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On the chemical distance exponent for the two-sided level set of the two-dimensional Gaussian free field

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Abstract. In this paper we study the two-sided level set of the two-dimensional discrete Gaussian free field (GFF), where a site is open if the absolute value of the GFF at this site is at most λ for a fixed parameter $\lambda > 0$. For the GFF on a box of size N with Dirichlet boundary conditions, we show that there exists $\epsilon > 0$ such that with probability tending to 1 as $N \to \infty$, all the open paths whose Euclidean diameters are of order N have lengths larger than $N^{1+\epsilon}$.

1. Introduction

The discrete Gaussian free field (GFF) on $\mathbb{Z}^d$ is a Gaussian random field with mean zero and covariance given by the Green function of the simple random walk. For $d \geq 3$, the (one-sided) level set (consisting of open sites, i.e., sites with GFF values exceeding h) percolation of the GFF has been extensively studied, and it has been shown that a non-trivial phase transition exists. More precisely, there exists a critical level $h_*(d) \in (0, \infty)$ such that the level set has a unique infinite open cluster for $h < h_*(d)$ and has only finite open clusters for $h > h_*(d)$ (see Bricmont et al., 1987; Rodriguez and Sznitman, 2013; Drewitz and Rodriguez, 2015; Drewitz et al., 2018 for a non-exhaustive list of references).

Since the Green function blows up in two dimensions, we instead consider the GFF on a box $V_N \subset \mathbb{Z}^2$ of side length $2N$. As an analogue of the existence of an infinite open cluster in higher dimensions, one can investigate the macroscopic connectivity property for the level set on V_N. It was shown in Ding and Li (2018) that for any level h, there are open paths connecting the boundaries of a macroscopic annulus (i.e., the two boundaries are at distance of order N) with non-vanishing probability as $N \to \infty$. Another interesting yet challenging question for percolation models is on the chemical distance, the intrinsic distance on the graph defined by open clusters. For many percolation models, physicists predicted that there exists an exponent $d_{\min}$ (called the chemical distance exponent) such that the chemical distance is comparable to the Euclidean distance

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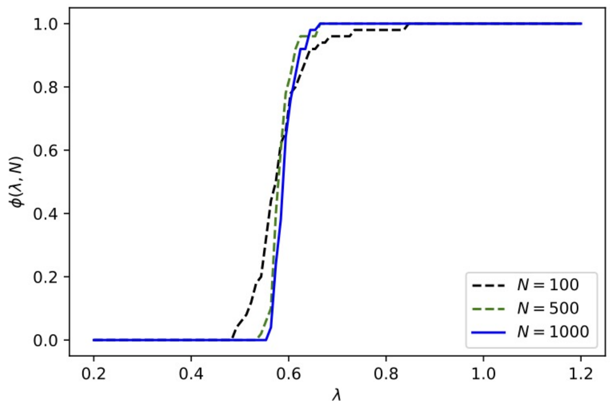


FIGURE 1.1. $\phi(\lambda, N)$ is the probability of existence of λ-open paths in $V_{N/2}$ w.r.t. the GFF on V_N connecting the left and the right boundaries of $V_{N/2}$.

to the power of $d_{\min}$ (see Havlin and Nossal, 1984; Herrmann and Stanley, 1988; Schrenk et al., 2013). Such chemical distance exponents have been studied in many percolation models, including the Bernoulli percolation models (see Aizenman and Burchard, 1999; Damron et al., 2017, 2021; Antal and Pisztora, 1996; Garet and Marchand, 2007) and the supercritical percolation models with long-range correlations on $\mathbb{Z}^d$ for $d \geq 3$ (see Drewitz et al., 2014). As for the level set of the two-dimensional GFF, it was shown in Ding and Li (2018); Ding and Wirth (2020) that the geodesics under the chemical distance have lengths $O(N(\log N)^{1/4})$, implying that the chemical distance exponent is equal to 1.

In this paper, we investigate the two-sided level set of the two-dimensional GFF, where a site is open if its GFF value is in $[-\lambda, \lambda]$. While it remains a challenge to show that for some fixed large λ with non-vanishing probability as $N \to \infty$ there exists a macroscopic path (i.e, a path with two ends at Euclidean distance of order N) in the two-sided level set, we believe that this is true by the simulation results[1] (see Figure 1.1). With this belief, the question on the chemical distance is meaningful and interesting, and we will show (see Theorem 1.1 below) that with probability tending to 1 as $N \to \infty$, open macroscopic paths, shall they exist, have lengths larger than $N^{1+\epsilon}$ for some $\epsilon > 0$. Therefore, the chemical distance exponent is strictly larger than 1, which is drastically different from that of the (one-sided) level set.

Next, we will define our model more precisely, and then state our main result. For each positive integer N, let V_N be the box of side length $2N$ centered at the origin, i.e., $V_N := [-N, N]^2 \cap \mathbb{Z}^2$. Denote by $\{\eta^{V_N}(x) : x \in V_N\}$ the GFF on V_N with Dirichlet boundary conditions. Concretely, it is a mean-zero Gaussian process with covariance given by

$$\mathbb{E}\eta^{V_N}(x)\eta^{V_N}(y) = G_N(x, y) \quad \text{for } x, y \in V_N,$$

where $G_N(x, y) := E_x \sum_{n=0}^{\tau-1} 1_{\{S_n = y\}}$ is the Green function of the 2D simple random walk $\{S_n : n = 0, 1, 2, \cdots\}$, and τ is the hitting time to the boundary $\partial V_N := \{x \in V_N : \text{there exists } y \in \mathbb{Z}^2 \setminus V_N \text{ such that } y \text{ is a neighbor of } x\}$. Let λ be positive and fixed. A site $x \in V_N$ is called *λ-open* if $|\eta^{V_N}(x)| \leq \lambda$. Define the two-sided level set $\Lambda_{\lambda,N}$ as the collection of λ-open sites in

[1] The GFF on V_N is numerically generated by using the codes in `https://github.com/sswatson/GaussianFreeFields.jl`. Then, we use the built-in function `measurements.label` in python to compute $\phi(\lambda, N)$. For each $\lambda = 0.2, 0.21, \cdots, 1.2$ and each $N = 100, 500, 1000$, the experiments are conducted 50 times to find $\phi(\lambda, N)$, respectively.

$V_{N/2} := [-N/2, N/2]^2 \cap \mathbb{Z}^2$, that is

$$\Lambda_{\lambda,N} := \{x \in V_{N/2} : |\eta^{V_N}(x)| \leq \lambda\}. \tag{1.1}$$

Suppose that P is a path in $\mathbb{Z}^2$, i.e., a sequence of sites $z_0, z_1, \cdots, z_n$ with $\|z_i - z_{i-1}\| = 1$ for $i = 1, \cdots, n$, where $\|\cdot\|$ denotes the Euclidean distance. We say that P is λ-open if so are all z_i's. Denote by $|P| := n$ the length of P, by $\|P\| := \|z_n - z_0\|$ the distance between the endpoints of P. For $\kappa, \epsilon > 0$, let $\mathcal{P}_N^{\kappa,\epsilon}$ be the family of macroscopic paths in $V_{N/2}$ with lengths at most $N^{1+\epsilon}$, i.e.,

$$\mathcal{P}_N^{\kappa,\epsilon} = \big\{P : P \text{ is a path in } V_{N/2} \text{ satisfying } \|P\| \geq \kappa N \text{ and } |P| \leq N^{1+\epsilon}\big\}. \tag{1.2}$$

Our main result is the following theorem.

Theorem 1.1. *For each $\lambda > 0$, there exists $\epsilon = \epsilon(\lambda) > 0$ such that for every $\kappa \in (0, 1)$,*

$$\lim_{N\to\infty} \mathbb{P}\big(P \textit{ is } \lambda\textit{-open for some } P \in \mathcal{P}_N^{\kappa,\epsilon}\big) = 0.$$

Remark 1.2. Actually, we have obtained the following more quantitative result (see Proposition 4.4), which in particular implies that all open macroscopic paths have super-linear lengths even when $\lambda = \lambda_N$ grows in N as long as $\lambda_N \leq c\sqrt{\log\log N}$ for some constant $c > 0$. It remains an interesting question to determine the tight bound for λ_N at which all macroscopic λ_N-open paths have super-linear lengths.

Remark 1.3. Theorem 1.1 also holds with $V_{N/2}$ being replaced with $V_{\delta N} = [-\delta N, \delta N]^2 \cap \mathbb{Z}^2$ for $\delta \in (0, 1)$.

1.1. *Notation conventions.* For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, denote

$$\|x - y\| := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \quad\text{and}\quad |x - y|_\infty := |x_1 - y_1| \vee |x_2 - y_2|,$$

where $a \vee b := \max\{a, b\}$. Denote $d(x, B) := \inf_{y\in B} \|x - y\|$, $d(B_1, B_2) := \inf_{x\in B_1} d(x, B_2)$, $d_\infty(x, B) := \inf_{y\in B} |x - y|_\infty$ and $d_\infty(B_1, B_2) := \inf_{x\in B_1} d_\infty(x, B_2)$.

A site is a point in $\mathbb{Z}^2$. In this paper, only boxes with sides parallel to the axes are involved. If B is a box in $\mathbb{R}^2$ and $B \cap \mathbb{Z}^2 \neq \emptyset$, we denote by z_B the lower left corner of $B \cap \mathbb{Z}^2$. For $x \in \mathbb{R}^2$ and $\ell > 0$, denote by $B(x, \ell)$ the ball of radius ℓ centered at x, and by $B_\infty(x, \ell)$ the box in $\mathbb{R}^2$ of side length 2ℓ centered at x, i.e.,

$$B(x, \ell) := \{y \in \mathbb{R}^2 : \|x - y\| \leq \ell\}, \quad B_\infty(x, \ell) := \{y \in \mathbb{R}^2 : |x - y|_\infty \leq \ell\}.$$

Denote by $V_\ell(x)$ the set of sites in $B_\infty(x, \ell/2)$, i.e.,

$$V_\ell(x) := B_\infty(x, \ell/2) \cap \mathbb{Z}^2,$$

which is a box in $\mathbb{Z}^2$ of side length at most ℓ.

Suppose that P is a path (in $\mathbb{Z}^2$), i.e., a sequence of sites $z_0, z_1, \cdots, z_n$ with $\|z_i - z_{i-1}\| = 1$, $i = 1, \cdots, n$ such that for $1 \leq i \leq n - 1$, $z_i \neq z_j$ for all $j \neq i$. Denote the endpoints, the end-to-end distance and the length of P respectively by

$$x_P := z_0, \quad y_P := z_n; \quad \|P\| := \|z_n - z_0\|, \quad |P| := n.$$

By sub-path of P, we mean a path with sites on P.

Let λ be positive and fixed. We will use a dyadic number $K = 2^k$ (for some positive integer k) as the scale parameter, which will be eventually chosen to depend on λ. Let κ be a fixed number in $(0, 1)$. For $a \geq 1$, let $\lfloor a \rfloor$ be the greatest integer less equal to a, and denote $[a] := \{1, \cdots, \lfloor a \rfloor\}$. Let $m = \lfloor \frac{1}{k} \log_2(\kappa N) \rfloor - 1$, which is the unique positive integer such that

$$K^{m+1} \leq \kappa N < K^{m+2}. \tag{1.3}$$

We will let $N \to \infty$, equivalently, $m \to \infty$. Let c, c', c'', $C_1, C_2, \cdots$ be positive universal constants. Furthermore, let C be a specific positive universal constant such that all the results in Section 2.2 hold for $K \geq C$.

1.2. *Outline of the proof.* Our proof strategy is based on a multi-scale analysis, which is a classic and powerful method in percolation theory; see for instance Chayes (1996); Orzechowski (1998); Sznitman (2015); Ding and Zhang (2019). As a typical instance when applying multi-scale analysis, the actual implementation is quite non-trivial and the main challenge is to design a proper induction procedure. We will next present a brief outline of our induction procedure by describing the content in each of the subsequent sections.

In Section 2, after reviewing some basic and standard facts about the GFF (see Section 2.1), we will review the following tree structure associated to a path as constructed in Ding and Zhang (2019) (see Section 2.2). For $j \geq 1$ and a path P in scale K^j (i.e., $\|P\|$ is comparable to K^j), a tree $\mathcal{T}_P$ of depth j is constructed in association with P, where the root corresponds to the path P and nodes at level r (for $r \geq 1$) in $\mathcal{T}_P$ correspond to disjoint sub-paths of P in scale K^{j-r}. In addition, the parent/child relation of nodes in $\mathcal{T}_P$ corresponds to path/child-path relation (see Figure 2.2). In each scale, we will consider the notion of tame/untamed paths, where roughly tame paths are like straight lines and untamed paths are like curves (see Definition 2.9). Then, a crucial property is that for all $P \in \mathcal{P}_N^{\kappa,\epsilon}$ (recall (1.2) for its definition), the untamed nodes, i.e., nodes correspond to untamed sub-paths, are rare in $\mathcal{T}_P$ (see Lemma 2.10).

In Section 3, we will deal with the initial step of the induction. We first bound the probability of the existence of an open crossing through a parallelogram (with aspect ratio 16) in the long direction, by relating such a crossing to a contour separating boundaries of an annulus (see Figure 3.4) via a standard path gluing argument. Since a tame path crosses through a parallelogram with aspect ratio of order K (from which we can extract order $\sqrt{K}$ well-separated parallelograms with aspect ratio 16), the probability of the existence of a tame and open path in any scale is at most $e^{-0.01\sqrt{K}}$ (see Proposition 3.2).

In Section 4, we will implement the induction analysis (analogous to Lemma 4.4 in Ding and Zhang, 2019). For $r \geq 0$, let $Y_{P,r}$ be the fraction of tame and open nodes (i.e., nodes corresponding to tame and open sub-paths) at level r in $\mathcal{T}_P$. For $r = 0$, $Y_{P,r}$ can be controlled by the aforementioned $e^{-0.01\sqrt{K}}$ decay in Proposition 3.2. For $r \geq 1$, let $P^{(i)}$'s be the child-paths of P, and we have the recursive relation that $Y_{P,r}$ is the average of $Y_{P^{(i)},r-1}$'s (see (4.15)). Assuming good control for $Y_{P^{(i)},r-1}$, we can then apply a large deviation analysis and recursively obtain good control for $Y_{P,r}$. One subtlety is that, our ultimate goal is to obtain a uniform control for all $P \in \mathcal{P}_N^{\kappa,\epsilon}$, and as a result in the preceding recursive analysis we need to apply a union bound in every step, which can be absorbed into our probability decay obtained from large deviation analysis (see Proposition 4.2 and Proposition 4.3, noting that ξ's therein bound Y's by (4.2) and (4.3)). Combined with Lemma 2.10, it yields that with high probability, for all $P \in \mathcal{P}_N^{\kappa,\epsilon}$, there exists a non-open sub-path of P (correspondingly, a non-open node in $\mathcal{T}_P$). This implies Theorem 1.1. In the actual proof, there are additional technical complications such as the fluctuations of the harmonic functions (defined in Lemma 2.2) that emerge in every step of the induction, which are assumed to be flat in the preceding discussion. In order to address this, we will employ standard regularity estimates for GFF with some carefully chosen parameters (see (4.4) and (4.5)).

2. Preliminaries

In Section 2.1, we will give some basic facts about the GFF. In Section 2.2, we will introduce some notations and results in Ding and Zhang (2019), which are needed in this paper. In Section 2.3, we will define some families of paths.

2.1. *Properties of the GFF.* Let $B \subseteq \mathbb{Z}^2$ be finite and non-empty. Denote by $\{\eta^B(x) : x \in B\}$ the GFF on B with Dirichlet boundary conditions, i.e., a mean-zero Gaussian process with covariance given by

$$\mathbb{E}\eta^B(x)\eta^B(y) = G_B(x,y) \quad \text{for } x, y \in B,$$

where $G_B(x,y) := E_x \sum_{n=0}^{\tau-1} 1_{\{S_n=y\}}$ is the Green function of the 2D simple random walk $\{S_n : n = 0,1,2,\cdots\}$, and τ is the hitting time to the boundary $\partial B = \{x \in B : \|x-y\| = 1 \text{ for some } y \in \mathbb{Z}^2 \setminus B\}$. Without loss of generality, $\eta^B(x) := 0$ for all $x \in \mathbb{Z}^2 \setminus B$.

Fix $\chi = \frac{1}{10}$. If $B \subseteq \mathbb{Z}^2$ is a box of side length L, define the box

$$B^\chi := \{z \in B : d_\infty(z, \partial B) > \chi L\}.$$

Lemma 2.1 (Equation (4) in Ding and Zhang, 2019)**.** *Suppose that $B \subseteq \mathbb{Z}^2$ is a box of side length L. There is a universal constant $C_1 > 0$ such that*

$$\Big|\mathbb{E}\big(\eta^B(x)\eta^B(y)\big) - \frac{2}{\pi}\log\frac{L}{|x-y|_\infty \vee 1}\Big| \le C_1 \quad \textit{for all } x, y \in B^\chi.$$

Lemma 2.2 (Markov Property)**.** *Let D be a finite subset of $\mathbb{Z}^2$, and $B \subseteq D$. Let η^D be the GFF on D. Denote*

$$H^B(x) := \mathbb{E}\big(\eta^D(x) \bigm| \sigma\{\eta^D(z) : z \in B^c \cup \partial B\}\big).$$

Then H^B is harmonic on B, with $H^B|_{\partial B} = \eta^D|_{\partial B}$. Moreover, the field

$$\eta^B := \eta^D - H^B$$

is a version of the GFF on B, which is independent of H^B. In other words, $\eta^D = \eta^B \oplus H^B$ is an orthogonal decomposition.

A proof of the above lemma can be found in the proof of Lemma 3.1 in Biskup (2020), and the following lemma is a direct consequence of Lemma 3.10 in Bramson et al. (2016).

Lemma 2.3. *In addition to the assumptions in Lemma 2.2, assume furthermore that B is a box of side length L. Then,*

$$\mathbb{E}\big(H^B(x) - H^B(y)\big)^2 \le C_2 \frac{|x-y|_\infty}{L} \qquad \textit{for all } x, y \in B^\chi,$$

where $C_2 > 0$ is a universal constant.

The next two results are about general Gaussian fields (i.e., not necessarily GFFs), which will be used to prove Lemma 2.6.

Lemma 2.4 (Dudley's inequality, Lemma 4.1 in Adler, 1990)**.** *Let $U \subseteq \mathbb{Z}^2$ be a box of side length ℓ and $\{G_w : w \in U\}$ be a mean zero Gaussian field satisfying*

$$\mathbb{E}(G_x - G_y)^2 \le |x-y|_\infty/\ell \qquad \textit{for all } x, y \in U.$$

Then $\mathbb{E}\max_{x\in U} G_x \le C_3$, where $C_3 > 0$ is a universal constant.

Lemma 2.5 (Borell-Tsirelson inequality, Lemma 7.1 in Talagrand, 2014)**.** *Let $\{G_x : x \in A\}$ be a centered Gaussian field on a finite index set A. Then,*

$$\mathbb{P}\Big(\Big|\max_{x\in A} G_x - \mathbb{E}\max_{x\in A} G_x\Big| \ge a\Big) \le 2e^{-\frac{a^2}{2\sigma^2}} \qquad \textit{for all } a > 0,$$

where $\sigma^2 = \max_{x\in A} \mathrm{Var}(G_x)$.

Lemma 2.6. *In addition to the assumptions in Lemma 2.3, assume furthermore that U is a box of side length ℓ in B^χ. Then, there exists a universal constant $C_4 > 0$ such that the following holds. Suppose $\varepsilon \ge C_4\sqrt{\frac{\ell}{L}}$. Then, for all $z \in U$,*

$$\mathbb{P}\Big(\big|H^B(x) - H^B(z)\big| \ge \varepsilon \textit{ for some } x \in U\Big) \le 4\exp\Big\{-\frac{\varepsilon^2 L}{8C_2\ell}\Big\}.$$

Proof: Note that $U \subseteq B^\chi \subseteq B \subseteq D$. Fix $z \in U$. For $x \in U$, denote $G_x := H^B(x) - H^B(z)$. By Lemma 2.3,

$$\mathbb{E}(G_x - G_y)^2 \le C_2 \frac{|y-x|_\infty}{L} \le \frac{C_2 \ell}{L} \cdot \frac{|y-x|_\infty}{\ell} \quad \text{for all } x, y \in U.$$

Combined with Lemma 2.4, it yields that $\mathbb{E}\max_{x\in U} G_x \le \frac{C_4}{2}\sqrt{\frac{\ell}{L}}$, where $C_4 := 2C_3\sqrt{C_2}$. Consequently, by the symmetry of the Gaussian distribution,

$$\mathbb{P}\big(|G_x| \ge \varepsilon \text{ for some } x \in U\big) \le 2\mathbb{P}\big(G_x \ge \varepsilon \text{ for some } x \in U\big)$$
$$\le \quad 2\mathbb{P}\Big(\max_{x\in U} G_x - \mathbb{E}\max_{x\in U} G_x \ge \frac{\varepsilon}{2}\Big) \le 4\exp\Big\{-\frac{\varepsilon^2}{8\sigma^2}\Big\},$$

where $\sigma^2 = \max_{x\in U} \mathrm{Var}(G_x)$, and in the second inequality we have used the assumption $\varepsilon \ge C_4\sqrt{\frac{\ell}{L}}$. Furthermore, by Lemma 2.3, $\sigma^2 \le \frac{C_2\ell}{L}$. Combining it with the above formula, we complete the proof of Lemma 2.6. □

The proof of the following lemma is similar to that of Lemma 1 in Ding and Li (2018), thus is omitted.

Lemma 2.7. *Let $\ell_1 > 0$, $\ell_2 \ge \ell_1 + 2$ and $z \in \mathbb{Z}^2$. Suppose $V_{\ell_1}(z) \subseteq D \subseteq V_{\ell_2}(z)$. Then, for all $x \in \partial V_{\ell_1}(z)$,*

$$\sum_{y\in\partial V_{\ell_1}(z)} G_D(x,y) \le \sum_{y\in\partial V_{\ell_1}(z)} G_{V_{\ell_2}(z)}(x,y) \le 2(\ell_2 - \ell_1).$$

2.2. *The tree structure associated with a path.* In this section, we will introduce the concepts of child-paths and associated trees constructed in Section 3 of Ding and Zhang (2019). All is about the geometric structure of a deterministic path, where probability is not involved.

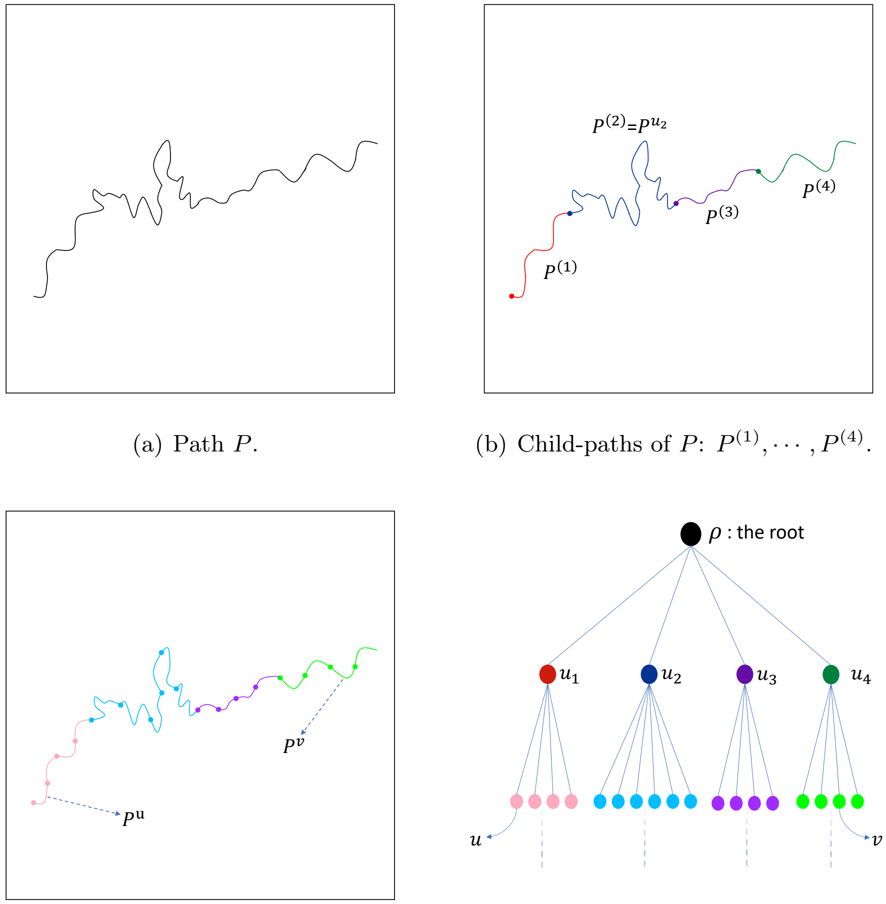
Suppose that P is a path from x to y. Recall $\|P\| := \|x-y\|$ and $x_P := x$. Recall that $B(x,\ell)$ is a ball centered at x and of radius ℓ. For $j \ge 1$, let $\mathcal{SL}_j$ be a family of paths, defined as follows:

$$\mathcal{SL}_j := \big\{P : 1 \le \|P\|/K^j \le 1 + 1/K \text{ and } P \subseteq B(x_P, \|P\|)\big\}. \tag{2.1}$$

Denote $\mathcal{SL}_0 := \mathbb{Z}^2$, where a site is regarded as a path with length 0. A path P is said to be in scale K^j if $P \in \mathcal{SL}_j$.

Recall that κ is positive and fixed, and that m is the unique positive integer such that $K^{m+1} \le \kappa N < K^{m+2}$. In this construction, we will only need to consider $\mathcal{SL}_j$ for $j \le m-1$. For each path P in $\mathcal{SL}_j$ (for $j \in [m-1]$) or P with $\|P\| \ge \kappa N$, in Section 3 of Ding and Zhang (2019) the authors extracted a collection of disjoint sub-paths $\{P^{(i)} : 1 \le i \le d_P\}$ for some $d_P \ge 1$, and these sub-paths are referred to as *child-paths* of P. Note that the child-paths of P are in $\mathcal{SL}_{m-1}$ if $\|P\| \ge \kappa N$; and are in $\mathcal{SL}_{j-1}$ if $P \in \mathcal{SL}_j$.

With definitions above at hand, we are now ready to associate each path P in scale K^j with a tree $\mathcal{T}_P$ of depth j in a recursive manner for $j = 1, \ldots, m-1$ (see Figure 2.2). As the base construction for $P \in \mathcal{SL}_1$, we identify P as the root and identify each child-path of P as a leaf, yielding a tree of depth 1 which is the desired $\mathcal{T}_P$. For $2 \le j \le m-1$, we suppose that each $Q \in \mathcal{SL}_{j-1}$ has been associated with a tree $\mathcal{T}_Q$ of depth $j-1$, and we consider $P \in \mathcal{SL}_j$. Let P be identified as the root, denoted by ρ, and the child-paths $P^{(i)}$'s be respectively identified as children of ρ, which are denoted by u_i's. By attaching the root of $\mathcal{T}_{P^{(i)}}$ to the node u_i, one obtains the tree $\mathcal{T}_P$ of depth j. Note that each node u in $\mathcal{T}_P$ is identified as a sub-path of P, which is denoted by P^u. Furthermore, if u is at level r in $\mathcal{T}_P$, P^u is in scale K^{j-r}. Similarly, a path P with $\|P\| \ge \kappa N$ is associated with a tree $\mathcal{T}_P$ of depth m, noting that the child-paths $P^{(i)}$'s are in scale K^{m-1} and thus are associated with trees $\mathcal{T}_{P^{(i)}}$'s of depth $m-1$.



(a) Path P.

(b) Child-paths of P: $P^{(1)}, \cdots, P^{(4)}$.

(c) Child-paths of $P^{(i)}$'s.

(d) The tree $\mathcal{T}_P$ associated with P.

FIGURE 2.2. $K = 4$. The root ρ is identified as the path P, say in scale K^j. The nodes u_i's correspond to child-paths $P^{(i)}$'s of P, shown in (b). u and v are two specific nodes at level 2, with P^u and P^v (in scale K^{j-2}) being pointed out in (c).

For integer $r \geq 0$, define a collection of boxes of side length K^r as follows:

$$\mathcal{BD}_r := \Big\{ \Big[aK^r - 1/2, (a+1)K^r - 1/2 \Big] \times \Big[bK^r - 1/2, (b+1)K^r - 1/2 \Big] : a, b \in \mathbb{Z} \Big\}.$$

Note that the boundaries of boxes in $\mathcal{BD}_r$ do not intersect with $\mathbb{Z}^2$. Thus, $\{(B \cap \mathbb{Z}^2) : B \in \mathcal{BD}_r\}$ forms a partition of $\mathbb{Z}^2$.

Proposition 2.8 (Proposition 3.1 in Ding and Zhang, 2019)**.** *There is a universal constant C such that the following holds for all $K \geq C$, and $j = 2, \cdots, m-1$. For $P \in \mathcal{SL}_j$, one has $d_P \geq K$, and that each box in $\mathcal{BD}_{j-1}$ is visited by at most 12 child-paths of P.*

Suppose $j \geq 1$, and P is a path from x_P to y_P in scale K^j. Let $E(P)$ be an ellipse, whose focuses are x_P and y_P, and whose ratio of width to height is K to 1. Let $\widetilde{E}(P)$ be a modification of $E(P)$, which we need for technical reasons (that will become clearer later). Concretely,

$$E(P) := \big\{ z \in \mathbb{R}^2 : \|x_P - z\| + \|y_P - z\| \leq (1 + 2K^{-2})\|P\| \big\},$$
$$\text{and} \quad \widetilde{E}(P) := \{ z \in \mathbb{R}^2 : d(z, E(P)) \leq 4K^{j-1} \}.$$

Definition 2.9. A path P is said to be tame if $P \subseteq \widetilde{E}(P)$, and untamed otherwise.

In each scale, a tame path is roughly a straight line, and, on the contrary an untamed path is in spirit a curve (for example, in (b) of Figure 2.2, $P^{(1)}$, $P^{(3)}$ and $P^{(4)}$ are tame, while $P^{(2)}$ is untamed). Note that untamed nodes, i.e., nodes corresponding to untamed sub-paths, have more children (see u_2 in (d) of Figure 2.2). Therefore, untamed nodes will bring in more leaves, which correspond to different sites in P. Thus, the fraction of untamed nodes can be bounded by the length of P. To this end, let θ_P be the unit uniform flow on $\mathcal{T}_P$ from the root ρ to the leaves, i.e., $\theta_P(\rho) = 1$ and $\theta_P(v) = \frac{1}{d_u}\theta_P(u)$ if v is one of the d_u many children of u. For $\delta \in (0,1)$ and $K = 2^k$, define a family of paths as follows:

$$\mathcal{P}_N^{\kappa,\delta,K} := \Big\{ P : P \text{ is a path in } V_{N/2}, \|P\| \geq \kappa N \text{ and } |P| \leq N^{1+\frac{\delta}{K^2 k}} \Big\}. \tag{2.2}$$

Lemma 2.10 (Proposition 3.6 in Ding and Zhang, 2019)**.** *For $\kappa, \delta \in (0,1)$ and $K \geq C$. There exists $C' = C'(\kappa,\delta) > 0$ such that for $N \geq e^{C'K^5}$ and $P \in \mathcal{P}_N^{\kappa,\delta,K}$,*

$$\sum_{u: 1 \leq L(u) \leq m-1} \theta_P(u) 1_{\{u \text{ is untamed}\}} \leq 2\delta m,$$

where $L(u)$ is the level of u in the associated tree $\mathcal{T}_P$.

2.3. *Families of paths.* Recall that $\mathcal{BD}_r$ consists of disjoint boxes of side length K^r. For $j \geq 1$, let END_j be a family of boxes, defined as

$$\begin{aligned} \mathrm{END}_1 &:= \big\{ \{z\} : z \in V_{N/2} \big\}, \\ \mathrm{END}_j &:= \big\{ B : B \in \mathcal{BD}_{j-2} \text{ and } B \cap V_{N/2} \neq \emptyset \big\} \text{ for all } j \geq 2. \end{aligned} \tag{2.3}$$

Boxes in END_j have side length K^{j-2}, thus are regarded as "points" in scale K^j. They play the role of endpoints of paths in scale K^j, and are called end-boxes.

For $j \geq 1$ and $B \in \mathrm{END}_j$, denote by $\mathcal{P}_j(B)$ paths in scale K^j started from the end-box B, and by $T_j(B)$ the tame ones, i.e.,

$$\mathcal{P}_j(B) := \{ P \in \mathcal{SL}_j : x_P \in B \}, \quad T_j(B) := \{ P \in \mathcal{P}_j(B) : P \text{ is tame} \}. \tag{2.4}$$

For a pair of end-boxes $B, B' \in \mathrm{END}_j$, let $T_j(B,B')$ be the family of tame paths from B to B', i.e.,

$$T_j(B,B') := \{ P \in \mathcal{SL}_j : x_P \in B, y_P \in B', \text{ and } P \text{ is tame} \}.$$

3. Tame paths are unlikely to be open

In this section, we will estimate the probability of existence of tame and open paths started from a fixed end-box (see Proposition 3.2). Recall that a site $x \in V_{N/2}$ is said to be λ-open if $|\eta^{V_N}(x)| \leq \lambda$.

Definition 3.1. For $V \subseteq V_N$ and $\alpha \in \mathbb{R}$, we say a site $x \in V$ is (V,λ,α)-open if $|\eta^V(x) + \alpha| \leq \lambda$. A path $P \subseteq V$ is said to be (V,λ,α)-open if so is every site in P.

Recall that END_j is the family of end-boxes of side length K^{j-2}, defined in (2.3). For an end-box $B \in \mathrm{END}_j$, $T_j(B)$ is the family of tame paths in scale K^j started from B. Denote by z_B the lower left corner of $B \cap \mathbb{Z}^2$. Recall that $V_\ell(x) := B_\infty(x,\ell/2) \cap \mathbb{Z}^2$ is a box of side length ℓ, defined in Section 1.1.

Proposition 3.2. *There exists a universal constant $c > 0$ such that the following holds for all $\lambda \geq 1$ and $K \geq \widehat{K}_0(\lambda) := e^{c\lambda^2}$. Suppose that $j \in [m-1]$, $B \in \mathrm{END}_j$, and $\alpha \in \mathbb{R}$. Then, for any box V satisfying $V_{4K^j}(z_B) \subseteq V \subseteq V_N$,*

$$\mathbb{P}\big(P \text{ is } (V,\lambda,\alpha)\text{-open for some } P \in T_j(B) \big) \leq e^{-0.01\sqrt{K}}.$$

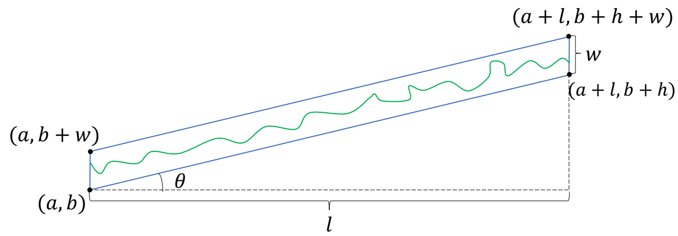


FIGURE 3.3. D and its crossing.

In Section 3.1, we will show that with positive probability, a *good* parallelogram (defined above Lemma 3.3) with aspect ratio $O(1)$ has no open crossings (see Lemma 3.3). In Section 3.2, we will first investigate tame paths from B to another end-box B', i.e., paths in $T_j(B, B')$. We will adjust the modified ellipse $\widetilde{E}(P)$ (defined above Definition 2.9) for all $P \in T_j(B, B')$ to a parallelogram D with aspect ratio of order K, and then extract $\sqrt{K}/8$ disjoint good parallelograms D_i's from D (see Figure 3.6). Then, we will show that with probability not exceeding $e^{-0.015\sqrt{K}}$ there exists a (V, λ, α)-open path in $T_j(B, B')$ (see Lemma 3.4). Finally, we obtain Proposition 3.2 by the union bound, where B' is taken over all possible end-boxes.

3.1. *Good parallelograms.* In this section, we consider a closed parallelogram D with corners (a, b), $(a + l, b + h)$, $(a + l, b + h + w)$ and $(a, b + w)$, where

$$a, l \in \mathbb{Z}; \quad b, w, h \in \mathbb{R}^2; \quad l > h \geq 0; \quad \text{and } l \geq w \geq 10$$

(Here 10 is a somewhat arbitrary choice). We call l and w respectively the length and width of D. By crossing of such a parallelogram D, we mean a path in D connecting its two short sides (see Figure 3.3). Define the crossing event as follows:

$$\mathcal{A}(D, V, \lambda, \alpha) := \{\text{there exists a } (V, \lambda, \alpha)\text{-open crossing of } D\}. \tag{3.1}$$

Denote $\theta = \arctan \frac{h}{l}$. Define the anchor of D as follows:

$$v_0 := \Big(\Big\lfloor \frac{a + h + l - 7w \sin^2 \theta}{2} \Big\rfloor, \Big\lfloor \frac{b + h - l + 7w \sin \theta \cos \theta}{2} \Big\rfloor \Big).$$

Note that $\theta \in [0, \frac{\pi}{4}]$ and $v_0 \in \mathbb{Z}^2$. We say that D is *good* if

$$l = 16w.$$

The proof of the following lemma is similar to that of Proposition 4 in Ding and Li (2018).

Lemma 3.3. *There exists a universal constant $c' > 0$ such that the following hold. Suppose $\lambda \geq 1$ and $L/w \geq e^{c'\lambda^2}$. Then, for each good parallelogram D with width w and anchor v_0, and $\alpha \in \mathbb{R}$, we have*

$$\mathbb{P}\Big(\mathcal{A}\big(D, V_L(v_0), \lambda, \alpha\big)\Big) \leq \frac{7}{8}. \tag{3.2}$$

Proof: For $j = 0, 1, 2, 3$, denote by D_j the rotation (counterclockwise) of D around the anchor v_0, with the angle of rotation being equal to $j \times \frac{\pi}{2}$ radians, respectively. Since D is good, by the definition of v_0, $\bigcup_{i=0}^{3} D_i$ forms an annulus centered at v_0, lying in $V_{4l}(v_0)$, and surrounding $V_{2w}(v_0)$. We denote this annulus by R (see Figure 3.4). Abbreviate $V := V_L(v_0)$ and $\mathcal{A}_j := \mathcal{A}\big(D_j, V, \lambda, \alpha\big)$. Then,

$$\text{on the event } \bigcap_{j=0}^{3} \mathcal{A}_j, \text{ there is a } (V, \lambda, \alpha)\text{-open contour in } R,$$

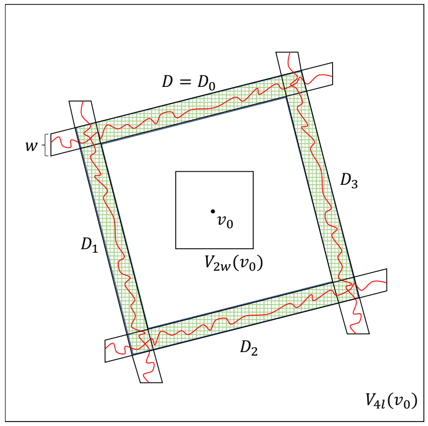


FIGURE 3.4. R is the (green) annulus. (Red) curves are open crossings of D_i's.

where by contour, we mean a path with two endpoints coinciding. We use $\mathbf{C}$ to stand for a contour. Denote by $\mathfrak{C}$ the collection of all contours in R, equipped with the partial order: $\mathbf{C}_1 \preceq \mathbf{C}_2$ if $\mathbf{C}_1^* \subseteq \mathbf{C}_2^*$, where $\mathbf{C}^*$ consists of sites surrounded by $\mathbf{C}$. Let $\mathscr{C}$ consist of all open contours in R, which can be regarded as a random subset of $\mathfrak{C}$. Then, there exists a unique maximal open contour in $\mathscr{C}$, which is denoted by $\mathcal{C}$. Then, it suffices to show

$$\mathbb{P}\big(\mathscr{C} \neq \emptyset\big) \leq \frac{1}{2}. \tag{3.3}$$

Indeed, assuming (3.3), we have

$$\begin{aligned}\mathbb{P}\big(\mathscr{C} \neq \emptyset\big) &\geq \mathbb{P}\big(\cap_{j=0}^{3} \mathcal{A}_j\big) = 1 - \mathbb{P}\big(\cup_{j=0}^{3} \mathcal{A}_j^c\big) \\ &\geq 1 - \sum_{j=0}^{3} \mathbb{P}(\mathcal{A}_j^c) = 1 - 4(1 - \mathbb{P}(\mathcal{A}_0)),\end{aligned}$$

where we have used the fact $\mathbb{P}(\mathcal{A}_j) = \mathbb{P}(\mathcal{A}_0)$ by rotation invariance. Consequently, we have $P(\mathcal{A}_0) \leq \frac{7}{8}$, i.e., Lemma 3.3 holds.

Next, we will show (3.3). Denote

$$X := \frac{1}{|\partial V_{2w}(v_0)|} \sum_{x \in \partial V_{2w}(v_0)} \big(\eta^V(x) + \alpha\big).$$

Let $c' \geq \pi C_1 + \log 2$, where C_1 is defined in Lemma 2.1. By the assumption $L/w \geq e^{c'\lambda^2}$ and Lemma 2.1,

$$\mathbb{E}\eta^V(x)\eta^V(y) \geq \frac{2}{\pi} \log\left(\frac{L}{2w}\right) - C_1 \geq \frac{1}{\pi} \log\left(\frac{L}{2w}\right) \quad \text{for all } x, y \in \partial V_{2w}(v_0).$$

It follows that

$$\operatorname{Var}(X) \geq \frac{1}{\pi} \log\left(\frac{L}{2w}\right). \tag{3.4}$$

For a deterministic contour $\mathbf{C}$ lying in the annulus R, let $\widehat{\mathbf{C}} = (V \setminus \mathbf{C}^*) \cup \mathbf{C}$ consist of sites outside $\mathbf{C}$ (including the boundary $\mathbf{C}$). Denote $\mathcal{F}_{\widehat{\mathbf{C}}} := \sigma\big\{\eta^V(x) : x \in \widehat{\mathbf{C}}\big\}$ and $Y := X - \mathbb{E}\big(X \mid \mathcal{F}_{\widehat{\mathbf{C}}}\big)$. Recall $\ell = 16w$ since D is a good parallelogram. By taking D, ℓ_1 and ℓ_2 in Lemma 2.7 respectively as $\mathbf{C}^*$, $2w$ and $4l$, one has $\sum_{y \in \partial V_{2w}(v_0)} G_{\mathbf{C}^*}(x, y) \leq 2(4l - 2w)$. Consequently,

$$\operatorname{Var}(Y) = \frac{1}{|\partial V_{2w}(v_0)|^2} \sum_{x,y \in \partial V_{2w}(v_0)} G_{\mathbf{C}^*}(x, y) \leq 16. \tag{3.5}$$

Note that for each $x \in \partial V_{2w}(v_0)$,

$$\mathbb{E}\big(\eta^V(x) \mid \mathcal{F}_{\widehat{\mathbf{C}}}\big) = \sum_{y\in\mathbf{C}} P_x(S_\tau = y) \cdot \eta^V(y), \tag{3.6}$$

where $\{S_n\}$ is the simple random walk started from x, and τ is the hitting time to $\mathbf{C}$. Recalling that $\mathcal{C}$ is the maximal, i.e., the outermost, open contour in $\mathscr{C}$, one has $\{\mathcal{C} = \mathbf{C}\} \in \mathcal{F}_{\widehat{\mathbf{C}}}$. On the event $\{\mathcal{C} = \mathbf{C}\}$, we have $|\eta^V(y) + \alpha| \le \lambda$ for all $y \in \mathbf{C}$. Combined with (3.6), it yields that for all $x \in \partial V_{2w}(v_0)$,

$$\big|\mathbb{E}\big(\eta^V(x) + \alpha \mid \mathcal{F}_{\widehat{\mathbf{C}}}\big)\big| \le \sum_{y\in\mathbf{C}} P_x(S_\tau = y) \cdot \big|\eta^V(y) + \alpha\big| \le \lambda.$$

This implies $\big|\mathbb{E}\big(X \mid \mathcal{F}_{\widehat{\mathbf{C}}}\big)\big| \le \lambda$. Consequently, $|X| = \big|\mathbb{E}\big(X \mid \mathcal{F}_{\widehat{\mathbf{C}}}\big) + Y\big| \le 2\lambda$ provided $|Y| \le \lambda$. Noting that Y and $\mathcal{F}_{\widehat{\mathbf{C}}}$ are independent, we have

$$\mathbb{P}\big(|X| \le 2\lambda \mid \mathcal{C} = \mathbf{C}\big) \ge \mathbb{P}\big(|Y| \le \lambda \mid \mathcal{C} = \mathbf{C}\big) = \mathbb{P}\big(|Y| \le \lambda\big).$$

It follows that

$$\mathbb{P}\big(\mathscr{C} \neq \emptyset\big) = \sum_{\mathbf{C}\in\mathfrak{C}} \mathbb{P}\big(\mathcal{C} = \mathbf{C}\big) = \sum_{\mathbf{C}\in\mathfrak{C}} \frac{\mathbb{P}(|X| \le 2\lambda, \mathcal{C} = \mathbf{C})}{\mathbb{P}(|X| \le 2\lambda \mid \mathcal{C} = \mathbf{C})} \le \frac{\mathbb{P}(|X| \le 2\lambda)}{\mathbb{P}(|Y| \le \lambda)}. \tag{3.7}$$

Let ϕ_{σ^2} be the probability density function of $N(0, \sigma^2)$. By (3.4) and (3.5), for $\lambda \ge 1$,

$$\mathbb{P}(|X| \le 2\lambda) \le 4\lambda\phi_{\sigma_1^2}(0), \quad \mathbb{P}(|Y| \le \lambda) \ge \mathbb{P}(|Y| \le 1) \ge 2\phi_{16}(1),$$

where $\sigma_1^2 := \frac{1}{\pi}\log\big(\frac{L}{2w}\big)$. These, combined with (3.7), imply that for all $\lambda \ge 1$,

$$\mathbb{P}\big(\mathscr{C} \neq \emptyset\big) \le \frac{4\lambda\phi_{\sigma_1^2}(0)}{2\phi_{16}(1)} \le \frac{\sqrt{2}}{\phi_{16}(1)} \cdot \frac{\lambda}{\sqrt{c'\lambda^2 - \log 2}}.$$

where we have used the assumption $L/w \ge e^{c'\lambda^2}$. Finally, pick a large constant c' such that $c' \ge \pi C_1 + \log 2$ and the right hand side above is less than $\frac{1}{2}$. Then, we obtain (3.3). This completes the proof of Lemma 3.3. □

3.2. *Proof of Proposition 3.2.* Recall that END_j consists of boxes of side length K^{j-2}, defined in (2.3), and $T_j(B, B')$ consists of tame paths in scale K^j from B to B', for $B, B' \in \mathrm{END}_j$. The essential ingredient of Proposition 3.2 is the following lemma.

Lemma 3.4. *Under the same assumptions of Proposition 3.2, for all $B' \in \mathrm{END}_j$ such that $T_j(B, B') \neq \emptyset$,*

$$\mathbb{P}\Big(P \text{ is } (V, \lambda, \alpha)\text{-open for some } P \in T_j(B, B')\Big) \le e^{-0.015\sqrt{K}}.$$

Assuming Lemma 3.4, we can obtain Proposition 3.2 via the union bound.

Proof of Proposition 3.2, assuming Lemma 3.4: Note that for $B \in \mathrm{END}_j$, one can find at most K^5 boxes B''s in END_j such that $T_j(B, B') \neq \emptyset$. By the union bound, for $K \ge \widehat{K}_0(\lambda)$,

$$\mathbb{P}\Big(P \text{ is } (V, \lambda, \alpha)\text{-open for some } P \in T_j(B)\Big) \le K^5 e^{-0.015\sqrt{K}} \le e^{-0.01\sqrt{K}}.$$

This completes the proof of Proposition 3.2. □

Next, we will concentrate on proving Lemma 3.4. In the rest of this section, we will assume that $j \in [m-1]$, B and B' are two fixed end-boxes such that $T_j(B, B') \neq \emptyset$, V is an arbitrary set satisfying $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, where z_B is the lower left corner of $B \cap \mathbb{Z}^2$, and α is an arbitrary real number. Note that

$$\bigcup_{P \in T_j(B,B')} \subseteq V_{4K^j}(z_B) \subseteq V.$$

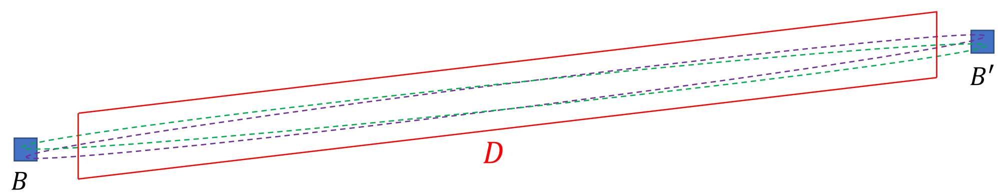


FIGURE 3.5. The parallelogram D covers the modified ellipse $\widetilde{E}(P)$ (except the area around the ends) for all $P \in T_j(B, B')$.

Let $(x, y), (x', y') \in \mathbb{R}^2$ be the lower left corners of B and B', respectively. Without loss of generality, suppose that $x' - x \geq y' - y \geq 0$. Then, it is not hard to check that the following geometric facts hold for all $K \geq 2^{32}$.

(G1) We can find a parallelogram D with width $w = 20K^{j-1}$ and length $K^j/4$ such that the following holds. Each path P in $T_j(B, B')$, i.e., a tame path in scale K^j from B to B', contains a crossing of D (see Figure 3.5);

(G2) We can extract good parallelograms D_i's, $i \in [\sqrt{K}/8]$ from D, with width $w := 20K^{j-1}$ and length $l = 16w$ such that the following holds. Denote by v_i the anchor of D_i, $L := K^{j-1/2}$, $U_i := V_L(v_i)$. Then, U_i's are disjoint, and $D_i \subseteq V_{4l}(v_i) \subseteq U_i$ for each i (see Figure 3.6).

(G3) Let $U := \bigcup_{i \in [\sqrt{K}/8]} U_i$. Then, $U \subseteq V_{4K^j}(z_B) \subseteq V$.

By (G1),

$$\{P \text{ is } (V, \lambda, \alpha)\text{-open for some } P \in T_j(B, B')\} \subseteq \mathcal{A}\big(D, V, \lambda, \alpha\big), \tag{3.8}$$

where $\mathcal{A}\big(D, V, \lambda, \alpha\big)$ is the crossing event defined in (3.1).

We now decompose the GFF η^V into independent GFFs on boxes U_i's and an independent harmonic function. Precisely, define

$$\mathcal{F}_\partial := \sigma\{\eta^V(z) : z \in (V \setminus U) \cup \partial U\},$$

$$H_\partial(z) := \mathbb{E}\big(\eta^V(z) \mid \mathcal{F}_\partial\big), \quad \eta^{U_i}(z) := \eta^V(z) - H_\partial(z) \text{ for all } z \in U_i.$$

By the Markov property (Lemma 2.2),

$$\eta^{U_i} = \{\eta^{U_i}(z) : z \in U_i\}$$

is a version of the GFF on U_i for each $i \in [\sqrt{K}/8]$. Moreover, η^{U_i}'s are mutually independent since U_i's are disjoint by (G2), and they are also independent of H_∂.

Denote

$$\varepsilon_0 := 100\sqrt{C_2},$$

where C_2 is defined in Lemma 2.3. Let $\mathcal{E}_0$ be the event that fluctuations of the harmonic function H_∂ are not small, defined as

$$\mathcal{E}_0 := \bigcup_{i \in [\sqrt{K}/8]} \mathcal{E}_{0,i}, \quad \text{where } \mathcal{E}_{0,i} := \Big\{\big|H_\partial(z) - H_\partial(v_i)\big| \geq \varepsilon_0 \text{ for some } z \in D_i\Big\}. \tag{3.9}$$

Lemma 3.5. *Let $K \geq C_5 := (2 \vee C_3)^{32}$ (recalling that C_3 is defined in Lemma 2.4). Then, $\mathbb{P}\big(\mathcal{E}_0\big) \leq e^{-0.5\sqrt{K}}$.*

Proof: Recall $w = 20K^{j-1}$, $l = 16w$, and $L = K^{j-1/2}$ by (G2). Recall that $C_4 := 2C_3\sqrt{C_2}$, which is defined in the proof of Lemma 2.6. By the assumption $K \geq C_5$, we have $C_4\sqrt{\frac{4l}{L}} \leq \varepsilon_0$. Setting

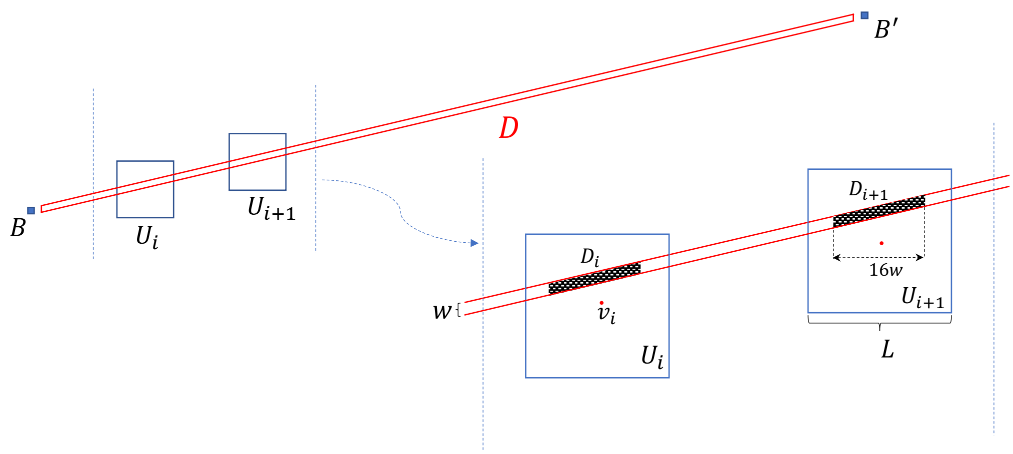


FIGURE 3.6. D (defined in (G1)) is the parallelogram in red, with aspect ratio of order K. U_i's are the boxes in blue. The hard parallelograms are D_i's, with aspect ratio of order 1.

$\ell = 4l$ and $\varepsilon = \varepsilon_0$ in Lemma 2.6, we have

$$\begin{aligned}\mathbb{P}\Big(\mathcal{E}_{0,i}\Big) &\leq \mathbb{P}\Big(\big|H_\partial(z) - H_\partial(v_i)\big| \geq \varepsilon_0 \text{ for some } z \in V_{4l}(v_i)\Big) \\ &\leq 4\exp\Big\{-\frac{\varepsilon_0^2 L}{32C_2 l}\Big\} \leq e^{-0.9\sqrt{K}},\end{aligned}$$

where we have used the fact that $D_i \subseteq V_{4l}(v_i) \subseteq U_i$ by (G2) in the first inequality, and $K \geq C_5 \geq 2^{32}$ in the last inequality. By the union bound, $\mathbb{P}(\mathcal{E}_0) \leq \frac{1}{8}\sqrt{K}e^{-0.9\sqrt{K}} \leq e^{-0.5\sqrt{K}}$, completing the proof. □

Proof of Lemma 3.4: Suppose that $\lambda \geq 1$, $j \in [m-1]$, $B, B' \in \mathrm{END}_j$ such that $T_j(B, B') \neq \emptyset$, $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\alpha \in \mathbb{R}$. Let D be the parallelogram defined in (G1). Let $\mathcal{A} = \mathcal{A}\big(D, V, \lambda, \alpha\big)$, which is the crossing event defined in (3.1). By (3.8), to prove Lemma 3.4, it suffices to show that

$$\mathbb{P}\big(\mathcal{A}\big) \leq e^{-0.015\sqrt{K}}. \tag{3.10}$$

Recall $w = 20K^{j-1}$ and $L = K^{j-1/2}$ by (G2). Let c be a constant such that for all $\lambda \geq 1$,

$$\widehat{K}_0(\lambda) := e^{c\lambda^2} \geq 400e^{2c'(\lambda+\varepsilon_0)^2} \vee C_5 \vee C, \tag{3.11}$$

where c', C, C_5, ε_0 are respectively defined in Lemma 3.3, Proposition 2.8, Lemma 3.5, and above (3.9). For $K \geq \widehat{K}_0(\lambda)$, we have $L/w \geq e^{c'(\lambda+\varepsilon_0)^2}$. Then, by (G2) and Lemma 3.3, for each α,

$$\mathbb{P}\big(\mathcal{A}(D_i, U_i, \lambda + \varepsilon_0, \alpha)\big) \leq \frac{7}{8} \quad \text{for all } i. \tag{3.12}$$

Note that for $z \in D_i$,

$$\eta^V(z) = \eta^{U_i}(z) + H_\partial(z) = \big(\eta^{U_i}(z) + H_\partial(v_i)\big) + \big(H_\partial(z) - H_\partial(v_i)\big).$$

By the triangle inequality, on the event $\{|H_\partial(z) - H_\partial(v_i)| \leq \varepsilon_0 \text{ for all } z \in D_i\}$, $\big|\eta^V(z) + \alpha\big| \leq \lambda$ implies $\big|\eta^{U_i}(z) + \alpha + H_\partial(v_i)\big| \leq \lambda + \varepsilon_0$ for each $z \in D_i$. Denote

$$\mathcal{A}_i := \mathcal{A}\big(D_i, U_i, \lambda + \varepsilon_0, \alpha + H_\partial(v_i)\big).$$

Then, on the event $\mathcal{E}_0^c$,

$$\mathcal{A} \subseteq \bigcap_{i\in[\sqrt{K}/8]} \mathcal{A}(D_i, V, \lambda, \alpha) \subseteq \bigcap_{i\in[\sqrt{K}/8]} \mathcal{A}_i.$$

Therefore,

$$\mathbb{P}(\mathcal{A}) \le \mathbb{P}\Big(\bigcap_{i\in[\sqrt{K}/8]} \mathcal{A}_i\Big) + \mathbb{P}(\mathcal{E}_0),$$

where $\mathbb{P}(\mathcal{E}_0) \le e^{-0.5\sqrt{K}}$ by Lemma 3.5. Recall that $\mathcal{F}_\partial$ is the sigma-algebra outside U, defined below (3.8). Since $\mathcal{A}_i$'s are conditionally independent given $\mathcal{F}_\partial$,

$$\mathbb{P}\Big(\bigcap_{i\in[\sqrt{K}/8]} \mathcal{A}_i\Big) = \mathbb{E}\mathbb{P}\Big(\bigcap_{i\in[\sqrt{K}/8]} \mathcal{A}_i \,\Big|\, \mathcal{F}_\partial\Big) = \mathbb{E}\Big(\prod_{i\in[\sqrt{K}/8]} \mathbb{P}(\mathcal{A}_i \mid \mathcal{F}_\partial)\Big).$$

By (3.12), $\mathbb{P}(\mathcal{A}_i \mid \mathcal{F}_\partial) \le \frac{7}{8}$ for all i, where $H_\partial(v_i)$'s are regarded as constants for being measurable with respect to $\mathcal{F}_\partial$. Collecting the above arguments, we conclude that for all $K \ge \widehat{K}_0(\lambda) \ge C_5 \ge 2^{32}$,

$$\mathbb{P}(\mathcal{A}) \le \Big(\frac{7}{8}\Big)^{\lfloor\sqrt{K}/8\rfloor} + e^{-0.5\sqrt{K}} \le e^{-0.015\sqrt{K}}.$$

Thus, (3.10) holds, which completes the proof of Lemma 3.4. □

4. Multi-scale analysis on the hierarchical structure of the path

Recall that $\mathcal{SL}_j$ consists of paths P with $\|P\|$ being of order K^j (see (2.1)), where $\|P\|$ is the Euclidean distance of the two ends of P. Recall that for P in scale K^j, i.e., $P \in \mathcal{SL}_j$, a tree $\mathcal{T}_P$ of depth j is associated with P (see Section 2.2). The root ρ of $\mathcal{T}_P$ corresponds to P, and a node u at level $r = L(u)$ of $\mathcal{T}_P$ corresponds to a sub-path P^u in scale K^{j-r}. Moreover, the node u is said to be tame/open if so is P^u.

Next, we will estimate the quantity of nodes in $\mathcal{T}_P$ which are both tame and open. The results are stated in Propositions 4.2 and 4.3, which will be respectively proved in Sections 4.1 and 4.2. On the one hand, for paths in $\mathcal{P}_N^{\kappa,\delta,K}$, untamed nodes are rare in the associated trees (see Lemma 2.10). On the other hand, Propositions 4.2 and 4.3 give upper bounds on the quantity of tame and open nodes. Combining them, we obtain Proposition 4.4. We will prove Theorem 1.1 by assuming Proposition 4.4, and defer the proof of Proposition 4.4 to Section 4.3.

Let $j \in [m-1]$, and $B \in \mathrm{END}_j$, which is a box of side length K^{j-2} (see (2.3)). Recall that $\mathcal{P}_j(B)$ consists of paths in scale K^j started from B (defined in (2.4)). Suppose $0 \le r \le j-1$. Let $\mathcal{T}_{P,r}$ be the collection of nodes at level r. For each $u \in \mathcal{T}_{P,r}$, there is a unique end-box in END_{j-r}, denoted by B_u, containing the starting site of P^u. Let $\mathscr{A}$ be the collection of all real functions $\bar{\alpha}$ defined on end-boxes, i.e.,

$$\bar{\alpha} : \bigcup_{i\in[m-1]} \mathrm{END}_i \longmapsto \mathbb{R}.$$

Suppose $\bar{\alpha} \in \mathscr{A}$. For all $P \in \mathcal{P}_j(B)$, $\bar{\alpha}$ induces a function on $\mathcal{T}_P$:

$$u \mapsto \bar{\alpha}_u := \bar{\alpha}(B_u), \qquad \text{for all } u \in \mathcal{T}_P. \tag{4.1}$$

Let θ_P be the unit uniform flow on $\mathcal{T}_P$ from the root ρ to the leaves (the definition is just before (2.2)). Recall that a path P is (V, λ, α)-open if $|\eta^V(x) + \alpha| \le \lambda$ for all $x \in P$ (see Definition 3.1). A node u is said to be (V, λ, α)-open if so is P^u. Suppose $\lambda > 0$, $\bar{\alpha} \in \mathscr{A}$, $j \in [m-1]$, $0 \le r \le j-1$, and $B \in \mathrm{END}_j$. Let V be a box such that $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, where z_B is the lower left corner of $B \cap \mathbb{Z}^2$. For $P \in \mathcal{P}_j(B)$, the fraction of tame and open nodes, i.e., the total flow through such nodes, at level r is defined as

$$Y_{P,r,\lambda,\bar{\alpha}} := \sum_{u \in \mathcal{T}_{P,r}} \theta_P(u) 1_{\{u \text{ is tame and } (V,\lambda,\bar{\alpha}_u)\text{-open}\}}, \tag{4.2}$$

which is bounded by

$$\xi_{r,\lambda,\bar{\alpha},j,B} := \max\left\{Y_{P,r,\lambda,\bar{\alpha}} : P \in \mathcal{P}_j(B)\right\}. \tag{4.3}$$

Note that $\bar{\alpha}_\rho \equiv \bar{\alpha}(B)$, which is a constant for all trees associated with paths in $\mathcal{P}_j(B)$. Thus, in the specific case $r = 0$,

$$\xi_{0,\lambda,\bar{\alpha},j,B} = 1_{\{P \text{ is } (V,\lambda,\bar{\alpha}_\rho)\text{-open for some } P \in \mathcal{T}_j(B)\}}.$$

Then, Proposition 3.2 can be repeated in terms of the unit uniform flow as follows.

Corollary 4.1. *Suppose $\lambda \geq 1$, $K \geq \widehat{K}_0(\lambda) := e^{c\lambda^2}$, $j \in [m-1]$, $B \in \mathrm{END}_j$, $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$. Then,*

$$\mathbb{P}(\xi_{0,\lambda,\bar{\alpha},j,B} > 0) \leq e^{-0.01\sqrt{K}}.$$

Next, we will investigate the case $r \geq 1$. Recall that C_2 is defined in Lemma 2.3. Define

$$\varepsilon_0 = 100\sqrt{C_2},\ \varepsilon_1 = 8\sqrt{C_2}, \ \text{ and } \beta = 2^{-9},\ \varepsilon_{r+1} = 4\sqrt{C_2}\beta^{r/2} \ \text{ for all } \ r \geq 1, \tag{4.4}$$

which will be used to bound the fluctuations of the harmonic functions. Define recursively

$$\widehat{K}_r(\lambda) := \widehat{K}_{r-1}(\lambda + \varepsilon_r) = \widehat{K}_0\Big(\lambda + \sum_{i=1}^r \varepsilon_i\Big) \ \text{ for all } r \geq 1.$$

Furthermore, we denote

$$c_r = (\beta K)^r \text{ for all } r \geq 1,$$

which will be used in the exponents in the upper bounds of probabilities. Denote

$$\Delta_1 = \frac{9 \log K}{\beta K^{1/8}}; \quad \Delta_{r+1} = \frac{\log(1 + 2c_r) + 9\beta^{-1}\log K}{c_r} \ \text{ for all } \ r \geq 1.$$

Define recursively

$$\delta_0 = 0, \quad \delta_1 = 1/2; \quad \delta_{r+1} = \delta_r + \Delta_r \ \text{ for all } r \geq 1, \tag{4.5}$$

which will be used to bound the fraction of tame and open nodes. We will prove the following Propositions 4.2, 4.3 and 4.4 in Sections 4.1, 4.2, and 4.3, respectively.

Proposition 4.2. *Suppose $2 \leq j \leq m-1$. Then, for $\lambda \geq 1$, $K \geq \widehat{K}_1(\lambda)$, $B \in \mathrm{END}_j$, $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$, we have*

$$\mathbb{P}(\xi_{1,\lambda,\bar{\alpha},j,B} > \delta) \leq e^{-K^{1/8}} \quad \text{for all } \delta \geq \delta_1 := \frac{1}{2}.$$

Proposition 4.3. *Suppose $2 \leq r < j \leq m-1$. Then, for $\lambda \geq 1$, $K \geq \widehat{K}_r(\lambda)$, $B \in \mathrm{END}_j$, $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$, we have*

$$\mathbb{P}(\xi_{r,\lambda,\bar{\alpha},j,B} > \delta) \leq 2e^{-c_{r-1}(\delta - \delta_r)} \quad \text{for all } \delta > \delta_r.$$

Proposition 4.4. *For all $\kappa \in (0,1)$, there exists $a = a(\kappa) > 0$ such that for all $\lambda \geq 1$,*

$$\mathbb{P}\big(P \text{ is } \lambda\text{-open for some } P \in \mathcal{P}_N^{\kappa, \exp\{-a\lambda^2\}}\big) \leq e^{-\left(\frac{\log N}{a\lambda^2} - a\lambda^2\right)},$$

for all $N \geq \exp\{e^{a\lambda^2}\}$.

Assuming Proposition 4.4, we will show Theorem 1.1.

Proof of Theorem 1.1, assuming Proposition 4.4: For $\lambda > 0, \epsilon > 0$, define

$$E_{\lambda,\epsilon,N} := \{P \text{ is } \lambda\text{-open for some } P \in \mathcal{P}_N^{\kappa,\epsilon}\}.$$

For $\lambda \geq 1$, letting $\epsilon = \epsilon_\lambda := e^{-a\lambda^2}$, by Proposition 4.4, $\mathbb{P}(E_{\lambda,\epsilon,N}) \to 0$ as $N \to \infty$. For $\lambda \in (0, 1]$, letting $\epsilon = e^{-a}$, noting that $E_{\lambda,\epsilon,N} \subseteq E_{1,\epsilon_1,N}$, we also have $\mathbb{P}(E_{\lambda,\epsilon,N}) \to 0$ as $N \to \infty$. Thus, we have completed the proof of Theorem 1.1. □

4.1. *Proof of Proposition 4.2.* We will apply a large deviation analysis on $Y_{P,1,\lambda,\bar{\alpha}}$ (see Lemma 4.5), noting that $Y_{P,1,\lambda,\bar{\alpha}}$ is the fraction of tame and open child-paths of P. This section is organized as follows. We will first introduce notations and show preliminary results. Next, we will show lemmas for preparation. Finally, we will prove Proposition 4.2.

Assume $2 \leq j \leq m-1$, $B \in \mathrm{END}_j$, V is a box such that $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$. Recall that $\mathcal{P}_j(B)$ consists of paths in scale K^j started from B. Suppose P has d child-paths $P^{(1)}, \cdots, P^{(d)}$, which are in $\mathcal{SL}_{j-1}$. Suppose $P^{(i)}$ is started from an end-box B_i in END_{j-1}. Recall that each end-box in END_{j-1} is of side length K^{j-3} for $j \geq 3$, and consists of a single site for $j = 2$ (see (2.3)). By Proposition 2.8, each $\widetilde{B}$ in $\mathcal{BD}_{j-1}$ contains at most 12 end-boxes in the sequence $\mathcal{S} := \{B_i\}_{i\in[d]}$. Denote by $\mathrm{END}_{j-1}^{(d)}(B)$ such sequences, and for each $\mathcal{S}$, denote by $\mathcal{P}_{j,\mathcal{S}}(B)$ paths whose d child-paths respectively started from B_i's, i.e.,

$$\mathcal{P}_{j,\mathcal{S}}(B) := \Big\{P \in \mathcal{SL}_j : \; P \text{ is started from } B, \text{ and has } d \text{ child-paths such that } P^{(i)} \in \mathcal{P}_{j-1}(B_i) \text{ for all } i \in [d]\Big\}, \tag{4.6}$$

$$\mathrm{END}_{j-1}^{(d)}(B) := \Big\{\mathcal{S} : \; B_i \in \mathrm{END}_{j-1} \text{ for each } i \in [d]; \quad \mathcal{P}_{j,\mathcal{S}}(B) \neq \emptyset; \text{ and } \#\{i : B_i \subseteq \widetilde{B}\} \leq 12 \text{ for each } \widetilde{B} \in \mathcal{BD}_{j-1}\Big\},$$

where # stands for the cardinality of a set. Then,

$$\mathcal{P}_j(B) \subseteq \bigcup_{d\geq K} \bigcup_{\mathcal{S}\in \mathrm{END}_{j-1}^{(d)}(B)} \mathcal{P}_{j,\mathcal{S}}(B). \tag{4.7}$$

Consequently, it boils down to analyzing properties of paths in $\mathcal{P}_{j,\mathcal{S}}(B)$.

Next, we will assume that $d \geq K$ and $\mathcal{S} := \{B_i\}_{i\in[d]} \in \mathrm{END}_{j-1}^{(d)}(B)$ for the rest of this section. Abbreviate

$$z_i := z_{B_i} \quad \text{and} \quad U_i := V_{K^{j-7/8}}(z_i) \quad \text{for all } i \in [d] \tag{4.8}$$

Note that $U_i \subseteq V_{4K^j}(z_B) \subseteq V$ for all $i \in [d]$ (see Figure 4.7). Suppose $P \in \mathcal{P}_{j,\mathcal{S}}(B)$. By (4.6), $P^{(i)} \in \mathcal{P}_{j-1}(B_i) \subseteq \mathcal{SL}_{j-1}$ for all $i \in [d]$. By the definition of $\mathcal{SL}_{j-1}$ (see (2.1)), $P^{(i)} \subseteq V_{4K^{j-1}}(z_i)$. Therefore, we conclude

$$P^{(i)} \subseteq V_{4K^{j-1}}(z_i) \subseteq U_i \subseteq V_{4K^j}(z_B) \subseteq V \quad \text{for all } i \in [d].$$

Let H_i be the conditional expectation of η^V given $\sigma\{\eta^V(z) : z \in U_i^c \cup \partial U_i\}$. By the Markov property (Lemma 2.2), $\eta^{U_i} := \eta^V - H_i$ is a version of GFF on U_i for each $i \in [d]$. For all $x \in P^{(i)}$, we write

$$\eta^V(x) = \big(\eta^{U_i}(x) + H_i(z_i)\big) + \big(H_i(x) - H_i(z_i)\big).$$

Recall that $\varepsilon_1 = 8\sqrt{C_2}$ is defined in (4.4). Denote by $\widehat{\mathcal{E}}_{\mathcal{S}}$ the event that fluctuations of some H_i is not small, i.e.,

$$\widehat{\mathcal{E}}_{\mathcal{S}} := \bigcup_{i\in[d]} \Big\{\big|H_i(x) - H_i(z_i)\big| \geq \varepsilon_1 \text{ for some } x \in V_{4K^{j-1}}(z_i)\Big\}. \tag{4.9}$$

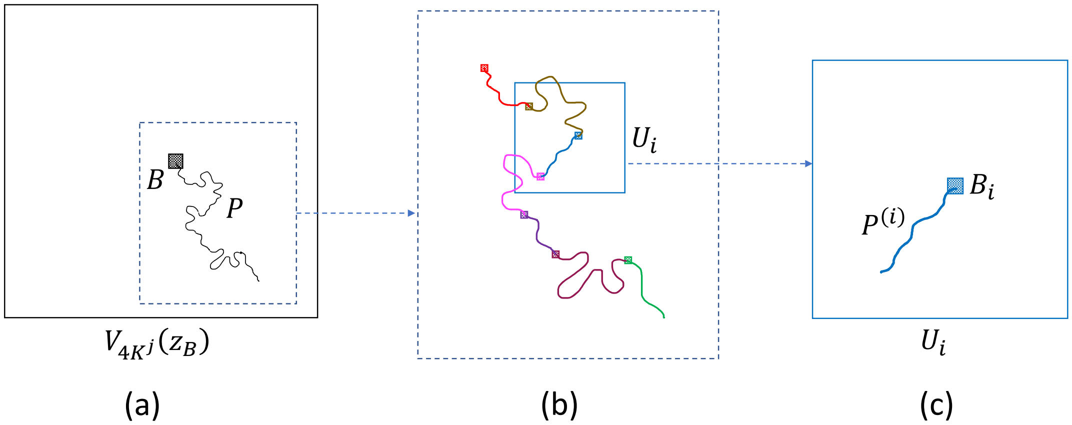


FIGURE 4.7. The solid boxes are end-boxes. $P \in \mathcal{SL}_j$, which is the curve in (a). The child-paths $P^{(i)}$'s are in scale K^{j-1}, and are distinguished from each other with different colors in (b). In (c), we zoom in the local of a specific U_i, which is a box of side length $K^{j-7/8}$.

Assume $\widehat{\mathcal{E}}_{\mathcal{S}}^c$ occurs. Denote $\alpha_i := \bar{\alpha}(B_i)$, recalling (4.1) for the definition of $\bar{\alpha}$. By the triangle inequality, if $P^{(i)}$ is (V, λ, α_i)-open, it is $(U_i, \lambda + \varepsilon_1, \alpha_i + H_i(z_i))$-open. Hence, the following event E_i occurs.

$$E_i := \{\text{there exists a tame and } (U_i, \lambda + \varepsilon_1, \alpha_i + H_i(z_i))\text{-open path in } \mathcal{P}_{j-1}(B_i)\}.$$

Recall that $Y_{P,1,\lambda,\bar{\alpha}}$ is the fraction of tame and $(V, \lambda, \bar{\alpha}_u)$-open nodes at level 1, defined in (4.2). Consequently, on the event $\widehat{\mathcal{E}}_{\mathcal{S}}^c$, for all $P \in \mathcal{P}_{j,\mathcal{S}}(B)$, we have

$$Y_{P,1,\lambda,\bar{\alpha}} = \frac{1}{d} \sum_{i=1}^{d} 1_{\{P^{(i)} \text{ is tame and } (V,\lambda,\alpha_i)\text{-open}\}} \le \frac{1}{d} \sum_{i=1}^{d} \zeta_i,$$

where $\zeta_i := 1_{E_i}$ is the indicator function of E_i. Denote

$$\zeta_{1,\mathcal{S}} := \max\left\{Y_{P,1,\lambda,\bar{\alpha}} : P \in \mathcal{P}_{j,\mathcal{S}}(B)\right\},$$

and we have

$$\zeta_{1,\mathcal{S}} \le \frac{1}{d} \sum_{i=1}^{d} \zeta_i, \qquad \text{on the event } \widehat{\mathcal{E}}_{\mathcal{S}}^c.$$

Consequently, for all $\delta > 0$ and $\mathcal{S} \in \mathrm{END}_{j-1}^{(d)}(B)$,

$$\mathbb{P}\big(\{\zeta_{1,\mathcal{S}} > \delta\} \cap \widehat{\mathcal{E}}_{\mathcal{S}}^c\big) \le \mathbb{P}\left(\frac{1}{d} \sum_{i=1}^{d} \zeta_i > \delta\right). \tag{4.10}$$

The proof of the following lemma is based on a large deviation analysis, similar to that of Lemma 4.4 in Ding and Zhang (2019).

Lemma 4.5. *Suppose $\lambda \ge 1$, $K \ge \widehat{K}_1(\lambda)$ and $d \ge K$. Then, for each $\mathcal{S} := \{B_i\}_{i \in [d]} \in \mathrm{END}_{j-1}^{(d)}(B)$, we have*

$$\mathbb{P}\left(\frac{1}{d} \sum_{i=1}^{d} \zeta_i > \delta\right) \le e^{-10^{-4} K^{1/4} \delta d} \qquad \textit{for all } \delta \ge \delta_1 = \frac{1}{2}.$$

Proof: Let $\beta_K = (48K^{1/4})^{-1}$. We will divide B_i's into β_K^{-1} groups in the following procedure, such that U_i's in each group are disjoint (see (4.8) for definition of U_i's). Recall that B_i is a box of side length K^{j-3} for $j \geq 3$, and consists a single site for $j = 2$. Recall that U_i is the box of side length $K^{j-7/8}$ centered at $z_i = z_{B_i}$. Thus, if $d_\infty(B_i, B_{i'}) \geq 1.5K^{j-7/8}$, then $d_\infty(U_i, U_{i'}) \geq K^{j-1}$. First, we will divide $\mathcal{BD}_{K^{j-1}}$ into $4K^{1/4} = (2K^{j-7/8}/K^{j-1})^2$ families $\widetilde{\mathcal{G}}_s$'s, where $\widetilde{\mathcal{G}}_1$ consists of boxes respectively containing $(2aK^{j-7/8}, 2bK^{j-7/8})$, $a, b \in \mathbb{Z}$ and other $\widetilde{\mathcal{G}}_s$'s are its shifts. Let

$$\mathcal{G}_s := \big\{B_i : i \in [d], \text{ and } B_i \subseteq \widetilde{B} \text{ for some } \widetilde{B} \in \widetilde{\mathcal{G}}_s\big\}.$$

Then, by Proposition 2.8, we can divide each $\mathcal{G}_s$ into 12 groups $\mathcal{G}_{s,t}, t \in [12]$, such that for each s, t, a box in $\widetilde{\mathcal{G}}_s$ contains at most one B_i in $\mathcal{G}_{s,t}$. Thus, U_i's in each group $\mathcal{G}_{s,t}$ are disjoint.

Let $U_{s,t} = \bigcup_{i:B_i \in \mathcal{G}_{s,t}} U_i$. Define the sigma-algebra

$$\mathcal{F}_{s,t} := \sigma\big\{\eta^V(x) : x \in (V \setminus U_{s,t}) \cup \partial U_{s,t}\big\}.$$

Then, conditioned on $\mathcal{F}_{s,t}$, ζ_i's in each group $\mathcal{G}_{s,t}$ are mutually independent. Denote

$$W_{s,t} := \prod_{B_i \in \mathcal{G}_{s,t}} e^{a\beta_K(\zeta_i - \delta)},$$

where $\delta \geq \delta_1 = \frac{1}{2}$ and a is a positive number to be set. Then, we have

$$\mathbb{E}W_{s,t}^{1/\beta_K} = \mathbb{E} \prod_{B_i \in \mathcal{G}_{s,t}} e^{a(\zeta_i - \delta)} = \mathbb{E} \prod_{B_i \in \mathcal{G}_{s,t}} \mathbb{E}\Big(e^{a(\zeta_i - \delta)} \,\Big|\, \mathcal{F}_{s,t}\Big).$$

Next, we will estimate $\mathbb{E}\big(e^{a(\zeta_i - \delta)} \mid \mathcal{F}_{s,t}\big)$. Let $g(K) = e^{-0.01\sqrt{K}}$. Since $K \geq \widehat{K}_1(\lambda) = \widehat{K}_0(\lambda + \varepsilon_1)$, by Corollary 4.1, ζ_i is a Bernoulli random variable with $\mathbb{P}(\zeta_i = 1 \mid \mathcal{F}_{s,t}) \leq g(K)$. Consequently,

$$\mathbb{E}\Big(e^{a(\zeta_i - \delta)} \,\Big|\, \mathcal{F}_{s,t}\Big) \leq e^{a(1-\delta)} g(K) + e^{-a\delta} \leq 2g(K)^\delta,$$

where in the last inequality we set $a = \log\Big(\frac{\delta}{1-\delta} g(K)^{-1}\Big)$. By the Cauchy-Schwarz inequality,

$$\mathbb{E}e^{a\beta_K \sum_{i=1}^d (\zeta_i - \delta)} = \mathbb{E} \prod_{s=1}^{4K^{1/4}} \prod_{t=1}^{12} W_{s,t} \leq \prod_{s=1}^{4K^{1/4}} \prod_{t=1}^{12} \Big(\mathbb{E}W_{s,t}^{1/\beta_K}\Big)^{\beta_K}.$$

Collecting the above results, we conclude that

$$\begin{aligned}\mathbb{P}\bigg(\frac{1}{d}\sum_{i=1}^d \zeta_i > \delta\bigg) &\leq \mathbb{E}e^{a\beta_K \sum_{i=1}^d (\zeta_i - \delta)} \leq \prod_{s=1}^{4K^{1/4}} \prod_{t=1}^{12} \prod_{B_i \in \mathcal{G}_{s,t}} \big(2g(K)^\delta\big)^{\beta_K} \\ &\leq \big(2g(K)^\delta\big)^{\beta_K d} \leq e^{-10^{-4}K^{1/4}\delta d}.\end{aligned}$$

This completes the proof of Lemma 4.5. □

Recall that $C_5 = (2 \vee C_3)^{32}$ is defined in Lemma 3.5. The following lemma is a deterministic geometric fact.

Lemma 4.6. *Suppose $K \geq C_5$, $j \in [2, m-1] \cap \mathbb{Z}$ and $B \in \mathrm{END}_j$. Let $\{B'_t : 1 \leq t \leq T\}$ consist of boxes in $\mathcal{BD}_{j-3}$ that intersect with some path in $\mathcal{P}_j(B)$. Then, $T \leq K^7$.*

Proof: Suppose $P \in \mathcal{P}_j(B)$, and it is started at x_P. By the definition of $\mathcal{SL}_j$ (see (2.1)),

$$K^j \leq \|P\| \leq K^j + K^{j-1}, \quad P \subseteq B(x_P, \|P\|), \quad \text{and } \|x_P - z_B\|_\infty \leq K^{j-2}.$$

If $B' \in \mathcal{BD}_{j-3}$ intersects with a path in $\mathcal{P}_j(B)$, one has $B' \subseteq B_\infty(z_B, K^j + K^{j-1} + K^{j-2} + K^{j-3})$. Note that boxes in $\mathcal{BD}_{j-3}$ are non-overlapping. For $K \geq C_5 \geq 2^{32}$, we have $T \leq \big(\frac{2K^j}{K^{j-3}}\big)^2 \leq K^7$. □

Recall that $\widehat{\mathcal{E}}_{\mathcal{S}}$ is the event that fluctuations of some H_i is not small, defined in (4.9). Define the event

$$\mathcal{E}_1 := \bigcup_{d\geq K} \bigcup_{\mathcal{S}\in \mathrm{END}^{(d)}_{j-1}(B)} \widehat{\mathcal{E}}_{\mathcal{S}}. \tag{4.11}$$

Next, we will estimate $\mathbb{P}(\mathcal{E}_1)$. Recall that $C_5 = (2 \vee C_3)^{32}$ is defined in Lemma 3.5.

Lemma 4.7. *Suppose $j \in [2, m-1] \cap \mathbb{Z}$, $B \in \mathrm{END}_j$, and $K \geq C_5$. Then, $\mathbb{P}\big(\mathcal{E}_1\big) \leq e^{-1.5K^{1/8}}$.*

Proof: The proof is analogous to that of Lemma 3.5. Let $\{B'_t : t \in [T]\}$ be those boxes given in Lemma 4.6. For each B'_t, abbreviate $z'_t = z_{B'_t}$ and $W_t = V_{K^{j-7/8}}(z'_t)$. Let H'_t be the conditional expectation of η^V given $\eta^V|_{W_t\cup\partial W_t}$. By Lemma 2.6, for $K \geq C_5$ and $t \in [T]$, we have $\mathbb{P}(\mathcal{E}_{1,t}) \leq 4e^{-2K^{1/8}}$, where

$$\mathcal{E}_{1,t} := \big\{\big|H'_t(x) - H'_t(z'_t)\big| \geq \varepsilon_1 \text{ for some } x \in V_{4K^{j-1}}(z'_t)\big\}.$$

Note that

$$\mathcal{E}_1 \subseteq \bigcup_{t\in[T]} \mathcal{E}_{1,t},$$

and $T \leq K^7$. We conclude

$$\mathbb{P}(\mathcal{E}_1) \leq 4K^7 e^{-2K^{1/8}} \leq e^{-1.5K^{1/8}},$$

which completes the proof. □

Finally, we will prove Proposition 4.2, by combining the above results .

Proof of Proposition 4.2: Suppose $\lambda \geq 1$, $K \geq \widehat{K}_1(\lambda)$, $j \in [2, m-1] \cap \mathbb{Z}$, $B \in \mathrm{END}_j$, $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$. By the decomposition of paths (see (4.7)),

$$\mathbb{P}\big(\xi_{1,\lambda,\bar{\alpha},j,B} > \delta\big) \leq \sum_{d=K}^{\infty} \sum_{\mathcal{S}\in \mathrm{END}^{(d)}_{j-1}(B)} \mathbb{P}\big(\{\zeta_{1,\mathcal{S}} > \delta\} \cap \widehat{\mathcal{E}}^c_{\mathcal{S}}\big) + \mathbb{P}\big(\mathcal{E}_1\big), \tag{4.12}$$

where $\xi_{1,\lambda,\bar{\alpha},j,B}$ and $\zeta_{1,\mathcal{S}}$ are defined in (4.3) and above (4.10), respectively. By Lemma 4.6, there are at most K^{7d} sequences in $\mathrm{END}^{(d)}_{j-1}(B)$. Combining (4.12), (4.10), Lemma 4.5 and Lemma 4.7, and using the union bound, we conclude that for $\delta \geq \frac{1}{2}$,

$$\mathbb{P}(\xi_{1,\lambda,\bar{\alpha},j,B} > \delta) \leq \sum_{d=K}^{\infty} K^{7d} e^{-10^{-4}K^{1/4}\delta d} + e^{-1.5K^{1/8}} \leq e^{-K^{1/8}}.$$

This completes the proof of Proposition 4.2. □

4.2. *Proof of Proposition 4.3.* The framework of this section is similar to that of the previous section, which deals with the specific case $r = 1$. In this section, we will apply an induction analysis on r, by using the recursive relation of $Y_{P,r,\lambda,\bar{\alpha}}$'s (see (4.15)). The proof is similar to that of Proposition 4.2, but suitable adjustments need to be made.

Assume $2 \leq r < j \leq m-1$, $B \in \mathrm{END}_j, V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$ in this section. Let $\mathcal{S} := \{B_i\}_{i\in[d]}$, which is a sequence of end-boxes of side length K^{j-3}. Compared with (4.8), we adjust the side length of U_i here, and define

$$z_i := z_{B_i} \text{ and } U_i := V_{4K^{j-1}}(z_i) \text{ for all } i \in [d]. \tag{4.13}$$

Note that $U_i \subseteq V$ for all i. Let H_i be the conditional expectation of η^V given $\eta^V|_{U_i^c\cup\partial U_i}$. Recall that $\beta = 2^{-9}$ and $\varepsilon_{r+1} = 4\sqrt{C_4}\beta^{r/2}$ are defined in (4.4). Define the events

$$\widehat{\mathcal{E}}_{r,\mathcal{S}} := \bigcup_{i\in[d]} \mathcal{E}_{r,i}, \quad \text{and} \quad \mathcal{E}_r := \bigcup_{d\geq K} \bigcup_{\mathcal{S}\in \mathrm{END}^{(d)}_{j-1}(B)} \widehat{\mathcal{E}}_{r,\mathcal{S}} \text{ for all } r \geq 2, \tag{4.14}$$

where

$$\mathcal{E}_{r,i} := \big\{\text{there exists a box } \widetilde{B} \in \mathrm{END}_{j-r} \text{ such that } \widetilde{B} \subseteq V_{3K^{j-1}}(z_i) \text{ and } \big|H_i(x) - H_i(z_{\widetilde{B}})\big| \geq \varepsilon_r \text{ for some } x \in V_{4K^{j-r}}(z_{\widetilde{B}})\big\}.$$

Note that if $\widetilde{B} \subseteq V_{3K^{j-1}}(z_i)$ then $V_{4K^{j-r}}(z_{\widetilde{B}}) \subseteq U_i$ for all $r \geq 2$ (recall (4.13)). Here, $\widehat{\mathcal{E}}_{r,s}$ and $\mathcal{E}_r$ play the roles of $\widehat{\mathcal{E}}_{\mathcal{S}}$ defined in (4.9) and $\mathcal{E}_1$ defined in (4.11), respectively.

Suppose $P \in \mathcal{P}_{j,\mathcal{S}}(B)$. Then, the i-th child-path $P^{(i)}$ of P is started from B_i. Suppose $u \in \mathcal{T}_{P^{(i)},r-1}$, i.e., u is a node at level $r-1$ in $\mathcal{T}_{P^{(i)}}$, and thus at level r in $\mathcal{T}_P$. Then, P^u is in scale K^{j-r}. Recall that B_u is the unique box in END_{j-r} containing the starting site of P^u, and $z_u := z_{B_u}$. Since P^u is a sub-path of $P^{(i)}$ and $P^{(i)}$ is started from B_i, we have

$$B_u \subseteq V_{3K^{j-1}}(z_i) \subseteq U_i, \quad \text{ and } \quad P^u \subseteq V_{4K^{j-r}}(z_u) \subseteq U_i.$$

For all $x \in P^u$, we write

$$\eta^V(x) = \big(\eta^{U_i}(x) + H_i(z_u)\big) + (H_i(x) - H_i(z_u)).$$

By the triangle inequality, on the event $\mathcal{E}_{r,i}^c$, u is $(U_i, \lambda+\varepsilon_r, \bar{\alpha}_u + H_i(z_u))$-open if it is $(V, \lambda, \bar{\alpha}_u)$-open. Recall that $Y_{P,r,\lambda,\bar{\alpha}}$ is the fraction of tame and $(V, \lambda, \bar{\alpha}_u)$-open nodes in $\mathcal{T}_P$, defined in (4.2). Then,

$$Y_{P,r,\lambda,\bar{\alpha}} = \frac{1}{d}\sum_{i=1}^{d} Y_{P^{(i)},r-1,\lambda,\bar{\alpha}}. \tag{4.15}$$

Furthermore, define

$$\zeta_{r,\mathcal{S}} := \max\big\{Y_{P,r,\lambda,\bar{\alpha}} : P \in \mathcal{P}_{j,\mathcal{S}}(B)\big\},$$

$$\zeta_{i,r-1,\lambda+\varepsilon_r,j-1} := \max_{Q \in \mathcal{P}_{j-1}(B_i)} \sum_{u \in \mathcal{T}_{Q,r-1}} \theta_Q(u) 1_{\{u \text{ is tame and } (U_i,\lambda+\varepsilon_r,\bar{\alpha}_u+H_i(z_u))\text{-open}\}}. \tag{4.16}$$

Here, $\zeta_{r,\mathcal{S}}$ and $\zeta_{i,r-1,\lambda+\varepsilon_r,j-1}$ play the roles of $\zeta_{1,\mathcal{S}}$ and ζ_i (defined above (4.10)) in the previous proof, respectively. Then, we have

$$\mathbb{P}\Big(\{\zeta_{r,\mathcal{S}} > \delta\} \cap \widehat{\mathcal{E}}_{r,\mathcal{S}}^c\Big) \leq \mathbb{P}\Big(\frac{1}{d}\sum_{i=1}^{d} \zeta_{i,r-1,\lambda+\varepsilon_r,j-1} > \delta\Big), \tag{4.17}$$

which is analogous to (4.10).

The proof of the following Lemma 4.8 is quite analogous to that of Lemma 4.7, thus is omitted.

Lemma 4.8. *Let $r \in [2, m-2] \cap \mathbb{Z}$, $j \in [r+1, m-1] \cap \mathbb{Z}$, $B \in \mathrm{END}_j$ and $K \geq C_5$. Then,* $\mathbb{P}(\mathcal{E}_r) \leq e^{-c_{r-1}}$.

Next, we will show Proposition 4.3. The idea of the proof is essentially similar to that of Lemma 4.5, while the whole proof is much more complicated here.

Proof of Proposition 4.3: We will prove that the following statements (i) and (ii) hold for all $2 \leq r < j \leq m-1$.

(i) Suppose $\lambda \geq 1$, $K \geq \widehat{K}_r(\lambda)$, $B \in \mathrm{END}_j$, $V_{4K^j}(z_B) \subseteq V \subseteq V_N$, and $\bar{\alpha} \in \mathscr{A}$. Then,

$$\mathbb{P}(\xi_{r,\lambda,\bar{\alpha},j,B} > \delta) \leq 2e^{-c_{r-1}(\delta-\delta_r)} \quad \text{ for all } \delta > \delta_r.$$

(ii) Suppose $\lambda \geq 1$, $K \geq \widehat{K}_{r+1}(\lambda)$, $B \in \mathrm{END}_{j+1}$, $d \geq K$, $\{B_i\}_{i\in[d]} \in \mathrm{END}_j^{(d)}(B)$, and $\bar{\alpha} \in \mathscr{A}$. Recall (4.16) for the definition of $\zeta_{i,r,\lambda+\varepsilon_{r+1},j}$. Then,

$$\mathbb{P}\Big(\frac{1}{d}\sum_{i=1}^{d} \zeta_{i,r,\lambda+\varepsilon_{r+1},j} > \delta\Big) \leq \Big(K^{-9} e^{-\beta c_{r-1}(\delta-\delta_{r+1})}\Big)^d \text{ for all } \delta > \delta_{r+1}. \tag{4.18}$$

In Step 1, we will show that (i) implies (ii) for the same r. In Step 2, we will show (i) for r being replaced with $r+1$ and all $j \in [r+2, m-1] \cap \mathbb{Z}$, provided that (ii) holds for all $j \in [r+1, m-1] \cap \mathbb{Z}$.

In Step 3, we will show (i) holds for $r = 2$ and $j \in [3, m-1] \cap \mathbb{Z}$, which serves as the induction's basis.

Step 1. Suppose (i) holds. We will prove (ii). The proof is analogous to that of Lemma 4.5. Concretely, we can divide $\{B_i\}_{i\in[d]}$ into $432(< 2^9 = \beta^{-1})$ groups $\mathcal{G}_t$'s such that U_i's in each group are disjoint, where $U_i = V_{4K^{j-1}}(z_i)$ (see (4.13)). Denote $\widetilde{U}_t = \bigcup_{i:B_i\in\mathcal{G}_t} U_i$. Then, define the sigma-algebra

$$\mathcal{F}_t := \sigma\big\{\eta^V(x) : x \in (V \setminus \widetilde{U}_t) \cup \partial\widetilde{U}_t\big\},$$

which plays the role of $\mathcal{F}_{s,t}$ in the proof of Lemma 4.5. Abbreviate $\zeta_i = \zeta_{i,r,\lambda+\varepsilon_{r+1},j}$.

For each $i \in [d]$, applying (i) to $\lambda + \varepsilon_{r+1}$, we have for all $K \geq \widehat{K}_{r+1}(\lambda) := \widehat{K}_r(\lambda + \varepsilon_{r+1})$,

$$\mathbb{P}(\zeta_i > \delta \mid \mathcal{F}_t) \leq 2e^{-c_{r-1}(\delta-\delta_r)} \quad \text{for all } \delta > \delta_r. \tag{4.19}$$

It follows that

$$\mathbb{E}\Big(e^{c_{r-1}(\zeta_i-\delta)} \;\Big|\; \mathcal{F}_t\Big) \leq (1 + 2c_{r-1})e^{-c_{r-1}(\delta-\delta_r)}.$$

By replacing $2g(K)$ in the proof of Lemma 4.5 with the right hand side in the above inequality, in an analogous way, we conclude that

$$\mathbb{P}\bigg(\frac{1}{d}\sum_{i=1}^{d}\zeta_i > \delta\bigg) \leq \Big((1 + 2c_{r-1})e^{-c_{r-1}(\delta-\delta_r)}\Big)^{\beta d} \quad \text{for all } \delta > \delta_r.$$

For $\delta > \delta_{r+1}$, we split $\delta - \delta_r$ into $(\delta - \delta_{r+1}) + \Delta_r$ on the right hand side of the above inequality, where Δ_r is defined above (4.5). Then, we obtain (4.18), finishing the first step.

Step 2. Assuming that (ii) holds for all $j \in [r+1, m-1]\cap\mathbb{Z}$, we will show (i) for r being replaced with $r+1$ and all $j \in [r+2, m-1]\cap\mathbb{Z}$. Note that $r+2 \leq j \leq m-1$ implies $r+1 \leq j-1 \leq m-2$. For $K \geq \widehat{K}_{r+1}(\lambda)$, we apply (ii) to $j-1$, and have

$$\mathbb{P}\bigg(\frac{1}{d}\sum_{i=1}^{d}\zeta_{i,r,\lambda+\varepsilon_{r+1},j-1} > \delta\bigg) \leq \Big(K^{-9}e^{-\beta c_{r-1}(\delta-\delta_{r+1})}\Big)^{d} \quad \text{for all } \delta > \delta_{r+1}.$$

Combining it with (4.17) and Lemma 4.8, we obtain that for all $\delta > \delta_{r+1}$,

$$\begin{aligned}\mathbb{P}\big(\xi_{r+1,\lambda,\bar{\alpha},j,B} > \delta\big) &\leq \sum_{d=K}^{\infty}\sum_{\mathcal{S}\in\mathrm{END}^{(d)}_{j-1}(B)} \mathbb{P}\big(\{\zeta_{r+1,\mathcal{S}} > \delta\} \cap \widehat{\mathcal{E}}^c_{r+1,\mathcal{S}}\big) + \mathbb{P}\big(\mathcal{E}_{r+1}\big)\\ &\leq \sum_{d=K}^{\infty} K^{7d}\Big(K^{-9}e^{-\beta c_{r-1}(\delta-\delta_{r+1})}\Big)^{d} + e^{-c_r} \leq 2e^{-c_r(\delta-\delta_{r+1})},\end{aligned}$$

via an argument similar to (4.12). This finishes Step 2.

Step 3. Applying Proposition 4.2 to $\lambda + \varepsilon_2$ and $K \geq \widehat{K}_2(\lambda) = \widehat{K}_1(\lambda + \varepsilon_2)$, we obtain that

$$\mathbb{P}(\zeta_{i,1,\lambda+\varepsilon_2,j-1} > \delta \mid \mathcal{F}_t) \leq \exp\{-K^{1/8}\} \quad \text{for all } \delta \geq \delta_1.$$

By a reasoning similar to Step 1, where (4.19) is replaced with the above formula, we obtain (i) for $r = 2$ and $j \in [3, m-1] \cap \mathbb{Z}$. This finishes Step 3.

As observed above, this completes the proof of Proposition 4.3. □

4.3. *Proof of Proposition 4.4.* In this section, we will suppose that $\lambda \geq 1$, $\delta \in (0,1)$, $K := 2^k \geq C$, $N \geq e^{C'K^5}$, where $C' = C'(\kappa,\delta)$ is defined in Lemma 2.10. Recall

$$\mathcal{P}^{\kappa,\delta,K}_N = \Big\{P : P \text{ is a path in } V_{N/2}, \|P\| \geq \kappa N \text{ and } |P| \leq N^{1+\frac{\delta}{K^{2k}}}\Big\},$$

defined in (2.2). Suppose $P \in \mathcal{P}^{\kappa,\delta,K}_N$. By the tree construction in Section 2.2, one can extract d_P child-paths $P^{(i)}$'s from P, which are in $\mathcal{SL}_{m-1}$.

Furthermore, P is identified as the root of the associated tree $\mathcal{T}_P$ (refer to the beginning of Section 2.2 for the construction of associated trees). For $u \in \mathcal{T}_{P^{(i)}}$, one has $L_i(u) = L(u) - 1$, where $L(u)$ and $L_i(u)$ are respectively the levels of u with respect to $\mathcal{T}_P$ and $\mathcal{T}_{P^{(i)}}$. Let θ_P and $\theta_{P^{(i)}}$ be the unit uniform flows on $\mathcal{T}_P$ and $\mathcal{T}_{P^{(i)}}$, respectively(see the paragraph just above (2.2) for the definition of such flows). By Lemma 2.10,

$$\sum_{i=1}^{d_P} \sum_{u:0 \le L_i(u) \le m-2} \frac{1}{d_P} \theta_{P^{(i)}}(u) 1_{\{u \text{ is untamed}\}} = \sum_{u:1 \le L(u) \le m-1} \theta_P(u) 1_{\{u \text{ is untamed}\}} \le 2\delta m.$$

This implies that there exists $i_0 \in [d_P]$ such that

$$\sum_{u:0 \le L_{i_0}(u) \le m-2} \theta_{P^{(i_0)}}(u) 1_{\{u \text{ is untamed}\}} \le 2\delta m.$$

Next, we will assume that P is λ-open for the rest of the proof. Then, it requires to investigate the fraction of tame and open nodes in the tree $\mathcal{T}_{\widetilde{P}}$ associated with a path $\widetilde{P}$ in scale K^{m-1}, where $\widetilde{P}$ actually stands for $P^{(i_0)}$. Suppose $B \in \mathrm{END}_{m-1}$ and $\widetilde{P} \in \mathcal{P}_{m-1}(B)$, which is a path in scale K^{m-1}. Recall that $v \in V_N$ is λ-open if $\left|\eta^{V_N}(v)\right| \le \lambda$, i.e., v is $(V_N, \lambda, 0)$-open. Set $V = V_N$, $\bar{\alpha} \equiv 0$, $j = m-1$ and $P = \widetilde{P}$ in (4.2) and (4.3) specifically. Then, the variables $Y_{P,r,\lambda,\bar{a}}$ and $\xi_{r,\lambda,\bar{\alpha},j,B}$ therein have the following specific expressions:

$$\widetilde{Y}_{\widetilde{P},r,\lambda} := \sum_{u \in \mathcal{T}_{\widetilde{P},r}} \theta_{\widetilde{P}}(u) 1_{\{u \text{ is tame and } \lambda\text{-open}\}},$$

$$\widetilde{\xi}_{r,\lambda,B} := \max\left\{\widetilde{Y}_{\widetilde{P},r,\lambda} : \widetilde{P} \in \mathcal{P}_{m-1}(B)\right\}.$$

By Proposition 4.3, for all $\lambda \ge 1$, $2 \le r \le m-2$, $K \ge \widehat{K}_r(\lambda)$,

$$\mathbb{P}\left(\widetilde{\xi}_{r,\lambda,B} > \delta\right) \le 2e^{-c_{r-1}(\delta - \delta_r)} \quad \text{for all } \delta > \delta_{r+1}, \tag{4.20}$$

where δ_r and c_r are respectively defined in and above (4.5).

Denote by B_{i_0} be the unique box in END_{m-1} from which $P^{(i_0)}$ is started. Then, we have

$$\begin{aligned} m-1 &= \sum_{u:0 \le L_{i_0}(u) \le m-2} \theta_{P^{(i_0)}}(u) 1_{\{u \text{ is } \lambda\text{-open}\}} \\ &= \sum_{r=0}^{m-2} \widetilde{Y}_{P^{(i_0)},r,\lambda} + \sum_{u:0 \le L_{i_0}(u) \le m-2} \theta_{P^{(i_0)}}(u) 1_{\{u \text{ is untamed}\}} \le \sum_{r=0}^{m-2} \widetilde{\xi}_{r,\lambda,B_{i_0}} + 2\delta m. \end{aligned}$$

By the above inequality,

$$\begin{aligned} &\mathbb{P}\left(P \text{ is } \lambda\text{-open for some } P \in \mathcal{P}_N^{\kappa,\delta,K}\right) \\ \le &\mathbb{P}\left(\sum_{r=0}^{m-2} \widetilde{\xi}_{r,\lambda,B} \ge m - 1 - 2\delta m \text{ for some } B \in \mathrm{END}_{m-1}\right). \end{aligned} \tag{4.21}$$

Recall $\widehat{K}_0(\lambda) = e^{c\lambda^2}$ (see (3.11)), and that ε_r's and $\widehat{K}_r(\lambda) = \widehat{K}_0(\lambda + \sum_{i=1}^r \varepsilon_i)$ are respectively defined in and below (4.4). Then, noting that $\sum_{i=1}^\infty \varepsilon_i < \infty$ and $\widehat{K}_0(\lambda)$ is increasing in λ, one has

$$\widehat{K}_\infty(\lambda) := \widehat{K}_0\left(\lambda + \sum_{i=1}^\infty \varepsilon_i\right) < \infty.$$

Recall (4.5) and see the paragraph just above it for the definitions of δ_r's, c_r's and Δ_r's. Furthermore, for all $\delta > 0$, there exists $b = b(\delta) > c$ such that for all $\lambda \ge 1$,

$$\widetilde{K}(\lambda, \delta) := e^{b\lambda^2} \ge \widehat{K}_\infty(\lambda),$$

and

$$\sum_{r=1}^{\infty}\Delta_r=\frac{9\log K}{\beta K^{1/8}}+\sum_{r=1}^{\infty}\frac{\log(1+2c_r)+9\beta^{-1}\log K}{c_r}\le\delta.$$

Next, we will assume $K\ge\widetilde{K}(\lambda,\delta)$ for the rest of the proof. It follows that $\delta_r\le\frac{1}{2}+\delta$ for all $r\ge 0$. Next, we will set $\delta\in(\frac{1}{32},\frac{3}{32})$ and assume $m\ge 32$. Then,

$$\sum_{r=0}^{m-2}\left(\delta_r+\frac{1}{2^{r+2}}(1-8\delta)m\right)<m-1-2\delta m.$$

Note that at most $(K^6/\kappa)^2$ different boxes lie in END_{m-1}, since $\kappa N<K^{m+2}$ by (1.3). Consequently, by the union bound,

$$\mathbb{P}\left(\sum_{r=0}^{m-2}\widetilde{\xi}_{r,\lambda,B}\ge m-1-2\delta m\text{ for some }B\in\mathrm{END}_{m-1}\right)\le\frac{K^{12}}{\kappa^2}\sum_{r=0}^{m-2}p_{m,r},\tag{4.22}$$

where

$$p_{m,r}:=\max_{B\in\mathrm{END}_{m-1}}\mathbb{P}\left(\widetilde{\xi}_{r,\lambda,B}>\delta_r+\frac{1}{2^{r+2}}(1-8\delta)m\right).$$

For $r=0,1$, we have $\frac{1}{2^{r+2}}(1-8\delta)m\ge 1$, which implies $p_{m,r}=0$. By (4.20), for $2\le r\le m-2$, we have

$$p_{m,r}\le 2\exp\left\{-\frac{1}{8}(1-8\delta)\left(\frac{\beta K}{2}\right)^{r-1}m\right\}.$$

Furthermore, $\beta K/2\ge 2^{22}$, and thus $\frac{1}{8}\left(\frac{\beta K}{2}\right)^{r-1}\ge r$ for all $r\ge 2$. Therefore,

$$\sum_{r=0}^{m-2}p_{m,r}\le 2\sum_{r=2}^{m-2}e^{-(1-8\delta)mr}\le\frac{2}{e^{(1-8\delta)m}-1}.\tag{4.23}$$

Take constants $\delta':=\frac{1}{16}$ and $b'=b'(\kappa)$ such that for all $\lambda\ge 1$,

$$K_0:=e^{b'\lambda^2}\ge\widetilde{K}(\lambda,\delta')\vee C'(\kappa,\delta')\vee(8\kappa^{-2}).$$

Denote $a:=100\times b'$. For $N\ge\exp\{e^{a\lambda^2}\}$, one has $\log N\ge C'(\kappa,\delta')K_0^5$. Consequently, the above assumption $m\ge 32$ is satisfied. Combining (4.21), (4.22) and (4.23), we obtain that for $N\ge\exp\{e^{a\lambda^2}\}$,

$$\begin{aligned}&\mathbb{P}\big(P\text{ is }\lambda\text{-open for some }P\in\mathcal{P}_N^{\kappa,\delta',K_0}\big)\\ \le&\frac{K_0^{12}}{\kappa^2}4e^{-\frac{m}{4}}\le 4\sqrt{e}\kappa^{-2-(4b'\lambda^2)^{-1}}e^{12b'\lambda^2}N^{-(4b'\lambda^2)^{-1}}\\ \le&8\kappa^{-2}e^{-\left(\frac{\log N}{4b'\lambda^2}-12b'\lambda^2\right)}\le e^{-\left(\frac{\log N}{a\lambda^2}-a\lambda^2\right)}.\end{aligned}$$

Finally, set $\epsilon:=\frac{\delta'}{K_0^2\log_2 K_0}$. Noting $\epsilon=(16\log_2 e)^{-1}\lambda^{-2}e^{-2b'\lambda^2}\ge e^{-a\lambda^2}$, we have $\mathcal{P}_N^{\kappa,\epsilon}\subseteq\mathcal{P}_N^{\kappa,\delta',K_0}$ (see (1.2) for the definition of $\mathcal{P}_N^{\kappa,\epsilon}$). It follows that

$$\mathbb{P}\big(P\text{ is }\lambda\text{-open for some }P\in\mathcal{P}_N^{\kappa,\epsilon}\big)\le e^{-\left(\frac{\log N}{a\lambda^2}-a\lambda^2\right)},$$

completing the proof of Proposition 4.4.

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