

# Spine decomposition and $L \log L$ criterion for superprocesses with non-local branching mechanisms

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**Abstract.** In this paper, we provide a pathwise spine decomposition for superprocesses with both local and non-local branching mechanisms under a martingale change of measure. This result complements earlier results established for superprocesses with purely local branching mechanisms and for multitype superprocesses. As an application of this decomposition, we obtain necessary/sufficient conditions for the limit of the fundamental martingale to be non-degenerate. In particular, we obtain extinction properties of superprocesses with non-local branching mechanisms as well as a Kesten-Stigum  $L \log L$  theorem for the fundamental martingale.

## 1. Introduction

The so-called *spine decomposition* for superprocesses was introduced in terms of a semigroup decomposition by [Evans \(1993\)](#). To be more specific, [Evans \(1993\)](#) described the semigroup of a superprocess with branching mechanism  $\psi(\lambda) = \lambda^2$  under a martingale change of measure in terms of the semigroup of an immortal particle (called the *spine*) and the semigroup of the original superprocess. Since then there has been a lot of interest in finding the spine decomposition for other types of superprocesses due to a variety of applications. For example, [Engländer and Kyprianou \(2004\)](#) used a similar semigroup decomposition to establish the  $L^1$ -convergence of martingales for

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superdiffusions with quadratic branching mechanisms. Later, [Kyprianou et al. \(2012\)](#); [Kyprianou and Murillo-Salas \(2013\)](#) obtained a pathwise spine decomposition for a one-dimensional super-Brownian motion with spatially-independent local branching mechanism, in which independent copies of the original superprocess immigrate along the path of the immortal particle, and they used this decomposition to establish the  $L^p$ -boundedness ( $p \in (1, 2]$ ) of martingales. A similar pathwise decomposition was obtained by [Liu et al. \(2009\)](#) for a class of superdiffusions in bounded domains with spatially-dependent local branching mechanisms, and it was used to establish a Kesten-Stigum  $L \log L$  theorem, which gives the sufficient and necessary condition for the martingale limit to be non-degenerate. In the set-up of branching Markov processes, such as branching diffusions and branching random walks, an analogous decomposition has been introduced and used as a tool to analyze branching Markov processes. See, for example, [Hardy and Harris \(2009\)](#) for a brief history of the spine approach for branching Markov processes. Until very recently such a spine decomposition for superprocesses was only available for superprocesses with local branching mechanisms. In the recent paper [Kyprianou and Palau \(2018\)](#), the authors established a spine decomposition for a multitype continuous-state branching process (MCSBP) and used it to study the extinction properties. Concurrently to their work, a similar decomposition has been obtained by [Chen et al. \(2019a\)](#) for a multitype superdiffusion. However, in both papers, only a very special kind of non-local branching mechanisms is considered. The first goal of this paper is to close the gap by establishing a pathwise spine decomposition for superprocesses with both local and general non-local branching mechanisms.

In this paper, the Schrödinger operator associated with the mean semigroup of the superprocess is characterised by its bilinear form. Then some technical assumptions (Assumptions 1-2 below) are imposed to ensure the existence of a principal eigenvalue  $\lambda_1$  and a positive ground state  $h$ , and hence to ensure the existence of a fundamental martingale (Theorem 3.2 below). These assumptions may look strong, but they hold for a large class of processes, and we illustrate this for several interesting examples, including MCSBP, in Section 7. Our result (Theorem 4.6 below) shows that, for a superprocess with both local and non-local branching, under a martingale change of measure, the spine runs as a copy of a conservative process, which can be constructed by concatenating copies of a subprocess of the  $h$ -transform of the original spatial motion via a transfer kernel determined by the non-local branching mechanism, and the general nature of the branching mechanism induces three different kinds of immigration: the *continuous*, *discontinuous* and *revival-caused* immigration. The concatenating procedure and revival-caused immigration are consequences of non-local branching, and they do not occur when the branching mechanism is purely local.

In connection with the limit theory, it is natural to ask whether or not the limit of the fundamental martingale is non-degenerate. Using the spine decomposition, we establish sufficient and necessary conditions for the martingale limit to be non-degenerate, respectively, in Theorem 5.1 and Theorem 6.2. A similar idea was used in [Engländer and Kyprianou \(2004\)](#); [Liu et al. \(2009\)](#) for (purely local branching) superdiffusions, and in [Kyprianou and Palau \(2018\)](#) for MCSBP. However, in this paper, we extend this idea much further by considering superprocesses where the spatial motion may be discontinuous and the branching mechanism is allowed to be generally non-local. Suppose that  $\{Z_n : n \geq 1\}$  is a Galton-Watson branching process with offspring distribution  $\{p_n : n \geq 0\}$ . Let  $L$  stand for a random variable with this offspring distribution. Let  $m := \sum_{n=0}^{+\infty} np_n$  be the mean of the offspring distribution. Then  $Z_n/m^n$  is a non-negative martingale. [Kesten and Stigum \(1966\)](#) proved that when  $1 < m < +\infty$ , the limit of  $Z_n/m^n$  is non-degenerate if and only if  $E(L \log^+ L) < +\infty$ . This result is usually referred to as the Kesten-Stigum  $L \log L$  theorem. Our Corollary 6.5 shows that, in the case of  $\lambda_1 < 0$ , the martingale limit is non-degenerate if and only if an  $L \log L$ -type condition holds. This result extends an earlier result obtained in [Liu et al. \(2009\)](#) for superdiffusions and can be viewed as a natural analogue of the Kesten-Stigum  $L \log L$  theorem for superprocesses. Our Corollary 6.4 says that, under suitable assumptions, the non-local branching superprocess exhibits weak local extinction if and only if  $\lambda_1 \geq 0$ . This result can be regarded as

a general non-local branching counterpart of Engländer and Kyprianou (2004, Theorem 3), where the same result is proved for a special class of superdiffusions in a domain  $D \subset \mathbb{R}^d$  (the branching mechanism considered in Engländer and Kyprianou (2004) is  $\psi(x, \lambda) = -\beta(x)\lambda + \alpha(x)\lambda^2$  with  $\alpha, \beta$  being Hölder continuous functions in  $D$  with order  $\eta \in (0, 1]$ ,  $\alpha > 0$  in  $D$  and  $\beta$  being bounded from above.)

In this paper we assume the spatial motion to be a symmetric Hunt process on a locally compact separable metric space. This assumption is not really necessary. An extension is possible. One direction is to assume the spatial motion to be a transient Borel right process on a Luzin space, whose Dirichlet form satisfies Silverstein’s sector condition. Definitions of smooth measures and Kato class can then be extended, while still preserving the properties used in this paper. We refer to Chen (2002); Chen and Song (2003a) for Kato class measures defined in this way. Nevertheless, we keep to the less general type of spatial motions to avoid unnecessary technicalities.

The rest of this paper is organized as follows. In Section 2 we review some basic definitions and properties of non-local branching superprocesses, including the definition of Kuznetsov measures which will be used later. In Section 3, we present our main working assumptions and the fundamental martingale. Section 4 provides the spine decomposition and its proof. The proof of Proposition 4.3 is postponed to the Appendix. In Sections 5 and 6 we use the spine decomposition to find sufficient and necessary conditions for the limit of the fundamental martingale to be non-degenerate respectively. In particular, we obtain extinction properties of the non-local branching superprocess as well as a Kesten-Stigum  $L \log L$  theorem for the martingale. In the last section, we give some concrete examples to illustrate our results.

Notation and basic setting: Throughout this paper we use “:=” as a definition. We always assume that  $E$  is a locally compact separable metric space with Borel  $\sigma$ -algebra  $\mathcal{B}(E)$  and  $m$  is a  $\sigma$ -finite measure on  $(E, \mathcal{B}(E))$  with full support. Let  $E_\partial := E \cup \{\partial\}$  be the one-point compactification of  $E$ . Any function  $f$  on  $E$  will be automatically extended to  $E_\partial$  by setting  $f(\partial) = 0$ . For a function  $f$  on  $E$ ,  $\|f\|_\infty := \sup_{x \in E} |f(x)|$  and  $\text{esssup}_{x \in E} f := \inf_{N: m(N)=0} \sup_{x \in E \setminus N} |f(x)|$ . Numerical functions  $f$  and  $g$  on  $E$  are said to be  $m$ -equivalent ( $f = g [m]$  in notation) if  $m(\{x \in E : f(x) \neq g(x)\}) = 0$ . If  $f(x, t)$  is a function on  $E \times [0, +\infty)$ , we say  $f$  is *locally bounded* if  $\sup_{t \in [0, T]} \sup_{x \in E} |f(x, t)| < +\infty$  for every  $T \in (0, +\infty)$ . We denote by  $f^t(\cdot)$  the function  $x \mapsto f(x, t)$ . Let  $\mathcal{M}(E)$  denote the space of finite Borel measures on  $E$  topologized by the weak convergence. Let  $\mathcal{M}(E)^0 := \mathcal{M}(E) \setminus \{0\}$  where  $0$  denotes the null measure on  $E$ . When  $\mu$  is a measure on  $\mathcal{B}(E)$  and  $f, g$  are measurable functions, let  $\langle f, \mu \rangle := \int_E f(x)\mu(dx)$  and  $(f, g) := \int_E f(x)g(x)m(dx)$  whenever the right hand sides make sense. Sometimes we also write  $\mu(f)$  for  $\langle f, \mu \rangle$ . We use  $\mathcal{B}_b(E)$  (respectively,  $\mathcal{B}^+(E)$ ) to denote the space of bounded (respectively, non-negative) measurable functions on  $(E, \mathcal{B}(E))$ . For  $a, b \in \mathbb{R}$ ,  $a \wedge b := \min\{a, b\}$ ,  $a \vee b := \max\{a, b\}$ , and  $\log^+ a := \log(a \vee 1)$ .

## 2. Preliminaries

2.1. *Superprocess with non-local branching mechanisms.* Let

$$\xi = (\Omega, \mathcal{H}, \mathcal{H}_t, \theta_t, \xi_t, \Pi_x, \zeta)$$

be an  $m$ -symmetric Hunt process on  $E$ . Here  $\{\mathcal{H}_t : t \geq 0\}$  is the minimal admissible filtration,  $\{\theta_t : t \geq 0\}$  the time-shift operator of  $\xi$  satisfying  $\xi_t \circ \theta_s = \xi_{t+s}$  for  $s, t \geq 0$ , and  $\zeta := \inf\{t > 0 : \xi_t = \partial\}$  the lifetime of  $\xi$ . Let  $\{\mathfrak{G}_t : t \geq 0\}$  be the transition semigroup of  $\xi$ , i.e., for any non-negative measurable function  $f$ ,

$$\mathfrak{G}_t f(x) := \Pi_x [f(\xi_t)].$$

For  $\alpha > 0$  and  $f \in \mathcal{B}^+(E)$ , let  $G_\alpha f(x) := \int_0^{+\infty} e^{-\alpha t} \mathfrak{G}_t f(x) dt$ . It is known by Chen and Fukushima (2012, Lemma 1.1.14) that  $\{\mathfrak{G}_t : t \geq 0\}$  can be uniquely extended to a strongly continuous contraction semigroup on  $L^2(E, m)$ , which we also denote by  $\{\mathfrak{G}_t : t \geq 0\}$ . By the theory of Dirichlet

forms, there exists a regular symmetric Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(E, m)$  associated with  $\xi$ :

$$\mathcal{F} = \left\{ u \in L^2(E, m) : \sup_{t>0} \frac{1}{t} \int_E (u(x) - \mathfrak{S}_t u(x)) u(x) m(dx) < +\infty \right\},$$

$$\mathcal{E}(u, v) = \lim_{t \rightarrow 0} \frac{1}{t} \int_E (u(x) - \mathfrak{S}_t u(x)) v(x) m(dx), \quad \forall u, v \in \mathcal{F}.$$

Moreover, for all  $f \in \mathcal{B}_b(E) \cap L^2(E, m)$  and  $\alpha > 0$ ,

$$G_\alpha f \in \mathcal{F} \text{ satisfies that } \mathcal{E}_\alpha(G_\alpha f, v) = (f, v) \quad \forall v \in \mathcal{F}, \tag{2.1}$$

where  $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)$ . We assume that  $\xi$  admits a transition density  $p(t, x, y)$  with respect to the measure  $m$ , which is symmetric in  $(x, y)$  for each  $t > 0$ . Under this absolute continuity assumption, ‘‘quasi everywhere’’ statements can be strengthened to ‘‘everywhere’’ ones. Moreover, we can define notions without exceptional sets, for example, *positive continuous additive functionals (PCAF in abbreviation) in the strict sense* (cf. Fukushima et al., 1994, Section 5.1). In this paper, we will only deal with notions in the strict sense and omit ‘‘in the strict sense’’.

It is well known (see Fukushima et al., 1994, Theorem A.3.21, for instance) there exist a kernel  $N(x, dy)$  on  $(E, \mathcal{B}(E))$  with  $N(x, \{x\}) = 0$  for all  $x \in E$  and a PCAF  $H$  of  $\xi$  with  $\int_E \Pi_x(H_t) \mu(dx) < \infty$  for all  $t \geq 0$  and probability measure  $\mu$  on  $(E, \mathcal{B}(E))$  such that for any  $x \in E$ , any  $t \geq 0$ , and any non-negative Borel function  $f$  on  $E \times E$  vanishing on the diagonal  $\{(y, y) : y \in E\}$ ,

$$\Pi_x \left[ \sum_{s \leq t} f(\xi_{s-}, \xi_s) \right] = \Pi_x \left[ \int_0^t \int_E f(\xi_s, y) N(\xi_s, dy) dH_s \right]. \tag{2.2}$$

The pair  $(N, H)$  is called a Lévy system of  $\xi$ .

In this paper, we consider a superprocess  $X := \{X_t : t \geq 0\}$  with spatial motion  $\xi$  and a non-local branching mechanism  $\psi$  given by

$$\psi(x, f) = \phi^L(x, f(x)) + \phi^{NL}(x, f) \quad \text{for } x \in E, f \in \mathcal{B}_b^+(E). \tag{2.3}$$

The first term  $\phi^L$  in (2.3) is called the local branching mechanism and takes the form

$$\phi^L(x, \lambda) = a(x)\lambda + b(x)\lambda^2 + \int_{(0, +\infty)} \left( e^{-\lambda\theta} - 1 + \lambda\theta \right) \Pi^L(x, d\theta), \quad x \in E, \lambda \geq 0, \tag{2.4}$$

where  $a \in \mathcal{B}_b(E)$ ,  $b \in \mathcal{B}_b^+(E)$  and  $(\theta \wedge \theta^2) \Pi^L(x, d\theta)$  is a bounded kernel from  $E$  to  $(0, +\infty)$ . The second term  $\phi^{NL}$  in (2.3) is called the non-local branching mechanism and takes the form

$$\phi^{NL}(x, f) = -c(x)\pi(x, f) - \int_{(0, +\infty)} \left( 1 - e^{-\theta\pi(x, f)} \right) \Pi^{NL}(x, d\theta), \quad x \in E, \tag{2.5}$$

where  $c(x)$  is a non-negative bounded measurable function on  $E$ ,  $\pi(x, dy)$  is a probability kernel on  $E$  with  $\pi(x, \{x\}) \neq 1$ ,  $\pi(x, f)$  stands for  $\int f(y)\pi(x, dy)$  and  $\theta \Pi^{NL}(x, d\theta)$  is a bounded kernel from  $E$  to  $(0, +\infty)$ . To be specific,  $X$  is an  $\mathcal{M}(E)$ -valued Markov process such that for every  $f \in \mathcal{B}_b^+(E)$  and every  $\mu \in \mathcal{M}(E)$ ,

$$P_\mu \left( e^{-\langle f, X_t \rangle} \right) = e^{-\langle u_f(\cdot, t), \mu \rangle} \quad \text{for } t \geq 0, \tag{2.6}$$

where  $u_f(x, t) := -\log P_{\delta_x} \left( e^{-\langle f, X_t \rangle} \right)$  is the unique non-negative locally bounded solution to the integral equation

$$\begin{aligned} u_f(x, t) &= \mathfrak{S}_t f(x) - \Pi_x \left[ \int_0^t \psi(\xi_s, u_f^{t-s}) ds \right] \\ &= \mathfrak{S}_t f(x) - \Pi_x \left[ \int_0^t \phi^L(\xi_s, u_f(t-s, \xi_s)) ds \right] - \Pi_x \left[ \int_0^t \phi^{NL}(\xi_s, u_f^{t-s}) ds \right]. \end{aligned} \tag{2.7}$$

We refer to the process described above as a  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess. Such a process is defined in Li (2011) via its log-Laplace functional. Another usual way of constructing the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess is as the high intensity limit of a sequence of branching particle systems, where whenever a particle dies, it chooses from two different branching types: the local branching type (when the particle dies at  $x$ , it is replaced by a random number of offspring situated at  $x$ ), and the non-local branching type (when the particle dies, it gives birth to a random number of particles in  $E$ , and the offspring then start to move from their locations of birth). We refer to Li (2011); Dawson et al. (2002) for such a construction.

We define for  $x \in E$ ,

$$\gamma(x, dy) := \left( c(x) + \int_{(0, +\infty)} \theta \Pi^{NL}(x, d\theta) \right) \pi(x, dy), \quad \gamma(x) := \gamma(x, 1). \tag{2.8}$$

Clearly,  $\gamma(x)$  is a non-negative bounded function on  $E$  and  $\gamma(x, dy)$  is a bounded kernel on  $E$ . Define  $A := \{x \in E : \gamma(x) > 0\}$ . Note that  $\phi^{NL}(x, \cdot) = 0$  for all  $x \in E \setminus A$ . If  $A = \emptyset$  (i.e.,  $\phi^{NL} \equiv 0$ ), we call  $\psi$  a (purely) local branching mechanism. Without loss of generality, we always assume that  $A \neq \emptyset$ . The arguments and results of this paper also work for (purely) local branching mechanisms.

It follows from Li (2011, Theorem 5.12) that the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess has a right realization in  $\mathcal{M}(E)$ . Let  $\mathcal{W}_0^+$  denote the space of right continuous paths from  $[0, +\infty)$  to  $\mathcal{M}(E)$  having zero as a trap. We may and do assume that  $X$  is the coordinate process in  $\mathcal{W}_0^+$  and that  $(\mathcal{F}_\infty, (\mathcal{F}_t)_{t \geq 0})$  is the natural filtration on  $\mathcal{W}_0^+$  generated by the coordinate process. The following proposition follows from Li (2011, Proposition 2.27 and Proposition 2.29).

**Proposition 2.1.** *For all  $\mu \in \mathcal{M}(E)$  and  $f \in \mathcal{B}_b(E)$ ,*

$$P_\mu(\langle f, X_t \rangle) = \langle \mathfrak{P}_t f, \mu \rangle,$$

where  $\mathfrak{P}_t f(x)$  is the unique locally bounded solution to the following integral equation:

$$\mathfrak{P}_t f(x) = \mathfrak{S}_t f(x) - \Pi_x \left[ \int_0^t a(\xi_s) \mathfrak{P}_{t-s} f(\xi_s) ds \right] + \Pi_x \left[ \int_0^t \gamma(\xi_s, \mathfrak{P}_{t-s} f) ds \right]. \tag{2.9}$$

Moreover, for all  $\mu \in \mathcal{M}(E)$ ,  $g \in \mathcal{B}_b^+(E)$  and  $f \in \mathcal{B}_b(E)$ ,

$$P_\mu(\langle f, X_t \rangle e^{-\langle g, X_t \rangle}) = e^{-\langle V_t g, \mu \rangle} \langle V_t^f g, \mu \rangle,$$

where  $V_t g(x) := u_g(x, t)$  is the unique non-negative locally bounded solution to (2.7) with initial value  $g$ , and  $V_t^f g(x)$  is the unique locally bounded solution to the following integral equation

$$V_t^f g(x) = \mathfrak{S}_t f(x) - \Pi_x \left[ \int_0^t \Psi(\xi_s, V_{t-s} g, V_{t-s}^f g) ds \right], \tag{2.10}$$

where

$$\begin{aligned} \Psi(x, f, g) := & g(x) \left( a(x) + 2b(x)f(x) + \int_{(0, +\infty)} \theta \left( 1 - e^{-f(x)\theta} \right) \Pi^L(x, d\theta) \right) \\ & - \pi(x, g) \left( c(x) + \int_{(0, +\infty)} \theta e^{-\theta \pi(x, f)} \Pi^{NL}(x, d\theta) \right). \end{aligned}$$

**2.2. Kuznetsov measures.** Let  $\{Q_t(\mu, \cdot) := P_\mu(X_t \in \cdot) : t \geq 0, \mu \in \mathcal{M}(E)\}$  be the transition kernel of the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess  $X$ . Then by (2.6), we have

$$\int_{\mathcal{M}(E)} e^{-\langle f, \nu \rangle} Q_t(\mu, d\nu) = \exp(-\langle V_t f, \mu \rangle) \quad \text{for } \mu \in \mathcal{M}(E) \text{ and } t \geq 0.$$

It implies that  $Q_t(\mu_1 + \mu_2, \cdot) = Q_t(\mu_1, \cdot) * Q_t(\mu_2, \cdot)$  for all  $\mu_1, \mu_2 \in \mathcal{M}(E)$ , and hence  $Q_t(\mu, \cdot)$  is an infinitely divisible probability measure on  $\mathcal{M}(E)$ . By the semigroup property of  $Q_t$ ,  $V_t$  satisfies that

$$V_s V_t = V_{t+s} \quad \text{for all } s, t \geq 0.$$

Moreover, by the infinite divisibility of  $Q_t$ , each operator  $V_t$  has the representation

$$V_t f(x) = \lambda_t(x, f) + \int_{\mathcal{M}(E)^0} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, d\nu), \quad t > 0, f \in \mathcal{B}_b^+(E), \quad (2.11)$$

where  $\lambda_t(x, dy)$  is a bounded kernel on  $E$  and  $(1 \wedge \nu(1))L_t(x, d\nu)$  is a bounded kernel from  $E$  to  $\mathcal{M}(E)^0$ . Let  $Q_t^0$  be the restriction of  $Q_t$  to  $\mathcal{M}(E)^0$ . Let

$$E_0 := \{x \in E : \lambda_t(x, E) = 0 \text{ for all } t > 0\}.$$

If  $x \in E_0$ , then we get from (2.11) that

$$V_t f(x) = \int_{\mathcal{M}(E)^0} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, d\nu) \quad \text{for } t > 0, f \in \mathcal{B}_b^+(E).$$

It then follows from Li (2011, Proposition 2.8 and Theorem A.40) that for every  $x \in E_0$ , the family of measures  $\{L_t(x, \cdot) : t > 0\}$  on  $\mathcal{M}(E)^0$  constitutes an entrance law for the restricted semigroup  $\{Q_t^0 : t \geq 0\}$ , and hence there corresponds a unique  $\sigma$ -finite measure  $\mathbb{N}_x$  on  $(\mathcal{W}_0^+, \mathcal{F}_\infty)$  such that  $\mathbb{N}_x(\{0\}) = 0$ , and that for any  $0 < t_1 < t_2 < \dots < t_n < +\infty$ ,

$$\mathbb{N}_x(X_{t_1} \in d\nu_1, X_{t_2} \in d\nu_2, \dots, X_{t_n} \in d\nu_n) = L_{t_1}(x, d\nu_1) Q_{t_2-t_1}^0(\nu_1, d\nu_2) \cdots Q_{t_n-t_{n-1}}^0(\nu_{n-1}, d\nu_n).$$

It immediately follows that for all  $t > 0$  and  $f \in \mathcal{B}_b^+(E)$ ,

$$\mathbb{N}_x \left(1 - e^{-\langle f, X_t \rangle}\right) = \int_{\mathcal{M}(E)^0} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, d\nu) = V_t f(x). \quad (2.12)$$

This measure  $\mathbb{N}_x$  is called the *Kuznetsov measure* corresponding to the entrance law  $\{L_t(x, \cdot) : t > 0\}$  or the *excursion law* for the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess. When  $\mu \in \mathcal{M}(E)$  is supported by  $E_0$  and  $N(dw)$  is a Poisson random measure on  $\mathcal{W}_0^+$  with intensity measure

$$\int_{E_0} \mu(dx) \mathbb{N}_x(\cdot),$$

the process defined by

$$\tilde{X}_0 = \mu; \quad \tilde{X}_t := \int_{\mathcal{W}_0^+} w_t N(dw), \quad t > 0,$$

is a realization of the superprocess  $(X, P_\mu)$ . We refer to Li (2011, section 8.4) for more details on the Kuznetsov measures. In the sequel, we assume that

**Assumption 0.**  $E_+ := \{x \in E : b(x) > 0\} \subset E_0$ .

Under this assumption, the Kuznetsov measure  $\mathbb{N}_x$  exists for every  $x \in E_+$  when  $E_+$  is nonempty. It is established in Chen et al. (2019b) that Assumption 0 is automatically true for superdiffusions with a (purely) local branching mechanism. In the general case, Li (2011, Theorem 8.6) gives the following sufficient condition for Assumption 0: If there is a spatially independent local branching mechanism  $\phi(\lambda)$  taking the form

$$\phi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0, +\infty)} \left(e^{-\lambda\theta} - 1 + \lambda\theta\right) n(d\theta) \quad \text{for } \lambda \geq 0,$$

where  $\alpha \in \mathbb{R}$ ,  $\beta \in \mathbb{R}^+$  and  $(\theta \wedge \theta^2)n(d\theta)$  is a bounded kernel on  $(0, +\infty)$ , such that  $\phi'(\lambda) \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ , and that the branching mechanism  $\psi$  of  $X$  is bounded below by  $\phi$  in the sense that

$$\psi(x, f) \geq \phi(f(x)) \quad \text{for all } x \in E \text{ and } f \in \mathcal{B}_b^+(E),$$

then  $E_0 = E$ .

### 3. Fundamental martingale and weak local extinction

In this section we will establish a fundamental martingale of the form  $e^{\lambda_1 t} \langle h, X_t \rangle$  for the superprocess  $X$  in terms of the principal eigenvalue  $\lambda_1$  and the corresponding positive eigenfunction  $h$  of the Schrödinger operator associated with the mean semigroup. For a MCSBP (resp. a multitype superdiffusion), if one considers the  $E$ -valued spatial motion on an enriched state space  $E \times I$ , where  $I$  is the finite or countable set of types, then the mutation in types is the jumps in the  $I$ -coordinate, and the associated mean semigroup is generated by a matrix (resp. a coupled elliptic system). So, the spectral theory of matrices (resp. the potential theory for elliptic systems) can be applied. See, for example, [Palau and Yang \(2020, Examples 3.7 and 3.8\)](#). For a general non-local branching superprocess considered in Subsection 2.1, the associated Schrödinger operator takes the form  $\mathcal{L} - a + \gamma$ , where  $\mathcal{L}$  is the generator of underlying spatial motion, and  $\gamma$  is an integral operator given by  $\gamma(f)(x) = \gamma(x, f)$ . Since the integral operator  $\gamma$  can be quite general, the method mentioned above is not applicable. Instead we characterize the Schrödinger operator in terms of the associated bilinear form, and impose some technical assumptions to ensure the existence of  $\lambda_1$  and  $h$ .

*Definition 3.1.* We call a non-negative measure  $\mu$  on  $E$  a smooth measure of  $\xi$  if there is a PCAF  $A_t^\mu$  of  $\xi$  such that

$$\int_E f(x) \mu(dx) = \lim_{t \rightarrow 0} \frac{1}{t} \Pi_m \left[ \int_0^t f(\xi_s) dA_s^\mu \right] \quad \text{for all } f \in \mathcal{B}^+(E).$$

Here  $\Pi_m(\cdot) := \int_E \Pi_x(\cdot) m(dx)$ . This measure  $\mu$  is called the Revuz measure of  $A_t^\mu$ . Moreover, we say that a smooth measure  $\mu$  belongs to the Kato class  $\mathbf{K}(\xi)$ , if

$$\limsup_{t \downarrow 0} \int_0^t \int_E p(s, x, y) \mu(dy) ds = 0.$$

A function  $q$  is said to be in the class  $\mathbf{K}(\xi)$  if  $q(x)m(dx)$  is in  $\mathbf{K}(\xi)$ .

Clearly all bounded measurable functions are included in  $\mathbf{K}(\xi)$ . It is known (see, e.g., [Albeverio and Ma, 1991, Proposition 2.1.\(i\)](#) and [Stollmann and Voigt, 1996, Theorem 3.1](#)) that if  $\nu \in \mathbf{K}(\xi)$ , then for every  $\varepsilon > 0$  there is some constant  $A_\varepsilon > 0$  such that

$$\int_E u(x)^2 \nu(dx) \leq \varepsilon \mathcal{E}(u, u) + A_\varepsilon \int_E u(x)^2 m(dx) \quad \forall u \in \mathcal{F}. \tag{3.1}$$

**Assumption 1.**  $\int_E \gamma(x, \cdot) m(dx) \in \mathbf{K}(\xi)$ , where  $\gamma(x, \cdot)$  is the kernel defined in (2.8).

Under Assumption 1, it follows from (3.1), the boundedness of  $\gamma(x)$  and the inequality

$$|u(x)u(y)| \leq \frac{1}{2}(u(x)^2 + u(y)^2)$$

that, for every  $\varepsilon > 0$ , there is a constant  $K_\varepsilon > 0$  such that

$$\int_E \int_E u(x)u(y) \gamma(x, dy) m(dx) \leq \varepsilon \mathcal{E}(u, u) + K_\varepsilon \int_E u(x)^2 m(dx) \quad \forall u \in \mathcal{F}.$$

It follows that the bilinear form  $(\mathcal{Q}, \mathcal{F})$  defined by

$$\mathcal{Q}(u, v) := \mathcal{E}(u, v) + \int_E a(x)u(x)v(x)m(dx) - \int_E \int_E u(y)v(x) \gamma(x, dy) m(dx), \quad u, v \in \mathcal{F},$$

is closed and that there are positive constants  $K$  and  $\beta_0$  such that  $\mathcal{Q}_{\beta_0}(u, u) := \mathcal{Q}(u, u) + \beta_0(u, u) \geq 0$  for all  $u \in \mathcal{F}$ , and

$$|\mathcal{Q}(u, v)| \leq K \mathcal{Q}_{\beta_0}(u, u)^{1/2} \mathcal{Q}_{\beta_0}(v, v)^{1/2} \quad \forall u, v \in \mathcal{F}.$$

It then follows from [Kunita \(1970\)](#) that for such a closed form  $(\mathcal{Q}, \mathcal{F})$  on  $L^2(E, m)$ , there are unique strongly continuous semigroups  $\{T_t : t \geq 0\}$  and  $\{\widehat{T}_t : t \geq 0\}$  on  $L^2(E, m)$  such that  $\|T_t\|_{L^2(E, m)} \leq e^{\beta_0 t}$ ,  $\|\widehat{T}_t\|_{L^2(E, m)} \leq e^{\beta_0 t}$ , and

$$(T_t f, g) = (f, \widehat{T}_t g) \quad \forall f, g \in L^2(E, m). \quad (3.2)$$

Let  $\{U_\alpha\}_{\alpha > \beta_0}$  and  $\{\widehat{U}_\alpha\}_{\alpha > \beta_0}$  be given by  $U_\alpha f := \int_0^{+\infty} e^{-\alpha t} T_t f dt$  and  $\widehat{U}_\alpha f := \int_0^{+\infty} e^{-\alpha t} \widehat{T}_t f dt$  respectively. Then  $\{U_\alpha\}_{\alpha > \beta_0}$  and  $\{\widehat{U}_\alpha\}_{\alpha > \beta_0}$  are strongly continuous pseudo-resolvents in the sense that they satisfy the resolvent equations

$$U_\alpha - U_\beta + (\alpha - \beta)U_\alpha U_\beta = 0, \quad \widehat{U}_\alpha - \widehat{U}_\beta + (\alpha - \beta)\widehat{U}_\alpha \widehat{U}_\beta = 0$$

for all  $\alpha, \beta > \beta_0$ , and

$$\mathcal{Q}_\alpha(U_\alpha f, g) = \mathcal{Q}_\alpha(g, \widehat{U}_\alpha f) = (f, g) \quad \forall f \in L^2(E, m), g \in \mathcal{F}. \quad (3.3)$$

Recall from [Proposition 2.1](#) that  $\mathfrak{P}_t$  is the mean semigroup of the  $(\mathfrak{S}_t, \Phi^L, \Phi^{NL})$ -superprocess, which satisfies the equation [\(2.9\)](#). Since  $\gamma(x, dy)$  is a bounded kernel on  $E$ , by [\(2.9\)](#), we have for every  $f \in \mathcal{B}_b(E)$ ,

$$\|\mathfrak{P}_t f\|_\infty \leq \|f\|_\infty + (\|a\|_\infty + \|\gamma(\cdot, 1)\|_\infty) \int_0^t \|\mathfrak{P}_{t-s} f\|_\infty ds.$$

By Gronwall's lemma,  $\|\mathfrak{P}_t f\|_\infty \leq e^{c_1 t} \|f\|_\infty$  for some constant  $c_1 > 0$ . For  $f \in \mathcal{B}_b(E)$  and  $\alpha > c_1$ , define  $R_\alpha f(x) := \int_0^{+\infty} e^{-\alpha t} \mathfrak{P}_t f(x) dt$ . By taking Laplace transform on both sides of [\(2.9\)](#), we get

$$R_\alpha f(x) = G_\alpha f(x) - G_\alpha(\alpha R_\alpha f)(x) + G_\alpha(\gamma(\cdot, R_\alpha f))(x), \quad (3.4)$$

where  $G_\alpha$  is the  $\alpha$ -resolvent of  $(\mathfrak{S}_t)_{t \geq 0}$ . A particular case is when  $a(x), \gamma(x) \in L^2(E, m)$ . In this case, for all  $f \in \mathcal{B}_b(E) \cap L^2(E, m)$  and  $\alpha$  sufficiently large, both  $a(x)R_\alpha f(x)$  and  $\gamma(x, R_\alpha f)$  are in  $\mathcal{B}_b(E) \cap L^2(E, m)$ . Then it follows from [\(2.1\)](#) that  $G_\alpha f, G_\alpha(\alpha R_\alpha f), G_\alpha(\gamma(\cdot, R_\alpha f)) \in \mathcal{F}$ , and then by [\(3.4\)](#), [\(2.1\)](#) and [\(3.3\)](#),

$$\mathcal{Q}_\alpha(R_\alpha f, v) = (f, v) = \mathcal{Q}_\alpha(U_\alpha f, v) \quad \text{for all } v \in \mathcal{F},$$

which implies that  $R_\alpha f$  is  $m$ -equivalent to  $U_\alpha f$  for  $\alpha$  sufficiently large. This indicates that there is some strong relation between  $\mathfrak{P}_t$  and  $T_t$ . In fact we will show in [Proposition 5.2](#) below that  $\mathfrak{P}_t f = T_t f [m]$  for every  $t > 0$  and every  $f \in \mathcal{B}_b(E) \cap L^2(E, m)$ . This means that  $\mathfrak{P}_t$  can be regarded as a bounded linear operator on the space of bounded measurable functions in  $L^2(E, m)$ , which is dense in  $L^2(E, m)$ . Hence  $T_t$  can be regarded as the unique bounded linear operator on  $L^2(E, m)$  which is an extension of  $\mathfrak{P}_t$ .

**Assumption 2.** There exist a constant  $\lambda_1 \in (-\infty, +\infty)$  and positive functions  $h, \widehat{h} \in \mathcal{F}$  with  $h$  bounded continuous,  $\|h\|_{L^2(E, m)} = 1$  and  $(h, \widehat{h}) = 1$  such that

$$\mathcal{Q}(h, v) = \lambda_1(h, v), \quad \mathcal{Q}(v, \widehat{h}) = \lambda_1(v, \widehat{h}) \quad \forall v \in \mathcal{F}. \quad (3.5)$$

In [Theorem 3.2](#) below, we will prove that  $e^{\lambda_1 t} \langle h, X_t \rangle$  is a non-negative martingale. To prove this we first prove that  $h$  is invariant for some semigroup, see [\(3.13\)](#) below. Since  $h \in \mathcal{F}$  is continuous, it follows from [Chen and Fukushima \(2012, Theorem 4.2.6\)](#) that for every  $x \in E$ ,  $\Pi_x$ -a.s.

$$h(\xi_t) - h(\xi_0) = M_t^h + N_t^h, \quad t \geq 0,$$

where  $M^h$  is a martingale additive functional of  $\xi$  having finite energy and  $N_t^h$  is a continuous additive functional of  $\xi$  having zero energy. The formula above is usually called Fukushima's decomposition. It follows from [\(3.5\)](#) and [Fukushima et al. \(1994, Theorem 5.4.2\)](#) that  $N_t^h$  is of bounded variation, and

$$N_t^h = -\lambda_1 \int_0^t h(\xi_s) ds + \int_0^t a(\xi_s) h(\xi_s) ds - \int_0^t \gamma(\xi_s, h) ds, \quad \forall t \geq 0.$$



Following the idea of [Chen et al. \(2004, Section 2\)](#), we define a local martingale on the random time interval  $[0, \zeta_p)$  by

$$M_t := \int_0^t \frac{1}{h(\xi_{s-})} dM_s^h, \quad t \in [0, \zeta_p), \quad (3.6)$$

where  $\zeta_p$  is the predictable part of the lifetime  $\zeta$  of  $\xi$ , that is,

$$\zeta_p = \begin{cases} \zeta & \text{if } \zeta < \infty \text{ and } \xi_{\zeta-} = \xi_{\zeta}, \\ \infty, & \text{otherwise,} \end{cases}$$

see [Sharpe \(1988, Theorem 44.5\)](#). Then the solution  $H_t$  of the stochastic differential equation

$$H_t = 1 + \int_0^t H_{s-} dM_s, \quad t \in [0, \zeta_p), \quad (3.7)$$

is a positive local martingale on  $[0, \zeta_p)$  and hence a supermartingale. Consequently, the formula

$$d\Pi_x^h = H_t d\Pi_x \quad \text{on } \mathcal{H}_t \cap \{t < \zeta\} \quad \text{for } x \in E$$

uniquely determines a family of subprobability measures  $\{\Pi_x^h : x \in E\}$  on  $(\Omega, \mathcal{H})$ . Hence we have

$$\Pi_x^h[f(\xi_t)] = \Pi_x[H_t f(\xi_t); t < \zeta], \quad t \geq 0, f \in \mathcal{B}^+(E).$$

Note that by (3.6), (3.7) and Doléan-Dade's formula,

$$H_t = \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t\right) \prod_{0 < s \leq t} \frac{h(\xi_s)}{h(\xi_{s-})} \exp\left(1 - \frac{h(\xi_s)}{h(\xi_{s-})}\right) \quad \forall t \in [0, \zeta_p), \quad (3.8)$$

where  $M^c$  is the continuous martingale part of  $M$ . Applying Ito's formula to  $\log h(\xi_t)$ , we obtain that for every  $x \in E$ ,  $\Pi_x$ -a.s. on  $[0, \zeta)$ ,

$$\begin{aligned} \log h(\xi_t) - \log h(\xi_0) &= M_t - \frac{1}{2}\langle M^c \rangle_t + \sum_{s \leq t} \left( \log \frac{h(\xi_s)}{h(\xi_{s-})} - \frac{h(\xi_s) - h(\xi_{s-})}{h(\xi_{s-})} \right) \\ &\quad - \lambda_1 t + \int_0^t a(\xi_s) ds - \int_0^t \frac{\gamma(\xi_s, h)}{h(\xi_s)} ds. \end{aligned} \quad (3.9)$$

Put

$$q(x) := \frac{\gamma(x, h)}{h(x)} \quad \text{for } x \in E. \quad (3.10)$$

By (3.8) and (3.9), we get

$$H_t = \exp\left(\lambda_1 t - \int_0^t a(\xi_s) ds + \int_0^t q(\xi_s) ds\right) \frac{h(\xi_t)}{h(\xi_0)}.$$

To emphasize, the process  $\xi$  under  $\{\Pi_x^h, x \in E\}$  will be denoted as  $\xi^h$ . For a measurable function  $g$ , we set

$$e_g(t) := \exp\left(-\int_0^t g(\xi_s) ds\right) \quad \forall t \geq 0,$$

whenever it is well defined. Then we have for all  $f \in \mathcal{B}^+(E)$  and  $t \geq 0$ ,

$$\mathfrak{S}_t^h f(x) := \Pi_x^h[f(\xi_t^h)] = \frac{e^{\lambda_1 t}}{h(x)} \Pi_x[e_{a-q}(t) h(\xi_t) f(\xi_t)]. \quad (3.11)$$

It follows from [Chen et al. \(2004, Theorem 2.6\)](#) that the transformed process  $\xi^h$  is a conservative and recurrent (in the sense of [Fukushima et al., 1994](#))  $\tilde{m}$ -symmetric right Markov process on  $E$  with  $\tilde{m}(dy) := h(y)^2 m(dy)$ . Thus

$$\mathfrak{S}_t^h 1 = 1[\tilde{m}] \quad \text{for all } t > 0. \quad (3.12)$$

Note that for all  $t > 0$  and  $x \in E$ , the measure  $\mathfrak{S}_t^h(x, \cdot) := \Pi_x^h(\xi_t^h \in \cdot)$  is absolutely continuous with respect to  $\tilde{m}$ , since  $\mathfrak{S}_t^h(x, \cdot)$  is absolutely continuous with respect to the measure  $\mathfrak{S}_t(x, \cdot) := \Pi_x(\xi_t \in \cdot)$  by (3.11) and the latter is absolutely continuous with respect to  $m$ . Moreover, by the right continuity of the sample paths of  $\xi^h$ , one can easily verify that both 1 and  $\mathfrak{S}_t^h 1(x)$  are excessive functions for  $\{\mathfrak{S}_t^h : t > 0\}$ . Thus by [Chen and Fukushima \(2012, Theorem A.2.17\)](#), (3.12) implies that

$$1 = \mathfrak{S}_t^h 1(x) = \frac{e^{\lambda_1 t}}{h(x)} \Pi_x [e_{a-q}(t)h(\xi_t)] \quad \text{for all } x \in E. \tag{3.13}$$

**Theorem 3.2.** *Suppose Assumptions 1-2 hold. Then for every  $\mu \in \mathcal{M}(E)$ ,  $W_t^h(X) := e^{\lambda_1 t} \langle h, X_t \rangle$  is a non-negative  $P_\mu$ -martingale with respect to the filtration  $\{\mathcal{F}_t : t \geq 0\}$ .*

*Proof:* By the Markov property of  $X$ , it suffices to prove that for all  $x \in E$  and  $t \geq 0$ ,

$$\mathfrak{P}_t h(x) = P_{\delta_x} (\langle h, X_t \rangle) = e^{-\lambda_1 t} h(x). \tag{3.14}$$

Let  $A(s, t) := -\int_s^t a(\xi_r)dr + \int_s^t q(\xi_r)dr$  and  $u(t, x) := \Pi_x [e^{A(0,t)}h(\xi_t)]$ . Clearly by (3.13),  $u(t, x) = e^{-\lambda_1 t}h(x)$ . Note that

$$e^{A(0,t)} - 1 = -(e^{A(t,t)} - e^{A(0,t)}) = \int_0^t (-a(\xi_s) + q(\xi_s)) e^{A(s,t)} ds. \tag{3.15}$$

By (3.15), Fubini's theorem and the Markov property of  $\xi$ , we have

$$\begin{aligned} u(t, x) &= \mathfrak{S}_t h(x) + \Pi_x [(e^{A(0,t)} - 1)h(\xi_t)] \\ &= \mathfrak{S}_t h(x) - \Pi_x \left[ \int_0^t a(\xi_s) e^{A(s,t)} h(\xi_t) ds \right] + \Pi_x \left[ \int_0^t q(\xi_s) e^{A(s,t)} h(\xi_t) ds \right] \\ &= \mathfrak{S}_t h(x) - \Pi_x \left[ \int_0^t a(\xi_s) u(t-s, \xi_s) ds \right] + \Pi_x \left[ \int_0^t \frac{\gamma(\xi_s, h)}{h(\xi_s)} u(t-s, \xi_s) ds \right] \\ &= \mathfrak{S}_t h(x) - \Pi_x \left[ \int_0^t a(\xi_s) u(t-s, \xi_s) ds \right] + \Pi_x \left[ \int_0^t \gamma(\xi_s, u^{t-s}) ds \right]. \end{aligned}$$

In the last equality above we use the fact that  $u(t-s, x) = e^{-\lambda_1(t-s)}h(x)$ . Thus  $u(t, x)$  is a locally bounded solution to (2.9) with initial value  $h$ . By the uniqueness of the solution, we get  $u(t, x) = \mathfrak{P}_t h(x) = P_{\delta_x} (\langle h, X_t \rangle)$ .  $\square$

For  $\mu \in \mathcal{M}(E)$ , we say that the process  $X$  exhibits *weak local extinction* (resp. *local extinction*) under  $P_\mu$  if for every nonempty relatively compact open subset  $B$  of  $E$ ,  $P_\mu (\lim_{t \rightarrow +\infty} X_t(B) = 0) = 1$  (resp.  $P_\mu (X_t(B) = 0$  for sufficiently large  $t) = 1$ ). It is proved in [Engländer and Pinsky \(1999\)](#) (see also [Engländer and Kyprianou, 2004](#)) that local extinction and weak local extinction coincide for superdiffusions in a domain  $D \subset \mathbb{R}^d$  with local branching mechanism  $\psi(x, \lambda) = -\beta(x)\lambda + \alpha(x)\lambda^2$ , where  $\alpha$  and  $\beta$  are Hölder continuous functions on  $D$  with order  $\eta \in (0, 1]$ ,  $\alpha > 0$  in  $D$  and  $\beta$  is bounded from above. However the two notions are different in general. In this paper we are only concerned with weak local extinction.

**Corollary 3.3.** *Suppose Assumptions 1-2 hold. For all  $\mu \in \mathcal{M}(E)$  and nonempty relatively compact open subset  $B$  of  $E$ ,*

$$P_\mu \left( \limsup_{t \rightarrow +\infty} e^{\lambda_1 t} X_t(B) < +\infty \right) = 1.$$

*In particular, if  $\lambda_1 > 0$ , then  $X$  exhibits weak local extinction under  $P_\mu$ .*

*Proof:* This corollary follows immediately from Theorem 3.2 and the fact that

$$e^{\lambda_1 t} X_t(B) \leq e^{\lambda_1 t} \left\langle \frac{h}{\inf_{x \in B} h(x)} 1_B, X_t \right\rangle \leq \frac{1}{\inf_{x \in B} h(x)} W_t^h(X).$$

□

*Remark 3.4.* Corollary 3.3 implies that the local mass of  $X_t$  grows subexponentially and the growth rate can not exceed  $-\lambda_1$ . However, when one considers the total mass process  $\langle 1, X_t \rangle$ , the growth rate may actually exceed  $-\lambda_1$ . We refer to Engländer and Kyprianou (2004) and Engländer et al. (2016) for more concrete examples.

#### 4. Spine decomposition

4.1. *Concatenation process.* We assume Assumptions 1-2 hold. It is well-known (see, e.g., Sharpe, 1988, p. 286) that for every  $x \in E$ , there is a unique (up to equivalence in law) right process  $((\hat{\xi}_t)_{t \geq 0}; \hat{\Pi}_x^h)$  on  $E$  with lifetime  $\hat{\zeta}$  and cemetery point  $\partial$ , such that

$$\hat{\Pi}_x^h(\hat{\xi}_t \in B) = \Pi_x^h[e_q(t); \xi_t^h \in B] \quad \forall B \in \mathcal{B}(E),$$

where  $q$  is the nonnegative function defined in (3.10).  $\hat{\xi}$  is called the  $e_q(t)$ -subprocess of  $\xi^h$ , which can be obtained by killing  $\xi^h$  with rate  $q$ . In fact, a version of the  $e_q(t)$ -subprocess can be obtained by the following method of curtailment of the lifetime. Let  $Z$  be an exponential random variable, of parameter 1, independent of  $\xi^h$ . Put

$$\hat{\zeta}(\omega) := \inf\{t \geq 0 : \int_0^t q(\xi_s^h(\omega)) ds \geq Z(\omega)\} (= +\infty, \text{ if such } t \text{ does not exist}),$$

and

$$\hat{\xi}_t(\omega) := \begin{cases} \xi_t^h(\omega) & \text{if } t < \hat{\zeta}(\omega), \\ \partial & \text{if } t \geq \hat{\zeta}(\omega). \end{cases}$$

Then the process  $((\hat{\xi}_t)_{t \geq 0}, \hat{\Pi}_x^h)$  is equal in law to the  $e_q(t)$ -subprocess of  $\xi^h$ . Now we define a probability on  $E$  by

$$\pi^h(x, dy) := \frac{h(y)\gamma(x, dy)}{\gamma(x, h)} = \frac{h(y)\pi(x, dy)}{\pi(x, h)} \quad \text{for } x \in E. \tag{4.1}$$

Let  $\tilde{\xi} := (\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathcal{G}}_t, \tilde{\theta}_t, \tilde{\xi}_t, \tilde{\Pi}_x, \tilde{\zeta})$  be the right process constructed from  $\hat{\xi}$  and the instantaneous distribution  $\kappa(\omega, dy) := \pi^h(\hat{\xi}_{\hat{\zeta}(\omega)-}(\omega), dy)$  by using the so-called “piecing out” procedure (cf. Ikeda et al., 1966). We will follow the terminology of Sharpe (1988, Section II.14) and call  $\tilde{\xi}$  a *concatenation* process defined from an infinite sequence of copies of  $\hat{\xi}$  and the transfer kernel  $\kappa(\omega, dy)$ . One can also refer to Li (2011, Section A.6) for a summary of concatenation processes. The intuitive idea of this concatenation is described as follows. The process  $\tilde{\xi}$  evolves as the process  $\xi^h$  until time  $\hat{\zeta}$ , it is then revived by means of the kernel  $\kappa(\omega, dy)$  and evolves again as  $\xi^h$  and so on, until a countably infinite number of revivals have occurred. Clearly in the case of purely local branching mechanism (i.e.  $\gamma(x) \equiv 0$  on  $E$ ), we have  $\hat{\zeta} = +\infty$  almost surely and hence  $\tilde{\xi}$  runs as a copy of  $\xi^h$ .

Let  $\tilde{\mathfrak{S}}_t$  be the transition semigroup of  $\tilde{\xi}$ . It satisfies the following renewal equation.

$$\tilde{\mathfrak{S}}_t f(x) = \Pi_x^h[e_q(t)f(\xi_t^h)] + \Pi_x^h\left[\int_0^t q(\xi_s^h)e_q(s)\pi^h(\xi_s^h, \tilde{\mathfrak{S}}_{t-s}f)ds\right], \quad f \in \mathcal{B}_b^+(E). \tag{4.2}$$

By Li (2011, Proposition 2.9), the above equation can be rewritten as

$$\tilde{\mathfrak{S}}_t f(x) = \Pi_x^h[f(\xi_t^h)] - \Pi_x^h\left[\int_0^t q(\xi_s^h)\tilde{\mathfrak{S}}_{t-s}f(\xi_s^h)ds\right] + \Pi_x^h\left[\int_0^t q(\xi_s^h)\pi^h(\xi_s^h, \tilde{\mathfrak{S}}_{t-s}f)ds\right].$$

**Proposition 4.1.** For all  $f \in \mathcal{B}_b^+(E)$ ,  $t \geq 0$  and  $x \in E$ ,

$$\tilde{\mathfrak{S}}_t f(x) = \frac{e^{\lambda_1 t}}{h(x)} \mathfrak{P}_t(fh)(x). \tag{4.3}$$

In particular,  $\tilde{\mathfrak{S}}_t 1(x) \equiv 1$ , and hence  $\tilde{\xi}$  has infinite lifetime. Moreover, for each  $t > 0$  and  $x \in E$ ,  $\tilde{\xi}$  has a transition density  $\tilde{p}(t, x, y)$  with respect to the probability measure  $\rho(dy) := h(y)\hat{h}(y)m(dy)$ .

*Proof:* By (4.2), (3.11), (2.8), (3.10) and (4.1), we have

$$\begin{aligned} \tilde{\mathfrak{S}}_t f(x) &= \frac{e^{\lambda_1 t}}{h(x)} \Pi_x [e_a(t)h(\xi_t)f(\xi_t)] \\ &\quad + \frac{e^{\lambda_1 t}}{h(x)} \Pi_x \left[ \int_0^t e_a(s)q(\xi_s)h(\xi_s)e^{-\lambda_1(t-s)}\pi^h(\xi_s, \tilde{\mathfrak{S}}_{t-s}f)ds \right] \\ &= \frac{e^{\lambda_1 t}}{h(x)} \Pi_x [e_a(t)h(\xi_t)f(\xi_t)] \\ &\quad + \frac{e^{\lambda_1 t}}{h(x)} \Pi_x \left[ \int_0^t e_a(s)\gamma(\xi_s, e^{-\lambda_1(t-s)}h\tilde{\mathfrak{S}}_{t-s}f)ds \right]. \end{aligned} \quad (4.4)$$

Let  $u(t, x) := e^{-\lambda_1 t}h(x)\tilde{\mathfrak{S}}_t f(x)$ . Clearly  $u(t, x)$  is a locally bounded function on  $[0, +\infty) \times E$ . Moreover, it follows from (4.4) and Li (2011, Proposition 2.9) that

$$\begin{aligned} u(t, x) &= \Pi_x [e_a(t)h(\xi_t)f(\xi_t)] + \Pi_x \left[ \int_0^t e_a(s)\gamma(\xi_s, u^{t-s})ds \right] \\ &= \Pi_x [h(\xi_t)f(\xi_t)] - \Pi_x \left[ \int_0^t a(\xi_s)u(t-s, \xi_s)ds \right] + \Pi_x \left[ \int_0^t \gamma(\xi_s, u^{t-s})ds \right]. \end{aligned}$$

This implies that  $u(t, x)$  is a locally bounded solution to (2.9) with initial value  $fh$ . Hence we get  $e^{-\lambda_1 t}h(x)\tilde{\mathfrak{S}}_t f(x) = u(t, x) = \mathfrak{P}_t(fh)(x)$  by the uniqueness of the solution. It then follows from (3.14) that  $\tilde{\mathfrak{S}}_t 1(x) \equiv 1$  on  $E$ .

To prove the second part of this proposition, it suffices to prove that for all  $t > 0$  and  $x \in E$ ,  $\tilde{\mathfrak{S}}_t 1_B(x) = 0$  for all  $B \in \mathcal{B}(E)$  with  $\rho(B) = 0$  (or equivalently,  $m(B) = 0$ ). Note that

$$\Pi_x [h1_B(\xi_t)] = \int_B h(y)p(t, x, y)m(dy) = 0.$$

It follows from the above argument that  $e^{-\lambda_1 t}h(x)\tilde{\mathfrak{S}}_t 1_B(x) = \mathfrak{P}_t(h1_B)(x)$  is the unique locally bounded solution to (2.9) with initial value 0. Thus  $\tilde{\mathfrak{S}}_t 1_B(x) \equiv 0$ .  $\square$

*Remark 4.2.* The formula (4.3) can be written as

$$\frac{\mathbb{P}_{\delta_x} [\langle fh, X_t \rangle]}{\mathbb{P}_{\delta_x} [\langle h, X_t \rangle]} = \tilde{\Pi}_x [f(\tilde{\xi}_t)] \quad \text{for } f \in \mathcal{B}_b^+(E) \text{ and } t \geq 0, \quad (4.5)$$

which enables us to calculate the first moment of the superprocess in terms of an auxiliary process. An analogous formula for a special class of non-local branching Markov processes, which is called a ‘‘many-to-one’’ formula, is established in Bansaye et al. (2011), but with a totally different method. By (3.14), we may rewrite (4.5) as

$$\mathbb{P}_{\delta_x} [\langle fh, X_t \rangle] = e^{\lambda_1 t}h(x)\tilde{\Pi}_x [f(\tilde{\xi}_t)] \quad \text{for } f \in \mathcal{B}_b^+(E) \text{ and } t \geq 0.$$

Let  $\tau_1$  be the first revival time of  $\tilde{\xi}$ . For  $n \geq 2$ , define  $\tau_n$  recursively by  $\tau_n := \tau_{n-1} + \tau_1 \circ \tilde{\theta}_{\tau_{n-1}}$ . Since  $\tilde{\xi}$  has infinite lifetime,  $\tilde{\Pi}_x (\lim_{n \rightarrow +\infty} \tau_n = +\infty) = 1$  for all  $x \in E$ .

**Proposition 4.3.** *For all  $f(s, x, y), g(s, x, y) \in \mathcal{B}^+([0, +\infty) \times E \times E)$ ,  $t > 0$  and  $x \in E$ , we have*

$$\tilde{\Pi}_x \left[ \sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] = \tilde{\Pi}_x \left[ \int_0^t ds \int_E \pi^h(\tilde{\xi}_s, dy)q(\tilde{\xi}_s)f(s, \tilde{\xi}_s, y) \right] \quad (4.6)$$

and

$$\begin{aligned}
 & \tilde{\Pi}_x \left[ \left( \sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right) \left( \sum_{\tau_j \leq t} g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \right) \right] \\
 &= \tilde{\Pi}_x \left[ \sum_{\tau_i \leq t} f g(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] + \tilde{\Pi}_x \left[ \int_0^t ds \int_E \pi^h(\tilde{\xi}_s, dy) q(\tilde{\xi}_s) f(s, \tilde{\xi}_s, y) \right. \\
 &\quad \left. \cdot \tilde{\Pi}_y \left( \int_0^{t-s} dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(s+r, \tilde{\xi}_r, z) \right) \right] \\
 &+ \tilde{\Pi}_x \left[ \int_0^t ds \int_E \pi^h(\tilde{\xi}_s, dy) q(\tilde{\xi}_s) g(s, \tilde{\xi}_s, y) \right. \\
 &\quad \left. \cdot \tilde{\Pi}_y \left( \int_0^{t-s} dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) f(s+r, \tilde{\xi}_r, z) \right) \right]. \tag{4.7}
 \end{aligned}$$

The proof of this proposition will be given in the Appendix below.

4.2. *Spine decomposition.* In this section we work under Assumptions 0-2. Recall from Theorem 3.2 that the process  $W_t^h(X)$  is a non-negative  $\mathbb{P}_\mu$ -martingale for every  $\mu \in \mathcal{M}(E)$ . We can define a new probability measure  $\mathbb{Q}_\mu$  for every  $\mu \in \mathcal{M}(E)^0$  by the following formula:

$$d\mathbb{Q}_\mu|_{\mathcal{F}_t} := \frac{1}{\langle h, \mu \rangle} W_t^h(X) d\mathbb{P}_\mu|_{\mathcal{F}_t} \quad \text{for all } t \geq 0.$$

It then follows from Proposition 2.1 that for any  $f \in \mathcal{B}_b^+(E)$  and  $t \geq 0$ ,

$$\mathbb{Q}_\mu \left( e^{-\langle f, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} \mathbb{P}_\mu \left( \langle h, X_t \rangle e^{-\langle f, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^h f, \mu \rangle,$$

where  $V_t^h f(x)$  is the unique locally bounded solution to (2.10) with initial value  $h$ . In this subsection we will establish the spine decomposition of  $X$  under  $\mathbb{Q}_\mu$ .

*Definition 4.4.* For all  $\mu \in \mathcal{M}(E)$  and  $x \in E$ , there is a probability space with probability measure  $\mathbb{P}_{\mu,x}$  that carries the following processes.

- (i)  $((\tilde{\xi}_t)_{t \geq 0}; \mathbb{P}_{\mu,x})$  is equal in law to  $\tilde{\xi}$ , a copy of the concatenation process starting from  $x$ ;
- (ii)  $(n; \mathbb{P}_{\mu,x})$  is a random measure such that, given  $\tilde{\xi}$  starting from  $x$ ,  $n$  is a Poisson random measure which issues  $\mathcal{M}(E)$ -valued processes  $X^{n,t} := (X_s^{n,t})_{s \geq 0}$  at space-time points  $(\tilde{\xi}_t, t)$  with rate

$$d\mathbb{N}_{\tilde{\xi}_t} \times 2b(\tilde{\xi}_t) dt.$$

Here for every  $y \in E_+ = \{z \in E : b(z) > 0\}$ ,  $\mathbb{N}_y$  denotes the Kuznetsov measure on  $\mathcal{W}_0^+$  corresponding to the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess, while for  $y \in E \setminus E_+$ ,  $\mathbb{N}_y$  denotes the null measure on  $\mathcal{W}_0^+$ . Note that, given  $\tilde{\xi}$ , immigration happens only at space-time points  $(\tilde{\xi}_t, t)$  with  $b(\tilde{\xi}_t) > 0$ . Let  $D^n$  denote the almost surely countable set of immigration times, and  $D_t^n := D^n \cap [0, t]$ . Given  $\tilde{\xi}$ , the processes  $\{X^{n,t} : t \in D^n\}$  are mutually independent.

- (iii)  $(m; \mathbb{P}_{\mu,x})$  is a random measure such that, given  $\tilde{\xi}$  starting from  $x$ ,  $m$  is a Poisson random measure which issues  $\mathcal{M}(E)$ -valued processes  $X^{m,t} := (X_s^{m,t})_{s \geq 0}$  at space-time points  $(\tilde{\xi}_t, t)$  with initial mass  $\theta$  at rate

$$\theta \Pi^L(\tilde{\xi}_t, d\theta) \times d\mathbb{P}_{\theta \delta_{\tilde{\xi}_t}} \times dt.$$

Here  $\mathbb{P}_{\theta \delta_x}$  denotes the law of the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess starting from  $\theta \delta_x$ . Let  $D^m$  denote the almost surely countable set of immigration times, and  $D_t^m := D^m \cap [0, t]$ . Given  $\tilde{\xi}$ , the processes  $\{X^{m,t} : t \in D^m\}$  are mutually independent, also independent of  $n$  and  $\{X^{n,t} : t \in D^n\}$ .

- (iv)  $\{((X_s^{r,i})_{s \geq 0}; \mathbb{P}_{\mu,x}), i \geq 1\}$  is a family of  $\mathcal{M}(E)$ -valued processes such that, given  $\tilde{\xi}$  starting from  $x$  (including its revival times  $\{\tau_i : i \geq 1\}$ ),  $X^{r,i} := (X_s^{r,i})_{s \geq 0}$  is equal in law to  $((X_s)_{s \geq 0}, \mathbb{P}_{\pi_i})$  where  $\mathbb{P}_{\pi_i}$  denotes the law of the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess starting from  $\pi_i(\cdot) := \Theta_i \pi(\tilde{\xi}_{\tau_i-}, \cdot)$  and  $\Theta_i$  is a  $[0, +\infty)$ -valued random variable with distribution  $\eta(\tilde{\xi}_{\tau_i-}, d\theta)$  given by

$$\eta(x, d\theta) := \left( \frac{c(x)}{\gamma(x)} 1_A(x) + 1_{E \setminus A}(x) \right) \delta_0(d\theta) + \frac{1}{\gamma(x)} 1_A(x) 1_{(0, +\infty)}(\theta) \theta \Pi^{NL}(x, d\theta). \tag{4.8}$$

Moreover, given  $\tilde{\xi}$  starting from  $x$  (including  $\{\tau_i : i \geq 1\}$ ),  $\{\Theta_i : i \geq 1\}$  are mutually independent,  $\{X^{r,i} : i \geq 1\}$  are mutually independent, also independent of  $n, m, \{X^{n,t} : t \in D^n\}$  and  $\{X^{m,t} : t \in D^m\}$ .

- (v)  $((X_t)_{t \geq 0}; \mathbb{P}_{\mu,x})$  is equal in law to  $((X_t)_{t \geq 0}; \mathbb{P}_\mu)$ , a copy of the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess starting from  $\mu$ . Moreover,  $((X_t)_{t \geq 0}; \mathbb{P}_{\mu,x})$  is independent of  $\tilde{\xi}, n, m$  and all the immigration processes.

We denote by

$$I_t^c := \sum_{s \in D_t^n} X_{t-s}^{n,s}, \quad I_t^d := \sum_{s \in D_t^m} X_{t-s}^{m,s} \quad \text{and} \quad I_t^r := \sum_{\tau_i \leq t} X_{t-\tau_i}^{r,i}$$

the continuous immigration, the discontinuous immigration and the revival-caused immigration, respectively. We define  $\Gamma_t$  by

$$\Gamma_t := X_t + I_t^c + I_t^d + I_t^r, \quad \forall t \geq 0.$$

The process  $\tilde{\xi}$  is called the *spine* process, and the process  $I_t := I_t^c + I_t^d + I_t^r$  is called the *immigration* process.

For any  $\mu \in \mathcal{M}(E)$  and any measure  $\nu$  on  $(E, \mathcal{B}(E))$  with  $0 < \langle h, \nu \rangle < +\infty$ , we randomize the law  $\mathbb{P}_{\mu,x}$  by replacing the deterministic choice of  $x$  with an  $E$ -valued random variable having distribution  $h(x)\nu(dx)/\langle h, \nu \rangle$ . We denote the resulting law by  $\mathbb{P}_{\mu,\nu}$ . That is to say,

$$\mathbb{P}_{\mu,\nu}(\cdot) := \frac{1}{\langle h, \nu \rangle} \int_E \mathbb{P}_{\mu,x}(\cdot) h(x) \nu(dx).$$

Clearly  $\mathbb{P}_{\mu,\delta_x} = \mathbb{P}_{\mu,x}$ . Since the laws of  $X$  and  $(\tilde{\xi}, I)$  under  $\mathbb{P}_{\mu,\nu}$  do not depend on  $\nu$  and  $\mu$  respectively, we sometimes write  $\mathbb{P}_{\mu,\cdot}$  or  $\mathbb{P}_{\cdot,\nu}$ . For simplicity we also write  $\mathbb{P}_\mu$  for  $\mathbb{P}_{\mu,\mu}$ . Here we take the convention that  $\mathbb{P}_0(\Gamma_t = 0 \forall t \geq 0) = 1$ .

For  $s \geq 0$ , define

$$\Lambda_s^m := \langle 1, X_0^{m,s} \rangle, \quad \text{if } s \in D^m \quad \text{and} \quad \Lambda_s^m := 0 \quad \text{otherwise.} \tag{4.9}$$

Then, given  $\tilde{\xi}, \{\Lambda_s^m, s \geq 0\}$  is a Poisson point process with characteristic measure  $\theta \Pi^L(\tilde{\xi}_s, d\theta)$ . Let  $\mathcal{G}$  be the  $\sigma$ -field generated by  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $\{\Theta_i : i \geq 1\}$ ,  $\{D_t^m : t \geq 0\}$ ,  $\{D_t^n : t \geq 0\}$ , and  $\{\Lambda_s^m, s \geq 0\}$ .

**Proposition 4.5.** *For  $\mu \in \mathcal{M}(E)^0, f \in \mathcal{B}_b^+(E)$  and  $t \geq 0$ , we have  $\mathbb{P}_\mu$ -a.s.*

$$\begin{aligned} \mathbb{P}_\mu[\langle f, \Gamma_t \rangle | \mathcal{G}] &= \langle \mathfrak{P}_t f, \mu \rangle + \sum_{s \in D_t^i} \mathfrak{P}_{t-s} f(\tilde{\xi}_s) \\ &+ \sum_{s \in D_t^m} \Lambda_s^m \mathfrak{P}_{t-s} f(\tilde{\xi}_s) + \sum_{\tau_i \leq t} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, \mathfrak{P}_{t-\tau_i} f). \end{aligned} \tag{4.10}$$

*Proof:* By (2.12), we have for every  $x \in E_+ = \{x \in E : b(x) > 0\}$ ,  $f \in \mathcal{B}_b^+(E)$  and  $t > 0$ ,

$$\mathbb{N}_x(\langle f, X_t \rangle) = \mathbb{P}_{\delta_x}(\langle f, X_t \rangle) = \mathfrak{P}_t f(x).$$

Let  $D_t^r := \{\tau_i : \tau_i \leq t\}$ . Then by the definition of  $\Gamma_t$ , under  $\mathbb{P}_\mu$ ,

$$\begin{aligned} \mathbb{P}_\mu[\langle f, \Gamma_t \rangle | \mathcal{G}] &= \mathbb{P}_\mu(\langle f, X_t \rangle) + \sum_{s \in D_t^r} \mathbb{P}_\mu[\langle f, X_{t-s}^{n,s} \rangle | \mathcal{G}] \\ &\quad + \sum_{s \in D_t^m} \mathbb{P}_\mu[\langle f, X_{t-s}^{m,s} \rangle | \mathcal{G}] + \sum_{s=\tau_i \in D_t^r} \mathbb{P}_\mu[\langle f, X_{t-s}^{r,i} \rangle | \mathcal{G}] \\ &= \mathbb{P}_\mu(\langle f, X_t \rangle) + \sum_{s \in D_t^r} \mathbb{N}_{\tilde{\xi}_s}(\langle f, X_{t-s} \rangle) \\ &\quad + \sum_{s \in D_t^m} \mathbb{P}_{\Lambda_s^m \delta_{\tilde{\xi}_s}}(\langle f, X_{t-s} \rangle) + \sum_{s=\tau_i \in D_t^r} \mathbb{P}_{\pi_i}(\langle f, X_{t-s} \rangle) \\ &= \langle \mathfrak{P}_t f, \mu \rangle + \sum_{s \in D_t^r} \mathfrak{P}_{t-s} f(\tilde{\xi}_s) \\ &\quad + \sum_{s \in D_t^m} \Lambda_s^m \mathfrak{P}_{t-s} f(\tilde{\xi}_s) + \sum_{s=\tau_i \in D_t^r} \Theta_i \pi(\tilde{\xi}_{s-}, \mathfrak{P}_{t-s} f). \quad \square \end{aligned}$$

The following is our main result on the spine decomposition for the  $(\mathfrak{S}_t, \phi^L, \phi^{NL})$ -superprocess. Its proof will be given in the next subsection.

**Theorem 4.6.** *Suppose that Assumptions 0-2 hold. For every  $\mu \in \mathcal{M}(E)^0$ , the process  $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$  is Markovian and has the same law as  $((X_t)_{t \geq 0}; \mathbb{Q}_\mu)$ .*

*Remark 4.7.* In the case of a purely local branching mechanism, the revival-caused immigration does not occur. To be more specific, in that case the spine runs as a copy of the  $h$ -transformed process  $\xi^h$  while only continuous and discontinuous immigration occur along the spine. The concatenating procedure and the revival-caused immigration are consequences of non-local branching. Similar phenomenon has been observed in Kyprianou and Palau (2018) for multitype continuous-state branching processes and in Chen et al. (2019a) for multitype superdiffusions.

*Remark 4.8.* The non-local branching mechanism  $\psi$  given by (2.3)-(2.5) is not the most general form that can be assumed to establish a spine decomposition. In fact, we can establish a spine decomposition for the class of branching mechanisms developed in Dawson et al. (2002):

$$\psi(x, f) = \phi^L(x, f(x)) + \phi^{NL}(x, f) \quad \text{for } x \in E, f \in \mathcal{B}_b^+(E),$$

where  $\phi^L$  takes the same form of (2.4) and

$$\begin{aligned} \phi^{NL}(x, f) &= - \int_{\mathcal{P}(E)} c(x, \pi) \pi(f) G(x, d\pi) \\ &\quad - \int_{\mathcal{P}(E)} \int_{(0, +\infty)} (1 - e^{-\theta \pi(f)}) \Pi^{NL}(x, \pi, d\theta) G(x, d\pi), \end{aligned}$$

where  $\mathcal{P}(E)$  denotes the space of probability measures on  $E$ ,  $G(x, d\pi)$  is a probability measure from  $E$  to  $\mathcal{P}(E)$ ,  $c(x, \pi)$  is a nonnegative bounded measurable function on  $E \times \mathcal{P}(E)$ , and  $\theta \Pi^{NL}(x, \pi, d\theta)$  is a bounded kernel from  $E \times \mathcal{P}(E)$  to  $(0, +\infty)$ . It is easy to see that  $\phi^{NL}$  has the form given in (2.5) when  $G(x, d\pi)$  is a Dirac measure on  $\mathcal{P}(E)$ .

To establish the spine decomposition, one should redefine  $\gamma(x, dy)$  as

$$\gamma(x, dy) = \int_{\mathcal{P}(E)} r(x, \pi) \pi(dy) G(x, d\pi),$$

where  $r(x, \pi) := c(x, \pi) + \int_{(0, +\infty)} \theta \Pi^{NL}(x, \pi, d\theta)$ . As a result, the instantaneous distribution, defined by (4.1), of the concatenation process (the spine)  $\tilde{\xi}$ , changes with  $\gamma$  accordingly. Regarding the spine decomposition for the above branching mechanism, there is a new feature of the revival-caused immigration, which is described as follows: Given the spine  $\tilde{\xi}$  (including its revival times), at each revival time  $\tau_i$ , a probability measure  $\pi_i$  is chosen from  $\mathcal{P}(E)$ , independently, according to the distribution  $G^*(\tilde{\xi}_{\tau_i-}, d\pi)$ , where

$$G^*(x, d\pi) := \frac{\pi(h)r(x, \pi)G(x, d\pi)}{\gamma(x, h)}.$$

An immigration  $(X_s^{r,i})_{s \geq 0}$  then occurs at  $\tau_i$ , and it is equal in law to the process  $((X_s)_{s \geq 0}, P_{\Theta_i, \pi_i})$ , where  $\Theta_i$  is an independent  $[0, +\infty)$ -valued random variable with distribution  $\eta(\tilde{\xi}_{\tau_i-}, \pi_i, d\theta)$  given by

$$\begin{aligned} \eta(x, \pi, d\theta) &:= \left( \frac{c(x, \pi)}{r(x, \pi)} 1_{\{r(x, \pi) > 0\}} + 1_{\{r(x, \pi) = 0\}} \right) \delta_0(dy) \\ &+ \frac{\theta 1_{\{\theta \in (0, +\infty)\}} \Pi^{NL}(x, \pi, d\theta)}{r(x, \pi)} 1_{\{r(x, \pi) > 0\}}. \end{aligned}$$

We omit the details of the proof here for brevity.

4.3. *Proof of Theorem 4.6.* In this subsection, we give the proof of Theorem 4.6. In order to do this, we prove a few lemmas first.

**Lemma 4.9.** *For all  $x \in E$ ,  $t \geq 0$  and  $f \in \mathcal{B}_b^+(E)$ ,*

$$\mathbb{P}_{\cdot, x} \left[ \exp \left( -\langle f, I_t^c + I_t^d \rangle \right) \mid \tilde{\xi}_s : 0 \leq s \leq t \right] = \exp \left( - \int_0^t \Phi(\tilde{\xi}_s, V_{t-s} f(\tilde{\xi}_s)) ds \right),$$

where  $\Phi(x, \lambda) := 2b(x)\lambda + \int_{(0, +\infty)} \theta (1 - e^{-\lambda\theta}) \Pi^L(x, d\theta)$  for  $x \in E$  and  $\lambda \geq 0$ .

*Proof:* This lemma follows from an argument which is almost identical to the one leading to (59)–(60) in [Kyprianou et al. \(2012\)](#). We omit the details here.  $\square$

**Lemma 4.10.** *Suppose  $f, l \in \mathcal{B}_b^+(E)$  and  $(x, s) \mapsto g_s(x)$  is a non-negative locally bounded measurable function on  $E \times [0, +\infty)$ . For all  $x \in E$  and  $t > 0$ , let*

$$e^{-w(x,t)} := \mathbb{P}_{\cdot, x} \left[ \exp \left( - \int_0^t g_{t-s}(\tilde{\xi}_s) ds - \langle f, I_t^c \rangle - l(\tilde{\xi}_t) \right) \right].$$

Then  $u(t, x) := e^{-\lambda_1 t} h(x) e^{-w(x,t)}$  satisfies the following integral equation:

$$\begin{aligned} u(t, x) &= \Pi_x \left[ e^{-l(\xi_t)} h(\xi_t) \right] + \Pi_x \left[ \int_0^t ds (\Phi(\xi_s, V_{t-s} f(\xi_s)) u(t-s, \xi_s) \right. \\ &\quad \left. - \Psi(\xi_s, V_{t-s} f, u^{t-s}) - g_{t-s}(\xi_s) u(t-s, \xi_s) \right), \end{aligned} \tag{4.11}$$

where  $\Psi$  and  $\Phi$  are defined in [Proposition 2.1](#) and [Lemma 4.9](#) respectively.

*Proof:* Following the idea of [Evans and O’Connell \(1994\)](#), it suffices to prove the result in the case when  $g$  does not depend on the time variable. Let  $\tau_1$  denote the first revival time of  $\tilde{\xi}$ . We have the following fundamental equation:

$$\begin{aligned} e^{-w(x,t)} &= \Pi_x^h \left[ e_{q+g}(t) e^{-l(\xi_t^h)} \right] + \Pi_x^h \left[ \int_0^t ds q(\xi_s^h) e_{q+g}(s) \right. \\ &\quad \left. \cdot \pi^h(\xi_s^h, e^{-w_{t-s}}) \int_{[0, +\infty)} e^{-\theta \pi(\xi_s^h, V_{t-s} f)} \eta(\xi_s^h, d\theta) \right]. \end{aligned}$$



The first term corresponds to the case when  $\tau_1 \geq t$ , and the second term corresponds to the case when the first revival happens at time  $s \in (0, t)$ . It then follows from Fubini's theorem and (3.11) that

$$e^{-\lambda_1 t} h(x) e^{-w(x,t)} = \Pi_x \left[ e_{a+g}(t) h(\xi_t) e^{-l(\xi_t)} \right] + \Pi_x \left[ \int_0^t ds e_{a+g}(s) q(\xi_s) h(\xi_s) \cdot \pi^h(\xi_s, e^{-\lambda_1(t-s)} e^{-w_{t-s}}) \int_{[0,+\infty)} e^{-\theta \pi(\xi_s, V_{t-s} f)} \eta(\xi_s, d\theta) \right].$$

We continue the above calculation by Li (2011, Proposition 2.9) and (4.8) to get

$$\begin{aligned} u(t, x) &= e^{-\lambda_1 t} h(x) e^{-w(x,t)} \\ &= \Pi_x \left[ h(\xi_t) e^{-l(\xi_t)} \right] - \Pi_x \left[ \int_0^t (a(\xi_s) + g(\xi_s)) e^{-\lambda_1(t-s)} h(\xi_s) e^{-w(\xi_s, t-s)} ds \right] \\ &\quad + \Pi_x \left[ \int_0^t ds q(\xi_s) h(\xi_s) e^{-\lambda_1(t-s)} \pi^h(\xi_s, e^{-w_{t-s}}) \int_{[0,+\infty)} e^{-\theta \pi(\xi_s, V_{t-s} f)} \eta(\xi_s, d\theta) \right] \\ &= \Pi_x \left[ h(\xi_t) e^{-l(\xi_t)} \right] - \Pi_x \left[ \int_0^t (a(\xi_s) + g(\xi_s)) e^{-\lambda_1(t-s)} h(\xi_s) e^{-w(\xi_s, t-s)} ds \right] \\ &\quad + \Pi_x \left[ \int_0^t ds \pi(\xi_s, e^{-\lambda_1(t-s)} h e^{-w_{t-s}}) \cdot \left( c(\xi_s) + \int_{(0,+\infty)} r e^{-\theta \pi(\xi_s, V_{t-s} f)} \Pi^{NL}(\xi_s, d\theta) \right) \right]. \end{aligned}$$

This directly leads to (4.11). □

**Lemma 4.11.** For all  $f, g \in \mathcal{B}_b^+(E)$ ,  $\mu \in \mathcal{M}(E)$ ,  $x \in E$  and  $t \geq 0$ ,

$$\mathbb{P}_{\mu, x} \left[ \exp \left( -\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t) \right) \right] = \frac{e^{\lambda_1 t}}{h(x)} e^{-\langle V_t f, \mu \rangle} V_t^{he^{-g}} f(x), \tag{4.12}$$

where  $V_t^{he^{-g}} f(x)$  is the unique locally bounded solution to (2.10) with initial value  $he^{-g}$ .

*Proof:* Recall from Definition 4.4 that  $(X; \mathbb{P}_{\mu, x})$  is independent of  $\tilde{\xi}$  and all the immigration processes. Moreover, given  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $I^r$  is independent of  $I^c$  and  $I^d$ . It then follows from Lemma 4.9 that

$$\begin{aligned} &\mathbb{P}_{\mu, x} \left[ \exp \left( -\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t) \right) \right] \\ &= \mathbb{P}_{\mu, x} \left[ \exp \left( -\langle f, X_t \rangle - \langle f, I_t^c + I_t^d \rangle - \langle f, I_t^r \rangle - g(\tilde{\xi}_t) \right) \right] \\ &= \mathbb{P}_{\mu, x} \left[ e^{-\langle f, X_t \rangle} \mathbb{P}_{\mu, x} \left\{ e^{-g(\tilde{\xi}_t)} \mathbb{P}_{\mu, x} \left[ \exp \left( -\langle f, I_t^c + I_t^d \rangle \right) \mid \tilde{\xi}_s : 0 \leq s \leq t \right] \right. \right. \\ &\quad \left. \left. \cdot \mathbb{P}_{\mu, x} \left[ \exp \left( -\langle f, I_t^r \rangle \right) \mid \{\tilde{\xi}_s : 0 \leq s \leq t\} \cup \{\tau_i : \tau_i \leq t\} \right] \right\} \right] \\ &= e^{-\langle V_t f, \mu \rangle} \mathbb{P}_{\mu, x} \left[ \exp \left( -\int_0^t \Phi(\tilde{\xi}_s, V_{t-s} f(\tilde{\xi}_s)) ds - \langle f, I_t^r \rangle - g(\tilde{\xi}_t) \right) \right]. \end{aligned} \tag{4.13}$$

Let  $v(t, x) := e^{-\lambda_1 t} h(x) \mathbb{P}_{\mu, x} \left[ \exp \left( -\int_0^t \Phi(\tilde{\xi}_s, V_{t-s} f(\tilde{\xi}_s)) ds - \langle f, I_t^r \rangle - g(\tilde{\xi}_t) \right) \right]$ . One can easily verify that  $(x, s) \mapsto g_s(x) := \Phi(x, V_s f(x))$  is a locally bounded function. Thus by Lemma 4.10,  $v(t, x)$  is a locally bounded solution to the equation (2.10) with initial value  $he^{-g}$ . By the uniqueness of such a solution, we have  $v(t, x) = V_t^{he^{-g}} f(x)$ . This and (4.13) lead to (4.12). □

**Proof of Theorem 4.6:** First we claim that for every  $\mu \in \mathcal{M}(E)^0$ ,  $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$  has the same one dimensional distribution as  $((X_t)_{t \geq 0}; \mathbb{Q}_\mu)$ . This would follow if for every  $f \in \mathcal{B}_b^+(E)$  and every  $t \geq 0$ ,

$$\mathbb{P}_\mu \left( e^{-\langle f, \Gamma_t \rangle} \right) = \mathbb{Q}_\mu \left( e^{-\langle f, X_t \rangle} \right). \tag{4.14}$$

By the definition of  $Q_\mu$  and Proposition 2.1,

$$Q_\mu \left( e^{-\langle f, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} P_\mu \left[ \langle h, X_t \rangle e^{-\langle f, X_t \rangle} \right] = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^h f, \mu \rangle, \quad (4.15)$$

where  $V_t^h f(x)$  is the unique locally bounded solution to (2.10) with initial value  $h$ . By Lemma 4.11, we have

$$\mathbb{P}_{\mu, x} [\exp(-\langle f, \Gamma_t \rangle)] = \exp(\lambda_1 t - \langle V_t f, \mu \rangle) h(x)^{-1} V_t^h f(x).$$

Thus

$$\begin{aligned} \mathbb{P}_\mu \left( e^{-\langle f, \Gamma_t \rangle} \right) &= \frac{1}{\langle h, \mu \rangle} \int_E \mathbb{P}_{\mu, x} \left( e^{-\langle f, \Gamma_t \rangle} \right) h(x) \mu(dx) \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^h f, \mu \rangle. \end{aligned} \quad (4.16)$$

Combining (4.15) and (4.16), we get (4.14). It follows that for every  $\mu \in \mathcal{M}(E)^0$ ,

$$\begin{aligned} \mathbb{P}_\mu(\Gamma_t = 0) &= Q_\mu(X_t = 0) \\ &= \frac{1}{\langle h, \mu \rangle} P_\mu \left( W_t^h(X); X_t = 0 \right) = 0 \quad \forall t > 0. \end{aligned} \quad (4.17)$$

It remains to prove the Markov property of  $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$ . To do this, we apply Evans and O'Connell (1994, Lemma 3.3) here. Recall that  $E_\partial = E \cup \{\partial\}$  where  $\partial$  is a cemetery point. We can extend the probability measure  $\mathbb{P}_{\mu, x}$  onto  $\mu \times \{\partial\}$  by defining that  $\mathbb{P}_{\mu, \partial}(\tilde{\xi}_t = \partial, I_t = 0 \forall t \geq 0) = 1$  for all  $\mu \in \mathcal{M}(E)$ . In the remainder of this proof, we call  $\mathfrak{J}$  a Markov kernel if  $\mathfrak{J}$  is a map from the measurable space  $(S, \mathcal{S})$  to the measurable space  $(S', \mathcal{S}')$  such that for every  $y \in S$ ,  $\mathfrak{J}(y, \cdot)$  is a probability measure on  $(S', \mathcal{S}')$ , and for every  $B \in \mathcal{S}'$ ,  $\mathfrak{J}(\cdot, B) \in b\mathcal{S}$  the space of bounded measurable functions on  $S$ . The kernel  $\mathfrak{J}$  will also be viewed as an operator taking  $f \in b\mathcal{S}'$  to  $\mathfrak{J}f \in b\mathcal{S}$  where  $\mathfrak{J}f(y) := \int_{S'} f(z) \mathfrak{J}(y, dz)$ .

Clearly  $((Z_t)_{t \geq 0} := ((\Gamma_t, \tilde{\xi}_t)_{t \geq 0}; \mathbb{P}_{\mu, x})$  is a Markov process on  $\mathcal{M}(E) \times E_\partial$ . Denote by  $S_t$  the transition semigroup of  $Z_t$ , by  $\mathfrak{K}$  the Markov kernel from  $\mathcal{M}(E) \times E_\partial$  to  $\mathcal{M}(E)$  induced by the projection from  $\mathcal{M}(E) \times E_\partial$  onto  $\mathcal{M}(E)$ , and by  $\mathfrak{Q}$  the Markov kernel from  $\mathcal{M}(E)$  to  $\mathcal{M}(E) \times E_\partial$  given by

$$\mathfrak{Q}(\nu_1, d(\nu_2 \times x)) := 1_{\{\nu_1 \neq 0\}} \delta_{\nu_1}(d\nu_2) \times 1_E(x) \frac{h(x) \nu_1(dx)}{\langle h, \nu_1 \rangle} + 1_{\{\nu_1 = 0\}} \delta_0(d\nu_2) \times \delta_\partial(dx).$$

Let  $R_t := \mathfrak{Q} S_t \mathfrak{K}$  for  $t \geq 0$ . One can easily verify that  $\mathfrak{Q} \mathfrak{K}$  is the identity kernel on  $\mathcal{M}(E)$  and  $R_t(\nu_1, d\nu_2) = \mathbb{P}_{\nu_1}(\Gamma_t \in d\nu_2)$  for all  $\nu_1 \in \mathcal{M}(E)$ . By Evans and O'Connell (1994, Lemma 3.3),  $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$  is Markovian as long as  $\mathfrak{Q} S_t = R_t \mathfrak{Q}$ . This would follow if for all  $f, g \in \mathcal{B}_b^+(E)$  and  $\nu_1 \in \mathcal{M}(E)$ ,

$$\begin{aligned} &\int_{\mathcal{M}(E)} \int_{\mathcal{M}(E) \times E_\partial} e^{-\langle f, \nu_3 \rangle - g(y)} \mathfrak{Q}(\nu_2, d(\nu_3 \times y)) R_t(\nu_1, d\nu_2) \\ &= \int_{\mathcal{M}(E) \times E_\partial} \int_{\mathcal{M}(E) \times E_\partial} e^{-\langle f, \nu_3 \rangle - g(y)} S_t(\nu_2 \times x, d(\nu_3 \times y)) \mathfrak{Q}(\nu_1, d(\nu_2 \times x)). \end{aligned} \quad (4.18)$$

By the above definitions, we have

$$\begin{aligned} \text{LHS of (4.18)} &= \mathbb{P}_{\nu_1} \left[ e^{-\langle f, \Gamma_t \rangle} \frac{\langle h e^{-g}, \Gamma_t \rangle}{\langle h, \Gamma_t \rangle} 1_{\{\Gamma_t \neq 0\}} \right] + \mathbb{P}_{\nu_1}(\Gamma_t = 0), \\ \text{RHS of (4.18)} &= \mathbb{P}_{\nu_1} \left[ e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} 1_{\{\nu_1 \neq 0\}} + 1_{\{\nu_1 = 0\}} \right]. \end{aligned}$$

In view of (4.17), to show (4.18), it suffices to show that for all  $\mu \in \mathcal{M}(E)^0$  and  $f, g \in \mathcal{B}_b^+(E)$ ,

$$\mathbb{P}_\mu \left[ e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] = \mathbb{P}_\mu \left[ e^{-\langle f, \Gamma_t \rangle} \frac{\langle h e^{-g}, \Gamma_t \rangle}{\langle h, \Gamma_t \rangle} 1_{\{\Gamma_t \neq 0\}} \right]. \tag{4.19}$$

It follows from Lemma 4.11 that

$$\begin{aligned} \mathbb{P}_\mu \left[ e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] &= \frac{1}{\langle h, \mu \rangle} \int_E \mathbb{P}_{\mu, x} \left[ e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] h(x) \mu(dx) \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^{h e^{-g}} f, \mu \rangle, \end{aligned} \tag{4.20}$$

where  $V_t^{h e^{-g}} f(x)$  is the unique locally bounded solution to (2.10) with initial value  $h e^{-g}$ . On the other hand, since  $(\Gamma_t, \mathbb{P}_\mu)$  and  $(X_t, \mathbb{Q}_\mu)$  are identically distributed for each  $t \geq 0$ , we have by the definition of  $\mathbb{Q}_\mu$  and Proposition 2.1 that

$$\begin{aligned} \mathbb{P}_\mu \left[ e^{-\langle f, \Gamma_t \rangle} \frac{\langle h e^{-g}, \Gamma_t \rangle}{\langle h, \Gamma_t \rangle} 1_{\{\Gamma_t \neq 0\}} \right] &= \mathbb{Q}_\mu \left[ e^{-\langle f, X_t \rangle} \frac{\langle h e^{-g}, X_t \rangle}{\langle h, X_t \rangle} 1_{\{X_t \neq 0\}} \right] \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} \mathbb{P}_\mu \left[ e^{-\langle f, X_t \rangle} \langle h e^{-g}, X_t \rangle \right] \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^{h e^{-g}} f, \mu \rangle. \end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21), we get (4.19). The proof is now complete.  $\square$

### 5. Sufficient conditions for a non-degenerate martingale limit

In this section, we will give sufficient conditions for the fundamental martingale to have a non-degenerate limit. We start with an assumption.

#### Assumption 3.

(i) Either one of the following conditions holds.

(1)  $a(x), \gamma(x) \in L^2(E, m)$ .

(2) The Lévy system  $(N, H)$  of  $\xi$  is of the form  $(N, t)$ , where  $N$  is given by

$$N(x, dy) = N(x, y) m(dy)$$

with  $N(x, y)$  being a symmetric Borel function on  $E \times E$ . The probability kernel  $\pi(x, dy)$  has a density  $\pi(x, y)$  with respect to the measure  $m$  such that

$$\gamma(x) \pi(x, y) = F(x, y) N(x, y) \quad \forall x, y \in E$$

for some non-negative bounded Borel function  $F(x, y)$  on  $E \times E$  vanishing on the diagonal.

(ii)  $(1_A \pi(\cdot, h), \widehat{h}) < +\infty$ .

(iii)  $x \mapsto \pi(x, h)/h$  is bounded from above on  $A$ .

It is easy to see that Assumption 3.(iii) implies Assumption 3.(ii). In this section we will use the first two items of this assumption. In the next section we will use items (i) and (iii) of this assumption. The following theorem, giving an  $L \log L$  type criterion for the martingale limit to be non-degenerate, is the main result of this section.

**Theorem 5.1.** *Suppose Assumptions 0–2 and 3.(i)–(ii) hold. Let  $W_\infty^h(X)$  be the almost sure limit of the non-negative martingale  $W_t^h(X)$ . Suppose that*

$$\left( \int_{(0, +\infty)} r h(\cdot) \log^+(r h(\cdot)) \Pi^L(\cdot, dr), \widehat{h} \right) + \left( \int_{(0, +\infty)} r \pi(\cdot, h) \log^+(r \pi(\cdot, h)) \Pi^{NL}(\cdot, dr), \widehat{h} \right) < +\infty. \tag{5.1}$$

We have

- (i) if  $\lambda_1 < 0$ , then  $W_t^h(X)$  converges to  $W_\infty^h(X)$  as  $t \rightarrow \infty$  in  $L^1(P_\mu)$  for every  $\mu \in \mathcal{M}(E)$ , and  $W_\infty^h(X)$  is non-degenerate in the sense that  $P_\mu(W_\infty^h(X) > 0) > 0$  for  $\mu \in \mathcal{M}(E)^0$ ;
- (ii) if  $\lambda_1 > 0$ , then  $W_\infty^h(X) = 0$   $P_\mu$ -a.s. for every  $\mu \in \mathcal{M}(E)$ .

In the remainder of this section we will assume Assumptions 0-2 hold. Additional conditions used are stated explicitly. To prove Theorem 5.1, we need a few preliminary results.

**Proposition 5.2.** *Suppose Assumption 3.(i) holds. For all  $f \in \mathcal{B}_b(E) \cap L^2(E, m)$  and  $s, t \in (0, +\infty)$ ,*

$$\lim_{s \rightarrow t} \mathfrak{P}_s f = \mathfrak{P}_t f \quad \text{in } L^2(E, m). \tag{5.2}$$

Moreover,

$$\mathfrak{P}_t f = T_t f [m] \quad \text{for all } t > 0, \tag{5.3}$$

where  $(T_t)_{t \geq 0}$  is the semigroup associated with the bilinear form  $(\mathcal{Q}, \mathcal{F})$  by (3.3).

*Proof:* First we suppose Assumption 3.(i2) holds. We note that  $\gamma(x, dy) = \gamma(x)\pi(x, dy) = \gamma(x)\pi(x, y)m(dy)$ . Then by Assumptions 1 and 3.(i2),

$$\int_E F(x, y)N(x, y)m(dx) = \int_E \gamma(x)\pi(x, y)m(dx) \in \mathbf{K}(\xi).$$

Thus  $F$  is in the class **J** defined in Chen and Song (2003a). Let  $\hat{F}(x, y) := F(y, x)$ . Then

$$\int_E \hat{F}(x, y)N(x, y)m(dx) = \int_E F(y, x)N(y, x)m(dx) = \int_E \gamma(y)\pi(y, x)m(dx) = \gamma(y).$$

Since  $\gamma$  is a bounded function on  $E$ , the above equation implies that  $\hat{F}$  is in the class **J**. One can also show easily that the functions  $\log(1 + F)$  and  $\log(1 + \hat{F})$  are in **J**. Define

$$A_{s,t} := - \int_s^t a(\xi_r)dr + \sum_{s < r \leq t} \log(1 + F(\xi_{r-}, \xi_r)) \quad \forall 0 \leq s < t < +\infty.$$

It follows from Chen and Song (2003a, Theorem 4.8) (see also Chen and Song, 2003b, p. 275) that the semigroup corresponding to the bilinear form  $(\mathcal{Q}, \mathcal{F})$  defined by (3.3) is

$$T_t f(x) = \Pi_x [e^{A_{0,t}} f(\xi_t)] \quad \forall t \geq 0, x \in E, f \in \mathcal{B}^+(E).$$

Furthermore, for any  $1 \leq p \leq +\infty$ ,  $(T_t)_{t \geq 0}$  is semigroup on  $L^p(E, m)$ , and for  $1 \leq p < +\infty$ ,  $(T_t)_{t \geq 0}$  is strongly continuous semigroup on  $L^p(E, m)$ . Similar to Chen and Song (2003a, (2.6)), we have

$$e^{A_{0,t}} - 1 = - \int_0^t e^{A_{s,t}} a(\xi_s)ds + \sum_{s \leq t} e^{A_{s,t}} F(\xi_{s-}, \xi_s).$$

Using this, the Markov property of  $\xi$ , (2.2) and Assumption 3.(i2), one can show that for any  $f \in \mathcal{B}_b(E)$  and  $x \in E$ ,

$$\begin{aligned} T_t f(x) &= \Pi_x [f(\xi_t)] - \Pi_x \left[ \int_0^t a(\xi_s)T_{t-s}f(\xi_s)ds \right] + \Pi_x \left[ \int_0^t \int_E T_{t-s}f(y)F(\xi_s, y)N(\xi_s, y)m(dy)ds \right] \\ &= \Pi_x [f(\xi_t)] - \Pi_x \left[ \int_0^t a(\xi_s)T_{t-s}f(\xi_s)ds \right] + \Pi_x \left[ \int_0^t \int_E T_{t-s}f(y)\gamma(\xi_s)\pi(\xi_s, y)m(dy)ds \right] \\ &= \Pi_x [f(\xi_t)] - \Pi_x \left[ \int_0^t a(\xi_s)T_{t-s}f(\xi_s)ds \right] + \Pi_x \left[ \int_0^t \int_E T_{t-s}f(y)\gamma(\xi_s, dy)ds \right]. \end{aligned}$$

This implies that  $T_t f(x)$  satisfies the integral equation (2.9). By uniqueness  $T_t f(x) = \mathfrak{P}_t f(x)$ , and thus we conclude the result of this proposition.

Now we suppose Assumption 3.(i1) holds. Fix  $f \in \mathcal{B}_b(E) \cap L^2(E, m)$ . We first prove (5.2). Without loss of generality, we assume  $0 < s < t < +\infty$ . Let  $F_r(x) := -a(x)\mathfrak{P}_r f(x) + \gamma(x, \mathfrak{P}_r f)$ . We have shown in the argument below (3.3) that  $\|\mathfrak{P}_r f\|_\infty \leq e^{c_1 r} \|f\|_\infty$  for some constant  $c_1 > 0$ .

Thus by definition,  $|F_r(x)| \leq (|a(x)| + \gamma(x))\|\mathfrak{P}_r f\|_\infty \leq e^{c_1 r}\|f\|_\infty(|a(x)| + \gamma(x))$ . Clearly by the boundedness of  $a(x)$  and  $\gamma(x)$ ,  $(x, r) \mapsto F_r(x)$  is locally bounded on  $E \times [0, +\infty)$  and by Assumption 3.(i),  $x \mapsto F_r(x) \in \mathcal{B}_b(E) \cap L^2(E, m)$ . By (2.9), we have

$$\begin{aligned} \mathfrak{P}_t f(x) - \mathfrak{P}_s f(x) &= \mathfrak{S}_t f(x) - \mathfrak{S}_s f(x) + \Pi_x \left[ \int_s^t F_r(\xi_{t-r}) dr \right] \\ &\quad + \Pi_x \left[ \int_0^s F_r(\xi_{t-r}) - F_r(\xi_{s-r}) dr \right]. \end{aligned} \quad (5.4)$$

Recall that  $\{\mathfrak{S}_t : t \geq 0\}$  is a strongly continuous contraction semigroup on  $L^2(E, m)$ . Thus

$$\begin{aligned} \|\mathfrak{S}_t f - \mathfrak{S}_s f\|_{L^2(E, m)} &= \|\mathfrak{S}_s (\mathfrak{S}_{t-s} f - f)\|_{L^2(E, m)} \\ &\leq \|\mathfrak{S}_{t-s} f - f\|_{L^2(E, m)} \rightarrow 0 \text{ as } s \rightarrow t. \end{aligned} \quad (5.5)$$

Note that

$$\left| \Pi_x \left[ \int_s^t F_r(\xi_{t-r}) dr \right] \right| \leq \int_s^t |\Pi_x [F_r(\xi_{t-r})]| dr = \int_s^t |\mathfrak{S}_{t-r} F_r(x)| dr. \quad (5.6)$$

We have by Minkowski's integral inequality and the contractivity of  $\mathfrak{S}_t$  that

$$\begin{aligned} \left\| \int_s^t |\mathfrak{S}_{t-r} F_r| dr \right\|_{L^2(E, m)} &\leq \int_s^t \|\mathfrak{S}_{t-r} F_r\|_{L^2(E, m)} dr \leq \int_s^t \|F_r\|_{L^2(E, m)} dr \\ &\leq \|f\|_\infty (\|a\|_{L^2(E, m)} + \|\gamma\|_{L^2(E, m)}) \int_s^t e^{c_1 r} dr \rightarrow 0 \end{aligned}$$

as  $s \rightarrow t$ . This together with (5.6) implies that

$$\lim_{s \rightarrow t} \left\| \Pi_x \left[ \int_s^t F_r(\xi_{t-r}) dr \right] \right\|_{L^2(E, m)} = 0. \quad (5.7)$$

Note that by the Markov property of  $\xi$ ,

$$\begin{aligned} \left| \Pi_x \left[ \int_0^s F_r(\xi_{t-r}) - F_r(\xi_{s-r}) dr \right] \right| &\leq \int_0^s \Pi_x [|\Pi_{\xi_{s-r}}(F_r(\xi_{t-s})) - \Pi_{\xi_{s-r}}(F_r(\xi_0))|] dr \\ &= \int_0^s \mathfrak{S}_{s-r} (|\mathfrak{S}_{t-s} F_r - F_r|)(x) dr. \end{aligned} \quad (5.8)$$

It follows from the strong continuity and contractivity of the semigroup  $\{\mathfrak{S}_t : t \geq 0\}$  that

$$\begin{aligned} \lim_{s \rightarrow t} \|\mathfrak{S}_{t-s} F_r - F_r\|_{L^2(E, m)} &= 0, \text{ and} \\ \|\mathfrak{S}_{t-s} F_r - F_r\|_{L^2(E, m)} &\leq 2\|F_r\|_{L^2(E, m)} \leq 2e^{c_1 r}\|f\|_\infty (\|a\|_{L^2(E, m)} + \|\gamma\|_{L^2(E, m)}). \end{aligned}$$

Thus by Minkowski's integral inequality and the dominated convergence theorem, we have

$$\begin{aligned} \left\| \int_0^s \mathfrak{S}_{s-r} (|\mathfrak{S}_{t-s} F_r - F_r|) dr \right\|_{L^2(E, m)} &\leq \int_0^s \|\mathfrak{S}_{s-r} (|\mathfrak{S}_{t-s} F_r - F_r|)\|_{L^2(E, m)} dr \\ &\leq \int_0^s \|\mathfrak{S}_{t-s} F_r - F_r\|_{L^2(E, m)} dr \\ &\rightarrow 0 \text{ as } s \rightarrow t. \end{aligned}$$

This together with (5.8) implies that

$$\lim_{s \rightarrow t} \left\| \Pi_x \left[ \int_0^s F_r(\xi_{t-r}) - F_r(\xi_{s-r}) dr \right] \right\|_{L^2(E, m)} = 0. \quad (5.9)$$

Combining (5.4)–(5.9), we arrive at (5.2). To prove (5.3), it suffices to prove that for every  $t > 0$  and every  $g \in L^2(E, m)$ ,

$$\int_E \mathfrak{P}_t f(x) g(x) m(dx) = \int_E T_t f(x) g(x) m(dx). \quad (5.10)$$

Note that by Hölder's inequality and (5.2), for  $s, t \in (0, +\infty)$ ,

$$\begin{aligned} & \left| \int_E \mathfrak{P}_t f(x) g(x) m(dx) - \int_E \mathfrak{P}_s f(x) g(x) m(dx) \right| \\ & \leq \|\mathfrak{P}_t f - \mathfrak{P}_s f\|_{L^2(E, m)} \|g\|_{L^2(E, m)} \rightarrow 0 \end{aligned} \quad (5.11)$$

as  $s \rightarrow t$ . This implies  $t \mapsto \int_E \mathfrak{P}_t f(x) g(x) m(dx)$  is a continuous function on  $(0, +\infty)$ . Similarly, using the strong continuity of  $\{T_t : t \geq 0\}$  on  $L^2(E, m)$ , one can prove that  $t \mapsto \int_E T_t f(x) g(x) m(dx)$  is also a continuous function on  $(0, +\infty)$ . By taking the Laplace transform of  $\int_E \mathfrak{P}_t f(x) g(x) m(dx)$  (resp.  $\int_E T_t f(x) g(x) m(dx)$ ), we get  $\int_E R_\alpha f(x) g(x) m(dx)$  (resp.  $\int_E U_\alpha f(x) g(x) m(dx)$ ). It has been shown in the argument below (3.3) that under Assumption 3.(i),  $R_\alpha f = U_\alpha f [m]$  for  $\alpha$  sufficiently large. So the Laplace transforms of both sides of (5.10) are identical for  $\alpha$  sufficiently large. Hence (5.10) follows from Post's inversion theorem for Laplace transforms.  $\square$

**Proposition 5.3.** *Under the assumptions of Proposition 5.2, the measure*

$$\rho(dx) = h(x) \widehat{h}(x) m(dx)$$

*is an invariant probability measure for the semigroup  $\{\widetilde{\mathfrak{S}}_t : t \geq 0\}$ , i.e., for all  $t \geq 0$  and  $f \in \mathcal{B}^+(E)$ ,*

$$\int_E \widetilde{\mathfrak{S}}_t f(x) \rho(dx) = \int_E f(x) \rho(dx). \quad (5.12)$$

*Proof:* By the monotone convergence theorem, we only need to prove (5.12) for  $f \in \mathcal{B}_b^+(E)$ . Clearly  $fh \in \mathcal{B}_b^+(E) \cap L^2(E, m)$ . It follows by (4.3), (5.3) and (3.2) that

$$\begin{aligned} \int_E \widetilde{\mathfrak{S}}_t f(x) \rho(dx) &= \int_E e^{\lambda_1 t} \mathfrak{P}_t(fh)(x) \widehat{h}(x) m(dx) \\ &= \int_E e^{\lambda_1 t} T_t(fh)(x) \widehat{h}(x) m(dx) \\ &= \int_E e^{\lambda_1 t} f(x) h(x) \widehat{T}_t \widehat{h}(x) m(dx) \\ &= \int_E f(x) \rho(dx). \end{aligned} \quad \square$$

**Lemma 5.4.** *The function  $g(x) := h(x)^{-1} \mathbb{P}_{\delta_x} [W_\infty^h(X)]$  satisfies that*

$$\mathbb{P}_\mu [W_\infty^h(X)] = \langle gh, \mu \rangle \quad \text{for all } \mu \in \mathcal{M}(E). \quad (5.13)$$

*Moreover,*

$$\widetilde{\mathfrak{S}}_t g(x) = g(x) \quad \text{for all } t \geq 0 \text{ and } x \in E. \quad (5.14)$$

*Proof:* To prove the first claim, we note that for an arbitrary constant  $\lambda > 0$ , by the bounded convergence theorem,

$$\begin{aligned} \mathbb{P}_\mu \left[ \exp \left( -\lambda W_\infty^h(X) \right) \right] &= \lim_{t \rightarrow +\infty} \mathbb{P}_\mu \left[ \exp \left( -\lambda W_t^h(X) \right) \right] \\ &= \lim_{t \rightarrow +\infty} \exp \left( -\langle l_\lambda(t, \cdot), \mu \rangle \right) \\ &= \exp \left( -\lim_{t \rightarrow +\infty} \langle l_\lambda(t, \cdot), \mu \rangle \right), \end{aligned} \quad (5.15)$$

where  $l_\lambda(t, x) := -\log P_{\delta_x} [\exp(-\lambda W_t^h(X))]$ . Let

$$l_\lambda(x) := \lim_{t \rightarrow +\infty} l_\lambda(t, x) = -\log P_{\delta_x} [\exp(-\lambda W_\infty^h(X))].$$

We have by Jensen's inequality that

$$l_\lambda(t, x) \leq \lambda P_{\delta_x} (W_t^h(X)) = \lambda e^{\lambda_1 t} \mathfrak{P}_t h(x) = \lambda h(x) \quad \text{for all } x \in E, t \geq 0.$$

Hence  $l_\lambda(x) \leq \lambda h(x)$  for all  $x \in E$ . This together with (5.15) and the dominated convergence theorem yields that

$$P_\mu [\exp(-\lambda W_\infty^h(X))] = e^{-\langle l_\lambda, \mu \rangle}. \tag{5.16}$$

Thus we get (5.13) by differentiating both sides of (5.16) with respect to  $\lambda$  and then letting  $\lambda \downarrow 0$ . Note that  $0 \leq g \leq 1$  by Fatou's lemma. By the Markov property of  $X$  and (5.13), we have for all  $t \geq 0$  and  $x \in E$ ,

$$\begin{aligned} g(x) &= \frac{1}{h(x)} P_{\delta_x} \left[ \lim_{s \rightarrow +\infty} e^{\lambda_1(t+s)} \langle h, X_{t+s} \rangle \right] \\ &= \frac{e^{\lambda_1 t}}{h(x)} P_{\delta_x} \left[ P_{X_t} \left( \lim_{s \rightarrow +\infty} W_s^h(X) \right) \right] = \frac{e^{\lambda_1 t}}{h(x)} P_{\delta_x} \left[ P_{X_t} (W_\infty^h(X)) \right] \\ &= \frac{e^{\lambda_1 t}}{h(x)} P_{\delta_x} [\langle gh, X_t \rangle] = \frac{e^{\lambda_1 t}}{h(x)} \mathfrak{P}_t(gh)(x) = \tilde{\mathfrak{S}}_t g(x). \end{aligned}$$

Here we used (4.3) in the last equality. □

**Lemma 5.5.** *Suppose Assumption 3.(i) holds. Let  $D^m$  and  $\Lambda_s^m$  be as in Definition 4.4.(iii) and (4.9) respectively. If condition (5.1) holds, then for  $m$ -almost every  $x \in E$ ,*

$$\lim_{D^m \ni s \rightarrow +\infty} \frac{\log^+(\Lambda_s^m h(\tilde{\xi}_s))}{s} = \lim_{i \rightarrow +\infty} \frac{\log^+ \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h)}{\tau_i} = 0 \quad \mathbb{P}_{\cdot, x}\text{-a.s.} \tag{5.17}$$

*Proof:* To prove (5.17), it suffices to prove that for any  $\varepsilon > 0$  sufficiently small,

$$\mathbb{P}_{\cdot, x} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} = +\infty \right) \text{ and } \mathbb{P}_{\cdot, x} \left( \sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} = +\infty \right) = 0. \tag{5.18}$$

For any  $B \in \mathcal{B}(E)$  with  $0 < m(B) < +\infty$ , let  $\mu_B(dx) := \hat{h}(x) 1_B(x) m(dx)$ . Clearly  $\mu_B \in \mathcal{M}(E)^0$ . Recall that given  $\tilde{\xi}$ ,  $\{\Lambda_s^m : s \geq 0\}$  is a Poisson point process with characteristic measure  $\lambda \Pi^L(\tilde{\xi}_s, d\lambda)$ . Thus by Fubini's theorem and the fact that  $\rho(dx) = h(x) \hat{h}(x) m(dx)$  is an invariant measure for  $\tilde{\mathfrak{S}}_t$ , we have

$$\begin{aligned} \mathbb{P}_{\mu_B} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \right) &= \mathbb{P}_{\mu_B} \left( \int_0^{+\infty} \int_{(0, +\infty)} \lambda 1_{\{\lambda h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, d\lambda) ds \right) \\ &= \frac{1}{\langle h, \mu_B \rangle} \int_B \mathbb{P}_{\mu_B, x} \left( \int_0^{+\infty} \int_{(0, +\infty)} \lambda 1_{\{\lambda h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, d\lambda) ds \right) \rho(dx) \\ &\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E \rho(dx) \int_{(0, +\infty)} \lambda 1_{\{\lambda h(x) > e^{\varepsilon s}\}} \Pi^L(x, d\lambda) \\ &= \frac{1}{\langle h, \mu_B \rangle} \left( \int_{(0, +\infty)} \lambda \Pi^L(\cdot, d\lambda) \int_0^{\log^+ \lambda h(x) / \varepsilon} ds, h \hat{h} \right) \\ &= \frac{1}{\varepsilon \langle h, \mu_B \rangle} \left( \int_{(0, +\infty)} \lambda h(\cdot) \log^+(\lambda h(\cdot)) \Pi^L(\cdot, d\lambda), \hat{h} \right). \end{aligned} \tag{5.19}$$

The right hand side of (5.19) is finite by (5.1). Thus we get

$$\mathbb{P}_{\mu_B} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = 1.$$

Note that

$$\mathbb{P}_{\mu_B} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = \rho(B)^{-1} \int_B \mathbb{P}_{\cdot, x} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) \rho(dx).$$

Thus  $\mathbb{P}_{\cdot, x} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = 1$  for  $m$ -almost every  $x \in B$ . Since  $B$  is arbitrary, the first equality of (5.18) holds for  $m$ -almost every  $x \in E$ .

Recall from Definition 4.4 that given  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $\Theta_i$  is distributed as  $\eta(\tilde{\xi}_{\tau_i-}, d\theta)$  given by (4.8). Thus by Fubini's theorem and (4.6),

$$\begin{aligned} & \mathbb{P}_{\mu_B} \left( \sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} \right) \\ &= \mathbb{P}_{\mu_B} \left[ \sum_{i=1}^{+\infty} \int_{\theta \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}} \eta(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\ &= \mathbb{P}_{\mu_B} \left[ \sum_{i=1}^{+\infty} \frac{1}{\gamma(\tilde{\xi}_{\tau_i-})} 1_{A(\tilde{\xi}_{\tau_i-})} \int_{\theta \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}} \theta \Pi^{NL}(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\ &= \mathbb{P}_{\mu_B} \left[ \int_0^{+\infty} q(\tilde{\xi}_s) \frac{1}{\gamma(\tilde{\xi}_s)} 1_{A(\tilde{\xi}_s)} ds \int_{\theta \pi(\tilde{\xi}_s, h) > e^{\varepsilon s}} \theta \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] \\ &= \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_B \mathbb{P}_{\mu, x} \left[ \frac{\pi(\tilde{\xi}_s, h)}{h(\tilde{\xi}_s)} 1_{A(\tilde{\xi}_s)} \int_{\theta \pi(\tilde{\xi}_s, h) > e^{\varepsilon s}} \theta \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] \rho(dx) \\ &\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E 1_A(x) \pi(x, h) \hat{h}(x) m(dx) \int_{\theta \pi(x, h) > e^{\varepsilon s}} \theta \Pi^{NL}(x, d\theta) \\ &= \frac{1}{\langle h, \mu_B \rangle} \int_E 1_A(x) \pi(x, h) \hat{h}(x) m(dx) \int_{(0, +\infty)} \theta \Pi^{NL}(x, d\theta) \int_0^{\frac{\log^+(\theta \pi(x, h))}{\varepsilon}} ds \\ &= \frac{1}{\varepsilon \langle h, \mu_B \rangle} \left( \pi(\cdot, h) \int_{(0, +\infty)} \theta \log^+(\theta \pi(\cdot, h)) \Pi^{NL}(\cdot, d\theta), 1_A \hat{h} \right). \end{aligned} \tag{5.20}$$

The right hand side of (5.20) is finite by (5.1). Thus we get

$$\mathbb{P}_{\mu_B} \left( \sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} < +\infty \right) = 1.$$

Using an argument similar to that at the end of the first paragraph of this proof, one can prove that the second equality of (5.18) holds for  $m$ -almost every  $x \in E$ .  $\square$

**Proof of Theorem 5.1:** Recall that, by assumption, Assumptions 0-2 and 3.(i)-(ii) hold.

(i) Suppose  $\lambda_1 < 0$ . Without loss of generality, we assume  $\mu \in \mathcal{M}(E)^0$ . Since  $W_t^h(X)$  is a non-negative martingale, to show it is a closed martingale, it suffices to prove

$$P_\mu \left[ W_\infty^h(X) \right] = \langle h, \mu \rangle. \tag{5.21}$$

First we claim that (5.21) is true for  $\mu_B(dy) := 1_B(y) \hat{h}(y) m(dy)$  with  $B \in \mathcal{B}(E)$  and  $0 < m(B) < +\infty$ . It is straightforward to see from the change of measure methodology (see, for example, Durrett,



2019, Theorem 5.3.3) that the proof for this claim is complete as soon as we can show that

$$\mathbb{Q}_{\mu_B} \left( \limsup_{t \rightarrow +\infty} W_t^h(X) < +\infty \right) = 1. \tag{5.22}$$

Since  $((X_t)_{t \geq 0}; \mathbb{Q}_{\mu_B})$  is equal in law to  $((\Gamma_t)_{t \geq 0}; \mathbb{P}_{\mu_B})$ , (5.22) is equivalent to that

$$\mathbb{P}_{\mu_B} \left( \limsup_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty \right) = 1. \tag{5.23}$$

In the remainder of this proof, we define a function  $\log^* \theta := \theta/e$  if  $\theta \leq e$  and  $\log^* \theta := \log \theta$  if  $\theta > e$ . Under the assumptions of Theorem 5.1, one can prove by elementary computation that (5.1) implies

$$\left( \int_{(0,+\infty)} rh(\cdot) \log^*(rh(\cdot)) \Pi^L(\cdot, dr), \widehat{h} \right) + \left( \int_{(0,+\infty)} r\pi(\cdot, h) \log^*(r\pi(\cdot, h)) \Pi^{NL}(\cdot, dr), 1_A \widehat{h} \right) < +\infty. \tag{5.24}$$

Recall that  $\mathcal{G}$  is the  $\sigma$ -field generated by  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $\{D_t^m : t \geq 0\}$ ,  $\{D_t^n : t \geq 0\}$ ,  $\{\Theta_i : i \geq 1\}$  and  $\{\Lambda_s^m : s \geq 0\}$ . By (4.10), for any  $t > 0$ ,

$$\begin{aligned} \mathbb{P}_{\mu_B} \left( W_t^h(\Gamma) \mid \mathcal{G} \right) &= e^{\lambda_1 t} \left( \langle \mathfrak{P}_t h, \mu_B \rangle + \sum_{s \in D_t^n} \mathfrak{P}_{t-s} h(\tilde{\xi}_s) + \sum_{s \in D_t^m} \Lambda_s^m \mathfrak{P}_{t-s} h(\tilde{\xi}_s) \right. \\ &\quad \left. + \sum_{\tau_i \leq t} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, \mathfrak{P}_{t-\tau_i} h) \right) \\ &= \langle h, \mu_B \rangle + \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) + \sum_{s \in D_t^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) + \sum_{\tau_i \leq t} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \\ &\leq \langle h, \mu_B \rangle + \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) + \sum_{s \in D_t^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) + \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h). \end{aligned} \tag{5.25}$$

We begin with the second term on the right hand side of (5.25). Let  $\varepsilon \in (0, -\lambda_1)$  be an arbitrary constant.

$$\sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) = \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} + \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} =: \text{I} + \text{II}.$$

Recall that given  $\tilde{\xi}$ , the random measure  $\sum_{s \in D_t^n} \delta_s(\cdot)$  on  $[0, +\infty)$  is a Poisson random measure with intensity  $2b(\tilde{\xi}_t)dt$ , and that  $\rho(dx) = h(x)\widehat{h}(x)m(dx)$  is an invariant probability measure for  $\tilde{\mathfrak{S}}_t$ . We have by Fubini's theorem,

$$\begin{aligned} \mathbb{P}_{\mu_B} (\text{II}) &= \mathbb{P}_{\mu_B} \left[ \mathbb{P}_{\mu_B} \left( \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \mid \tilde{\xi}_r : r \geq 0 \right) \right] \\ &= \mathbb{P}_{\mu_B} \left( \int_0^{+\infty} 2b(\tilde{\xi}_s) e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} ds \right) \\ &= \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_B \mathbb{P}_{\mu, x} \left[ b(\tilde{\xi}_s) h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \right] \rho(dx) \\ &\leq \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E b(x) h(x) 1_{\{h(x) \leq e^{\varepsilon s}\}} \rho(dx) \\ &\leq \frac{2\|b\|_\infty}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{(\lambda_1 + \varepsilon)s} ds \int_E \rho(dx) < +\infty. \end{aligned}$$

Thus we have  $\mathbb{P}_{\mu_B}(\text{II} < +\infty) = 1$ . On the other hand,

$$\begin{aligned}
\mathbb{P}_{\mu_B} \left( \sum_{s \in D^n} 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \right) &= \mathbb{P}_{\mu_B} \left( \int_0^{+\infty} 2b(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} ds \right) \\
&= \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_B \mathbb{P}_{\mu, x} \left[ b(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \right] \rho(dx) \\
&\leq \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E b(x) 1_{\{h(x) > e^{\varepsilon s}\}} \rho(dx) \\
&= \frac{2}{\langle h, \mu_B \rangle} \int_E b(x) \rho(dx) \int_0^{\frac{\log^+ h(x)}{\varepsilon}} ds \\
&\leq \frac{2}{\varepsilon \langle h, \mu_B \rangle} \|b\|_\infty \log^+ \|h\|_\infty < +\infty.
\end{aligned}$$

This implies that I is the sum of finitely many terms. Thus  $\mathbb{P}_{\mu_B}(\text{I} < +\infty) = 1$ . For the third term in (5.25), we have

$$\begin{aligned}
\sum_{s \in D^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) &= \sum_{s \in D^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} + \sum_{s \in D^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \\
&=: \text{III} + \text{IV}.
\end{aligned}$$

In view of Definition 4.4.(iii), for III, we have

$$\begin{aligned}
\mathbb{P}_{\mu_B}(\text{III}) &= \mathbb{P}_{\mu_B} \left( \int_0^{+\infty} \int_E e^{\lambda_1 s} r^2 h(\tilde{\xi}_s) 1_{\{rh(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, dr) ds \right) \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_B \mathbb{P}_{\mu, x} \left( \int_E r^2 h(\tilde{\xi}_s) 1_{\{rh(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, dr) \right) \rho(dx) \\
&\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E \rho(dx) \int_E r^2 h(x) 1_{\{rh(x) \leq e^{\varepsilon s}\}} \Pi^L(x, dr) \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_E \rho(dx) \int_{(0, +\infty)} r^2 h(\cdot) \Pi^L(\cdot, dr) \int_{\log^+ rh(\cdot)/\varepsilon}^{+\infty} e^{\lambda_1 s} ds \\
&= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left( \int_{(0, +\infty)} r^2 h^2(\cdot) (rh(\cdot) \vee 1)^{\lambda_1/\varepsilon} \Pi^L(\cdot, dr), \hat{h} \right) \\
&= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left( \int_{(0, +\infty)} \Pi^L(\cdot, dr) rh(\cdot) \log^*(rh(\cdot)) \cdot \left( \frac{rh(\cdot)}{(rh(\cdot) \vee 1)^{-\lambda_1/\varepsilon} \log^*(rh(\cdot))} \right), \hat{h} \right). \quad (5.26)
\end{aligned}$$

Note that the function  $r \mapsto \frac{r}{(r \vee 1)^{-\lambda_1/\varepsilon} \log^* r}$  is bounded from above on  $(0, +\infty)$ . This together with (5.24) implies that the right hand side of (5.26) is finite. It follows that  $\mathbb{P}_{\mu_B}(\text{III} < +\infty) = 1$ . It has been shown by (5.19) that

$$\mathbb{P}_{\mu_B} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = 1.$$

This implies that IV is the sum of finitely many terms. Thus we have  $\mathbb{P}_{\mu_B}(\text{IV} < +\infty) = 1$ . The fourth term on the right hand side of (5.25) can be dealt with similarly. In fact, we have

$$\begin{aligned} \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) &= \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \leq e^{\varepsilon \tau_i}\}} \\ &\quad + \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} \\ &=: \text{V} + \text{VI}. \end{aligned}$$

Recall that given  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $\Theta_i$  is distributed according to  $\eta(\tilde{\xi}_{\tau_i-}, d\theta)$  given by (4.8). Thus by Fubini's theorem and (4.6),

$$\begin{aligned} \mathbb{P}_{\mu_B}(\text{V}) &= \mathbb{P}_{\mu_B} \left[ \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \pi(\tilde{\xi}_{\tau_i-}, h) \int_{[0, +\infty)} \theta 1_{\{\theta \pi(\tilde{\xi}_{\tau_i-}, h) \leq e^{\varepsilon \tau_i}\}} \eta(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\ &= \mathbb{P}_{\mu_B} \left[ \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \frac{\pi(\tilde{\xi}_{\tau_i-}, h)}{\gamma(\tilde{\xi}_{\tau_i-})} 1_A(\tilde{\xi}_{\tau_i-}) \int_{\{0 < \theta \pi(\tilde{\xi}_{\tau_i-}, h) \leq e^{\varepsilon \tau_i}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\ &= \frac{1}{\langle h, \mu_B \rangle} \int_B \rho(dx) \mathbb{P}_{\mu, x} \left[ \int_0^{+\infty} q(\tilde{\xi}_s) e^{\lambda_1 s} \frac{\pi(\tilde{\xi}_s, h)}{\gamma(\tilde{\xi}_s)} 1_A(\tilde{\xi}_s) ds \cdot \int_{\{0 < \theta \pi(\tilde{\xi}_s, h) \leq e^{\varepsilon s}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] \\ &= \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_B \rho(dx) \mathbb{P}_{\mu, x} \left[ \frac{\pi(\tilde{\xi}_s, h)^2}{h(\tilde{\xi}_s)} 1_A(\tilde{\xi}_s) \cdot \int_{\{0 < \theta \pi(\tilde{\xi}_s, h) \leq e^{\varepsilon s}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] \\ &\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E 1_A(x) \pi(x, h)^2 \hat{h}(x) m(dx) \int_{\{0 < \theta \pi(x, h) \leq e^{\varepsilon s}\}} \theta^2 \Pi^{NL}(x, d\theta) \\ &= \frac{1}{\langle h, \mu_B \rangle} \int_E 1_A(x) \pi(x, h)^2 \hat{h}(x) m(dx) \int_{(0, +\infty)} \theta^2 \Pi^{NL}(x, d\theta) \int_{\frac{\log^+(\theta \pi(x, h))}{\varepsilon}}^{+\infty} e^{\lambda_1 s} ds \\ &= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left( \int_{(0, +\infty)} 1_A(\cdot) \pi(\cdot, h)^2 \theta^2 (\pi(\cdot, h) \theta \vee 1)^{\lambda_1/\varepsilon} \Pi^{NL}(\cdot, d\theta), \hat{h} \right) \\ &= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left( \int_{(0, +\infty)} \Pi^{NL}(\cdot, d\theta) 1_A(\cdot) \pi(\cdot, h) \theta \log^*(\pi(\cdot, h) \theta) \right. \\ &\quad \cdot \left. \left( \frac{\pi(\cdot, h) \theta}{(\pi(\cdot, h) \theta \vee 1)^{-\lambda_1/\varepsilon} \log^*(\pi(\cdot, h) \theta)} \right), \hat{h} \right). \end{aligned}$$

Since  $\theta \mapsto \frac{\theta}{(\theta \vee 1)^{-\lambda_1/\varepsilon} \log^* \theta}$  is bounded from above on  $(0, +\infty)$ , we get  $\mathbb{P}_{\mu_B}(\text{V}) < +\infty$  by (5.24), and hence  $\mathbb{P}_{\mu_B}(\text{V} < +\infty) = 1$ . We have shown in (5.20) that

$$\mathbb{P}_{\mu_B} \left( \sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} < +\infty \right) = 1.$$

Thus VI is the sum of finitely many terms and  $\mathbb{P}_{\mu_B}(\text{VI} < +\infty) = 1$ . The above arguments show that the right hand side of (5.25) is finite a.s., and hence

$$\limsup_{t \rightarrow +\infty} \mathbb{P}_{\mu_B} \left( W_t^h(\Gamma) \mid \mathcal{G} \right) < +\infty \quad \mathbb{P}_{\mu_B}\text{-a.s.}$$

By Fatou's lemma,  $\mathbb{P}_{\mu_B} \left( \liminf_{t \rightarrow +\infty} W_t^h(\Gamma) \mid \mathcal{G} \right) < +\infty$   $\mathbb{P}_{\mu_B}$ -a.s. Let

$$A_n := \left\{ \mathbb{P}_{\mu_B} \left( \liminf_{t \rightarrow +\infty} W_t^h(\Gamma) \mid \mathcal{G} \right) \leq n \right\} \in \mathcal{G} \quad \text{for } n \geq 1.$$

Then  $\mathbb{P}_{\mu_B}(\cup_{n=1}^{+\infty} A_n) = 1$ . Since

$$\int_{A_n} \liminf_{t \rightarrow +\infty} W_t^h(\Gamma) d\mathbb{P}_{\mu_B} = \int_{A_n} \mathbb{P}_{\mu_B} \left( \liminf_{t \rightarrow +\infty} W_t^h(\Gamma) \mid \mathcal{G} \right) d\mathbb{P}_{\mu_B} \leq n,$$

we get  $\liminf_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty$   $\mathbb{P}_{\mu_B}$ -a.s on  $A_n$  for all  $n \geq 1$ . Thus

$$\mathbb{P}_{\mu_B} \left( \liminf_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty \right) = 1.$$

Note that by [Harris and Roberts \(2009, Proposition 2\)](#)  $W_t^h(\Gamma)^{-1}$  is a non-negative  $\mathbb{P}_{\mu_B}$ -supermartingale, which implies that  $\lim_{t \rightarrow +\infty} W_t^h(\Gamma)^{-1}$  exists  $\mathbb{P}_{\mu_B}$ -a.s. It follows that

$$\mathbb{P}_{\mu_B} \left( \limsup_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty \right) = 1.$$

This proves (5.23) and consequently  $\mathbb{P}_{\mu_B} [W_\infty^h(X)] = \langle h, \mu_B \rangle$ . Since  $\mathbb{P}_{\mu_B} [W_\infty^h(X)] = \langle gh, \mu_B \rangle$ , where  $g(x) = h(x)^{-1} \mathbb{P}_{\delta_x} [W_\infty^h(X)]$ , we have

$$\langle gh, \mu_B \rangle = \langle h, \mu_B \rangle. \quad (5.27)$$

Note that  $0 \leq g(x) \leq 1$  for every  $x \in E$ . We get by (5.27) that  $g(x) = 1$   $m$ -a.e. on  $B$ . Since  $B$  is arbitrary,  $g(x) = 1$   $m$ -a.e. on  $E$ . It then follows from (5.14) that  $g(x) = \tilde{\Theta}_t g(x) = \int_E \tilde{p}(t, x, y) g(y) \rho(dy) = 1$  for every  $x \in E$ . Therefore by (5.13),  $\mathbb{P}_\mu [W_\infty^h(X)] = \langle h, \mu \rangle$  holds for all  $\mu \in \mathcal{M}(E)$ . This completes the proof for Theorem 5.1.(i).

(ii) Suppose  $\lambda_1 > 0$ . Clearly  $\mathbb{P}_\mu (W_\infty^h(X) = 0) = 1$  if and only if  $\mathbb{P}_\mu [W_\infty^h(X)] = 0$ . By (5.13), this would follow if  $g(x) = 0$  for every  $x \in E$ . Recall that  $g(x) = \tilde{\Theta}_t g(x) = \int_E \tilde{p}(t, x, y) g(y) \rho(dy)$ . It suffices to prove that  $g(x) = 0$  for  $m$ -almost every  $x \in E$ , or equivalently,

$$\mathbb{P}_{\delta_x} [W_\infty^h(X)] = 0 \quad \text{for } m\text{-a.e. } x \in E. \quad (5.28)$$

By the change of measure methodology (see, for example, [Durrett, 2019](#), Theorem 5.3.3), (5.28) would follow if

$$\mathbb{P}_{\delta_x} (\limsup_{t \rightarrow +\infty} W_t^h(\Gamma) = +\infty) = 1 \quad \text{for } m\text{-a.e. } x \in E. \quad (5.29)$$

By the definition of  $\Gamma_t$ , we have

$$W_s^h(\Gamma) = e^{\lambda_1 s} \langle h, \Gamma_s \rangle \geq e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) \quad \text{for } s \in D^m,$$

and

$$W_{\tau_i}^h(\Gamma) \geq e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \quad \text{for } i \geq 1.$$

Thus under  $\mathbb{P}_{\delta_x}$ ,

$$\begin{aligned} \limsup_{t \rightarrow +\infty} W_t^h(\Gamma) &\geq \limsup_{D^m \ni s \rightarrow +\infty} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \geq 1\}} \\ &\quad \vee \limsup_{i \rightarrow +\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq 1\}}. \end{aligned} \quad (5.30)$$

Lemma 5.5 implies that for  $m$ -a.e.  $x \in E$ , both  $\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq 1\}}$  and  $\Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \geq 1\}}$  grow subexponentially. Thus when  $\lambda_1 > 0$ , the right hand side of (5.30) goes to infinity. Hence we get (5.29) for  $m$ -a.e.  $x \in E$ .  $\square$

## 6. Necessary conditions for a non-degenerate martingale limit

In this section we will give necessary conditions for the fundamental martingale to have a non-degenerate limit. Recall from Proposition 4.1 that  $\tilde{p}(t, x, y)$  is the transition density of the spine  $\tilde{\xi}$  with respect to the measure  $\rho$ . We start with the following assumption.

**Assumption 4.**

$$\lim_{t \rightarrow +\infty} \sup_{x \in E} \operatorname{esssup}_{y \in E} |\tilde{p}(t, x, y) - 1| = 0.$$

**Proposition 6.1.** *Suppose that Assumptions 0–4 hold. Then  $\rho$  is an ergodic measure for  $(\tilde{\mathfrak{G}}_t)_{t \geq 0}$ , that is,  $\rho$  is an invariant probability measure for  $(\tilde{\mathfrak{G}}_t)_{t \geq 0}$  and for any invariant set  $B$ , either  $\rho(B) = 0$  or  $\rho(B) = 1$ .*

*Proof:* Recall from Proposition 5.3 that  $\rho$  is an invariant probability measure for  $(\tilde{\mathfrak{G}}_t)_{t \geq 0}$ . By Da Prato and Zabczyk (1996, Theorem 3.2.4), it suffices to prove that for any  $\varphi \in L^2(E, \rho)$ ,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \tilde{\mathfrak{G}}_s \varphi ds = \langle \varphi, \rho \rangle \quad \text{in } L^2(E, \rho). \quad (6.1)$$

It follows from Assumption 4 that for any  $\varepsilon > 0$ , there is  $t_0 > 0$  such that

$$\sup_{x \in E} \operatorname{esssup}_{y \in E} |\tilde{p}(s, x, y) - 1| \leq \varepsilon \quad \text{for all } s \geq t_0. \quad (6.2)$$

For  $x \in E$  and  $t > t_0$ ,

$$\frac{1}{t} \int_0^t \tilde{\mathfrak{G}}_s \varphi ds - \langle \varphi, \rho \rangle = \frac{1}{t} \int_0^{t_0} \tilde{\mathfrak{G}}_s \varphi ds - \frac{t_0}{t} \langle \varphi, \rho \rangle + \frac{1}{t} \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1) \varphi(y) \rho(dy). \quad (6.3)$$

By (6.2) and Jensen's inequality, we have

$$\begin{aligned} & \left\| \frac{1}{t} \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1) \varphi(y) \rho(dy) \right\|_{L^2(E, \rho)}^2 \\ &= \frac{1}{t^2} \int_E \rho(dx) \left( \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1) \varphi(y) \rho(dy) \right)^2 \\ &\leq \frac{t - t_0}{t^2} \int_E \rho(dx) \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1)^2 \varphi(y)^2 \rho(dy) \\ &\leq \frac{(t - t_0)^2}{t^2} \varepsilon^2 \|\varphi\|_{L^2(E, \rho)}^2. \end{aligned} \quad (6.4)$$

Moreover, by Jensen's inequality and (5.12),

$$\begin{aligned} \left\| \frac{1}{t} \int_0^{t_0} \tilde{\mathfrak{G}}_s \varphi ds \right\|_{L^2(E, \rho)}^2 &= \frac{1}{t^2} \int_E \rho(dx) \left( \int_0^{t_0} \tilde{\mathfrak{G}}_s \varphi(x) ds \right)^2 \\ &\leq \frac{t_0}{t^2} \int_E \rho(dx) \int_0^{t_0} \tilde{\mathfrak{G}}_s (\varphi^2)(x) ds \\ &= \frac{t_0^2}{t^2} \int_E \varphi(x)^2 \rho(dx) = \frac{t_0^2}{t^2} \|\varphi\|_{L^2(E, \rho)}^2. \end{aligned} \quad (6.5)$$

By (6.3)–(6.5), we have

$$\begin{aligned} \left\| \frac{1}{t} \int_0^t \tilde{\mathfrak{G}}_s \varphi ds - \langle \varphi, \rho \rangle \right\|_{L^2(E, \rho)} &\leq \frac{t_0}{t} \|\varphi\|_{L^2(E, \rho)} + \frac{t_0}{t} |\langle \varphi, \rho \rangle| + \frac{t - t_0}{t} \varepsilon \|\varphi\|_{L^2(E, \rho)} \\ &\leq \frac{2t_0}{t} \|\varphi\|_{L^2(E, \rho)} + \frac{t - t_0}{t} \varepsilon \|\varphi\|_{L^2(E, \rho)}. \end{aligned}$$

Letting  $t \rightarrow +\infty$  and then  $\varepsilon \rightarrow 0$ , we get (6.1). □

Define

$$E_1 := \{x \in E : \text{supp } \Pi^L(x, \cdot) \supseteq [N, +\infty) \text{ for some } N \geq 0\}, \tag{6.6}$$

$$E_2 := \{x \in A : \text{supp } \Pi^{NL}(x, \cdot) \supseteq [N, +\infty) \text{ for some } N \geq 0\}. \tag{6.7}$$

The main result of this section is the following theorem.

**Theorem 6.2.** *Suppose that Assumptions 0-4 hold. Then  $\mathbb{P}_\mu(W_\infty^h(X) = 0) = 1$  for all  $\mu \in \mathcal{M}(E)$  if either (5.1) fails or the following conditions hold.*

$$\lambda_1 \geq 0 \text{ and } m(E_1 \cup E_2) > 0.$$

To prove Theorem 6.2, we need the following lemma.

**Lemma 6.3.** *Suppose that Assumptions 0-4 hold.*

(i) *If  $m(E_1 \cup E_2) > 0$ , then for  $m$ -almost every  $x \in E$ ,*

$$\limsup_{D^m \ni s \rightarrow +\infty} \Lambda_s^m h(\tilde{\xi}_s) \vee \limsup_{i \rightarrow +\infty} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.}; \tag{6.8}$$

(ii) *if (5.1) fails, then for  $m$ -almost every  $x \in E$ ,*

$$\limsup_{D^m \ni s \rightarrow +\infty} \frac{\log^+ \Lambda_s^m h(\tilde{\xi}_s)}{s} \vee \limsup_{i \rightarrow +\infty} \frac{\log^+ \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h)}{\tau_i} = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.} \tag{6.9}$$

*Proof:* It is easy to see that (6.9) is equivalent to saying that for  $m$ -almost every  $x \in E$  and all  $\lambda < 0$ ,

$$\limsup_{D^m \ni s \rightarrow +\infty} e^{\lambda s} \Lambda_s^m h(\tilde{\xi}_s) \vee \limsup_{i \rightarrow +\infty} e^{\lambda \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.}$$

We divide the conditions of this lemma into two cases, and prove the results separately.

Case I: Suppose either one of the following conditions holds:

(I.a)  $m(E_1) > 0$ ;

(I.b)  $\left( \int_{(0, +\infty)} rh(\cdot) \log^+(rh(\cdot)) \Pi^L(\cdot, dr), \hat{h} \right) = +\infty$ .

Let  $\lambda < 0$  be an arbitrary constant. To prove (6.8) ( resp. (6.9)) under condition (I.a) (resp. (I.b)), it suffices to prove that for  $m$ -a.e.  $x \in E$  and any  $M \geq 1$ ,

$$\begin{aligned} & \mathbb{P}_{\cdot, x} \left( \sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \geq M\}} = +\infty \right) = 1 \\ & \text{(resp. } \mathbb{P}_{\cdot, x} \left( \sum_{s \in D^m} 1_{\{e^{\lambda s} \Lambda_s^m h(\tilde{\xi}_s) \geq M\}} = +\infty \right) = 1 \text{)}. \end{aligned} \tag{6.10}$$

For  $0 \leq s \leq t < +\infty$ ,  $\theta \leq 0$  and  $M \geq 1$ , let

$$I_\theta(s, t) := \int_s^t dr \int_{(0, +\infty)} u 1_{\{e^{\theta r} u h(\tilde{\xi}_r) \geq M\}} \Pi^L(\tilde{\xi}_r, du),$$

and  $I_\theta(t) := I_\theta(0, t)$ . Recall that, given  $\tilde{\xi}$ , for any  $T > 0$ ,  $\#\{s \in D_T^m : e^{\theta s} \Lambda_s^m h(\tilde{\xi}_s) \geq M\}$  is a Poisson random variable with parameter  $I_\theta(T)$ . Hence (6.10) would follow if for  $m$ -a.e.  $x \in E$ ,

$$\mathbb{P}_{\cdot, x} (I_0(\infty) = +\infty) = 1 \quad \text{(resp. } \mathbb{P}_{\cdot, x} (I_\lambda(\infty) = +\infty) = 1) \tag{6.11}$$

under condition (I.a) (resp. (I.b)). Let  $\nu(dx) := \widehat{h}(x)m(dx)$ . Clearly  $\mathbb{P}_{\cdot,\nu} = \int_E \mathbb{P}_{\cdot,x} \rho(dx)$ . Recall that  $\rho$  is an invariant measure for  $\widetilde{\mathfrak{S}}_t$ . By Fubini's theorem,

$$\begin{aligned} \mathbb{P}_{\cdot,\nu}(\mathbf{I}_\theta(T)) &= \int_E \mathbb{P}_{\cdot,x} \left[ \int_0^T dr \int_{(0,+\infty)} u 1_{\{e^{\theta r} u h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, du) \right] \rho(dx) \\ &= \int_0^T dr \int_E \mathbb{P}_{\cdot,x} \left[ \int_{(0,+\infty)} u 1_{\{e^{\theta r} u h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, du) \right] \rho(dx) \\ &= \int_0^T dr \int_E \rho(dx) \int_{(0,+\infty)} u 1_{\{e^{\theta r} u h(x) \geq M\}} \Pi^L(x, du). \end{aligned} \quad (6.12)$$

By the boundedness of  $h$  and  $x \mapsto \int_{(0,+\infty)} (u \wedge u^2) \Pi^L(x, du)$ , we have

$$\begin{aligned} \mathbb{P}_{\cdot,\nu}(\mathbf{I}_\theta(T)) &\leq T \int_E \widehat{h}(x)m(dx) \int_{u \geq M/h(x)} h(x)u \Pi^L(x, du) \\ &= T \int_E \widehat{h}(x)m(dx) \int_{u \geq M/h(x)} h(x) \left(1 \vee \frac{1}{u}\right) (u \wedge u^2) \Pi^L(x, du) \\ &\leq T \left(1 \vee \frac{\|h\|_\infty}{M}\right) \left\| \int_{(0,+\infty)} (u \wedge u^2) \Pi^L(x, du) \right\|_\infty \int_E h(x) \widehat{h}(x)m(dx) < +\infty. \end{aligned}$$

Thus  $\mathbb{P}_{\cdot,\nu}(\mathbf{I}_\theta(T) < +\infty) = 1$ . On the other hand, by the Markov property of  $\widetilde{\xi}$  and (6.12),

$$\begin{aligned} \mathbb{P}_{\cdot,\nu}(\mathbf{I}_\theta(T)^2) &= \int_E \mathbb{P}_{\cdot,x}(\mathbf{I}_\theta(T)^2) \rho(dx) \\ &= 2 \int_E \rho(dx) \mathbb{P}_{\cdot,x} \left[ \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(\widetilde{\xi}_s) \geq M\}} \Pi^L(\widetilde{\xi}_s, du) \right. \\ &\quad \cdot \left. \int_s^T dr \int_{(0,+\infty)} v 1_{\{e^{\theta r} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right] \\ &= 2 \int_E \rho(dx) \mathbb{P}_{\cdot,x} \left[ \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(\widetilde{\xi}_s) \geq M\}} \Pi^L(\widetilde{\xi}_s, du) \right. \\ &\quad \cdot \left. \mathbb{P}_{\cdot,\widetilde{\xi}_s} \left( \int_0^{T-s} dr \int_{(0,+\infty)} v 1_{\{e^{\theta(r+s)} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right) \right] \\ &= 2 \int_E \rho(dx) \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(x) \geq M\}} \Pi^L(x, du) \\ &\quad \cdot \mathbb{P}_{\cdot,x} \left( \int_0^{T-s} dr \int_{(0,+\infty)} v 1_{\{e^{\theta(r+s)} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right) \\ &\leq 2 \int_E \rho(dx) \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(x) \geq M\}} \Pi^L(x, du) \\ &\quad \cdot \mathbb{P}_{\cdot,x} \left( \int_0^T dr \int_{(0,+\infty)} v 1_{\{e^{\theta r} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right) \\ &= 2 \int_E \rho(dx) \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(x) \geq M\}} \Pi^L(x, du) \mathbb{P}_{\cdot,x}(\mathbf{I}_\theta(T)). \end{aligned} \quad (6.13)$$

Assumption 4 implies that there are constants  $t_1, \delta > 0$  such that

$$\sup_{x \in E} \operatorname{ess\,sup}_{y \in E} \widetilde{p}(t, x, y) \leq 1 + \delta \quad \text{for all } t \geq t_1. \quad (6.14)$$

Using Fubini's theorem, (6.14) and (6.12), we have for  $T > t_1$ ,

$$\begin{aligned} \mathbb{P}_{\cdot, x} [\mathbf{I}_\theta(t_1, T)] &= \int_{t_1}^T dr \int_E \tilde{p}(r, x, y) \rho(dy) \int_{(0, +\infty)} v 1_{\{e^{\theta r} v h(y) \geq M\}} \Pi^L(y, dv) \\ &\leq (1 + \delta) \int_{t_1}^T dr \int_E \rho(dy) \int_{(0, +\infty)} v 1_{\{e^{\theta r} v h(y) \geq M\}} \Pi^L(y, dv) \\ &\leq (1 + \delta) \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T)). \end{aligned} \quad (6.15)$$

On the other hand, for  $x \in E$ ,

$$\begin{aligned} \mathbb{P}_{\cdot, x} (\mathbf{I}_\theta(t_1)) &= \mathbb{P}_{\cdot, x} \left[ \int_0^{t_1} dr \int_{(0, +\infty)} v 1_{\{e^{\theta r} v h(\tilde{\xi}_r) \geq M\}} \Pi^L(\tilde{\xi}_r, dv) \right] \\ &= \int_0^{t_1} dr \int_E \tilde{p}(r, x, y) \rho(dy) \int_{(0, +\infty)} v 1_{\{e^{\theta r} v h(y) \geq M\}} \Pi^L(y, dv) \\ &\leq \int_0^{t_1} dr \int_E \tilde{p}(r, x, y) \rho(dy) \int_{v \geq M/h(y)} \left( 1 \vee \frac{1}{v} \right) (v \wedge v^2) \Pi^L(y, dv) \\ &\leq t_1 \left( 1 \vee \frac{\|h\|_\infty}{M} \right) \left\| \int_{(0, +\infty)} (v \wedge v^2) \Pi^L(\cdot, dv) \right\|_\infty =: c_1 < +\infty. \end{aligned} \quad (6.16)$$

It follows from (6.15) and (6.16) that for  $T > t_1$ ,

$$\mathbb{P}_{\cdot, x} (\mathbf{I}_\theta(T)) = \mathbb{P}_{\cdot, x} (\mathbf{I}_\theta(t_1)) + \mathbb{P}_{\cdot, x} (\mathbf{I}_\theta(t_1, T)) \leq c_1 + (1 + \delta) \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T)).$$

This together with (6.12) and (6.13) implies that

$$\mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T)^2) \leq 2c_1 \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T)) + 2(1 + \delta) \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T))^2.$$

Hence by the Cauchy-Schwarz inequality, we have

$$\mathbb{P}_{\cdot, \nu} \left( \mathbf{I}_\theta(T) \geq \frac{1}{2} \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T)) \right) \geq \frac{\mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T))^2}{4 \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T)^2)} \geq \frac{\mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T))}{8c_1 + 8(1 + \delta) \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\theta(T))}. \quad (6.17)$$

Note that  $\mathbb{P}_{\cdot, \nu} (\mathbf{I}_0(T)) = T \int_E \rho(dx) \int_{u \geq h(x)/M} u \Pi^L(x, du)$ . Condition (I.a) implies that the integral on the right hand side is positive. Hence  $\mathbb{P}_{\cdot, \nu} (\mathbf{I}_0(T)) \rightarrow +\infty$  as  $T \rightarrow +\infty$ . On the other hand, note that by (6.12) and Fubini's theorem, for  $\lambda < 0$ ,

$$\begin{aligned} \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\lambda(T)) &\leq \int_E \hat{h}(x) m(dx) \int_{(0, +\infty)} h(x) u \Pi^L(x, du) \int_0^T 1_{\{s \leq \frac{\log^+(h(x)u) - \log M}{-\lambda}\}} ds \\ &= \int_E \hat{h}(x) m(dx) \int_{(0, +\infty)} h(x) u \left( T \wedge \frac{\log^+(h(x)u) - \log M}{-\lambda} \right)^+ \Pi^L(x, du). \end{aligned}$$

Clearly condition (I.b) implies that  $\lim_{T \rightarrow +\infty} \mathbb{P}_{\cdot, \nu} (\mathbf{I}_\lambda(T)) = +\infty$ . Thus by letting  $T \rightarrow +\infty$  in (6.17), we get  $\mathbb{P}_{\cdot, \nu} (\mathbf{I}_0(\infty) = +\infty) > 0$  (resp.  $\mathbb{P}_{\cdot, \nu} (\mathbf{I}_\lambda(\infty) = +\infty) > 0$ ) under condition (I.a) (resp. (I.b)). Since  $\{\mathbf{I}_0(\infty) = +\infty\}$  (resp.  $\{\mathbf{I}_\lambda(\infty) = +\infty\}$ ) is an invariant event of the canonical dynamic system associated with  $(\tilde{\mathfrak{S}}_t)_{t \geq 0}$  and ergodic measure  $\rho$ , it follows from [Da Prato and Zabczyk \(1996, Theorem 1.2.4\)](#) that  $\mathbb{P}_{\cdot, \nu} (\mathbf{I}_0(\infty) = +\infty) = 1$  (resp.  $\mathbb{P}_{\cdot, \nu} (\mathbf{I}_\lambda(\infty) = +\infty) = 1$ ) under condition (I.a) (resp. (I.b)). Hence we prove (6.11).

Case II. Suppose either one of the following conditions holds:

(II.a)  $m(E_2) > 0$ ;

(II.b)  $\left( \int_{(0, +\infty)} \pi(\cdot, h) r \log^+(\pi(\cdot, h) r) \Pi^{NL}(\cdot, dr), \hat{h} \right) = +\infty$ .



Let  $\lambda < 0$  be an arbitrary constant. To prove (6.8) ( resp. (6.9)) under condition (II.a) (resp. (II.b)), it suffices to prove that for  $m$ -a.e.  $x \in E$  and any  $M \geq 1$ ,

$$\begin{aligned} & \mathbb{P}_{\cdot, x} \left( \sum_{i=1}^{+\infty} \mathbf{1}_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-, h}) \geq M\}} = +\infty \right) = 1 \\ & \text{(resp. } \mathbb{P}_{\cdot, x} \left( \sum_{i=1}^{+\infty} \mathbf{1}_{\{e^{\lambda \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-, h}) \geq M\}} = +\infty \right) = 1 \text{)}. \end{aligned} \quad (6.18)$$

The main idea of this proof is similar to that of Case I. For all  $T > 0$ ,  $\theta \leq 0$  and  $M \geq 1$ , let  $\mathbb{II}_\theta(T) := \sum_{\tau_i \leq T} \mathbf{1}_{\{e^{\theta \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-, h}) \geq M\}}$ . For any  $s \geq 0$  and  $x \in E$ , define

$$\begin{aligned} f_\theta(s, x) &:= \int_{[0, +\infty)} \mathbf{1}_{\{e^{\theta s} u \pi(x, h) \geq M\}} \eta(x, du) \\ &= \frac{1}{\gamma(x)} \mathbf{1}_A(x) \int_{(0, +\infty)} \mathbf{1}_{\{e^{\theta s} u \pi(x, h) \geq M\}} u \Pi^{NL}(x, du), \\ g_\theta(s, x) &:= q(x) f_\theta(s, x) \\ &= \frac{\pi(x, h)}{h(x)} \mathbf{1}_A(x) \int_{(0, +\infty)} \mathbf{1}_{\{e^{\theta s} u \pi(x, h) \geq M\}} u \Pi^{NL}(x, du). \end{aligned}$$

Recall from Definition 4.4 that, given  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $\Theta_i$  is distributed according to  $\eta(\tilde{\xi}_{\tau_i-}, dr)$ . By (4.6), we have for  $x \in E$ ,

$$\begin{aligned} \mathbb{P}_{\cdot, x}(\mathbb{II}_\theta(T)) &= \mathbb{P}_{\cdot, x} \left[ \sum_{\tau_i \leq T} \int_{[0, +\infty)} \mathbf{1}_{\{e^{\theta \tau_i} r \pi(\tilde{\xi}_{\tau_i-, h}) \geq M\}} \eta(\tilde{\xi}_{\tau_i-}, dr) \right] \\ &= \mathbb{P}_{\cdot, x} \left[ \sum_{\tau_i \leq T} f_\theta(\tau_i, \tilde{\xi}_{\tau_i-}) \right] = \mathbb{P}_{\cdot, x} \left[ \int_0^T q(\tilde{\xi}_s) f_\theta(s, \tilde{\xi}_s) ds \right] \\ &= \mathbb{P}_{\cdot, x} \left[ \int_0^T g_\theta(s, \tilde{\xi}_s) ds \right]. \end{aligned} \quad (6.19)$$

We still use  $\nu$  to denote the measure  $\hat{h}(x)m(dx)$ . Since  $\rho$  is an invariant measure for  $\tilde{\mathfrak{S}}_t$ , by Fubini's theorem,

$$\begin{aligned} \mathbb{P}_{\cdot, \nu}(\mathbb{II}_\theta(T)) &= \int_E \mathbb{P}_{\cdot, x}(\mathbb{II}_\theta(T)) \rho(dx) = \int_0^T ds \int_E \mathbb{P}_{\cdot, x}(g_\theta(s, \tilde{\xi}_s)) \rho(dx) \\ &= \int_0^T ds \int_E g_\theta(s, x) \rho(dx) \end{aligned} \quad (6.20)$$

$$= \int_A \hat{h}(x)m(dx) \int_{(0, +\infty)} \pi(x, h)r \Pi^{NL}(x, dr) \int_0^T \mathbf{1}_{\{e^{\theta s} r \pi(x, h) \geq M\}} ds. \quad (6.21)$$

It then follows by Assumption 3.(ii) that

$$\text{RHS of (6.21)} \leq T \left\| \int_{(0, +\infty)} y \Pi^{NL}(\cdot, dy) \right\|_\infty \int_A \pi(x, h) \hat{h}(x)m(dx) < +\infty.$$

Therefore  $\mathbb{P}_{\cdot,\nu}(\mathbb{II}_\theta(T) < +\infty) = 1$ . Recall that given  $\tilde{\xi}$  (including  $\{\tau_i : i \geq 1\}$ ),  $\{\Theta_i : i \geq 1\}$  are mutually independent, we have

$$\begin{aligned} & \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)^2) - \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)) \\ &= \mathbb{P}_{\cdot,x} \left[ \sum_{\tau_i, \tau_j \leq T, i \neq j} \mathbf{1}_{\{e^{\theta\tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \mathbf{1}_{\{e^{\theta\tau_j} \Theta_j \pi(\tilde{\xi}_{\tau_j-}, h) \geq M\}} \right] \\ &= \mathbb{P}_{\cdot,x} \left[ \sum_{\tau_i, \tau_j \leq T, i \neq j} \int_{[0,+\infty)} \mathbf{1}_{\{e^{\theta\tau_i} y \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_i-}, dy) \right. \\ & \quad \left. \cdot \int_{[0,+\infty)} \mathbf{1}_{\{e^{\theta\tau_j} z \pi(\tilde{\xi}_{\tau_j-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_j-}, dz) \right] \\ &= \mathbb{P}_{\cdot,x} \left[ \sum_{\tau_i, \tau_j \leq T, i \neq j} f_\theta(\tau_i, \tilde{\xi}_{\tau_i-}) f_\theta(\tau_j, \tilde{\xi}_{\tau_j-}) \right]. \end{aligned}$$

Thus by (4.7),

$$\begin{aligned} & \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)^2) - \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)) \\ &= 2\mathbb{P}_{\cdot,x} \left[ \int_0^T q(\tilde{\xi}_s) f_\theta(s, \tilde{\xi}_s) ds \int_E \tilde{\Pi}_y \left( \int_0^{T-s} q(\tilde{\xi}_r) f_\theta(s+r, \tilde{\xi}_r) dr \right) \pi^h(\tilde{\xi}_s, dy) \right] \\ &= 2\mathbb{P}_{\cdot,x} \left[ \int_0^T g_\theta(s, \tilde{\xi}_s) ds \int_E \tilde{\Pi}_y \left( \int_0^{T-s} g_\theta(s+r, \tilde{\xi}_r) dr \right) \pi^h(\tilde{\xi}_s, dy) \right]. \end{aligned} \quad (6.22)$$

Note that for all  $x \in E$  and  $\theta \leq 0$ ,  $s \mapsto g_\theta(s, x)$  is non-increasing. Thus it follows from (6.22) that

$$\mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)^2) \leq 2\mathbb{P}_{\cdot,x} \left[ \int_0^T g_\theta(s, \tilde{\xi}_s) ds \int_E \tilde{\Pi}_y \left( \int_0^T g_\theta(r, \tilde{\xi}_r) dr \right) \pi^h(\tilde{\xi}_s, dy) \right] + \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)). \quad (6.23)$$

By Fubini's theorem, (6.14) and (6.20), we have for  $y \in E$  and  $T > t_1$ ,

$$\begin{aligned} \tilde{\Pi}_y \left( \int_{t_1}^T g_\theta(s, \tilde{\xi}_s) ds \right) &= \int_{t_1}^T ds \int_E \tilde{p}(s, y, z) g_\theta(s, z) \rho(dz) \\ &\leq (1 + \delta) \int_{t_1}^T ds \int_E g_\theta(s, z) \rho(dz) \\ &\leq (1 + \delta) \mathbb{P}_{\cdot,\nu}(\mathbb{II}_\theta(T)). \end{aligned} \quad (6.24)$$

On the other hand, by Assumption 3.(iii),

$$\begin{aligned} \sup_{y \in E} \tilde{\Pi}_y \left( \int_0^{t_1} g_\theta(s, \tilde{\xi}_s) ds \right) &= \sup_{y \in E} \int_0^{t_1} ds \int_A \tilde{p}(s, y, z) \frac{\pi(z, h)}{h(z)} \rho(dz) \int_{(0,+\infty)} r \mathbf{1}_{\{e^{\theta s} r \pi(z, h) \geq M\}} \Pi^{NL}(z, dr) \\ &\leq \left\| \frac{\pi(\cdot, h)}{h} \mathbf{1}_A \right\|_\infty \left\| \int_{(0,+\infty)} r \Pi^{NL}(\cdot, dr) \right\|_\infty \sup_{y \in E} \int_0^{t_1} ds \int_A \tilde{p}(s, y, z) \rho(dz) \\ &\leq c_2 t_1 =: c_3 < +\infty. \end{aligned}$$

This and (6.24) imply that

$$\tilde{\Pi}_y \left( \int_0^T g_\theta(s, \tilde{\xi}_s) ds \right) \leq c_3 + (1 + \delta) \mathbb{P}_{\cdot,\nu}(\mathbb{II}_\theta(T)) \quad \text{for all } y \in E \text{ and } T > t_1.$$

This together with (6.23) and (6.19) implies that

$$\mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)^2) \leq (1 + 2c_3) \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)) + 2(1 + \delta) \mathbb{P}_{\cdot,\nu}(\mathbb{II}_\theta(T)) \mathbb{P}_{\cdot,x}(\mathbb{II}_\theta(T)).$$

Consequently,

$$\mathbb{P}_{\cdot, \nu} (\mathbb{II}_\theta(T)^2) = \int_E \mathbb{P}_{\cdot, x} (\mathbb{II}_\theta(T)^2) \rho(dx) \leq (1 + 2c_3) \mathbb{P}_{\cdot, \nu} (\mathbb{II}_\theta(T)) + 2(1 + \delta) \mathbb{P}_{\cdot, \nu} (\mathbb{II}_\theta(T))^2.$$

Recall that  $\mathbb{P}_{\cdot, \nu} (\mathbb{II}_0(T)) = T \int_A \pi(x, h) \widehat{h}(x) m(dx) \int_{r \geq M/\pi(x, h)} r \Pi^{NL}(x, dr)$ . Condition (II.a) implies that the integral on the right hand side is positive. Thus  $\mathbb{P}_{\cdot, \nu} (\mathbb{II}_0(T)) \rightarrow +\infty$  as  $T \rightarrow +\infty$ . On the other hand, note that by (6.21) and Fubini's theorem, for  $\lambda < 0$ ,

$$\mathbb{P}_{\cdot, \nu} (\mathbb{II}_\lambda(T)) = \int_A \widehat{h}(x) m(dx) \int_{(0, +\infty)} \Pi^{NL}(x, dr) \pi(x, h) r \left( \frac{\log^+(\pi(x, h)r) - \log M}{-\lambda} \wedge T \right)^+.$$

Clearly condition (II.b) implies that  $\lim_{T \rightarrow +\infty} \mathbb{P}_{\cdot, \nu} (\mathbb{II}_\lambda(T)) = +\infty$ . Similarly by using the Cauchy-Schwarz inequality and letting  $T \rightarrow +\infty$ , we get  $\mathbb{P}_{\cdot, \nu} (\mathbb{II}_0(\infty) = +\infty) > 0$  (resp.  $\mathbb{P}_{\cdot, \nu} (\mathbb{II}_\lambda(\infty) = +\infty) > 0$ ) under condition (II.a) (resp. (II.b)).

For each  $n \geq 1$ , we denote by  $\widetilde{\mathcal{G}}_n$  the  $\sigma$ -field generated by  $\xi$  up to time  $\tau_n$  (including  $\{\tau_1, \dots, \tau_n\}$ ) and  $\{\Theta_i : i \leq n\}$ . Obviously for each  $i \geq 1$ , both

$$\mathbb{1}_{\{e^{\theta\tau_i} \Theta_i \pi(\widetilde{\xi}_{\tau_i-, h}) \geq M\}} \quad \text{and} \quad \int_{[0, +\infty)} \mathbb{1}_{\{e^{\theta\tau_i} r \pi(\widetilde{\xi}_{\tau_i-, h}) \geq M\}} \eta(\widetilde{\xi}_{\tau_i-, dr})$$

are  $\widetilde{\mathcal{G}}_i$ -measurable. Moreover, for every  $x \in E$ , under  $\mathbb{P}_{\cdot, x}$ ,

$$\mathbb{P}_{\cdot, x} \left( \mathbb{1}_{\{e^{\theta\tau_{i+1}} \Theta_{i+1} \pi(\widetilde{\xi}_{\tau_{i+1}-, h}) \geq M\}} \mid \widetilde{\mathcal{G}}_i \right) = \mathbb{P}_{\cdot, x} \left( \int_{[0, +\infty)} \mathbb{1}_{\{e^{\theta\tau_{i+1}} r \pi(\widetilde{\xi}_{\tau_{i+1}-, h}) \geq M\}} \eta(\widetilde{\xi}_{\tau_{i+1}-, dr}) \mid \widetilde{\mathcal{G}}_i \right).$$

Applying the second Borel-Cantelli lemma (see, for example, [Durrett, 2019](#), Corollary 5.3.2) to both sides of the above equality, we get that

$$\left\{ \sum_{i=1}^{+\infty} \mathbb{1}_{\{e^{\theta\tau_i} \Theta_i \pi(\widetilde{\xi}_{\tau_i-, h}) \geq M\}} = +\infty \right\} = \left\{ \sum_{i=1}^{+\infty} \int_{[0, +\infty)} \mathbb{1}_{\{e^{\theta\tau_i} r \pi(\widetilde{\xi}_{\tau_i-, h}) \geq M\}} \eta(\widetilde{\xi}_{\tau_i-, dr}) = +\infty \right\}$$

under  $\mathbb{P}_{\cdot, x}$ . It is easy to see from the above representation that  $\{\mathbb{II}_0(\infty) = +\infty\}$  (resp.  $\{\mathbb{II}_\lambda(\infty) = +\infty\}$ ) is an invariant event of the canonical dynamic system associated with  $(\widetilde{\mathcal{G}}_t)_{t \geq 0}$  and ergodic measure  $\rho$ , so it follows from [Da Prato and Zabczyk \(1996, Theorem 1.2.4\)](#) that  $\mathbb{P}_{\cdot, \nu} (\mathbb{II}_0(\infty) = +\infty) = 1$  (resp.  $\mathbb{P}_{\cdot, \nu} (\mathbb{II}_\lambda(\infty) = +\infty) = 1$ ) under condition (II.a) (resp. (II.b)). Thus (6.18) is valid.  $\square$

**Proof of Theorem 6.2.** Applying the same argument as in the beginning of the proof of Theorem 5.1.(ii) here, we only need to show that under the assumptions of Theorem 6.2,

$$\mathbb{P}_{\delta_x} (\limsup_{t \rightarrow +\infty} W_t^h(\Gamma) = +\infty) = 1 \quad \text{for } m\text{-a.e. } x \in E.$$

In view of (5.30), this would follow if for  $m$ -a.e.  $x \in E$ ,

$$\limsup_{D^m \ni s \rightarrow +\infty} e^{\lambda_1 s} \Lambda_s^m h(\widetilde{\xi}_s) \vee \limsup_{i \rightarrow +\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\widetilde{\xi}_{\tau_i-, h}) = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.}$$

which, under the assumptions of this theorem, is automatically true by Lemma 6.3. Hence we complete the proof.  $\square$

The following corollaries follow directly from Theorem 5.1 and Theorem 6.2.

**Corollary 6.4.** *Suppose that Assumptions 0-4 hold and that  $m(E_1 \cup E_2) > 0$  with  $E_1$  and  $E_2$  defined in (6.6) and (6.7) respectively. For every  $\mu \in \mathcal{M}(E)^0$ ,  $W_\infty^h(X)$  is non-degenerate if and only if  $\lambda_1 < 0$  and condition (5.1) holds. Moreover,  $X_t$  under  $\mathbb{P}_\mu$  exhibits weak local extinction if  $\lambda_1 \geq 0$ .*

**Corollary 6.5.** *Suppose Assumptions 0-4 hold and  $\lambda_1 < 0$ . For every  $\mu \in \mathcal{M}(E)^0$ ,  $W_\infty^h(X)$  is non-degenerate if and only if condition (5.1) holds.*

*Remark 6.6.* Note that in the case of purely local branching mechanism, Assumption 4 can be written as

$$\lim_{t \rightarrow +\infty} \sup_{x \in E} \operatorname{esssup}_{y \in E} |p^h(t, x, y) - 1| = 0,$$

where  $p^h(t, x, y)$  denotes the transition density function of  $\xi^h$  with respect to the measure  $\rho$ . If  $E$  is a bounded domain in  $\mathbb{R}^d$ ,  $m$  is the Lebesgue measure on  $\mathbb{R}^d$  and  $\xi$  is a symmetric diffusion on  $E$ , then  $a(x) \in \mathcal{B}_b(E) \subset \mathbf{K}(\xi) \cap L^2(E, m)$ . Hence for the class of superdiffusions with local branching mechanisms considered in Liu et al. (2009), our Assumptions 0-4 hold naturally and Corollary 6.5 generalizes Liu et al. (2009, Theorem 1.1).

### 7. Examples

In this section, we will give examples satisfying Assumptions 0-4. We will not try to give the most general examples possible.

*Example 7.1.* Suppose  $E = \{1, 2, \dots, K\}$  ( $K \geq 2$ ),  $m$  is the counting measure on  $E$  and  $\mathfrak{S}_t f(i) = f(i)$  for all  $i \in E$ ,  $t \geq 0$  and  $f \in \mathcal{B}^+(E)$  (that is, there is no spatial motion). Suppose

$$\begin{aligned} \phi^L(i, \lambda) &:= a(i)\lambda + b(i)\lambda^2 + \int_{(0, +\infty)} (e^{-\lambda r} - 1 + \lambda r) \Pi^L(i, dr), \\ \phi^{NL}(i, f) &:= -c(i)\pi(i, f) - \int_{(0, +\infty)} (1 - e^{-r\pi(i, f)}) \Pi^{NL}(i, dr), \end{aligned}$$

where for each  $i \in E$ ,  $a(i) \in (-\infty, +\infty)$ ,  $b(i), c(i) \geq 0$ ,  $(r \wedge r^2)\Pi^L(i, dr)$  and  $r\Pi^{NL}(i, dr)$  are bounded kernels from  $E$  to  $(0, +\infty)$  with  $\{i \in E : \int_{(0, +\infty)} r\Pi^{NL}(i, dr) > 0\} \neq \emptyset$ , and  $\pi(i, dj)$  is a probability kernel on  $E$  with  $\pi(i, \{i\}) = 0$  for every  $i \in E$ . As a special case of the model given in Section 2.1, we have a non-local branching superprocess  $\{X_t : t \geq 0\}$  in  $\mathcal{M}(E)$  with transition probabilities given by

$$P_\mu [\exp(-\langle f, X_t \rangle)] = \exp(-\langle V_t f, \mu \rangle) \quad \text{for } \mu \in \mathcal{M}(E), t \geq 0 \text{ and } f \in \mathcal{B}_b^+(E),$$

where  $V_t f(i)$  is the unique non-negative locally bounded solution to the following integral equation:

$$V_t f(i) = f(i) - \int_0^t (\phi^L(i, V_s f(i)) + \phi^{NL}(i, V_s f)) ds \quad \text{for } t \geq 0, i \in E.$$

For every  $i \in E$  and  $\mu \in \mathcal{M}(E)$ , we define  $\mu^{(i)} := \mu(\{i\})$ . The map  $\mu \mapsto (\mu^{(1)}, \dots, \mu^{(K)})^T$  is clearly a homeomorphism between  $\mathcal{M}(E)$  and the  $K$ -dimensional product space  $[0, +\infty)^K$ . Hence  $\{(X_t^{(1)}, \dots, X_t^{(K)})^T : t \geq 0\}$  is a Markov process in  $[0, +\infty)^K$ , which is called a  $K$ -type continuous-state branching process. (Clearly the 1-type continuous-state branching process defined in a similar way coincides with the classical one-dimensional continuous-state branching process, see, for example, Li, 2011, Chapter 3.) For simplicity, we assume  $b(i) \equiv 0$ . For  $i, j \in E$ , let  $p_{ij} := \pi(i, \{j\})$  and  $\gamma(i) := c(i) + \int_{(0, +\infty)} r\Pi^{NL}(i, dr)$ . Define the  $K \times K$  matrix  $\mathbf{M}(t) = (M(t)_{ij})_{ij}$  by  $M(t)_{ij} := P_{\delta_i} [X_t^{(j)}]$  for  $i, j \in E$ . Let  $\mathfrak{P}_t$  denote the mean semigroup of  $X$ , that is

$$\mathfrak{P}_t f(i) := P_{\delta_i} [\langle f, X_t \rangle] = \sum_{j=1}^K M(t)_{ij} f(j) \quad \text{for } i \in E, t \geq 0 \text{ and } f \in \mathcal{B}^+(E).$$

By the Markov property and (2.9),  $\mathbf{M}(t)$  satisfies that

$$\mathbf{M}(0) = \mathbf{I}, \quad \mathbf{M}(t + s) = \mathbf{M}(t)\mathbf{M}(s) \quad \text{for } t, s \geq 0,$$

$$\text{and } M(t)_{ij} = \delta_j(i) - a(i) \int_0^t M(s)_{ij} ds + \gamma(i) \sum_{k=1}^K p_{ik} \int_0^t M(s)_{kj} ds$$

for  $i, j \in E$ . This implies that  $\mathbf{M}(t)$  has a formal matrix generator  $\mathbf{A} := (A_{ij})_{ij}$  given by

$$\mathbf{M}(t) = e^{\mathbf{A}t} = \sum_{n=0}^{+\infty} t^n \frac{\mathbf{A}^n}{n!}, \quad \text{and } A_{ij} = \gamma(i)p_{ij} - a(i)\delta_i(j) \text{ for } i, j \in E.$$

We assume  $\mathbf{A}$  is an irreducible matrix. It then follows by [Barczy and Pap \(2016, Lemma A.1\)](#) that  $M(t)_{ij} > 0$  for all  $t > 0$  and  $i, j \in E$ . Let  $\Lambda := \sup_{\lambda \in \sigma(\mathbf{A})} \operatorname{Re}(\lambda)$  where  $\sigma(\mathbf{A})$  denotes the set of eigenvalues of  $\mathbf{A}$ . The Perron-Frobenius theory (see, for example, [Barczy and Pap, 2016, Lemma A.3](#)) tells us that for every  $t > 0$ ,  $e^{\Lambda t}$  is a simple eigenvalue of  $\mathbf{M}(t)$ , and there exist a unique positive right eigenvector  $\mathbf{u} = (u_1, \dots, u_K)^T$  and a unique positive left eigenvector  $\mathbf{v} = (v_1, \dots, v_K)^T$  such that

$$\sum_{i=1}^K u_i = \sum_{i=1}^K u_i v_i = 1, \quad \mathbf{M}(t)\mathbf{u} = e^{\Lambda t}\mathbf{u}, \quad \mathbf{v}^T \mathbf{M}(t) = e^{\Lambda t}\mathbf{v}.$$

Moreover it is known by [Barczy and Pap \(2016, Lemma A.3\)](#) that for each  $i, j \in E$ ,

$$e^{-\Lambda t} M(t)_{ij} \rightarrow u_i v_j \quad \text{as } t \rightarrow +\infty. \quad (7.1)$$

One can easily verify that Assumptions 0-3 hold with  $\lambda_1 = -\Lambda$ ,  $h(i) = cu_i$  and  $\hat{h}(i) = c^{-1}v_i$ , where  $c := \left(\sum_{j=1}^K u_j^2\right)^{-1/2}$  is a positive constant. Thus  $W_t^h(X) := ce^{-\Lambda t} \sum_{i=1}^K u_i X_t^{(i)}$  is a non-negative martingale. Applying [Theorem 4.6](#) here, we can deduce that under the martingale change of measure the spine process  $\tilde{\xi}$  is a continuous-time Markov process on  $E$  with intensity matrix  $\mathbf{Q} = (q_{ij})_{ij}$  given by

$$q_{ii} := -\frac{\gamma(i) \sum_{j=1}^K p_{ij} u_j}{u_i} = -(\Lambda + a(i)), \quad q_{ij} := \frac{\gamma(i) p_{ij} u_j}{u_i} \quad \text{for } i \neq j.$$

Let  $\rho(dj) := u_j v_j m(dj) = \sum_{i=1}^K u_j v_j \delta_i(dj)$ . Let  $\tilde{\mathfrak{S}}_t$  denote the transition semigroup of the spine  $\tilde{\xi}$  and  $\tilde{p}(t, i, j)$  denote its transition density with respect to  $\rho$ . It follows by [Proposition 4.1](#) that for each  $i, j \in E$ ,

$$\begin{aligned} \tilde{p}(t, i, j) u_j v_j &= \int_E \tilde{p}(t, i, k) \delta_j(k) \rho(dk) = \tilde{\mathfrak{S}}_t \delta_j(i) \\ &= \frac{e^{-\Lambda t}}{h(i)} \mathfrak{P}_t(h\delta_j)(i) = \frac{e^{-\Lambda t}}{u_i} M(t)_{ij} u_j. \end{aligned} \quad (7.2)$$

Thus  $\tilde{p}(t, i, j) = e^{-\Lambda t} (u_i v_j)^{-1} M(t)_{ij}$ . By [\(7.1\)](#), we have for each  $i, j \in E$

$$\tilde{p}(t, i, j) \rightarrow 1 \quad \text{as } t \rightarrow +\infty.$$

Hence Assumption 4 also holds for this example. Applying [Corollary 6.4](#) here, we conclude that for every non-trivial  $\mu \in \mathcal{M}(E)$ , the martingale limit

$$W_\infty^h(X) := \lim_{t \rightarrow +\infty} W_t^h(X) = \lim_{t \rightarrow +\infty} ce^{-\Lambda t} \sum_{i=1}^K u_i X_t^{(i)}$$

is non-degenerate if and only if  $\Lambda > 0$  and

$$\sum_{i=1}^K u_i v_i \int_{(0, +\infty)} r \log^+(ru_i) \Pi^L(i, dr) + \sum_{i=1}^K \sum_{j=1}^K p_{ij} u_j v_i \int_{(0, +\infty)} r \log^+(r \sum_{k=1}^K p_{ik} u_k) \Pi^{NL}(i, dr) < +\infty.$$

Using elementary computation, one can reduce the above condition to

$$\int_{(0,+\infty)} r \log^+ r \Pi^L(i, dr) + \int_{(0,+\infty)} r \log^+ r \Pi^{NL}(i, dr) < +\infty \quad \forall i \in E. \quad (7.3)$$

In particular, under condition (7.3),  $P_\mu \left( \lim_{t \rightarrow +\infty} X_t^{(i)} = 0 \right) = 1$  for every  $i \in E$  and every non-trivial  $\mu \in \mathcal{M}(E)$  if and only if  $\Lambda \leq 0$ . This result is also proved in [Kyprianou and Palau \(2018, Theorem 6\)](#).

Now we give some other examples.

*Example 7.2.* Suppose that  $E$  is a bounded  $C^3$  domain in  $\mathbb{R}^d$  ( $d \geq 1$ ),  $m$  is the Lebesgue measure on  $E$  and that  $\xi = (\xi_t, \Pi_x)$  is the killed Brownian motion in  $E$ . Suppose that  $\phi^L$  and  $\phi^{NL}$  are as given in Subsection 2.1. We assume Assumption 0 holds. We further assume that the probability kernel  $\pi(x, dy)$  has a bounded density with respect to the Lebesgue measure  $m$ , i.e.,  $\pi(x, dy) = \pi(x, y)dy$  with  $\pi(x, y)$  being bounded on  $E \times E$ . Assumption 1 and Assumption 3.(i1) are trivially satisfied. Let  $(\mathfrak{P}_t)_{t \geq 0}$  be the semigroup on  $\mathcal{B}_b(E)$  uniquely determined by the integral equation (2.9). It follows from [Hering \(1978, Theorem\)](#) that Assumption 2, Assumption 3.(ii) are satisfied, and that  $(\mathfrak{P}_t)_{t \geq 0}$  is *uniformly primitive* in the sense of [Hering \(1978\)](#). Thus for all  $t > 0$ ,  $f \in \mathcal{B}_b^+(E)$  and  $x \in E$ ,

$$\left| \mathfrak{P}_t f(x) - e^{-\lambda_1 t} (f, \widehat{h}) h(x) \right| \leq c_t e^{-\lambda_1 t} (f, \widehat{h}) h(x), \quad (7.4)$$

where  $c_t \geq 0$  satisfying  $c_t \downarrow 0$  as  $t \uparrow +\infty$ ,  $\lambda_1$  is the constant in Assumption 2, and  $h, \widehat{h}$  are the functions in Assumption 2. Let  $\widetilde{\mathfrak{S}}_t f(x) := e^{\lambda_1 t} h(x)^{-1} \mathfrak{P}_t(fh)(x)$  for  $f \in \mathcal{B}^+(E)$ ,  $t \geq 0$  and  $x \in E$ . Let  $\widetilde{p}(t, x, y)$  be the density of  $\widetilde{\mathfrak{S}}_t P_t$  with respect to the measure  $\rho(dy) := h(y)\widehat{h}(y)dy$  on  $E$ . By (7.4), we have for every  $t > 0$ ,  $f \in \mathcal{B}_b^+(E)$  and  $x \in E$ ,

$$\left| \widetilde{\mathfrak{S}}_t f(x) - \langle f, \rho \rangle \right| = \left| \int_E (\widetilde{p}(t, x, y) - 1) f(y) \rho(dy) \right| \leq c_t \langle f, \rho \rangle.$$

It follows from this that

$$\sup_{x \in E} \operatorname{esssup}_{y \in E} |\widetilde{p}(t, x, y) - 1| \leq c_t \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Hence Assumption 4 is satisfied. Assumption 3.(iii) will be satisfied if the function  $\pi(x, y)$  satisfies

$$\int_E \pi(x, y) h(y) dy \leq c h(x) \quad \forall x \in \{z \in E : \gamma(z) > 0\}$$

for some constant  $c > 0$ , where  $h$  is the function in Assumption 2 and  $\gamma(z)$  is as given in Subsection 2.1.

*Example 7.3.* Suppose that  $E$  is a bounded  $C^{1,1}$  open set in  $\mathbb{R}^d$  ( $d \geq 1$ ),  $m$  is the Lebesgue measure on  $E$ ,  $\alpha \in (0, 2)$ ,  $\beta \in [0, \alpha \wedge d)$  and that  $\xi = (\xi_t, \Pi_x)$  is an  $m$ -symmetric Hunt process on  $E$  satisfying the following conditions: (1)  $\xi$  has a Lévy system  $(N, t)$  where  $N = N(x, dy)$  is a kernel given by

$$N(x, dy) = \frac{C_1}{|x - y|^{d+\alpha}} dy \quad x, y \in E$$

for some constant  $C_1 > 0$ . (2)  $\xi$  admits a jointly continuous transition density  $p(t, x, y)$  with respect to the Lebesgue measure and that there exists a constant  $C_2 > 1$  such that

$$C_2^{-1} q_{\alpha, \beta}(t, x, y) \leq p(t, x, y) \leq C_2 q_{\alpha, \beta}(t, x, y) \quad \forall (t, x, y) \in (0, 1] \times E \times E,$$

where

$$q_{\alpha, \beta}(t, x, y) = \left( 1 \wedge \frac{\delta_E(x)}{t^{1/\alpha}} \right)^\beta \left( 1 \wedge \frac{\delta_E(y)}{t^{1/\alpha}} \right)^\beta \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \quad (7.5)$$

Here  $\delta_E(x)$  stands for the Euclidean distance between  $x$  and the boundary of  $E$ . Suppose that  $\phi^L$  and  $\phi^{NL}$  are as given in Subsection 2.1. We assume Assumption 0 holds. We further assume that the probability kernel  $\pi(x, dy)$  has a density  $\pi(x, y)$  with respect to the Lebesgue measure  $m$  satisfying the condition

$$\pi(x, y) \leq C_3|x - y|^{\epsilon-d} \quad \forall x, y \in E$$

for some positive constants  $C_3$  and  $\epsilon$ . Define

$$F(x, y) := C_1^{-1}|x - y|^{d+\alpha}\gamma(x)\pi(x, y) \quad \forall x, y \in E.$$

One can show easily that Assumption 1 and Assumption 3.(i2) are satisfied. Define

$$F^*(x, y) := \log(1 + F(x, y)) \quad \forall x, y \in E.$$

It is obvious that there exists  $C_4 > 0$  such that

$$0 \leq F^*(x, y) \leq C_4(|x - y|^{\epsilon+\alpha} \wedge 1) \quad \forall x, y \in E,$$

and thus, by [Chen et al. \(2015, Proposition 4.2\)](#),  $F^*$  belongs to the Kato class  $\mathbf{J}_{\alpha,\beta}$  defined in [Chen et al. \(2015\)](#), i.e.,  $\lim_{t \downarrow 0} N_{F^*}^{\alpha,\beta}(t) = 0$ , where

$$N_{F^*}^{\alpha,\beta}(t) := \sup_{x \in E} \int_0^t \int_{E \times E} q_{\alpha,\beta}(s, x, z) \left(1 + \frac{|z - y| \wedge t^{1/\alpha}}{|x - z|}\right)^\beta \frac{F^*(y, z) + F^*(z, y)}{|z - y|^{d+\alpha}} dy dz ds.$$

The measure  $\mu(dx) := -a(x)dx$  obviously belongs to the Kato class  $\mathbf{K}_{\alpha,\beta}$  defined in [Chen et al. \(2015\)](#), i.e.,  $\lim_{t \downarrow 0} N_\mu^{\alpha,\beta}(t) = 0$ , where

$$N_\mu^{\alpha,\beta}(t) = \sup_{x \in E} \int_0^t \int_E q_{\alpha,\beta}(s, x, y) |a|(y) dy ds,$$

since  $a$  is a bounded function. For  $0 \leq t < +\infty$ , let  $A_t := -\int_0^t a(\xi_r) dr + \sum_{0 < r \leq t} F^*(\xi_{r-}, \xi_r)$ . Let  $(T_t)_{t \geq 0}$  be the Feynman-Kac semigroup of  $\xi$  given by

$$T_t f(x) := \Pi_x[\exp(A_t) f(\xi_t)], \quad t \geq 0, x \in E, f \in \mathcal{B}^+(E).$$

Now it follows from [Chen et al. \(2015, Theorem 1.3\)](#) that the semigroup  $(T_t)_{t \geq 0}$  has a jointly continuous density  $q(t, x, y)$  with respect to the Lebesgue measure and there exists a constant  $C_5 > 1$  such that

$$C_5^{-1}q_{\alpha,\beta}(t, x, y) \leq q(t, x, y) \leq C_5q_{\alpha,\beta}(t, x, y) \quad \forall (t, x, y) \in (0, 1] \times E \times E. \tag{7.6}$$

Let  $(\widehat{T}_t)_{t \geq 0}$  be the dual semigroup of  $(T_t)_{t \geq 0}$ . By (7.6), one can easily show that for any  $f \in \mathcal{B}_b(E)$ ,  $T_t f$  and  $\widehat{T}_t f$  are bounded continuous functions on  $E$ , that  $T_t$  and  $\widehat{T}_t$  are bounded operators from  $L^2(E, m)$  into  $L^\infty(E, m)$ , and that  $(T_t)_{t \geq 0}$  and  $(\widehat{T}_t)_{t \geq 0}$  are strongly continuous semigroups on  $L^2(E, m)$ . Let  $\mathbf{L}$  and  $\widehat{\mathbf{L}}$  be the generators of  $(T_t)_{t \geq 0}$  and  $(\widehat{T}_t)_{t \geq 0}$  respectively. Let  $\sigma(\mathbf{L})$  and  $\sigma(\widehat{\mathbf{L}})$  denote the spectrum of  $\mathbf{L}$  and  $\widehat{\mathbf{L}}$  respectively. It follows from (7.6) and Jentzsch's theorem ([Schaefer, 1974, Theorem V.6.6, p. 337](#)) that the common value  $-\lambda_1 := \sup \text{Re}(\sigma(\mathbf{L})) = \sup \text{Re}(\sigma(\widehat{\mathbf{L}}))$  is an eigenvalue of multiplicity 1 for both  $\mathbf{L}$  and  $\widehat{\mathbf{L}}$ , and that an eigenfunction  $h$  of  $\mathbf{L}$  associated with  $-\lambda_1$  is bounded continuous and can be chosen strictly positive on  $E$  and satisfies  $\|h\|_{L^2(E, m)} = 1$ , and that an eigenfunction  $\widehat{h}$  of  $\widehat{\mathbf{L}}$  associated with  $-\lambda_1$  is bounded continuous and can be chosen strictly positive on  $E$  and satisfies  $(h, \widehat{h}) = 1$ . Thus Assumption 2 and 3.(ii) are satisfied. It follows from (7.6) and the equations  $e^{-\lambda_1 t} h = T_1 h$ ,  $e^{-\lambda_1 t} \widehat{h} = \widehat{T}_1 h$  that there exists a constant  $C_6 > 1$  such that

$$C_6^{-1}\delta_E(x)^\beta \leq h(x) \leq C_6\delta_E(x)^\beta, \quad C_6^{-1}\delta_E(x)^\beta \leq \widehat{h}(x) \leq C_6\delta_E(x)^\beta \quad \forall x \in E.$$

It follows from this, (7.6) and the semigroup property that the semigroups  $(T_t)_{t \geq 0}$  and  $(\widehat{T}_t)_{t \geq 0}$  are intrinsically ultracontractive. For the definition of intrinsic ultracontractivity, see [Kim and Song](#)

(2008). Let  $\tilde{\mathfrak{S}}_t f(x) := e^{\lambda_1 t} h(x)^{-1} T_t(fh)(x)$  for  $f \in \mathcal{B}^+(E)$ ,  $t \geq 0$  and  $x \in E$ . Then  $\tilde{\mathfrak{S}}_t$  admits a density  $\tilde{p}(t, x, y)$  with respect to the probability measure  $h(y)\hat{h}(y)dy$  which is related to  $q(t, x, y)$  by

$$\tilde{p}(t, x, y) = \frac{e^{\lambda_1 t} q(t, x, y)}{h(x)\hat{h}(y)} \quad \forall (t, x, y) \in (0, +\infty) \times E \times E.$$

Now it follows from Kim and Song (2008, Theorem 2.7) that Assumption 4 is satisfied. As in the previous example, Assumption 3.(iii) will be satisfied if the function  $\pi(x, y)$  satisfies

$$\int_E \pi(x, y)h(y)dy \leq ch(x) \quad \forall x \in \{z \in E : \gamma(z) > 0\}$$

for some constant  $c > 0$ , where  $\gamma(z)$  is as given in Subsection 2.1.

One concrete example of  $\xi$  is the killed symmetric  $\alpha$ -stable process in  $E$ . In this case, (7.5) is satisfied with  $\beta = \alpha/2$ , a fact which was first proved in Chen et al. (2010a).

Another concrete example of  $\xi$  is the censored symmetric  $\alpha$ -stable process in  $E$  introduced in Bogdan et al. (2003) when  $\alpha \in (1, 2)$ . In this case, (7.5) is satisfied with  $\beta = \alpha - 1$ , a fact which was first proved in Chen et al. (2010b).

In fact, by using Chen et al. (2015), one could also include the case when  $E$  is a  $d$ -set,  $\alpha \in (0, 2)$  and  $\xi$  is an  $\alpha$ -stable-like process in  $E$  introduced in Chen and Kumagai (2003). We omit the details.

*Example 7.4.* Suppose that  $E = \mathbb{R}^d$ ,  $m$  is the Lebesgue measure on  $\mathbb{R}^d$ ,  $\alpha \in (0, 2)$  and that  $\xi = (\xi_t, \Pi_x)$  is a Markov process corresponding to the Feynman-Kac transform of a  $d$ -dimensional isotropic  $\alpha$ -stable process with killing potential  $\eta(x) = |x|^\beta$  ( $\beta > 0$ ). Let  $J(x) = J(|x|)$  be the Lévy density of the isotropic  $\alpha$ -stable process, i.e.,  $J(x) = c(d, \alpha)|x|^{-d-\alpha}$  for some positive constant  $c(d, \alpha)$  depending only on  $d$  and  $\alpha$ . It is known that  $\xi$  has a Lévy system  $(N, t)$  where  $N(x, dy) = 2J(y - x)dy$ . Let  $(\mathcal{E}, \mathcal{F})$  be the Dirichlet form of  $\xi$ . Then  $\mathcal{E}$  has the following form

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y))J(y - x)dx dy + \int_{\mathbb{R}^d} u(x)v(x)|x|^\beta dx$$

for all  $u, v \in \mathcal{F}$ . Suppose that the branching mechanisms  $\phi^L$  and  $\phi^{NL}$  are as given in Subsection 2.1. For simplicity we assume  $a(x) \equiv 0$  and Assumption 0 holds. Let  $\pi(x) = \pi(|x|)$  be a probability density on  $\mathbb{R}^d$  such that the function  $\pi(x)/J(x)$  is bounded from above. We assume that the probability kernel  $\pi(x, dy)$  has a density  $\pi(x, y) = \pi(y - x)$  with respect to the Lebesgue measure and that the function  $\gamma(x) \equiv \gamma$  is a constant. Define  $\gamma(x, y) := \gamma\pi(y - x)$ . Then  $\int_{\mathbb{R}^d} \gamma(x, y)dx = \gamma \int_{\mathbb{R}^d} \pi(y - x)dx = \gamma$ , and Assumption 1 is trivially satisfied. Define

$$F(x, y) := \frac{\gamma(x, y)}{2J(y - x)} = \frac{\gamma\pi(y - x)}{2J(y - x)} \quad \forall x, y \in \mathbb{R}^d.$$

Then  $F$  is a bounded function on  $\mathbb{R}^d \times \mathbb{R}^d$  vanishing on the diagonal, and thus Assumption 3.(i2) is satisfied. Define

$$\begin{aligned} \tilde{A}_t &:= \sum_{s \leq t} \log(1 + F(\xi_{s-}, \xi_s)) - 2 \int_0^t \int_{\mathbb{R}^d} F(\xi_s, y)J(y - \xi_s)dy ds \\ &= \sum_{s \leq t} \log(1 + F(\xi_{s-}, \xi_s)) - \gamma \int_0^t \int_{\mathbb{R}^d} \pi(y - \xi_s)dy ds \\ &= \sum_{s \leq t} \log(1 + F(\xi_{s-}, \xi_s)) - \gamma t. \end{aligned}$$

It follows from Chen and Song (2003a, Theorem 4.8) (see also Chen and Song, 2003b, p. 275) that the bilinear form corresponding to the symmetric semigroup

$$\tilde{T}_t f(x) = \Pi_x \left[ e^{\tilde{A}_t} f(\xi_t) \right] \quad \forall t \geq 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d)$$



is

$$\begin{aligned} \tilde{\mathcal{Q}}(u, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y))(1 + F(x, y))J(y - x)dx dy \\ &\quad + \int_{\mathbb{R}^d} u(x)v(x)|x|^\beta dx \quad \forall u, v \in \mathcal{F}. \end{aligned}$$

This is the bilinear form of the stable-like Lévy process killed with the potential  $\eta(x) = |x|^\beta$ . Now we apply [Kaleta and Lőrinczi \(2015, Examples 4.5 and 4.8\)](#) to get that the symmetric semigroup  $(\tilde{T}_t)_{t \geq 0}$  is intrinsically ultracontractive. Define

$$A_t := \sum_{s \leq t} \log(1 + F(\xi_{s-}, \xi_s)).$$

Again by [Chen and Song \(2003a, Theorem 4.8\)](#) (see also [Chen and Song, 2003b, p. 275](#)) that the bilinear form corresponding to the symmetric semigroup

$$T_t f(x) = \Pi_x [e^{A_t} f(\xi_t)] \quad \forall t \geq 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d)$$

is

$$\begin{aligned} \mathcal{Q}(u, v) &= \mathcal{E}(u, v) - 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(y)v(x)F(x, y)J(x - y)dx dy \\ &= \mathcal{E}(u, v) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(y)v(x)\gamma(x, y)dx dy \quad \forall u, v \in \mathcal{F}. \end{aligned}$$

We observe that  $T_t f = e^{\gamma t} \tilde{T}_t f$ . So the semigroup  $(T_t)_{t \geq 0}$  is also intrinsically ultracontractive. Applying similar argument as in [Example 7.3](#), one can show that Assumptions 2 and 4 are satisfied. Finally, Assumption 3(iii) (and thus Assumption 3(ii)) will be satisfied if the function  $\pi(x)$  satisfies that

$$\int_{\mathbb{R}^d} \pi(y - x)h(y)dy \leq ch(x) \quad \forall x \in \mathbb{R}^d$$

for some positive constant  $c$ . Here  $h$  is the strictly positive eigenfunction associated with the principal eigenvalue of the generator of the semigroup  $(T_t)_{t \geq 0}$ .

### Appendix

*Proof of Proposition 4.3:* We will prove (4.6) first. We claim that

$$\tilde{\Pi}_x [f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1})1_{\{\tau_1 \leq t\}}] = \tilde{\Pi}_x \left[ \int_0^{t \wedge \tau_1} q(\tilde{\xi}_s)ds \int_E f(s, \tilde{\xi}_s, y)\pi^h(\tilde{\xi}_s, dy) \right]. \tag{7.7}$$

It is easy to see from the construction of  $\tilde{\xi}$  that

$$\text{LHS of (7.7)} = \Pi_x^h \left[ \int_0^t q(\xi_s^h)e_q(s)ds \int_E f(s, \xi_s^h, y)\pi^h(\xi_s^h, dy) \right]. \tag{7.8}$$

On the other hand, by Fubini's theorem, we have

$$\begin{aligned} \text{RHS of (7.7)} &= \int_0^t ds \tilde{\Pi}_x \left[ \int_E q(\tilde{\xi}_s)f(s, \tilde{\xi}_s, y)\pi^h(\tilde{\xi}_s, dy)1_{\{s < \tau_1\}} \right] \\ &= \int_0^t ds \Pi_x^h \left[ \int_E q(\hat{\xi}_s)f(s, \hat{\xi}_s, y)\pi^h(\hat{\xi}_s, dy)1_{\{s < \hat{\zeta}\}} \right] \\ &= \int_0^t ds \Pi_x^h \left[ e_q(s) \int_E q(\xi_s^h)f(s, \xi_s^h, y)\pi^h(\xi_s^h, dy) \right] \\ &= \Pi_x^h \left[ \int_0^t q(\xi_s^h)e_q(s)ds \int_E f(s, \xi_s^h, y)\pi^h(\xi_s^h, dy) \right]. \end{aligned} \tag{7.9}$$

Combining (7.8) and (7.9) we arrive at the claim (7.7). Note that applying the shift operator  $\tilde{\theta}_{\tau_n}$  to  $f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1})1_{\{\tau_1 \leq t\}}$  gives  $f(\tau_{n+1}, \tilde{\xi}_{\tau_{n+1}-}, \tilde{\xi}_{\tau_{n+1}})1_{\{\tau_{n+1} \leq t\}}$ . Using the strong Markov property of  $\tilde{\xi}$  and Fubini's theorem, we can prove by induction that for all  $n \geq 2$ ,

$$\tilde{\Pi}_x \left[ f(\tau_n, \tilde{\xi}_{\tau_n-}, \tilde{\xi}_{\tau_n})1_{\{\tau_n \leq t\}} \right] = \tilde{\Pi}_x \left[ \int_{t \wedge \tau_{n-1}}^{t \wedge \tau_n} q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right].$$

Thus by the above equality, Fubini's theorem and the fact that  $\tilde{\Pi}_x(\lim_{n \rightarrow +\infty} \tau_n = +\infty) = 1$ , we have

$$\begin{aligned} \tilde{\Pi}_x \left[ \sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] &= \tilde{\Pi}_x \left[ \sum_{i=1}^{+\infty} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i})1_{\{\tau_i \leq t\}} \right] \\ &= \tilde{\Pi}_x \left[ \lim_{n \rightarrow +\infty} \int_0^{t \wedge \tau_n} q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right] \\ &= \tilde{\Pi}_x \left[ \int_0^t q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right]. \end{aligned}$$

Hence we have proved (4.6). We next show (4.7). It is easy to see that

$$\begin{aligned} &\tilde{\Pi}_x \left[ \left( \sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right) \left( \sum_{\tau_j \leq t} g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \right) \right] \\ &= \tilde{\Pi}_x \left[ \sum_{\tau_i \leq t} f g(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] \\ &\quad + \sum_{i=1}^{+\infty} \sum_{j=i+1}^{+\infty} \left\{ \tilde{\Pi}_x \left[ f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \right. \\ &\quad \left. + \tilde{\Pi}_x \left[ g(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) f(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \right\}. \quad (7.10) \end{aligned}$$

By the strong Markov property and (7.8), we have for  $j \geq 2$ ,

$$\begin{aligned} &\tilde{\Pi}_x \left[ f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\ &= \tilde{\Pi}_x \left[ f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) 1_{\{\tau_1 \leq t\}} \tilde{\Pi}_{\tilde{\xi}_{\tau_1}} \left( g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} + s \leq t\}} \right) \Big|_{s=\tau_1} \right] \\ &= \Pi_x^h \left[ \int_0^t ds \int_E \pi^h(\xi_s^h, dy) q(\xi_s^h) e_q(s) f(s, \xi_s^h, y) \right. \\ &\quad \left. \cdot \tilde{\Pi}_y \left( g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} \leq t-s\}} \right) \right], \quad (7.11) \end{aligned}$$

and for  $j > i \geq 2$ ,

$$\begin{aligned} &\tilde{\Pi}_x \left[ f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\ &= \tilde{\Pi}_x \left[ 1_{\{\tau_1 \leq t\}} \cdot \tilde{\Pi}_{\tilde{\xi}_{\tau_1}} \left( f(\tau_{i-1} + s, \tilde{\xi}_{\tau_{i-1}-}, \tilde{\xi}_{\tau_{i-1}}) g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} + s \leq t\}} \right) \Big|_{s=\tau_1} \right] \\ &= \Pi_x^h \left[ \int_0^t ds \int_E \pi^h(\xi_s^h, dy) q(\xi_s^h) e_q(s) \right. \\ &\quad \left. \cdot \tilde{\Pi}_y \left( f(\tau_{i-1} + s, \tilde{\xi}_{\tau_{i-1}-}, \tilde{\xi}_{\tau_{i-1}}) g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} \leq t-s\}} \right) \right]. \quad (7.12) \end{aligned}$$

By (7.11), Fubini's theorem, the strong Markov property of  $\tilde{\xi}$ , (4.6) and (7.8),

$$\begin{aligned}
 & \sum_{j=2}^{+\infty} \tilde{\Pi}_x \left[ f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \mathbf{1}_{\{\tau_j \leq t\}} \right] \\
 &= \Pi_x^h \left[ \int_0^t ds \int_E \pi^h(\xi_s^h, dy) q(\xi_s^h) e_q(s) f(s, \xi_s^h, y) \right. \\
 & \quad \left. \cdot \tilde{\Pi}_y \left( \sum_{j=2}^{+\infty} g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) \mathbf{1}_{\{\tau_{j-1} \leq t-s\}} \right) \right], \\
 &= \Pi_x^h \left[ \int_0^t ds \int_E \pi^h(\xi_s^h, dy) q(\xi_s^h) e_q(s) f(s, \xi_s^h, y) \tilde{\Pi}_y \left( \sum_{\tau_k \leq t-s} g(\tau_k + s, \tilde{\xi}_{\tau_k-}, \tilde{\xi}_{\tau_k}) \right) \right] \\
 &= \Pi_x^h \left[ \int_0^t ds \int_E \pi^h(\xi_s^h, dy) q(\xi_s^h) e_q(s) f(s, \xi_s^h, y) \right. \\
 & \quad \left. \cdot \tilde{\Pi}_y \left( \int_0^{t-s} dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(r + s, \tilde{\xi}_r, z) \right) \right] \\
 &= \tilde{\Pi}_x \left[ f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) \tilde{\Pi}_{\tilde{\xi}_{\tau_1}} \left( \int_0^{t-s} dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(r + s, \tilde{\xi}_r, z) \right) \Big|_{s=\tau_1} \mathbf{1}_{\{\tau_1 \leq t\}} \right] \\
 &= \tilde{\Pi}_x \left[ f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) \int_{\tau_1}^t dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(r, \tilde{\xi}_r, z) \mathbf{1}_{\{\tau_1 \leq t\}} \right].
 \end{aligned}$$

Similarly, by (7.12), Fubini's theorem, the strong Markov property of  $\tilde{\xi}$ , (4.6) and (7.8), we can prove by induction that for  $i \geq 1$ ,

$$\begin{aligned}
 & \sum_{j=i+1}^{+\infty} \tilde{\Pi}_x \left[ f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \mathbf{1}_{\{\tau_j \leq t\}} \right] \\
 &= \tilde{\Pi}_x \left[ f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \int_{\tau_i}^t dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(r, \tilde{\xi}_r, z) \mathbf{1}_{\{\tau_i \leq t\}} \right].
 \end{aligned}$$

By this, Fubini's theorem, the strong Markov property of  $\tilde{\xi}$  and (4.6), we get

$$\begin{aligned}
 & \sum_{i=1}^{+\infty} \sum_{j=i+1}^{+\infty} \tilde{\Pi}_x \left[ f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \mathbf{1}_{\{\tau_j \leq t\}} \right] \\
 &= \tilde{\Pi}_x \left[ \sum_{i=1}^{+\infty} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \int_{\tau_i}^t dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(r, \tilde{\xi}_r, z) \mathbf{1}_{\{\tau_i \leq t\}} \right] \\
 &= \tilde{\Pi}_x \left[ \sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \int_{\tau_i}^t dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(r, \tilde{\xi}_r, z) \right] \\
 &= \tilde{\Pi}_x \left[ \sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \tilde{\Pi}_{\tilde{\xi}_{\tau_i}} \left( \int_0^{t-s} q(\tilde{\xi}_r) dr \int_E g(s+r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \Big|_{s=\tau_i} \right] \\
 &= \tilde{\Pi}_x \left[ \int_0^t ds \int_E \pi^h(\tilde{\xi}_s, dy) q(\tilde{\xi}_s) f(s, \tilde{\xi}_s, y) \cdot \tilde{\Pi}_y \left( \int_0^{t-s} dr \int_E \pi^h(\tilde{\xi}_r, dz) q(\tilde{\xi}_r) g(s+r, \tilde{\xi}_r, z) \right) \right]. \quad (7.13)
 \end{aligned}$$

Combining (7.10) and (7.13), we arrive at (4.7).  $\square$

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