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The Loewner difference equation and convergence of loop-erased random walk

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Abstract. We revisit the convergence of loop-erased random walk, LERW, to SLE_2 when the curves are parametrized by capacity. We construct a Markovian coupling of the driving processes and Loewner chains for the chordal version of LERW and chordal SLE_2 based on the Green's function for LERW as martingale observable and using an elementary discrete-time Loewner "difference" equation. We keep track of error terms and obtain power-law decay. This coupling is different than the ones previously considered in this context, e.g., in that each of the processes has the domain Markov property at mesoscopic capacity time increments, given the sigma algebra of the coupling. At the end of the paper we discuss in some detail a version of Skorokhod embedding. Our recent work on the convergence of LERW parametrized by length to SLE_2 parameterized by Minkowski content uses specific features of the coupling constructed here.

1. Introduction, set-up, and main results

1.1. *Introduction.* Loop-erased random walk (LERW) is the random self-avoiding path one gets after erasing the loops in the order they form from a simple random walk. In the plane, which is the only case we consider here, it was proved in Lawler et al. (2004) that LERW has a conformally invariant lattice size scaling limit, namely the Schramm-Loewner evolution with parameter 2, SLE_2. In this paper we revisit this in the case of chordal LERW, proving the result in a slightly different framework than Lawler et al. (2004). A major motivation for doing this work is that we need the theorem in this form for our proof of convergence of LERW to SLE_2 in the *natural parametrization* Lawler and Viklund (2021). That is, we prove in Lawler and Viklund (2021) that LERW (viewed

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as a continuous curve) parametrized by renormalized length converges in the lattice size scaling limit to SLE_2 parametrized by 5/4-dimensional Minkowski content Lawler and Viklund (2021). Prior to our work, all SLE convergence results consider the discrete curve reparametrized by an appropriate "capacity" so that it can be directly described by the Loewner equation (see below) and the convergence takes place in this parametrization. While this is useful for technical purposes, the capacity parametrization is not natural from the point of view of the discrete process. Indeed, useful information is lost when reparametrizing and for several applications one needs to consider the discrete curve parametrized by length. We refer to the introduction of Lawler and Viklund (2021) for further motivation and discussion. In order to further describe our results we will first discuss in more detail the work in Lawler et al. (2004); Lawler and Viklund (2021) and then elaborate on the results of this paper.

The proof in Lawler et al. (2004) (as well as other SLE convergence results) is based on a description of LERW viewed as a continuous curve in terms of Loewner's differential equation, see e.g. Lawler (2005, Chapter 4). In the case of SLE_κ the Loewner driving process is $\sqrt{\kappa}$ times a standard Brownian motion. The main step is to show that the LERW driving process converges to Brownian motion with variance parameter 2. The way this is done is by first identifying a *martingale observable*. This is a lattice function which for a fixed lattice point is approximately a martingale with respect to the LERW. One needs to be able to approximate the observable well in rough domains by some continuum quantity with conformal symmetries. In Lawler et al. (2004) a discrete Poisson kernel was used as observable, converging in the scaling limit to a conformally invariant version of the usual Poisson kernel. The martingale property translates via the Loewner equation to an approximate martingale property of the Loewner process. The argument produces an estimate on the variance of the increments and from this information one can couple with Brownian motion using Skorokhod embedding.

Our proof here follows the same basic idea but is based on a different observable: the LERW Green's function, that is, the probability that the LERW passes through a given vertex inside the domain. (Since LERW is a self-avoiding walk this probability is also equal to the expected number of visits to the vertex, hence the terminology.) By the domain Markov property the Green's function evaluated at a fixed vertex is a LERW martingale. The approximation result, which is also important for Lawler and Viklund (2021), was proved in Beneš et al. (2016). More precisely, that paper proves that the LERW Green's function properly renormalized converges with a power-law convergence rate in the scaling limit to the SLE_2 Green's function, which is conformally covariant and explicitly known. The theorem does not need assumptions on boundary regularity. Recall that the SLE_κ Green's function is the limit as $\varepsilon \to 0$ of the renormalized probability that an SLE_κ curve gets within distance ε of a given point inside the domain. The observable used in Lawler et al. (2004) is specific to LERW but the Green's function is not. Many of the estimates given here apply to other models as well, assuming one has established convergence to the appropriate SLE_κ Green's function with sufficient control of error terms. However, such a convergence result is presently known only for LERW.

LERW is a random self-avoiding walk on a lattice (we use $\mathbb{Z}^2$) and as such can be viewed either as a continuous curve traced edge by edge or as a sequence of Jordan domains obtained by removing the faces touched (and disconnected from the target point) when walking along the LERW. These viewpoints are of course essentially equivalent but other considerations may make one more convenient than the other. For example, Jordan domains can be easier to work with analytically. In this paper we adopt the second point of view. We exploit a fundamental robustness of Loewner's equation: the analysis is based on a difference version of Loewner's equation which uses only mesoscopic scale information about the growth process. The difference equation does not require the conformal maps to come from a curve, only that the sequence of maps is generated by composing maps corresponding to small hulls of controlled diameter and capacity. There is still a discrete "Loewner process" representing the growth on a mesoscopic scale, up to a uniform multiplicative error. (But this process does not uniquely determine the evolution.) The resulting argument is in a sense more elementary. We explain how to compare solutions to the difference equation corresponding to nearby Loewner processes, and write down formulas for some of the usual important processes such as the derivative and conformal radius.

Given the results we obtain here and some additional but not difficult regularity estimates for LERW it is not too hard to derive convergence of the LERW path to SLE_2 as curves in the half-plane capacity parametrization, see for instance Lawler et al. (2004). We have chosen to not discuss this here, and instead focus on the more novel parts of the argument. We have tried to provide a good amount of detail and to make the paper self-contained with the hope that it will be read not only by experts but also as an introduction to these techniques.

1.2. *Discrete quantities.* We now discuss the discrete quantities we will use. We want the setup to match that of Lawler and Viklund (2021), so in this section there will necessarily be some overlap in the presentation.

- Let A be a finite subset of $\mathbb{Z}^2$, and write $\partial_e A$ for the edge boundary of A, that is, the set of edges of $\mathbb{Z}^2$ with exactly one endpoint in A. We specify elements of $\partial_e A$ by a, the midpoint of the edge; this is unique up to the orientation. Given an edge $a \in \partial_e A$, we write a_-, a_+ for the two vertices connected by a with the convention that $a_- \in \mathbb{Z}^2 \setminus A$ and $a_+ \in A$. Note that
$$a_-, b_- \in \partial A := \{z \in \mathbb{Z}^2 \setminus A : \mathrm{dist}(z, A) = 1\},$$
$$a_+, b_+ \in \partial_i A := \{z \in A : \mathrm{dist}(z, \partial A) = 1\}.$$
We also write $e_a = [a_-, a_+], e_b = [b_-, b_+]$ for the edges oriented from the outside to the inside.
- Let $\mathcal{A}$ denote the set of triples (A, a, b) where A is a finite, simply connected subset of $\mathbb{Z}^2$ containing the origin, and a, b are elements of $\partial_e A$ with $a_- \neq b_-$. We allow $a_+ = b_+$. Sometimes we slightly abuse notation and write $A \in \mathcal{A}$ when A is a simply connected subset of $\mathbb{Z}^2$ containing the origin.
- let $\mathcal{S} = \{x + iy \in \mathbb{C} : |x|, |y| \leqslant 1/2\}$ be the closed square of side length one centered at the origin and $\mathcal{S}_z = z + \mathcal{S}$. If $(A, a, b) \in \mathcal{A}$, let D_A be the corresponding simply connected domain defined as the interior of
$$\bigcup_{z \in A} \mathcal{S}_z.$$
This is a simply connected Jordan domain whose boundary is a subset of the edge set of the dual graph of $\mathbb{Z}^2$. Note that $a, b \in \partial D_A$. We refer to D_A as a "union of squares" domain, slightly abusing terminology.
- Let $F = F_{A,a,b}$ denote a conformal map from D_A onto $\mathbb{H}$ with $F(a) = 0, F(b) = \infty$. This map is defined only up to a dilation, but in our arguments we always fix one particular choice. Note that F and F^{-1} extend continuously to the boundary of the domain (with the appropriate definition of continuity at infinity).
- For $z \in D_A$, we define the important conformal invariants
$$\theta_{A,a,b}(z) = \arg F(z), \quad S_{A,a,b}(z) = \sin \theta_{A,a,b}(z),$$
which are independent of the choice of F, since F is unique up to scaling. Also for $z \in \mathbb{H}$, we write
$$S(z) = \sin[\arg(z)].$$
Note that $(\arg z)/\pi$ is the harmonic measure in $\mathbb{H}$ of the negative real line and $\sin[\arg z]$ is comparable to the minimum of the harmonic measures of the positive and negative real lines.

- We write $r_A(z) = r_{D_A}(z)$ for the conformal radius of D_A with respect to z. This is usually defined for any simply connected domain D as $r_D(z) = \varphi'(z)^{-1}$ where $\varphi : D \to \mathbb{D}$ is the Riemann map with $\varphi(z) = 0, \varphi'(z) > 0$. We can also compute it from F by
$$r_A(z) = 2\,\frac{\operatorname{Im} F(z)}{|F'(z)|},$$
which is independent of the choice of F.
- Let $(A, a, b) \in \mathcal{A}$. If a conformal transformation $F : D_A \to \mathbb{H}, F(a) = 0, F(b) = \infty$ as above has been fixed we can consider half-plane capacity with respect to F as follows. Let $K \subset D_A$ be a half-plane hull, that is, a relatively closed set such that $D_A \setminus K$ is simply connected. The half-plane capacity of K (with respect to F) is defined by the usual half-plane capacity of $F(K)$ in $\mathbb{H}$, see Section 2. It is also convenient to define $R = R_{A,a,b,F} = 4|(F^{-1})'(2i)|$ which is the conformal radius of D_A seen from $F^{-1}(2i)$.
- We will state our main convergence result in a fixed domain. For simplicity we will make a rather strong assumption about its boundary regularity. The coupling results about Loewner chains do not use this assumption. Suppose D is an analytic simply connected domain containing 0 as an interior point. Let $N > 1$. We sometimes want to consider a lattice approximation of D with mesh N^{-1}, and we define it as follows. We take $A = A(N, D) \in \mathcal{A}$ to be the largest discrete simply connected set such that $D_A \subset N \cdot D$. We write
$$\check{D} = N^{-1} D_A$$
for the scaled domain. Then $\check{D}$ is a simply connected Jordan domain which approximates D from the inside and converges to D in the Carathéodory sense (with respect to 0) as $N \to \infty$. If $a, b \in \partial_e A$ are given, we write $\check{a}, \check{b} \in \partial\check{D}$ for $N^{-1}a, N^{-1}b$, respectively. If $a', b' \in \partial D$ are given, we typically choose $a, b \in \partial_e A$ among the edges closest to $N \cdot a, N \cdot b$, respectively.
- A *walk* $\omega = [\omega_0, \dots, \omega_n]$ is a sequence of nearest neighbors in $\mathbb{Z}^2$. The length $|\omega| = n$ is by definition the number of traversed edges.
- If $A \in \mathcal{A}$ and $z, w \in A$, we write $\mathcal{K}_A(z, w)$ for the set of walks ω starting at z, ending at w, and otherwise staying in A.
- The simple random walk measure p assigns to each walk measure $p(\omega) = 4^{-|\omega|}$. The two-variable function
$$G_A(z, w) := p\left(\mathcal{K}_A(z, w)\right)$$
is the simple random walk *Green's function*.
- If $a, b \in \partial_e A$, there is an obvious bijection between $\mathcal{K}_A(a_+, b_+)$ and $\mathcal{K}_A(a, b)$, the set of walks starting with edge e_a, ending with e_b^R and otherwise staying in A. Here we write ω^R for the reversal of the path ω, that is, if $\omega = [\omega_0, \omega_1, \dots, \omega_k]$, then $\omega^R = [\omega_k, \omega_{k-1}, \dots, \omega_0]$. We sometimes write $\omega : a \to b$ for walks in $\mathcal{K}_A(a, b)$ with the condition to stay in A implicit.
- We write $H_{\partial A}(a, b)$ for the total random walk measure of $\mathcal{K}_A(a, b)$. It is easy to see that $H_{\partial A}(a, b) = G_A(a_+, b_+)/16$ (this is sometimes called a last-exit decomposition). The factor of $1/16 = (1/4)^2$ comes from the p-measure of the edges e_a, e_b. $H_{\partial A}(a, b)$ is called the *boundary Poisson kernel*.
- A *self-avoiding walk* (SAW) is a walk visiting each point at most once. We write $\mathcal{W}_A(z, w) \subset \mathcal{K}_A(z, w)$ for the set of SAWs from z to w staying in A. We will write ω for general walks and reserve η for SAWs. We write $\mathcal{W}_A(a, b)$ similarly when a, b are boundary edges.
- The loop-erasing procedure takes a walk ω as input and outputs a SAW $\eta = \mathrm{LE}[\omega]$, the *loop-erasure* of ω. Given a walk $\omega = [\omega_0, \dots, \omega_n]$, we define $\mathrm{LE}[\omega] = [\mathrm{LE}[\omega]_0, \dots, \mathrm{LE}[\omega]_k]$ as follows.
  - If ω is self-avoiding, set $\mathrm{LE}[\omega] = \omega$.
  - Otherwise, define $s_0 = \max\{j \leqslant n : \omega_j = \omega_0\}$ and let $\mathrm{LE}[\omega]_0 = \omega_{s_0}$.

– For $i \geqslant 0$, if $s_i < n$, define $s_{i+1} = \max\{j \leqslant n : \omega_j = \omega_{s_i+1}\}$ and set $\mathrm{LE}[\omega]_{i+1} = \omega_{s_i+1} = \omega_{s_{i+1}}$.

Note that if $e_a \oplus \omega \oplus e_b^R \in \mathcal{K}_A(a,b)$, then $\mathrm{LE}[e_a \oplus \omega \oplus e_b^R] = e_a \oplus \mathrm{LE}[\omega] \oplus e_b^R$.

- Given a measure on walks, the loop-erasing procedure induces a natural measure on SAWs. We define $\hat{P}_{A,a,b}$, the "loop-erased" p-measure, on $\mathcal{W}_A(a,b)$ by

$$\hat{P}_{A,a,b}(\eta) = \sum_{\omega \in \mathcal{K}_A(a,b):\, \mathrm{LE}(\omega)=\eta} p(\omega).$$

This can also be written

$$\hat{P}_{A,a,b}(\eta) = p(\eta)\Lambda_A(\eta), \tag{1.1}$$

where $m(\eta; A) = \log \Lambda_A(\eta)$ is the loop-measure (using p) of loops intersecting η and staying in A, see, e.g., Beneš et al. (2016, Section 2). This does not define a probability measure; indeed the total mass $\hat{P}_{A,a,b}[\mathcal{W}_A(z,w)] = H_{\partial A}(a,b)$. Let

$$\mathbf{P}_{A,a,b} = \frac{\hat{P}_{A,a,b}}{H_{\partial A}(a,b)}$$

denote the probability measure obtained by normalization. This is the probability law of (chordal) *loop-erased random walk* (LERW) in A from a to b.

With these definitions in place, we can state the main result from Beneš et al. (2016), which we will make significant use of in this paper. We emphasize that no assumptions about the discrete domain A are made.

Theorem 1.1. There exists $\hat{c} > 0$ and $u > 0$ such that the following holds. Suppose $(A,a,b) \in \mathcal{A}$ and that $\zeta \in A$ is such that $S_{A,a,b}(\zeta) \geqslant r_A(\zeta)^{-u}$, then

$$\mathbf{P}_{A,a,b}\{\zeta \in \eta\} = \hat{c}\, r_A(\zeta)^{-3/4} S^3_{A,a,b}(\zeta) \left[1 + O\left(r_A(\zeta)^{-u} S^{-1}_{A,a,b}(\zeta)\right)\right]. \tag{1.2}$$

We have not estimated u except $u > 0$. For the rest of the paper we fix a value of u such that (1.2) holds and we may assume that $u < 1$. We can also write (1.2) using the SLE_2 Green's function for (D_A, a, b) which is further discussed in Section 1.3. Let

$$G_{D_A}(\zeta; a, b) = \tilde{c}\, r_A(\zeta)^{-3/4}\, S^3_{A,a,b}(\zeta),$$

for a specific (but unknown) constant $\tilde{c} > 0$ that will be defined later. We may rewrite (1.2) as

$$\mathbf{P}_{A,a,b}\{\zeta \in \eta\} = c_*\, G_{D_A}(\zeta; a, b) \left[1 + O\left(r_A(\zeta)^{-u}\right)\ S^{-1}_{A,a,b}(\zeta)\right], \tag{1.3}$$

where $c_* = \hat{c}/\tilde{c}$ is a positive constant whose exact value is presently unknown.

1.3. *Schramm-Loewner evolution.* Recall that chordal SLE_κ in $\mathbb{H}$ is a random continuous curve $\gamma(t), t \geqslant 0$, constructed by first solving the Loewner differential equation

$$\partial_t g_t(z) = \frac{2/\kappa}{g_t(z) - B_t}, \quad g_0(z) = z \in \mathbb{H}.$$

Here B_t is standard Brownian motion. (Note that we are parametrizing time so that the SLE_κ hull at time t has half-plane capacity $2t/\kappa$, which is slightly different but equivalent to the usual way to parametrize the equation.) We shall only consider $0 < \kappa < 8$ in this paper, and primarily $\kappa = 2$. The conformal maps $g_t(z)$ can be expanded at infinity as

$$g_t(z) = z + \frac{(2/\kappa)t}{z} + O(|z|^{-2}).$$

Then for each $t \geqslant 0$ we define the SLE_κ curve and trace by

$$\gamma(t) = \lim_{y \to 0+} g_t^{-1}(U_t + iy), \quad \gamma_t := \gamma[0,t].$$

This limit is known to almost surely exist for each t and to define a continuous curve $t \mapsto \gamma(t)$ in $\mathbb{H}$ growing from 0 to ∞. This defines SLE_κ in the reference domain $\mathbb{H}$ with marked boundary points $0, \infty$ and we extend the definition to any simply connected domain D with two marked boundary points (more precisely prime ends) a, b (we write (D, a, b) for such a triple) by transferring the curve by a Riemann map taking $\mathbb{H}$ to D, 0 to a, and ∞ to b. Using Brownian scaling, one can see that this is well defined if one allows for a linear time reparametrization.

The (Euclidean) Green's function for SLE_κ in a domain (D, a, b) is defined by

$$G_D(z, a, b) = \lim_{\varepsilon \to 0} \varepsilon^{d-2} \mathbf{P} \{ \mathrm{dist}(z, \gamma_\infty) \leqslant \varepsilon \} = \tilde{c}\, r_D^{d-2}(z) S^\beta_{D,a,b}(z),$$

where γ is chordal SLE_κ in D from a to b,

$$d = 1 + \frac{\kappa}{8}, \qquad \beta = \frac{8}{\kappa} - 1$$

is the dimension of the SLE_κ trace, and the SLE_κ boundary exponent, respectively, and $\tilde{c} \in (0, \infty)$ is a constant whose exact value is not known. Here r_D and $S_{D,a,b}$ are defined in the same manner as for the union of squares domains D_A discussed in the previous subsection. In one replaces distance by conformal radius in the probability, the limit also exists and is the same but with a (different) constant that is computable.

1.4. *Overview and main results.* In Section 2 we introduce the Loewner difference equation in both forward and reverse settings, with related quantities, and derive the needed estimates on the derivative of the conformal maps. In Section 2.3 we compute how the SLE Green's function changes when growing a hull of small capacity. Section 3 contains the main results which gives the coupling of the LERW and SLE Loewner chains, see in particular Theorem 3.6 and Proposition 3.8. The appendix discusses a Markovian version of Skorokhod embedding that we use in Section 3.

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2. Discrete and continuous time Loewner chains

2.1. *Forward-time Loewner chain.* The Loewner differential equation is a continuous limit of Loewner difference estimates. The difference estimates hold for sets more general than curves, and since we are dealing with "union of squares" domains, we will use the difference formulation. Here we will review the basics from Lawler (2005, Section 3.4) and then we will give some extensions. It is important for us to be careful with the error terms.

We recall that a set $K \subset \mathbb{H}$ is a *(compact $\mathbb{H}$-) hull*, or half-plane hull, if K is relatively closed and $H_K := \mathbb{H} \setminus K$ is a simply connected domain. Let $h_K = \mathrm{hcap}(K)$ be the (half-plane) capacity which can be defined in two equivalent ways:

- If B_t is a complex Brownian motion and $\tau = \inf\{t : B_t \in \mathbb{R} \cup K\}$, then
$$h_K = \lim_{y \to \infty} y\, \mathbf{E}^{iy} \left[\mathrm{Im}\left[B_\tau \right] \right].$$
- If $g_K : H_K \to \mathbb{H}$ is the unique conformal transformation (the Loewner map) with $g_K(z) = z + o_K(1)$ as $z \to \infty$, then
$$g_K(z) = z + \frac{h_K}{z} + O_K(|z|^{-2}), \quad z \to \infty.$$

We write the error terms here as o_K, O_K to emphasize that they depend on K; the error terms without subscript that we write below will be uniform over all K. Note that $h_K \leqslant r_K^2$ where

$$r_K = \mathrm{diam}(K).$$

(We are slightly abusing notation here; we are also writing r_A, r_D for conformal radius. Which is meant will be clear from context and that we use K only to denote half-plane hulls.) Let

$$\Upsilon_K(z) = \frac{\mathrm{Im}\,[g_K(z)]}{|g_K'(z)|},$$

and recall that $2\Upsilon_K(z)$ is the conformal radius of H_K seen from z. The following estimate is at the foundation of the Loewner theory, see Lawler (2005, Proposition 3.46) and also Section 2.2: If $0 \in \overline{K}$ and $|z| \geqslant 2r_K$ then

$$g_K(z) = z + \frac{h_K}{z} + O\left(\frac{r_K\, h_K}{|z|^2}\right). \tag{2.1}$$

Note that the error term depends only on $r_K, h_K, |z|$ and not on the exact shape of K. Note that if $U \in \mathbb{R}$ then $g_{K+U}(z) = g_K(z-U) + U$, where $K + U := \{z : z - U \in K\}$. By applying the Cauchy integral formula to $f_K(z) = g_K(z) - z - h_K/z$, we see that

$$g_K'(z) = 1 - \frac{h_K}{z^2} + O\left(\frac{r_K\, h_K}{|z|^3}\right), \quad |z| \geqslant 2\, r_K. \tag{2.2}$$

This is the starting point for the next lemma.

Lemma 2.1. *There exists $c < \infty$ such that the following holds. Suppose $U \in \mathbb{R}$; K is a hull with $0 \in \overline{K}$ and $r_K < 1/2$; $z = x + iy$; and let g, r, h, Υ denote*

$$g_{K+U},\ r_K,\ h_K = h_{K+U},\ \text{and } \Upsilon_{K+U},$$

respectively. Then $\mathrm{Im}\,[g(z)] \leqslant y$ and $\Upsilon(z) \leqslant y$. Moreover, if

$$\delta = r^{1/4}, \quad h \leqslant \delta r, \quad \delta \leqslant y,$$

then

$$\begin{aligned} \left| g(z) - z - \frac{h}{z-U} \right| &\leqslant ch\delta^2, \\ \left| g'(z) - 1 + \frac{h}{(z-U)^2} \right| &\leqslant ch\delta, \\ \left| \mathrm{Im}\,[g(z)] - y\, \left[1 - \frac{h}{|z-U|^2}\right] \right| &\leqslant cyh\delta, \\ \left| \Upsilon(z) - y \left[1 - \frac{2h \sin^2 \arg(z-U)}{|z-U|^2}\right] \right| &\leqslant cyh\delta, \end{aligned} \tag{2.3}$$

In particular, if $\sin \arg(z-U) \geqslant \nu$, then

$$\frac{\Upsilon(z)}{y} \leqslant \left(\frac{\mathrm{Im}\,(g(z))}{y}\right)^{2\nu^2} [1 + O(h\delta)]\,. \tag{2.4}$$

Proof: Since $g_{K+U}(z) = g_K(z-U) + U$, it suffices to prove the result when $U = 0$ which we will assume from now on. The first two inequalities follow immediately from (2.1) and (2.2), respectively. Write $\theta = \arg z \in [-\pi, \pi]$. Taking imaginary parts in the first inequality and using $|z| \geqslant \mathrm{Im}\,(y) \geqslant \delta$, we get

$$\begin{aligned} \mathrm{Im}\,[g(z)] &= y\, \left[1 - \frac{h}{|z|^2}\right] + O\left(h\delta^2\right) \\ &= y \left[1 - \frac{h\,(\cos^2\theta + \sin^2\theta)}{|z|^2}\right] + O\left(h\delta^2\right) \end{aligned}$$

and since $y \geqslant \delta$ we get the third inequality. Since

$$\left|1 - \frac{h}{z^2}\right| = 1 - \mathrm{Re}\left[\frac{h}{z^2}\right] + O\left(\frac{h^2}{|z|^4}\right) = 1 + \frac{h\,(\sin^2\theta - \cos^2\theta)}{|z|^2} + O\left(\frac{h^2}{|z|^4}\right),$$

and $h/|z| \leqslant r$, we get

$$|g_K'(z)|^{-1} = 1 + \frac{h\,(\cos^2\theta - \sin^2\theta)}{|z|^2} + O\left(\frac{hr}{|z|^3}\right),$$

Combining, we get

$$\Upsilon_K(z) = y\left[1 - \frac{2h\,\sin^2\theta}{|z|^2} + O\left(\frac{h\delta^2}{y}\right)\right].$$

□

Suppose now we have a sequence of hulls $K_1, K_2, \ldots$ each of small diameter and such that $0 \in \overline{K}_j$ and locations $U_1, U_2, \ldots \in \mathbb{R}$ determining a “Loewner process”, so that, roughly speaking $K_j + U_j$ is near U_j. Let

$$r_j = r_{K_j}, \quad h_j = h_{K_j}, \quad g^j = g_{K_j + U_j}$$

and let

$$g_j = g^j \circ \cdots \circ g^1.$$

If $z \in \mathbb{H}$, we define

$$z_j = x_j + iy_j = g_j(z).$$

This is defined up to the first j such that $z_j - U_j \in K_j$. (Recall that K_j is located near 0.) If we have two sequences for which the capacity increments and Loewner processes, h_j and U_j, are close, then from the basic Loewner estimate (2.1) we would expect the corresponding functions g_n to be close for points which are away from the real line. We give a precise formulation of this in the next proposition. To illustrate the idea of the proof, let us sketch a continuous-time argument first. Suppose $U_t, \tilde{U}_t$ are continuous, real-valued function, defined on $[0, T]$, with $T < \infty$ fixed, and write $\varepsilon := \sup_{t\in[0,T]} |U_t - \tilde{U}_t|$. Write $g_t, \tilde{g}_t$ for the corresponding Loewner chains (run at speed 1) and $z_t = g_t(z) - U_t$ and $\tilde{z}_t = \tilde{g}_t(z) - \tilde{U}_t$. Suppose we know that $\delta > 0$ is such that $\delta \leqslant \min\{\mathrm{Im}\ z_T, \mathrm{Im}\ \tilde{z}_T\}$. If $G_t = g_t(z) - \tilde{g}_t(z)$, then

$$\dot{G}_t = \psi_t[-G_t + (U_t - \tilde{U}_t)], \quad G_0 = 0, \quad \text{where } \psi_t = \frac{1}{z_t \tilde{z}_t}.$$

By solving the ODE and using the definition of ε we have

$$|G_t| = \left|\int_0^t e^{-\int_s^t \psi_r dr} \psi_s (U_s - \tilde{U}_s) ds\right| \leqslant \varepsilon \int_0^t e^{\int_s^t |\psi_r| dr} |\psi_s| ds.$$

From here we integrate and then proceed by applying Cauchy-Schwarz’ inequality: if $y = \mathrm{Im}\ z$, then

$$\left(\int_0^t |\psi_r| dr\right)^2 \leqslant \int_0^t \frac{1}{|z_r|^2} dr \int_0^t \frac{1}{|\tilde{z}_r|^2} dr = \log\frac{\mathrm{Im}\ z}{\mathrm{Im}\ z_t} \log\frac{\mathrm{Im}\ z}{\mathrm{Im}\ \tilde{z}_t} \leqslant \left(\log\frac{y}{\delta}\right)^2.$$

The identity comes from taking the imaginary part of the Loewner equation and the last estimate uses the definition of δ. Hence we get the estimate

$$|g_t(z) - \tilde{g}_t(z)| = |G_t| \leqslant c\,(\varepsilon/\delta)\,(y \wedge 1),$$

where c depends only on T. It is possible to estimate in terms of other norms relating U_t and $\tilde{U}_t$ and, as we will see, continuity is not necessary to assume.

Proposition 2.1. *There exists $1 < c < \infty$ such that the following holds. Suppose (K_1, U_1), $(K_2, U_2) \ldots$ and $(\tilde{K}_1, \tilde{U}_1), (\tilde{K}_2, \tilde{U}_2), \ldots$ are two sequences as above with corresponding r_j, h_j, g^j, g_j and $\tilde{r}_j, \tilde{h}_j, \tilde{g}^j, \tilde{g}_j$. Let*

$$0 < h < r^2 < \varepsilon^2 < \delta^8 < 1/c,$$

and $n \leqslant 1/h$ and suppose that for all $j = 1, \ldots, n$,

$$|h_j - h| \leqslant hr/\delta, \quad |\tilde{h}_j - h| \leqslant hr/\delta,$$

$$r_j, \tilde{r}_j \leqslant r,$$

$$|U_j - \tilde{U}_j| \leqslant \varepsilon.$$

Suppose $z = x + iy \in \mathbb{H}$ and let $z_n = x_n + iy_n = g_n(z), \tilde{z}_n = \tilde{x}_n + i\tilde{y}_n = \tilde{g}_n(z)$. Then, if $y_n, \tilde{y}_n \geqslant \delta$,

$$|g_n(z) - \tilde{g}_n(z)| \leqslant c\,(\varepsilon/\delta)\,(y \wedge 1). \tag{2.5}$$

Moreover, if we assume that $y_n \geqslant 2\delta$ and make no a priori assumptions on $\tilde{y}_n$, then $\tilde{y}_n \geqslant \delta$ holds, and hence (2.5) follows in this case, too.

Proof: Without loss of generality, we will assume that $y \leqslant 3$; for $y \geqslant 3$, we can use the fact that $g_n - \tilde{g}_n$ is a bounded holomorphic function on $\{\mathrm{Im}\,(w) > 3\}$ that goes to zero as $w \to \infty$, and hence

$$|g_n(z) - \tilde{g}_n(z)| \leqslant \max\{|g_n(s + 3i) - \tilde{g}_n(s + 3i)| : s \in \mathbb{R}\}.$$

Using Lemma 2.1, and that $r < \delta^4$, we see that for $j = 0, \ldots, n - 1$,

$$z_{j+1} = z_j + \frac{h}{z_j - U_j} + O\left(h\delta^2\right), \tag{2.6}$$

$$y_{j+1} = y_j \left[1 - \frac{h}{|z_j - U_j|^2} + O\left(h\delta\right)\right],$$

and similarly for $\tilde{z}_j, \tilde{y}_j$.

Hence

$$y_n = y \prod_{j=0}^{n-1} \left[1 - \frac{h}{|z_j - U_j|^2} + O\left(h\delta\right)\right] = y\,[1 + O(\delta)]\,\exp\left\{-\sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|^2}\right\}.$$

Since $y_n \geqslant \delta$ and $y \leqslant 3$, it follows that

$$\sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|^2} \leqslant \log(y/\delta) + O(\delta), \tag{2.7}$$

and similarly for $(\tilde{z}_j, \tilde{U}_j)$. Using the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|\,|\tilde{z}_j - \tilde{U}_j|} &\leqslant \left[\sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|^2}\right]^{1/2} \left[\sum_{j=0}^{n-1} \frac{h}{|\tilde{z}_j - \tilde{U}_j|^2}\right]^{1/2} \\ &\leqslant \log(y/\delta) + O(\delta).\end{aligned} \tag{2.8}$$

Let $\Delta_j = z_j - \tilde{z}_j$. Let us first assume that $|\Delta_j| \leqslant \delta/2$. By subtracting the expressions in (2.6) for z_j and $\tilde{z}_j$, we see that

$$\Delta_{j+1} = \Delta_j + \frac{h\,(U_j - \tilde{U}_j - \Delta_j)}{(z_j - U_j)\,(\tilde{z}_j - \tilde{U}_j)} + O\left(h\delta^2\right).$$

This implies that there exists c such that

$$|\Delta_{j+1}| \leqslant |\Delta_j|\,[1 + \rho_j] + c\,\varepsilon\,\rho_j,$$

where

$$\rho_j = \frac{h}{|z_j - U_j|\,|\tilde{z}_j - \tilde{U}_j|}.$$

It follows that if $Y_j := |\Delta_j| + c\varepsilon$ (with the same c), then

$$Y_{j+1} \leqslant Y_j(1+\rho_j).$$

Hence, from (2.8),

$$|\Delta_n| \leqslant c\varepsilon \prod_{j=0}^{n-1}(1+\rho_j) \leqslant c'\varepsilon\frac{y}{\delta}$$

provided that the right-hand side is less than $\delta/2$. Since $y \leqslant 3$ and $\varepsilon \leqslant \delta^4$, this will be true if δ is sufficiently small.

For the final assertion, suppose that j is such that $\tilde{y}_j \geqslant \delta$. Then since $\varepsilon \leqslant \delta^4$, we can use (2.5) to see that $|y_j - \tilde{y}_j| \leqslant c(\varepsilon/\delta)y \leqslant O(\delta^4)$. Since $y_j \geqslant 2\delta$, it follows that $\tilde{y}_j \geqslant 2\delta(1-O(\delta^3))$. But $|\tilde{y}_{j+1} - \tilde{y}_j| \leqslant c'h_j/y_j \leqslant O(\delta^7)$. Consequently, as long as δ is sufficiently small, taking c larger if necessary, we can continue until $j = n$.

□

Corollary 2.2. *Suppose we make the assumptions of the previous proposition, but replace the condition $y_n \geqslant 2\delta$ with*

$$\Upsilon_n(z), \tilde{\Upsilon}_n(z) \geqslant 2(2\delta)^{2\nu^2},$$

where

$$\nu = \min_{0 \leqslant j \leqslant n} \{\sin[\arg(g_j(z) - U_j)]\}.$$

Then the results still hold for δ sufficiently small.

Proof: Using (2.4), there is a constant c such that for δ sufficiently small

$$\Upsilon_n(z), \tilde{\Upsilon}_n(z) \leqslant c y_n^{2\nu^2}.$$

□

The next proposition, which is important for Lawler and Viklund (2021), gives a familiar representation of the derivative of the uniformizing map and a related geometric estimate.

Proposition 2.3. *There exists $1 < c < \infty$ such that the following holds. Suppose (K_1, U_1), $(K_2, U_2)\ldots$ is a sequence as above with corresponding r_j, h_j, g^j, g_j. Let*

$$0 < h < r^2 < \delta^8 < 1/c,$$

and $n \leqslant 1/h$ and suppose that for all $j = 1, \ldots, n$,

$$|h_j - h| \leqslant hr/\delta, \quad r_j \leqslant r.$$

Suppose $z = x + iy \in \mathbb{H}$ and let $z_n = x_n + iy_n = g_n(z)$. Then if $y_n \geqslant \delta$,

$$|g_n'(z)| = \exp\left\{-\sum_{j=0}^{n-1} \operatorname{Re}\frac{h}{(z_j - U_j)^2}\right\}(1 + O(\delta)). \tag{2.9}$$

In particular, there is a constant c such that if

$$\nu = \min_{0 \leqslant j \leqslant n} \{\sin[\arg(g_j(z) - U_j)]\}, \tag{2.10}$$

then,

$$|g_n'(z)| \geqslant c\left(\frac{y_n}{y}\right)^{1-2\nu^2}. \tag{2.11}$$

Proof: By the chain rule and Lemma 2.1 we have

$$\begin{aligned}\log|g_n'(z)| &= \sum_{j=1}^{n}\log|(g^j)'(z_{j-1})| \\ &= \sum_{j=0}^{n-1}\log\left|1-\frac{h}{(z_j-U_j)^2}+O(h\delta)\right| \\ &= -\sum_{j=0}^{n-1}\left(\text{Re }\frac{h}{(z_j-U_j)^2}+O(h\delta)\right).\end{aligned}$$

This proves the first claim. For the second assertion, note that (2.10) implies

$$-\text{Re }\frac{h}{(z_j-U_j)^2} = -\left(1-2S_j^2\right)\frac{h}{|z_j-U_j|^2} \geqslant -\left(1-2\nu^2\right)\frac{h}{|z_j-U_j|^2},$$

where

$$S_j = \sin\left[\arg(g_j(z)-U_j\right].$$

But in the proof of Proposition 2.1 we saw that

$$\exp\left\{-\sum_{j=0}^{n}\frac{h}{|z_j-U_j|^2}\right\} = (y_n/y)\left(1+O(\delta)\right).$$

Combining these estimates finishes the proof. □

2.2. *Reverse-time Loewner chain.* In this section we consider a reverse-time version of the discrete Loewner chain. The estimates are analogous to the forward-time case discussed above and indeed could be concluded almost directly from them.

Let K be a half-plane hull with $r_K < 1/2$. We associate with K a conformal map,

$$f_K : \mathbb{H} \to H_K, \quad f_K(z) = z - \frac{h_K}{z} + o(|z|^{-1}),$$

and of course, $f_K = g_K^{-1}$. Consider the symmetrized hull $K^R = \overline{K} \cup \{z : \text{Re } z - i\,\text{Im } z \in K\}$. There is a minimal interval

$$I_K = [x_-, x_+]$$

such that f_K extends by Schwarz reflection to a conformal bijection $f_K^R : \mathbb{C} \smallsetminus I_K \to H_K^R$, where $H_K^R = \mathbb{C} \smallsetminus K^R$. If $0 \in \overline{K}$ then $I_K \subset [-2r_K, 2r_K]$. The basic *reverse-time* Loewner estimate can be given as follows.

Lemma 2.2. *Suppose K is a half-plane hull with $0 \in \overline{K}$. If $|\text{Im } z| \geqslant 4r_K$, then*

$$f_K(z) = z - \frac{h_K}{z} + O\left(\frac{r_K h_K}{|z|^2}\right).$$

Proof: Let $v(z) = \text{Im}\,(f_K(z) - z)$. Then $v(z)$ is a positive and bounded harmonic function on $\mathbb{H}$ such that $v(z) \to 0$ as $z \to \infty$. We can use the Poisson kernel to write

$$v(z) = \frac{1}{\pi}\int_{I_K} v(\xi)\,\text{Im }\frac{-1}{z-\xi}d\xi, \quad z \in \mathbb{H},$$

and since $h_K = \lim_{y\to+\infty} yv(iy)$, by dominated convergence,

$$\frac{1}{\pi}\int_{I_K} v(\xi)d\xi = h_K.$$

Note that if $\xi \in I_K$ and $|z| \geqslant 3r_K$, then

$$\mathrm{Im}\ \frac{-1}{z-\xi} = \mathrm{Im}\ \frac{-1}{z} \cdot \left(1 + O\left(\frac{r_K}{|z|}\right)\right).$$

Hence if $|z| \geqslant 3r_K$,

$$|v(z) - h_K\, \mathrm{Im}\ \frac{-1}{z}| \leqslant c\frac{r_K h_K\, \mathrm{Im}\ z}{|z|^3},$$

with a universal constant. Consequently, if $|z| \geqslant 4r_K$, then using the Poisson integral,

$$|\partial_x(v(z) + h_K\, \mathrm{Im}\ \frac{1}{z})| + |\partial_y(v(z) + h_K\, \mathrm{Im}\ \frac{1}{z})| \leqslant c\frac{r_K h_K}{|z|^3}.$$

Since $v(z)$ tends to 0 at ∞ and is the imaginary part of a holomorphic function whose derivative is controlled by the partial derivatives of v, we can integrate along $z + i\mathbb{R}_+$ to conclude that there is a universal constant c such that if $|\,\mathrm{Im}\ z| \geqslant 4r_K$, then

$$|f_K(z) - z + \frac{h_K}{z}| \leqslant c\frac{r_K h_K}{|z|^2},$$

which is what we wanted to prove. □

The next lemma follows from this estimate as in the previous section after noting that $f_{K+U}(z) = f_K(z-U) + U$.

Lemma 2.3. *There exists $c < \infty$ such that the following holds. Suppose $U \in \mathbb{R}$, K is a hull such that $0 \in \overline{K}$ and $r_K < 1/2$, $z = x + iy$, and write f, r, h for $f_{K+U}, r_K, h_K = h_{K+U}$ respectively. Then if $\delta = r^{1/4}$ and $y \geqslant \delta$,*

$$\left|f(z) - z + \frac{h}{z-U}\right| \leqslant ch\delta^2,$$

$$\left|f'(z) - 1 - \frac{h}{(z-U)^2}\right| \leqslant ch\delta,$$

$$\left|\mathrm{Im}\ [f(z)] - y\left[1 + \frac{h}{|z-U|^2}\right]\right| \leqslant cyh\delta, \tag{2.12}$$

We will consider sequences (K_j, U_j), where we center the hulls by requiring $0 \in \overline{K}_j$ as above and $U_j \in \mathbb{R}$ are the locations of the hulls. Let

$$r_j = r_{K_j}, \quad h_j = h_{K_j}, \quad f^j = f_{K_j + U_j}.$$

We assume $r_j < 1/2$. Also let

$$f_j = f^1 \circ \cdots \circ f^j.$$

If $z \in \mathbb{H}$, we define

$$z_j = x_j + iy_j = f_j(z).$$

This is defined for all positive j.

Proposition 2.4. *There exists $1 < c < \infty$ such that the following holds. Suppose (K_1, U_1), $(K_2, U_2)\ldots$ and $(\tilde{K}_1, \tilde{U}_1), (\tilde{K}_2, \tilde{U}_2), \ldots$ are two sequences as above with corresponding r_j, h_j, f^j, f_j and $\tilde{r}_j, \tilde{h}_j, \tilde{f}^j, \tilde{f}_j$. Let*

$$0 < h < r^2 < \varepsilon^2 < \delta^8 < 1/c,$$

and $n \leqslant 1/h$ and suppose that for all $j = 1, \ldots, n$,

$$|h_j - h| \leqslant hr/\delta, \quad |\tilde{h}_j - h| \leqslant hr/\delta,$$

$$r_j, \tilde{r}_j \leqslant r,$$

$$|U_j - \tilde{U}_j| \leqslant \varepsilon.$$

Suppose $z = x + iy \in \mathbb{H}$. Then if $1 \geqslant y \geqslant \delta$, it holds that

$$\left| f_n(z) - \tilde{f}_n(z) \right| \leqslant c(\varepsilon/\delta) \tag{2.13}$$

and

$$\left| y|f_n'(z)| - y|\tilde{f}_n'(z)| \right| \leqslant c(\varepsilon/\delta)$$

Proof: Write $y_j = \operatorname{Im} f_j(z)$ and similarly for $\tilde{y}_j$. Since $n \leqslant 1/h$ there is a constant c such that $y_j, \tilde{y}_j \leqslant c$ for $j = 0, \dots, n$. As in the proof of Proposition 2.1 but using the previous lemma, we have

$$y_n = y\,[1 + O(\delta)] \exp\left\{ \sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|^2} \right\}.$$

So as in (2.8),

$$\sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|^2} \leqslant \log(c/y) + O(\delta)$$

and

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|\,|\tilde{z}_j - \tilde{U}_j|} &\leqslant \left[\sum_{j=0}^{n-1} \frac{h}{|z_j - U_j|^2} \right]^{1/2} \left[\sum_{j=0}^{n-1} \frac{h}{|\tilde{z}_j - \tilde{U}_j|^2} \right]^{1/2} \\ &\leqslant \log(c/y) + O(\delta). \end{aligned} \tag{2.14}$$

Set $\Delta_j = z_j - \tilde{z}_j$. Then we have

$$\Delta_{j+1} = \Delta_j + \frac{h\,(\tilde{U}_j - U_j + \Delta_j)}{(z_j - U_j)\,(\tilde{z}_j - \tilde{U}_j)} + O\left(h\delta^2\right)$$

and so there exists c such that

$$|\Delta_{j+1}| + c\,\varepsilon \leqslant (|\Delta_j| + c\,\varepsilon)\;[1 + \rho_j]\,,$$

where

$$\rho_j = \frac{h}{|z_j - U_j|\,|\tilde{z}_j - \tilde{U}_j|}.$$

We can then integrate using $\Delta_0 = 0$ to find

$$|\Delta_n| \leqslant c\,(\varepsilon/\delta).$$

The last estimate follows from (2.13) using the Cauchy integral formula. □

Proposition 2.5. *There exists $1 < c < \infty$ such that the following holds. Suppose (K_1, U_1), $(K_2, U_2) \dots$ is a sequence as above with corresponding r_j, h_j, f^j, f_j. Let*

$$0 < h < r^2 < \delta^8 < 1/c,$$

and $n \leqslant 1/h$ and suppose that for all $j = 1, \dots, n$,

$$|h_j - h| \leqslant hr/\delta, \quad r_j \leqslant r.$$

Suppose $z = x + iy \in \mathbb{H}$ and let $z_n = f_n(z)$. Then if $y \geqslant \delta$,

$$|f_n'(z)| = \exp\left\{ \sum_{j=0}^{n-1} \operatorname{Re} \frac{h}{(z_j - U_j)^2} \right\} (1 + O(\delta))\,. \tag{2.15}$$

In particular, there is a constant c such that if

$$\nu = \min_{0 \leqslant j \leqslant n} \left\{ \sin\left[\arg\left(z_j - U_j\right)\right] \right\}, \tag{2.16}$$

then,

$$|f_n'(z)| \leqslant c \left(\frac{y_n}{y}\right)^{1-2\nu^2}. \tag{2.17}$$

2.3. *Expansion of the SLE Green's function.* We consider now the SLE$_\kappa$ Green's function which in the case $\kappa = 2$ equals

$$G_D(z,a,b) := \tilde{c}\, r_D^{-3/4}(z) S^3_{D,a,b}(z).$$

Note that in the half-plane case we have simply $G_{\mathbb{H}}(z,0,\infty) = \tilde{c}\,(2\,\mathrm{Im}\ z)^{-3/4}\sin^3(\arg z)$. We shall later use the LERW analog as an observable to help prove convergence to SLE$_2$. For this, we need to understand how the scaling limit, that is, the SLE Green's function, changes if the domain is perturbed by growing a small hull. The computation is no more difficult for general κ so we will not assume $\kappa = 2$ here.

Let $z_\pm = i \pm 1$. Then

$$\sin[\arg(z_\pm)] = \frac{\sqrt{2}}{2}.$$

Lemma 2.4. *Suppose K is a half-plane hull such that $0 \in \overline{K}$ and $r = r_K < 1/2, h = h_K =$ $\mathrm{hcap}(K) < 1/2$. Write $g = g_K$ for the Loewner map. Then if $z_\pm = i \pm 1$,*

$$\mathrm{Im}\,[g(z_\pm)] = 1 - \frac{h}{2} + O(hr),$$

$$|g'(z_\pm)| = 1 + O(hr).$$

Suppose $\xi \in I_K$, where I_K is as in Section 2.2. Then,

$$\sin\left[\arg(g(z_\pm) - \xi)\right] = \frac{\sqrt{2}}{2}\left[1 \pm \frac{\xi}{2} + \frac{\xi^2}{8} - \frac{h}{2} + O(|\xi|^3) + O(hr)\right].$$

Remark. Note that the assumption that $\xi \in I_K$ implies that $|\xi| \leqslant cr$ for a universal constant c, so we could have written $O(r^3)$ instead of $O(|\xi|^3)$.

Proof: We will prove the result for z_+; the argument for z_- is identical. Let us write

$$w = g(z_+) = x + iy = |w|\, e^{i \arg w},$$

where $\arg w \in [0,\pi]$. Using (2.1),

$$x = 1 + \frac{h}{2} + O(hr), \qquad y = 1 - \frac{h}{2} + O(hr), \quad |w| = \sqrt{2} + O(hr).$$

Moreover,

$$\sin\arg w = \frac{y}{|w|} = \frac{1}{\sqrt{2}} - \frac{h}{2\sqrt{2}} + O(hr), \qquad \arg w = \frac{\pi}{4} - \frac{h}{2} + O(hr).$$

Using (2.3) and the fact that $z_\pm^2$ is purely imaginary, we have

$$|g'(z_\pm)| = 1 + O(hr).$$

We now want to expand $\arg(g(z_+) - \xi) = \arg(w - \xi)$. Proceeding directly by Taylor expansion becomes a bit involved, so we will first exploit the harmonicity. For the moment, let us assume that $\xi \geqslant 0$. Let $\psi_\xi(\zeta) = \arg(\zeta - \xi) - \arg(\zeta)$ and note that $\psi_\xi(\zeta)$ equals π times the harmonic measure of $[0,\xi]$ in $\mathbb{H}$. Since $\xi \in I_K$ we know that $|\psi_\xi(z_+)| \leqslant cr$. Moreover, since ψ_ξ is a positive harmonic function, $|z_+ - w| = O(h)$, and the distance to the boundary from z_+ is larger than a constant,

$$|\psi_\xi(z_+) - \psi_\xi(w)| \leqslant ch\psi_\xi(z_+) = O(h|\xi|) = O(hr),$$

that is, $\psi_\xi(w) = \psi_\xi(z_+) + O(hr)$. Hence, using the Poisson kernel for $\mathbb{H}$,

$$\begin{aligned}\arg(w-\xi) &= \psi_\xi(w) + \arg(w) \\ &= \psi_\xi(z_+) + \frac{\pi}{4} - \frac{h}{2} + O(hr) \\ &= \int_0^\xi \frac{dt}{(1-t)^2+1} + \frac{\pi}{4} - \frac{h}{2} + O(hr) \\ &= \frac{\pi}{4} - \frac{h}{2} + \frac{\xi}{2} + \frac{\xi^2}{4} + O(|\xi|^3) + O(hr).\end{aligned}$$

If $\xi < 0$, we need to consider the probability of hitting the boundary in $[\xi, 0]$, but the same basic argument shows that in this case

$$\begin{aligned}\arg(w-\xi) &= \arg(w) - \int_\xi^0 \frac{dt}{(1-t)^2+1} \\ &= \frac{\pi}{4} - \frac{h}{2} + \frac{\xi}{2} + \frac{\xi^2}{4} + O(|\xi|^3) + O(hr).\end{aligned}$$

Doing the analogous computation with $z = z_-$ we get

$$\arg(g(z_\pm) - \xi) = (2 \mp 1)\frac{\pi}{4} - \frac{h}{2} + \frac{\xi}{2} \pm \frac{\xi^2}{4} + O(|\xi|^3) + O(hr).$$

Finally we use the elementary formulas

$$\sin\left(\frac{\pi}{4} + \varepsilon\right) = \sin(\pi/4)\left[1 + \varepsilon - \frac{\varepsilon^2}{2} + O(\varepsilon^3)\right],$$

and

$$\sin\left(\frac{3\pi}{4} + \varepsilon\right) = \sin(3\pi/4)\left[1 - \varepsilon - \frac{\varepsilon^2}{2} + O(\varepsilon^3)\right].$$

We conclude

$$\sin(\arg(g(z_\pm) - \xi)) = \frac{\sqrt{2}}{2}\left[1 \pm \frac{\xi}{2} + \frac{\xi^2}{8} - \frac{h}{2} + O(|\xi|^3) + O(hr)\right].$$

□

The expansion of the observable is an immediate consequence. We will use this result only with $\kappa = 2$, but we state it so that it can be applied to other discrete models converging to SLE_κ for $0 < \kappa < 8$ if the analog of (1.2) is known.

Proposition 2.6. *Suppose we are in the setting of Lemma 2.4. If* $0 < \kappa < 8$ *and*

$$\alpha = \frac{\kappa}{8} - 1, \quad \beta = \frac{8}{\kappa} - 1,$$

then

$$\Upsilon(z_\pm)^\alpha \sin^\beta(\arg(g(z_\pm) - \xi)) = \left(\frac{\sqrt{2}}{2}\right)^\beta \left(1 \pm A_\kappa\,\xi + B_\kappa\left[\xi^2 - \frac{h\kappa}{2}\right] + O_\kappa(|\xi|^3) + O_\kappa(hr)\right), \tag{2.18}$$

where

$$A_\kappa = \frac{4}{\kappa} - \frac{1}{2}, \quad B_\kappa = \frac{8}{\kappa^2} - \frac{2}{\kappa} + \frac{1}{8}, \quad \Upsilon(z_\pm) = \frac{\operatorname{Im}[g(z_\pm)]}{|g'(z_\pm)|}.$$

3. Coupling the Loewner processes and Loewner chains

In this section we derive the basic coupling results relating the Loewner processes and the corresponding Loewner chains. The method we follow is the same as in Lawler et al. (2004) but we work with a different observable, namely the LERW Green's function, and with the discrete Loewner equation. In order to be able to use the results in Lawler and Viklund (2021) we also need to be more careful with measurability properties and the resulting coupling is a bit different from the one of Lawler et al. (2004). We will give some quantitative estimates (in terms however of the unknown exponent u chosen so that (1.2) holds), but when we do we have not bothered to optimize exponents.

3.1. *Loewner process.* We start with $(A, a, b) \in \mathcal{A}$, so that A is a lattice domain with marked boundary edges a, b. At this point we do not assume A is taken to approximate a particular domain D. In particular, in this sense we make no assumptions about "boundary regularity" on A. Recall that we write $F : D_A \to \mathbb{H}$ for a conformal transformation with $F(a) = 0, F(b) = \infty$. As we have noted before, there is a one-parameter family of such transformations F, so we will now fix one of them. Define

$$R = R_{A,a,b,F} = 4|(F^{-1})'(2i)|$$

and note that R equals the conformal radius of D_A seen from $F^{-1}(2i)$. We will prove facts for (A, a, b, F) with R sufficiently large and we will not always be explicit about this.

Fix a mesoscopic scale h, defined by

$$h = R^{-2u/3}, \tag{3.1}$$

where $0 < u < 1$ is fixed and chosen so that (1.2) holds. This is somewhat arbitrary, but we will use that $R^{-u} = O(h^{6/5})$.

Before going into detailed estimates, let us pause here and give an overview of the argument. Given (A, a, b, F) we grow a piece of a LERW in A from a to b of capacity h; more precisely, we will stop the path the first time the image of its discrete hull (the squares touched by the LERW together with the ones disconnected from b) in $\mathbb{H}$ has reached capacity h or diameter $h^{2/5}$. (In Lawler et al. (2004) the analogous stopping time is defined slightly differently, in terms of the capacity increment and the driving term displacement.) But we shall prove that with very large probability the latter event does not occur. Indeed, since LERW is unlikely to "creep" along the boundary we expect the diameter of the increment in $\mathbb{H}$ to be of order $h^{1/2}$. So, we have a mesoscopic piece η_h of LERW whose discrete hull is of half-plane capacity (very near) h. The domain Markov property of LERW implies that for ζ sufficiently far away from η_h,

$$p(\zeta) = \mathbf{E}\left[\mathbf{E}\left[p(\zeta) \mid \eta_h\right]\right] = \mathbf{E}\left[p_h(\zeta)\right],$$

where $p(\zeta) = \mathbf{P}_{A,a,b}\left\{\zeta \in \eta\right\}$ and $p_h(\zeta) = \mathbf{P}_{A',a',b}\left\{\zeta \in \eta\right\}$ is computed in the smaller domain $(A', a', b) \in \mathcal{A}$ with the LERW hull removed and with marked edges the "tip" a' of η_h and b. Using (1.2) we can express both sides of the equation in terms of the SLE_2 Green's function for D_A and $D_{A'}$ (both Jordan domains), and using Proposition 2.6 we can can expand $p_h(\zeta)$ in terms of the discrete Loewner process displacement ξ. By doing this for two different choices of ζ we get two independent equations which allow us to show that $\mathbf{E}[\xi] = 0$ and $\mathbf{E}[\xi^2 - \text{hcap}[\eta_h]] = 0$ up to a very small error of $O(h^{6/5})$. These are the two critical estimates.

This argument can be iterated thanks to the domain Markov property. We do so enough times to build a macroscopic piece of LERW with very large probability. The outputs are uniform estimates on the conditional expectations and conditional variances of the Loewner process displacements in the sense of a sequence of hull increments and "positions", exactly as in Section 2. The position displacements nearly form a discrete martingale (with a controlled error), and can, with some work, be coupled with Brownian motion using Skorokhod embedding. From the estimate on the variance of the displacement, we conclude that it is a standard Brownian motion, meaning $\kappa = 2$.

3.1.1. *One step.* We begin by discussing the estimates for one mesoscopic increment of the LERW. Suppose $\eta = \{\eta_0, \eta_1, \dots, \eta_{|\eta|}\}$ is a SAW chosen from the LERW probability measure $\mathbf{P}_{A,a,b}$. Let $A_0 = A, a_0 = a$, which coincides with the first edge $[\eta_0, \eta_1]$. For integer $j \geqslant 1$, let $a_j = [\eta_j, \eta_{j+1}]$ (or viewed as a point, the midpoint of that edge) and set

$$A_j = A \smallsetminus \eta^j, \quad \eta^j := \{\eta_0, \eta_1, \dots, \eta_j\},$$

where if needed we take the connected component having b as a boundary edge. Then $a_j, b \in \partial_e A_j$ and D_{A_j} is a simply connected domain which is a subset of D_A and we also have $a_j, b \in \partial D_{A_j}$. Note that (A_j, a_j, b) is measurable with respect to observing the first $j+1$ vertices (in order) on the LERW. We will consider sequences of configurations of the form (A_j, a_j, b) coming from the LERW. We write $\mathcal{F}_j$ for the filtration generated by $\{(A_k, a_k, b) : k = 0, \dots, j\}$. Let $K_j := F(D_A \smallsetminus D_{A_j})$ be the half-plane hull generated by A_j. Note that $0 \in \overline{K}_j$ and that (viewed as a point) $F(a_j) \in K_j$. Let $g_{K_j} : \mathbb{H} \smallsetminus K_j \to \mathbb{H}$ be the uniformizing Loewner map as in Section 2.1. With this set up we introduce a stopping time m as follows:

$$m = \min\left\{ j \geqslant 0 : \ \mathrm{hcap}\,[K_j] \geqslant h \ \text{or} \ \mathrm{diam}\,[K_j] \geqslant h^{2/5} \right\}. \tag{3.2}$$

This stopping time is finite almost surely for R sufficiently large. Note that m depends implicitly on the choice of F. We will write

$$t_j := \mathrm{hcap}[K_j],$$

for $j = 0, 1, \dots$. Using the Beurling estimate, we have the easy upper bounds

$$t_m \leqslant h + O(R^{-1}), \quad \mathrm{diam}[K_m] \leqslant h^{2/5} + O(R^{-1/2}).$$

We expect however that t_m is very close to h and that $\mathrm{diam}[K_m]$ is in fact very close to $h^{1/2}$. Indeed, we have the following lemma.

Lemma 3.1. *There exist $0 < \alpha, c < \infty$ and $R_0 < \infty$ such that for any choice of (A, a, b, F) as above with $R_{A,a,b,F} \geqslant R_0$, for $L > 1$,*

$$\mathbf{P}_{A,a,b}\left\{ \mathrm{diam}[K_m] \geqslant L\, h^{1/2} \right\} \leqslant c\, e^{-\alpha L}.$$

Proof: We sketch the proof here; for details see Section 4. We let m' be the first j such that $\mathrm{hcap}[K_j] \geqslant h$ or $\mathrm{diam}[K_j] \geqslant 4\sqrt{h}$. The key step is to show that there exists $\rho > 0$ such for uniformly for all (A, a, b, F) with R sufficiently large, with probability at least ρ, we have $\mathrm{diam}[K_{m'}] < 4\sqrt{h}$. This uses results from Kozdron and Lawler (2005). If this happens we have reached capacity h and we stop; otherwise, we keep going, stopping again the first time the capacity reaches h or the diameter of the hull increment reaches $4\sqrt{h}$. The probability of doing this J times without success is at most $(1-\rho)^J$. If we have succeeded, i.e., reached capacity h, within J steps then $\mathrm{diam}[K_m] \leqslant O(J\sqrt{h})$. □

Let m be as in (3.2). Define

$$\xi = g_{K_m}(a_m) \in \mathbb{R}.$$

Note that because of the bound on $\mathrm{diam}[K_m]$, a harmonic measure estimate shows that there is a constant $c < \infty$ such that $|\xi| \leqslant c\, h^{2/5}$ for R large enough.

Lemma 3.2. *There exist $0 < \beta, c < \infty$ and $R_0 < \infty$ such that for any choice of (A, a, b, F) as above with $R_{A,a,b,F} \geqslant R_0$, it holds that*

$$|\mathbf{E}_{A,a,b}\,[\xi]| \leqslant c h^{6/5}, \qquad \left| \mathbf{E}_{A,a,b}\left[\xi^2 - h\right] \right| \leqslant c h^{6/5},$$

and

$$\mathbf{E}_{A,a,b}\left[\exp\left\{\beta\,\xi\,h^{-1/2}\right\}\right] \leqslant c. \tag{3.3}$$

Proof: Write $z_\pm = 2(i \pm 1)$ and $H = F^{-1}$. Then H is a conformal map of $\mathbb{H}$ onto D_A. Let $w, \zeta_+, \zeta_- \in \mathbb{Z}^2$ be lattice points in A closest to $H(2i), H(z_+), H(z_-)$, respectively. In case of ties, we choose arbitrarily. The domain Markov property for LERW implies that

$$\mathbf{P}_{A,a,b}\{\zeta_\pm \in \eta\} = \mathbf{E}_{A,a,b}\left[\mathbf{P}_{A_m,a_m,b}\{\zeta_\pm \in \eta\}\right]. \tag{3.4}$$

We will estimate the two sides of this equation. To keep the notation simpler we will write $z = z_\pm$ and $\zeta = \zeta_\pm$. We begin with the left-hand side for which we can use (1.2) directly. Recall that $R = R_{A,a,b,F} = 4|H'(2i)|$. By distortion estimates we know that

$$|F(w) - 2i| + |F(\zeta) - z| \leqslant O(R^{-1})$$

and

$$|F'(\zeta)|^{-1} = |H'(z)|\left(1 + O(R^{-1})\right).$$

Hence,

$$r_A(\zeta) = 4|H'(z)|\left(1 + O(R^{-1})\right),$$

and

$$\sin(\arg F(\zeta)) = \sin(\arg z) + O(R^{-1}) = \frac{\sqrt{2}}{2} + O(R^{-1}).$$

It follows from (1.2) that

$$\begin{aligned}\mathbf{P}_{A,a,b}\{\zeta \in \eta\} &= \hat{c}\,4^{-3/4}|H'(z)|^{-3/4}\left(\left(\frac{\sqrt{2}}{2}\right)^3 + O(h^{6/5})\right)\\ &= \hat{c}\,2^{-3}\,|H'(z)|^{-3/4}\left(1 + O(h^{6/5})\right),\end{aligned}$$

where we used that $R^{-u} = O(h^{6/5})$ (and $u < 1$). We now estimate the right-hand side of (3.4). By the chain rule and distortion estimates, with $g = g_{K_m}$,

$$r_{A_m}(\zeta) = 2\frac{\mathrm{Im}\ g(z)}{|g'(z)|}|H'(z)|(1 + O(R^{-1})),$$

$$\sin\left(\arg\left[g \circ F(\zeta) - \xi\right]\right) = \sin\left[\arg\left(g(z) - \xi\right)\right] + O(R^{-1}).$$

So, using (1.2) for (A_m, a_m, b),

$$\begin{aligned}\mathbf{P}_{A_m,a_m,b}\{\zeta \in \eta\} &= \hat{c}\,2^{-3/4}|H'(z)|^{-3/4}\left(\frac{\mathrm{Im}\ g(z)}{|g'(z)|}\right)^{-3/4}\left(\sin^3\left[\arg(g(z) - \xi)\right] + O(h^{6/5})\right)\\ &= 2^{9/4}\mathbf{P}_{A,a,b}\{\zeta \in \eta\}\left(\frac{\mathrm{Im}\ g(z)}{|g'(z)|}\right)^{-3/4}\left(\sin^3\left[\arg(g(z) - \xi)\right] + O(h^{6/5})\right).\end{aligned} \tag{3.5}$$

Note that $r = \mathrm{diam}(K_m) \leqslant h^{2/5} + O(R^{-1})$ so there is a constant c such that $|\xi| \leqslant ch^{2/5}$ for h sufficiently small. Hence $O(hr + |\xi|^3) = O\left(h^{6/5}\right)$. We can now apply Proposition 2.6 (with $\kappa = 2$) after rescaling: write $g_{K_m}(z) = 2g_{\tilde{K}}(z/2)$ where $\tilde{K} = K_m/2$ and apply the result to $g_{\tilde{K}}(1 \pm i)$ with $r_{\tilde{K}} = r/2$, $h_{\tilde{K}} = t_m/4$. This gives

$$2^{9/4}\left(\frac{\mathrm{Im}\ g(z)}{|g'(z)|}\right)^{-3/4}\sin^3\left[\arg(g(z) - \xi)\right] = 1 \pm \frac{3}{2 \cdot 2}\,\xi + \frac{9}{8 \cdot 4}\,\left(\xi^2 - t_m\right) + O(h^{6/5}).$$

Using this, by combining (3.4) with (3.5), we see that

$$\mathbf{E}_{A,a,b}\left[\pm\frac{3}{2 \cdot 2}\,\xi + \frac{9}{8 \cdot 4}\,\left(\xi^2 - t_m\right)\right] = O(h^{6/5}).$$

These equations imply

$$|\mathbf{E}_{A,a,b}[\xi]| = O(h^{6/5}), \quad \left|\mathbf{E}_{A,a,b}\left[\xi^2 - t_m\right]\right| = O(h^{6/5}).$$

Recall that $|\xi| \leqslant 2r$. Using Lemma 3.1 with $L = h^{-1/10}$ we conclude that $\mathbf{E}_{A,a,b}[t_m] = h + o(h^{6/5})$ and we also get the final assertion of the lemma. □

Proposition 3.1. *There exist $0 < \alpha, c < \infty$ such that one can define on the same probability space a random variable ξ with the distribution $\mathbf{P}_{A,a,b}$ and a standard Brownian motion W_t, and a stopping time τ for W_t such that $\xi - \mu = W_\tau$ where $\mu = \mathbf{E}_{A,a,b}[\xi]$. Moreover,*

$$\mathbf{E}[\tau] = \mathbf{E}_{A,a,b}\left[(\xi-\mu)^2\right] = h + O(h^{6/5}),$$

and if

$$W^* = \max\{|W_t| : t \leqslant \tau\},$$

then

$$\mathbf{E}\left[\exp\left\{\alpha W^* h^{-1/2}\right\}\right] \leqslant c.$$

Proof: This can be seen using Lemma 3.2 from the construction via Skorokhod embedding. The last inequality uses (3.3). See Lemma A.3 in Appendix A for details.

□

3.1.2. *Sequence of steps.* We start with (A, a, b) and F as before, and having chosen a mesoscopic scale h. We have defined a step $(A, a, b) \to (A_m, a_m, b)$ which corresponds to a mesoscopic capacity increment of the LERW. Using the domain Markov property, this process can be continued to define a sequence of steps. The estimates of Lemma 3.2 will hold as long as the conformal radii (seen from the preimage of $2i$) of the decreasing domains are comparable to that of A, allowing if necessary for changing constants.

Let us be more precise. Let $\eta = \{\eta_0, \eta_1, \dots, \eta_{|\eta|}\}$ be LERW in A from a to b as in the previous subsection. Recall the definitions of the configurations (A_j, a_j, b) for $j = 0, 1, \dots$ generated by η. Associated with each A_j we also have a conformal transformation $F_j : D_{A_j} \to \mathbb{H}$ defined by $F_j := g_{K_j} \circ F$, where g_{K_j} is the Loewner map of the half-plane hull K_j. Note that the normalizations of the F_j are determined by the global choice of normalization of F.

We inductively define a sequence of stopping times $m_n, n = 0, 1, 2, \dots$ for $\mathcal{F}_j$, the filtration generated by $\{(A_k, a_k, b) : k = 0, \dots, j\}$. First set $m_0 = 0, m_1 = m$, where $m = m(A, a, b)$ is as in (3.2). Given $(A_{m_{n-1}}, a_{m_{n-1}}, b)$ and $F_{m_{n-1}}$, m_n is then defined in the same way as m in (3.2) but replacing (A, a, b, F) by $(A_{m_{n-1}}, a_{m_{n-1}}, b, F_{m_{n-1}})$ and taking the smallest $j \geqslant m_{n-1}$ such that the capacity increases by h or the diameter of the hull increment (after uniformizing) increases by $h^{2/5}$. (Note that the normalizations of the maps are determined from the initial choice of F.) That is, writing $g_{m_{n-1}}$ for the Loewner map of $K_{m_{n-1}}$ we let

$$m_n = \min\{j \geqslant m_{n-1} : \operatorname{hcap}\left[g_{m_{n-1}}(K_j \smallsetminus K_{m_{n-1}})\right] \geqslant h \\ \text{or } \operatorname{diam}\left[g_{m_{n-1}}(K_j \smallsetminus K_{m_{n-1}})\right] \geqslant h^{2/5}\}.$$

Informally, the LERW makes a capacity increment of h between m_{n-1} and m_n and we expect the total capacity of K_{m_n} to be about nh.

We can now define the "Loewner process"

$$U_n = F_{m_n}(a_{m_n}), \tag{3.6}$$

with increments

$$\xi_n = U_n - U_{n-1}.$$

Note that $U_0 = 0$. We choose the term Loewner process over the more standard "driving process/term" since while the SAW determines the U_n process, the converse is not true. Write also

$$H_n = F(D_{m_n}) \subset \mathbb{H}$$

for the complement of K_{m_n}. Let n_0 be the integer part of $3/(2h)$. Then $n_0 \asymp R^{2u/3}$ and $\operatorname{hcap} K_{m_n} \leqslant \operatorname{hcap} K_{m_{n_0}} \leqslant 3/2 + O(R^{2u/3-1})$ for all $n \leqslant n_0$. Hence for such n we have (with implied universal constants)

$$|(F_{m_n}^{-1})'(2i)| \asymp |(F^{-1})'(2i)| = R/4$$

for R large enough.

The next lemma shows that with very large probability, after n_0 iterations, we have built a hull of capacity at least 1 (actually very near 3/2). Let $\overline{\mathcal{F}}_n = \mathcal{F}_{m_n}$ denote the σ-algebra of the LERW configurations.

Lemma 3.3. *There exist $0 < c, \alpha < \infty$ and $R_0 < \infty$ such that for all (A, a, b, F) with $R \geqslant R_0$,*

$$\mathbf{P}\left\{t_{m_{n_0}} < 1\right\} \leqslant c h^{-1} e^{-\alpha h^{-1/10}}.$$

Proof: By Lemma 3.1 there are constants α, c such that for $n = 1, \ldots, n_0$,

$$\mathbf{P}\left[t_{m_n} - t_{m_{n-1}} < h \mid \overline{\mathcal{F}}_{n-1}\right] \leqslant c e^{-\alpha h^{-1/10}}.$$

Since $n_0 = O(1/h)$, summing over n up to n_0 gives the lemma. □

Lemma 3.4. *There exist $0 < c, R_0 < \infty$ such that the following holds. For any (A, a, b, F) with $R_{A,a,b,F} \geqslant R_0$ there is a coupling of a LERW η with law $\mathbf{P}_{A,a,b}$ and a standard Brownian motion $(W_t, \tilde{\mathcal{F}}_t)$ with a sequence of stopping times $\{\tau_n\}$ for $(W_t, \tilde{\mathcal{F}}_t)$ for which the following estimates hold:*

(i.)

$$\mathbf{P}\left\{\max_{1 \leqslant n \leqslant n_0} |\tau_n - nh| > c h^{1/5}\right\} \leqslant c h^{1/5}, \tag{3.7}$$

(ii.)

$$\mathbf{P}\left\{\max_{1 \leqslant n \leqslant n_0} |W_{\tau_n} - U_n| > c h^{1/10}\right\} \leqslant c h^{1/10},$$

(iii.)

$$\mathbf{P}\left\{\max_{1 \leqslant n \leqslant n_0} \max_{\tau_{n-1} \leqslant t \leqslant \tau_n} |W_t - W_{\tau_{n-1}}| > c h^{2/5}\right\} \leqslant c h^{1/10},$$

(iv.)

$$\mathbf{P}\left\{\max_{0 \leqslant t \leqslant \tau_{n_0}} \max_{t - h^{1/5} \leqslant s \leqslant t} |W_t - W_s| > c h^{1/12}\right\} \leqslant c h^{1/10}.$$

Moreover, if $\mathcal{G}_n$ denotes the σ-algebra generated by $\overline{\mathcal{F}}_n = \mathcal{F}_{m_n}$ and $\tilde{\mathcal{F}}_{\tau_n}$ (i.e., $\mathcal{G}_n = \mathcal{F}_{m_n} \vee \tilde{\mathcal{F}}_{\tau_n}$), then $t \mapsto W_{t+\tau_n} - W_{\tau_n}$ is independent of $\mathcal{G}_n$ and the distribution of the LERW given $\mathcal{G}_n$ is the same as the distribution given $\overline{\mathcal{F}}_n$.

Proof: Using Lemma 3.2 and the domain Markov property we see that there is a constant $c < \infty$ such if R is large enough, for $n \leqslant n_0$,

$$\left|\mathbf{E}\left[\xi_n \mid \overline{\mathcal{F}}_{n-1}\right]\right| \leqslant c h^{6/5},$$

$$\left|\mathbf{E}\left[\xi_n^2 - (t_{m_n} - t_{m_{n-1}}) \mid \overline{\mathcal{F}}_{n-1}\right]\right| \leqslant c h^{6/5},$$

$$\xi_n^4 \leqslant c h^{8/5}.$$

Let $\delta_0 = 0$ and for $n = 1, 2, \ldots, n_0$,

$$\delta_n = \xi_n - \mathbf{E}[\xi_n \mid \overline{\mathcal{F}}_{n-1}].$$

This is clearly a martingale difference sequence. We use the Skorokhod embedding theorem (see Proposition 3.1 and Appendix A, in particular Theorem A.2) to define a standard Brownian motion W_t, generating the filtration $\tilde{\mathcal{F}}_t$, and a sequence of stopping times $0 = \tau_0 < \tau_1 < \dots$ for W such that the Brownian increments satisfy

$$W_{\tau_n} - W_{\tau_{n-1}} = \delta_n.$$

It is important that this coupling has the property that it does not look "into the future of the LERW". That is to say, if $\mathcal{G}_n$ denotes the σ-algebra generated by $\tilde{\mathcal{F}}_{\tau_n}$ and $\overline{\mathcal{F}}_n$, then the Brownian motion $t \mapsto W_{t+\tau_n} - W_{\tau_n}$ is independent of $\mathcal{G}_n$ and the distribution of the LERW in the future given $\mathcal{G}_n$ is the same as the distribution given $\overline{\mathcal{F}}_n$.

From $U_n = \sum_{j=1}^n \xi_j = \sum_{j=1}^n (\delta_n + \mathbf{E}[\xi_j \mid \overline{\mathcal{F}}_{j-1}])$ we have

$$|U_n - W_{\tau_n}| \leqslant \sum_{j=1}^n |\mathbf{E}[\xi_j \mid \overline{\mathcal{F}}_{j-1}]|.$$

So since $n_0 = O(h^{-1})$,

$$\mathbf{E}\left[\sum_{j=1}^{n_0} |\mathbf{E}[\xi_j \mid \overline{\mathcal{F}}_{j-1}]|\right] = O(h^{1/5}).$$

Hence by the Markov inequality,

$$\mathbf{P}\left\{\sum_{j=1}^{n_0} |\mathbf{E}[\xi_j \mid \overline{\mathcal{F}}_{j-1}]| \geqslant h^{1/10}\right\} = O(h^{1/10}).$$

Therefore, except for an event of probability $O(h^{1/10})$,

$$|U_n - W_{\tau_n}| \leqslant ch^{1/10} \qquad \text{for all } n \leqslant n_0. \tag{3.8}$$

This gives (ii). We will now compare the capacity increments. Using $\delta_n = \xi_n - \mathbf{E}[\xi_n \mid \overline{\mathcal{F}}_{n-1}]$, we have

$$\mathbf{E}[\delta_n^2 - (t_{m_n} - t_{m_{n-1}}) \mid \mathcal{G}_{n-1}] = O(h^{6/5})$$

and by construction, since $\delta_n = W_{\tau_n} - W_{\tau_{n-1}}$,

$$\mathbf{E}[\delta_n^2 - (\tau_n - \tau_{n-1}) \mid \mathcal{G}_{n-1}] = 0.$$

So we expect that the τ increments are close to the capacity increments which in turn are deterministic with very large probability. We will show the first part of this by looking at a suitable martingale. For this, note that if

$$\mu_n = t_{m_n} - t_{m_{n-1}}, \quad \nu_n = \tau_n - \tau_{n-1},$$

then the last two estimates show that

$$\mathbf{E}[\mu_n - \nu_n \mid \mathcal{G}_{n-1}] = O(h^{6/5}).$$

Consider the martingale

$$M_n = \sum_{j=1}^n Y_j,$$

where

$$Y_j = \mu_j - \nu_j - \mathbf{E}[\mu_j - \nu_j \mid \mathcal{G}_{j-1}].$$

We know that $\mu_n^2 \leqslant ch^2$ and moreover,

$$3\mathbf{E}[\nu_n^2 \mid \mathcal{G}_{n-1}] = \mathbf{E}[(W_{\tau_n} - W_{\tau_{n-1}})^4 \mid \mathcal{G}_{n-1}] = \mathbf{E}[\delta_n^4 \mid \mathcal{G}_{n-1}] = O(h^{8/5}),$$

where the last estimate uses that $\delta_n = \xi_n - \mathbf{E}[\xi_n \mid \overline{\mathcal{F}}_{n-1}]$ and $|\xi_n|^4 \leqslant ch^{8/5}$. Hence,

$$\mathbf{E}[\mu_n^2 + \nu_n^2 \mid \mathcal{G}_{n-1}] = O(h^{8/5}),$$

and we can sum these estimates (using Jensen's inequality) to see that

$$\mathbf{E}[M_{n_0}^2] = \sum_{j=1}^{n_0} \mathbf{E}[Y_j^2] = O(h^{3/5}).$$

Using Doob's maximal inequality,

$$\mathbf{P}\left\{ \max_{1 \leqslant n \leqslant n_0} |M_n| \geqslant h^{1/5} \right\} \leqslant c h^{-2/5}\, \mathbf{E}[M_{n_0}^2] = O(h^{1/5}).$$

Since

$$\max_{1 \leqslant n \leqslant n_0} |t_{m_n} - \tau_n| \leqslant \max_{1 \leqslant n \leqslant n_0} |M_n| + c h^{1/5},$$

we see that except on an event of probability $O(h^{1/5})$ we have

$$\max_{1 \leqslant n \leqslant n_0} |t_{m_n} - \tau_n| \leqslant c h^{1/5}. \tag{3.9}$$

By Lemma 3.1 we know that except on an event of probability $o(h^{1/5})$,

$$\max_{1 \leqslant n \leqslant n_0} |t_{m_n} - nh| \leqslant c h^{1/5},$$

and so we conclude that except on an event of probability $O(h^{1/5})$,

$$\max_{1 \leqslant n \leqslant n_0} |\tau_n - nh| \leqslant c h^{1/5}. \tag{3.10}$$

This gives (i). For (iii) we can use the last estimate of Proposition 3.1 together with Chebyshev's inequality and (iv) follows from (i) and a modulus of continuity estimate for Brownian motion (see Lemma A.1). □

We rephrase the coupling result as follows.

Theorem 3.2. There exist $0 < c, R_0 < \infty$ such that the following holds. For any (A, a, b) with $R_{A,a,b,F} \geqslant R_0$ we can define a LERW domain configuration sequence

$$\{(A_j, a_j, b), \quad j = 0, 1, \ldots, J\},$$

stopping times $m_n, n = 0, \ldots, n_0$, for the LERW, a standard Brownian motion $W_t, 0 \leqslant t \leqslant 1$, and a sequence of increasing stopping times $\tau_n, n = 0, \ldots, n_0$, for the Brownian motion, on the same probability space such that the following holds.

- The distribution of $\{(A_{m_n}, a_{m_n}, b)\}$ is that of the LERW domains corresponding to $\mathbf{P}_{A,a,b}$ sampled at mesoscopic capacity increments, as described above.
- Let $\mathcal{G}_n$ denote the σ-algebra generated by $\{(A_j, a_j, b) : n = 0, \ldots, m_n\}$ and $\{W_t : t \leqslant \tau_n\}$. Then,
$$\{(A_j, a_j, b) :\ j > m_n\},$$
$$\{W_{t+\tau_n} - W_{\tau_n} : t \geqslant 0\}$$
are conditionally independent of $\mathcal{G}_n$ given (A_{m_n}, a_{m_n}, b).
- There exists a stopping time $n_* \leqslant n_0$ with respect to $\{\mathcal{G}_n\}$ such that
$$\mathbf{P}\{n_* < n_0\} \leqslant c\, h^{1/10},$$
and such that for $n < n_*$,
$$|W_{\tau_n} - U_n| \leqslant c\, h^{1/10};$$
$$|\tau_n - nh| \leqslant c\, h^{1/5};$$
$$\max_{\tau_{n-1} \leqslant t \leqslant \tau_n} |W_t - W_{\tau_{n-1}}| \leqslant c\, h^{2/5};$$
$$\max_{t \leqslant \tau_n} \max_{t - h^{1/5} \leqslant s \leqslant t} |W_t - W_s| \leqslant c\, h^{1/12}.$$

- For $n \leqslant n_*$, $\operatorname{hcap}\left[g_{m_{n-1}}(K_{m_n} \smallsetminus K_{m_{n-1}})\right] \leqslant h + h^2$.
  Moreover, for $n < n_*$, $\operatorname{hcap}\left[g_{m_{n-1}}(K_{m_n} \smallsetminus K_{m_{n-1}})\right] \geqslant h$.

Proof of Theorem 3.2: Let c be as in Lemma 3.4. We define n_* to be the minimum of n_0 and the first n such that either of

$$|W_{\tau_n} - U_n| > ch^{1/10};$$

$$|\tau_n - nh| > ch^{1/5};$$

$$\max_{\tau_{n-1} \leqslant t \leqslant \tau_n} |W_t - W_{\tau_{n-1}}| > ch^{2/5};$$

$$\max_{t \leqslant \tau_n} \max_{t - h^{1/5} \leqslant s \leqslant t} |W_t - W_s| > ch^{1/12},$$

$$t_{m_n} - t_{m_{n-1}} < h$$

occurs. Note that if $t_{m_n} - t_{m_{n-1}} < h$, then the diameter of $g_{m_{n-1}}(K_{m_n} \smallsetminus K_{m_{n-1}}) \geqslant ch^{2/5}$. Hence using Lemma 3.4 and Lemma 3.1 we see that $\mathbf{P}\left\{n_* < n_0\right\} = O(h^{1/10})$. $\square$

3.2. *Loewner chains and coarsening.* Given the Brownian motion W_t of Theorem 3.2, there is a corresponding SLE_2 Loewner chain (g_t^{SLE}) obtained by solving the Loewner differential equation with W_t as driving term. The Loewner chain is generated by an SLE_2 path in $\mathbb{H}$ that we denote by $\gamma(t)$. Let $\hat{\gamma}(t) = F^{-1} \circ \gamma(t)$ which is an SLE_2 path from a to b in D_A parametrized by capacity in $\mathbb{H}$. (This parametrization depends on F but we have fixed F.) We write

$$F_{\tau_n}^{\mathrm{SLE}}(z) = (g_{\tau_n}^{\mathrm{SLE}} \circ F)(z) - W_{\tau_n}$$

and

$$F_{m_n}^{\mathrm{LERW}}(z) = (g_{m_n} \circ F)(z) - U_n.$$

We would now like to apply Proposition 2.1, but we can not directly do so with "microscopic" capacity scale h since the error in our estimate on the capacity increment is too large. (Recall that that proposition requires uniformly $|h_j - h| = O(hr/\delta)$, where $r/\delta = o(1)$, which we do not have in this case.) However, the estimate we do have is uniform on the integrated capacities, so we can instead consider a coarser scale in the same coupling.

Lemma 3.5. *There exist $0 < c, R_0 < \infty$ such that the following holds. Consider the setting and coupling of Theorem 3.2. Set*

$$u_n = \lceil nh^{-9/10} \rceil, \qquad \hat{n}_0 = \max\{n : u_n \leqslant n_0\},$$

and for $n = 0, 1, \ldots, \hat{n}_0$,

$$\hat{m}_n = m_{u_n}, \qquad \hat{\tau}_n = \tau_{u_n}.$$

Then except on an event of probability at most $ch^{1/10}$, if $\zeta \in A$ and $n < \hat{n}_0$ are such that $\operatorname{Im} F_{\hat{\tau}_n}^{SLE}(\zeta) \geqslant h^{1/100}$, then

$$\left|F_{\hat{m}_n}^{LERW}(\zeta) - F_{\hat{\tau}_n}^{SLE}(\zeta)\right| \leqslant ch^{1/25}.$$

Moreover, if $z = x + iy \in \mathbb{H}$, $h^{1/100} \leqslant y \leqslant 1$ and $f_{m_n}^{LERW} = g_{m_n}^{-1}$, $f_{\tau_n}^{SLE} = (g_{\tau_n}^{SLE})^{-1}$, then for all $n < \hat{n}_0$,

$$\left|f_{\hat{m}_n}^{LERW}(z) - f_{\hat{\tau}_n}^{SLE}(z)\right| \leqslant ch^{1/25}$$

and

$$\left|y|(f_{\hat{m}_n}^{LERW})'(z)| - y|(f_{\hat{\tau}_n}^{SLE})'(z)|\right| \leqslant ch^{1/25}.$$

Proof: Let $m_n, \tau_n, n_0, n_*, c, R_0$ be as in Theorem 3.2. Set

$$\hat{h} := h^{1/10},$$

where h is as in (3.1) and assume $R \geqslant R_0$. Then in the coupling of Theorem 3.2 there is an event E such that $\mathbf{P}(E^c) \leqslant c\hat{h}$, and on E we have $n_* \geqslant n_0$,

$$\max_{n<n_0} |U_n - W_{\tau_n}| \leqslant c\hat{h},$$

and

$$\max_{n<n_0} |\tau_n - nh| \leqslant c\hat{h}^2, \qquad \max_{n<n_0} |t_{m_n} - nh| \leqslant c\hat{h}^2.$$

Now set

$$u_n = \lceil nh^{-9/10} \rceil, \qquad \hat{n}_0 = \min\{n : u_n \geqslant n_0\},$$

and for $n = 0, 1, \ldots, \hat{n}_0 - 1$, define

$$\hat{m}_n = m_{u_n}, \qquad \hat{\tau}_n = \tau_{u_n}.$$

Then $n\hat{h} \leqslant u_n h \leqslant n\hat{h} + \hat{h}^{10}$ and

$$\max_{n<\hat{n}_0} \left| \hat{\tau}_n - n\hat{h} \right| \leqslant c\hat{h}^2, \qquad \max_{n<\hat{n}_0} \left| t_{\hat{m}_n} - n\hat{h} \right| \leqslant c\hat{h}^2.$$

We now want to apply Proposition 2.1 with

$$h = \hat{h}, \quad \delta = \hat{h}^{1/10}, \quad \varepsilon = \delta^4/2, \quad r = \varepsilon/2$$

on an event of large probability. For this we first need to know that

$$\max_{n<\hat{n}_0} \operatorname{diam} \left[g_{\hat{m}_{n-1}}(K_{\hat{m}_n} \smallsetminus K_{\hat{m}_{n-1}}) \right] \leqslant r. \tag{3.11}$$

Since $r \asymp \hat{h}^{2/5}$ we can use Lemma 3.1 (and the Markovian property of LERW) to see that (3.11) holds on an event $E' \subset E$ of probability at least $1 - c\hat{h}$. Since $\hat{\tau}_{n+1} - \hat{\tau}_n = \hat{h} + O(\hat{h}^2)$ and $r \asymp \hat{h}^{2/5}$ on E', we can use a modulus of continuity estimate for Brownian motion (see Lemma A.1) to see that

$$\max_{n<\hat{n}_0} \operatorname{diam} \left[g^{\mathrm{SLE}}_{\hat{\tau}_{n-1}}(\gamma[\hat{\tau}_{n-1}, \hat{\tau}_n]) \right] \leqslant r$$

on an event $E'' \subset E'$ of probability at least $1 - c\hat{h}$. Indeed, for each n,

$$g^{\mathrm{SLE}}_{\hat{\tau}_{n-1}}(\gamma[\hat{\tau}_{n-1}, \hat{\tau}_n]) - W_{\hat{\tau}_{n-1}}$$

is a hull attached at 0 generated by $W_t, t \in [\hat{\tau}_{n-1}, \hat{\tau}_n]$. Its maximal distance from 0 is therefore bounded by $c(\sqrt{\hat{\tau}_n - \hat{\tau}_{n-1}} + \sup_{t\in[\hat{\tau}_{n-1}, \hat{\tau}_n]} |W_t - W_{\hat{\tau}_{n-1}}|)$.

Finally, on the event E'' we apply Proposition 2.1 with the above parameters to get the estimate

$$|g_{\hat{m}_n}(z) - g^{\mathrm{SLE}}_{\hat{\tau}_n}(z)| \leqslant c\varepsilon/\delta \leqslant c\delta^3 = O(h^{3/100})$$

for all z, n such that $\operatorname{Im} g^{\mathrm{SLE}}_{\hat{\tau}_n}(z) \geqslant \delta$.

For the statement concerning the reverse flow, we may apply Proposition 2.4 with the same parameters. □

Proposition 3.3. *There exist $0 < c, R_0 < \infty$ such that the following holds. Consider the setting and coupling of Theorem 3.2. Then except on an event of probability at most $ch^{1/10}$, if $\zeta \in A$ and $n < n_0$ are such that* $\operatorname{Im} F^{SLE}_{\tau_n}(\zeta) \geqslant h^{1/100}$, *then*

$$\left| F^{LERW}_{m_n}(\zeta) - F^{SLE}_{\tau_n}(\zeta) \right| \leqslant ch^{1/25}.$$

Moreover, if $z = x + iy \in \mathbb{H}$, $h^{1/100} \leqslant y \leqslant 1$ and $f^{LERW}_{m_n} = g^{-1}_{m_n}$, $f^{SLE}_{\tau_n} = (g^{SLE}_{\tau_n})^{-1}$, then for all $n < n_0$,

$$\left| f^{LERW}_{m_n}(z) - f^{SLE}_{\tau_n}(z) \right| \leqslant ch^{1/25}$$

and

$$\big||y|(f^{LERW}_{m_n})'(z)| - y|(f^{SLE}_{\tau_n})'(z)|\big| \leqslant ch^{1/25}.$$

Proof: By Lemma 3.5 the conclusions hold with the coarsened sequence of stopping times, $\hat{m}_n$ and $\hat{\tau}_n$ for $n < \hat{n}_0$. We further know that except for an event of probability $O(h^{1/10})$, for $n < \hat{n}_0$, the half-plane capacity increments of the LERW process satisfy $t_{\hat{m}_n} - t_{\hat{m}_{n-1}} = O(h^{1/10})$ and the SLE half-plane capacity increments satisfy $\hat{\tau}_n - \hat{\tau}_{n-1} = O(h^{1/10})$. For $n < n_0$ let $k = k(n) < \hat{n}_0$ be the largest integer such that $u_k \leqslant n$. Then it follows that $t_{m_n} - t_{\hat{m}_k} = O(h^{1/10})$ and $\tau_n - \hat{\tau}_k = O(h^{1/10})$. If Im $F^{\mathrm{SLE}}_{\tau_n}(\zeta) \geqslant h^{1/100}$, then the Loewner equation therefore implies

$$|F^{\mathrm{SLE}}_{\tau_n}(\zeta) - F^{\mathrm{SLE}}_{\hat{\tau}_k}(\zeta) + W_{\tau_n} - W_{\hat{\tau}_k}| = o(h^{1/25}).$$

Moreover, a modulus of continuity estimate for Brownian motion shows that $|W_{\tau_n} - W_{\hat{\tau}_k}| = O(h^{1/25})$ with probability at least $1 - o(h^{1/10})$. From the coupling it follows that $|U_{m_n} - U_{\hat{m}_k}| = O(h^{1/25})$ with probability at least $1 - O(h^{1/10})$. Moreover, the Loewner difference equation implies

$$|F^{\mathrm{LERW}}_{m_n}(\zeta) - F^{\mathrm{LERW}}_{\hat{m}_k}(\zeta) + U_{m_n} - U_{\hat{m}_k}| = o(h^{1/25}).$$

The statements about $f^{\mathrm{LERW}}_{m_n}, f^{\mathrm{SLE}}_{\tau_n}$ follow similarly. □

4. Proof of Lemma 3.1

Lemma 4.1. *There exists $c > 0$ such that the following holds. Let σ_r be the first index j such that* $\mathrm{Im}\,[F(\eta_j)] \geqslant 2r$*. Then for $R^{-1/4} \leqslant r \leqslant c$,*

$$\mathbf{P}_{A,a,b}\{-r \leqslant \mathrm{Re}\,[\eta_j] \leqslant r \text{ for all } j \leqslant \sigma_r\} \geqslant c.$$

We note that $\mathrm{hcap}\,(\eta[0, \sigma_r]) \geqslant r^2$.

Proof: Let $\tilde{\omega}$ denote the excursion in A so that $\eta = \mathrm{LE}[\tilde{\omega}]$, and for ease of notation let use write $\omega_k = F[\tilde{\omega}_k]$.

We first consider the following event for the random walk excursion. Let ρ be the first j with $\mathrm{Im}\,[\omega_j] \geqslant 4r$ and consider the event that

$$-r \leqslant \mathrm{Re}\,[\omega_j] \leqslant r, \quad 0 \leqslant j \leqslant \rho,$$
$$\mathrm{Im}\,[\omega_j] \geqslant 3r, \quad \rho \leqslant j < \infty.$$

Note that on this event, if η is the loop-erasure of ω, then

$$-r \leqslant \mathrm{Re}\,[\eta_j] \leqslant r, \quad 0 \leqslant j \leqslant \sigma_r.$$

Hence, we need to show that this event on excursions has positive probability. The hard work was done in Kozdron and Lawler (2005, Proposition 3.14) where it is shown that there exists c' such that with positive probability, if ρ is the first time j that the excursion reaches $\{\mathrm{Im}\,(z) \geqslant c'r\}$, then $\max\{|\mathrm{Re}\,(\omega_j)| : 0 \leqslant j \leqslant \rho\} \leqslant r/2$. (That paper considers the map to the unit disk rather than the upper half plane, but the result can easily be adapted by mapping the disk to the half plane.) Given this event, the remainder of the path can be extended using the invariance principle. Indeed, this follows from the following facts about the Poisson kernel. Let us consider

$$V = V(A, h) = \{\zeta \in A : F(\zeta) \in \{|z| \leqslant 5r\}.$$
$$V_- = V_-(A, r) = \{\zeta \in V : \mathrm{Im}\,[F(\zeta)] \leqslant r\},$$
$$V_+ = V_+(A, r) = \{\zeta \in V : \mathrm{Im}\,[F(\zeta)] \geqslant 2r\}.$$

Then by combining (1) and (41) of Kozdron and Lawler (2005) , we can see that for R sufficiently large and $R^{-1/4} \leqslant r \leqslant R^{-\varepsilon}$, we have for all $\zeta_+ \in V_+, \zeta_- \in V_-$,

$$H_A(\zeta_+, b) \geqslant \frac{3}{2}\, H_A(\zeta_-, b). \tag{4.1}$$

In fact, one can show that there is $u > 0$ such that

$$\frac{H_A(\zeta_+, b)}{H_A(\zeta_-, b)} = \frac{\text{Im } F(\zeta_+)}{\text{Im } F(\zeta_-)} \left(1 + O(R^{-u})\right)$$

so, allowing for the small error, the quotient is at least $3/2$. This estimate implies that the probability that an excursion starting at $\zeta \in V_+$ with probability at least $1/3$ does not visit V_-.

□

We now complete the proof of Lemma 3.1. Let ξ_1 be the first j such that $|F(\eta_j)| \geqslant 4r$. Using the Beurling estimate, we have $|F(\eta_j)| \leqslant 4r + O(R^{-1/2}) \leqslant 5r$. Let $F_1 = g_1 \circ F$ where $g_1 : F(D_{A_{\xi_1}}) \to \mathbb{H}$ with $g(a_1) = 0$ and $g_1(z) \sim z$ as $z \to \infty$. Inductively, we define ξ_k to be the first $j = j_k$ such that $|F_{k-1}(\eta_j)| \geqslant 4r$, and define F_k in the same way. Let J be the first k such that

$$\text{Im } [F_{k-1}(\eta_{j_k})] \geqslant 2r.$$

Using the previous lemma, we see that

$$\mathbf{P}\{J \geqslant k\} \leqslant e^{-\alpha k},$$

for some $\alpha > 0$. In particular, for R sufficiently large,

$$\mathbf{P}\{J \geqslant r^{-1/15}\} \leqslant \exp\{-\alpha \lfloor r^{-1/15} \rfloor\} \leqslant \exp\{r^{-1/20}\}.$$

Note that $\text{hcap}[F(\eta_{\xi_J})] \geqslant \text{hcap}\,[F_{J-1}(\eta_J)] \geqslant r^2$. We also claim that there exists a universal $c_1 < \infty$ such that

$$\text{diam}\,[F(\eta[0, \xi_J])] \leqslant c_1 J r.$$

This is a fact about the Loewner equation. More generally, suppose that $K_1 \subset K_2 \subset \cdots$ is an increasing sequence of connected hulls in $\mathbb{H}$ with corresponding maps $g_j : \mathbb{H} \smallsetminus K_j \to \mathbb{H}$. Suppose also that for each j, $g_{j-1}(K_j \smallsetminus K_{j-1})$ is connected. For any connected hull K (see Lawler (2005, (3.14))) we compare the diameter with the (potential theoretic) capacity:

$$\text{diam}(K) \asymp \text{cap}_{\mathbb{H}}(K) := \lim_{y \to \infty} y\, \mathbf{P}^{iy}\{B_T \in K\},$$

where B is a complex Brownian motion and

$$T = T_K = \inf\{t : B_t \in K \cup \mathbb{R}\}.$$

If $T_j = T_{K_j}$ with $T_0 = T_\emptyset$, then

$$\mathbf{P}^{iy}(K_k) = \mathbf{P}^{iy}\{T_k < T_0\} \leqslant \sum_{j=1}^{k} \mathbf{P}^{iy}\{T_j < T_{j-1}\}.$$

Using conformal invariance of Brownian motion and the fact that $g_{j-1}(iy) = iy + O(1)$, we can see that

$$\begin{aligned} \lim_{y \to \infty} y\, \mathbf{P}^{iy}\{T_j < T_{j-1}\} &= \lim_{y \to \infty} y\, \mathbf{P}^{g_{j-1}(iy)}\{B(T_{g_{j-1}(K_j \smallsetminus K_{j-1})}) \notin \mathbb{R}\} \\ &= \text{cap}_{\mathbb{H}}[g_{j-1}(K_j \smallsetminus K_{j-1})], \end{aligned}$$

and hence,

$$\begin{aligned} \text{diam}(K_k) \leqslant c\, \text{cap}_{\mathbb{H}}(K_k) &\leqslant c \sum_{j=1}^{k} \text{cap}_{\mathbb{H}}[g_{j-1}(K_j \smallsetminus K_{j-1})] \\ &\leqslant c \sum_{j=1}^{k} \text{diam}\,[g_{j-1}(K_j \smallsetminus K_{j-1})]\,. \end{aligned}$$

This concludes the proof.

Appendix A. Skorokhod embedding

This appendix discusses a version of Skorokhod embedding used above. Much of this may be known, but we will give the argument as we are not making the standard assumptions. Since we only need to couple martingales with discrete distributions, we will restrict our consideration to such here.

A.1. *Preliminary results.* For the convenience of the reader we will first state a standard modulus of continuity estimate that we have used repeatedly in the paper, see Lemma 1.2.1 of Csörgő and Révész (1981) for the proof.

Lemma A.1. *Let B_t be standard Brownian motion. For each $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that for every $v, T > 0$, and $0 < h < T$,*

$$\mathbf{P}\left\{\sup_{t\in[0,T-h]}\sup_{s\in(0,h]}|B_{t+s}-B_t| \geqslant v\sqrt{h}\right\} \leqslant \frac{CT}{h}e^{-\frac{v^2}{2+\varepsilon}}.$$

Lemma A.2. *There exists $c < \infty$ such that if B_t is a standard Brownian motion starting at the origin, $0 < y \leqslant x$ and*

$$\tau = \tau_{(x,y)} = \inf\{t : B_t \in \{x, -y\}\},$$

then

$$\mathbf{P}\{B_\tau = x\} = \frac{y}{x+y}, \quad \mathbf{E}[\tau] = xy,$$

and for all positive integers $n > 2$,

$$\mathbf{E}[\tau^n] \leqslant c\,\pi^{-2n}\,n!\,y\,(x+y)^{2n-1}. \tag{A.1}$$

If $M = \max\{|B_t|, 0 \leqslant t \leqslant \tau\}$, and $a > 0$, then

$$\mathbf{E}[e^{aM}] < 4 + \frac{1}{2}e^{ay} + \frac{2\,y}{x+y}e^{ax}.$$

Proof: The first two are standard results obtained by stopping the martingales B_t and $B_t^2 - t$ at time τ.

By scaling it suffices to prove the third when $x + y = \pi$. If $x = y = \pi/2$, then, e.g., by solving the appropriate PDE by separation of variables, we have

$$\mathbf{P}\{\tau > t\} \sim c\,e^{-t},$$

and hence

$$\mathbf{E}[\tau^n] \leqslant c\int_0^\infty t^n\,e^{-t}\,dt = c\,n!.$$

For $y < \pi/2$, we use the following version of gambler's ruin,

$$\mathbf{P}\{\tau > t\} \leqslant \frac{c\,y}{\sqrt{t}},$$

to see that

$$\mathbf{E}[(\tau\wedge 1)^n] \leqslant \mathbf{E}[\tau\wedge 1] = \int_0^1 \mathbf{P}\{\tau \geqslant t\}\,dt \leqslant c\,y.$$

Also,

$$\mathbf{E}[(\tau-1)^n; \tau \geqslant 1] = \mathbf{P}\{\tau \geqslant 1\}\,\mathbf{E}[(\tau-1)^n \mid \tau \geqslant 1] \leqslant c\,y\,n!.$$

The estimate now follows from the Minowski inequality using

$$\tau = (\tau\wedge 1) + (\tau - 1)\,1\{\tau \geqslant 1\}.$$

For the last assertion, we start by noting that the gambler's ruin estimate implies that

$$\mathbf{P}\{M = y\} = \frac{1}{2}, \quad \mathbf{P}\{M = x\} = \frac{x}{x+y},$$

and for $y < s < x$, the density of M is given by $y/(s+y)^2$. Therefore,

$$\mathbf{E}[e^{aM}] = \frac{1}{2}\, e^{ay} + \frac{y}{x+y}\, e^{ax} + \int_y^x \frac{y}{(s+y)^2}\, e^{as}\, ds.$$

The function $f(s) = (s+y)^{-2}\, e^{as}$ has a single minimum for $s > 0$ at $s_0 = \frac{2}{a} - y$. It is increasing after this time. Therefore,

$$\int_y^{y \vee s_0} \frac{y}{(s+y)^2}\; e^{as}\, ds \leqslant e^{2-ay} \int_y^\infty \frac{y}{(s+y)^2}\, ds \leqslant \frac{e^2}{2} < 4.$$

Finally, note that

$$\int_{y\vee s_0}^x \frac{y}{(s+y)^2}\, e^{as}\, ds \leqslant \frac{x\,y}{(x+y)^2}\, e^{ax} \leqslant \frac{y}{x+y}\, e^{ax}.$$

□

Lemma A.3 (Skorokhod embedding)**.** *There exists $c < \infty$ such that the following holds. Suppose Z is a mean zero discrete random variable with $\mathbf{E}[Z^2] = \sigma^2$. Then we can find a Brownian motion B_t and a stopping time τ defined on a probability space $(\Omega, \mathcal{F})$ such that B_τ has the same distribution as Z. Moreover, $\mathbf{E}[\tau] = \sigma^2$; for all positive integers n,*

$$\mathbf{E}[\tau^n] \leqslant c\,(2/\pi)^{2n}\, n!\, \mathbf{E}[Z^{2n}], \tag{A.2}$$

and for all $\alpha > 0$, if $M = \max\{|B_t| : 0 \leqslant t \leqslant \tau\}$, then

$$\mathbf{E}\left[e^{\alpha M}\right] \leqslant 11\, \mathbf{E}\left[e^{\alpha|Z|}\right].$$

Proof: This is standard; we review the proof with "extra randomness". For ease we will assume that Z takes values in a countable set V that does not include the origin; it is easy to adapt to the case where the origin gets positive probability, We enumerate $V_+ = V \cap [0,\infty) = \{x_j\}, V_- = V \cap (-\infty, 0] = \{y_k\}$ (not necessarily in increasing order), and let

$$p_j = \mathbf{P}\{Z = x_j\}, \quad q_k = \mathbf{P}\{Z = -y_k\}.$$

The assumptions imply

$$\sum_j p_j + \sum_k q_k = 1,$$

and

$$\sum_j p_j\, x_j^2 + \sum_k q_k\, y_k^2 = \sigma^2.$$

Next, set

$$b := \sum_j p_j\, x_j = \sum_k q_k\, y_k$$

and define π on $U := V_+ \times V_-$ by

$$\pi_{jk} := \pi(x_j, y_k) = \frac{p_j\, q_k\, (x_j + y_k)}{b}.$$

Note that

$$\sum_{j,k} \pi_{jk} = \sum_k q_k + \sum_j p_j = 1.$$

Therefore π is a probability measure on U. Add to our probability space an independent U-valued random variable $Q = (X, Y)$ with distribution function $\{\pi_{jk}\}$. Let $\tau = \tau_Q$ where, as above,

$$\tau_{(x,y)} = \inf\{t : B_t = x \text{ or } B_t = -y\}.$$

Note that

$$\mathbf{P}\{B_\tau = x_j\} = \sum_k \pi_{jk}\, \frac{y_k}{x_j + y_k} = b^{-1} \sum_k p_j\, q_k\, y_k = p_j,$$

and similarly, $\mathbf{P}\{B_\tau = -y_k\} = q_k$. Therefore, B_τ has the same distribution as Z. Also,

$$\begin{aligned}\mathbf{E}[\tau] &= \sum_{j,k} \pi_{jk}\, \mathbf{E}[\tau \mid Q = (x_j, y_k)] \\ &= \sum_{j,k} \frac{p_j\, q_k\, (x_j + y_k)}{b}\, x_j\, y_k \\ &= \sum_j p_j\, x_j^2 + \sum_k q_k\, y_k^2 = \sigma^2.\end{aligned}$$

The estimate (A.1) and $x + y \leqslant 2(x \vee y)$ show that

$$\begin{aligned}\frac{(\pi/2)^{2n}}{n!} \mathbf{E}\,[\tau^n] &\leqslant c\, 2^{-2n} \sum_{j,k} \pi_{jk} [x_j \wedge y_k]\, [x_j + y_k]^{2n-1} \\ &\leqslant c\, b^{-1} \sum_{j,k} p_j\, q_k\, [x_j \wedge y_k]\, [x_j \vee y_k]^{2n} \\ &\leqslant c\, b^{-1} \sum_{j,k} p_j\, q_k\, [x_j^{2n}\, y_k + y_k^{2n}\, x_k] \\ &= c \left[\sum_j x_j^{2n}\, p_j + \sum_k y_k^{2n}\, q_k\right] \\ &= c\, \mathbf{E}[Z^{2n}].\end{aligned}$$

Given $Q = (x, y)$, we know that

$$\mathbf{E}[e^{\alpha M}] \leqslant \frac{9}{2}\, e^{\alpha(x \wedge y)} + \frac{2\,(x \wedge y)}{x + y}\, e^{\alpha(x \vee y)}.$$

Hence

$$\mathbf{E}[e^{\alpha M}] \leqslant \sum_{j,k} \pi_{jk} \left[\frac{9}{2}\, e^{\alpha(x_j \wedge y_k)} + \frac{2(x_j \wedge y_k)}{x_j + y_k}\, e^{\alpha(x_j \vee y_k)}\right].$$

We split the sum into

$$\sum_{j,k} = \sum_{x_j \leqslant y_k} + \sum_{x_j > y_k}.$$

If $x_j \leqslant y_k$,

$$\pi_{jk} \left[\frac{9}{2}\, e^{\alpha(x_j \wedge y_k)} + \frac{2\,(x_j \wedge y_k)}{x_j + y_j}\, e^{\alpha(x_j \vee y_k)}\right] \leqslant \frac{p_j\, q_k}{b}\, \left[9\, y_k\, e^{\alpha x_j} + 2\, x_j\, e^{\alpha y_k}\right],$$

$$\sum_{j,k} \frac{p_j\, q_k}{b}\, 9\, y_k\, e^{\alpha x_j} \leqslant 9 \sum_j p_j\, e^{\alpha x_j} = 9\, \mathbf{E}[e^{\alpha Z}; Z > 0],$$

$$\sum_{j,k} \frac{p_j\, q_k}{b}\, 2\, x_j\, e^{\alpha y_k} = 2 \sum_k q_k\, e^{\alpha y_k} = 2\, \mathbf{E}[e^{\alpha |Z|}; Z < 0].$$

Combining this with the $y_k < x_j$ terms we get

$$\mathbf{E}[e^{\alpha M}] \leqslant 11\, \mathbf{E}[e^{\alpha |Z|}].$$

□

We remark that now that we know the distribution, we could do the construction backwards. To be precise, we could start with a realization of the random variable Z; then choose Q from the conditional distribution given Z, which then would define a stopping time τ_Q; and then choose a stopped Brownian motion from the distribution of Brownian motion conditioned so that B_{τ_Q} takes the value Z.

Lemma A.4. *For every positive integer k, there exists $c_k < \infty$ such that the following is true. Suppose*

$$M_n = Z_1 + \cdots + Z_n, \quad M_0 = 0$$

is a martingale with respect to $\{\mathcal{F}_n\}$ and suppose that there exists $C < \infty$ such that for all positive integers n, k,

$$\mathbf{E}[Z_n^{2k} \mid \mathcal{F}_{n-1}] \leqslant C^{2k}.$$

Then for all n,

$$\mathbf{E}[M_n^{2k}] \leqslant c_k^{2k}\, C^{2k}\, n^k.$$

In particular for every $\delta > 0$,

$$\mathbf{P}\left\{\max_{j=1,\ldots,n} |M_j| \geqslant n^{(1+\delta)/2}\right\} \leqslant c_k^{2k}\, C^{2k}\, n^{-k\delta}.$$

Proof: The last assertion follows from the previous by the maximal inequality applied to the submartingale $|M_j|^{2k}$, so we will focus on the bound for $\mathbf{E}[M_n^{2k}]$. Without loss of generality, we will assume that $C = 1$ for otherwise we can consider M_n/C. We will prove the result by induction on k. For $k = 1$, orthogonality of martingale increments gives

$$\mathbf{E}[M_n^2] = \mathbf{E}[Z_1^2] + \cdots + \mathbf{E}[Z_n^2] \leqslant n.$$

We now suppose the result is true for $j \leqslant k$ and assume that $\mathbf{E}[Z_n^{2k+2} \mid \mathcal{F}_{n-1}] \leqslant 1$. By Hölder's inequality, $\mathbf{E}[|Z_n|^r \mid \mathcal{F}_{n-1}] \leqslant 1$ for $0 \leqslant r \leqslant 2k+2$. Also, for $r \leqslant 2k$,

$$\mathbf{E}\left[|M_n|^r\right] \leqslant \mathbf{E}\left[M_n^{2k}\right]^{r/2k} \leqslant c_k^r\, n^{r/2}.$$

The martingale property gives

$$\mathbf{E}\left[M_n^{2k+1}\, Z_{n+1}\right] = \mathbf{E}\left[M_n^{2k+1}\, \mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)\right] = 0.$$

For $2 \leqslant j \leqslant 2k+2$.

$$\begin{aligned}
\left|\mathbf{E}\left[M_n^{2k+2-j}\, Z_{n+1}^j\right]\right| &= \left|\mathbf{E}\left[M_n^{2k+2-j}\, \mathbf{E}(Z_{n+1}^j \mid \mathcal{F}_n)\right]\right| \\
&\leqslant \mathbf{E}\left[|M_n|^{2k+2-j}\, \mathbf{E}(|Z_{n+1}|^j \mid \mathcal{F}_n)\right] \\
&\leqslant \mathbf{E}\left[|M_n|^{2k+2-j}\right] \\
&\leqslant c_{2k}^{2k+2-j}\, n^{(2k+2-j)/2}
\end{aligned}$$

If we write

$$\mathbf{E}[M_{n+1}^{2k+2}] = \mathbf{E}\left[\mathbf{E}((M_n + Z_{n+1})^{2k+2} \mid \mathcal{F}_n)\right],$$

expand the product, and use the relations above, we will see that there exists β_k (which can be found explicitly but we do not need to do so) such that

$$\mathbf{E}[M_{n+1}^{2k+2}] \leqslant \mathbf{E}[M_n^{2k+2}] + \beta_k\, n^k,$$

from which we conclude the result. $\square$

A.2. *Coupling a martingale with Brownian motion.* We will now couple a discrete martingale with a Brownian motion using Skorokhod embedding.

- Suppose
$$M_n = Z_1 + Z_2 + \cdots + Z_n$$
is a square integrable martingale with respect to the filtration $\{\mathcal{F}_n\}$.
- Let $Q_1, Q_2, \ldots$ be U-valued random variables where $Q_n = (X_n, Y_n)$ is obtained as follows. The distribution of Q_n given $\mathcal{F}_{n-1}$ is that of the Q in the proof of Lemma A.3 where we choose Z to have the conditional distribution of Z_n given $\mathcal{F}_{n-1}$. Since the random variables Z_n have discrete distributions there are no technical issues in defining this conditional expectation. We use independent randomness to find a realization of Q_n given $\mathcal{F}_{n-1}$ and Z_n.
- Given Q_n we take an independent Brownian motion $B_t^{(n)}$ stopped at time τ_{Q_n} conditioned so that $B^{(n)}_{\tau_{Q_n}} = Z_n$.
- Define
$$\tau_n = \tau_{n-1} + \tau_{Q_n}$$
and
$$B_t = B_{\tau_{n-1}} + B^{(n)}_{t-\tau_{n-1}}, \quad \tau_{n-1} \leqslant t \leqslant \tau_n.$$
- Let $\mathcal{G}_n$ be the smallest σ-algebra containing $\mathcal{F}_n$ and such that $Q_1, \ldots, Q_n$ and $B_t, 0 \leqslant t \leqslant \tau_n$ are measurable.
- Recall from Lemma A.3 that
$$\mathbf{E}\left[\tau_{Q_n} \mid \mathcal{F}_{n-1}\right] = \mathbf{E}\left[Z_n^2 \mid \mathcal{F}_{n-1}\right].$$
Let
$$\Delta_n = \tau_{Q_n} - \mathbf{E}\left[\tau_{Q_n} \mid \mathcal{F}_{n-1}\right] = \tau_{Q_n} - \mathbf{E}\left[Z_n^2 \mid \mathcal{F}_{n-1}\right],$$
$$J_n = \sum_{j=1}^n \Delta_j, \quad V_n = \sum_{j=1}^n \mathbf{E}\left[Z_n^2 \mid \mathcal{F}_{n-1}\right].$$
Note that $\tau_n = V_n + J_n$ and J_n is a martingale with respect to $\{\mathcal{G}_n\}$.

This has the same distribution as the following.

- Given $\mathcal{G}_{n-1}$ choose Q_n using the appropriate conditional distribution and independent randomness. This gives τ_{Q_n}. Let $\tau_n = \tau_{n-1} + \tau_{Q_n}$.
- Take an independent Brownian motion and observe $B_t^{(n)}, 0 \leqslant t \leqslant \tau_{Q_n}$.
- Set $Z_n = B_{\tau_{Q_n}}$
- Set $B_t = B_{\tau_{n-1}} + B^{(n)}_{t-\tau_{n-1}}, \ \ \tau_{n-1} \leqslant t \leqslant \tau_n$.

A.3. *Application.* We will make the following moment assumption, keeping notation from the previous subsection.

- **Moment Assumption** There exist $\theta < \infty$ such that for all positive integers k, n,
$$\mathbf{E}\left[Z_n^{2k} \mid \mathcal{F}_{n-1}\right] \leqslant (\pi\theta/2)^{2k} (2k)!. \tag{A.3}$$

This is implied by a stronger assumption about exponential moments, given here:

- **Exponential Moment Assumption.** There exists $t > 0$ and $C' < \infty$ such that for all positive integers n,
$$\mathbf{E}\left[\exp\{t|Z_n|\} \mid \mathcal{F}_{n-1}\right] \leqslant C'.$$

Under the moment assumption (A.3) and (A.2), we can see that

$$\mathbf{E}\left[\Delta_n^{2k}\right] \leqslant c\,\theta^{4k}\,(2k)!\,(4k)!.$$

Using Lemma A.4, we see that

$$\mathbf{E}\left[J_n^{2k}\right] \leqslant \rho_{2k}\,n^k, \quad \rho_{2k} = c_k^{2k}\,c\,\theta^{4k}\,(2k)!\,(4k)!,$$

and hence for every $\delta > 0$,

$$\mathbf{P}\left\{\max_{1\leqslant j\leqslant n} |J_n| \geqslant n^{(1+\delta)/2}\right\} \leqslant \rho_{2k}\,n^{-k\delta}.$$

Theorem A.1. Suppose

$$M_n = Z_1 + Z_2 + \cdots + Z_n$$

is a square integrable martingale with respect to the filtration $\{\mathcal{F}_n\}$ satisfying the moment assumption (A.3). Suppose also that there exists c_1 such that for all n, with probability one,

$$|\mathbf{E}[Z_n^2 \mid \mathcal{F}_{n-1}] - 1| \leqslant c_1\,n^{-1/2}. \tag{A.4}$$

Then we can define the martingale and a Brownian motion B_t on the same probability space such that the following holds. For every $\varepsilon > 0, K < \infty$, there exists c depending only on ε, c_1 and the constants in (A.3) such that except for an event of probability cn^{-K},

$$\max_{0\leqslant j\leqslant n} |B_j - M_j| \leqslant n^{\frac{1}{4}+\varepsilon}.$$

Proof: Under the construction $M_j = B_{\tau_j}$. The modulus of continuity estimate for Brownian motion implies that, except for an event whose probability decays faster than every power of n, if $s, t \leqslant 2n$ with $|s-t| \leqslant 2n^{\frac{1}{2}+\varepsilon}$,

$$|B_s - B_t| \leqslant n^{\frac{1}{4}+\varepsilon}.$$

Hence, it suffices to show that

$$\mathbf{P}\left\{\max_{0\leqslant j\leqslant n} |\tau_j - j| \geqslant 2n^{\frac{1}{2}+\varepsilon}\right\} \leqslant c\,n^{-K},$$

and given (A.4), it suffices to show that

$$\mathbf{P}\left\{\max_{0\leqslant j\leqslant n} |J_j| \geqslant n^{\frac{1}{2}+\varepsilon}\right\} \leqslant c\,n^{-K}.$$

This follows from (A.3) and Lemma A.4.

□

The last proof only used the fact that for each k, the conditional $(2k)$th moment was uniformly bounded. It did not need the stronger form in (A.4).

We will consider one more assumption. Let us assume that we have a martingale M_n as above with respect to $\{\mathcal{F}_n\}$ satisfying (A.4). Let

$$M_i^{(j)} = M_{(j-1)m+i} - M_{(j-1)m}, \quad i = 0, \ldots, m.$$

Using the last lemma we can see that for each j, we can find a Brownian motion $B_t^{(j)}$ such that, except for an event of probability at most $c\,m^{-K}$,

$$\max_{0\leqslant i\leqslant m} |M_i^{(j)} - B_i^{(j)}| \leqslant m^{\frac{1}{4}+\varepsilon}.$$

The Brownian motion can be combined into a single Brownian motion,

$$B_t = B_t^{(1)}, \quad 0 \leqslant t \leqslant m,$$

$$B_t = B_{(j-1)m} + B_{t-(j-1)m}^{(j)}, \quad (j-1)m \leqslant t \leqslant jm.$$

Let $\mathcal{G}_n$ denote the σ-algebra generated by $\mathcal{F}_n$ and $\{B_t : 0 \leqslant t \leqslant n\}$. The construction has the following property:

- The conditional distribution of $M_{jm+1,jm+2,\ldots}$ given $\mathcal{G}_{jm}$ is the same as the conditional distribution of given $\mathcal{F}_{jm}$. Indeed the extra information added to $\mathcal{F}_{jm}$ to get $\mathcal{G}_{jm}$ is all randomness independent of the martingale.
- The Brownian motion $\tilde{B}_t = B_{t+jm} - B_{jm}$ is independent of $\mathcal{G}_{jm}$.
- Except for an event of probability at most cjm^{-K}, we have

$$\max_{0\leqslant i\leqslant jm} |M_i - B_i| \leqslant j\, m^{\frac{1}{4}+\varepsilon}.$$

- In particular if $n = jm \sim m^{1+\delta}$ with $\delta \leqslant 1/2$, then, except for an event of probability at most $cn^{-K'}$,

$$\max_{0\leqslant i\leqslant n} |M_i - B_i| \leqslant m^{\frac{1}{4}+\varepsilon+\delta} = n^{\frac{1}{4}+u}, \quad u = \frac{\varepsilon + \frac{3}{4}\delta}{1+\delta}.$$

Theorem A.2. Suppose

$$M_n = Z_1 + Z_2 + \cdots + Z_n$$

is a square integrable discrete martingale with respect to the filtration $\{\mathcal{F}_n\}$ satisfying (A.3) and (A.4). For every $0 < \delta \leqslant 1/2$ and $K < \infty$, there exists c depending only on δ, K and the constants in (A.3) and (A.4) such that the following holds. We can define the martingale and a Brownian motion B_t on the same probability space such that except for an event of probability cn^{-K},

$$\max_{0\leqslant j\leqslant n} |B_j - M_j| \leqslant n^{\frac{1}{4}+\delta}.$$

Moreover, if $m = \lfloor n^{1/(1+\delta)} \rfloor$, then for each $j \leqslant n^{\delta/(1+\delta)}$, $\{B_{t+jm} - B_{jm} : t \geqslant 0\}$ is independent of the σ-algebra generated by $\{M_k : k \leqslant jm\}$ and $\{B_t : t \leqslant jm\}$.

Proof: We will actually show a little more than we state.

Let

$$M_i^{(j)} = M_{(j-1)m+i} - M_{(j-1)m}, \quad i = 0, \ldots, m.$$

Using the last lemma we can see that for each j, we can find a Brownian motion $B_t^{(j)}$ such that, except for an event of probability at most $c\, m^{-K}$,

$$\max_{0\leqslant i\leqslant m} |M_i^{(j)} - B_i^{(j)}| \leqslant m^{\frac{1}{4}+\varepsilon}.$$

The Brownian motion can be combined into a single Brownian motion,

$$B_t = B_t^{(1)}, \quad 0 \leqslant t \leqslant m,$$

$$B_t = B_{(j-1)m} + B_{t-(j-1)m}^{(j)}, \quad (j-1)m \leqslant t \leqslant jm.$$

Let $\mathcal{G}_n$ denote the σ-algebra generated by $\mathcal{F}_n$ and $\{B_t : 0 \leqslant t \leqslant n\}$. The construction has the following property:

- The conditional distribution of $M_{jm+1,jm+2,\ldots}$ given $\mathcal{G}_{jm}$ is the same as the conditional distribution of given $\mathcal{F}_{jm}$. Indeed the extra information added to $\mathcal{F}_{jm}$ to get $\mathcal{G}_{jm}$ is all randomness independent of the martingale.
- The Brownian motion $\tilde{B}_t = B_{t+jm} - B_{jm}$ is independent of $\mathcal{G}_{jm}$.
- Except for an event of probability at most cjm^{-K}, we have

$$\max_{0\leqslant i\leqslant jm} |M_i - B_i| \leqslant j\, m^{\frac{1}{4}+\varepsilon}.$$

- In particular if $n = jm \sim m^{1+\delta}$ with $\delta \leqslant 1/2$, then, except for an event of probability at most $cn^{-K'}$,

$$\max_{0 \leqslant i \leqslant n} |M_i - B_i| \leqslant m^{\frac{1}{4}+\varepsilon+\delta} = n^{\frac{1}{4}+u}, \quad u = \frac{\varepsilon + \frac{3}{4}\delta}{1+\delta}.$$

□

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