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Factorization and discrete-time representation of multivariate CARMA processes

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Abstract. In this paper we show that stationary and non-stationary multivariate continuous-time ARMA (MCARMA) processes have the representation as a sum of multivariate complex-valued Ornstein-Uhlenbeck processes under some mild assumptions. The proof benefits from properties of rational matrix polynomials. A conclusion is an alternative description of the autocovariance function of a stationary MCARMA process. Moreover, that representation is used to show that the discrete-time sampled MCARMA(p,q) process is a weak VARMA$(p,p-1)$ process if second moments exist. That result complements the weak VARMA$(p,p-1)$ representation derived in Chambers and Thornton (2012). In particular, it relates the right solvents of the autoregressive polynomial of the MCARMA process to the right solvents of the autoregressive polynomial of the VARMA process; in the one-dimensional case the right solvents are the zeros of the autoregressive polynomial. Finally, a factorization of the sample autocovariance function of the noise sequence is presented which is useful for statistical inference.

1. Introduction

A multivariate continuous-time ARMA (MCARMA) process is a continuous-time version of the well-known vector ARMA (VARMA) process in discrete time. They are applied in diversified fields as, e.g., signal processing and control (cf. Garnier and Wang (2008); Larsson et al. (2006)), high-frequency financial econometrics (cf. Todorov (2009)) and financial mathematics (cf. Andresen et al. (2014)). The driving process of a MCARMA process is a Lévy process $L=(L(t))_{t\geq 0}$ which is an $\mathbb{R}^m$-valued stochastic process with $L(0)=0_m$ $\mathbb{P}$-a.s., stationary and independent increments and càdlàg sample paths. The idea is then that a d-dimensional MCARMA(p,q) process ($p>q$ positive integers) is the solution of the stochastic differential equation

$$A(D)Y(t)=B(D)DL(t) \quad \text{for } t\geq 0, \tag{1.1}$$

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where D is the differential operator with respect to t,

$$A(\lambda) := I_d\lambda^p + A_1\lambda^{p-1} + \ldots + A_p \quad \text{and} \quad B(\lambda) := B_0\lambda^q + \ldots + B_{q-1}\lambda + B_q \tag{1.2}$$

is the autoregressive and the moving average polynomial, respectively with $A_1, \ldots, A_p \in \mathbb{R}^{d\times d}$ and $B_0, \ldots, B_q \in \mathbb{R}^{d\times m}$. The matrix I_d denotes the $d \times d$-dimensional identity matrix and $0_{d\times m}$ denotes the $d \times m$-dimensional matrix whose entries are all zero in the following. In contrast, in discrete time the differential operator is replaced by the backshift operator and the differential of the Lévy process $DL(t)$ is replaced by a weak white noise. Since a Lévy process is not differentiable, the question arises what is the formal definition of a MCARMA process. We can interpret (1.1) via linear continuous-time state space models as in Marquardt and Stelzer (2007). Therefore, define

$$A^* := \begin{pmatrix} 0_{d\times d} & I_d & 0_{d\times d} & \cdots & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & I_d & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0_{d\times d} \\ 0_{d\times d} & \cdots & \cdots & 0_{d\times d} & I_d \\ -A_p & -A_{p-1} & \cdots & \cdots & -A_1 \end{pmatrix} \in \mathbb{R}^{pd\times pd}, \tag{1.3}$$

$C^* := (I_d, 0_{d\times d}, \ldots, 0_{d\times d}) \in \mathbb{R}^{d\times pd}$ and $B^* := (\beta_1^\mathsf{T} \cdots \beta_p^\mathsf{T})^\mathsf{T} \in \mathbb{R}^{pd\times m}$ with $\beta_1 := \ldots := \beta_{p-q-1} := 0_{d\times m}$ and

$$\beta_{p-j} := -\sum_{i=1}^{p-j-1} A_i\beta_{p-j-i} + B_{q-j}, \quad j = 0, \ldots, q.$$

Then the $\mathbb{R}^d$-valued *MCARMA*(p, q) *process* $Y := (Y(t))_{t\geq 0}$ is defined by the state space equation

$$Y(t) = C^*X(t) \quad \text{and} \quad \mathrm{d}X(t) = A^*X(t)\,\mathrm{d}t + B^*\,\mathrm{d}L(t). \tag{1.4}$$

Note that, if we define

$$A^{\#} = \begin{pmatrix} I_d & 0_{d\times d} & \cdots & 0_{d\times d} \\ A_1 & I_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{d\times d} \\ A_{p-1} & \cdots & A_1 & I_d \end{pmatrix} \in \mathbb{R}^{pd\times pd}, \quad B^{\#} = \begin{pmatrix} 0_{(p-(q+1))d\times m} \\ B_0 \\ \vdots \\ B_q \end{pmatrix} \in \mathbb{R}^{pd\times m}, \tag{1.5}$$

then

$$A^{\#}B^* = B^{\#}. \tag{1.6}$$

The class of MCARMA processes is very rich. Under the constrain of finite second moments, Schlemm and Stelzer (2012a), Corollary 3.4, show that the class of stationary MCARMA processes and the class of stationary state space models are equivalent (see Fasen-Hartmann and Scholz (2020) for cointegrated MCARMA processes).

The aim of the paper is to present sufficient criteria for stationary and non-stationary MCARMA(p, q) processes to have a representation as a sum of p multivariate Ornstein-Uhlenbeck processes (which are MCAR(1) = MCARMA(1,0) processes). In the one-dimensional case $d = 1$, under the assumption of distinct zeros $r_1, \ldots, r_p$ with strictly negative real parts of $A(\lambda)$, it is well-known that

$$Y(t) = \sum_{k=1}^{p} Y_k(t) \quad \text{with} \quad Y_k(t) = \int_{-\infty}^{t} \mathrm{e}^{r_k(t-u)}\,\frac{B(r_k)}{A'(r_k)}\,\mathrm{d}L(u) \tag{1.7}$$

is a stationary solution of the state space model (1.4) and hence, a CARMA process (see Brockwell et al. (2011), Proposition 2). The term $B(r_k)/A'(r_k)$ is the residue of $A(\lambda)^{-1}B(\lambda)$ at r_k. In the present paper we extend this finding to the multivariate setup for both stationary and non-stationary MCARMA processes. The zero r_k of $A(\lambda)$ in the one-dimensional case is replaced by a $d \times d$ matrix

R_k in the multivariate case, which is as well a kind of multivariate "zeros" of $A(\lambda)$, the so called right solvent satisfying $A_R(R_k) := R_k^p + A_1 R_k^{p-1} + \ldots + A_p = 0_{d\times d}$; the statement is derived in Theorem 3.2. Essential for our proof are basic principles from rational matrix polynomials coming from linear algebra, which are not necessary in dimension $d = 1$. A main feature of our outcome is that Y has a representation as a sum of multivariate Ornstein-Uhlenbeck processes and not only as a linear combination of multivariate Ornstein-Uhlenbeck processes. Since matrix multiplication is not commutative this is not trivial. That is different to the one-dimensional case where any linear combination of stationary Ornstein-Uhlenbeck processes is as well a sum of stationary Ornstein-Uhlenbeck processes. A straightforward consequence of our result is an alternative representation of the autocovariance function of a stationary MCARMA process in Proposition 3.7.

Although we consider in this paper a continuous-time model, the corresponding discrete-time models are of special interest. Despite having a continuous-time model, a reason for this is that one often observes the process only at discrete time points as, e.g, in the context of high-frequency data. Hence, we use the representation of a MCARMA(p, q) process as a sum of p multivariate Ornstein-Uhlenbeck processes to derive a vector-valued ARMA (VARMA)$(p, p-1)$ representation for the low frequency sampled MCARMA process $(Y(nh))_{n\in\mathbb{N}}$ ($h > 0$ fixed) in Theorem 3.9. For the proof of this theorem a representation of Y as a linear combination of multivariate Ornstein-Uhlenbeck processes is not sufficient. The statement is a direct extension of the ARMA$(p, p-1)$ representation of discretely sampled CARMA processes in Brockwell et al. (2011, Proposition 3) whose autoregressive polynomial $\prod_{k=1}^{p}(\lambda - \mathrm{e}^{-r_k h})$ of the ARMA representation has zeros $\mathrm{e}^{-r_1 h}, \ldots, \mathrm{e}^{-r_p h}$. In analogy, in the multivariate setup of this paper, the autoregressive polynomial of the VARMA representation has right solvents $\mathrm{e}^{-R_1 h}, \ldots, \mathrm{e}^{-R_p h}$.

In the econometric literature, the VARMA$(p, p-1)$ representation of a discretely sampled MCARMA process is well-known, see, e.g., Chambers and Thornton (2012), Corollary 1; a nice overview on this topic is presented in Chambers et al. (2018). In contrast to us, Chambers and Thornton (2012) assume some kind of observability and controllability conditions on submatrices of $\mathrm{e}^{A^{**}}$, where A^{**} is constructed form A^* by reflecting the entries of A^* at the diagonal from the left lower corner to the right upper corner. There, the coefficients of the autoregressive polynomial in the VARMA representation are complicated functions of these submatrices. The current paper presents an alternative and simpler representation of the VARMA parameters and in particular, it connects the autoregressive polynomial in the MCARMA representation to the autoregressive polynomial in the VARMA representation due to the solvents. Our proof is an alternative proof requiring only assumptions on the right solvents of $A(\lambda)$. In the multivariate setting, Schlemm and Stelzer (2012a), Proposition 5.1, proved that a MCARMA process has a representation as a multivariate linear combination of pd dependent one-dimensional Ornstein-Uhlenbeck processes. In the present paper, we will have multivariate Ornstein-Uhlenbeck processes and p instead of pd Ornstein-Uhlenbeck processes.

Similarly, as in the above mentioned papers our conclusions are advantageously for statistical inference of MCARMA processes. Brockwell and Lindner (2019) use the representation (1.7) to solve both the sampling and the embedding problem for CARMA processes. In the first case, they deduce the explicit parameters of the ARMA representation of $(Y(nh))_{n\in\mathbb{N}}$. In the second case, they present conditions for an ARMA(p, q) process to be embedded in a CARMA$(p, p-1)$ process. Therefore, we think that our results might be helpful for a multivariate version of the sampling and embedding problem as well. But this is outside the scope of the present paper. Moreover, our findings are helpful to derive probabilistic properties of a MCARMA process. Brockwell and Lindner (2009), for example, use the ARMA$(p, p-1)$ representation of a CARMA process to derive necessary and sufficient conditions for the existence of a CARMA process.

The paper is structured on the following way. In Section 2, we present preliminary results on matrix polynomials and rational matrix polynomials which lay the background for the upcoming results. The main results of the paper are given in Section 3.

2. Preliminaries

In this section, we review main results on matrix polynomials and rational matrix functions. References about matrix analysis and matrix polynomials are, e.g., the textbooks of Bernstein (2009), Horn and Johnson (2013) and Kailath (1980). The aim is to receive matrix valued "roots" of a matrix polynomial which help to define linear factors of a matrix polynomial. However, a challenge is that there does not exist the Fundamental Theorem of Algebra for matrix polynomials and matrix multiplication is not commutative.

Definition 2.1.

(a) A *λ-matrix $A:\mathbb{C}\to\mathbb{C}^{d\times m}$ of degree p and order (d,m)* is defined as

$$A(\lambda) \quad = \quad A_0\lambda^p + A_1\lambda^{p-1} + \ldots + A_{p-1}\lambda + A_p, \quad \lambda\in\mathbb{C},$$

where $A_k \in \mathbb{C}^{d\times m}$ for $k=0,\ldots,p$. If additionally, $d=m$ we say shortly that $A(\lambda)$ is of degree p and order d, and define the spectrum of $A(\lambda)$ as $\sigma(A(\cdot)) := \{\lambda\in\mathbb{C} : \det(A(\lambda))=0\}$. If $\sigma(A(\cdot))$ lies in the complement of the closed unit disc, then $A(\lambda)$ is called *Schur-stable*. The λ-matrix $A(\lambda)$ is called *monic λ-matrix of degree p and order d* if $A_0 := I_d$.

(b) Suppose $Z\in\mathbb{C}^{d\times d}$ and $d=m$. Then the *right matrix polynomial* $A_R:\mathbb{C}^{d\times d}\to\mathbb{C}^{d\times d}$ of the λ-matrix $A(\lambda)$ is defined as

$$A_R(Z) \quad := \quad A_0Z^p + A_1Z^{p-1} + \ldots + A_{p-1}Z + A_p.$$

Next, we extend the definition of a root to the matrix polynomial case.

Definition 2.2. For a monic λ-matrix $A(\lambda)$ of degree p and order d we define

$$A^{(k)}(\lambda) := \frac{d^k}{d\lambda^k}A(\lambda), \quad k=1,\ldots p.$$

A matrix $R\in\mathbb{C}^{d\times d}$ is defined to be a *right solvent of $A(\lambda)$ with multiplicity* $\nu\in\{1,\ldots,p\}$ if

$$A_R(R)=0_{d\times d}, \quad A_R^{(1)}(R)=0_{d\times d}, \quad \ldots, \quad A_R^{(\nu-1)}(R)=0_{d\times d} \quad \text{and} \quad A_R^{(\nu)}(R)\neq 0_{d\times d}.$$

If $A_R(R)=0_{d\times d}$ we simply say that R is a *right solvent* of $A(\lambda)$. A right solvent R of $A(\lambda)$ is called *regular* if $\sigma(R)\cap\sigma(A^{(1)}(\cdot))=\emptyset$, where $A^{(1)}(\lambda)$ is a monic λ-matrix of degree $p-1$ satisfying $A(\lambda)=A^{(1)}(\lambda)(\lambda I_d - R)$.

Definition 2.3. A set of right solvents $R_1,\ldots,R_\mu\in\mathbb{C}^{d\times d}$ of the λ-matrix $A(\lambda)$ of degree p is called *complete* if $\sigma(A(\cdot))=\bigcup_{j=1}^{\mu}\sigma(R_j)$, where $\sigma(R_j)=\{\lambda\in\mathbb{C}:\det(\lambda I_d - R_j)=0\}$ is the spectrum of R_j.

The Vandermonde matrix is extended in the next definition.

Definition 2.4. Suppose $R_1,\ldots,R_\mu$ are a complete set of right solvents of the matrix polynomial $A(\lambda)$ with multiplicities $\nu_1,\ldots,\nu_\mu$, respectively. We define the *confluent Vandermonde matrix* $W:=W(R_1,\ldots,R_\mu)\in\mathbb{C}^{pd\times pd}$ by $W=[W_1,\ldots,W_\mu]$, where for $k=1,\ldots,\mu$,

$$W_k = \begin{pmatrix} I_d & 0_d & \ldots & 0_d \\ R_k & I_d & & \vdots \\ R_k^2 & 2R_k & \ddots & 0_d \\ \vdots & \vdots & & I_d \\ \vdots & \vdots & & \vdots \\ R_k^{p-1} & (p-1)R_k^{p-2} & \ldots & \binom{p-1}{\nu_k-1}R_k^{p-\nu_k} \end{pmatrix} \in \mathbb{C}^{pd\times\nu_k d}.$$

In the case $\mu=p$ and $\nu_1=\ldots=\nu_p=1$, the confluent Vandermonde matrix reduces to the classical block Vandermonde matrix $V(R_1,\ldots,R_p)=W(R_1,\ldots,R_p)$.

Lemma 2.5 (Maroulas (1985), Theorem 3.4)**.** *Let $R_1, \ldots, R_\mu$ be right solvents of a monic λ-matrix $A(\lambda)$ of multiplicities $\nu_1, \ldots, \nu_\mu$, respectively. Then $W(R_1, \ldots, R_\mu)$ is non-singular if and only if*

$$\sigma(A(\cdot)) = \bigcup_{j=1}^{\mu} \sigma(R_j) \quad \text{and} \quad \sigma(R_j) \cap \sigma(R_i) = \emptyset \quad \text{for } j, i = 1, \ldots, \mu,\ j \neq i.$$

Thus, we have the following relation between the solvents of the λ-matrix $A(\lambda)$ and the coefficient matrices $A_1, \ldots, A_p$ of $A(\lambda)$.

Lemma 2.6 (Maroulas (1985))**.** *Let $R_1, \ldots, R_p$ be a complete set of regular right solvents of the monic λ-matrix $A(\lambda) = I_d\lambda^p + A_1\lambda^{p-1} + \ldots + A_{p-1}\lambda + A_p$. Then*

$$\begin{aligned} [A_p, \ldots, A_1] &= -[R_1^p, \ldots, R_p^p]V^{-1}(R_1, \ldots, R_p) \qquad \text{and} \\ A(\lambda) &= (\lambda I_d - R_p^*) \cdots (\lambda I_d - R_2^*)(\lambda I_d - R_1), \end{aligned} \tag{2.1}$$

where for $k = 2, \ldots, p$,

$$R_k^* = M_k(R_k)R_kM_k^{-1}(R_k) \quad \text{and} \quad M_k(R_k) = (\lambda I_d - R_{k-1}) \cdots (\lambda I_d - R_1).$$

Interesting is that in the multivariate setting R_k^* is not necessarily equal to R_k for $k = 1, \ldots, p$, as in the one-dimensional case $d = 1$. Moreover, not every monic λ-matrix has a linear factorization of the kind (2.1). Necessary and sufficient criteria for linear factorizations of λ-matrices are given in Beitia and Zaballa (1989).

Definition 2.7. A *strictly proper rational left λ-matrix* $F(\lambda)$ with degree p and order (d, m) has the representation

$$F(\lambda) = A(\lambda)^{-1}B(\lambda),$$

where $A(\lambda)$ is a monic λ-matrix of degree p and order d, and $B(\lambda)$ is a λ-matrix of degree $p-1$ and order (d, m). The rational λ-matrix $F(\lambda)$ is called *irreducible* if $A(\lambda)$ and $B(\lambda)$ are left coprime. If $F(\lambda)$ is irreducible and R is a *regular* right solvent of $A(\lambda)$ then the *residue of the rational λ-matrix* $F(\lambda)$ at R is defined by

$$\mathrm{Res}[F, R] := \frac{1}{2\pi i} \oint_{\Gamma_R} F(\lambda)\, d\lambda,$$

where Γ_R is a simple closed contour such that $\sigma(R)$ is contained in the interior of Γ_R and $\sigma(A(\cdot)) \setminus \sigma(R)$ is contained in the exterior of Γ_R.

The next result characterizes a rational left matrix function. However, although Tsay and Shieh (1982) assume that $d = m$, it is straightforward to extend the result to the case $d \neq m$ (cf. Leyva-Ramos (1991)).

Theorem 2.8 (Tsay and Shieh (1982), Theorem 4.1)**.** *Let $F(\lambda) = A(\lambda)^{-1}B(\lambda)$ be a irreducible strictly proper rational left λ-matrix of degree p and order (d, m), and $A(\lambda)$ has a complete set of regular right solvents $\{R_k : k = 1, \ldots, p\}$. Then*

$$F(\lambda) = \sum_{k=1}^{p} (\lambda I_d - R_k)^{-1}\, \mathrm{Res}[F, R_k].$$

Theorem 2.8 assumes that the right solvents are regular which excludes right solvents with multiplicities.

A formula for the calculation of a matrix residue is given in Leyva-Ramos (1991, Section 6, eq. (6.13)): Suppose the strictly proper left λ-matrix $F(\lambda) = A(\lambda)^{-1}B(\lambda)$ is irreducible and $A(\lambda)$ has a complete set of regular right solvents $\{R_k : k = 1, \ldots, p\}$. Notice, the matrix $A^{\#}$ as defined in (1.5) is non-singular because $A^{\#}$ has the only eigenvalue 1. Then, due to Lemma 2.5, the Vandermonde matrix $V(R_1,\ldots,R_p)$ is non-singular (cf. Leyva-Ramos (1991, Definition 4)) and

$$\begin{pmatrix}\mathrm{Res}[F,R_1]\\ \vdots \\ \mathrm{Res}[F,R_p]\end{pmatrix} = V(R_1,\ldots,R_p)^{-1}[A^{\#}]^{-1}B^{\#}. \tag{2.2}$$

Finally, the question arises how to calculate the right solvents of the λ-matrix $A(\lambda)$. A possibility to characterize a right solvent is by right latent roots and latent vectors as is done in Dennis et al. (1976).

Definition 2.9. Let $A(\lambda)$ be a λ-matrix of order d. If $\lambda_i\in\mathbb{C}$ satisfies $\det(A(\lambda_i))=0$, then λ_i is called *latent root* of $A(\lambda)$. A vector $p_i\in\mathbb{C}^d$ satisfying $A(\lambda_i)p_i=0_d$ is called *right latent vector* of $A(\lambda)$ associated to the latent root λ_i.

Theorem 2.10. *Suppose the monic λ-matrix $A(\lambda)$ has distinct latent roots $\lambda_1,\ldots,\lambda_{pd}$ with corresponding right latent vectors $p_1,\ldots,p_{pd}$, respectively. Define $P_k:=(p_{(k-1)d+1},\ldots,p_{kd})\in\mathbb{C}^{d\times d}$ and $\Lambda_k:=diag(\lambda_{(k-1)d+1},\ldots,\lambda_{kd})$ for $k=1,\ldots,p$.*

(a) *Then $R_k:=P_k\Lambda_kP_k^{-1}$ for $k=1,\ldots,p$ is a complete set of regular right solvents of $A(\lambda)$.*

(b) *Suppose the strictly proper left λ-matrix $F(\lambda)=A(\lambda)^{-1}B(\lambda)$ is irreducible, then the residue of $F(\lambda)$ can be calculated as in (2.2) and*

$$F(\lambda)=\sum_{k=1}^{p}(\lambda I_d-R_k)^{-1}\,\mathrm{Res}[F,R_k].$$

Proof: (a) is proven in Dennis et al. (1976), Theorem 4.5. (b) follows from (a) and Theorem 2.8. □

3. Results

In this section we present criteria for a MCARMA process to be a sum of multivariate Ornstein-Uhlenbeck processes. For the rest of the paper we will assume the following:

Assumption A. Let $A(\lambda)$, $B(\lambda)$ be defined as in (1.2) and $F(\lambda)=A(\lambda)^{-1}B(\lambda)$ be irreducible. Assume further that $A(\lambda)$ has a complete set of regular right solvents $\{R_k:k=1,\ldots,p\}$.

We want to comment on the severity of Assumption A.

Remark 3.1.

(a) Instead of assuming that the right solvents $\{R_k:k=1,\ldots,p\}$ are complete and regular, it is equivalent to assume that $V(R_1,\ldots,R_p)$ is non-singular (see Lemma 2.5). A sufficient condition for $A(\lambda)$ to have a complete set of regular right solvents is that A^* as defined in (1.3) has distinct eigenvalues. Indeed, $\sigma(A^*):=\{\lambda\in\mathbb{C}:\det(A^*-\lambda I_{pd})=0\}=\sigma(A(\cdot))$, due to Marquardt and Stelzer (2007), Lemma 3.8, such that by Theorem 2.10 the statement follows.

(b) In general it is possible to approximate the λ-matrix $F(\lambda)=A(\lambda)^{-1}B(\lambda)$ by a sequence of λ-matrices $F^{(n)}(\lambda)=A^{(n)}(\lambda)^{-1}B^{(n)}(\lambda)$ where the λ-matrices $A^{(n)}(\lambda)$ have pd different latent roots and hence, a complete set of regular right solvents satisfying Assumption A. But the degree of the λ-matrix $B^{(n)}(\lambda)$ is not necessarily q anymore. We will give some examples.

(c) Let the λ-matrix $A(\lambda)$ has the form $A(\lambda)=(\lambda I_d-R_p)\cdots(\lambda I_d-R_1)$ with $R_1,\ldots,R_p\in\mathbb{C}^{d\times d}$. Then there exists a sequence of matrices $R_1^{(n)},\ldots,R_p^{(n)}\in\mathbb{C}^{d\times d}$ where $\bigcup_{j=1}^{p}\sigma(R_j^{(n)})$ has pd different elements $\lambda_1^{(n)},\ldots,\lambda_{pd}^{(n)}$ different to the latent roots of $B(\lambda)$ and $\max_{k=1,\ldots,p}\|R_k-$

$R_k^{(n)}\| \to 0$ as $n \to \infty$. Define $A^{(n)}(\lambda) := (\lambda I_d - R_p^{(n)}) \cdots (\lambda I_d - R_2^{(n)})(\lambda I_d - R_1^{(n)})$. Then $\sigma(A^{(n)}(\cdot)) = \{\lambda_1^{(n)}, \ldots, \lambda_{pd}^{(n)}\}$. A conclusion of Theorem 2.10 is that $A^{(n)}(\lambda)$ has a complete set of p different regular right solvents. Of course, $A^{(n)}(\lambda)$ converges uniformly to $A(\lambda)$ as $n \to \infty$ on a compact set. Finally, $F^{(n)}(\lambda) = A^{(n)}(\lambda)^{-1}B(\lambda)$ converges uniformly to $F(\lambda) = A(\lambda)^{-1}B(\lambda)$ as $n \to \infty$ on a compact set and $F^{(n)}(\lambda)$ satisfies Assumption A.

(d) Suppose the λ-matrix $A(\lambda)$ has a complete set of regular right solvents $\{R_1, \ldots, R_\mu\}$ with multiplicities $\nu_1, \ldots, \nu_\mu$ and $W(R_1, \ldots, R_k)$, $k = 1, \ldots, \mu$ is invertible. As notation we use $\widetilde{R}_1, \ldots, \widetilde{R}_p$ for $R_1, \ldots, R_1, R_2, \ldots, R_\mu$ taking the multiplicities of the right solvents into account. Due to Maroulas (1985) there exist matrices $R_1^* \ldots, R_p^*$ with R_k^* similar to $\widetilde{R}_k$ such that $A(\lambda) = (\lambda I_d - R_p^*) \cdots (\lambda I_d - R_1^*)$. Then the assumptions in (c) are satisfied and $F(\lambda) = A(\lambda)^{-1}B(\lambda)$ can be approximated by a sequence $F^{(n)}(\lambda) = A^{(n)}(\lambda)^{-1}B(\lambda)$ uniformly on a compact set as $n \to \infty$, where $A^{(n)}(\lambda)$ has a complete set of p regular right solvents satisfying Assumption A.

Theorem 3.2. *Define for $k = 1, \ldots, p$ the multivariate complex-valued Ornstein-Uhlenbeck process*

$$Y_k(t) \;=\; \mathrm{e}^{R_k t} Y_k(0) + \int_0^t \mathrm{e}^{R_k(t-u)} \operatorname{Res}[F, R_k] \, \mathrm{d}L(u), \quad t \geq 0, \tag{3.1}$$

with some initial condition $Y_k(0)$ in $\mathbb{C}^d$ such that $V(R_1, \ldots, R_p)[Y_1(0)^\top, \ldots, Y_p(0)^\top]^\top \in \mathbb{R}^{pd}$. Then $Y(t) = \sum_{k=1}^p Y_k(t)$ is an $\mathbb{R}^d$-valued solution of the state space model (1.4) and hence, a MCARMA(p,q)-process.

Proof: Of course,

$$Y(t) = C^* \mathrm{e}^{A^* t} X(0) + \int_0^t C^* \mathrm{e}^{A^*(t-u)} B^* \, \mathrm{d}L(u)$$

is an $\mathbb{R}^d$-valued solution of the state space model (1.4) with some initial condition $X(0) \in \mathbb{R}^{pd}$. Define

$$E^* := [I_d, \ldots, I_d] \in \mathbb{R}^{d \times pd}, \;\; F^* = \begin{pmatrix} \operatorname{Res}[F, R_1] \\ \vdots \\ \operatorname{Res}[F, R_p] \end{pmatrix} \in \mathbb{C}^{pd \times d} \;\text{ and }\; R^* := \operatorname{diag}(R_1, \ldots, R_p) \in \mathbb{C}^{pd \times pd}$$

as a block diagonal matrix. Due to (2.2) and (1.6) the relation

$$F^* = V(R_1, \ldots, R_p)^{-1} [A^\#]^{-1} B^\# = V(R_1, \ldots, R_p)^{-1} B^*$$

holds. A further inspection of the matrices give

$$A^* V(R_1, \ldots, R_p) = V(R_1, \ldots, R_p) R^* \quad \text{and} \quad C^* V(R_1, \ldots, R_p) = E^*,$$

where we used that R_k is a right solvent of $A(\lambda)$. Therefore, define $T := V(R_1, \ldots, R_p)$, $Y^*(0) := [Y_1(0)^\top, \ldots, Y_p(0)^\top]^\top$ and $X^*(0) := TY^*(0) \in \mathbb{R}^{pd}$ such that

$$A^* = TR^*T^{-1}, \quad B^* = TF^* \quad \text{and} \quad C^* = E^* T^{-1}.$$

In particular, $\mathrm{e}^{A^*t} = T\mathrm{e}^{R^*t}T^{-1}$, $t \in \mathbb{R}$. Then for $t \geq 0$,

$$\begin{aligned} Y(t) &= C^*\mathrm{e}^{A^*t}X^*(0) + \int_0^t C^*\mathrm{e}^{A^*(t-u)}B^*\,\mathrm{d}L(u) \\ &= E^*\mathrm{e}^{R^*t}T^{-1}TY^*(0) + \int_0^t E^*\mathrm{e}^{R^*(t-u)}F^*\,\mathrm{d}L(u) \\ &= \sum_{k=1}^p \left[\mathrm{e}^{R_k t}Y_k(0) + \int_0^t \mathrm{e}^{R_k(t-u)}\operatorname{Res}[F,R_k]\,\mathrm{d}L(u)\right] \\ &= \sum_{k=1}^p Y_k(t) \end{aligned}$$

is $\mathbb{R}^d$-valued. □

Remark 3.3.

(a) If $\sigma(A^*)$ has only distinct eigenvalues then Theorem 2.10 gives the possibility to calculate a complete set of regular right solvents. Due to Section 2 we are able to calculate the residues as well. Thus, we obtain via (3.1) a representation of the MCARMA process as sum of Ornstein-Uhlenbeck processes.

(b) Since the solvents $R_1, \ldots, R_p$ are not unique, the representation of Y as sum of Ornstein-Uhlenbeck processes is not unique as well (cf. Example 3.6), only in the case $d = 1$ we have uniqueness.

(c) Any linear combination $\left(\sum_{k=1}^p \alpha_k Y_k(t)\right)_{t\geq 0}$ of $\mathbb{R}^d$-valued multivariate Ornstein-Uhlenbeck processes $Y_1, \ldots, Y_p$, where $\alpha_1, \ldots, \alpha_p \in \mathbb{R}^{d\times d}$, is a MCARMA$(p, p-1)$-process. But the exponent R_k in the definition of Y_k is not necessarily a right solvent of the autoregressive polynomial of the MCARMA process. But this is essential to derive a VARMA representation of the discrete-time sampled MCARMA process later on.

Corollary 3.4. *Suppose $\sigma(A(\cdot)) \subset \{(-\infty, 0) + i\mathbb{R}\}$ and $\mathbb{E}[\log(\max(1, \|L(1)\|))] < \infty$. Define for $k = 1, \ldots, p$ the multivariate complex-valued Ornstein-Uhlenbeck processes*

$$Y_k(t) = \int_{-\infty}^t \mathrm{e}^{R_k(t-u)}\operatorname{Res}[F,R_k]\,\mathrm{d}L(u), \quad t \in \mathbb{R},$$

Then $Y(t) = \sum_{k=1}^p Y_k(t) = \int_{-\infty}^t \sum_{k=1}^p \mathrm{e}^{R_k(t-u)}\operatorname{Res}[F,R_k]\,\mathrm{d}L(u)$, $t \in \mathbb{R}$, is a stationary $\mathbb{R}^d$-valued solution of the state space model (1.4) and hence, a MCARMA(p,q)-process.

Due to Sato and Yamazato (1984), Theorem 4.1, the stationary Ornstein-Uhlenbeck processes Y_k are well-defined.

Remark 3.5.

(a) Let Γ_k be a simple closed contour such that $\sigma(R_k)$ lies in the interior of Γ_k and the residuary spectrum $\sigma(A(\cdot)) \setminus \sigma(R_k)$ lies in the exterior of Γ_k and $\Gamma := \bigcup_{k=1}^p \Gamma_k$. Due to Cauchy's integral formula (see Lax (2002), Theorem 17.5), and Theorem 2.8 we obtain for $t \geq 0$,

$$\sum_{k=1}^p \mathrm{e}^{tR_k}\operatorname{Res}[F,R_k] = \frac{1}{2\pi i}\sum_{k=1}^p \oint_\Gamma \mathrm{e}^{t\lambda}(\lambda I_d - R_k)^{-1}\operatorname{Res}[F,R_k]\mathrm{d}\lambda = \frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{t\lambda}F(\lambda)\,\mathrm{d}\lambda.$$

In particular, if $\sigma(A(\cdot)) \subset \{(-\infty, 0) + i\mathbb{R}\}$ then the kernel function satisfies

$$\sum_{k=1}^p \mathrm{e}^{tR_k}\operatorname{Res}[F,R_k] = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{e}^{ti\omega}F(i\omega)\,\mathrm{d}\omega, \quad t \geq 0.$$

(b) In the case of repeated right solvents, Y has not the representation as a sum of multivariate Ornstein Uhlenbeck processes. Indeed, due to the representation of $F(\lambda) = \sum_{k=1}^{\mu}\sum_{j=1}^{\nu_k}(\lambda I_d - R_k)^{-j}F_{k,j}$ in Shieh et al. (1986), in general we have

$$\sum_{k=1}^{p} \mathrm{e}^{tR_k}\,\mathrm{Res}[F,R_k] \neq \frac{1}{2\pi i}\oint_{\Gamma} \mathrm{e}^{t\lambda}F(\lambda)\,\mathrm{d}\lambda$$

(cf. Brockwell and Lindner (2009), Lemma 2.4, in the case of one-dimensional CARMA processes).

Example 3.6. Let

$$A(\lambda) = I_d\lambda^2 + \begin{pmatrix} -11 & 22 \\ -12 & 21 \end{pmatrix}\lambda + \begin{pmatrix} -42 & 52 \\ -36 & 44 \end{pmatrix} \quad \text{and} \quad B(\lambda) = I_d, \quad \lambda \in \mathbb{C},$$

be given. Then

$$R_1 = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} -3 & -2 \\ 0 & -4 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} -7 & 6 \\ -3 & 2 \end{pmatrix}, \qquad R_4 = \begin{pmatrix} -3 & 0.5 \\ 0 & -2 \end{pmatrix}$$

are right solvents of $A(\lambda)$. The pair R_1, R_2 and the pair R_3, R_4, respectively build a complete set of regular right solvents of $A(\lambda)$. Then Theorem 3.2 and the formula for the residues (2.2) give that both $Y_1(t) + Y_2(t)$ with

$$\begin{aligned} Y_1(t) &= \mathrm{e}^{R_1 t}Y_1(0) + \int_0^t \mathrm{e}^{R_1(t-u)}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\mathrm{d}L(u), \\ Y_2(t) &= \mathrm{e}^{R_2 t}Y_2(0) + \int_0^t \mathrm{e}^{R_2(t-u)}\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}\mathrm{d}L(u), \end{aligned}$$

and $Y_3(t) + Y_4(t)$ with

$$\begin{aligned} Y_3(t) &= \mathrm{e}^{R_3 t}Y_3(0) + \int_0^t \mathrm{e}^{R_3(t-u)}\begin{pmatrix} 8 & -11 \\ 6 & -8 \end{pmatrix}\mathrm{d}L(u), \\ Y_4(t) &= \mathrm{e}^{R_4 t}Y_4(0) + \int_0^t \mathrm{e}^{R_4(t-u)}\begin{pmatrix} -8 & 11 \\ -6 & 8 \end{pmatrix}\mathrm{d}L(u), \end{aligned}$$

are MCARMA$(2,0)$ processes with AR polynomial $A(\lambda)$ and MA polynomial $B(\lambda)$.

For the rest of the paper we assume:

Assumption B. Y has the representation as given in Theorem 3.2, $\mathbb{E}\|L(1)\|^2 < \infty$ and $\mathbb{E}L(1) = 0_m$.

Now, we are able to present an alternative representation of the covariance function of a stationary MCARMA process. As notation we write Z^{H} for the transposed complex conjugated of a matrix $Z \in \mathbb{C}^{d\times d}$.

Proposition 3.7. *Suppose the setting of Corollary 3.4. The covariance function $(\gamma_Y(l))_{l\in\mathbb{N}_0} = (\mathrm{Cov}(Y(t+l), Y(t)))_{l\in\mathbb{N}_0}$ of Y has the representation*

$$\gamma_Y(l) = \sum_{i=1}^{p} \mathrm{e}^{lR_i}\Sigma_i, \quad l \in \mathbb{N}_0, \quad \text{where} \quad \Sigma_i := \sum_{j=1}^{p}\int_0^\infty \mathrm{e}^{uR_i}\,\mathrm{Res}[F,R_i]\Sigma_L\,\mathrm{Res}[F,R_j]^{\mathsf{H}}\mathrm{e}^{uR_j^{\mathsf{H}}}\,\mathrm{d}u.$$

Proof: An application of Corollary 3.4 gives

$$\begin{aligned} \gamma_Y(l) &= \sum_{i,j=1}^{p} \mathrm{Cov}\left(\int_{-\infty}^{t+l} \mathrm{e}^{R_i(t+l-u)}\,\mathrm{Res}[F,R_i]\,\mathrm{d}L(u), \int_{-\infty}^{t} \mathrm{e}^{R_j(t-u)}\,\mathrm{Res}[F,R_j]\,\mathrm{d}L(u)\right) \\ &= \sum_{i,j=1}^{p} \mathrm{e}^{lR_i}\int_0^\infty \mathrm{e}^{uR_i}\,\mathrm{Res}[F,R_i]\Sigma_L\,\mathrm{Res}[F,R_j]^{\mathsf{H}}\mathrm{e}^{uR_j^{\mathsf{H}}}\,\mathrm{d}u, \end{aligned}$$

which completes the proof. □

A final aim is to derive a VARMA representation for a MCARMA process observed at discrete time-points. To distinguish the notation between the continuous-time process and the sampled discrete-time process, we write $Y_n^{(h)}$ for $Y(nh)$ in the following and accordingly $Y_{k,n}^{(h)}$ for $Y_k(nh)$ for some fixed $h > 0$. Let us first state an auxiliary lemma.

Lemma 3.8. *For any $k = 1, \ldots, p$, $n \geq p$, $l = 0, \ldots, n$ and any matrices $C_1, \ldots, C_l \in \mathbb{C}^{d\times d}$ it holds that*

$$Y_{k,n}^{(h)} = \sum_{r=1}^{l} C_r Y_{k,n-r}^{(h)} + \left(\mathrm{e}^{hlR_k} - \sum_{r=1}^{l} C_r \mathrm{e}^{h(l-r)R_k}\right) Y_{k,n-l}^{(h)} + \sum_{r=0}^{l-1}\left(\mathrm{e}^{hrR_k} - \sum_{j=1}^{r} C_j \mathrm{e}^{h(r-j)R_k}\right) N_{k,n-r}^{(h)},$$

where $N_{k,n}^{(h)} = \int_{(n-1)h}^{nh} \mathrm{e}^{R_k(nh-u)}\,\mathrm{Res}[F, R_k]\,\mathrm{d}L(u)$.

Proof: The proof goes in the same vein as the proof of equation (2.8) in Brockwell and Lindner (2009) for scalars $c_1, \ldots, c_l$ instead of matrices $C_1, \ldots, C_l$, since Y_k is a multivariate Ornstein-Uhlenbeck process. □

Eventually, we obtain a VARMA$(p, p-1)$ representation for the sampled version of a MCARMA(p, q) process based on the ideas of Brockwell and Lindner (2009), Lemma 2.1, for CARMA(p, q) processes.

Theorem 3.9. *Define*

$$\Psi_0 := I_d, \qquad [\Psi_p, \ldots, \Psi_1] := -[\mathrm{e}^{-phR_1}, \ldots, \mathrm{e}^{-phR_p}]V^{-1}(\mathrm{e}^{-hR_1}, \ldots, \mathrm{e}^{-hR_p}) \in \mathbb{C}^{p\times pd}$$

and the λ-matrix $\Phi(\lambda) := I_d - \Phi_1\lambda - \ldots - \Phi_p\lambda^p$ of degree p and order d with $\Phi_j := -\Psi_p^{-1}\Psi_{p-j}$ for $j = 1, \ldots, p$. Then there exists a λ-matrix $\Theta(\lambda) = I_d + \Theta_1\lambda + \ldots + \Theta_{p-1}\lambda^{p-1}$ of degree $p-1$ and order d such that

$$\Phi(B)Y_n^{(h)} = \Theta(B)\varepsilon_n^{(h)}, \quad n \geq p, \tag{3.2}$$

where B denotes the backshift operator (i.e. $B^j Y_n^{(h)} = Y_{n-j}^{(h)}$ for $j \in \mathbb{N}$) and $(\varepsilon_n^{(h)})_{n\geq p}$ is a d-dimensional weak white noise. Thus, $(Y_n^{(h)})_{n\geq p}$ admits a weak $VARMA(p, p-1)$ representation.

Proof: First, we will show that $\Phi(\lambda)$ is well-defined and has the complete set of regular right solvents $\mathrm{e}^{-hR_1}, \ldots, \mathrm{e}^{-hR_p}$. Due to Assumption A and Lemma 2.5, the Vandermonde matrix $V(\mathrm{e}^{-hR_1}, \ldots, \mathrm{e}^{-hR_p})$ is non-singular and finally, $\Psi_1, \ldots, \Psi_p$ is well-defined. A conclusion of Assumption A and Lemma 2.6 is then that $\mathrm{e}^{-hR_1}, \ldots, \mathrm{e}^{-hR_p}$ is a complete set of regular right solvents of $\Psi(\lambda) = I_d\lambda^p + \Psi_1\lambda^{p-1} + \ldots + \Psi_p$. Note that $\Psi_p = (-1)^p \mathrm{e}^{-hR_p^*} \cdot \ldots \cdot \mathrm{e}^{-hR_2^*} \cdot \mathrm{e}^{-hR_1}$ where $R_2^*, \ldots, R_p^*$ are defined as in Lemma 2.6. Since the eigenvalues of $\mathrm{e}^{-hR_k^*}$, $k = 2, \ldots, p$ and e^{-hR_1} are non-zero, the matrix Ψ_p is non-singular. Finally, $\Phi(\lambda) = \Psi_p^{-1}\Psi(\lambda)$ is well-defined and has the complete set of regular right solvents $\mathrm{e}^{-hR_1}, \ldots, \mathrm{e}^{-hR_p}$.

Due to (3.1) we obtain $Y_n^{(h)} = \sum_{k=1}^{p} Y_{k,n}^{(h)}$ for $n \geq p$, where

$$Y_{k,n}^{(h)} = \mathrm{e}^{hR_k} Y_{k,n-1}^{(h)} + N_{k,n}^{(h)} \quad \text{and} \quad N_{k,n}^{(h)} = \int_{(n-1)h}^{nh} \mathrm{e}^{R_k(nh-u)}\,\mathrm{Res}[F, R_k]\,\mathrm{d}L(u)$$

(cf. Schlemm and Stelzer (2012a), Lemma 5.2). An application of Lemma 3.8 with $l = p$ and $C_r = \Phi_r$ for $r = 1, \ldots, p$ gives

$$Y_{k,n}^{(h)} = \sum_{r=1}^{p} \Phi_r Y_{k,n-r}^{(h)} + \left[\mathrm{e}^{hpR_k} - \sum_{r=1}^{p} \Phi_r \mathrm{e}^{h(p-r)R_k}\right] Y_{k,n-p}^{(h)} + \sum_{r=0}^{p-1}\left[\mathrm{e}^{hrR_k} - \sum_{j=1}^{r} \Phi_j \mathrm{e}^{h(r-j)R_k}\right] N_{k,n-r}^{(h)}.$$

The fact that e^{-hR_k} is a right solvent of $\Phi(\lambda)$ implies that

$$\mathrm{e}^{hpR_k} - \sum_{r=1}^{p} \Phi_r \mathrm{e}^{h(p-r)R_k} = \Phi(\mathrm{e}^{-hR_k})\mathrm{e}^{phR_k} = 0_{d\times d}, \quad k = 1,\dots,p.$$

Hence, we obtain

$$\Phi(B)Y_{k,n}^{(h)} = Y_{k,n}^{(h)} - \sum_{r=1}^{p} \Phi_r Y_{k,n-r}^{(h)} = \sum_{r=0}^{p-1} \left[\mathrm{e}^{hrR_k} - \sum_{j=1}^{r} \Phi_j \mathrm{e}^{h(r-j)R_k} \right] N_{k,n-r}^{(h)} =: U_n^{(h)}. \tag{3.3}$$

Define for $r = 0,\dots,p-1$ the iid sequence $(W_{r,n}^{(h)})_{n\in\mathbb{Z}}$ in $\mathbb{C}^d$ as

$$W_{r,n}^{(h)} := \int_{(n-1)h}^{nh} \sum_{k=1}^{p} \left[\mathrm{e}^{hrR_k} - \sum_{j=1}^{r} \Phi_j \mathrm{e}^{h(r-j)R_k} \right] \mathrm{e}^{R_k(nh-u)} \operatorname{Res}[F, R_k] \mathrm{d}L(u). \tag{3.4}$$

Summation over k and rearranging leads to

$$\Phi(B)Y_n^{(h)} = U_n^{(h)} = \sum_{r=0}^{p-1} W_{r,n-r}^{(h)}, \quad n \geq p. \tag{3.5}$$

Since $([W_{0,n}^{(h)\top},\dots,W_{p-1,n}^{(h)\top}]^\top)_{n\in\mathbb{Z}}$ is a sequence of iid random vectors, the d-dimensional sequence $(U_n^{(h)})_{n\in\mathbb{Z}} := \left(\sum_{r=0}^{p-1} W_{r,n-r}^{(h)}\right)_{n\in\mathbb{Z}}$ is $(p-1)$-dependent. Define

$$\varepsilon_n^{(h)} := U_n^{(h)} - \mathcal{P}_{\mathcal{M}_{n-1}} U_n^{(h)}, \quad n \in \mathbb{Z},$$

where $\mathcal{P}_{\mathcal{M}_{n-1}}$ denotes the orthogonal projection on $\mathcal{M}_{n-1} := \overline{\mathrm{sp}}\{U_j^{(h)} : -\infty < j \leq n-1\}$ and the closure is taken in the Hilbert space of square integrable complex random vectors with inner product $(U_1, U_2) \mapsto \mathbb{E}(U_1^{\mathsf{H}} U_2)$ for random vectors U_1, U_2 in $\mathbb{C}^d$. Then $\Theta_1,\dots\Theta_{p-1}$ is given as the solution of the equation

$$\mathcal{P}_{\overline{\mathrm{sp}}\{\varepsilon_{n-p+1}^{(h)},\dots,\varepsilon_{n-1}^{(h)}\}} U_n^{(h)} = \Theta_1 \varepsilon_{n-1}^{(h)} + \ldots + \Theta_{p-1} \varepsilon_{n-p+1}^{(h)}.$$

As in the proof of Brockwell and Davis (2006), Proposition 3.2.1, for one-dimensional $(p-1)$-dependent processes we can follow then the statement. □

Remark 3.10.

(a) The λ-matrix $\Psi(\lambda)$ has the complete set of right solvents $\mathrm{e}^{-hR_1},\dots,\mathrm{e}^{-hR_p}$ but due to Lemma 2.6, $\Psi(\lambda)$ has not necessarily the representation as $\prod_{k=1}^{p}(\lambda I_d - \mathrm{e}^{-hR_k})$. Thus, the λ-matrix $\Phi(\lambda)$ is not necessarily $\psi_p^{-1}\prod_{k=1}^{p}(\lambda \mathrm{e}^{hR_k} - I_d)$. This differs to the one-dimensional case where multiplication is commutative and $\Phi(\lambda) = \prod_{k=1}^{p}(\lambda - \mathrm{e}^{-hr_k})$ where $r_1,\dots,r_p$ are the one-dimensional zeros of $A(\lambda)$. However, $\Psi(\lambda)$ is the unique λ-matrix with right solvents $\mathrm{e}^{-hR_1},\dots,\mathrm{e}^{-hR_p}$ and $\Psi(0) = I_d$.

(b) If $\sigma(A(\cdot)) \subset \{(-\infty, 0) + i\mathbb{R}\}$ holds then

$$\sigma(\Psi(\cdot)) = \bigcup_{k=1}^{p} \sigma(\mathrm{e}^{-hR_k}) = \{\mathrm{e}^{-h\lambda} : \lambda \in \bigcup_{k=1}^{p} \sigma(R_k)\} = \{\mathrm{e}^{-h\lambda} : \lambda \in \sigma(A)\},$$

is outside the closed unit disc. Hence, $\Psi(\lambda)$ is Schur-stable.

Remark 3.11. Suppose the assumptions of Corollary 3.4 hold such that Y is stationary. Then Φ is Schur-stable and there exist matrices $K_j \in \mathbb{C}^{d\times d}$, $j \in \mathbb{N}$, with

$$Y_n^{(h)} = U_n^{(h)} + \sum_{j=1}^{\infty} K_{n-j} U_j^{(h)}, \quad n \in \mathbb{Z}. \tag{3.6}$$

Finally, (3.5) and (3.6) imply that $\mathcal{M}_{n-1} = \overline{\text{sp}}\{Y_j^{(h)} : -\infty < j \leq n-1\}$ and

$$\varepsilon_n^{(h)} = U_n^{(h)} - \mathcal{P}_{\mathcal{M}_{n-1}} U_n^{(h)} = Y_n^{(h)} - \mathcal{P}_{\mathcal{M}_{n-1}} Y_n^{(h)}, \quad n \in \mathbb{Z}.$$

This means that the white noise process $(\varepsilon_n^{(h)})_{n\in\mathbb{Z}}$ are the real-valued linear innovations of $(Y_n^{(h)})_{n\in\mathbb{Z}}$. Schlemm and Stelzer (2012a) present sufficient criteria for $(\varepsilon_n^{(h)})_{n\in\mathbb{Z}}$ to be exponentially completely regular.

Finally, we state the covariance function of the series $U^{(h)} := (U_n^{(h)})_{n\geq p}$ given in (3.3). The second-order properties of the series $U^{(h)}$ are of interest for indirect estimation as is done, e.g., in Fasen-Hartmann and Kimmig (2020) for CARMA processes. The basic idea of the indirect estimation approach is that the VARMA parameters of $(Y(nh))_{n\in\mathbb{N}}$ are estimated by standard techniques. Taking identifiability issues into account the autoregressive parameters of the continuous-time process are then estimated from the autoregressive parameters of the discrete-time VARMA process. Finally, a comparison of the autocorrelation function of $U^{(h)}$ for the estimated and the parametric model gives the moving average parameters of the MCARMA process.

Identifiability problems may arise because different equidistantly sampled MCARMA processes may have the same VARMA representation. However, Schlemm and Stelzer (2012b), Section 3.4, present sufficient criteria for the MCARMA process to be identifiable from its discrete-time observations and hence, from the VARMA$(p, p-1)$ parameters in (3.2).

Proposition 3.12. *Let $(U_n^{(h)})_{n\geq p}$ be the d-dimensional time series defined as $\Phi(B)Y_n^{(h)} = U_n^{(h)}$, and $(\gamma_{U^{(h)}}(l))_{l\in\mathbb{N}_0} = (Cov(U_{n+l}^{(h)}, U_n^{(h)}))_{l\in\mathbb{N}_0}$ denotes the autocovariance function. Then for $l = 0, \ldots, p-1$:*

$$\gamma_{U^{(h)}}(l) = \sum_{\nu=1}^{p} \mathrm{e}^{hlR_\nu} \left[\sum_{r=0}^{p-l-1} \sum_{\mu=1}^{p} \left(\mathrm{e}^{hrR_\nu} - \sum_{j=1}^{r+l} \Phi_j \mathrm{e}^{h(r-j)R_\nu} \right) \Sigma_{\nu,\mu}^{(h)} \left(\mathrm{e}^{hrR_\mu} - \sum_{j=1}^{r} \Phi_j \mathrm{e}^{h(r-j)R_\mu} \right)^{\mathsf{H}} \right],$$

and $\gamma_{U^{(h)}}(l) = 0_{d\times d}$ for $l \geq p$, where

$$\Sigma_{\nu,\mu}^{(h)} := \text{Cov}\left(N_{\nu,1}^{(h)}, N_{\mu,1}^{(h)}\right) = \int_0^h \mathrm{e}^{R_\nu u} \,\text{Res}[F, R_\nu]\; \Sigma_L\; \text{Res}[F, R_\mu]^{\mathsf{H}} \mathrm{e}^{R_\mu^{\mathsf{H}} u}\, \mathrm{d}u.$$

Proof: For lag $l \in \{0, \ldots, p-1\}$ we receive due to (3.4):

$$\begin{aligned}
\gamma_{U^{(h)}}(l) &= \text{Cov}\left(W_{0,n+l}^{(h)} + \ldots + W_{p-1,n+l-p+1}^{(h)}, W_{0,n}^{(h)} + \ldots + W_{p-1,n-p+1}^{(h)}\right) \\
&= \sum_{r=0}^{p-l-1} \text{Cov}\left(W_{r+l,n-r}^{(h)}, W_{r,n-r}^{(h)}\right) \\
&= \sum_{r=0}^{p-l-1} \Bigg[\sum_{\nu=1}^{p} \sum_{\mu=1}^{p} \left(\mathrm{e}^{h(r+l)R_\nu} - \sum_{j=1}^{r+l} \Phi_j \mathrm{e}^{h(r+l-j)R_\nu} \right) \\
&\quad \cdot \text{Cov}\left(N_{\nu,n-r}^{(h)}, N_{\mu,n-r}^{(h)}\right) \left(\mathrm{e}^{hrR_\mu} - \sum_{j=1}^{r} \Phi_j \mathrm{e}^{h(r-j)R_\mu} \right)^{\mathsf{H}} \Bigg],
\end{aligned}$$

where $\text{Cov}\left(N_{\nu,n-r}^{(h)}, N_{\mu,n-r}^{(h)}\right) = \Sigma_{\nu,\mu}^{(h)}$, and finally, the assertion follows. □

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References

Andresen, A., Benth, F. E., Koekebakker, S., and Zakamulin, V. The CARMA interest rate model. *Int. J. Theor. Appl. Finance*, **17** (2), 1450008, 27 (2014). MR3198712.

Beitia, M. A. and Zaballa, I. Factorization of the matrix polynomial $A(\lambda) = A_0\lambda^t + A_1\lambda^{t-1} + \cdots + A_{t-1}\lambda + A_t$. *Linear Algebra Appl.*, **121**, 423–432 (1989). MR1011750.

Bernstein, D. S. *Matrix mathematics. Theory, facts, and formulas.* Princeton University Press, Princeton, NJ, second edition (2009). ISBN 978-0-691-14039-1. MR2513751.

Brockwell, P. J. and Davis, R. A. *Time series: theory and methods.* Springer Series in Statistics. Springer, New York (2006). ISBN 978-1-4419-0319-8; 1-4419-0319-8. MR2839251.

Brockwell, P. J., Davis, R. A., and Yang, Y. Estimation for non-negative Lévy-driven CARMA processes. *J. Bus. Econom. Statist.*, **29** (2), 250–259 (2011). MR2807879.

Brockwell, P. J. and Lindner, A. Existence and uniqueness of stationary Lévy-driven CARMA processes. *Stochastic Process. Appl.*, **119** (8), 2660–2681 (2009). MR2532218.

Brockwell, P. J. and Lindner, A. Sampling, embedding and inference for CARMA processes. *J. Time Series Anal.*, **40** (2), 163–181 (2019). MR3915525.

Chambers, M., McCrorie, J., and Thornton, M. Continuous time modelling based on an exact discrete time representation. In van Montfort, K., Oud, J. H. L., and Voelkle, M. C., editors, *Continuous Time Modeling in the Behavioral and Related Sciences*, pp. 317–357. Springer (2018). DOI: 10.1007/978-3-319-77219-6_14.

Chambers, M. J. and Thornton, M. A. Discrete time representation of continuous time ARMA processes. *Econometric Theory*, **28** (1), 219–238 (2012). MR2899219.

Dennis, J. E., Jr., Traub, J. F., and Weber, R. P. The algebraic theory of matrix polynomials. *SIAM J. Numer. Anal.*, **13** (6), 831–845 (1976). MR432675.

Fasen-Hartmann, V. and Kimmig, S. Robust estimation of stationary continuous-time ARMA models via indirect inference. *J. Time Series Anal.*, **41** (5), 620–651 (2020). MR4176167.

Fasen-Hartmann, V. and Scholz, M. Cointegrated continuous-time linear state-space and MCARMA models. *Stochastics*, **92** (7), 1064–1099 (2020). MR4156002.

Garnier, H. and Wang, L. *Identification of continuous-time models from sampled data.* Advances in Industrial Control. Springer London, 1 edition (2008). ISBN 978-1-84800-160-2; 978-1-84996-740-2. DOI: 10.1007/978-1-84800-161-9.

Horn, R. A. and Johnson, C. R. *Matrix analysis.* Cambridge University Press, Cambridge, second edition (2013). ISBN 978-0-521-54823-6. MR2978290.

Kailath, T. *Linear systems.* Prentice-Hall Information and System Sciences Series. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1980). ISBN 0-13-536961-4. MR569473.

Larsson, E. K., Mossberg, M., and Söderström, T. An overview of important practical aspects of continuous-time ARMA system identification. *Circuits Systems Signal Process.*, **25** (1), 17–46 (2006). MR2205975.

Lax, P. D. *Functional analysis.* Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York (2002). ISBN 0-471-55604-1. MR1892228.

Leyva-Ramos, J. Partial-fraction expansion in system analysis. *Internat. J. Control*, **53** (3), 619–639 (1991). MR1093288.

Maroulas, J. Factorization of matrix polynomials with multiple roots. *Linear Algebra Appl.*, **69**, 9–32 (1985). MR798364.

Marquardt, T. and Stelzer, R. Multivariate CARMA processes. *Stochastic Process. Appl.*, **117** (1), 96–120 (2007). MR2287105.

Sato, K.-i. and Yamazato, M. Operator-self-decomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type. *Stochastic Process. Appl.*, **17** (1), 73–100 (1984). MR738769.

Schlemm, E. and Stelzer, R. Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes. *Bernoulli*, **18** (1), 46–63

(2012a). MR2888698.

Schlemm, E. and Stelzer, R. Quasi maximum likelihood estimation for strongly mixing state space models and multivariate Lévy-driven CARMA processes. *Electron. J. Stat.*, **6**, 2185–2234 (2012b). MR3020261.

Shieh, L., Chang, F., and McInnis, B. The block partial fraction expansion of a matrix fraction description with repeated block poles. *IEEE Trans. Automat. Control*, **31** (3), 236–239 (1986). DOI: 10.1109/TAC.1986.1104253.

Todorov, V. Estimation of continuous-time stochastic volatility models with jumps using high-frequency data. *J. Econometrics*, **148** (2), 131–148 (2009). MR2500652.

Tsay, Y. T. and Shieh, L. S. Some applications of rational matrices to problems in systems theory. *Internat. J. Systems Sci.*, **13** (12), 1319–1337 (1982). MR705091.