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Second Order Expansions for Sample Median with Random Sample Size

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Abstract. In the paper, second-order Chebyshev-Edgeworth expansions are proved for the sample median when the sample size has negative binomial or discrete Pareto-like distributions. The limiting distributions of the scaled sample median depend not only on the sample size distribution but also on the chosen scaling factor. The limiting distributions are the generalized Laplace, the normal and the scaled Student distributions, depending on the random, non-random or mixed scaling factor. Second order Cornish-Fisher expansions are also derived and the negative moments of the random sample sizes are calculated.

1. Introduction

In classical statistical inference, the number of observations is normally a known parameter. If the data are collected in a fixed period, then the number of observations is typically random. For example, patients with flu within a week, the number of call options not exercised on the expiration

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date, or the number of the α particles detected by a radiation source with a Geiger-Müller counter during an hour, such experiments lead to models with random numbers of observations.

A survey of statistical inference with a random number of observations can be found, for example, in Esquível et al. (2016) and the references there. In the aforementioned paper, the inference for the mean and variance in the normal model is investigated. ANOVA models based on samples with Poisson or binomial distributed number of observations were investigated in Nunes et al. (2019a,b,c) for the analysis of one-way fixed effects to avoid false rejection.

Denote the real axis, the positive numbers and the indicator function as follows

$$\mathbb{R} = (-\infty\,,\,\infty), \quad \mathbb{N} = \{1,2,...\} \quad \text{and} \quad \mathbb{I}_A = \mathbb{I}_A(x) = \begin{cases} 1, & x \in A \subset \mathbb{R} \\ 0, & x \notin A \subset \mathbb{R} \end{cases},$$

respectively. Moreover, if x is a real value, then $[x]$ denotes the greatest integer less than or equal to x.

It is assumed that all random variables considered in the following are defined on one probability space $(\Omega, \mathbb{A}, \mathbb{P})$. Let $X, X_1, X_2, \ldots \in \mathbb{R}$ be independent identically distributed random variables. In statistics the random variables $X_1, X_2, \ldots$ are observations.

Consider the statistics $T_m := T_m\,(X_1, \ldots, X_m)$ of a sample with sample size m. We write T_m has asymptotic normality $\mathcal{AN}(\mu_m, \sigma_m^2)$ where μ_m and $\sigma_m^2 > 0$ are sequences of constants if

$$\sup_x \left| \mathbb{P}\Big(\frac{T_m - \mu_m}{\sigma_m} \le x\Big) - \Phi(x) \right| \to 0 \quad \text{as} \quad m \to \infty\,.$$

Here $\Phi(x)$ is the distribution function with density $\varphi(x)$ of the standard normal Y:

$$\mathbb{P}(Y \le x) = \Phi(x) \quad \text{and} \quad \varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R}. \tag{1.1}$$

The notation $\mathcal{N}(\mu, \sigma^2)$ will be used to denote either the normal distribution with mean μ and variance σ^2 or a random variable having this distribution. Hence, Y is $\mathcal{N}(0,1)$ in (1.1).

Along with $X_1, X_2, \ldots$, consider now a sequence of integer-valued positive random sample sizes $N_n \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ the random variables N_n are independent of the sample $X_1, X_2, \ldots$ and that $N_n \to \infty$ in probability as $n \to \infty$. Furthermore, let the existence of a distribution function $H(y)$ with $H(0) = 0$ and a sequence $0 < g_n \uparrow \infty$ be assumed such that

$$\sup\nolimits_{y \ge 0} |\mathbb{P}(N_n/g_n \le y) - H(y)| \to 0 \quad \text{as} \quad n \to \infty. \tag{1.2}$$

Let $T_m := T_m\,(X_1, \ldots, X_m)$ be some statistic of a sample with non-random sample size $m \in \mathbb{N}$. Define the random variable T_{N_n} for every $n \in \mathbb{N}$:

$$T_{N_n}(\omega) := T_{N_n(\omega)}\left(X_1(\omega), \ldots, X_{N_n(\omega)}\right), \;\; \omega \in \Omega,$$

i.e. T_{N_n} is some statistic obtained from a random sample $X_1, X_2, \ldots, X_{N_n}$.

Many models lead to random sums S_{N_n} and random means $T^*_{N_n}$:

$$S_{N_n} = \sum\nolimits_{k=1}^{N_n} X_k \quad \text{and} \quad T^*_{N_n} = \frac{1}{N_n} \sum\nolimits_{k=1}^{N_n} X_k = \frac{1}{N_n} S_{N_n}. \tag{1.3}$$

Wald's identity for random sums $\mathbb{E}(S_{N_n}) = \mathbb{E}(N_n)\mathbb{E}(X_1)$, when N_n and X_1 have finite expectations, is a powerful tool in statistical inference, particularly in sequential analysis, see e.g. Wald (1945) and Kolmogorov and Prokhorov (1949). Robbins (1948) proved that the asymptotic normality of both the sum S_m and the sample size N_n lead to asymptotic normality of the random sum S_{N_n}.

In Döbler (2015), a basic overview to asymptotic distributions of random sums was given. Using Stein's method, quantitative Berry-Esseen bounds of random sums were proved in Chen et al. (2011, Theorem 10.6), Döbler (2015, Theorems 2.5 and 2.7) and Pike and Ren (2014, Theorem 1.3) in case of approximation by normal and Laplace distributions. Kalashnikov (1997) studied applications of geometric random sums when N_n is geometrically distributed.

The randomness of the sample size may crucially change asymptotic properties of random sums, see e.g. Gnedenko (1989) or Gnedenko and Korolev (1996). Appropriate scaling factors by random sums S_{N_n} or random means $T^*_{N_n}$ affect the type of limit distributions.

If the statistic T_m is asymptotically normal, then the limiting laws of appropriate scaled statistics T_{N_n} are scale mixtures $\int_0^\infty \Phi(x\,y^\gamma)dH(y)$ of normal distribution $\Phi(x\,y^\gamma)$ with a real constant γ and the mixture distribution $H(y)$, determined by N_n and given in (1.2). Scale mixtures of normal distributions are often employed as an important class for statistical treatments for symmetric data, see e.g. Andrews and Mallows (1974) and Fujikoshi et al. (2010, Chapter 13).

Example 1.1. Let $X_1, X_2, ...$ be independent $\mathcal{N}(0,1)$ then $T^*_m = (X_1 + ... + X_m)/m$ is $\mathcal{N}\big(0, 1/m\big)$. Let $T^*_{N_n(1)}$ be random mean given in (1.3) where $N_n(1) \in \mathbb{N}$ be geometrically distributed as special case of negative binomially distributed $N_n(r)$ (see (5.1) below) for $r = 1$:

$$\mathbb{P}(N_n(1) = j) = \left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^{j-1}, \;\; j, n = 1, 2, ... \text{ with } \mathbb{E}(N_n(1)) = n.$$

Assume that for each $n \in \mathbb{N}$ the random variable $N_n(1)$ is independent of the sequence $X_1, X_2, ...$ Then statement (1.2) holds with $g_n = n$ and limit exponential distribution $H(y) = (1 - e^{-y})\,\mathbb{I}_{\{y \ge 0\}}$. The following conclusions can be drawn:

$$\begin{aligned}
&\mathbb{P}\left(\sqrt{N_n(1)}\,T^*_{N_n(1)} \le x\right) = \int_0^\infty \Phi(x)dH(y) = \Phi(x) \quad \text{for all } n \in \mathbb{N},\\
&\lim_{n\to\infty} \mathbb{P}\left(\sqrt{\mathbb{E}(N_n(1))}\,T^*_{N_n(1)} \le x\right) = \int_0^\infty \Phi(x\,y^{1/2})dH(y) = \int_{-\infty}^x \left(2 + u^2\right)^{-3/2} du,\\
&\lim_{n\to\infty} \mathbb{P}\left(\frac{N_n(1)}{\sqrt{\mathbb{E}(N_n(1))}}\,T^*_{N_n(1)} \le x\right) = \int_0^\infty \Phi(x\,y^{-1/2})dH(y) = \int_{-\infty}^x \frac{1}{\sqrt{2}} e^{-\sqrt{2}\,|u|} du.
\end{aligned}$$

Three different limit distributions occur. The scaled random mean $T^*_{N_n(1)}$ is standard normally distributed or tends to the Student distribution with 2 degrees of freedom depending on whether the random scaling factor $\sqrt{N_n(1)}$ or the non-random scaling factor $\sqrt{\mathbb{E}(N_n(1))}$ were chosen. Moreover, the Laplace distribution with variance 1 is the limiting distribution when scaling with the mixing factor $N_n(1)/\sqrt{\mathbb{E}(N_n(1))}$.

The first statement above follows from conditioning and stability of the normal law. Student distribution as a limit for statistics from samples with a random sample size are proved e.g. in Gnedenko (1989), Bening and Korolev (2004) and Schluter and Trede (2016). The Laplace limit law follows e.g. from Bening and Korolev (2008) or Schluter and Trede (2016).

Bening et al. (2013) presented a general transfer theorem for asymptotic expansions of the distribution of statistics T_m from samples with non-random sample size m to statistics T_{N_n} from samples with random sample size N_n. The authors applied corresponding expansions for both the normalized statistic T_m and the appropriate scaled random sample size N_n. In the aforementioned paper, first order expansions of the random mean $T^*_{N_n}$ are proved if the sample size N_n is negative binomial distributed with success probability $1/n$ or N_n is the maximum of n independent identically distributed discrete Pareto random variables with tail index 1. For the mean $T^*_m = (X_1 + ... + X_m)/m$, first order Chebyshev-Edgeworth expansions were applied. For random sample size N_n, the rate of convergence in (1.2) with Cn^{-b} for $0 < b \le 1$ are used. Therefore, the convergence rates for the random mean $T^*_{N_n}$ cannot be better than C/n. To improve the convergence rates, in Christoph et al. (2020, Theorems 1 and 4) second-order asymptotic expansions were proved for suitably normalized sample sizes N_n in cases mentioned above. Moreover, second order Chebyshev-Edgeworth expansions for $T^*_{N_n}$ were constructed for the first time, with which the relevant results in Bening et al. (2013) were improved. See also Fujikoshi and Ulyanov (2020, Chapter 9).

Analogous results were obtained in Christoph and Ulyanov (2020, 2021b) for the three most important geometric statistics of Gaussian vectors with random dimension N_n, the length of a vector, the distance between two independent vectors and the angle between these vectors associated with their sample correlation coefficient. Moreover, Chebyshev-Edgeworth expansions for T_{N_n} based on random sample size N_n are presented in Christoph and Ulyanov (2021a) when the statistic T_m is asymptotically chi-squared distributed.

Burnashev (1997) proved second-order Chebyshev-Edgeworth expansions for the median of a sample $\{X_1, ..., X_m\}$ of independent identically distributed random variables with common continuous distribution function $F_X(x)$ and symmetric probability density $p_X(x)$, where $m \in \mathbb{N}$ is the non-random sample size.

Using these results, in present paper we construct second-order Chebyshev-Edgeworth expansions for the median of a sample with the random sample sizes N_n mentioned above.

The structure of the paper is the following. Order statistics are considered in Section 2 with special attention to the median. In Section 3 we clarify the result of Burnashev (1997) in the sense that the closeness between the sample median M_m and the corresponding second order expansion is estimated by inequalities for any integer $m \geq 1$ instead of some $\mathcal{O}$-order as $m \to \infty$. In Section 4 we give a transition proposition from non-random to random sample sizes. Sections 5 and 6 consider the cases of negative binomial and discrete Pareto-like sample sizes N_n. In Section 7 the Cornish-Fisher expansions for the quantiles of sample medians M_{N_n} and M_m are derived from the corresponding Chebyshev-Edgeworth-type expansions. Finally, the proofs are collected in Section 8.

2. Elements of Order Statistics

Let $\{X_1, X_2, ..., X_m\}$ be a sample of independent observations with common distribution function $F_X(x)$ and the density function $p_X(x)$. The ordered sample

$$X_{m:1} \leq X_{m:2} \leq ... \leq X_{m:m}$$

define the *order statistics* of the sample. Special cases are sample maximum $X_{m:m}$, sample minimum $X_{m:1}$ and median M_m, defined by

$$M_m = \begin{cases} X_{m:(m+1)/2}, & \text{for odd} \quad m, \\ (X_{m:m/2} + X_{m:(m+2)/2})/2, & \text{for even} \quad m, \end{cases} \qquad m \in \mathbb{N}\,. \tag{2.1}$$

The distribution function of the kth order statistics $X_{m,k}$ is simple to find, but tedious to calculate:

$$\mathbb{P}\left(X_{m,k} \leq x\right) = \sum\nolimits_{i=k}^{m} \binom{m}{i} \left(F(x)\right)^i \left(1 - F(x)\right)^{m-i}, \quad x \in \mathbb{R}, \quad 1 \leq k \leq m\,.$$

It is easily seen that for the maximum and minimum values $X_{m:m}$ and $X_{m:1}$ one has

$$\mathbb{P}\left(X_{m,m} \leq x\right) = F^m(x) \quad \text{and} \quad \mathbb{P}\left(X_{m,1} \leq x\right) = 1 - \left(1 - F(x)\right)^m.$$

Extreme value analysis as an important branch of statistics deals with largest and smallest values of samples. It has its own special asymptotic theory with the three non-degenerate limit distribution families: *Weibull, Gumbel and Fréchet laws*, see e.g. Embrechts et al. (1997), Nevzorov (2001, Lectures 10–12), de Haan and Ferreira (2006) and Ahsanullah et al. (2013, Chapter 11).

Many important monographs deal with asymptotic theory for order statistics, see e.g. Serfling (1980, Chapters 2 and 3), Balakrishnan and Rao (1998b,a), Reiss (1989, Section 4.1), van der Vaart (1998, Chapter 21), Nevzorov (2001, Lectures 8 and 9), David and Nagaraja (2003, Chapters 10 and 11) and Ahsanullah et al. (2013, Chapter 10).

There are different definitions of single sample quantiles in statistical literature, based on rounding or on linear interpolation. Let $0 < p < 1$. The single *sample* pth-*quantile* is defined as $X_{m:[m\,p]+1}$, except the median M_m if $p = 1/2$, see (2.1). Define

$$X^*_{m;p} = \begin{cases} M_m, & \text{if} \quad p = 1/2, \\ X_{m:[m\,p]+1}, & \text{if} \quad p \neq 1/2, \end{cases} \qquad m \in \mathbb{N}, \quad 0 < p < 1\,.$$

The asymptotic normality of the normalized *sample quantiles* $X^*_{m;p}$ is well known, see e.g. Cramér (1946, Chapter 28.5): Let x_p be the pth *population quantile*, $p \in (0,1)$, of the continuous distribution function $F_X(x)$, i.e. $F_X(x_p) = p$. If the density $p_X(x)$ is continuous in some neighborhood of $x = x_p$ and $p_X(x_p) > 0$ then

$$R_m = \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{p_X(x_p)\sqrt{m}}{\sqrt{p\,(1-p)}} (M^*_{m;p} - x_p) \le x \right) - \Phi(x) \right| \to 0 \text{ as } m \to \infty. \tag{2.2}$$

In Cramér (1946, Chapter 28.5) it is additionally assumed that the density $p_x(x)$ has a continuous derivative $p'_X(x)$ in some neighborhood of $x = x_p$. This condition can be omitted as shown e.g. in Serfling (1980, Section 2.3.3), Reiss (1989, Theorem 4.1.4) and van der Vaart (1998, Section 5.3). Furthermore, an example is presented in Serfling (1980, Section 2.3.3) that the median is not asymptotically normally distributed if the continuity of $p_X(x)$ in $x = x_{1/2}$ is violated. Based on Serfling's example, we study the following

Example 2.1. Consider the distribution $F^*_X(x)$ with symmetric density $p^*_X(x)$ which is discontinuous at $x_{1/4}$ and $x_{3/4}$:

$$F^*_X(x) = \begin{cases} x + 3/4, & x \in [-3/4, -1/2), \\ (x+1)/2, & x \in [-1/2, 1/2), \\ x + 1/4, & x \in [1/2, 3/4], \end{cases} \quad \text{and} \quad p^*_X(x) = \begin{cases} 1, & x \in [-3/4, -1/2), \\ 1/2, & x \in [-1/2, 1/2), \\ 1, & x \in [1/2, 3/4]. \end{cases} \tag{2.3}$$

At the population quantiles x_p , $0 < p < 1$, except at the first quartile $x_{1/4} = -1/2$ and third quartile $x_{3/4} = 1/2$, the density $p^*_X(x)$ is continuous and the sample median $M^*_{m;p}$ is $\mathcal{AN}\Big(x_p, \dfrac{p\,(1-p)}{(p^*_X(x_p))^2\, m}\Big)$, e.g. the sample median M^*_m is $\mathcal{AN}(0, 1/m)$. At the discontinuous points $x = \pm 1/2$ of $p^*_X(x)$ both for $p = 1/4$ and $p = 3/4$ the sample quartiles $X_{m,[mp]+1}$ are not asymptotically normally distributed. We can nevertheless use Theorem A in Serfling (1980, Section 2.3.3) to approximate the probability $\mathbb{P}(m^{1/2}(X_{m,[mp]+1} - x_p) \le t)$ by the normal distributions $\mathcal{N}(0, 3/16)$ if $t < 0$ and $\mathcal{N}(0, 3/64)$ if $t > 0$ for $p = 1/4$ and $\mathcal{N}(0, 3/64)$ if $t < 0$ and $\mathcal{N}(0, 3/16)$ if $t > 0$ for $p = 3/4$.

Theorem C in Serfling (1980, Section 2.3.3) gives a convergence rate $R_m = \mathcal{O}(m^{-1/2})$ for (2.2) if in the neighborhood of x_p, $F_X(x)$ possesses a positive continuous density $p_X(x)$ and a bounded second derivative $F''_X(x)$.

More general expansions of distributions for central order statistics $X_{m,k}$ were established in Reiss (1989, Section 4.2) which differ from the classical Chebyshev-Edgeworth expansions since the higher order terms are given by integrals of polynomials with respect to the normal distribution depending in a non-trivial way on sample size m and on the index k of the order statistic $X_{m,k}$. As special cases expansions of the distributions for the order statistics from uniform and exponential random variables are given.

The remainder $\sup_x R^*_m(x)$ in approximations of normalized order statistics by asymptotic expansions usually meets order condition $\sup_x R^*_m(x) = \mathcal{O}(m^{-k/2})$ as $m \to \infty$ for some $k > 1$. In the equivalent condition $\sup_x R^*_m(x) \le Cm^{-k/2}$ for all $m \ge M$ the values $C > 0$ and $M > 0$ are unknown.

However, for the transfer proposition from the non-random to the random sample size in Section 4, estimates of $\sup_x R^*_m(x)$ are required in the form of inequalities for each $m \in \mathbb{N}$.

For sample median M_m and symmetric densities $p_X(x)$ qualitative difference between rates of convergence in (2.2) was shown in Burnashev (1997, Section 5). For smooth densities with $p'_X(0) = 0$ the convergence rate has order m^{-1}. However, when $p'_X(+0) \neq 0$, the order is $m^{-1/2}$.

Huang (1999) discussed the even-odd phenomenon for the median in statistical literature and gave a counterexample which contradicts the statistical folklore: "It never pays to base the median on an odd number of observations".

To perform statistical analysis of large data sets Minsker (2019) presents new results for the *median-of-means estimator* using new algorithms for distributed statistical estimations that exploit *divide-and-conquer approach.*

In Peña et al. (2019) confidence regions for median of X in the nonparametric measurement error model are constructed and several applications are given when a confidence interval about the center of a distribution is desired.

To estimate the location parameter of a distribution function $F_X(x)$ one could use also the random mean $T_m^* = (X_1 + ... + X_m)/m$. If X is normally distributed with mean μ and variance σ^2 then T_m^* is normally distributed with mean μ and variance σ^2/m, whereas sample median M_m is $\mathcal{AN}(\mu, \pi\,\sigma^2/(2\,m))$ and $\pi\,\sigma^2/(2\,m) \approx 1.57\sigma^2/m$.

Although the method of the median is less effective compared to the method of the arithmetic mean, Kolmogorov (2019) advises "*when the distribution law is unknown and can deviate markedly from the normal law, it is safer to use the method of the median.*" For example, the median provides better confidence intervals for the Laplace distribution, while the mean works better for normally distributed observations. For heavy tailed distributions, sample median is often preferable to sample mean. For illustration, to estimate the location parameter μ of a Cauchy distribution $F_X(x)$, with density $p_X(x) = (\pi + \pi(x-\mu)^2)^{-1}$, $x \in \mathbb{R}$, the sample mean T_m^* is not a consistent estimator of the location parameter μ due to the stability property of the Cauchy law: T_m^* also has Cauchy distribution function $F_X(x)$. However, sample median M_m is $\mathcal{AN}(\mu, \pi^2/(4\,m))$, see Serfling (1980, Section 2.3.5).

3. Non-Asymptotic Expansions for Sample Median

The regularity conditions for density $p_X(x)$ in Burnashev (1997) are as follows:

Assumption A: *The density $p_X(x)$ is continuous and symmetric around zero, i.e., $p_X(-x) = p_X(x)$, $x \in \mathbb{R}$ and $p_X(0) > 0$. Moreover, the density $p_X(x)$ has three continuous bounded derivatives in some interval $(0, x_0)$, $x_0 > 0$.*

Define $\quad p_0 = p_X(0) > 0, \quad p_1 = p_X'(0+) \quad$ and $\quad p_2 = p_X''(0+)$.

The regularity conditions in Assumption A are fulfilled, for example, for

- normal density (1.1),
- heavy tailed Student's t-distribution $S_\nu(x)$ with density function

$$s_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad \nu > 0, \quad x \in \mathbb{R}, \tag{3.1}$$

including Cauchy distribution in case $\nu = 1$, where the degree of freedom parameter $\nu > 0$ determines the heaviness of the distribution tail,

- the triangular distribution with density

$$t_a(x) = \frac{a - |x|}{a^2}\mathbb{I}_{(-a\,,\,a)}(x), \;\; a > 0\,, \tag{3.2}$$

- the continuous uniform distribution or rectangular distribution with density

$$u_a(x) = \frac{1}{2\,a}\mathbb{I}_{(-a\,,\,a)}(x), \;\; a > 0 \tag{3.3}$$

- symmetric Laplace distribution $L_\mu(x)$ having density

$$l_\mu(x) = \frac{1}{\sqrt{2}\,\mu}\,e^{-\sqrt{2}\,|x|/\mu}, \quad x \in \mathbb{R}, \quad \mu > 0, \quad x \in \mathbb{R}. \tag{3.4}$$

- and the distribution $F_X^*(x)$ with density $p_X^*(x)$ defined by (2.3) in Example 2.1.

The corresponding coefficients p_0, p_1, and p_2 in these examples are:

- $\varphi(x)$: $p_0 = 1/\sqrt{2\,\pi}$, $p_1 = 0$, $p_2 = -1/\sqrt{2\,\pi}$,
- $s_\nu(x)$: $p_0 = \dfrac{\Gamma((\nu+1)/2)}{\sqrt{v\pi}\,\Gamma(v/2)}$, $p_1 = 0$, $p_2 = -\dfrac{\Gamma((\nu+3)/2)}{\sqrt{v\pi}\,\Gamma((v+2)/2)}$,
- $t_a(x)$: $p_0 = a^{-1}$, $p_1 = -a^{-2}$, $p_2 = 0$,
- $u_a(x)$: $p_0 = (2\,a)^{-1}$, $p_1 = 0$, $p_2 = 0$,
- $l_\mu(x)$: $p_0 = 1/(\sqrt{2}\,\mu)$, $p_1 = -\mu^{-2}$, $p_2 = \sqrt{2}\mu^{-3}$
- $p_X^*(x)$: $p_0 = 1/2$, $p_1 = 0$, $p_2 = 0$.

Under Assumption A Burnashev (1997, Theorem 1) proved in relation (2.2) an asymptotic expansion in terms of orders $m^{-1/2}$ and m^{-1} with remainder $\mathcal{O}(m^{-3/2})$ as $m \to \infty$. Only a direct combinatorial approach and no limit theorems were used in the proof. Therefore, the remainder can be estimated by an inequality. Define

$$m^* = 2\,[m/2] = \begin{cases} m & \text{for even m,} \\ m-1 & \text{for odd m.} \end{cases} \tag{3.5}$$

Proposition 3.1. *Let Assumption A be satisfied, then for all $m \geq 2$:*

$$R_m^* = \sup_{x\in\mathbb{R}} \left| \mathbb{P}(2p_0\sqrt{m^*}\,M_m \leq x) - \Phi(x) - \frac{f_1(x)}{\sqrt{m^*}} - \frac{f_2(x)}{m^*} \right| \leq \frac{C_1}{m^{3/2}}, \tag{3.6}$$

where C_1 does not depend on m,

$$f_1(x) = \frac{p_1 x|x|}{4p_0^2}\varphi(x) \quad \text{and} \quad f_2(x) = \frac{x}{4}\Big(3 + x^2 + \frac{p_2 x^2}{6p_0^3} - \frac{p_1^2 x^4}{8p_0^4}\Big)\varphi(x). \tag{3.7}$$

Since $0 < (m-1)^{-\alpha} - m^{-\alpha} \leq 2\,m^{-3/2}$ for $m \geq 2$ and $\alpha = 1/2$ or $\alpha = 1$ an immediate consequence of inequality (3.6) is

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\Big(2p_0\sqrt{m^*}\,M_m \leq x\Big) - \Phi(x) - \frac{f_1(x)}{\sqrt{m}} - \frac{f_2(x)}{m} \right| \leq \frac{C_2}{m^{3/2}}, \tag{3.8}$$

where (3.8) for $m = 1$ is trivial and C_2 does not dependent on m.

Remark 3.2. If the parent distributions of the sample $\{X_1, ..., X_m\}$ have normal density (1.1), Student's t-density (3.1), uniform density (3.3) or the density $p_X^*(x)$ in (2.3) then the convergence rate for the median M_m in (2.2) is of order m^{-1}. The triangular density (3.2) and the Laplace density (3.4) have discontinuous derivatives at $x = 0$, nevertheless $p_1 > 0$ and the convergence rate in (2.2) has the order $m^{-1/2}$.

Remark 3.3. As in Burnashev (1997) the natural normalizing factor in (3.6) is m^*, i.e., $\sqrt{m-1}$ for odd $m \geq 3$ and $\sqrt{m}$ for even m. He proved also for all $m \geq 2$

$$\Big| \mathbb{P}(2p_0\sqrt{m^*}\,M_{m^*} \leq x) - \mathbb{P}(2p_0\sqrt{m^*}\,M_{m^*+1} \leq x) \Big| \leq Cm^{-3/2}.$$

Hence, for sample median M_m each odd observation adds an *amount of information* of order $m^{-3/2}$ and not m^{-1} as usual with normalizing factor $\sqrt{m}$ by M_m.

Remark 3.4. The advantage of second-order approximations is proven by numerical calculations in Burnashev (1997, Section 4). For Laplace density the remaining term R_m in (2.2) contributes less than 10 % of the actual value only for sample sizes $m > 250$. On the other hand, the remaining term R_m^* from the approximation (3.6) contributes less than 10 % of the actual value starting with the sample size $m = 8$ for the Laplace density and $m = 11$ for smooth heavy tailed Cauchy density.

Remark 3.5. The restriction to symmetrical densities can certainly be removed by requirements on derivatives of $p_X(x)$ in some open intervals $(x_{1/2}, x_{1/2}+\varepsilon)$ and $(x_{1/2}-\varepsilon, x_{1/2})$. However, the asymptotic expansions with remainder term estimations become technically much more complicated. For example, in Kotz et al. (2001, Chapter 3) for asymmetric Laplace distributions the population quantiles and the median are given, which depends on an additional parameter asymmetry. The population median is located at the discontinuity point only for the symmetric Laplace density.

4. Transfer Proposition from Non-Random to Random Sample Sizes

Let the Assumption A be satisfied, then (3.8) holds for all integer $m \ge 1$. Suppose that distribution functions of the random sample size N_n satisfy the following condition.

Assumption B: *There exist a distribution function $H(y)$ with $H(0+) = 0$, a function of bounded variation $h_2(y)$ with $h_2(0) = h_2(\infty) = 0$, a sequence $0 < g_n \uparrow \infty$ and real numbers $b > 0$ and $C_3 > 0$ such that for all $n \in \mathbb{N}$*

$$\left.\begin{array}{ll} \sup_{y\ge 0} \left|\mathbb{P}\big(g_n^{-1} N_n \le y\big) - H(y)\right| \le C_3 n^{-b}, & 0 < b \le 1 \\ \sup_{y\ge 0} \left|\mathbb{P}\big(g_n^{-1} N_n \le y\big) - H(y) - n^{-1} h_2(y)\right| \le C_3 n^{-b}, & b > 1 \end{array}\right\} \tag{4.1}$$

Remark 4.1. The negative binomial and discrete Pareto-like sample sizes fulfill Assumption B, see Propositions 5.1 and 6.1 below. For example, in the articles Döbler (2015); Esquível et al. (2016); Nunes et al. (2019a,b,c) mentioned in the introduction, the binomial or Poisson distributions as random sample sizes N of observations are considered. If $N = N_n$ is binomial (with parameters n and $0 < p < 1$) or Poisson (with rate λn, $\lambda > 0$) distributed, then $\mathbb{P}(N_n \le \mathbb{E}N_n x)$ tends to the degenerated in 1 distribution as $n \to \infty$. Second-order expansion in the case of a degenerate limiting distribution could not be found. On the other hand N_n is $\mathcal{AN}(\mathbb{E}N_n, \mathrm{Var}(N_n))$. Berry-Esseen inequalities for Poisson and binomial random sums are proved in Döbler (2015); Korolev and Shevtsova (2012); Sunklodas (2014), but Chebyshev–Edgeworth expansions for these lattice distributed random variables exist so far only with bounds of small-o or large-$\mathcal{O}$ orders, see e.g. Kolassa and McCullagh (1990). For (4.1) in Assumption B, non-asymptotic error bounds C_3 are required because in the Theorem 4.2 one term in (4.2) depends on C_3. Therefore, we cannot apply Theorem 4.2 and 4.5 to samples with binomial or Poisson sample sizes. About non-asymptotic bounds and large-$\mathcal{O}$ order conditions, see Fujikoshi and Ulyanov (2020, Chapter 1).

Theorem 4.2. *Let $\gamma \in \{-1/2, 0, 1/2\}$ and both Assumptions A and B be satisfied. Then the following inequality holds for all $n \in \mathbb{N}$:*

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}\Big(2p_0 (g_n/N_n)^\gamma \sqrt{N_n^*}\; M_{N_n} \le x\Big) - G_n(x, 1/g_n)\right| \\ \le C_2\, \mathbb{E}\left(N_n^{-3/2}\right) + (C_3 D_n(\gamma) + C_4)\, n^{-b}, \tag{4.2}$$

$$G_n(x, 1/g_n) = \int_{1/g_n}^{\infty} \Big(\Phi(xy^\gamma) + \frac{f_1(xy^\gamma)}{\sqrt{g_n y}} + \frac{f_2(xy^\gamma)}{g_n y}\Big) d\Big(H(y) + \frac{h_2(y)}{n}\Big), \tag{4.3}$$

$$D_n(\gamma) = \sup_x D_n(x;\gamma) \le D(\gamma) < \infty \tag{4.4}$$

$$D_n(x;\gamma) = \int_{1/g_n}^{\infty} \left|\frac{\partial}{\partial y}\left(\Phi(xy^\gamma) + \frac{f_1(xy^\gamma)}{\sqrt{y g_n}} + \frac{f_2(xy^\gamma)}{y g_n}\right)\right| dy,$$

where $f_1(z), f_2(z), h_2(y)$ are given in (3.7) and (4.1) and

$$N_n^* = 2\,[N_n/2] = \begin{cases} N_n & \text{for even realizations of } N_n, \\ N_n - 1 & \text{for odd realizations of } N_n. \end{cases} \tag{4.5}$$

The positive constants C_2, C_3, C_4, D do not depend on n.

Remark 4.3. The scaling factor $(g_n/N_n)^\gamma\,\sqrt{N_n^*}$ seems to be the natural one in case of the median of a sample with a random sample size N_n since the distribution of N_n/g_n has a known limit distribution and N_n^* the same structure as m^* in Burnashev (1997), see (3.5).

Remark 4.4. The lower bound of the integral in (4.3) depends on g_n which can affect the coefficients at $1/\sqrt{g_n}$ and $1/g_n$ in the approximation. For example the proof of Theorem 5.6 in Section 8 shows that some integrals tend to infinity as $g_n \to \infty$, see (8.18):

$$g_n^{-1}\int_{1/g_n}^\infty \left|\frac{f_2(x\sqrt{y})}{y}\right| dH(y) \le c g_n^{-b} \quad \text{if} \quad b<1 \quad \text{and} \quad \gamma=1/2.$$

Theorem 4.5. *Under the conditions of Theorem 4.2 and the additional conditions to functions $H(.)$ and $h_2(.)$, depending on the convergence rate $b>0$ in (4.1):*

$$\left.\begin{array}{lll} i: & H(1/g_n) \le c_1(b)\, g_n^{-b} & \textit{for} \quad b>0,\\ ii: & \int_0^{1/g_n} y^{-1/2} dH(y) \le c_2(b)\, g_n^{-b+1/2} & \textit{for} \quad b>1/2,\\ iii: & \int_0^{1/g_n} y^{-1} dH(y) \le c_3(b)\, g_n^{-b+1} & \textit{for} \quad b>1, \end{array}\right\} \tag{4.6}$$

$$\left.\begin{array}{lll} i: & h_2(0)=0, \quad \textit{and} \quad |h_2(1/g_n)| \le c_4(b)\, n\, g_n^{-b} & \textit{for} \quad b>1,\\ ii: & \int_0^{1/g_n} y^{-1}|h_2(y)|dy \le c_5(b)\, n\, g_n^{-b} & \textit{for} \quad b>1, \end{array}\right\} \tag{4.7}$$

we obtain for the function $G_n(x,1/g_n)$ defined in (4.3):

$$\sup_x \left|G_n(x,1/g_n) - G_{n,2}(x) - I_1(x,n) - I_2(x,n)\right| \le C\, g_n^{-b} \tag{4.8}$$

with

$$G_{n,2}(x) = \begin{cases} \int_0^\infty \Phi(xy^\gamma)dH(y), & 0<b\le 1/2,\\ \int_0^\infty \Big(\Phi(xy^\gamma) + \dfrac{f_1(xy^\gamma)}{\sqrt{g_n y}}\Big)dH(y) =: G_{n,1}(x), & 1/2<b\le 1,\\ G_{n,1}(x) + \int_0^\infty \dfrac{f_2(xy^\gamma)}{g_n y}dH(y) + \int_0^\infty \dfrac{\Phi(xy^\gamma)}{n}dh_2(y), & b>1, \end{cases} \tag{4.9}$$

$$I_1(x,n) = \int_{1/g_n}^\infty \Big(\frac{f_1(xy^\gamma)\ \mathbb{I}_{(0,1/2]}(b)}{\sqrt{g_n y}} + \frac{f_2(xy^\gamma)}{g_n\, y}\Big)dH(y) \quad \textit{for} \quad b\le 1 \tag{4.10}$$

and

$$I_2(x,n) = \int_{1/g_n}^\infty \Big(\frac{f_1(xy^\gamma)}{n\sqrt{g_n y}} + \frac{f_2(xy^\gamma)}{n\, g_n y}\Big)dh_2(y) \quad \textit{for} \quad b>1. \tag{4.11}$$

Remark 4.6. If $b>1/2$ then (4.6ii) implies (4.6i). If $b>1$ then (4.6iii) implies (4.6ii) and (4.6i). Conditions (4.6) and (4.7) guarantee to extend the integration range of the integrals in (4.9) from $[1/g_n,\infty)$ to $(0,\infty)$ which ensures (4.8). The length of the asymptotic expansion is defined by (4.9).

Remark 4.7. The limit distributions $\int_0^\infty \Phi(xy^\gamma)dH(y)$ in (4.9) are scale mixtures of normal distribution $\Phi(x\,y^\gamma)$ with a real constant γ and mixture distribution $H(y)$.

In the next two sections we use both Theorems 4.2 and 4.5 when the scale mixture $G(x) = \int_0^\infty \Phi(x\,y^\gamma)dH(y)$ as limiting distribution of M_{N_n} can be expressed in terms of the well-known distributions. We obtain non-asymptotic results like in Proposition 3.1 for the sample median M_{N_n}, using second order approximations for both the statistic M_m and for the random sample size N_n. In both cases the jumps of the distribution function of the random sample size N_n only affect the function $h_2(y)$ in formula (4.1).

5. Sample Size has Negative Binomial Distribution

Let the sample size $N_n(r)$ the negative binomially distributed (shifted by 1) with parameters $1/n$ and $r > 0$, having probability mass function

$$\mathbb{P}(N_n(r) = j) = \frac{\Gamma(j + r - 1)}{\Gamma(j)\,\Gamma(r)} \left(\frac{1}{n}\right)^r \left(1 - \frac{1}{n}\right)^{j-1}, \quad j = 1, 2, ... \tag{5.1}$$

with $g_n = \mathbb{E}(N_n(r)) = r\,(n-1) + 1$. Schluter and Trede (2016, Section 2.1) pointed out that the negative binomial distribution is one of the two leading cases for count models, it accommodates the over-dispersion typically observed in count data (which the Poisson model cannot) and they showed in a general unifying framework

$$\lim_{n\to\infty} \sup_x |\mathbb{P}(N_n(r)/g_n \le x) - G_{r,r}(x)| = 0, \tag{5.2}$$

where $G_{r,r}(x)$ is the Gamma distribution function with the shape parameter which coincides with the scale parameter and equals $r > 0$, having density

$$g_{r,r}(x) = \frac{r^r}{\Gamma(r)} x^{r-1} e^{-rx}\, \mathbb{I}_{(0\,,\,\infty)}(x), \quad x \in \mathbb{R}. \tag{5.3}$$

The statement (5.2) was proved earlier in Bening and Korolev (2004, Lemma 2.2).

The convergence rate in (5.2) for $r > 0$ is given in Bening et al. (2013, Formula (21)) or Gavrilenko et al. (2017, Formula (17)):

$$\sup_x |\mathbb{P}(N_n(r)/g_n \le x) - G_{r,r}(x)| \le C_r n^{-\min\{r,1\}}. \tag{5.4}$$

In Schluter and Trede (2016) and Gavrilenko et al. (2017) the negative binomial random variable $\tilde{N}_n(r)$ is not shifted: $\tilde{N}_n(r) = N_n(r) - 1 \in \{0, 1, 2, ...\}$ with $\mathbb{E}\tilde{N}_n(r) = r(n-1)$. Then we have $\mathbb{P}(\tilde{N}_n(r) \le 0) - G_{r,r}(0) = n^{-r} \to 0$ as $n \to \infty$ instead of $\mathbb{P}(N_n(r) \le 0) - G_{r,r}(0) = 0$. Moreover

$$\mathbb{P}\left(\frac{\tilde{N}_n(r))}{r(n-1)} \le x\right) = \mathbb{P}\left(\frac{N_n(r)}{g_n} \le x + \frac{1-x}{g_n}\right).$$

The statements (5.2) and (5.4) still hold when $\tilde{N}_n(r)$ is shifted by a fixed integer. From Taylor expansion with Lagrange remainder term it follows that for $r > 1$

$$\left| G_{r,r}\left(x + \frac{1-x}{g_n}\right) - G_{r,r}(x) - g_{r,r}(x)\frac{1-x}{g_n}\right| \le C\, g_n^{-\min\{r,2\}}.$$

Hence, for $r > 1$ shifting $\tilde{N}_n(r)$ has influence of a term by g_n^{-1}. Second order asymptotic expansions for $N_n(r)$ where proved in Christoph et al. (2020, Theorem 1):

Proposition 5.1. *Let $r > 0$, discrete random variable $N_n(r)$ have probability mass function (5.1) and $g_n := \mathbb{E}N_n(r) = r(n-1) + 1$. For $x > 0$ and all $n \in \mathbb{N}$ there exists a real number $C_3(r) > 0$ such that*

$$\sup_{x\ge0}\left|\mathbb{P}\left(\frac{N_n(r)}{g_n} \le x\right) - G_{r,r}(x) - \frac{h_{2;r}(x)}{n}\right| \le C_3(r)\, n^{-\min\{r,2\}}, \tag{5.5}$$

where

$$h_{2;r}(x) = \begin{cases} 0 & \text{for } r \le 1, \\ \dfrac{g_{r,r}(x)\left((x-1)(2-r) + 2Q_1(g_n\, x)\right)}{2\,r} & \text{for } r > 1, \end{cases}$$

$$Q_1(y) = 1/2 - (y - [y]) \text{ and } [.] \text{ denotes the integer part of a number.}$$

Remark 5.2. The jumps of the sample size $N_n(r)$ have an effect only on the function $Q_1(.)$ in the term $h_{2;r}(x)$. The function $Q_1(y)$ is periodic with period 1, it is right-continuous with jump height 1 at each integer point y.

In Theorem 4.2 an estimate for the negative moment $\mathbb{E}(N_n)^{-3/2}$ of the random sample size N_n is required. Proposition 5.1 is used in Bening (2020, Corollary 2) to obtain an asymptotic expansion of negative moments $\mathbb{E}(N_n(r))^{-p}$ for $1 < p+1 \le r \le 2$. Such expansions are applied in the mentioned paper to analyze asymptotic deficiencies and risk functions of estimates based on random-size samples. An improved result with leading term and remainder estimation is given here:

Theorem 5.3. *Let $r > 0$ and $p > 0$. Then the following expansions for negative moments hold for all $n \ge 2$:*

$$\mathbb{E}(N_n(r))^{-p} = \begin{cases} R^*_{1;n}, \quad 0 < r < p \le 2, & \\ \dfrac{r^r\,\ln(g_n)}{\Gamma(r)\,g_n^r} + R^*_{2;n}, & r = p \le 2, \\ \dfrac{r^p\,\Gamma(r-p)}{\Gamma(r)\,g_n^p} + R^*_{3;n}, & \max\{0, r-1\} < p < r \le 2, \\ \dfrac{r^p\,\Gamma(r-p)}{\Gamma(r)\,g_n^p} - \dfrac{(2-r)\,p\,r^p(p+1)\,\Gamma(r-p)}{2\,\Gamma(r)\,(r-p-1)\,r\,n\,g_n^p} + R^*_{4;n}, & p+1 < r \le 2, \\ \dfrac{r^p\,\Gamma(r-p)}{\Gamma(r)\,g_n^p} - \dfrac{(2-r)\,p\,r^p\,\ln(g_n)}{2\,\Gamma(r)\,r\,n\,g_n^p} + R^*_{5;n}, & p+1 = r \le 2, \\ R^*_{6;n}, & \min\{p, r\} > 2, \end{cases} \tag{5.6}$$

where $|R^*_{k;n}| \le c^*_k(p,r)\,g_n^{-\min\{r,2\}}$ for some constants $c^*_k(p,r) < \infty$, $k = 1, 2, ..., 6$.

Corollary 5.4. *The leading terms in (5.6) and the bound (5.5) lead to the estimate*

$$\mathbb{E}\big(N_n(r)\big)^{-p} \le C(r,p) \begin{cases} n^{-\min\{r,p,2\}}, & p \ne \min\{r,2\} \\ \ln(n)\,n^{-\min\{r,p,2\}}, & p = \min\{r,2\} \end{cases} \tag{5.7}$$

Assume the statistic T_m is asymptotically normal and $H(y) = G_{r,r}(y)$ is the limit distribution for $N_n(r)/\mathbb{E}(N_n(r))$. As in Example 1.1 the limit distributions of the scaled statistics $T_{N_n(r)}$ with random size $N_n(r)$ and scaling factors $(g_n/N_n(r))^\gamma\sqrt{N^*_n(r)})$ are again the scale mixtures

$$V_\gamma(x) = \int_0^\infty \Phi(xy^\gamma)dG_{r,r}(y) \quad \text{with} \quad \gamma \in \{-1/2, 0, 1/2\}.$$

For the densities $v_\gamma(x)$ of $V_\gamma(x)$ then the following apply:

$$v_\gamma(x) = \frac{r^r}{\sqrt{2\,\pi}\,\Gamma(r)} \int_0^\infty y^{\gamma+r-1} e^{-(x^2\,y^{2\,\gamma}/2+r\,y)}dy \quad \text{with} \quad \gamma \in \{-1/2, 0, 1/2\}.$$

The gamma function (8.1) with $\alpha = r + 1/2$ and $p = (r + x^2/2)$ for $\gamma = 1/2$ and (8.2) with $m = 1$, $p = r$ and $q = x^2/2$ for $\gamma = -1/2$ and $r = 2$ lead to

$$v_\gamma(x) = \begin{cases} s_{2\,r}(x) = \dfrac{\Gamma(r+1/2)}{\sqrt{2\,r\pi}\,\Gamma(r)}\left(1 + \dfrac{x^2}{2\,r}\right)^{-(r+1/2)}, & \gamma = 1/2, \\ \varphi(x) = \dfrac{1}{\sqrt{2\,\pi}} e^{-x^2/2}, & \gamma = 0, \\ l_2(x) = \left(\dfrac{1}{2} + |x|\right) e^{-2\,|x|}, \quad \text{for} \quad r = 2, & \gamma = -1/2, \end{cases} \tag{5.8}$$

Hence, the scale mixtures $V_\gamma(x)$ are the Student's t-distribution with $2\,r$ degrees of freedom $S_{2r}(x)$ if $\gamma = 1/2$, the normal law $\Phi(x)$ if $\gamma = 0$ and for $\gamma = -1/2$ the second order generalized Laplace distribution:

$$L_2(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sign}(x)\left(1 - (1 + |x|)\,e^{-2\,|x|}\right), \quad x \in \mathbb{R}. \tag{5.9}$$

For arbitrary $r > 0$ Macdonald functions $K_{r-1/2}(x)$ occur in the densities $l_r(x)$ of $L_2(x)$. Both $L_2(x)$ and $l_2(x)$ can be calculated in closed forms for integer values of r. The standard Laplace density with variance 1 is $l_1(x) = \dfrac{1}{\sqrt{2}}\,\mathrm{e}^{-\sqrt{2}\,|x|}$. These functions are discussed in more detail with references in Christoph and Ulyanov (2020, Section 5.1.3).

Remark 5.5. In Gnedenko et al. (1984) it was shown that under the conditions of Example 1.1 the median of a sample with geometrically distributed sample size $N_n(1)$ tends to the Student t-distribution $S_2(x)$ for $n \to \infty$.

Theorem 5.6. *Let $r > 0$. Consider the sample median $M_{N_n(r)}$ when random sample size $N_n(r)$ has probability mass function (5.1) and $g_n = \mathbb{E}N_n(r) = r(n-1)+1$. If inequalities (3.8) and (5.5) hold for the median $M_m(X_1,...,X_m)$ and the random sample size $N_n(r)$, respectively, then the following expansions apply for all $n \in \mathbb{N}$ uniformly in $x \in \mathbb{R}$:*
i: Scaling factor $\sqrt{g_n N_n^*(r)/N_n(r)}$ for the sample median $M_{N_n(r)}$ leads to Student's t-approximation:

$$\left|\mathbb{P}\Big(2p_0\sqrt{\frac{g_n N_n^*(r)}{N_n(r)}}\,M_{N_n(r)} \le x\Big) - S_{2r;2}(x;n))\right| \le \begin{cases} C_r n^{-\min\{r,3/2\}}, & r \neq 3/2, \\ C_r \ln(n)\ n^{-3/2}, & r = 3/2, \end{cases} \tag{5.10}$$

where $N_n^*(r)$ is defined in (4.5),

$$S_{2r;2}(x;n) = S_{2r}(x) + s_{2r}(x)\left(\frac{A_{1;r}(x)}{\sqrt{g_n}} + \frac{A_{2;r}(x)}{g_n} + \frac{(2-r)(x^3+x)}{4\,r\,n(2r-1)}\mathbb{I}_{\{r>1\}}\right), \tag{5.11}$$

$$A_{1;r}(x) = \frac{p_1\,x\,|x|}{4\,p_0^2}\,\mathbb{I}_{\{r>1/2\}} \qquad \text{and}$$

$$A_{2;r}(x) = \frac{x}{4}\left(\frac{3(2r+x^2)}{2r-1} + x^2\Big(1+\frac{p_2}{6p_0^3}\Big) - \frac{x^4 p_1^2(2r+1)}{8p_0^4(2r+x^2)}\right)\mathbb{I}_{\{r>1\}}.$$

ii: Normal approximation is obtained with random scaling factor $\sqrt{N_n^*(r)}$ for the sample median $M_{N_n(r)}$:

$$\left|\mathbb{P}(\sqrt{N_n^*(r)}\,M_{N_n(r)} \le x) - \Phi_{n,2}(x)\right| \le C_r \begin{cases} n^{-\min\{r,3/2\}}, & r \neq 3/2, \\ \ln(n)\,n^{-3/2}, & r = 3/2, \end{cases}$$

where with $f_1(x)$ and $f_2(x)$ given in (3.7)

$$\begin{aligned} \Phi_{n,2}(x) \quad &= \quad \Phi(x) + \frac{f_1(x)}{\sqrt{g_n}}\left(\ln(g_n)\,\mathbb{I}_{\{r=1/2\}} + \frac{r^{1/2}\,\Gamma(r-1/2)}{\Gamma(r)}\,\mathbb{I}_{\{r>1/2\}}\right) \\ &\quad + \frac{f_2(x)}{g_n}\left(\ln(n)\,\mathbb{I}_{\{r=1\}} + \frac{r}{r-1}\,\mathbb{I}_{\{r>1\}}\right), \end{aligned} \tag{5.12}$$

iii: If $r = 2$, mixed scaling factor $\sqrt{N_n^*(2)\,N_n(2)/g_n}$ for the statistic $M_{N_n(2)}$ leads to generalized Laplace approximation:

$$\left|\mathbb{P}\left(\sqrt{N_n^*(2)\,N_n(2)/g_n}\,M_{N_n(2)} \le x\right) - L_2(x) - l_{n;2}(x)\right| \le C_2\,n^{-3/2} \tag{5.13}$$

where $L_2(x)$ is generalized distribution Laplace, defined in (5.9) and

$$l_{n;2}(x) = e^{-2|x|}\left\{\frac{p_1\,x\,|x|}{2\,p_0^2\sqrt{g_n}} + \frac{x}{g_n}\left(\frac{3}{2} + |x|\Big(1+\frac{p_2}{6p_0^3}\Big) + \frac{p_1^2}{8\,p_0^4}\left(2\,x^2 + |x|\,\right)\right)\right\}.$$

Remark 5.7. Under (5.4) with $r > 1/2$ first order expansions of $\mathbb{P}\big(2p_0 g_n^\gamma\,M_{N_n} \le x\big)$ for $\gamma \in \{0, 1/2\}$ were announced in the conference paper Bening et al. (2016). The convergence rates in Theorems 3.1 and 3.2 as well as in Corollaries 3.1 and 3.2 in case $1/2 < r < 1$ have to be $\mathcal{O}(n^{-r})$ instead of $\mathcal{O}(n^{-1})$ as announced in Bening et al. (2016). Moreover, in case $r = 1$ the bound c/n in (5.10) improves the corresponding estimate $\mathcal{O}\big(\ln(n)\,n^{-1}\big)$ which was stated in the aforementioned paper .

6. Sample Size N_n is Pareto-Like Distributed

Let $Y(s) \in \mathbb{N}$ be discrete Pareto II distributed with parameter $s > 0$, having probability mass and distribution functions

$$\mathbb{P}(Y(s) = k) = \frac{s}{s+k-1} - \frac{s}{s+k} \quad \text{and} \quad \mathbb{P}\big(Y(s) \le k\big) = \frac{k}{s+k},\ k \in \mathbb{N}, \tag{6.1}$$

which is a particular class of a general model of discrete Pareto distributions, obtained by discretization continuous Pareto II (Lomax) distributions on positive integers, see Buddana and Kozubowski (2014).

Now, let $Y_1(s), Y_2(s), ...,$ be independent random variables with the same distribution (6.1). Define for $n \in \mathbb{N}$ and $s > 0$ the random variable

$$N_n(s) = \max_{1 \le j \le n} Y_j(s) \quad \text{with} \quad \mathbb{P}(N_n(s) \le k) = \left(\frac{k}{s+k}\right)^n, \quad n \in \mathbb{N}. \tag{6.2}$$

The distribution of $N_n(s)$ is extremely spread out on the positive integers.

Christoph et al. (2020) proved the following Chebyshev-Edgeworth expansion:

Proposition 6.1. *Let the discrete random variable $N_n(s)$ have distribution function (6.2). For $y > 0$, fixed $s > 0$ and all $n \in \mathbb{N}$ then there exists a real number $C_2(s) > 0$ such that*

$$\sup_{y>0} \left| \mathbb{P}\left(\frac{N_n(s)}{n} \le y\right) - H_s(y) - \frac{h_{2;s}(y)}{n} \right| \le \frac{C_2(s)}{n^2}, \tag{6.3}$$

$$H_s(y) = e^{-s/y} \text{ and } h_{2;s}(y) = s\, e^{-s/y}\left(s - 1 + 2Q_1(n\,y)\right)/\left(2\,y^2\right),\ y > 0\,, \tag{6.4}$$

where $Q_1(y)$ is defined in (5.1).

Remark 6.2. Lyamin (2010) proved a first order bound in (6.3) for integer $s \ge 1$:

$$\left| \mathbb{P}\left(\frac{N_n(s)}{n} \le x\right) - e^{-s/x} \right| \le \frac{C_s}{n},\ C_s = \begin{cases} 8e^{-2}/3 = 0.36..., & s = 1, n \ge 2 \\ 2e^{-2} = 0.27..., & s \ge 2, n \ge 1 \end{cases}. \tag{6.5}$$

In case $n = 1$ and $s = 1$ we have $\mathbb{P}\left(N_1(1) \le x\right) = 0$ for $0 < x < 1$ and

$$\sup_{0<x<1} \left| \mathbb{P}\left(N_1(1) \le x\right) - e^{-1/x} \right| = \sup_{0<x<1} e^{-1/x} = e^{-1} = 0.367\ldots.$$

Remark 6.3. The continuous function $H_s(y) = e^{-s/y}\mathbb{I}_{\{y>0\}}$ with parameter $s > 0$ is the distribution function of the inverse exponential random variable $W(s) = 1/V(s)$, where $V(s)$ is exponentially distributed with rate parameter $s > 0$. Both $H_s(y)$ and $\mathbb{P}(N_n(s) \le y)$ are heavy tailed with shape parameter 1.

Therefore $\mathbb{E}\big(N_n(s)\big) = \infty$ for all $n \in \mathbb{N}$ and $\mathbb{E}\big(W(s)\big) = \infty$. Moreover:

- First absolute pseudo moment $\nu_1 = \int_0^\infty x \left| d\left(\mathbb{P}\big(N_n(s) \le n\,x\big) - e^{-s/x}\right)\right| = \infty$,
- Absolute difference moment $\chi_u = \int_0^\infty x^{u-1} \left|\mathbb{P}\big(N_n(s) \le n\,x\big) - e^{-s/x}\right| dx < \infty$ for $1 \le u < 2$.

These statements are proved in Christoph et al. (2020, Lemma 2). On pseudo moments and some of their generalizations see e.g. Christoph and Wolf (1992, Chapter 2).

Next we estimate the negative moment $\mathbb{E}(N_n(s))^{-p}$, $p > 0$, for the random sample size $N_n(s)$:

Theorem 6.4. *Let $s > 0$ and $p > 0$. Then for all $n \ge 2$ the following statements hold for negative moments:*

$$\mathbb{E}(N_n(s))^{-p} = \begin{cases} \dfrac{\Gamma(p+1)}{s^p\, n^p} + \dfrac{(s-1)\, p\, \Gamma(p+2)}{2\, s^{p+1} n^{p+1}} + R^*_{1;n}, & 0 < p < 1, \\ \dfrac{\Gamma(p+1)}{s^p\, n^p} + R^*_{2;n}, & 1 \le p < 2, \\ R^*_{3;n}, & p \ge 2, \end{cases} \tag{6.6}$$

where $|R^*_{k;n}| \le c^*_k(p)\, n^{-2}$ *for some constants* $c^*_k(p) < \infty, \;\; k = 1, 2, 3.$

Corollary 6.5. *The leading terms in* (6.6) *and the bound* (6.3) *lead to the estimate*

$$\mathbb{E}\big(N_n(s)\big)^{-p} \le C(p)\, n^{-\min\{p,2\}}, \tag{6.7}$$

where for $0 < p \le 2$ *the order of the bound is optimal.*

Remark 6.6. In Bening (2020, Corollary 3) the expansion (6.6) for $0 < p < 1$ is given with an additional term at n^{-p-1}, which however has order n^{-2}, see (8.23).

Assume the statistic T_m is asymptotically normal and $H(y) = H_s(y) = e^{-s/y}$, $y > 0$, is the limit distribution for $N_n(s)/n$. The limit distributions of the scaled statistics $T_{N_n(s)}$ with random size $N_n(s)$ and scaling factors $(n/N_n(s))^\gamma \sqrt{N^*_n(s)})$ are the scale mixtures

$$V_\gamma(x) = \int_0^\infty \Phi(x y^\gamma) dH_s(y) \quad \text{with} \quad \gamma \in \{-1/2, 0, 1/2\},$$

see also Christoph and Ulyanov (2020). The densities $v_\gamma(x)$ of $V_\gamma(x)$ are then given by

$$v_\gamma(x) = \frac{s}{\sqrt{2\,\pi}} \int_0^\infty y^{\gamma-2} e^{-\left(x^2\, y^{2\,\gamma}/2 + s/y\right)} dy \quad \text{with} \quad \gamma \in \{-1/2, 0, 1/2\}.$$

The use of (8.3) with $m = 1$, $p = x^2/2$ and $q = s$ for $\gamma = 1/2$ and the substitution $z = 1/y$ and (8.1) with $\alpha = 3/2$ and $p = (s + x^2/2)$ for $\gamma = -1/2$ then lead to

$$v_\gamma(x) = \begin{cases} l_{1/\sqrt{s}}(x) = \dfrac{\sqrt{2\,s}}{2}\, e^{-\sqrt{2\,s}|x|}, & \gamma = 1/2, \\ \varphi(x) = \dfrac{1}{\sqrt{2\,\pi}} e^{-x^2/2}, & \gamma = 0, \\ s^*_2(x;\sqrt{s}) = \dfrac{1}{2\,\sqrt{2\,s}}\, \left(1 + \dfrac{x^2}{2\,s}\right)^{-3/2}, & \gamma = -1/2. \end{cases} \tag{6.8}$$

Hence, the scale mixtures $V_\gamma(x)$ are the Laplace distribution $L_{1/\sqrt{s}}(x)$ with scale parameter $1/\sqrt{s}$ if $\gamma = 1/2$, the normal law $\Phi(x)$ if $\gamma = 0$ and for $\gamma = -1/2$ the *scaled Student's t-distribution* $S^*_2(x;\sqrt{s})$ *with 2 degrees of freedom* and density $s^*_2(x;\sqrt{s})$. If Z has density $s^*_2(x;\sqrt{s})$ then $Z/\sqrt{s}$ has a classic Student's t-density with 2 degrees of freedom (3.1) with $\nu = 2$.

Theorem 6.7. *Let* $s > 0$. *Consider the sample median* M_{N_n} *with random sample size* $N_n = N_n(s)$ *having distribution function* (6.2). *If inequalities* (3.8) *and* (6.3) *hold for the median* $M_m(X_1, ..., X_m)$ *and the random sample size* $N_n(s)$, *respectively, then the following expansions apply for all* $n \in \mathbb{N}$:
i: Let $\gamma = 1/2$. *The scaling factor* $\sqrt{nN^*_n(s)/N_n(s)}$ *by the sample median* $M_{N_n(s)}$ *leads to Laplace approximation:*

$$\sup_x \left|\mathbb{P}\Big(2p_0\sqrt{nN^*_n(s)/N_n(s)}\, M_{N_n(s)} \le x\Big) - L_{1/\sqrt{s}}(x;n)\right| \le C_s\, n^{-3/2}\,, \tag{6.9}$$

where N^*_n *is defined in* (4.5)

$$\begin{aligned} L_{1/\sqrt{s}}(x;n) \;&=\; L_{1/\sqrt{s}}(x) + l_{1/\sqrt{s}}(x) \left\{A_{1;s}(x)\, n^{-1/2} + A_{2;s}(x)\, n^{-1}\right\}, \qquad (6.10) \\ A_{1;s}(x) \;&=\; \frac{p_1\, x\, |x|}{4\, p_0^2} \qquad \textit{and} \\ A_{2;s}(x) \;&=\; \frac{(4-s)x(1+\sqrt{2s}|x|)}{8\, s} + \frac{x^3}{4}\Big(1 + \frac{p_2}{6p_0^3}\Big) - \frac{p_1^2 x^3 |x| \sqrt{2s}}{32\, p_0^4}. \end{aligned}$$

ii: If $\gamma = 0$, the normal approximation is obtained with random scaling factor $\sqrt{N_n^*(s)}$ at the sample median $M_{N_n(s)}$ with $f_1(x)$ and $f_2(x)$ given in (3.7):

$$\sup_x \left| \mathbb{P}(\sqrt{N_n^*(s)}\, M_{N_n(s)} \le x) - \Phi(x) - \frac{\sqrt{\pi}\, f_1(x)}{2\sqrt{s}\sqrt{g_n}} - \frac{f_2(x)}{s\, g_n} \right| \le C_s n^{-3/2}.$$

iii: If $\gamma = -1/2$, the mixed scaling factor $\sqrt{N_n^*(s)\, N_n(s)/n}$ by statistic $M_{N_n(s)}$ leads to scaled Student's t-approximation:

$$\sup_x \left| \mathbb{P}\left(\sqrt{N_n^*(s)\, N_n(s)/n}\, M_{N_n(s)} \le x\right) - S_2^*(x;\sqrt{s}) - s_{n;2}^*(x;\sqrt{s}) \right| \le C_2\, n^{-3/2}$$

where $S_2^*(x;\sqrt{s})$ is scaled Student's distribution with density $s_2^*(x;\sqrt{s})$ defined in (6.8) and

$$\begin{aligned} s_{n;2}^*(x;\sqrt{s}) \quad &= \quad s_2^*(x;\sqrt{s}) \left\{ \frac{3\, p_1\, x\, |x|}{4\, p_0^2 \sqrt{n}\, (x^2+2s)} \right. \\ &+ \left. \frac{x}{4\, n}\left(\frac{9+3\,(s-1)}{x^2+2s} + \left(1+\frac{p_2}{6p_0^3}\right) \frac{15\, x^2}{(x^2+2s)^2} - \frac{p_1^2\, 105\, x^4}{8\, p_0^4\, (x^2+2s)^3} \right) \right\}. \end{aligned}$$

Remark 6.8. Under the condition (6.5) a first order expansions for $\gamma \in \{0, 1/2\}$ was announced in the conference paper Bening et al. (2016), where in Theorem 4.1 and Corollary 4.1 the limit distribution has to be $L_{1/\sqrt{s}}(x)$.

7. Cornish-Fisher Expansions for Quantiles of M_m and M_{N_n}

In statistical inference it is of fundamental importance to obtain the quantiles of the distribution of statistics under consideration. Statistical applications and modeling with quantile functions are discussed extensively by Gilchrist (2000). There are very few quantile functions which can be expressed in closed form. The Cornish-Fisher expansions provide tools to approximate the quantiles of probability laws.

Let $F_n(x)$ be a distribution function admitting a Chebyshev-Edgeworth expansion in powers of $g_n^{-1/2}$ with $0 < g_n \uparrow \infty$ as $n \to \infty$:

$$F_n(x) = G(x) + g(x)\left(a_1(x) g_n^{-1/2} + a_2(x) g_n^{-1}\right) + R(g_n),\ R(g_n) = \mathcal{O}(g_n^{-3/2}), \tag{7.1}$$

where $g(x)$ is the density of a three times differentiable limit distribution $G(x)$.

Proposition 7.1. *Let $F_n(x)$ be given by (7.1) and let $x(u)$ and u be quantiles of distributions F_n and G with the same order α, i.e. $F_n(x(u)) = G(u) = \alpha$. Then the following relation holds for $n \to \infty$:*

$$x(u) = u + b_1(u) g_n^{-1/2} + b_2(u) g_n^{-1} + R^*(g_n), \quad R^*(g_n) = \mathcal{O}(g_n^{-3/2}),$$

with

$$b_1(u) = -a_1(u) \quad \textit{and} \quad b_2(u) = \frac{g'(u)}{2\, g(u)} a_1^2(u) + a_1'(u) a_1(u) - a_2(u)\,.$$

Proposition 7.1 is a direct consequence of more general statements, see e.g. Ulyanov (2011, p. 311–315), Fujikoshi et al. (2010, Chapter 5.6.1) or Ulyanov et al. (2016) and the references therein.

First we consider random median M_{N_n} if sample size $N_n = N_n(r)$ is negative binomial distributed with probability mass function (5.1) and Student's t-distribution $S_{2r}(x)$ is the limit law. The second order expansion (5.10) in Theorem 5.6 admits a relation like (7.1) with $g_n = r(n-1)+1$ and $a_k(x) = A_{k;r}(x)$, $k = 1, 2$. The transfer Proposition 7.1 implies the following statement:

Corollary 7.2. *Suppose $r > 0$. Let $x = x_\alpha$ and $u = u_\alpha$ be α-quantiles of standardized statistic $\mathbb{P}\Big(2p_0\sqrt{g_n N_n^*(r)/N_n(r)}\, M_{N_n(r)} \le x\Big)$ and of the limit Student's t-distribution $S_{2r}(u)$, respectively. Then with previous definitions the following Cornish-Fischer expansion holds as $n \to \infty$:*

$$x = u - \frac{p_1 u|u|}{4p_0^2\sqrt{g_n}}\mathbb{I}_{\{r>1/2\}} + \frac{B_2(u)}{g_n}\mathbb{I}_{r>1\}} + \begin{cases} \mathcal{O}(n^{-\min\{r,3/2\}}), & r \neq 3/2 \\ \mathcal{O}(\ln(n) n^{-3/2}), & r = 3/2 \end{cases},$$

where $\quad B_2(u) = \dfrac{p_1^2\, u^3}{8\, p_0^4} - \dfrac{(5-r)\, u^3 \,+\, (5r+2)u)}{4(2\,r\,-\,1)} - \dfrac{u^3}{4}\Big(1 + \dfrac{p_2}{6\, p_0^3}\Big).$

Next we study the approximation of quantiles for the random mean M_{N_n} if sample size $N_n = N_n(s)$ is based on discrete Pareto distributions with probability mass function (6.2) and Laplace distribution $L_{1/\sqrt{s}}(x)$ is the limit law. Relation (6.9) in Theorem 6.7 admits a expansion like (7.1) with $g_n = n$ and $a_k(x) = A_{k;s}(x)$, $k = 1, 2$. The transfer Proposition 7.1 leads now to:

Corollary 7.3. *Suppose $s > 0$. Let $x = x_\alpha$ and $u = u_\alpha$ be α-quantiles of standardized statistic $\mathbb{P}\Big(2p_0\sqrt{n N_n^*(s)/N_n(s)}\, M_{N_n(s)} \le x\Big)$ and of the limit Laplace distribution $L_{1/\sqrt{s}}(u)$, respectively. Then with previous definitions the following Cornish-Fisher expansion holds*

$$x = u - \frac{p_1 u|u|}{4p_0^2\sqrt{n}} + \frac{B_2(u)}{n} + \mathcal{O}(n^{-3/2}), \quad \textit{as} \quad n \to \infty,$$

where $\quad B_2(u) = \dfrac{p_1^2\, u^3}{8\, p_0^4} + \dfrac{(4-s)\, u\,(1 + \sqrt{2s}|u|)}{8\, s} - \dfrac{u^3}{4}\Big(1 + \dfrac{p_2}{6p_0^3}\Big).$

For the sake of completeness let us consider the Cornish-Fischer expansion for the median M_m, too. With (3.8) where $a_k(x) = f_k(x)$, $k = 1, 2$ are defined in (3.7), then holds

Corollary 7.4. *Let $x = x_\alpha$ and $u = u_\alpha$ be α-quantiles of standardized statistic $\mathbb{P}\Big(2p_0\sqrt{m^*}\, M_m \le x\Big)$ and of the limit normal distribution $\Phi(u)$, respectively. Then with previous definitions the classical Cornish-Fischer expansion holds as $m \to \infty$:*

$$x = u - \frac{p_1 u|u|}{4p_0^2\sqrt{m}} + \frac{1}{m}\left(\frac{p_1^2\, u^3}{8\, p_0^4} - \frac{u}{4} - \frac{u^3}{4}\Big(1 + \frac{p_2}{6\, p_0^3}\Big)\right) + \mathcal{O}(m^{-3/2}).$$

8. Proofs

In order to be able to present the occurring integrals in closed forms, we use the following formulas in Prudnikov et al. (1986) 2.3.3.1, 2.3.16.2, 2.3.16.3 and 2.5.31.4:

$$\int_0^\infty z^{\alpha-1}\, e^{-p\,z} dz \overset{z=1/y}{=} \int_0^\infty y^{-\alpha-1}\, e^{-p/y} dy = \Gamma(\alpha)\, p^{-\alpha}, \qquad \alpha > 0, \quad p > 0, \tag{8.1}$$

$$\int_0^\infty y^{m-1/2}\, e^{-py-q/y} dy = (-1)^m \sqrt{\pi}\, \frac{\partial^m}{\partial p^m}\left(p^{-1/2}\, e^{-2\sqrt{p\,q}}\right), \quad m = 0, 1, 2, \ldots, \tag{8.2}$$

$$\int_0^\infty y^{-m-1/2}\, e^{-py-q/y} dy = (-1)^m \frac{\sqrt{\pi}}{\sqrt{p}}\, \frac{\partial^m}{\partial q^m}\left(e^{-2\sqrt{p\,q}}\right), \qquad m = 0, 1, 2, \ldots, \tag{8.3}$$

$$\int_0^\infty y^{\alpha-1}\, e^{-py}\, \sin(b\,y) dy = \frac{\Gamma(\alpha)\, \sin(\alpha)\, \arctan(b/p)}{(b^2+p^2)^{\alpha/2}}, \qquad \alpha > -1 \tag{8.4}$$

and the Fourier series expansion of $Q_1(y)$ at all non-integer points y, see formula 5.4.2.9 with $a = 0$:

$$Q_1(y) = 1/2 - (y - [y]) = \sum_{k=1}^\infty \frac{\sin(2\,\pi\,k\,y)}{k\,\pi}, \quad y \neq [y]. \tag{8.5}$$

Proof of Proposition 3.1: Following the proof of Burnashev (1997, Theorem 1) one has to change Stirling's formula of the Gamma functions $\Gamma(z)$ and $1/\Gamma(z)$ as $z \to \infty$ by inequalities, proved in

Nemes (2015, Theorem 1.3):

$$\left.\begin{array}{rcl}\Gamma(z) &=& \sqrt{2\,\pi}\, z^{z-1/2}\, e^{-z}\,(1+\dfrac{1}{12z}+\dfrac{1}{288z^2}+R_3(z)),\\ \dfrac{1}{\Gamma(z)} &=& \dfrac{1}{\sqrt{2\,\pi}}\, z^{-z+1/2}\, e^{z}\,(1-\dfrac{1}{12z}+\dfrac{1}{288z^2}+\tilde{R}_3(z)),\end{array}\right\}\quad z>0,$$

with $\{|R_3(z)|, |\tilde{R}_3(z)| \le c z^{-3}$ and $c = \dfrac{(1+\zeta(3))\Gamma(3)(2\sqrt{3}+1)}{2\,(2\pi)^4} \le 0.006.$

Here $\zeta(z)$ is the Riemann zeta function with $\zeta(3) \approx 1.202...$

Finally, when ever Taylor's formula is used with remainder in big $\mathcal{O}$ notation, then the remainder has to be estimated in Lagrange form by an inequality. The constants $C_1, C_2 > 0$ in (3.6) and (3.8) depend only on p_0, p_1, p_2 and the upper bound of $p_X'''(x)$ in some interval $(0, x_0),\ x_0 > 0$. □

Proof of Theorem 4.2: The proof follows along the similar arguments of the more general transfer theorem in Bening et al. (2013, Theorem 3.1) for $\gamma \ge 0$ under conditions of our Theorem 4.2. Then conditioning on N_n, we have

$$\begin{aligned}\mathbb{P}\Big(2p_0\,(g_n/N_n)^\gamma\,\sqrt{N_n^*}\;\,M_{N_n} \le x\Big) &= \mathbb{P}\Big(2p_0\sqrt{N_n^*}\,M_{N_n} \le x\,(N_n/g_n)^\gamma\Big)\\ &= \sum\nolimits_{m=1}^{\infty} \mathbb{P}\Big(2p_0\sqrt{m^*}\,M_m \le x\,(m/g_n)^\gamma\Big)\,\mathbb{P}(N_n = m).\end{aligned}$$

Using now (3.8) with $\Phi_m(z) := \Phi(z) + m^{-1/2} f_1(z) + m^{-1} f_2(z)$:

$$\begin{aligned}&\sup\nolimits_x \sum\nolimits_{m=1}^{\infty} \Big|\mathbb{P}\Big(2p_0\sqrt{m^*}\,M_m \le x\,(m/g_n)^\gamma\Big) - \Phi_m(x\,(m/g_n)^\gamma)\Big|\;\mathbb{P}(N_n = m)\\ &\overset{(3.8)}{\le} C_2 \sum\nolimits_{m=1}^{\infty} m^{-3/2}\,\mathbb{P}(N_n = m) = C_2\,\mathbb{E}(N_n^{-3/2}).\end{aligned} \tag{8.6}$$

Taking into account $\mathbb{P}\Big(N_n/g_n < 1/g_n\Big) = \mathbb{P}\Big(N_n < 1\Big) = 0$ we obtain

$$\begin{aligned}\sum\nolimits_{m=1}^{\infty} \Phi_m(x\,(m/g_n)^\gamma)\mathbb{P}(N_n = m) &= \mathbb{E}\,(\Phi_{N_n}(x\,(N_n/g_n)^\gamma))\\ &= \int_{1/g_n}^{\infty} \Delta_n(x, y; \gamma)d\mathbb{P}(N_n/g_n \le y) = G_n(x, 1/g_n) + I(\gamma),\end{aligned}$$

where $\Delta_n(x, y; \gamma) := \Phi(x\,y^\gamma) + f_1(x\,y^\gamma)/\sqrt{g_n y} + f_2(x\,y^\gamma)/(g_n y),\ G_n(x, 1/g_n)$ is defined in (4.3) and

$$I(\gamma) = \int_{1/g_n}^{\infty} \Delta_n(x, y; \gamma)d\Big(\mathbb{P}(N_n/g_n \le y) - H(y) - \frac{h_2(y)}{n}\Big).$$

Estimating integral $I(\gamma)$ we use integration by parts for Lebesgue-Stieltjes integrals:

$$\begin{aligned}|I(\gamma)| \;\le\;& \sup\nolimits_x \lim_{L\to\infty} |\Delta_n(x, y; \gamma)|\,\Big|\mathbb{P}\big(N_n/g_n \le y\big) - H(y) - n^{-1}h_2(y)\Big|\Big|_{y=1/g_n}^{y=L}\\ &+ \sup\nolimits_x \int_{1/g_n}^{\infty} \Big|\frac{\partial}{\partial\, y}\Delta_n(x, y; \gamma)\Big|\,\Big|\mathbb{P}\big(N_n/g_n \le y\big) - H(y) - n^{-1}h_2(y)\Big|\,dy.\end{aligned}$$

First we calculate $(\partial/\partial\, y)\Delta_n(x, y; \gamma)$ for $\gamma \in \{0, \pm 1/2\}$. We get

$$\frac{\partial}{\partial\, y}\Big(\frac{f_1(x\,y^\gamma)}{\sqrt{y}}\Big) = \frac{\mathrm{sign}(x)\,q_1(x\,y^\gamma)}{4\,y^{3/2}} \quad\text{and}\quad \frac{\partial}{\partial\, y}\Big(\frac{f_2(x\,y^\gamma)}{y}\Big) = \frac{q_2(x\,y^\gamma)}{4\,y^2} \tag{8.7}$$

with $q_1(z) = a_0(2\gamma - 1/2 - \gamma\, z^2)z^2\,\varphi(z)$ and
$q_2(z) = \Big(\gamma\, a_2 z^6 - (\gamma a_1 + (5\gamma - 1)a_2)z^4 + ((3\gamma - 1)a_1 - 3\gamma)z^2 + 3(\gamma - 1)\Big)z\varphi(z),$
where $a_0 = p_1/p_0^2,\ a_1 = 1 + p_2/(6p_0^3)$ and $a_2 = p_1^2/(8p_0^4)$, see (3.7).

The functions $f_k(z)$ and $q_k(z)$, $k=1,2$, are bounded, we suppose

$$\sup_z |f_k(z)| \le c_k^* < \infty \quad \text{and} \quad \sup_z |q_k(z)| \le c_k^{**} < \infty, \qquad k=1,2. \tag{8.8}$$

To estimate $D_n(\gamma)$ defined in (4.4) we look at $D_n(x;\gamma)$ for $x \neq 0$ since $D_n(0;\gamma)=0$. Using (8.7) together with $\int_{1/g_n}^{\infty}(\partial/\partial\, y)\Phi(xy^\gamma)dy = 1-\Phi(x/g_n^\gamma) \le 1/2$ for $x>0$ and $\int_{1/g_n}^{\infty}|(\partial/\partial\, y)\Phi(xy^\gamma)|dy = \Phi(x/g_n^\gamma) \le 1/2$ for $x<0$ in case of $\gamma=\pm 1/2$ and $\frac{\partial}{\partial\, y}\Phi(x\, y^\gamma)=0$ for $\gamma=0$, then we find $D_n(x;\gamma) \le 1/2 + c_1^{**}/2 + c_2^{**}/4 = D(\gamma) < \infty$ and (4.4) holds. It follows from (4.1) and (8.8) that $|I(\gamma)| \le (C_4 + D(\gamma)\, C_3)\, n^{-b}$ with $C_4 = (1+c_1^*+c_2^*)C_3$. Together with (8.6), Theorem 4.2 is proved. □

Proof of Theorem 4.5: Using condition (4.6i) we find

$$\int_0^{1/g_n} \Phi(xy^\gamma)dH(y) \le \int_0^{1/g_n} dH(y) = H(1/g_n) - H(0) \le c_1 g_n^{-b}.$$

It follows from (8.8), (4.6ii) and (4.6iii) that for $k=1,2$

$$\int_0^{1/g_n} |f_k(xy^\gamma)| y^{-k/2} dH(y) \le c_k^* \int_0^{1/g_n} y^{-k/2} dH(y) \le c_k^*\, c_{k+1}(r) g_n^{-b+k/2}.$$

Integration by parts, $|z|\varphi(z)/2 \le c^* = (8\,\pi\, e)^{-1/2}$, (4.7i) and (4.7ii) lead to

$$\left| \int_0^{1/g_n} \Phi(xy^\gamma) dh_2(y) \right| \le |h_2(1/g_n)| + c^* \int_0^{1/g_n} y^{-1}|h_2(y)|dy \le (c_4 + c^* c_5) n\, g_n^{-b}.$$

Taking into account (4.3), (4.9), (4.10) and (4.11) we obtain (4.8). □

Proof of Theorem 5.3: Integrating by parts and substituting $y/g_n = x$, we obtain

$$\begin{aligned} \mathbb{E}(N_n(r))^{-p} &= \int_1^\infty \frac{1}{y^p} d\mathbb{P}\big(N_n(r) < y\big) = \frac{p}{g_n^p} \int_{1/g_n}^\infty \frac{1}{x^{p+1}} \mathbb{P}\left(\frac{N_n(r)}{g_n} < x\right) dx \\ &= \frac{p}{g_n^p} \int_{1/g_n}^\infty \frac{1}{x^{p+1}} \left(G_{r,r}(x) + \frac{h_{2;r}(x)}{n} \right) dx + R_1(n) = I_1 + I_2 + R_1(n), \end{aligned} \tag{8.9}$$

where (5.5) of the Proposition 5.1 gives

$$|R_1(n)| \le \frac{p}{g_n^p} \int_{1/g_n}^\infty \frac{1}{x^{p+1}} \left| \mathbb{P}\left(\frac{N_n(r)}{g_n} < x\right) - G_{r,r}(x) - \frac{h_{2;r}(x)}{n} \right| dx \le \frac{C_3(r)}{n^{\min\{r,2\}}}.$$

The integral I_1 in (8.9) we calculate with integration by parts:

$$I_1 = \frac{p}{g_n^p} \int_{1/g_n}^\infty \frac{G_{r,r}(x)}{x^{p+1}} dx = I_{1;p}(n) + R_2(n)$$

with $R_2(n) = G_{r,r}(1/g_n) \le r^r g_n^{-r}/\Gamma(r+1)$

$$I_{1;p}(n) = \frac{r^r}{\Gamma(r)\, g_n^p} \int_{1/g_n}^\infty \frac{e^{-r\,x}}{x^{p+1-r}} dx = \begin{cases} \dfrac{r^p\, \Gamma(r-p)}{\Gamma(r)\, g_n^p} + R_3(n), & p<r, \\ \dfrac{r^r\, \ln(g_n)}{\Gamma(r)\, g_n^r} + R_4(n), & p=r, \\ R_5(n), & p>r, \end{cases} \tag{8.10}$$

where for $p<r$

$$|R_3(n)| = \frac{r^r}{\Gamma(r)\, g_n^p} \int_0^{1/g_n} x^{r-p-1} e^{-r\,x} dx \le \frac{r^r}{\Gamma(r)\,(r-p)\, g_n^r}.$$

In case $p = r$ we split the integral in $I_{1;p}(n)$ into three parts, the first one leads to the leading term in (8.10):

$$\int_{1/g_n}^{\infty} \frac{e^{-r\,x}}{x} dx = \int_{1/g_n}^{1/r} x^{-1} dx - \int_{1/g_n}^{1/r} \frac{1-e^{-r\,x}}{x} dx + \int_{1/r}^{\infty} \frac{e^{-r\,x}}{x} dx. \tag{8.11}$$

Then we obtain

$$|R_4(n)| = \left| I_{1;p}(n) - \frac{r^r \ln(g_n)}{\Gamma(r)\, g_n^r} \right| \le \frac{r^r\,(1+r+e^{-1})}{\Gamma(r)\, g_n^r}.$$

For $p > r$ we have

$$R_5(n) = \frac{r^r}{\Gamma(r)\, g_n^p} \int_{1/g_n}^{\infty} \frac{e^{-r\,x}}{x^{p+1-r}} dx \le \frac{r^r}{\Gamma(r)\,(p-r)\, g_n^r}.$$

Now we calculate the integral I_2 in (8.9) in case of $r > 1$:

$$I_2 = \frac{p}{g_n^p\, n} \int_{1/g_n}^{\infty} \frac{h_{2;r}(x)}{x^{p+1}} dx = \frac{p\, r^r\,(2-r)}{2\, r\, \Gamma(r)\, g_n^p\, n} \int_{1/g_n}^{\infty} \frac{e^{-r\,x}}{x^{p+2-r}} (x-1) dx + I_{2,p}(n). \tag{8.12}$$

Define $I_{3,p}(n) = I_2 - I_{2,p}(n)$. Since the integrals in I_1 and $I_{3,p}(n)$ have the same structure, one get with the above method for $r > 1$

$$I_{3,p}(n) = \begin{cases} \dfrac{p\, r^p\,(2-r)}{2\, r\, \Gamma(r) g_n^p\, n}\Big(\Gamma(r-p) - r\,\Gamma(r-p-1)\Big) + R_6(n), & p < r-1, \\ \dfrac{p\, r^r\,(2-r)}{2\, r\, \Gamma(r) g_n^{r-1}\, n}(-r \ln(g_n)) + R_7(n), & p = r-1, \\ R_8(n), & p > r-1, \end{cases} \tag{8.13}$$

where $|R_k(n)| \le c_k(r,p) g_n^{-r}$, with some constants $c_k(r,p)$, $k = 6, 7, 8$.

It remains to show that integral $I_{2,p}(n)$ in (8.12) has the order of the remainder:

$$|I_{2,p}(n)| = \frac{p\, r^{r-1}}{\Gamma(r)\, g_n^p\, n} \left| \int_{1/g_n}^{\infty} \frac{e^{-r\,x}}{x^{p+2-r}} Q_1\big(g_n\, x\big) dx \right| \le c(p,r) g_n^{-r} \quad \text{for} \quad r > 1. \tag{8.14}$$

Let $0 < p < r-1$. Then

$$I_{2,p}(n) = \frac{p\, r^{r-1}}{\Gamma(r)\, g_n^p\, n} \int_0^{\infty} \frac{e^{-r\,x}}{x^{p+2-r}} Q_1\big(g_n\, x\big) dx + R_9(n) = \frac{p\, r^{r-1}}{\Gamma(r)\, g_n^p\, n} J^*_{2;p}(n) + R_9(n)$$

where, since $g_n \le r\, n$ for $r > 1$ and $|Q_1(g_n x)| \le 1/2$, applies

$$|R_9(n)| = \frac{p\, r^{r-1}}{\Gamma(r)\, g_n^p\, n} \left| \int_0^{1/g_n} \frac{e^{-r\,x}}{x^{p+2-r}} Q_1(g_n\, x) dx \right| \le \frac{p\, r^r}{2\Gamma(r) g_n^{p+1}} \int_0^{1/g_n} \frac{dx}{x^{p+2-r}} = \frac{c_{p,r}}{g_n^r}.$$

Considering (8.5) and interchange integral and sum we find

$$J^*_{2;p}(n) = \int_0^{\infty} \frac{e^{-r\,x}}{x^{p+2-r}} Q_1\big(g_n\, x\big) dx = \sum\nolimits_{k=1}^{\infty} \frac{1}{k\,\pi} \int_0^{\infty} \frac{e^{-r\,x}}{x^{p+2-r}} \sin(2\,\pi\, k\, g_n\, x) dx.$$

Applying (8.4) with $\alpha = r-p-1$, $p = r$ and $b = 2\pi k g_n$ then

$$\begin{aligned} J^*_{2;p}(n) &= \sum\nolimits_{k=1}^{\infty} \frac{1}{k\,\pi} \int_0^{\infty} x^{r-p-2} e^{-r\,x} \sin(2\,\pi\, k\, g_n\, x) dx \\ &= \frac{\Gamma(r-p-1)}{\pi} \sum\nolimits_{k=1}^{\infty} \frac{\sin\big((r-p-1)\arctan\,(2\pi k g_n/r)\big)}{k\,\big(\big(2\pi k g_n\big)^2 + r^2\big)^{(r-p-1)/2}}. \end{aligned}$$

Hence, for $0 < p < r-1$ we have

$$|J^*_{2;p}(n)| \le \frac{\Gamma(r-p-1)}{\pi} \sum\nolimits_{k=1}^{\infty} \frac{1}{k\,(2\pi\, k\, g_n)^{r-p-1}} \le \frac{\Gamma(r-p-1)}{\pi(2\pi)^{r-p-1}}\, \zeta(r-p)\, g_n^{-r+p+1}$$

with Riemann zeta function $\zeta(r-p) < \infty$ since $r-p>1$ and (8.14) holds.

In case $0<p=r-1$ the Fourier series expansion (8.5) of $Q_1(y)$ and integration by parts lead to

$$I_{2,p}(n) = \frac{p\,r^{r-1}}{\Gamma(r)\,g_n^p\,n}\sum_{k=1}^{\infty}\frac{1}{k^2\,2\,\pi^2\,g_n}\left(\frac{g_n}{e^{r/g_n}} - \int_{1/g_n}^{\infty}\Big(\frac{1}{x^2}+\frac{1}{x}\Big)e^{-r\,x}\cos(2\,\pi\,k\,g_n\,x)dx\right)$$

and

$$|I_{2,p}(n)| \le \frac{p\,r^{r-1}}{\Gamma(r)\,g_n^p\,n}\sum\nolimits_{k=1}^{\infty}\frac{2\,g_n+g_n/r}{k^2\,2\,\pi^2\,g_n} \le \frac{p\,r^{r-1}(2\,r+1)}{2\Gamma(r)\pi^2}\,\zeta(2)\,g_n^{-r}, \quad p=r-1.$$

If $p>r-1$ using $|Q_1(y)\le 1/2$ we find

$$|I_{2,p}(n)| \le \frac{p\,r^{r-1}}{\Gamma(r)\,g_n^p\,n}\int_{1/g_n}^{\infty}x^{-r+p+2}dx \le \frac{p\,r^r}{2\,\Gamma(r)(p+1-r)}g_n^{-r}, \quad p>r-1,$$

and (8.14) is proved. Estimates (8.9), (8.10), (8.13) and (8.14) lead to (5.6) and Theorem 5.3 is proved. □

Proof of Theorem 5.6: We use Theorems 4.2 and 4.5 with $H(y)=G_{r,r}(y)$, $h_2(y)=h_{2;r}(y)$, $g_n = r(n-1)+1$ and $b=\min\{r\,,\,2\}$ defined in Proposition 5.1 .

It follows from (5.7) with $p=3/2$ that

$$\mathbb{E}\big(N_n(r)\big)^{-3/2} \le C(r)\begin{cases} n^{-\min\{r,3/2\}}, & r\ne 3/2\\ \ln(n)\,n^{-3/2}, & r=3/2.\end{cases} \tag{8.15}$$

Next we check conditions (4.6) and (4.7). Using (5.3) we find for $k=0,1,2$

$$\int_0^{1/g_n} y^{-k/2}dG_{r,r}(y) = \frac{r^r}{\Gamma(r)}\int_0^{1/g_n} y^{r-k/2-1}\,e^{-ry}dy \le \frac{r^r}{\Gamma(r)\,(r-k/2)}g_n^{-r+k/2}.$$

Hence we obtain $c_{k+1}(r) = r^r\,(\Gamma(r)\,(r-k/2)^{-1}$ if $r>k/2$ for $k=0,1,2$ in (4.6).

Let $r>1$ and define

$$c_3^* = \frac{r^{r-1}}{2\,\Gamma(r)}\sup_y\{e^{-r\,y}\,(|y-1||2-r|+1)\} < \infty. \tag{8.16}$$

In this case we find $g_{r,r}(0)=0$, $h_{2;r}(0)=0$ and $g_n\le r\,n$. Hence (4.7i) and (4.7ii) hold with $c_4(r)=c_3^*$ and $c_5(r)=c_3^*/(r-1)$.

Now we estimate the integrals (4.10) and (4.11). Using (8.8) for $f_1(z)$ and $f_2(z)$ defined in (3.7) we find for $0<r<1/2$ and $\gamma\in\{0,1/2\}$:

$$J_{1;r;n}(x;\gamma) = \int_{1/g_n}^{\infty}\frac{|f_1(x\,y^\gamma)|}{\sqrt{y}}dG_{r,r}(y) \le \frac{c_1^*\,r^r}{\Gamma(r)}\int_{1/g_n}^{\infty}y^{r-3/2}dy = \frac{c_1^*\,r^r\,g_n^{r-1/2}}{\Gamma(r)\,(1/2-r)}$$

If $r=1/2$ and $\gamma=1/2$ then with $x^2/(1+x^2)\le 1$ we have

$$J_{1;1/2;n}(x;1/2) = \frac{x^2|p_1|}{8\,\pi p_0^2}\int_{1/g_n}^{\infty}e^{-((1+x^2)/2)\,y}dy \le \frac{x^2|p_1|}{8\,\pi p_0^2\,(1+x^2)/2} \le \frac{2|p_1|}{8\,\pi p_0^2}.$$

In the case of $r=1/2$ and $\gamma=0$ using (8.11) we find for first integral in (4.10)

$$\left|\frac{f_1(x)}{\sqrt{2\pi}\,\sqrt{g_n}}\int_{1/g_n}^{\infty}\frac{e^{-y/2}}{y}dy - \frac{f_1(x)\,\ln(g_n)}{\sqrt{2\pi}\,\sqrt{g_n}}\right| \le \frac{c_1^*\,(3/2+e^{-1})}{\sqrt{2\pi}\,\sqrt{g_n}}. \tag{8.17}$$

Consider the second term in (4.10). Let $r<1$ and $\gamma\in\{0,1/2\}$, then

$$J_{1;r;n}^*(x;\gamma) = \int_{1/g_n}^{\infty}\frac{|f_2(x\,y^\gamma)|}{y}dG_{r,r}(y) \le \frac{c_2^*\,r^r}{(1-r)\,\Gamma(r)}\,g_n^{1-r}. \tag{8.18}$$

If $r=1$ and $\gamma=1/2$ we define the polynomial $P_4(z)$ by $f_2(z)=P_4(z)z\,\varphi(z)$ with $z=x\sqrt{y}$ and put $c_4^*=\sup_z\{|P_4(z)|\varphi(z/\sqrt{2})\}<\infty$. Then $|f_2(z)|\le c_4^*|z|\,e^{-z^2/4}$, $g_n=n$ and

$$J^*_{1;1;n}(x;1/2)\le c_4^*\,|x|\int_{1/n}^\infty y^{-1/2}e^{-(x^2/4+1)\,y}\,dy\le\frac{c_4^*\,|x|\,\sqrt{\pi}}{(x^2/4+1)^{1/2}}\le c_4^*\,2\,\sqrt{\pi}.$$

If $r=1$ and $\gamma=0$ using (8.11) we obtain in the same way as in (8.17)

$$\left|\frac{f_2(x)}{n}\int_{1/n}^\infty\frac{e^{-y/2}}{y}dy-\frac{f_2(x)\,\ln(n)}{n}\right|\le\frac{c_2^*\,(2+e^{-1})}{n}. \tag{8.19}$$

Hence, for $0<r\le 1$ and $\gamma\in\{0,1/2\}$ we have

$$\left|I_1(x,n)-\frac{f_1(x)\,\ln(g_n)}{\sqrt{2\pi}\,\sqrt{g_n}}\mathbb{I}_{\{r=1/2,\,\gamma=0\}}-\frac{f_2(x)\,\ln(n)}{n}\,\mathbb{I}_{\{r=1,\,\gamma=0\}}\right|\le C_r g_n^{-r}.$$

In the case of $\gamma=-1/2$, the integral $I_1(x,n)$ does not occur because we only consider only the case $r=2$.

Now we estimate $I_2(x,n)$ in (4.11) for $r>1$ and $\gamma\in\{0,\pm1/2\}$. Integration by parts for Lebesgue-Stieltjes integrals, (8.7) and (4.7i) lead to

$$\begin{aligned}|I_2(x,n)|\ &\le\ \frac{1}{n}(f_1(x/g_n^\gamma)+f_2(x/g_n^\gamma)\,|h_{2;r}(1/g_n)|+I_2^*(x,n)\\ &\le\ (c_1^*+c_2^*)c_4(r)\,g_n^{-r}+I_2^*(x,n)\end{aligned} \tag{8.20}$$

with

$$I_2^*(x,n)\ =\ \int_{1/g_n}^\infty\Big(\frac{|q_1(x\,y^{g_n^\gamma})|}{4\,n\,\sqrt{g_n}y^{3/2}}+\frac{|q_2(x\,y^\gamma)|}{4\,n\,g_ny^2}\Big)\,|h_{2;r}(y)|dy, \tag{8.21}$$

where for $k=1,2$ functions $f_k(z)$ and $q_k(z)$ are bounded, see (8.8).

Moreover $g_ny^2\ge\sqrt{g_n}y^{3/2}$ for $y\ge 1/g_n$ and $g_n\le n\,r$ for $r>1$.

If $1<r<3/2$ with c_3^* defined in (8.16) we find

$$|I_2^*(x,n)|\le\frac{(c_1^{**}+c_2^{**})c_3^*}{4\,n\,\sqrt{g_n}}\int_{1/g_n}^\infty y^{r-5/2}dy=\frac{(c_1^{**}+c_2^{**})\,r\,c_3^*}{4\,(3/2-r)}\,g_n^{-r}.$$

If $r>3/2$ with $c_4^*=\dfrac{r^{r-1}}{2\,\Gamma(r)}\,\sup_y\{(e^{-r\,y/2}\,(|y-1|\,|2-r|+1)\}<\infty$ we obtain

$$|I_2^*(x,n)|\le\frac{(c_1^{**}+c_2^{**})c_4^*}{4\,n\,\sqrt{g_n}}\int_{1/g_n}^\infty y^{r-5/2}e^{-r\,y/2}dy\le\frac{(c_1^{**}+c_2^{**})\,r\,c_4^*\Gamma(r-3/2)}{4(r/2)^{r-3/2}g_n^{3/2}}.$$

For $r=3/2$ the above estimates of $|I_2^*(x,n)|$ lead to an exponential integral:

$$\begin{aligned}|I_2^*(x,n)|\ &\le\ \frac{(c_1^{**}+c_2^{**})c_4^*}{4\,n\,\sqrt{g_n}}\left(\int_{1/g_n}^1 y^{-1}dy+\int_1^\infty e^{-3\,y/2}dy\right)\\ &\le\ \frac{(c_1^{**}+c_2^{**})\,r\,c_4^*}{4}\left(\ln(g_n)+\frac{2}{3}\,e^{-3/2}\right)\,g_n^{-3/2}.\end{aligned}$$

In the latter case $r=3/2$ the bound $|I_2^*(x,n)|\le C\,g_n^{-3/2}$ may be obtained for $\gamma=1/2$ with an analogous procedure as for estimating the above integral $|I_1(x,n)|$ for $r=1$ in (8.18). This proof is omitted because the rate of convergence in Theorem 5.6, see (5.10), is determined by the negative moment (8.15), where the term $\ln(n)$ cannot be omitted.

To obtain (5.11) we calculate the integrals in (4.9), which are similar to that in the proof of Theorem 2 in Christoph et al. (2020). The densities of the limit laws are given in (5.8) for $\gamma\in\{0,\pm1/2\}$.

The calculation of the further integrals in (4.9) depends on γ.

For $\gamma = 1/2$ and $r > 1/2$ we find with $f_1(x)$ defined in (3.7) and $\alpha = 1/2$ in (8.1)

$$\int_0^\infty \frac{f_1(x\sqrt{y})}{\sqrt{y}} dG_{r,r}(y) = \frac{p_1\, x|x|\, r^r}{4\, p_0^2\, \Gamma(r)\, \sqrt{2\,\pi}} \int_0^\infty \frac{y^{r-1/2}}{e^{(r+x^2/2)y}} dy = \frac{p_1\, x|x|}{4\, p_0^2} s_{2r}(x).$$

Analogously, if $r > 1$ we obtain for $f_2(x)$ with $\alpha = -1/2, 1/2, 3/2$ in (8.1)

$$\int_0^\infty \frac{f_2(x\sqrt{y})}{y} dG_{r,r}(y) = \frac{x}{4} \left\{ \frac{3\,(2r + x^2)}{2r - 1} + \left(1 + \frac{p_2}{6p_0^3}\right) x^2 - \frac{p_1^2 x^4 (2r + 1)}{8p_0^4\,(2r + x^2)} \right\} s_{2r}(x).$$

The integral $\int_0^\infty \Phi(x\sqrt{y}) dh_{2;r}(y)$ in (4.9) is the same as the integral $J_4(x)$ in the proof of Theorem 2 in Christoph et al. (2020) where is shown:

$$\frac{1}{n} \sup_x \left| \int_0^\infty \Phi(x\sqrt{y}) dh_{2;r}(y) - \frac{(2 - r)x(x^2 + 1)}{4r(2r - 1)}\, s_{2r}(x) \right| \le c(r)\, n^{-r}$$

which proves the the Theorem 5.6 for $\gamma = 1/2$.

If $\gamma = 0$ then $f_k(xy^\gamma) = f_k(x)$, together with (8.17), (8.19), (8.1) with $\alpha = r - k/2$ and $p = r$ for $k = 1, 2$, we proved (5.12).

If $\gamma = -1/2$ and $r = 2$ then with (8.3) with $m = 0, 1, 2$, $p = x^2/2$ and $q = 2$ we find (5.13). Since for $r = 2$ we have $h_{2;2}(y) = g_{2;2}(y)\, Q_1(g_n y)/2$. Integration by parts leads to

$$\left| \int_0^\infty \Phi(x/\sqrt{y}) dh_{2;2}(y) \right| = \frac{|x|}{\sqrt{2\pi}} \left| \int_0^\infty y^{-1/2}\, e^{-x^2/(2y) - 2y}\, Q_1(g_n y) dy \right| \le \frac{\pi}{12\, e\, g_n},$$

where the last inequality was shown in Christoph and Ulyanov (2020) for $J^*_{3;2,2}$ in formula (A12). Theorem 5.6 is proved. □

Proof of Theorem 6.4: As in the beginning of the proof of Theorem 5.3 we obtain

$$\mathbb{E}(N_n(s))^{-p} = \frac{p}{n^p} \int_{1/n}^\infty \frac{1}{x^{p+1}} \left(H_s(x) + \frac{h_{2;s}(x)}{n} \right) dx + R_1(n) = I_1 + I_2 + I_3 + R_1(n),$$

where, with substitution $x = 1/y$ we find

$$I_1 = \frac{p}{n^p} \int_0^\infty \frac{e^{-s/x}}{x^{p+1}} dx + R_2(n) = \frac{p}{n^p} \int_0^\infty y^{p-1}\, e^{-s\,y} dy + R_2(n) = \frac{\Gamma(p + 1)}{s^p\, n^p} + R_2(n),$$

$$I_2 = \frac{p\, s\,(s - 1)}{2\, n^{p+1}} \int_0^\infty \frac{e^{-s/x}}{x^{p+3}} dx + R_3(n) = \frac{(s - 1)\, p\, \Gamma(p + 2)}{2\, s^{p+1} n^{p+1}} + R_3(n),$$

considering (8.5)

$$I_3 = \frac{p\, s}{n^{p+1}} \int_{1/n}^\infty \frac{e^{-s/x}\, Q_1(n\, x)}{x^{p+3}} dx = \frac{p\, s}{n^{p+1}} \sum\nolimits_{k=1}^\infty \frac{1}{k\,\pi} \int_{1/n}^\infty \frac{e^{-s/x}\, \sin(2\,\pi\, k\, n\, x)}{x^{p+3}} dx$$

and with (6.3) of Theorem 6.1

$$|R_1(n)| \le \frac{p}{n^p} \int_{1/n}^\infty \frac{1}{x^{p+1}} \left| \mathbb{P}\left(\frac{N_n(s)}{n} < x \right) - H_s(x) - \frac{h_{2sr}(x)}{n} \right| dx \le \frac{C_3(s)}{n^2}.$$

Since for $\alpha > 0$ and $0 < \beta \le 2$

$$\alpha \int_0^{1/n} \frac{e^{-s/x}}{x^{\alpha+1}} dx \le e^{-s\,n} n^\alpha \le C(\alpha, \beta, s)\, n^{-\beta}, \quad C(\alpha, \beta, s) = \left(\frac{\alpha + \beta}{s\, e} \right)^{\alpha+\beta} \tag{8.22}$$

we find with $c_2 = p\, C(0, 2, s)$ and $c_3 = sp|s - 1|\, C(2, 2, s)/2$

$$R_2(n) = \frac{p}{n^p} \int_0^{1/n} \frac{e^{-s/x}}{x^{p+1}} dx \le \frac{c_2}{n^2} \text{ and } |R_3(n)| = \frac{p\, s\, |s - 1|}{2\, n^{p+1}} \int_0^{1/n} \frac{e^{-s/x}}{x^{p+3}} dx \le \frac{c_3}{n^2}.$$

It remains to estimate I_3. Partial integration leads to

$$\int_{1/n}^{\infty} \frac{\sin(2\,\pi\,k\,n\,x)}{e^{s/x}\,x^{p+3}}dx = \frac{n^{p+3}}{e^{s\,n}2\,\pi\,k\,n} - \int_{1/n}^{\infty}\left(\frac{p+3}{x^{p+4}} - \frac{s}{x^{p+5}}\right)\frac{\cos(2\,\pi\,k\,n\,x)}{e^{s/x}2\,\pi\,k\,n}dx.$$

Considering the second inequality of (8.22) and $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$ we obtain

$$|I_3| \le p\,s\left(n\,e^{-s\,n} + \frac{2\Gamma(p+4)}{s^{p+3}\,n^{p+2}}\right) \le c(s,p)n^{-2}. \tag{8.23}$$

Theorem 6.4 is proved. ☐

Proof of Theorem 6.7: We use Theorems 4.2 and 4.5 with $H(y) = H_s(y)$ and $h_2(y) = h_{2;s}(y)$ defined in (6.4), $b = 2$ and $g_n = n$.

It follows from (6.7) with $p = 3/2$ that $\mathbb{E}\big(N_n(s)\big)^{-3/2} \le C(r)\,n^{-3/2}$.

The three conditions in (4.6) for $k = 0,1,2$ and $s > 0$ follow from

$$\int_0^{1/n} y^{-k/2}dH_s(y) = \int_0^{1/n} y^{-k/2}\frac{s}{y^2}e^{-s/y}dy = \int_n^{\infty} z^{k/2}e^{-s\,z}dz \le c_{k+1}(s)n^{-2+k/2}$$

with $c_{k+1}(s) = 2s^{-2}$. Moreover, (4.7) is valid, since $h_{2;s}(0) = \lim_{y\downarrow 0} h_{2;s}(y) = 0$,

$$|h_{2;s}(1/n)| \le \frac{1}{2}s(|s-1|+1)\,n^2e^{-s\,n} \le \frac{27\,(|s-1|+1)}{2\,s^2e^3\,n} = c_4(s)\,n^{-1}, \tag{8.24}$$

and

$$\int_0^{1/n}\frac{|h_{2;s}(y)|}{y}dy \le \frac{s(|s-1|+1)}{2}\int_0^{1/n}\frac{e^{-s/y}}{y^3}dy \le \frac{s|s-1|+1}{4}n^2e^{-sn} = c_5(s)\,n^{-1}.$$

It is worth to mention that in conditions (4.6) and (4.7) the functions $H_s(1/n)$ and $h_{2;s}(1/n)$ and the corresponding integrals decrease even exponentially with order $n\,e^{-s\,n}$ or $n^2\,e^{-s\,n}$, $s > 0$.

It remains to estimate $I_2(x,n)$ given in (4.11). Changing only $h_{2;r}(y)$ by $h_{2;s}(y)$ in the estimations (8.20) and (8.21) of the corresponding $I_2(x,n)$ in the proof of Theorem 5.6, using partial integration, the estimates (8.24), (8.8) and $ny^2 \ge \sqrt{n}y^{3/2}$ for $y \ge 1/n$, then we obtain

$$|I_2(x;n)| \le (c_1^* + c_2^*)\,c_4(s)n^{-2} + \frac{(4\,c_1^{**} + c_2^{**})\,(|s-1|+1)\,\Gamma(5/2)}{16\,s^{3/2}n^{3/2}} = c_(s)n^{-3/2}.$$

To obtain (6.10) we calculate integrals in (4.9) for $b = 3/2$ and $\gamma \in \{0, \pm 1/2\}$.

Let $\gamma = 1/2$. With (8.3) for $p = x^2/2 > 0$, $s > 0$, $m = 0,1,2$:

$$\int_0^{\infty}\frac{\varphi(x\sqrt{y})}{y^{m-3/2}}dH_s(y) = \int_0^{\infty}\frac{s\,e^{-x^2y/2-s/y}}{\sqrt{2\pi}\,y^{m+1/2}}dy = (-1)^m\,\frac{s}{|x|}\frac{\partial^m}{\partial s^m}e^{-\sqrt{2\,s}|x|}, \tag{8.25}$$

where

$$(-1)\frac{s}{|x|}\frac{\partial}{\partial s}e^{-\sqrt{2\,s}|x|} = l_{1/\sqrt{s}}(x) \;\text{ and }\; \frac{s}{|x|}\frac{\partial^2}{\partial s^2}e^{-\sqrt{2\,s}|x|} = \Big(\frac{|x|}{\sqrt{2\,s}} + \frac{1}{2\,s}\Big)l_{1/\sqrt{s}}(x).$$

Using (8.25) for $m = 1$,

$$\int_0^{\infty}\frac{f_1(x\sqrt{y})}{\sqrt{y}}dH_s(y) = \frac{p_1x|x|s}{4p_0^2\sqrt{2\,\pi}}\int_0^{\infty}\frac{e^{-(x^2/2)y-s/y}}{y^{3/2}}dy = \frac{p_1x|x|}{4p_0^2}\,l_{1/\sqrt{s}}(x)$$

and with (8.25) for $m = 0,1,2$, we calculate

$$\int_0^{\infty}\frac{f_2(x\sqrt{y})}{y}dH_s(y) = \frac{x}{4}\left\{3\,\frac{1+\sqrt{2\,s}}{2\,s} + \Big(1 + \frac{p_2}{6\,p_0^3}\Big)x^2 - \frac{p_1^2\,x^2\,|x|\,\sqrt{2\,s}}{8\,p_0^4}\right\}\,l_{1/\sqrt{s}}(x).$$

Finally, in the proof of Theorem 5 in Christoph et al. (2020), see J_4 and J_4^*, it was shown that

$$\sup_x \left| \int_0^\infty \Phi(x\sqrt{y}) dh_{2;s}(y) - \frac{(1-s)x(1+\sqrt{2s}|x|)}{8\,s} l_{1/\sqrt{s}}(x) \right| \le c(s) n^{-1/2}.$$

Let now $\gamma = 0$. Since $f_k(.)$ for $k = 1,2$ do not depend on y we get (5.12) with

$$s \int_0^\infty f_k(x)\, y^{-k/2-2} e^{-s/y} dy = f_k(x)\, s \int_0^\infty z^{k/2}\, e^{-s\,z} dz = f_k(x)\, s^{-k/2}\, \Gamma(k/2+1).$$

If $\gamma = -1/2$, we calculate the integrals with $f_1(.)$ and $f_2(.)$ in (4.9) with the second equation in (8.1) for $\alpha = 5/2, 7/2, 9/2$. The last integral in (4.9) is identical to $J_{3;s}^*(x) = J_{4;s}^*(x) + J_{5;s}^*(x)$ in the proof of Theorem 8 in Christoph and Ulyanov (2020) and the integrals were calculated there. Hence

$$n^{-1} \sup_x \left| \int_0^\infty \Phi(x/\sqrt{y}) dh_{2;s}(y) - \frac{3\,(s-1)\,x}{4\,(x^2+2s)}\, s_2^*(x;\sqrt{s}) \right| \le C(s) n^{-2}$$

and Theorem 6.7 is proved. □

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