next identities will be in force

$$r(x) = \int_{\mathbb{R}^3} \exp(i\langle \mathbf{k}, x\rangle)\, \frac{f(|\mathbf{k}|)}{|\mathbf{k}|^2}\, d\mathbf{k} = 4\pi \int_{\mathbb{R}^+} \frac{\sin(\rho|x|)}{\rho|x|}\, f(\rho)\, d\rho,$$

with normalization $\int_{\mathbb{R}^3} \frac{1}{|\mathbf{k}|^2} f(|\mathbf{k}|)\, d\mathbf{k} = 4\pi \int_{\mathbb{R}^+} f(\rho)\, d\rho = 1$.

2.1. *Bargmann-Fock model.* Let us take $f(\rho) = (2\pi)^{-3/2}\rho^2 e^{-\rho^2/2}$, $\rho \in \mathbb{R}^+$ as spectral density. In this case,

$$r(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp(i\langle \mathbf{k}.x\rangle)\, e^{-|\mathbf{k}|^2/2}\, d\mathbf{k} = e^{-|x|^2/2}, \; x \in \mathbb{R}^3.$$

2.2. *Gamma type.* Let us take $f(\rho) = \frac{\beta^{p+1}}{4\pi p!}\rho^p e^{-\beta\rho}$, $\rho \in \mathbb{R}^+$ with p a positive integer and β some positive real constant. We remark that $f(\rho) = \frac{\beta}{4\pi p}\rho\gamma(\rho)$, where γ is the probability density function of a $\Gamma(p, \beta)$-distribution. We then write the covariance function as

$$r(x) = \frac{\beta}{p|x|} \int_{\mathbb{R}^+} \sin(\rho|x|)\, \gamma(\rho)\, d\rho = \frac{\beta}{p|x|}\, Im\big(\widehat{\gamma}(|x|)\big),$$

where Im stands for the imaginary part of any complex number and $\widehat{\gamma}$ stands for the characteristic function of the distribution γ. Since $\widehat{\gamma}(t) = (1 - i\frac{|t|}{\beta})^{-p}$, we get

$$r(x) = \frac{1}{p}\left(1 + \frac{|x|^2}{\beta^2}\right)^{-p} \sum_{1 \le j \le p;\, j\ odd} (-1)^{(j-1)/2} \binom{p}{j} \beta^{-(j-1)}\, |x|^{j-1}.$$

2.3. *Black-Body radiation.* The *Black-Body model* is prescribed by $f(\rho) = \frac{c\rho^3}{e^\rho - 1}$, being c a convenient constant. According to Equation (6.8) in Berry and Dennis (2000), see also Formula 2 in Section 3.911 of Gradshteyn and Ryzhik (2015),

$$r(x) = \frac{c_1}{|x|^2} - \frac{c_2|x|\cosh(|x|)}{\sinh(|x|)^2}.$$

2.4. *Power law model.* Assume that $f(\rho) = \frac{1-\beta}{4\pi}\rho^{-\beta}\,\mathbb{I}_{(0,1)}(\rho)$ with $0 < \beta < 1$. The covariance function of this model is given by

$$r(x) = (1-\beta)\,|x|^{\beta-1}\int_0^{|x|}\rho^{-\beta-1}\,\sin\rho\,d\rho, \quad x \in \mathbb{R}^3.$$

2.5. *Berry's monochromatic random waves model.* Assume that the power spectrum Π is uniformly distributed on the two-dimensional sphere $\mathbb{S}^2$. For this isotropic model, relation (2.4) holds with Π^{rad} proportional to the Dirac mass at 1, *i.e.* $\Pi^{rad} = \frac{1}{4\pi}\delta_1$. Thus, the covariance function is given by

$$r(x) = \operatorname{sinc}(|x|) = \frac{\sin(|x|)}{|x|}, \quad x \in \mathbb{R}^3.$$

3. Main results

The dislocation lines $\{x \in \mathbb{R}^3 : |\psi(x)| = 0\}$ have Hausdorff dimension one. For any bounded domain Q in $\mathbb{R}^3$, we introduce

$$\mathcal{Z}(Q) = \{x \in Q : |\psi(x)| = 0\}, \quad \ell(\mathcal{Z}(Q)) = \operatorname{length}(\mathcal{Z}(Q)).$$

We now present our main results.

3.1. *The expectation.* Here Q is arbitrary but fixed.

Theorem 3.1. *Let ψ be defined as in (2.1) and assume that $\psi'(0)$ is non degenerated. Let λ_i, $i = 1, 2, 3$ be the eigenvalues of the covariance matrix $-r''(0)$ and $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. Hence,*

$$\mathbb{E}(\ell(\mathcal{Z}(Q))) = \frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{2\pi}\mathbb{E}|D^{-\frac{1}{2}}(N \wedge N')|\operatorname{vol}(Q),$$

being (N, N') a standard normal random vector in $\mathbb{R}^6$ and $\wedge$ the usual cross product of vectors in $\mathbb{R}^3$.

Next, we specialize this result to the isotropic case and compare it with the *almost* isotropic case.

Corollary 3.2. *In the same conditions as above,*

(i) *if $\lambda_i = \lambda = -r''_{11}(0)$, $i = 1, 2, 3$, we have*

$$\mathbb{E}(\ell(\mathcal{Z}(Q))) = \frac{\lambda}{\pi}\operatorname{vol}(Q)\ ;$$

(ii) *for $\lambda > 0$ fixed, as $\max_i|\lambda_i - \lambda| \to 0$, we have the following expansion*

$$\mathbb{E}(\ell(\mathcal{Z}(Q))) = \frac{\lambda}{\pi}\operatorname{vol}(Q)\left(1 + (-1 + \frac{2}{3}\sqrt{\lambda})\sum_{i=1}^{3}(\lambda_i - \lambda)\right) + O(\max_i|\lambda_i - \lambda|^2).$$

The proofs of Theorem 3.1 and Corollary 3.2 are postponed to Section 4. The first item in the corollary is coherent with (3.14) in Berry and Dennis (2000) taking into account that $\lambda = k_3/3$ in Berry and Dennis notation. Besides, we have

$$\mathbb{E}(\ell(\mathcal{Z}(Q))) = \frac{\mathbb{E}|\xi'(0) \wedge \eta'(0)|}{2\pi}\operatorname{vol}(Q),$$

where $\xi'(0) \wedge \eta'(0)$ is the so-called *vorticity*, see (2.2) in Berry and Dennis (2000).

3.2. *Asymptotic variance and distribution.* We restrict ourselves to the isotropic case.

Theorem 3.3 and Proposition 3.5 below consider the case where the radial component Π^{rad} of the power spectrum Π admits a density whereas in Proposition 3.4, Π^{rad} is an atom. Furthermore, they are concerned with two different types of behaviour at infinity of the covariance function. Theorem 3.3 and Proposition 3.4 lead to the same order of magnitude of the normalization term and to the same limit distribution. Proposition 3.5 differs in both aspects.

Let

$$R(x) = \max\left\{|r(x)|, |r_i'(x)|, |r_{ij}''(x)| \,:\, 1 \le i, j \le 3\right\},\ x \in \mathbb{R}^3. \tag{3.1}$$

Theorem 3.3. *Let ψ be an isotropic Berry's random wave defined as in (2.1) and (2.4) such that Π^{rad} admits a density with respect to Lebesgue measure. Assume that $R(x) \to 0$ whenever $|x| \to \infty$ and that $R \in L^2(\mathbb{R}^3)$. Finally, let $Q_n = [-n, n]^3$. Hence,*

(i) *there exists $0 < V < \infty$ such that*

$$\lim_{n\to\infty} \frac{\mathrm{Var}(\ell(\mathcal{Z}(Q_n)))}{\mathrm{vol}(Q_n)} = V;$$

(ii) *as $n \to \infty$, the distribution of*

$$\frac{\ell(\mathcal{Z}(Q_n)) - \mathbb{E}(\ell(\mathcal{Z}(Q_n)))}{\mathrm{vol}(Q_n)^{1/2}}$$

converges towards the centered normal distribution with variance V.

Theorem 3.3 includes Bargmann-Fock, Gamma type and Black-Body models (see Examples 2.1, 2.2 and 2.3) as shown in Section 6.1.

Next proposition concerns Berry's monochromatic model (see Example 2.5). We stress that r, and hence R, is not square integrable on $\mathbb{R}^3$ and Theorem 3.3 does not apply. Nevertheless a similar CLT holds.

Proposition 3.4. *Let ψ be the isotropic Berry's random wave defined as in (2.1) and (2.4) such that $\Pi^{rad} = \frac{1}{4\pi}\delta_1$. Assume also that $Q_n = [-n, n]^3$. Then,*

(i) *there exists $V \in [0, +\infty)$ such that* $\lim_{n\to\infty} \frac{\mathrm{Var}(\ell(\mathcal{Z}(Q_n)))}{\mathrm{vol}(Q_n)} = V$*;*

(ii) *as $n \to \infty$, the distribution of*

$$\frac{\ell(\mathcal{Z}(Q_n)) - \mathbb{E}(\ell(\mathcal{Z}(Q_n)))}{\mathrm{vol}(Q_n)^{1/2}}$$

converges towards the centered normal distribution with variance V.

In the proof of this proposition, as for the proof of Theorem 3.3, we will decompose $\mathrm{Var}(\ell(\mathcal{Z}(Q_n)))$ as the sum of the variances of the so called chaotic components $I_{2q}(Q_n) : q \ge 1$, see (5.4) below. Let us mention that, as observed in the 2-dimensional case Nourdin et al. (2019), the normalized variance of the second chaotic component tends to 0 as $n \to \infty$. In order to prove that $V > 0$ one can study the fourth chaotic component $I_4(Q_n)$ as $n \to \infty$ but the computations become heavy.

In case that $V > 0$ holds true, item (i) in Proposition 3.4 states that the variance of the nodal length on the domain $Q_n \subset \mathbb{R}^3$ grows up to infinity with the same order of magnitude as the volume of Q_n. We believe that the variances of the components $I_{2q}(Q_n) : q \geq 2$ are of the same order as $n \to \infty$. Let us recall that in the 2-dimensional case the variance of the nodal length on a domain $Q \subset \mathbb{R}^2$ is asymptotically proportional to $area(Q) \log(area(Q))$ as Q grows up to $\mathbb{R}^2$ (see Berry (2002); Nourdin et al. (2019)).

Finally, we consider the power law case (see Example 2.4). It follows that $r(x) \approx |x|^{\beta-1}$ as $x \to \infty$. Hence, $r \notin L^2(\mathbb{R}^3)$ and one cannot apply Theorem 3.3. Nevertheless, for $0 < \beta < 1/4$ an asymptotic behaviour can be established as stated in the next proposition.

Proposition 3.5. *Let ψ be the isotropic Berry's random wave defined as in (2.1) and (2.4) such that $\Pi^{rad}(d\rho) = \frac{1-\beta}{4\pi}\rho^{-\beta}\,\mathbb{I}_{(0,1)}(\rho)\,d\rho$ with parameter $\beta \in (0, 1/4)$. Assume also that $Q_n = [-n, n]^3$. Then,*

(i) $\displaystyle \lim_{n\to\infty} \frac{\mathrm{Var}(\ell(\mathcal{Z}(Q_n)))}{\mathrm{vol}(Q_n)^{2(\beta+2)/3}} = V \in (0, +\infty)$

(ii) *as $n \to \infty$, $\displaystyle \frac{\ell(\mathcal{Z}(Q_n)) - \mathbb{E}(\ell(\mathcal{Z}(Q_n)))}{\mathrm{vol}(Q_n)^{(\beta+2)/3}}$ converges in distribution towards a non-Gaussian distribution represented by the double Wiener integral (6.5) below.*

Note the unusual normalizing power of $\mathrm{vol}(Q_n)$ in the first item of Proposition 3.5. Note also that a non-Gaussian limit is appearing in the second item, which is in hard constrast with the preceeding examples.

While proving Proposition 3.5, we will show that the behaviour $\ell(\mathcal{Z}(Q_n)) - \mathbb{E}(\ell(\mathcal{Z}(Q_n)))$ is governed by its projection on the second Wiener chaos and that its limit distribution belongs to this chaos. This fact gives the non-Gaussianity of the limit distribution.

We end this section with some remarks.

Remark 3.6. Performing the isotropic space scaling $x \mapsto \kappa x$ in $\mathbb{R}^3$ for some $\kappa > 0$ yields the next equivalence.

If ψ is an in Theorem 3.3 and if ψ_κ is defined as $\psi_\kappa = \psi(\kappa\,\cdot)$ then, the distribution of

$$\frac{\mathrm{length}(\psi_\kappa^{-1}(0) \cap [-1,1]^3) - \mathbb{E}(\mathrm{length}(\psi_\kappa^{-1}(0) \cap [-1,1]^3))}{\kappa^{1/2}}$$

converges as κ tends to $+\infty$ towards a centered normal distribution with some variance V.

One can see this asymptotics either as an *infill statistics* statement since the performed scaling is nothing but a zooming (see Canzani and Hanin, 2020), or as a *high energy* statement (see Nourdin et al., 2019) since the second spectral moment λ_κ of ψ_κ is such that $\lambda_\kappa = \kappa^2\lambda$ and hence tends to $+\infty$.

Remark 3.7. Most part of our analysis can be carried out similarly in higher dimension, but our motivation (Berry and Dennis, 2000; Dennis, 2007) and examples are 3-dimensional. Equation (2.5) shall be adapted in higher dimension.

4. Expected nodal length

In this section we compute the mean length of the dislocation lines and prove Theorem 3.1 and Corollary 3.2.

We need some further notations. For any $x \in \mathbb{R}^3$, let $Z(x) = (\xi'(x), \eta'(x))$ where $Z(x)$ is sometimes considered as a vector in $\mathbb{R}^6$ and sometimes as a 2×3 matrix.

We also denote

$$\det{}^{\perp} Z(x) = \det{}^{\perp} \begin{pmatrix} \xi_1'(x) & \xi_2'(x) & \xi_3'(x) \\ \eta_1'(x) & \eta_2'(x) & \eta_3'(x) \end{pmatrix},$$

where for any real matrix M, $\det^{\perp} M$ stands for $\det(MM^{\top})$. Routine computation shows that

$$\det{}^{\perp} Z(x) = |\xi'(x) \wedge \eta'(x)|^2. \tag{4.1}$$

This equality is a particular case of the well known Binet-Cauchy formula.

The expectation of $\ell(\mathcal{Z}(Q))$ is given by Rice formula,

$$\begin{aligned} \mathbb{E}[\ell(\mathcal{Z}(Q))] &= \int_Q \mathbb{E}\big[(\det{}^{\perp} Z(x))^{1/2} | \xi(x) = \eta(x) = 0\big] p_{\xi(x),\eta(x)}(0,0)\, dx \\ &= \operatorname{vol}(Q) \frac{1}{2\pi} \mathbb{E}\big[(\det{}^{\perp} Z(0))^{1/2}\big], \end{aligned}$$

where we have used stationarity and independence to get the second line as well as the fact that $\xi(0)$ and $\eta(0)$ are independent standard Gaussian random variables.

Formula (4.1) gives

$$\mathbb{E}[\ell(\mathcal{Z}(Q))] = \frac{\operatorname{vol}(Q)}{2\pi} \mathbb{E}|\xi'(0) \wedge \eta'(0)|. \tag{4.2}$$

Recall that $\operatorname{Cov}(\xi_i'(0), \xi_j'(0)) = \operatorname{Cov}(\eta_i'(0), \eta_j'(0)) = -r_{ij}''(0)$. Without loss of generality (see Adler and Taylor, 2007) we only study the case where

$$-r''(0) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = D.$$

We write $\xi'(0) = D^{\frac{1}{2}} N$ and $\eta'(0) = D^{\frac{1}{2}} N'$, being N and N' two independent $N(0, I_3)$ vectors. Then, using the following algebraic property of the cross product,

$$D^{\frac{1}{2}} N \wedge D^{\frac{1}{2}} N' = (\det D^{\frac{1}{2}}) D^{-\frac{1}{2}} (N \wedge N'),$$

it holds

$$\mathbb{E}[\ell(\mathcal{Z}(Q))] = \operatorname{vol}(Q) \frac{\sqrt{\lambda_1 \lambda_2 \lambda_3}}{2\pi} \mathbb{E}|D^{-\frac{1}{2}} (N \wedge N')|.$$

This proves Theorem 3.1. We now move to the corollary.

(i) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then $D = \lambda I_3$. Furthermore, from Azaïs et al. (2011) page 34, we know that $\mathbb{E}|N \wedge N'| = 2$, thus $\mathbb{E}[\ell(\mathcal{Z}(Q))] = \operatorname{vol}(Q) \frac{\lambda}{\pi}$.

(ii) Let $\lambda > 0$ be fixed and consider $\lambda^* = (\lambda, \lambda, \lambda)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$.

Recall that $Z(0) = (\xi'(0), \eta'(0)) \sim N(0, \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_1, \lambda_2, \lambda_3))$. Then, from (4.2) we have

$$\mathbb{E}[\ell(\mathcal{Z}(Q))] = \frac{\operatorname{vol}(Q)}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |y \wedge y'| p_{\boldsymbol{\lambda}}(y) p_{\boldsymbol{\lambda}}(y') dy dy',$$

where $p_{\boldsymbol{\lambda}}(y) = (2\pi)^{-3/2} (\lambda_1 \lambda_2 \lambda_3)^{-1/2} e^{-\frac{1}{2} \sum_{i=1}^3 (\lambda_i)^{-1/2} y_i^2}$. Hence, for $i = 1, 2, 3$ we get

$$\partial_{\lambda_i} \big(p_{\boldsymbol{\lambda}}(y)\big) = \Big(-\frac{1}{2\lambda_i} + \frac{1}{4(\lambda_i)^{3/2}} y_i^2 \Big) p_{\boldsymbol{\lambda}}(y)$$

and so

$$\partial_{\lambda_i} \mathbb{E}(|\xi' \wedge \eta'|)\big|_{\boldsymbol{\lambda} = \lambda^*} = -\mathbb{E}(|N \wedge N'|) + \frac{\sqrt{\lambda}}{2} \mathbb{E}(|N \wedge N'| (N_i)^2) = -2 + \frac{4}{3} \sqrt{\lambda},$$

being (N, N') a standard normal vector in $\mathbb{R}^6$. Taylor formula allows one to terminate the proof of Corollary 3.2.

5. Hermite expansion and chaotic decomposition

In this section, we introduce preliminary materials that will be useful in the sequel. It mainly deals with Hermite expansion which yields Itô-Wiener's standard chaotic decomposition.

We introduce Hermite polynomials by $H_0(x) = 1$, $H_1(x) = x$ for $x \in \mathbb{R}$ and for $n \geq 2$ by

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x), \quad x \in \mathbb{R}.$$

They form a complete orthogonal system in $L^2(\varphi(dx))$, being φ the standard normal density function in $\mathbb{R}$. More precisely, for standard normal X, Y with covariance ρ it holds

$$\mathbb{E}(H_p(X)H_q(Y)) = \delta_{pq} p! \rho^p, \tag{5.1}$$

being δ_{pq} Kronecker's delta function.

The multi-dimensional Hermite polynomials are tensorial products of their one-dimensional versions. That is, for $\boldsymbol{\alpha} = (\alpha_i)_i \in \mathbb{N}^m$ and $\boldsymbol{y} = (y_i)_i \in \mathbb{R}^m$,

$$\tilde{H}_{\boldsymbol{\alpha}}(\boldsymbol{y}) = \prod_{i=1}^{m} H_{\alpha_i}(y_i).$$

In this case, Hermite polynomials form a complete orthogonal system of $L^2(\varphi_m(d\boldsymbol{y}))$ being φ_m the standard normal density function in $\mathbb{R}^m$. In other words, if $f \in L^2(\varphi_m(d\boldsymbol{y}))$, then f can be written in the L^2-sense as

$$f(\boldsymbol{y}) = \sum_{q=0}^{\infty} \sum_{\boldsymbol{\alpha} \in \mathbb{N}^m,\, |\boldsymbol{\alpha}|=q} f_{\boldsymbol{\alpha}} \tilde{H}_{\boldsymbol{\alpha}}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{R}^m,$$

with $|\boldsymbol{\alpha}| = \sum_{i=1}^{m} \alpha_i$ and

$$f_{\boldsymbol{\alpha}} = \frac{1}{\boldsymbol{\alpha}!} \int_{\mathbb{R}^m} f(\boldsymbol{y}) \tilde{H}_{\boldsymbol{\alpha}}(\boldsymbol{y}) \varphi_m(d\boldsymbol{y}),$$

with $\boldsymbol{\alpha}! = \prod_{i=1}^{m} \alpha_i$.

We are now ready to state the Hermite expansion of the length of the zero set. From now on, we restrict our model to the isotropic case and assume that the second spectral moment λ is positive.

Denote

$$\overline{Y}(x) = \left(\xi(x), \eta(x), \frac{\xi'(x)}{\sqrt{\lambda}}, \frac{\eta'(x)}{\sqrt{\lambda}} \right) \in \mathbb{R}^8.$$

Let also $c_{\boldsymbol{\alpha}} = b_{\alpha_1} b_{\alpha_2} a_{(\alpha_3,\ldots,\alpha_8)}$ being

$$b_\alpha = \frac{1}{\alpha! \sqrt{2\pi}} H_\alpha(0) \tag{5.2}$$

and $a_{(\alpha_3,\ldots,\alpha_8)}$ the Hermite coefficient of $\boldsymbol{y} \in \mathbb{R}^6 \mapsto \det{}^{\perp}(\boldsymbol{y})^{1/2}$.

Proposition 5.1. *With the above notations, it holds in the L^2-sense that*

$$\ell(\mathcal{Z}(Q)) - \mathbb{E}(\ell(\mathcal{Z}(Q))) = \lambda \sum_{q \geq 1} I_{2q}(Q),$$

where

$$I_{2q}(Q) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^8, |\boldsymbol{\alpha}|=2q} c_{\boldsymbol{\alpha}} \int_Q \tilde{H}_{\boldsymbol{\alpha}}(\overline{Y}(x))dx.$$

The proof of this proposition is based on the following standard lemma.

Lemma 5.2. *Consider a compactly supported positive even kernel $h:\mathbb{R}\to\mathbb{R}$ such that $\int h = 1$. For $\varepsilon > 0$, let $h_\varepsilon(x) = \frac{1}{\varepsilon}h(x/\varepsilon)$. Set $\overline{h}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ by $\overline{h}_\varepsilon(x,y) = h_\varepsilon(x)h_\varepsilon(y)$. Define*

$$\ell_\varepsilon = \frac{1}{\varepsilon^2}\int_Q \overline{h}_\varepsilon(\xi(x),\eta(x))(\det{}^\perp(Z(x)))^{1/2}dx.$$

Hence, ℓ_ε converge to $\ell(\mathcal{Z}(Q))$ almost surely and in L^2. Besides, ℓ_ε admits the Hermite (L^2) expansion

$$\ell_\varepsilon = \lambda\sum_{q=1}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathbb{N}^8, |\boldsymbol{\alpha}|=2q} c^\varepsilon_{\boldsymbol{\alpha}} \int_Q \tilde{H}_{\boldsymbol{\alpha}}(\overline{Y}(x))dx, \tag{5.3}$$

being $c^\varepsilon_{\boldsymbol{\alpha}} = b^\varepsilon_{\alpha_1}b^\varepsilon_{\alpha_2}a_{(\alpha_3,\dots,\alpha_8)}$ with $a_{(\alpha_3,\dots,\alpha_8)}$ as above and b^ε_α the Hermite coefficients of h_ε.

Proof: We prove briefly the L^2 convergence.

Let us introduce the level sets

$$\mathcal{Z}^y(Q) = \{x\in\mathbb{R}^3 \,:\, \psi(x) = y\},$$

for $y\in R^2$ and denote its length by

$$\ell(\mathcal{Z}^y(Q)) = \text{length}(\mathcal{Z}^y(Q)).$$

In the first place, the following second order Kac-Rice formula holds (see Azaïs and Wschebor, 2009, Theorem 6.9).

$$\begin{aligned}&\mathbb{E}[\ell(\mathcal{Z}^{y_1}(Q))\ell(\mathcal{Z}^{y_2}(Q))]\\ &= \int_{Q\times Q}\mathbb{E}[(\det{}^\perp Z(x_1)\det{}^\perp Z(x_2))^{\frac{1}{2}}|\psi(x_1)=y_1,\psi(x_2)=y_2]p_{\psi(x_1),\psi(x_2)}(y_1,y_2)dx_1dx_2.\end{aligned}$$

Moreover, the process $\psi(x)$ satisfies all the hypotheses of Theorem 4.1 of Azaïs and León (2020) implying that these expressions are finite and continuous with respect to the level variables y_1, y_2. Using the coarea formula we can write

$$\ell_\varepsilon = \int_{\mathbb{R}^2} h_\varepsilon(y)\ell(\mathcal{Z}(Q))dy = \int_{\mathbb{R}^2} h(u)\ell(\mathcal{Z}^{\varepsilon u}(Q))du.$$

In this manner we can compute

$$\begin{aligned}&\mathbb{E}[(\ell(\mathcal{Z}(Q))) - \ell_\varepsilon)^2]\\ &= \int_{Q\times Q}\mathbb{E}[(\det{}^\perp Z(x_1)\det{}^\perp Z(x_2))^{\frac{1}{2}}|\psi(x_1)=\psi(x_2)=0]p_{\psi(x_1),\psi(x_2)}(0,0)dx_1dx_2\\ &\qquad -2\int_{Q\times Q}\int_{\mathbb{R}^2} h(u)\mathbb{E}[(\det{}^\perp Z(x_1)\det{}^\perp Z(x_2))^{\frac{1}{2}}|\psi(x_1)=0\,\psi(x_2)=\varepsilon u]\\ &\qquad\qquad p_{\psi(x_1),\psi(x_2)}(0,\varepsilon u)dx_1dx_2du\\ &+\int_{Q\times Q}\int_{\mathbb{R}^2\times\mathbb{R}^2} h(u_1)h(u_2)\mathbb{E}[(\det{}^\perp Z(x_1)\det{}^\perp Z(x_2))^{\frac{1}{2}}|\psi(x_1)=\varepsilon u_1\,\psi(x_2)=\varepsilon u_2]\\ &\qquad\qquad p_{\psi(x_1),\psi(x_2)}(\varepsilon u_1,\varepsilon u_2)dx_1dx_2du_1du_2.\end{aligned}$$

The continuity quoted above yields that

$$\mathbb{E}[(\ell(\mathcal{Z})(Q)) - \ell_\varepsilon)^2] \to 0,$$

and the L^2 convergence follows. To obtain the expansion (5.3) one can proceed as in Estrade and León (2016). □

Then Proposition 5.1 is obtained by taking limit when $\varepsilon \to 0$ in L^2.

The orthogonality of the chaotic decomposition given by Proposition 5.1 yields the following expansion for the variance of the zero set length,

$$\mathrm{Var}(\ell(\mathcal{Z}(Q))) = \lambda^2 \sum_{q\geq 1} \mathrm{Var}(I_{2q}(Q)). \tag{5.4}$$

We state a lemma concerning the asymptotic behaviour of this series as $Q \uparrow \mathbb{R}^3$. Recall that function R is defined in (3.1).

Lemma 5.3. *Let $Q_n = [-n,n]^3$. If the covariance function r is isotropic, if $R(x) \to 0$ as $|x| \to \infty$ and R belongs to $L^{2q_0}(\mathbb{R}^3)$ for some positive integer q_0, then there exists $V_{2q_0} \in [0,+\infty)$ such that*

$$\lim_{n\to\infty} \frac{\sum_{q\geq q_0} \mathrm{Var}(I_{2q}(Q_n))}{\mathrm{vol}(Q_n)} = V_{2q_0}.$$

Proof: For simplicity, we normalize R as

$$R(x) = \max\left\{|r(x)|, \frac{|r_i'(x)|}{\sqrt{\lambda}}, \frac{|r_{ij}''(x)|}{\lambda} : 1 \leq i,j \leq 3\right\}. \tag{5.5}$$

The proof follows the same lines as that of Proposition 2.1 in Estrade and León (2016) with minor modifications.

We only detail the part that needs to be adapted. For fixed $q \geq q_0$ we write

$$\mathrm{Var}(I_{2q}(Q_n)) = \lambda^2 \sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=2q} c_{\boldsymbol{\alpha}} c_{\boldsymbol{\beta}} \int_{\mathbb{R}^3} \mathrm{vol}(Q_n \cap Q_n - x) \mathbb{E}[\tilde{H}_{\boldsymbol{\alpha}}(\overline{Y}(0))\tilde{H}_{\boldsymbol{\beta}}(\overline{Y}(x))]\, dx.$$

Using Mehler's formula (see Lemma 10.7 in Azaïs and Wschebor, 2009), we get the next upper bound for any $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathbb{N}^8$ such that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = 2q$,

$$\mathbb{E}[\tilde{H}_{\boldsymbol{\alpha}}(\overline{Y}(0))\tilde{H}_{\boldsymbol{\beta}}(\overline{Y}(x))] = \sum_{\Lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \boldsymbol{\alpha}!\boldsymbol{\beta}! \prod_{1\leq i,j\leq 8} \frac{\mathrm{Cov}(\overline{Y}_i(0)\overline{Y}_j(x))^{d_{ij}}}{d_{ij}!} \leq K_q\, R(x)^{2q},$$

where $\Lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \{d_{ij} \geq 0 : \sum_i d_{ij} = \alpha_j, \sum_j d_{ij} = \beta_i\}$. Here we have used that

$$|\mathrm{Cov}(\overline{Y}_i(0), \overline{Y}_j(x))| \leq R(x), \quad \text{for any } x \in \mathbb{R}^3,$$

and that $\sum_{i,j} d_{ij} = 2q$. Thus, it follows that for any $q \geq q_0$, $\frac{\mathrm{Var}(I_{2q}(Q_n))}{\mathrm{vol}(Q_n)}$ has a finite limit as $n \to \infty$.

The end of the proof is exactly as in Estrade and León (2016). □

In the sequel, the second chaotic component $I_2(Q)$ appears repeatedly. Hence, we end this section analyzing it in detail.

In the next lemma, we do not assume any restrictive condition on the covariance function r, except it is isotropic.

We denote by $e_j \in \mathbb{N}^8$ the j-th canonical vector, that is, the vector all of whose entries are zero but the j-th which is one.

Lemma 5.4. *With the previous notation and assuming r is isotropic, we have*

$$I_2(Q) = \sum_{1\leq k\leq 8} c_{2e_k} \int_Q \tilde{H}_{2e_k}(\overline{Y}(x))dx,$$

with $c_{2e_k} = -\frac{1}{2\pi}$ for $k = 1, 2$ and $c_{2e_k} = \frac{1}{6\pi}$ for $k = 3, \ldots, 8$.

Moreover

$$\mathrm{Var}(I_2(Q)) = \frac{1}{\pi^2} \int_{\mathbb{R}^3} \mathrm{vol}(Q \cap Q - x)\mathcal{D}r(x)dx, \tag{5.6}$$

where the functional $\mathcal{D}$ is defined by

$$\mathcal{D}r(x) = r(x)^2 - \frac{2}{3\lambda}\sum_{j=1}^{3}(r'_j(x))^2 + \frac{1}{9\lambda^2}\sum_{j,l=1}^{3}(r''_{j,l}(x))^2, \quad x \in \mathbb{R}^3,$$

or equivalently, writing $r(x) = \gamma(|x|)$ for some map $\gamma : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{D}r(x) = \gamma(|x|)^2 + \frac{2}{3\lambda}(\frac{1}{3\lambda|x|^2} - 1)\gamma'(|x|)^2 + \frac{1}{9\lambda^2}\gamma''(|x|)^2, \quad x \in \mathbb{R}^3. \tag{5.7}$$

Proof: From Proposition 5.1 we have

$$\begin{aligned} I_2(Q) &= 2\sum_{1\leq i<j\leq 8} c_{e_i+e_j}\int_Q \tilde{H}_{e_i+e_j}(\overline{Y}(x))dx + \sum_{1\leq k\leq 8} c_{2e_k}\int_Q \tilde{H}_{2e_k}(\overline{Y}(x))dx \\ &:= 2\, I_2^{(1)} + I_2^{(2)}, \end{aligned}$$

where we recall that $c_{\boldsymbol{\alpha}} = b_{\alpha_1} b_{\alpha_2} a_{(\alpha_3,\ldots,\alpha_8)}$.

Let us first show that $c_{e_i+e_j} = 0$ for all $1 \leq i < j \leq 8$. This will imply that $I_2^{(1)} = 0$ and hence that $I_2(Q) = I_2^{(2)}$.

From Equation (5.2) it follows that $b_1 = 0$. Thus, $c_{e_i+e_j} = 0$ for $i \leq 2$ and any $j > i$.

Consider i and j in $\{3, \ldots, 8\}$ with $i < j$ and $j - i \neq 3$. Then,

$$\begin{aligned} a_{e_i+e_j} &= \int_{\mathbb{R}^6} (\det{}^\perp \boldsymbol{y})^{1/2} H_1(y_i)H_1(y_j)\varphi_6(\boldsymbol{y})d\boldsymbol{y} \\ &= \mathbb{E}\big(|(N_3, N_4, N_5) \wedge (N_6, N_7, N_8)|N_iN_j\big), \end{aligned}$$

for $N = (N_3, \ldots, N_8)$ standard normal random vector in $\mathbb{R}^6$. Denote by $N' = (N'_3, \ldots, N'_8)$ the vector obtained from N replacing N_i and N_{i+3} by $-N_i$ and $-N_{i+3}$ respectively. It is easy to check that $|(N_3, N_4, N_5) \wedge (N_6, N_7, N_8)| = |(N'_3, N'_4, N'_5) \wedge (N'_6, N'_7, N'_8)|$. Since N and N' are equally distributed, we have

$$\begin{aligned} a_{e_i+e_j} &= \mathbb{E}\Big(|(N'_3, N'_4, N'_5) \wedge (N'_6, N'_7, N'_8)|N'_iN'_j\Big) \\ &= \mathbb{E}\Big(|(N_3, N_4, N_5) \wedge (N_6, N_7, N_8)|(-N_i)N_j\Big) = -a_{e_i+e_j}. \end{aligned}$$

Thus, $a_{e_i+e_j} = c_{e_i+e_j} = 0$ if $i < j \in \{3, \ldots, 8\}$ with $j - i \neq 3$. The same argument but replacing N by $N' = (-N_3, -N_4, -N_5, N_6, N_7, N_8)$ yields $c_{e_3+e_6} = c_{e_4+e_7} = c_{e_5+e_8} = 0$.

Besides, the coefficients c_{2e_k} in $I_2^{(2)}$, $k = 1, \ldots, 8$, can be obtained by routine computations via a change to spherical coordinates.

Finally, we compute the variance of $I_2(Q)$. Note that in $I_2^{(2)}$, the random variables corresponding to $k \in \{1, 3, 4, 5\}$ are independent (and equally distributed) of those corresponding to $k \in \{2,6,7,8\}$. Thus, we consider one of these two blocks.

$$\begin{aligned}\mathrm{Var}(I_2(Q)) &= 2 \sum_{j,l \in \{1,3,4,5\}} c_{2e_j} c_{2e_l} \int_{Q\times Q} \mathbb{E}(H_2(\overline{Y}_j(s))H_2(\overline{Y}_l(t)))dsdt \\ &= 4 \sum_{j,l \in \{1,3,4,5\}} c_{2e_j} c_{2e_l} \int_{Q\times Q} (\mathbb{E}\overline{Y}_j(s)\overline{Y}_l(t))^2 dsdt \\ &= 4 \int_{\mathbb{R}^3} \mathrm{vol}(Q \cap Q - x)\Big(\sum_{j,l \in \{1,3,4,5\}} c_{2e_j} c_{2e_l} (\mathbb{E}\overline{Y}_j(0)\overline{Y}_l(x))^2\Big) dx,\end{aligned}$$

where we have used (5.1) and the stationarity of $\overline{Y}(x)$. Since, the covariances among the coordinates of $\overline{Y}(x)$ are the corresponding derivatives of $r(x)$, the result follows. □

6. Asymptotic variance and limit theorems

In this section, we prove the results of Section 3.2.

6.1. *Square integrable case.* We first prove Theorem 3.3 and we exhibit the examples afterwards.

Proof of Theorem 3.3. Let us assume that the conditions of Theorem 3.3 are satisfied. In order to simplify notations, we write Q instead of $Q_n = [-n,n]^3$ and $Q \uparrow \mathbb{R}^3$ instead of $n \to \infty$. Note that the second spectral moment λ does not vanish since it is equal to $\int_{\mathbb{R}^3} (k_1)^2 \frac{f(\mathbf{k})}{|\mathbf{k}|^2} d\mathbf{k}$, being f the spectral density.
(i) The upper bound for the asymptotic variance follows from Lemma 5.3 with $q_0 = 1$. Thus, it remains to prove that the limit variance is strictly positive.

Recall that Proposition 5.1 yields

$$\mathrm{Var}(\ell(\mathcal{Z}(Q))) = \lambda^2 \sum_{q\geq 1} \mathrm{Var}(I_{2q}(Q)) \geq \lambda^2 \mathrm{Var}(I_2(Q)),$$

and that $\mathrm{Var}(I_{2q}(Q))$ is given by (5.6) in Lemma 5.4.

Since $R \in L^2(\mathbb{R}^3)$ it follows that $\mathcal{D}r \in L^1(\mathbb{R}^3)$. Thus, by Lebesgue's dominated convergence theorem,

$$\lim_{Q\uparrow\mathbb{R}^3} \frac{\mathrm{Var}(I_2(Q))}{\mathrm{vol}(Q)} = \frac{1}{\pi^2} \int_{\mathbb{R}^3} \mathcal{D}r(x)dx.$$

Denoting by f the density of Π, Equation (2.3) now reads

$$r(x) = \int_{\mathbb{R}^3} e^{i\langle \mathbf{k},x\rangle} \frac{f(\mathbf{k})}{|\mathbf{k}|^2} d\mathbf{k}, \quad x \in \mathbb{R}^3.$$

Taking derivatives, we get

$$r'_j(x) = \int_{\mathbb{R}^3} ik_j e^{i\langle \mathbf{k},x\rangle} \frac{f(\mathbf{k})}{|\mathbf{k}|^2} d\mathbf{k}; \quad r''_{j,l}(x) = -\int_{\mathbb{R}^3} k_j k_l e^{i\langle \mathbf{k},x\rangle} \frac{f(\mathbf{k})}{|\mathbf{k}|^2} d\mathbf{k}.$$

Hence, using Plancherel identity, we get

$$\int_{\mathbb{R}^3} \mathcal{D}r(x)dx = \int_{\mathbb{R}^3} \left(1 + \frac{|\mathbf{k}|^2}{3\lambda}\right)^2 \frac{f(\mathbf{k})^2}{|\mathbf{k}|^4} d\mathbf{k} > 0.$$

Statement *(i)* follows.

(ii) From item *(i)*, we know that

$$\frac{\mathrm{Var}(\ell(\mathcal{Z}(Q)))}{\mathrm{vol}(Q)} = \sum_{q\geq 0} \mathrm{Var}\left(\frac{I_{2q}(Q)}{\sqrt{\mathrm{vol}(Q)}}\right) \underset{Q\uparrow\mathbb{R}^3}{\longrightarrow} V_2 < +\infty.$$

Furthermore, in the same form as Proposition 2.1 in Estrade and León (2016), one can prove that

$$\lim_{N\to\infty} \sup_{Q\subset\mathbb{R}^3} \sum_{q\geq N} \mathrm{Var}\left(\frac{I_{2q}(Q)}{\sqrt{\mathrm{vol}(Q)}}\right) = 0.$$

Hence, to establish the CLT for $\ell(\mathcal{Z}(Q))$, it is sufficient to prove the asymptotic normality of each normalized component $I_{2q}(Q)/\sqrt{\mathrm{vol}(Q)}$ as $Q \uparrow \mathbb{R}^3$, see Peccati and Taqqu (2011, Th. 11.8.3). We do this in two steps.

Step 1: We translate the Hermite expansion obtained so far to the framework of isonormal processes, see Peccati and Taqqu (2011, Ch.8) for the details.

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\mathcal{H}_i = L^2(\mathbb{R}^3, \frac{\Pi(d\mathbf{k})}{|\mathbf{k}|^2})$, $i = 1, 2$, endowed with the inner product

$$\langle c \oplus s, c' \oplus s' \rangle_{\mathcal{H}} = \langle c, c' \rangle_{\mathcal{H}_1} + \langle s, s' \rangle_{\mathcal{H}_2} .$$

We also set $I_1^B : \mathcal{H} \to L^2(B) = L^2_{\mathbb{R}}(W)$ by

$$I_1^B(c \oplus s) = I_1^{W_1}(c) + I_1^{W_2}(s),$$

being W_1 and W_2 the real and the imaginary parts of W respectively. It follows that

$$\mathbb{E}\big(I_1^B(c \oplus s)\, I_1^B(c' \oplus s')\big) = \langle c \oplus s, c' \oplus s' \rangle_{\mathcal{H}} .$$

Thus B is a Gaussian isonormal process.

Now, let $h_{i,x}(\mathbf{k}) = c_{i,x}(\mathbf{k}) \oplus s_{i,x}(\mathbf{k}) \in \mathcal{H}$ be such that $\overline{Y}_i(x) = I_1^B(h_{i,x}(\mathbf{k}))$, $i = 1, \ldots, 8$. For instance, since $\overline{Y}_1(x) = \xi(x)$, we have $h_{1,x}(\mathbf{k}) = c_{1,x}(\mathbf{k}) \oplus s_{1,x}(\mathbf{k})$ with

$$c_{1,x}(\mathbf{k}) = \frac{\cos(\mathbf{k}\cdot x)}{|\mathbf{k}|} \quad \text{and} \quad s_{1,x}(\mathbf{k}) = -\frac{\sin(\mathbf{k}\cdot x)}{|\mathbf{k}|}.$$

Let $h_{i,x}(\mathbf{k}) \otimes h_{j,y}(\mathbf{k}') = (c_{i,x} \otimes c_{j,y}(\mathbf{k}, \mathbf{k}')) \oplus (s_{i,x} \otimes s_{j,y}(\mathbf{k}, \mathbf{k}'))$. By definition of the $2q$-folded multiple Wiener integral with respect to B, we get

$$\tilde{H}_{\boldsymbol{\alpha}}(\overline{Y}(x)) = \prod_{i=1}^{8} H_{\alpha_i}(\overline{Y}_i(x)) = I_{2q}^B\left(\otimes_{i=1}^8 h_{i,x}^{\alpha_i}\right),$$

where $|\boldsymbol{\alpha}| = 2q$ and $\otimes_{i=1}^8 h_{i,x}^{\alpha_i} = \otimes_{i=1}^8 h_{i,x}^{\alpha_i}(\mathbf{K})$ stands for the tensorial products of the kernels $h_{i,x}$ for $\mathbf{K} = (\mathbf{k}_1, \ldots, \mathbf{k}_{2q}) \in (\mathbb{R}^3)^{2q}$.

Therefore,

$$I_{2q}(Q) = I_{2q}^B\left(g_{2q}\right),$$

with

$$g_{2q}(\mathbf{K}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^8, |\boldsymbol{\alpha}|=2q} c_{\boldsymbol{\alpha}} \int_Q \otimes_{i=1}^8 h_{i,x}^{\alpha_i}(\mathbf{K}) dx.$$

Step 2: Once that $I_{2q}(Q)$ has been written as a multiple integral, thanks to the fourth moment Theorem (Nourdin and Peccati, 2012, Th. 6.3.1), to establish its asymptotic normality, it suffices to prove that the 2-norms of the so-called contractions of the normalized kernels tend to 0.

Remark 6.1. There are other ways of proving the asymptotic normality of a sequence of random variables living in a fixed chaos, see Nourdin and Peccati (2012); Peccati and Taqqu (2011) for details. We choose contractions since the computations are straightforward in the present case.

Let us recall that for $p \in \mathbb{N}$, symmetric $f, g \in \mathcal{H}^{\otimes p}$ and $1 \le n \le p$, the n-th contraction is defined as

$$f \otimes_n g = \sum_{i_1,\dots,i_n=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle_{\mathcal{H}^{\otimes n}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle_{\mathcal{H}^{\otimes n}} ,$$

being $\{e_i\}_i$ a complete orthogonal system in $\mathcal{H}$. The definition does not depend on the choice of the basis $\{e_i\}_i$. Since each covariance is bounded by R, in order to avoid messy notations we do not symmetrize the kernels in the next lines.

Note that in the case that $f = \otimes_{i=1}^p f_i$ and $g = \otimes_{i=1}^p g_i$, then

$$f \otimes_n g = \prod_{i=1}^{n} \langle f_i, g_i \rangle_{\mathcal{H}} \left(\otimes_{i=1}^{p-n} f_i \otimes \otimes_{i=p-n+1}^{2p-2n} g_i \right). \tag{6.1}$$

In our case, $p = 2q$ and

$$g_{2q} \otimes_n g_{2q} = \sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\alpha}'|=2q} c_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}'} \int_{Q\times Q} \big(\otimes_{i=1}^{8} h_{i,x}^{\alpha_i} \otimes_n \otimes_{i=1}^{8} h_{i,x'}^{\alpha'_i} \big) dx dx'.$$

Besides, from (6.1) we see that the contraction in the last integral yields n inner products (using n kernels with x and n kernels with x') that, since I_1^B is an isonormal process, equal the covariances of the corresponding elements of $\overline{Y}(x)$ and $\overline{Y}(x')$. For instance, $\langle h_{1,x}, h_{1,x'} \rangle_{\mathcal{H}} = \mathbb{E}(\xi(x)\xi(x')) = r(x - x')$. and $\langle h_{1,x}, h_{3,x'} \rangle_{\mathcal{H}} = \mathbb{E}(\xi(x)\xi'_1(x')) = r'_1(x - x')$, etc. Furthermore, it remains 'un-used' $2q - n$ kernels of x and $2q - n$ of x'.

Recall that $R(x) \to 0$ as $|x| \to \infty$ and that $R \in L^2(\mathbb{R}^3)$.

Taking $\mathcal{H}^{4q-2n}$ norms and using the fact that all the covariances of $\overline{Y}$ are bounded by R, we get

$$\left\| \frac{g_{2q}}{\sqrt{\mathrm{vol}(Q)}} \otimes_n \frac{g_{2q}}{\sqrt{\mathrm{vol}(Q)}} \right\|^2 \\ \le \frac{C_q}{\mathrm{vol}(Q)^2} \int_{Q^4} R^n(x - x') R^n(y - y') R^{2q-n}(x - y) R^{2q-n}(x' - y') dx dx' dy dy',$$

where C_q is some constant which takes into account the coefficients $c_{\boldsymbol{\alpha}}$ and the number of terms in the sums.

Now, we make the isometric change of variables $(u_1, u_2, u_3, u_4) \mapsto (x - x', y - y', x - y, x')$. Next, we enlarge the domain of integration to $\widetilde{Q}^4$ so that it includes the image of Q^4 under the change of variables and $\mathrm{vol}(\widetilde{Q}) = c\mathrm{vol}(Q)$ for some constant

c. Hence, we get

$$\begin{aligned}&\left\|\frac{g_{2q}}{\sqrt{\mathrm{vol}(Q)}}\otimes_n\frac{g_{2q}}{\sqrt{\mathrm{vol}(Q)}}\right\|^2\\ &\leq\frac{C_q}{\mathrm{vol}(Q)^2}\int_{\widetilde{Q}^4}R^n(u_1)R^n(u_2)R^{2q-n}(u_3)R^{2q-n}(u_2+u_3-u_1)du_1du_2du_3du_4\\ &\qquad\leq\frac{C_q}{\mathrm{vol}(Q)}\int_{\widetilde{Q}^3}R^n(u_1)R^n(u_2)R^{2q-n}(u_3)R^{2q-n}(u_2+u_3-u_1)du_1du_2du_3.\end{aligned}$$

If $1<n<2q-1$ (thus $q>1$), since $R\in L^2$ it follows that the contractions tend to 0.

Now, assume that $n=1$ and $q=1$ which is the most difficult case. By Cauchy-Schwarz, for fixed u_3 and u_1, $\int_{\widetilde{Q}}R(u_2)R(u_2+u_3-u_1)du_2$ is bounded. Hence, it suffices to prove that as $Q\uparrow\mathbb{R}^3$

$$\frac{1}{\sqrt{\mathrm{vol}(Q)}}\int_Q R(u)du\to 0. \tag{6.2}$$

To see this, take $Q'_n\subset Q_n$ such that $Q'_n\uparrow\mathbb{R}^3$ with $\mathrm{vol}(Q'_n)=o(\sqrt{\mathrm{vol}(Q_n)})$. Thus

$$\begin{aligned}\frac{1}{\sqrt{\mathrm{vol}(Q_n)}}\int_{Q_n}R(u)du&=\frac{1}{\sqrt{\mathrm{vol}(Q_n)}}\int_{Q_n\setminus Q'_n}R(u)du+o(1)\\ &=\frac{\mathrm{vol}(Q_n\setminus Q'_n)}{\sqrt{\mathrm{vol}(Q_n)}}\int_{Q_n\setminus Q'_n}R(u)\frac{du}{\mathrm{vol}(Q_n\setminus Q'_n)}+o(1)\\ &\leq\sqrt{\frac{\mathrm{vol}(Q_n\setminus Q'_n)}{\mathrm{vol}(Q_n)}}\left[\int_{Q_n\setminus Q'_n}R^2(u)du\right]^{1/2}+o(1)\\ &\leq\left[\int_{(Q'_n)^c}R^2(u)du\right]^{1/2}+o(1)\to_n 0,\end{aligned}$$

where the first inequality is due to Jensen's inequality. Hence, (6.2) follows. The remaining cases are similar and easier.

Hence,

$$\left\|\frac{g_{2q}}{\sqrt{\mathrm{vol}(Q)}}\otimes_n\frac{g_{2q}}{\sqrt{\mathrm{vol}(Q)}}\right\|^2\to_{Q\uparrow\mathbb{R}^3}0.$$

This completes the proof of the CLT assertion in Theorem 3.3. □

In order to illustrate the results, we end this section by giving three examples of random waves models that enter in the square integrable case.

Bargmann-Fock model. (see Example 2.1) Here $r(x)=e^{-|x|^2/2}$. Since the covariance function as well as all its derivatives belong to all $L^p(\mathbb{R}^3)$, Theorem 3.3 applies.

Gamma type. (see Example 2.2) For $\beta>0$ and $p\in\mathbb{N}$, we have

$$r(x)=\frac{1}{p}\Big(1+\frac{|x|^2}{\beta^2}\Big)^{-p}\sum_{1\leq j\leq p;\,j\ odd}(-1)^{(j-1)/2}\binom{p}{j}\beta^{-(j-1)}|x|^{j-1}.$$

Concerning integrability properties of r, we note that as $|x| \to \infty$,

$$|r(x)| \approx |x|^{-(p+1)} \text{ if } p \text{ is odd } \; ; \approx |x|^{-(p+2)} \text{ if } p \text{ is even,}$$

where we denote $f(x) \approx g(x)$ for the existence of a positive constant c such that $\lim_{|x|\to\infty} \frac{f(x)}{g(x)} = c$. In the same vein, for odd p, $|r'(x)| \approx |x|^{-(p+2)}$ and $|r''(x)| \approx |x|^{-(p+3)}$ whereas for even p, $|r'(x)| \approx |x|^{-(p+3)}$ and $|r''(x)| \approx |x|^{-(p+4)}$. Hence, for $p \geq 1$, it is clear that r and its derivatives belong to $L^2(\mathbb{R}^3)$ and Theorem 3.3 again applies.

Black-Body radiation. (see Example 2.3) We have

$$r(x) = \frac{c_1}{|x|^2} - \frac{c_2|x|\cosh(|x|)}{\sinh(|x|)^2}.$$

This implies that r and its derivatives are in $L^2(\mathbb{R}^3)$ and Theorem 3.3 once more applies.

6.2. *Berry's monochromatic random waves model.* (see Example 2.5) In this subsection we prove Proposition 3.4.

Proof: (*i*) We write $Q_n = \{nx : x \in Q_1\}$ and

$$\frac{\mathrm{vol}(Q_n \cap (Q_n - x))}{\mathrm{vol}(Q_n)} = \frac{\mathrm{vol}(Q_1 \cap (Q_1 - n^{-1}x))}{\mathrm{vol}(Q_1)} = c(n^{-1}x),$$

where $c : y \in \mathbb{R}^3 \mapsto c(y) := \frac{\mathrm{vol}(Q_1 \cap Q_1 - y)}{\mathrm{vol}(Q_1)}$ is continuous and compactly supported. Then, on the one hand, by Lemma 5.4

$$\frac{\mathrm{Var}(I_2(Q_n))}{\mathrm{vol}(Q_n)} = \frac{1}{\pi^2}\int_{\mathbb{R}^3} c(n^{-1}x)\mathcal{D}r(x)dx = \frac{4}{\pi}\int_{\mathbb{R}^+} C(n^{-1}\rho)D(\rho)\rho^2 d\rho, \tag{6.3}$$

where we have changed to polar coordinates and have set $\mathcal{D}r(x) = D(|x|)$ and $C(\rho) = \frac{1}{4\pi}\int_{\mathbb{S}^2} c(\rho u)d\sigma(u)$. Let us remark that C is compactly supported and that $C(0) = 1$.

On the other hand, since $\lambda = 1/3$ in that case, one can write from (5.7)

$$D(y)\,y^2 = -2\cos(2y) + 4\frac{\sin(2y)}{y} + 6F(y), \tag{6.4}$$

where

$$F(y) = \frac{1}{y^2}(\cos(2y) - \frac{\sin(2y)}{y} + \frac{\sin^2(y)}{y^2})$$

is an integrable function on $\mathbb{R}^+$. We now use (6.4) to split the integral in r.h.s. of (6.3) into three terms:

- Integrating twice by parts the first term yields

$$-2\int_{\mathbb{R}^+} C(n^{-1}y)\cos(2y)\,dy = \frac{1}{n}\big(C'(0) + \int_{\mathbb{R}^+}\cos(2ny)\,C''(y)\,dy\big) \underset{n\to\infty}{\longrightarrow} 0,$$

  where we have used that C' and C'' are compactly supported.
- For the second term, writing $\frac{2\sin(y)}{y}$ as the Fourier transform of the indicator function of $[-1,1]$ and using Parseval identity, one can prove that

$$4\int_{\mathbb{R}^+} C(n^{-1}y)\frac{\sin(2y)}{y}dy \underset{n\to\infty}{\longrightarrow} 4C(0)\frac{\pi}{2} = 2\pi.$$

- We use Lebesgue dominated convergence theorem to get the limit of the last term as n goes to ∞:

$$\int_{\mathbb{R}^+} C(n^{-1}y)\, F(y)\, dy \to \int_{\mathbb{R}^+} F(y)\, dy := J,$$

where a tricky integration by part allows one to get that $J = -\frac{\pi}{3}$.

Finally, we conclude that

$$\int_{\mathbb{R}^+} C(n^{-1}\rho)D(\rho)\rho^2 d\rho \underset{n\to\infty}{\longrightarrow} 0 + 2\pi - 6\frac{\pi}{3} = 0.$$

and hence Part (i) of Proposition 3.4, is now established.

(ii) Let us remark that $R(x)$ behaves like $1/|x|$ as $|x| \to \infty$, so that $R(x) \to 0$ and R belongs to $L^4(\mathbb{R}^3)$. Hence, thanks to Lemma 5.3, we get

$$\lim_{n\to\infty} \frac{\sum_{q\geq 2} \operatorname{Var}(I_{2q}(Q_n))}{\operatorname{vol}(Q_n)} = V_4 \in [0, +\infty).$$

Since $\operatorname{Var}(\ell(\mathcal{Z}(Q)) = \sum_{q\geq 1} \operatorname{Var}(I_{2q}(Q))$, applying (i), we get that $\frac{\operatorname{Var}\ell(\mathcal{Z}(Q_n))}{\operatorname{vol}(Q_n)} \to V_4 < +\infty$. In order to prove the CLT result, we use a similar procedure as for the proof of item (ii) of Theorem 3.3. The difference relies on the fact that the second component $I_2(Q)$ in the chaotic expansion of $\ell(\mathcal{Z}(Q))$ is now negligible with respect to $\sqrt{\operatorname{vol}(Q)}$, so we must only consider the contractions $\frac{g_{2q}}{\sqrt{\operatorname{vol}(Q)}} \otimes_n \frac{g_{2q}}{\sqrt{\operatorname{vol}(Q)}}$ as above for $q > 1$. Since R belongs to $L^4(\mathbb{R}^3)$, the same arguments allow us to conclude. □

6.3. *Power law model.* (see Example 2.4) In this subsection we prove Proposition 3.5.

Proof: (i) Since $r(x) \approx |x|^{\beta-1}$, we have that $\mathcal{D}r(x) \approx |x|^{2\beta-2}$. So, we get for $B(0,n)$ the Euclidean ball in $\mathbb{R}^3$, $\int_{B(0,n)} \mathcal{D}r(x)dx \approx n^{2\beta+1}$ and hence Lemma 5.4 yields

$$\operatorname{Var}(I_2(B(0,n))) \approx (\operatorname{vol}(B(0,n))^{(2\beta+4)/3},\ n \to +\infty.$$

Replacing the ball $B(0,n)$ by the rectangle $[-n,n]^3$ does not change the order of magnitude.

(ii) We now deal with the asymptotic distribution.

On the one hand, since $0 < \beta < 1/4$, one has $R \in L^4(\mathbb{R}^3)$ and Lemma 5.3 does apply with $q_0 = 2$. Then, $\frac{\operatorname{Var}(\sum_{q\geq 2} I_{2q}(Q_n))}{\operatorname{vol}(Q_n)^{(2\beta+4)/3}}$ tends to 0 and hence, in view of the distribution limit of the normalized length, only the second chaotic component is relevant.

On the other hand, by Lemma 5.4, $I_2(Q_n)$ is equal to the sum of two independent random variables with the same distribution. So we only consider one of these terms, namely $\sum_{k=1,3,4,5} c_{2e_k} \int_{Q_n} \tilde{H}_{2e_k}(\overline{Y}(x))dx$.

Thus, the first addend is constructed by using

$$\xi(x) = \int_{\mathbb{R}^3} e^{i<x,\mathbf{k}>} \sqrt{f(|\mathbf{k}|)} \frac{1}{|\mathbf{k}|} dW(\mathbf{k}),$$

being W a standard complex Brownian noise. In particular

$$H_2(\xi(x)) = \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i<x,\mathbf{k}+\mathbf{k}'>} \sqrt{f(|\mathbf{k}|)f(|\mathbf{k}'|)}\frac{1}{|\mathbf{k}|}\frac{1}{|\mathbf{k}'|}dW(\mathbf{k})dW(\mathbf{k}').$$

Considering the derivatives of ξ, which can be written as

$$\xi_j'(x) = i\int_{\mathbb{R}^3} e^{i<x,\mathbf{k}>}\mathbf{k}_j\sqrt{f(|\mathbf{k}|)}\frac{1}{|\mathbf{k}|}dW(\mathbf{k}),\;\; j=1,2,3,$$

we get

$$H_2(\frac{\xi_j'(x)}{\sqrt{\lambda}}) = -\frac{1}{\lambda}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i<x,\mathbf{k}+\mathbf{k}'>}\mathbf{k}_j\mathbf{k}_j'\sqrt{f(|\mathbf{k}|)f(|\mathbf{k}'|)}\frac{1}{|\mathbf{k}|}\frac{1}{|\mathbf{k}'|}dW(\mathbf{k})dW(\mathbf{k}').$$

Introducing the notation

$$g(\mathbf{k},\mathbf{k}') = -\frac{1}{2\pi}(1+\frac{1}{3\lambda}\sum_{j=1}^{3}\mathbf{k}_j\mathbf{k}_j')\sqrt{f(|\mathbf{k}|)f(|\mathbf{k}'|)}\frac{1}{|\mathbf{k}||\mathbf{k}'|},$$

the term of our interest is

$$\begin{aligned} I_2(Q_n) &= \int_{Q_n}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i<x,\mathbf{k}+\mathbf{k}'>}\, g(\mathbf{k},\mathbf{k}')\, dW(\mathbf{k})dW(\mathbf{k}')dx \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3}\Big(\int_{Q_n} e^{i<x,\mathbf{k}+\mathbf{k}'>}dx\Big)\, g(\mathbf{k},\mathbf{k}')\, dW(\mathbf{k})dW(\mathbf{k}') \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} 8n^3\prod_{j=1}^{3}\operatorname{sinc}(n(\mathbf{k}_j+\mathbf{k}_j'))\, g(\mathbf{k},\mathbf{k}')\, dW(\mathbf{k})dW(\mathbf{k}') \\ &\overset{d}{=} \int_{\mathbb{R}^3\times\mathbb{R}^3} 8\prod_{j=1}^{3}\operatorname{sinc}(\mathbf{k}_j+\mathbf{k}_j')\, g\Big(\frac{\mathbf{k}}{n},\frac{\mathbf{k}'}{n}\Big)\, dW(\mathbf{k})dW(\mathbf{k}'), \end{aligned}$$

where the change of variable $(\mathbf{k},\mathbf{k}')\to(n\mathbf{k},n\mathbf{k}')$ as well as the usual scaling property for Brownian measure allowed us to obtain the last identity.

Then, keeping in mind that $f(\rho)=\frac{1-\beta}{4\pi}\rho^{-\beta}\,\mathbb{I}_{(0,1)}(\rho)$, we have

$$n^{-(2+\beta)}\, g\Big(\frac{\mathbf{k}}{n},\frac{\mathbf{k}'}{n}\Big) \underset{n\to\infty}{\to} -\frac{1-\beta}{8\pi^2}(|\mathbf{k}||\mathbf{k}'|)^{-1-\beta/2}.$$

Hence, Theorem 1' of Dobrushin and Major (1979) yields the convergence in distribution of $n^{-(2+\beta)}I_2(Q_n)$ towards

$$-\frac{1-\beta}{\pi^2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\prod_{j=1}^{3}\operatorname{sinc}(\mathbf{k}_j+\mathbf{k}_j')(|\mathbf{k}||\mathbf{k}'|)^{-1-\beta/2}dW(\mathbf{k})dW(\mathbf{k}'). \tag{6.5}$$

This concludes the proof. □

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