



Refined Large Deviation Principle for Branching Brownian Motion Conditioned to Have a Low Maximum

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Abstract. Conditioning a branching Brownian motion to have an atypically low maximum leads to a suppression of the branching mechanism. In this note, we consider a branching Brownian motion conditioned to have a maximum below $\sqrt{2}\alpha t$ ($\alpha < 1$). We study the precise effects of an early/late first branching time and a low/high first branching location under this condition. We do so by imposing additional constraints on the first branching time and location. We obtain large deviation estimates, as well as the optimal first branching time and location given the additional constraints.

1. Introduction

In this note, we contribute to a deeper understanding of how spatial branching processes or log-correlated Gaussian processes realize unlikely events, such as having a low maximum. For a continuous time branching processes, the particles can refrain from branching with only exponential costs. As a result, when a continuous time branching process is conditioned to behave atypically, there is an interesting interplay between two factors: the first branching time, and the first branching location.

A branching Brownian motion (BBM) X can be constructed as follows. Starting from the origin at time 0, one particle performs one dimensional standard Brownian motion. After an exponentially distributed time with parameter one, the initial particle splits into two particles. From this branching location, the new particles follow independent Brownian paths and are subject to the same splitting

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rule. We denote the number of particles at time t by $n(t)$ and the particle positions by $\{X_k(t) : 1 \leq k \leq n(t)\}$.

We study the aforementioned two-factor interplay for BBM by asking the maximum to be a linear order below its typical value, while constraining the first branching time and location. When putting repulsive constraints, the branching mechanism is suppressed and the BBM has fewer particles. This seems to be a universal phenomenon. The repulsive constraints could come through spatial inhomogeneities [Engländer and den Hollander \(2003\)](#); [Engländer and Kyprianou \(2004\)](#); [Engländer \(2008\)](#); [Öz and Engländer \(2019\)](#), or a direct repulsive interaction through the center of mass (as conjectured in [Engländer, 2010](#)), or a polymer-type change of measure [Bovier and Hartung \(2021\)](#), or a requirement of an unusual maximum [Derrida and Shi \(2017\)](#); [Chen et al. \(2020\)](#).

The extreme values of branching Brownian motion have been studied extensively over the last 40 year (see, e.g. [Bramson, 1978, 1983](#); [Arguin et al., 2011, 2013](#); [Aïdékon et al., 2013](#); [Cortines et al., 2019](#)), and it is well known that the order of the maximum is given by

$$m_t := \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t. \quad (1.1)$$

Let $X_{max}(t) := \max_{k \leq n(t)} X_k(t)$ denote the position of the maximal particle at time t . The probability of BBM having a maximum smaller than $\sqrt{2}\alpha t$, $\alpha < 1$, was first analyzed by [Derrida and Shi \(2017\)](#). They showed that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left(\mathbb{P} \left(X_{max}(t) \leq \sqrt{2}\alpha t \right) \right) = -\psi(\alpha), \quad \psi(\alpha) := \begin{cases} 2\rho(1-\alpha), & \text{if } \alpha \in [-\rho, 1), \\ 1 + \alpha^2, & \text{if } \alpha \in (-\infty, -\rho], \end{cases} \quad (1.2)$$

where $\rho := \sqrt{2} - 1$. Moreover, they pointed out that if τ denotes the first branching time and y denotes the particle location at time τ , then the optimal choices are

$$\tau = \left(\frac{1-\alpha}{\sqrt{2}} \wedge 1 \right) t + o(t) \quad (1.3)$$

and

$$y = \sqrt{2}\alpha t - \sqrt{2}(t - \tau) + o(t) = \begin{cases} -\rho(1-\alpha)t + o(t), & \text{if } \alpha \in [-\rho, 1), \\ \sqrt{2}\alpha t + o(t), & \text{if } \alpha \in (-\infty, -\rho]. \end{cases} \quad (1.4)$$

These results were further refined by [Chen et al. \(2020\)](#). They gave precise constant and polynomial prefactors of the probability in (1.2), and proved the convergence in distribution of the first branching time and first branching location conditioned on the BBM having a low maximum. Moreover, they showed convergence of the extremal process under this conditioning.

The particular case $\alpha = 0$, which restricts the field to be below zero, has also received some attention for models related to BBM, such as the $2d$ discrete Gaussian free field and the binary branching random walk. In [Bolthausen et al. \(2001\)](#), Bolthausen, Deuschel, and Giacomin analyzed the $2d$ discrete Gaussian free field conditioned to be below zero away from the boundary. In [Roy \(2018\)](#), Roy studied a binary branching random walk conditioned to be below zero and gave bounds on the height of a typical vertex under the conditioning. Both models fall into the same universality class as BBM on the level of extremes.

Although this paper focuses on the BBM whose maximum is unusually low, it is worth noting that the probability of BBM to have an unusually high maximum, denoted as $u(t, x(t) + \sqrt{2}t) \equiv \mathbb{P}(X_{max}(t) \geq x(t) + \sqrt{2}t)$, has been studied as well (see, e.g. [Bramson, 1983](#); [Lalley and Sellke, 1987](#); [Chauvin and Rouault, 1988, 1990](#); [Harris, 1999](#); [Arguin et al., 2011](#); [Derrida et al., 2016](#)). $u(t, x)$ solves the Kolmogorov-Petrovsky-Piscounov or Fisher (F-KPP) equation [Fisher \(1937\)](#); [Kolmogorov et al. \(1996\)](#) (see [Ikeda et al., 1968a,b, 1969](#); [McKean, 1975](#)). The asymptotics of $u(t, x(t) + \sqrt{2}t)$ have been obtained for different ranges of $x(t)$ based on Bramson's analysis [Bramson \(1983\)](#). If

$x(t) = o(t)$, then (see [Bovier and Hartung, 2020](#), Proposition 2.1)

$$\lim_{t \rightarrow \infty} \frac{t^{\frac{3}{2}}}{\frac{3}{2\sqrt{2}} \ln(t)} e^{\sqrt{2}x(t) + \frac{x(t)^2}{2t}} u\left(t, x(t) + \sqrt{2t}\right) = C, \tag{1.5}$$

and if $x(t) = at + o(t)$, $a > 0$, then (see [Bovier and Hartung, 2014](#), Proposition 3.1)

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}} e^{\sqrt{2}x(t) + \frac{x(t)^2}{2t}} u\left(t, x(t) + \sqrt{2t}\right) = C(a), \tag{1.6}$$

where C and $C(a)$ are strictly positive constants.

1.1. *Main results.* Intuitively, three types of initial behaviors of BBM may lead to it having a low maximum: the initial particle branches at a late time, the initial particle travels to a low position before branching, or (recursively) the two independent BBMs starting from the first branching position both have low maxima. In this paper, we are interested in the interplay of these effects and aim at quantifying the exact scale of the decay in the large deviation estimates, with restrictions on the first branching time and location.

Set $\tau := \inf\{0 \leq s \leq t, n(s) > 1\}$ to be the first branching time and $y := X_1(\tau)$ the first branching location. Define the events

$$T_A := \{\tau \in A\}, \quad L_B := \{y \in B\}, \tag{1.7}$$

where $A \subset [0, t]$ and $B \subset (-\infty, \infty)$. We estimate probabilities of the form

$$\mathbb{P}\left(\left\{X_{max}(t) \leq \sqrt{2}\alpha t\right\} \wedge T_A \wedge L_B\right). \tag{1.8}$$

First, we give a large deviation estimate for the probability that the maximum of a BBM is below $\sqrt{2}\alpha t$, with $\alpha \in (-\infty, 1)$, $\tau \in [0, \gamma t]$, and $\gamma \in (0, 1]$.

Theorem 1.1. *For all $\alpha \in (-\infty, 1)$ and $\gamma \in (0, 1]$,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}\left(\left\{X_{max}(t) \leq \sqrt{2}\alpha t\right\} \wedge T_{[0, \gamma t]}\right) = -\psi_1^\alpha(\gamma), \tag{1.9}$$

where

$$\psi_1^\alpha(\gamma) := \begin{cases} -\gamma + \frac{2\alpha^2}{1+\gamma} + 2, & \text{if } \gamma \in \left(0, -\frac{\alpha+\rho}{\rho} \wedge 1\right], \\ -(4\sqrt{2}\rho - 1)\gamma + 4\rho(1 - \alpha), & \text{if } \gamma \in \left(-\frac{\alpha+\rho}{\rho} \wedge 1, \frac{1-\alpha}{2\sqrt{2}-1} \wedge 1\right], \\ \gamma + \frac{(\alpha-(1-\gamma))^2}{\gamma}, & \text{if } \gamma \in \left(\frac{1-\alpha}{2\sqrt{2}-1} \wedge 1, \frac{1-\alpha}{\sqrt{2}} \wedge 1\right], \\ 2\rho(1 - \alpha), & \text{if } \gamma \in \left(\frac{1-\alpha}{\sqrt{2}} \wedge 1, 1\right]. \end{cases} \tag{1.10}$$

Note that not all cases in (1.9) occur for α fixed, as

$$-\frac{\alpha + \rho}{\rho} > 0 \iff \alpha < -\rho, \quad -\frac{\alpha + \rho}{\rho} < 1 \iff \alpha > -2\rho, \tag{1.11}$$

$$\frac{1 - \alpha}{2\sqrt{2} - 1} < 1 \iff \alpha > -2\rho, \quad \frac{1 - \alpha}{\sqrt{2}} < 1 \iff \alpha > -\rho. \tag{1.12}$$

The proof of Theorem 1.1 in Section 3.1 shows that the optimal strategy is to make the first branching happen at time

$$\tau(\gamma) = \left(\gamma \wedge \frac{1 - \alpha}{\sqrt{2}}\right) t + o(t), \tag{1.13}$$

and at position

$$y(\gamma) = \begin{cases} \frac{2\sqrt{2}\alpha\gamma}{1+\gamma}t + o(t), & \text{if } \gamma \in \left(0, -\frac{\alpha+\rho}{\rho} \wedge 1\right], \\ -2\sqrt{2}\rho\gamma t + o(t), & \text{if } \gamma \in \left[-\frac{\alpha+\rho}{\rho} \wedge 1, \frac{1-\alpha}{2\sqrt{2}-1} \wedge 1\right], \\ \sqrt{2}\alpha t - \sqrt{2}(1-\gamma)t + o(t), & \text{if } \gamma \in \left(\frac{1-\alpha}{2\sqrt{2}-1} \wedge 1, \frac{1-\alpha}{\sqrt{2}} \wedge 1\right], \\ -\rho(1-\alpha)t + o(t), & \text{if } \gamma \in \left(\frac{1-\alpha}{\sqrt{2}} \wedge 1, 1\right]. \end{cases} \tag{1.14}$$

See Figure 1.1 for plots of $\psi_1^\alpha(\gamma)$ and $y(\gamma)$ as illustrations. By comparing the time-constrained optimal choices of τ in (1.13) with the unconstrained ones in (1.3), we see that to obtain a low maximum the first branching happens as late as possible, until the unrestricted optimal branching time is smaller than γt . The time-constrained optimal choices for y , as shown in (1.14), depend on the values of α and γ .

Next, we impose the restriction $X \in T_{[\gamma t, t]}$ with $\gamma \in \left(\frac{1-\alpha}{\sqrt{2}}, 1\right]$ to understand how an unusually late first branching time affects the large deviation estimates when $\alpha \in (-\rho, 1)$. This is done in the following theorem.

Theorem 1.2. *For all $\alpha \in (-\rho, 1)$ and $\gamma \in \left(\frac{1-\alpha}{\sqrt{2}}, 1\right]$,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left\{ X_{max}(t) \leq \sqrt{2}\alpha t \right\} \wedge T_{[\gamma t, t]} \right) = -\psi_2^\alpha(\gamma), \tag{1.15}$$

where

$$\psi_2^\alpha(\gamma) := \begin{cases} \gamma + \frac{(\alpha - (1-\gamma))^2}{\gamma}, & \text{if } \gamma \in \left(\frac{1-\alpha}{\sqrt{2}}, (1-\alpha) \wedge 1\right), \\ \gamma, & \text{if } \gamma \in [(1-\alpha) \wedge 1, 1]. \end{cases} \tag{1.16}$$

The proof of Theorem 1.2 in Section 3.1 shows that the optimal strategy is to let the first branching happens at time

$$\tau(\gamma) = \gamma t + o(t), \text{ for all } \gamma \in \left(\frac{1-\alpha}{\sqrt{2}}, 1\right], \tag{1.17}$$

and at position

$$y(\gamma) = \begin{cases} \sqrt{2}\alpha t - \sqrt{2}(1-\gamma)t + o(t), & \text{if } \gamma \in \left(\frac{1-\alpha}{\sqrt{2}}, (1-\alpha) \wedge 1\right), \\ o(t), & \text{if } \gamma \in [(1-\alpha) \wedge 1, 1]. \end{cases} \tag{1.18}$$

See Figure 1.2 for plots of $\psi_2^\alpha(\gamma)$ and $y(\gamma)$ as illustrations. Note that if $\gamma > 1-\alpha$, the probability with the late-first-branching constraint is of order $e^{-\gamma t + o(t)}$, which is of the same order as the probability that a BBM does not branch in $[0, \gamma t]$.

After studying how restrictions on the first branching time affect the large deviation estimates, we turn our attention to the effects of a constrained first branching location. In Theorems 1.3 and 1.4, we fix the first branching time to be in the interval $[(\gamma - \epsilon)t, \gamma t]$ where ϵ is positive and small, and impose restrictions on the first branching location to be either below or above $\sqrt{2}\alpha t - \sqrt{2}(t - \tau)$.

Note that $\sqrt{2}(t - \tau)$ is the leading order of the maximum of a BBM running for time $t - \tau$. If the maximum of the BBM (running for time t) has to stay below $\sqrt{2}\alpha t$ and the first branching location is below $\sqrt{2}\alpha t - \sqrt{2}(t - \tau)$, the two BBMs starting from the initial branching position do not need to have an unusually low maxima. The following theorem gives the large deviation estimates when the first branching location is forced to be below $\sqrt{2}\alpha t - \sqrt{2}(t - \tau)$.

Theorem 1.3. *For all $\alpha \in (-\infty, 1)$, $\gamma \in (0, 1]$, and $\beta \in [1, \infty)$,*

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left\{ X_{max}(t) \leq \sqrt{2}\alpha t \right\} \wedge T_{[(\gamma-\epsilon)t, \gamma t]} \wedge L_{(-\infty, \sqrt{2}\alpha t - \sqrt{2}\beta(t-\tau))} \right) = -\psi_3^{\alpha, \gamma}(\beta), \tag{1.19}$$

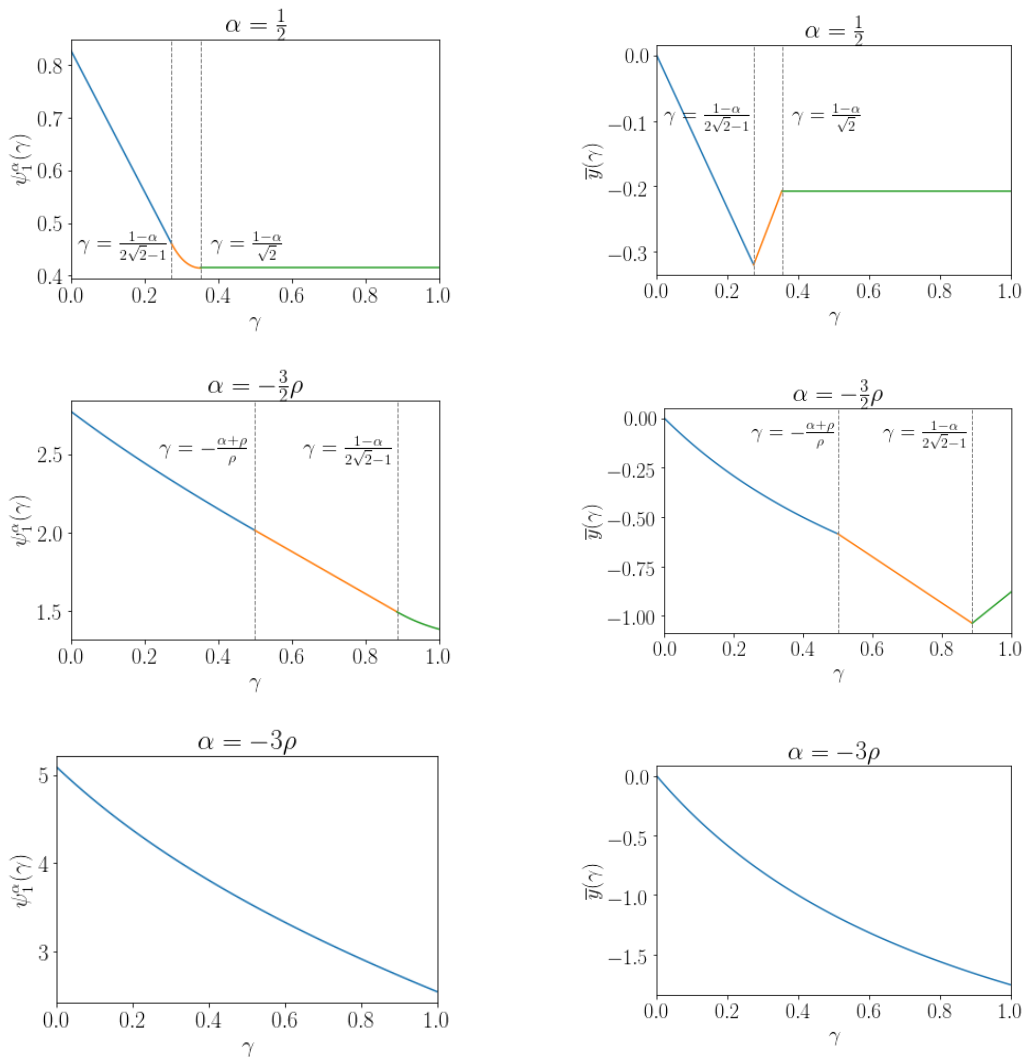


FIGURE 1.1. The plots in the left column show the rate function $\psi_1^\alpha(\gamma)$ in (1.10) in Theorem 1.1. Correspondingly, the plots in the right column depict the normalized optimal first branching locations $\bar{y}(\gamma) := \lim_{t \rightarrow \infty} y(\gamma)/t$, where $y(\gamma)$ is recorded in (1.14). We choose one representative α value in each of the ranges $[-\rho, 1)$, $(-2\rho, -\rho)$, and $(-\infty, -2\rho]$.

where

$$\psi_3^{\alpha,\gamma}(\beta) := \begin{cases} \gamma, & \text{if } \beta \in [1, \beta_1^\alpha(\gamma) \vee 1], \\ \gamma + \frac{(\alpha - (1-\gamma)\beta)^2}{\gamma}, & \text{if } \beta \in (\beta_1^\alpha(\gamma) \vee 1, \infty), \end{cases} \quad \text{with } \beta_1^\alpha(\gamma) := \begin{cases} \frac{\alpha}{1-\gamma}, & \text{if } 0 < \gamma < 1, \\ +\infty, & \text{if } \gamma = 1. \end{cases} \tag{1.20}$$

The proof of Theorem 1.3 in Section 3.2 shows that the optimal strategy for the initial particle is to split at location

$$y(\beta) = \begin{cases} o(t), & \text{if } \beta \in [1, \beta_1^\alpha(\gamma) \vee 1], \\ \sqrt{2}\alpha t - \sqrt{2}\beta(1-\gamma)t + o(t), & \text{if } \beta \in (\beta_1^\alpha(\gamma) \vee 1, \infty). \end{cases} \tag{1.21}$$

See Figure 1.3 for plots of $\psi_3^{\alpha,\gamma}(\beta)$ and $y(\beta)$ as illustrations. Note that since $\beta_1^\alpha(\gamma) > 1 \iff \gamma > 1 - \alpha$, the $\beta \in [1, \beta_1^\alpha(\gamma) \vee 1]$ case occurs only when $\alpha \in (0, 1)$ and $\gamma \in (1 - \alpha, 1]$. In this situation,

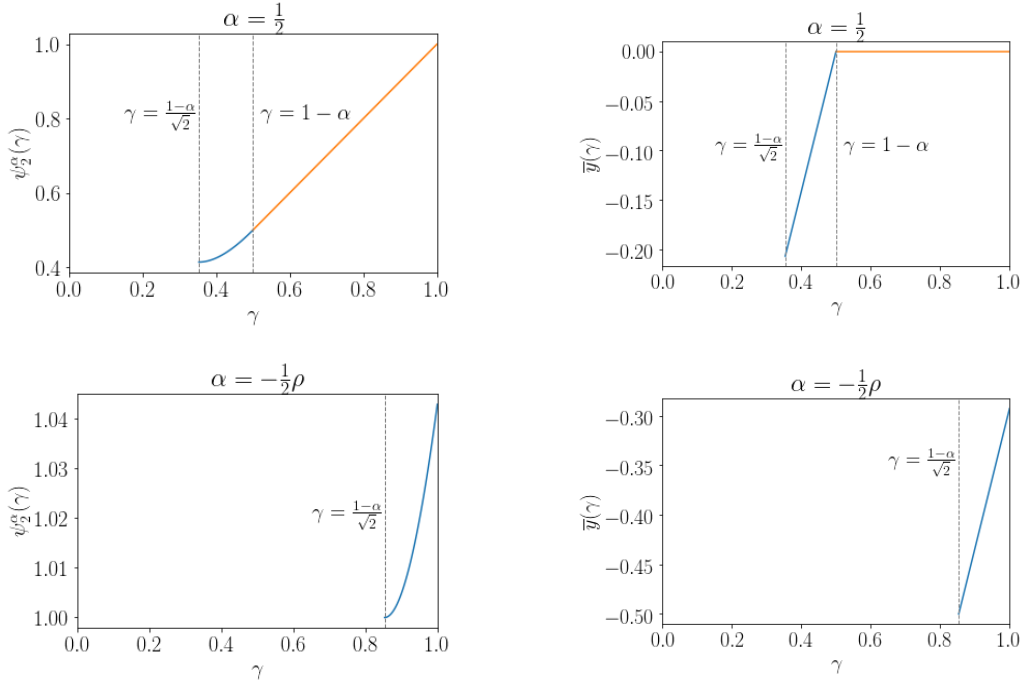


FIGURE 1.2. The plots in the left column show the rate function $\psi_2^\alpha(\gamma)$ in (1.16) in Theorem 1.2. Correspondingly, the plots in the right column depict the normalized optimal first branching locations $\bar{y}(\gamma) = \lim_{t \rightarrow \infty} y(\gamma)/t$, where $y(\gamma)$ is recorded in (1.18). We choose one representative α value in each of the two ranges $(0, 1)$ and $(-\rho, 0]$.

the additional location restriction has no additional impact and the values of the rate functions $\psi_2^\alpha(\gamma)$ and $\psi_3^{\alpha,\gamma}(\beta)$ agree. In all other cases, the location constraint does affect the large deviation estimates, and the optimal strategy for the initial particle is to split at the highest possible position $y \sim \sqrt{2}t - \sqrt{2}\beta(1 - \gamma)t$.

Next, we consider the case where the first branching location is restricted to be above $\sqrt{2}\alpha t - \sqrt{2}(t - \tau)$.

Theorem 1.4. For all $\alpha \in (-\infty, 1)$, $\gamma \in (0, 1]$, and $\beta \in (-\infty, 1]$,

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(\left\{ X_{max}(t) \leq \sqrt{2}\alpha t \right\} \wedge T_{[(\gamma-\epsilon)t, \gamma t]} \wedge L_{[\sqrt{2}\alpha t - \sqrt{2}\beta(t-\tau), \infty)} \right) = -\psi_4^{\alpha,\gamma}(\beta), \quad (1.22)$$

where,

- for $\gamma \in \left(0, -\frac{\alpha+\rho}{\rho} \wedge 1\right)$, $\psi_4^{\alpha,\gamma}(\beta)$ is equal to

$$\begin{cases} -(1 + \beta^2)\gamma + \frac{(\alpha-\beta)^2}{\gamma} + 2(\alpha\beta + 1), & \text{if } \beta \in \left(-\infty, \frac{\alpha}{1+\gamma}\right), \\ -\gamma + \frac{2\alpha^2}{1+\gamma} + 2, & \text{if } \beta \in \left[\frac{\alpha}{1+\gamma}, 1\right]; \end{cases} \quad (1.23)$$

- for $\gamma \in \left[-\frac{\alpha+\rho}{\rho} \wedge 1, 1\right]$, $\psi_4^{\alpha,\gamma}(\beta)$ is equal to

$$\begin{cases} -(1 + \beta^2)\gamma + \frac{(\alpha-\beta)^2}{\gamma} + 2(\alpha\beta + 1), & \text{if } \beta \in (-\infty, -\rho], \\ -(4\rho(1 - \beta) - 1 - \beta^2)\gamma + \frac{(\alpha-\beta)^2}{\gamma} + 4\rho(1 - \beta) + 2(\alpha - \beta)\beta, & \text{if } \beta \in (-\rho, \beta_2^\alpha(\gamma) \wedge 1), \\ -(4\sqrt{2}\rho - 1)\gamma + 4\rho(1 - \alpha), & \text{if } \beta \in [\beta_2^\alpha(\gamma) \wedge 1, 1], \end{cases} \quad (1.24)$$

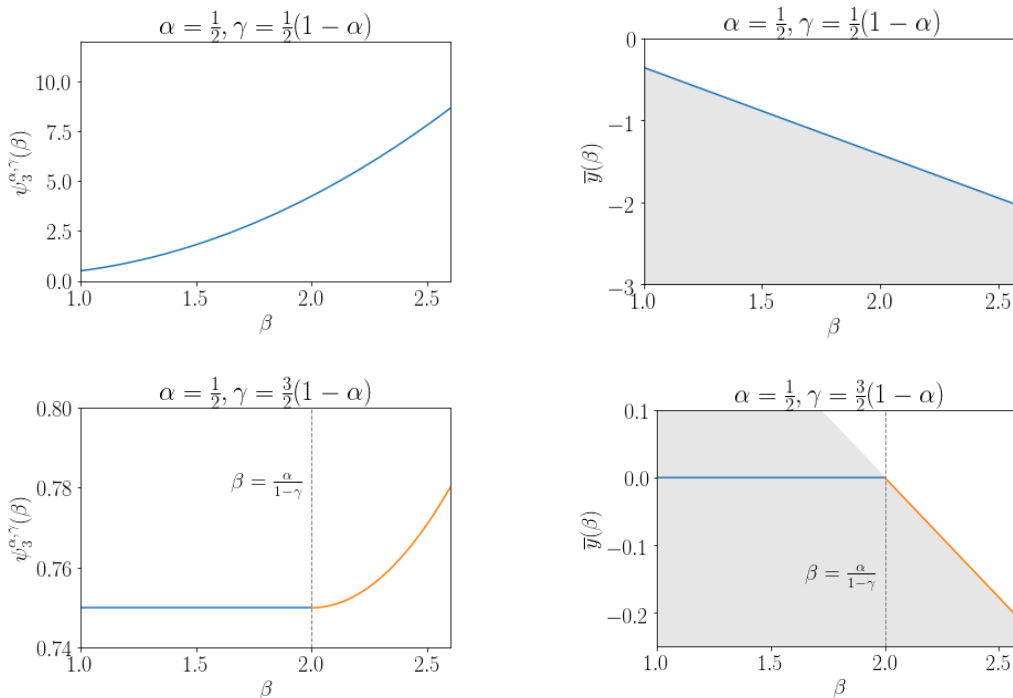


FIGURE 1.3. The plots in the left column show the rate function $\psi_3^{\alpha,\gamma}(\beta)$ in (1.20) in Theorem 1.3. Correspondingly, the plots in the right column depict the normalized optimal first branching locations $\bar{y}(\beta) = \lim_{t \rightarrow \infty} y(\beta)/t$, where $y(\beta)$ is recorded in (1.21). Representative values are chosen for α and γ . The shaded areas in the right plots indicate the area where the first branching is allowed to happen.

with

$$\beta_2^\alpha(\gamma) := \begin{cases} \frac{\alpha+2\rho\gamma}{1-\gamma}, & \text{if } 0 < \gamma < 1, \\ -\infty, & \text{if } \gamma = 1 \text{ and } \alpha < -2\rho, \\ +\infty, & \text{if } \gamma = 1 \text{ and } \alpha \geq -2\rho. \end{cases} \tag{1.25}$$

The proof of Theorem 1.4 in Section 3.2 shows that the optimal first branching location for $\gamma \in (0, -\frac{\alpha+\rho}{\rho} \wedge 1)$ is given by

$$y(\beta) = \begin{cases} \sqrt{2}\alpha t - \sqrt{2}\beta(1-\gamma)t + o(t), & \text{if } \beta \in (-\infty, \frac{\alpha}{1+\gamma}), \\ \frac{2\sqrt{2}\alpha\gamma}{1+\gamma}t + o(t), & \text{if } \beta \in [\frac{\alpha}{1+\gamma}, 1], \end{cases} \tag{1.26}$$

while for $\gamma \in [-\frac{\alpha+\rho}{\rho} \wedge 1, 1]$, it is equal to

$$y(\beta) = \begin{cases} \sqrt{2}\alpha t - \sqrt{2}\beta(1-\gamma)t + o(t), & \text{if } \beta \in (-\infty, \beta_2^\alpha(\gamma) \wedge 1), \\ -2\sqrt{2}\rho\gamma t + o(t), & \text{if } \beta \in [\beta_2^\alpha(\gamma) \wedge 1, 1]. \end{cases} \tag{1.27}$$

See Figure 1.4 for plots of $\psi_4^{\alpha,\gamma}(\beta)$ and $y(\beta)$ as illustrations. Note that, by (1.11), for $\alpha \in (-\infty, -2\rho]$ and for $\alpha \in [-\rho, 1)$, (1.23) resp. (1.24) define the rate function for all $\gamma \in (0, 1)$, while for $\alpha \in (-2\rho, -\rho)$, both (1.23) and (1.24) apply in appropriate γ ranges. Moreover, as $\beta_2^\alpha(\gamma) = \frac{\alpha+2\rho\gamma}{1-\gamma}$ when $0 < \gamma < 1$ and

$$\frac{\alpha + 2\rho\gamma}{1 - \gamma} < 1 \iff \gamma < \frac{1 - \alpha}{2\sqrt{2} - 1}, \tag{1.28}$$

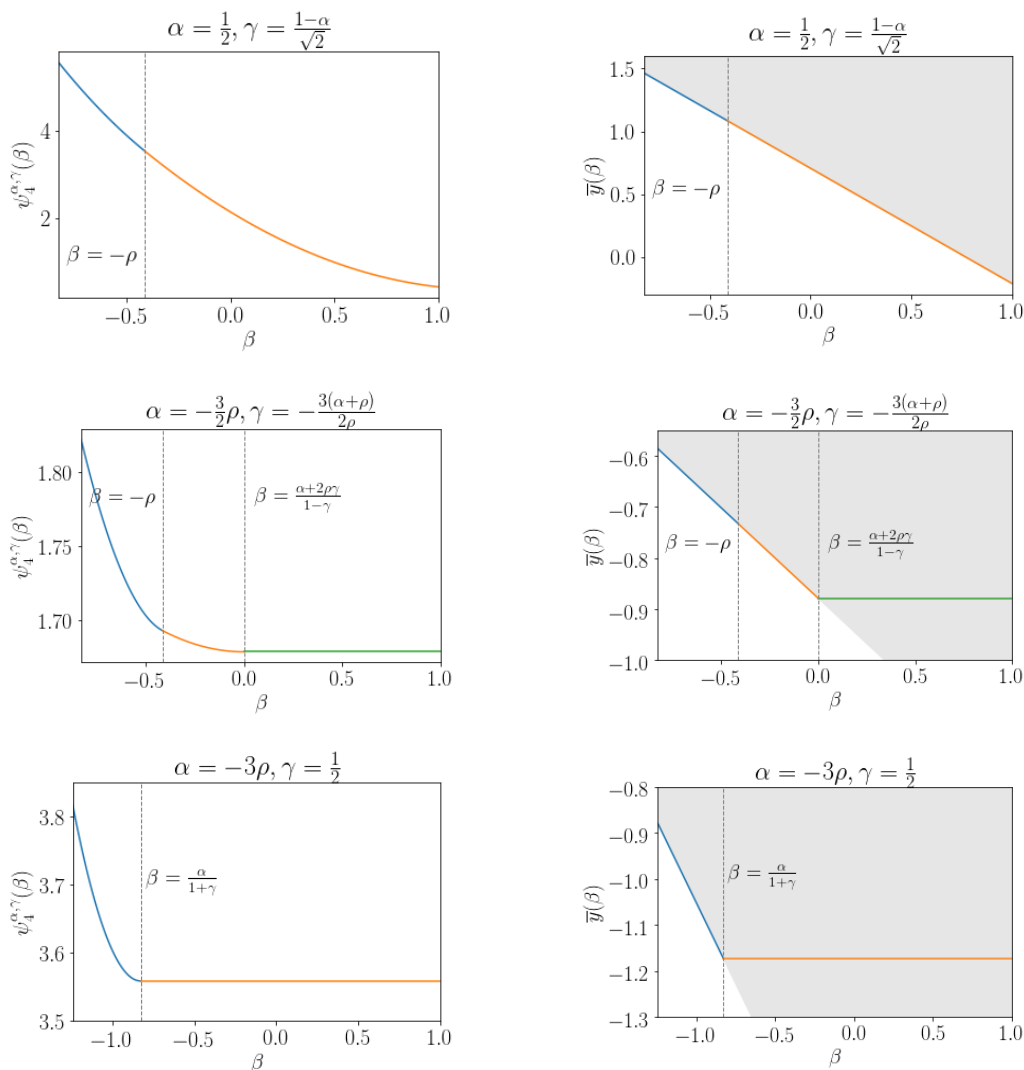


FIGURE 1.4. The plots in the left column show the rate function $\psi_4^{\alpha,\gamma}(\beta)$ in (1.23) and (1.24) in Theorem 1.3. Correspondingly, the plots in the right column depict the normalized optimal first branching locations $\bar{y}(\beta) = \lim_{t \rightarrow \infty} y(\beta)/t$, where $y(\beta)$ is recorded in (1.26) and (1.27). Representative values are chosen for α and γ . The shaded areas in the right plots indicate again the area where the first branching is allowed to happen.

the last case of (1.24) only occurs when the first branching happens early enough. It is also worth noticing that, if the range of β includes 1 (see the second case in (1.23) and the last two cases in (1.24)), the location constraint in Theorem 1.4 has no additional impact and the rate functions $\psi_4^{\alpha,\gamma}(\beta)$ and $\psi_1^{\alpha,\gamma}(\beta)$ coincide. In all other cases, the effect of the location constraint is evident, and the optimal strategy requires the initial particle to split at the lowest possible position $y \sim \sqrt{2}\alpha t - \sqrt{2}\beta(1-\gamma)t$ to minimize such an effect.

In Theorems 1.1-1.4, we have considered the probabilities of joint events, which are in the form $\{X_{max}(t) \leq \sqrt{2}\alpha t\} \wedge T_A \wedge L_B$. We comment that the asymptotic rate of the exponential decay of the conditional probability $\mathbb{P}(T_A \wedge L_B | X_{max}(t) \leq \sqrt{2}\alpha t)$ can be easily obtained from our results, since the asymptotic rate of the exponential decay of $\mathbb{P}(X_{max}(t) \leq \sqrt{2}\alpha t)$ is known from Derrida

and Shi (2017) as displayed in (1.2). We demonstrate how to achieve this for Theorem 1.1 in the following corollary, and comment that similar corollaries can be derived for Theorems 1.2-1.4.

Corollary 1.5. For all $\alpha \in (-\infty, 1)$ and $\gamma \in (0, 1]$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(T_{[0, \gamma t]} \mid X_{max}(t) \leq \sqrt{2\alpha t} \right) = \psi(\alpha) - \psi_1^\alpha(\gamma), \tag{1.29}$$

where $\psi(\alpha)$ is as in (1.2) and $\psi_1^\alpha(\gamma)$ is as in (1.10).

Proof: By Bayes' Theorem, the left-hand side of (1.29) is equal to

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\mathbb{P} \left(T_{[0, \gamma t]} \wedge \{ X_{max}(t) \leq \sqrt{2\alpha t} \} \right)}{\mathbb{P} \left(X_{max}(t) \leq \sqrt{2\alpha t} \right)} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(T_{[0, \gamma t]} \wedge \{ X_{max}(t) \leq \sqrt{2\alpha t} \} \right) - \lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P} \left(X_{max}(t) \leq \sqrt{2\alpha t} \right), \end{aligned} \tag{1.30}$$

which is equal to the right-hand side of (1.29) by (1.2) and Theorem 1.1. □

Outline of the paper. In Section 2, we first use the branching property to decompose the probability of a BBM to have a maximum below $\sqrt{2\alpha t}$ with constrained first branching time and location. The decomposition is stated in Lemma 2.2. In Lemmas 2.4 to 2.6, we analyze the resulting terms of this decomposition separately. In Section 3 we then prove our main results Theorems 1.1 to 1.4, based on the preparatory lemmas from Section 2.

2. Preparatory Lemmas

2.1. *Decomposition of the refined large deviation probabilities.* We rewrite the estimates from Derida and Shi (2017) (see also (1.2)) in a form that is convenient for us.

Corollary 2.1. For any $\tau \in [0, t)$, $\alpha, y \in \mathbb{R}$, and $K \in (1, \infty)$, as $t \rightarrow \infty$ and $t - \tau \rightarrow \infty$,

$$\mathbb{P} \left(X_{max}(t - \tau) \leq \sqrt{2\alpha t} - y \right) = \begin{cases} 1 - o(1), & \text{if } y \in I_1, \\ e^{-\sqrt{2}\rho(\sqrt{2}(1-\alpha)t - \sqrt{2}\tau + y) + o(t)}, & \text{if } y \in I_2, \\ e^{-(t-\tau) - \frac{(\sqrt{2\alpha t} - y)^2}{2(t-\tau)} + o(t)}, & \text{if } y \in I_{3,K}, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} I_1 &:= \left(-\infty, \sqrt{2\alpha t} - \sqrt{2}(t - \tau) \right), \\ I_2 &:= \left[\sqrt{2\alpha t} - \sqrt{2}(t - \tau), \sqrt{2\alpha t} + \sqrt{2}\rho(t - \tau) \right], \\ I_{3,K} &:= \left[\sqrt{2\alpha t} + \sqrt{2}\rho(t - \tau), \sqrt{2\alpha t} + \sqrt{2}K(t - \tau) \right], \end{aligned} \tag{2.4}$$

and

$$\mathbb{P} \left(X_{max}(t - \tau) \leq \sqrt{2\alpha t} - y \right) \leq e^{-K^2(t-\tau) + o(t)}, \quad \text{if } y \in I_{0,K}, \tag{2.5}$$

where

$$I_{0,K} := \left[\sqrt{2\alpha t} + \sqrt{2}K(t - \tau), \infty \right). \tag{2.6}$$

Proof: By rewriting the probability in the corollary as

$$\mathbb{P} \left(X_{max}(t - \tau) \leq \sqrt{2} \frac{\sqrt{2\alpha t} - y}{\sqrt{2}(t - \tau)} (t - \tau) \right), \tag{2.7}$$

distinguishing whether $\frac{\sqrt{2\alpha t} - y}{\sqrt{2}(t - \tau)}$ is in the range $[-\rho, 1]$ or $[-K, -\rho]$, and applying the estimate (1.2) on the respective compact intervals, (2.2) and (2.3) follow. Note that the convergence in (1.2) is uniform

on compact intervals by Dini’s theorem, as $\frac{1}{t} \ln(\mathbb{P}(X_{max}(t) \leq \sqrt{2\alpha t}))$ increases monotonically in α and the limit function $-\psi(\alpha)$ is continuous.

For (2.1), we let

$$x = \sqrt{2\alpha t} - \sqrt{2}(t - \tau) - y, \tag{2.8}$$

apply (1.5) when $x = o(t)$ and (1.6) when $x = at + o(t)$, $a > 0$ to one minus the probability in (2.7), and obtain that one minus the probability in (2.7) is asymptotically equal to

$$C(t - \tau)^{-\frac{3}{2}} \ln(t - \tau) \exp\left(-\sqrt{2}\left(\sqrt{2\alpha t} - \sqrt{2}(t - \tau) - y\right) - \frac{(\sqrt{2\alpha t} - \sqrt{2}(t - \tau) - y)^2}{2(t - \tau)}\right), \tag{2.9}$$

resp.

$$C(t - \tau)^{-\frac{1}{2}} \exp\left(-\sqrt{2}\left(\sqrt{2\alpha t} - \sqrt{2}(t - \tau) - y\right) - \frac{(\sqrt{2\alpha t} - \sqrt{2}(t - \tau) - y)^2}{2(t - \tau)}\right), \tag{2.10}$$

both of order $o(1)$. This implies (2.1).

For the remaining (2.5), since $y \in I_{0,K}$, the left-hand side is less than or equal to

$$\begin{aligned} \mathbb{P}\left(X_{max}(t - \tau) \leq -\sqrt{2}K(t - \tau)\right) &\leq \mathbb{P}\left(B_{t-\tau} \leq -\sqrt{2}K(t - \tau)\right) \\ &= \int_{-\infty}^{-\sqrt{2}K(t-\tau)} \frac{e^{-\frac{y^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} dy. \end{aligned} \tag{2.11}$$

where $B_{t-\tau}$ is a standard Brownian motion. Then (2.5) follows from Gaussian tail bounds. □

Recall that in Theorems 1.1 to 1.4, the refined large deviation probabilities take the form

$$\mathbb{P}\left(\left\{X_{max}(t) \leq \sqrt{2\alpha t}\right\} \wedge T_A \wedge L_B\right), \tag{2.12}$$

where $A \subset [0, t)$ and $B \subset (-\infty, \infty)$. We rewrite (2.12) by disintegrating at the first branching time and using the branching property of BBM. In the following lemma, note that Y_1, Y_2 , and Y_3 depend on τ .

Lemma 2.2. *Let $\alpha \in (-\infty, 1)$, $K \in (1, \infty)$, $A \subset [0, (1 - \epsilon)t]$ where $0 < \epsilon < 1$, and $B \subset (-\infty, \sqrt{2\alpha t} + \sqrt{2}K(t - \tau)]$. As $t \rightarrow \infty$,*

$$\mathbb{P}\left(\left\{X_{max}(t) \leq \sqrt{2\alpha t}\right\} \wedge T_A \wedge L_B\right) = \int_A (Y_1(B \cap I_1) + Y_2(B \cap I_2) + Y_3(B \cap I_{3,K})) d\tau, \tag{2.13}$$

where

$$Y_1(B \cap I_1) := \int_{B \cap I_1} e^{-\tau+o(1)} \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{y^2}{2\tau}\right) dy, \tag{2.14}$$

$$Y_2(B \cap I_2) := \int_{B \cap I_2} e^{(4\sqrt{2}\rho-1)\tau-4\rho(1-\alpha)t+o(t)} \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(y + 2\sqrt{2}\rho\tau)^2}{2\tau}\right) dy, \tag{2.15}$$

$$Y_3(B \cap I_{3,K}) := \int_{B \cap I_{3,K}} \sqrt{\frac{t-\tau}{t+\tau}} e^{-2t+\tau-\frac{2\alpha^2 t^2}{t+\tau}+o(t)} \frac{1}{\sqrt{2\pi\tau\frac{t-\tau}{t+\tau}}} \exp\left(-\frac{(y - \frac{2\sqrt{2\alpha t\tau}}{t+\tau})^2}{2\tau\frac{t-\tau}{t+\tau}}\right) dy. \tag{2.16}$$

Proof: By the branching property of BBM at the first branching time τ and location y , the probability in (2.13) is equal to

$$\int_A \int_B \frac{1}{\sqrt{2\pi\tau}} e^{-\tau-\frac{y^2}{2\tau}} \mathbb{P}\left(X_{max}(t - \tau) \leq \sqrt{2\alpha t} - y\right)^2 dy d\tau. \tag{2.17}$$

As the probability term and the remaining terms in the above integrand are bounded below by 0 and above by 1, we can exchange limit and integration by the dominated convergence theorem. As $t \rightarrow \infty$, $t - \tau$ also goes to infinity and thus by Corollary 2.1, (2.17) can be rewritten as

$$\begin{aligned} & \int_A \int_{B \cap I_1} (1 - o(1)) \frac{1}{\sqrt{2\pi\tau}} e^{-\tau - \frac{y^2}{2\tau}} dy d\tau \\ & + \int_A \int_{B \cap I_2} \frac{1}{\sqrt{2\pi\tau}} e^{-\tau - \frac{y^2}{2\tau} - 2\sqrt{2}\rho(\sqrt{2}(1-\alpha)t - \sqrt{2}\tau + y) + o(t)} dy d\tau \\ & + \int_A \int_{B \cap I_{3,K}} \frac{1}{\sqrt{2\pi\tau}} e^{-\tau - \frac{y^2}{2\tau} - 2(t-\tau) - \frac{(\sqrt{2}\alpha t - y)^2}{(t-\tau)} + o(t)} dy d\tau. \end{aligned} \tag{2.18}$$

Completing the squares for y , rearranging the terms, and letting $t \rightarrow \infty$, we obtain (2.13). □

We then show that if K is large enough, the probability $\mathbb{P}(\{X_{max}(t) \leq \sqrt{2}\alpha t\} \wedge T_A \wedge L_{I_{0,K}})$ can be made arbitrarily small and thus negligible.

Lemma 2.3. *Let $\alpha \in (-\infty, 1)$, $K \in (1, \infty)$, $A' \subset [0, 1 - \epsilon]$ where $0 < \epsilon < 1$, and $A = \{\lambda_\tau t : \lambda_\tau \in A'\}$. For any $M > 0$, there exists K large enough such that*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{P}(\{X_{max}(t) \leq \sqrt{2}\alpha t\} \wedge T_A \wedge L_{I_{0,K}}) \leq -M. \tag{2.19}$$

Proof: Similarly to the proof of Lemma 2.2, the probability in (2.19) is equal to

$$\int_A \int_{I_{0,K}} \frac{1}{\sqrt{2\pi\tau}} e^{-\tau - \frac{y^2}{2\tau}} \mathbb{P}(X_{max}(t - \tau) \leq \sqrt{2}\alpha t - y)^2 dy d\tau, \tag{2.20}$$

which, after bounding the probability in (2.20) by Corollary 2.1 and using Gaussian tail bounds, is bounded above by

$$\int_A \exp\left((K^2 - 1)\tau - (\alpha + K)^2 \frac{t^2}{\tau} + 2\alpha Kt + o(t)\right) d\tau. \tag{2.21}$$

By a change of variables $\lambda_\tau = \tau/t$, the bound becomes

$$\int_{A'} \exp\left(t \left[(K^2 - 1)\lambda_\tau - \frac{(\alpha + K)^2}{\lambda_\tau} + 2\alpha K \right] + o(t)\right) d\lambda_\tau. \tag{2.22}$$

Notice that the coefficient of t in the exponent, when regarded as a function of λ_τ , strictly increases for all $\lambda_\tau > 0$. By the Laplace's method, as $t \rightarrow \infty$, (2.22) is asymptotically

$$\exp\left(t \left[(K^2 - 1)\lambda'_\tau - \frac{(\alpha + K)^2}{\lambda'_\tau} + 2\alpha K \right] + o(t)\right), \quad \lambda'_\tau := \sup_{\lambda_\tau} A'. \tag{2.23}$$

Thus the left-hand side of (2.19) is bounded above by

$$(K^2 - 1)\lambda'_\tau - \frac{(\alpha + K)^2}{\lambda'_\tau} + 2\alpha K = -\frac{1 - \lambda'^2_\tau}{\lambda'_\tau} \left(K + \frac{\alpha}{\lambda'_\tau + 1}\right)^2 - \frac{2\alpha^2}{\lambda'_\tau + 1} - \lambda'_\tau, \tag{2.24}$$

which is negative and can be arbitrarily small once we choose K large enough. The claim of the lemma then follows. □

2.2. *First estimates on $Y_1, Y_2,$ and Y_3 .* The decomposition in Lemma 2.2 suggests that we need to obtain good approximations for $Y_1(B \cap I_1), Y_2(B \cap I_2),$ and $Y_3(B \cap I_{3,K})$. This is done in Lemma 2.4, Lemma 2.5, and Lemma 2.6.

Lemma 2.4. *For all $t > 0$ and $\alpha \in (-\infty, 1),$ suppose that there exist some $\lambda_\tau \in (0, 1]$ and $\lambda_u \in \mathbb{R}$ such that $\tau = \lambda_\tau t$ and $u = \sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t$. Then, as $t \rightarrow \infty,$*

$$Y_1((-\infty, u]) = \begin{cases} \exp(-tI_{11}(\lambda_\tau; \lambda_u) + o(t)), & \text{if } \lambda_u > \beta_1^\alpha(\lambda_\tau), \\ \exp(-tI_{12}(\lambda_\tau) + o(t)), & \text{if } \lambda_u \leq \beta_1^\alpha(\lambda_\tau), \end{cases} \tag{2.25}$$

where

$$I_{11}(x; y) := x + \frac{(\alpha - y(1 - x))^2}{x}, \quad I_{12}(x) := x. \tag{2.26}$$

Proof: Since $\tau = \lambda_\tau t$ and $u = \sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t,$ $Y_1((-\infty, u])$ is equal to

$$\int_{-\infty}^{\sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t} e^{-\lambda_\tau t} \frac{1}{\sqrt{2\pi\lambda_\tau t}} \exp\left(-\frac{y^2}{2\lambda_\tau t}\right) dy. \tag{2.27}$$

When $\sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t < 0 \iff \lambda_u(1 - \lambda_\tau) > \alpha,$ by Gaussian tail estimates (2.27) is approximately

$$\exp\left(-\lambda_\tau t - \frac{(\alpha - \lambda_u(1 - \lambda_\tau))^2}{\lambda_\tau} t + o(t)\right). \tag{2.28}$$

On the other hand, when $\lambda_u(1 - \lambda_\tau) \leq \alpha,$ the y integral in (2.27) is bounded from below by $\frac{1}{2}$ and from above by 1. Combining these two cases, (2.25) follows. \square

Lemma 2.5. *For all $t > 0$ and $\alpha \in (-\infty, 1),$ suppose that there exist some $0 < \lambda_\tau \leq 1$ and $-\infty < \lambda_v < \lambda_u < \infty,$ such that $\tau = \lambda_\tau t, u = \sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t,$ and $v = \sqrt{2}\alpha t - \sqrt{2}\lambda_v(1 - \lambda_\tau)t$. Then, as $t \rightarrow \infty,$*

$$Y_2([u, v]) = \begin{cases} \exp(-tI_{21}(\lambda_\tau; \lambda_v) + o(t)), & \text{if } \lambda_v > \beta_2^\alpha(\lambda_\tau), \\ \exp(-tI_{22}(\lambda_\tau) + o(t)), & \text{if } \lambda_v \leq \beta_2^\alpha(\lambda_\tau) \leq \lambda_u, \\ \exp(-tI_{21}(\lambda_\tau; \lambda_u) + o(t)), & \text{if } \lambda_u < \beta_2^\alpha(\lambda_\tau), \end{cases} \tag{2.29}$$

where

$$I_{21}(x; y) := -(4\rho(1 - y) - 1 - y^2)x + \frac{(\alpha - y)^2}{x} + (4\rho(1 - y) + 2y(\alpha - y)), \tag{2.30}$$

$$I_{22}(x) := -(4\sqrt{2}\rho - 1)x + 4\rho(1 - \alpha).$$

Proof: Plugging $\tau = \lambda_\tau t, u = \sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t,$ and $v = \sqrt{2}\alpha t - \sqrt{2}\lambda_v(1 - \lambda_\tau)t$ into (2.15), $Y_2([u, v])$ becomes

$$\int_{\sqrt{2}\alpha t - \sqrt{2}\lambda_u(1 - \lambda_\tau)t}^{\sqrt{2}\alpha t - \sqrt{2}\lambda_v(1 - \lambda_\tau)t} e^{(4\sqrt{2}\rho - 1)\lambda_\tau t - 4\rho(1 - \alpha)t + o(t)} \frac{1}{\sqrt{2\pi\lambda_\tau t}} \exp\left(-\frac{(y + 2\sqrt{2}\rho\lambda_\tau t)^2}{2\lambda_\tau t}\right) dy. \tag{2.31}$$

With a change of variable $z = y + 2\sqrt{2}\rho\lambda_\tau t,$ this is equal to

$$\int_{(\sqrt{2}\alpha - \sqrt{2}\lambda_u(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau)t}^{(\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau)t} e^{(4\sqrt{2}\rho - 1)\lambda_\tau t - 4\rho(1 - \alpha)t + o(t)} \frac{1}{\sqrt{2\pi\lambda_\tau t}} \exp\left(-\frac{z^2}{2\lambda_\tau t}\right) dz. \tag{2.32}$$

When $\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau < 0,$ (2.32) can be estimated by Gaussian tail asymptotics as

$$\exp\left(\left(4\sqrt{2}\rho - 1\right)\lambda_\tau t - 4\rho(1 - \alpha)t - \frac{(\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau)^2 t}{2\lambda_\tau} + o(t)\right), \tag{2.33}$$

which, after some rearrangements, is equal to the first term on the right-hand side of (2.29).

When $\sqrt{2}\alpha - \sqrt{2}\lambda_u(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau > 0$, we apply again Gaussian tail estimates to (2.32) and obtain

$$\exp\left(\left(4\sqrt{2}\rho - 1\right)\lambda_\tau t - 4\rho(1 - \alpha)t - \frac{(\sqrt{2}\alpha - \sqrt{2}\lambda_u(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau)^2 t}{2\lambda_\tau} + o(t)\right), \tag{2.34}$$

which, after some rearrangements, is equal to the third term on the right-hand side of (2.29).

When $\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau \geq 0$ and $\sqrt{2}\alpha - \sqrt{2}\lambda_u(1 - \lambda_\tau) + 2\sqrt{2}\rho\lambda_\tau \leq 0$, the z integral in (2.32) is bounded below by $\frac{1}{2}$ and above by 1. Then (2.32) is equal to the second term on the right-hand side of (2.29).

Combining these three cases, (2.29) follows. □

Lemma 2.6. *For all $t > 0$ and $\alpha \in (-\infty, 1)$, let $K > |\alpha|$, and suppose that there exist some $\lambda_\tau \in (0, 1]$ and $\lambda_v \in (-K, \infty)$ such that $\tau = \lambda_\tau t$ and $v = \sqrt{2}\alpha t - \sqrt{2}\lambda_v(1 - \lambda_\tau)t$. Then, as $t \rightarrow \infty$,*

$$Y_3\left(\left[v, \sqrt{2}\alpha t + \sqrt{2}K(1 - \lambda_\tau)t\right]\right) = \begin{cases} \exp(-tI_{31}(\lambda_\tau; \lambda_v) + o(t)), & \text{if } \lambda_v < \frac{\alpha}{1+\lambda_\tau}, \\ \exp(-tI_{32}(\lambda_\tau) + o(t)), & \text{if } \lambda_v \geq \frac{\alpha}{1+\lambda_\tau}, \end{cases} \tag{2.35}$$

where

$$\begin{aligned} I_{31}(x; y) &:= -(1 + y^2)x + \frac{(\alpha - y)^2}{x} + 2(\alpha y + 1), \\ I_{32}(x) &:= -x + \frac{2\alpha^2}{1 + x} + 2. \end{aligned} \tag{2.36}$$

Proof: With $\tau = \lambda_\tau t$ and $v = \sqrt{2}\alpha t - \sqrt{2}\lambda_v(1 - \lambda_\tau)t$, $Y_3([v, \infty))$ is equal to

$$\int_{\sqrt{2}\alpha t - \sqrt{2}\lambda_v(1 - \lambda_\tau)t}^{\sqrt{2}\alpha t + \sqrt{2}K(1 - \lambda_\tau)t} \sqrt{\frac{1 - \lambda_\tau}{1 + \lambda_\tau}} e^{-2t + \lambda_\tau t - \frac{2\alpha^2 t}{1 + \lambda_\tau} + o(t)} \frac{1}{\sqrt{2\pi\lambda_\tau t \frac{1 - \lambda_\tau}{1 + \lambda_\tau}}} \exp\left(-\frac{(y - \frac{2\sqrt{2}\alpha\lambda_\tau t}{1 + \lambda_\tau})^2}{2\lambda_\tau t \frac{1 - \lambda_\tau}{1 + \lambda_\tau}}\right) dy. \tag{2.37}$$

With a change of variable $z = y - \frac{2\sqrt{2}\alpha\lambda_\tau t}{1 + \lambda_\tau}$, this becomes

$$\int_{\left(\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) - \frac{2\sqrt{2}\alpha\lambda_\tau}{1 + \lambda_\tau}\right)t}^{\left(\sqrt{2}\alpha + \sqrt{2}K(1 - \lambda_\tau) - \frac{2\sqrt{2}\alpha\lambda_\tau}{1 + \lambda_\tau}\right)t} \sqrt{\frac{1 - \lambda_\tau}{1 + \lambda_\tau}} e^{-2t + \lambda_\tau t - \frac{2\alpha^2 t}{1 + \lambda_\tau} + o(t)} \frac{1}{\sqrt{2\pi\lambda_\tau t \frac{1 - \lambda_\tau}{1 + \lambda_\tau}}} \exp\left(-\frac{z^2}{2\lambda_\tau t \frac{1 - \lambda_\tau}{1 + \lambda_\tau}}\right) dz. \tag{2.38}$$

Notice that $\sqrt{2}\alpha + \sqrt{2}K(1 - \lambda_\tau) - \frac{2\sqrt{2}\alpha\lambda_\tau}{1 + \lambda_\tau} > 0$ since we set $K > |\alpha| > \frac{|\alpha|}{1 + \lambda_\tau}$.

If $\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) - \frac{2\sqrt{2}\alpha\lambda_\tau}{1 + \lambda_\tau} > 0$, by Gaussian tail estimates (2.38) is equal to

$$\exp\left(-2t + \lambda_\tau t - \frac{2\alpha^2 t}{1 + \lambda_\tau} - \frac{\left(\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) - \frac{2\sqrt{2}\alpha\lambda_\tau}{1 + \lambda_\tau}\right)^2 t}{2\lambda_\tau \frac{1 - \lambda_\tau}{1 + \lambda_\tau}} + o(t)\right), \tag{2.39}$$

which, after some rearrangements, is equal to the first term on the right-hand side of (2.35).

For $\sqrt{2}\alpha - \sqrt{2}\lambda_v(1 - \lambda_\tau) - \frac{2\sqrt{2}\alpha\lambda_\tau}{1 + \lambda_\tau} \leq 0$, the z integral in (2.38) is bounded below by $\frac{1}{2}$ and above by 1. After some rearrangements, (2.38) is equal to the second term on the right-hand side of (2.35).

Combining these two cases, (2.35) follows. □

Remark 2.7. In addition to the estimates on Y_1, Y_2 , and Y_3 , to compute (2.13) in Lemma 2.2 we still need to evaluate the asymptotics of the integrals with respect to τ (or $\lambda_\tau = \tau/t$ after a change

of variables). One can check that the integral (2.13) with the estimates on Y plugged in is of the form

$$\int_a^b \exp(tf(\lambda_\tau) + o(t))d\lambda_\tau, \tag{2.40}$$

where $-\infty \leq a \leq b \leq \infty$ and $f(\lambda_\tau)$ is a real, differentiable function with a unique maximum at $\lambda_\tau^{\max} \in [a, b]$. As $t \rightarrow \infty$, asymptotics of (2.40) can be derived by the Laplace’s method (see for reference, e.g. de Bruijn (1981)), being $C \exp(tf(\lambda_\tau^{\max}) + o(t))$ where C is some positive constant. We omit the proof as this is a quite standard technique.

3. Proofs of the main theorems

3.1. *Proofs for the time-constrained probabilities.* In this subsection, we prove Theorem 1.1 and 1.2 using the estimates from Section 2. We first prove Theorem 1.1.

Proof of Theorem 1.1: We rewrite (1.9), the equation to be proved, in the following three cases with different ranges of α .

(i) Given $\alpha \in [-\rho, 1)$, as $t \rightarrow \infty$,

$$\mathbb{P}\left(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[0,\gamma t]}\right) = \begin{cases} e^{-tI_{22}(\gamma)+o(t)}, & \text{if } \gamma \in \left(0, \frac{1-\alpha}{2\sqrt{2}-1}\right], \\ e^{-tI_{11}(\gamma;1)+o(t)}, & \text{if } \gamma \in \left(\frac{1-\alpha}{2\sqrt{2}-1}, \frac{1-\alpha}{\sqrt{2}}\right], \\ e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}};1\right)+o(t)}, & \text{if } \gamma \in \left(\frac{1-\alpha}{\sqrt{2}}, 1\right]. \end{cases} \tag{3.1}$$

(ii) Given $\alpha \in (-2\rho, -\rho)$, as $t \rightarrow \infty$,

$$\mathbb{P}\left(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[0,\gamma t]}\right) = \begin{cases} e^{-tI_{32}(\gamma)+o(t)}, & \text{if } \gamma \in \left(0, -\frac{\alpha+\rho}{\rho}\right], \\ e^{-tI_{22}(\gamma)+o(t)}, & \text{if } \gamma \in \left(-\frac{\alpha+\rho}{\rho}, \frac{1-\alpha}{2\sqrt{2}-1}\right], \\ e^{-tI_{11}(\gamma;1)+o(t)}, & \text{if } \gamma \in \left(\frac{1-\alpha}{2\sqrt{2}-1}, 1\right]. \end{cases} \tag{3.2}$$

(iii) Given $\alpha \in (-\infty, -2\rho]$, as $t \rightarrow \infty$,

$$\mathbb{P}\left(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[0,\gamma t]}\right) = e^{-tI_{32}(\gamma)+o(t)} \text{ for all } \gamma \in (0, 1]. \tag{3.3}$$

Applying Lemma 2.2 with $A = [0, \gamma t]$ and $B = (-\infty, \sqrt{2}\alpha t + \sqrt{2}K(t - \tau)]$, and noticing that the remaining $L_{I_0,K}$ part is negligible for K large enough by Lemma 2.3, we rewrite the targeted probability as

$$\int_0^{\gamma t} Y_1(I_1) d\tau + \int_0^{\gamma t} Y_2(I_2) d\tau + \int_0^{\gamma t} Y_3(I_{3,K}) d\tau, \tag{3.4}$$

which, after the change of variable $\lambda_\tau = \tau/t$, becomes

$$\begin{aligned} & \int_0^\gamma tY_1\left(\left(-\infty, \sqrt{2}\alpha t - \sqrt{2}(1 - \lambda_\tau)t\right]\right) d\lambda_\tau \\ & + \int_0^\gamma tY_2\left(\left[\sqrt{2}\alpha t + \left[-\sqrt{2}(1 - \lambda_\tau)t, \sqrt{2}\rho(1 - \lambda_\tau)t\right]\right) d\lambda_\tau \\ & + \int_0^\gamma tY_3\left(\left[\left[\sqrt{2}\alpha t + \sqrt{2}\rho(1 - \lambda_\tau)t, \sqrt{2}\alpha t + \sqrt{2}K(1 - \lambda_\tau)t\right]\right) d\lambda_\tau. \end{aligned} \tag{3.5}$$

In the remainder of the proof, we will estimate the three summands in (3.5), and then compare their orders. Using Lemma 2.4 with $\lambda_u = 1$, we can rewrite the first summand of (3.5) as

$$\int_0^{\gamma \wedge (1-\alpha)} e^{-tI_{11}(\lambda_\tau;1)+o(t)}d\lambda_\tau + \int_{\gamma \wedge (1-\alpha)}^\gamma e^{-tI_{12}(\lambda_\tau)+o(t)}d\lambda_\tau. \tag{3.6}$$

Observe that when regarded as a function of λ_τ , $-I_{11}(\lambda_\tau; 1)$ strictly increases as $\lambda_\tau < \frac{1-\alpha}{\sqrt{2}}$ and strictly decreases as $\lambda_\tau > \frac{1-\alpha}{\sqrt{2}}$, while $-I_{12}(\lambda_\tau)$ strictly decreases for all λ_τ . By Laplace’s method, we know that (3.6) is equal to

$$\begin{cases} e^{-tI_{11}(\gamma; 1)+o(t)}, & \text{if } 0 < \gamma \leq \frac{1-\alpha}{\sqrt{2}}, \\ e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)}, & \text{if } \frac{1-\alpha}{\sqrt{2}} < \gamma \leq 1-\alpha, \\ e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)} + e^{-tI_{12}(1-\alpha)+o(t)}, & \text{if } 1-\alpha < \gamma \leq 1, \end{cases} \tag{3.7}$$

which, as $-I_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right) > -I_{12}(1-\alpha)$ since $2\rho < 1$, is equal to

$$\begin{cases} e^{-tI_{11}(\gamma; 1)+o(t)}, & \text{if } 0 < \gamma \leq \frac{1-\alpha}{\sqrt{2}}, \\ e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)}, & \text{if } \frac{1-\alpha}{\sqrt{2}} < \gamma \leq 1. \end{cases} \tag{3.8}$$

Using Lemma 2.5 with $\lambda_u = 1$ and $\lambda_v = -\rho$, we rewrite the second summand of (3.5) as

$$\int_0^{\gamma \wedge \left(-\frac{\alpha+\rho}{\rho}\right)} e^{-tI_{21}(\lambda_\tau; -\rho)+o(t)} d\lambda_\tau + \int_{\gamma \wedge \left(-\frac{\alpha+\rho}{\rho}\right)}^{\gamma \wedge \left(\frac{1-\alpha}{2\sqrt{2}-1}\right)} e^{-tI_{22}(\lambda_\tau)+o(t)} d\lambda_\tau + \int_{\gamma \wedge \left(\frac{1-\alpha}{2\sqrt{2}-1}\right)}^\gamma e^{-tI_{21}(\lambda_\tau; 1)+o(t)} d\lambda_\tau. \tag{3.9}$$

Notice that the first and the second exponents in (3.9), regarded as functions of λ_τ , both strictly increase for all $\lambda_\tau > 0$. Since $I_{21}(\cdot; 1) = I_{11}(\cdot; 1)$, the monotonicity of the third exponent has been described below (3.6). By Laplace’s method, (3.9) is equal to

$$\begin{cases} e^{-tI_{21}(\gamma; -\rho)+o(t)}, & \text{if } 0 < \gamma \leq -\frac{\alpha+\rho}{\rho}, \tag{3.10} \\ e^{-tI_{21}\left(-\frac{\alpha+\rho}{\rho}; -\rho\right)+o(t)} + e^{-tI_{22}(\gamma)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} < \gamma \leq \frac{1-\alpha}{2\sqrt{2}-1}, \tag{3.11} \\ e^{-tI_{21}\left(-\frac{\alpha+\rho}{\rho}; -\rho\right)+o(t)} + e^{-tI_{22}\left(\frac{1-\alpha}{2\sqrt{2}-1}\right)+o(t)} + e^{-tI_{21}(\gamma; 1)+o(t)}, & \text{if } \frac{1-\alpha}{2\sqrt{2}-1} < \gamma \leq \frac{1-\alpha}{\sqrt{2}}, \tag{3.12} \\ e^{-tI_{21}\left(-\frac{\alpha+\rho}{\rho}; -\rho\right)+o(t)} + e^{-tI_{22}\left(\frac{1-\alpha}{2\sqrt{2}-1}\right)+o(t)} + e^{-tI_{21}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)}, & \text{if } \frac{1-\alpha}{\sqrt{2}} < \gamma \leq 1. \tag{3.13} \end{cases}$$

Observe that the difference

$$-I_{21}\left(-\frac{\alpha+\rho}{\rho}; -\rho\right) - (-I_{22}(\gamma)) = -(4\sqrt{2}\rho - 1)\gamma - 4\sqrt{2}(\alpha + \rho) + \frac{\alpha + \rho}{\rho} \tag{3.14}$$

is negative when $\gamma > -\frac{\alpha+\rho}{\rho}$, which implies that the second exponential terms in (3.11)-(3.13) are of larger order than the first terms. Moreover, the third term in (3.12) dominates the second, since

$$\begin{aligned} & -I_{21}(\gamma; 1) - \left(-I_{22}\left(\frac{1-\alpha}{2\sqrt{2}-1}\right)\right) \\ &= \frac{2}{\gamma} \left(\left(\gamma - \frac{11-4\sqrt{2}}{4(2\sqrt{2}-1)}(1-\alpha)\right)^2 - \left(\left(\frac{11-4\sqrt{2}}{4(2\sqrt{2}-1)}\right)^2 - \frac{1}{2}\right)(1-\alpha)^2 \right), \end{aligned} \tag{3.15}$$

which is negative when

$$\frac{1-\alpha}{2\sqrt{2}-1} < \gamma < \left(\sqrt{2} - \frac{1}{2}\right)(1-\alpha), \tag{3.16}$$

which is the case for $\gamma \in \left(\frac{1-\alpha}{2\sqrt{2}-1}, \frac{1-\alpha}{\sqrt{2}}\right]$ in (3.12). The third term in (3.13) also dominates the second, since

$$-I_{21}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right) - \left(-I_{22}\left(\frac{1-\alpha}{2\sqrt{2}-1}\right)\right) = \frac{\rho^2}{2\sqrt{2}-1}(1-\alpha) > 0. \tag{3.17}$$

Hence, we conclude that the second term in (3.5) is of order

$$\begin{cases} e^{-tI_{21}(\gamma; -\rho)+o(t)}, & \text{if } 0 < \gamma \leq -\frac{\alpha+\rho}{\rho}, \\ e^{-tI_{22}(\gamma)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} < \gamma \leq \frac{1-\alpha}{2\sqrt{2}-1}, \\ e^{-tI_{21}(\gamma; 1)+o(t)}, & \text{if } \frac{1-\alpha}{2\sqrt{2}-1} < \gamma \leq \frac{1-\alpha}{\sqrt{2}}, \\ e^{-tI_{21}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)}, & \text{if } \frac{1-\alpha}{\sqrt{2}} < \gamma \leq 1. \end{cases} \tag{3.18}$$

Applying Lemma 2.6 with $\lambda_v = -\rho$, we rewrite the third summand in (3.5) as

$$\int_0^{\gamma \wedge \left(-\frac{\alpha+\rho}{\rho}\right)} e^{-tI_{32}(\lambda_\tau)+o(t)} d\lambda_\tau + \int_{\gamma \wedge \left(-\frac{\alpha+\rho}{\rho}\right)}^\gamma e^{-tI_{31}(\lambda_\tau; -\rho)+o(t)} d\lambda_\tau. \tag{3.19}$$

Notice that the two exponents in (3.19), regarded as functions of λ_τ , strictly increase when $\lambda_\tau > 0$. Thus by Laplace’s method, (3.19) is equal to

$$\begin{cases} e^{-tI_{32}(\gamma)+o(t)}, & \text{if } 0 < \gamma \leq -\frac{\alpha+\rho}{\rho}, \\ e^{-tI_{32}\left(-\frac{\alpha+\rho}{\rho}\right)+o(t)} + e^{-tI_{31}(\gamma; -\rho)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} < \gamma \leq 1. \end{cases} \tag{3.20}$$

Notice that the second term in (3.21) dominates, since

$$-I_{31}(\gamma; -\rho) - \left(-I_{32}\left(-\frac{\alpha+\rho}{\rho}\right)\right) = \frac{1+\rho^2}{\gamma} \left(\left(\gamma + \frac{\alpha+\rho}{2\rho(1+\rho^2)}\right)^2 - \left(\frac{(2\rho^2+1)(\alpha+\rho)}{2\rho(1+\rho^2)}\right) \right), \tag{3.22}$$

which is positive if

$$\gamma < \frac{\rho(\alpha+\rho)}{1+\rho^2} \text{ or } \gamma > -\frac{\alpha+\rho}{\rho}, \tag{3.23}$$

satisfied by the range of γ in (3.21). Thus the third term in (3.5) is of order

$$\begin{cases} e^{-tI_{32}(\gamma)+o(t)}, & \text{if } 0 < \gamma \leq -\frac{\alpha+\rho}{\rho}, \\ e^{-tI_{31}(\gamma; -\rho)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} < \gamma \leq 1. \end{cases} \tag{3.24}$$

So far, we have obtained the estimates of the three summands in (3.5), which are shown in (3.8), (3.18), and (3.24). Next, we compare them for different ranges of α and γ .

Case (i). Since $\alpha \in [-\rho, 1)$, we have

$$-\frac{\alpha+\rho}{\rho} \leq 0 < \frac{1-\alpha}{2\sqrt{2}-1} < \frac{1-\alpha}{\sqrt{2}} < 1-\alpha < 1. \tag{3.25}$$

Adding (3.8), (3.18), and (3.24) together, (3.5) is equal to

$$\begin{cases} e^{-tI_{11}(\gamma; 1)+o(t)} + e^{-tI_{22}(\gamma)+o(t)} + e^{-tI_{31}(\gamma; -\rho)+o(t)} = e^{-tI_{22}(\gamma)+o(t)}, & \text{if } 0 < \gamma \leq \frac{1-\alpha}{2\sqrt{2}-1}, \\ 2e^{-tI_{11}(\gamma; 1)+o(t)} + e^{-tI_{31}(\gamma; -\rho)+o(t)} = e^{-tI_{11}(\gamma; 1)+o(t)}, & \text{if } \frac{1-\alpha}{2\sqrt{2}-1} < \gamma \leq \frac{1-\alpha}{\sqrt{2}}, \\ 2e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)} + e^{-tI_{31}(\gamma; -\rho)+o(t)} = e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right)+o(t)}, & \text{if } \frac{1-\alpha}{\sqrt{2}} < \gamma \leq 1, \end{cases} \tag{3.26}$$

$$\tag{3.27}$$

$$\tag{3.28}$$

where the equality in (3.26) holds because

$$-I_{22}(\gamma) - (-I_{11}(\gamma; 1)) = \frac{9-4\sqrt{2}}{\gamma} \left(\gamma - \frac{1-\alpha}{2\sqrt{2}-1}\right)^2 \geq 0 \text{ for all } 0 < \gamma \leq 1, \tag{3.29}$$

and

$$-I_{22}(\gamma) - (-I_{31}(\gamma; -\rho)) = \frac{1}{\gamma}(\rho\gamma + \alpha + \rho)^2 > 0 \text{ for all } 0 < \gamma \leq 1, \tag{3.30}$$

the equality in (3.27) holds because

$$-I_{11}(\gamma; 1) - (-I_{31}(\gamma; -\rho)) = -\frac{6 - 2\sqrt{2}}{\gamma} \left(\left(\gamma - \frac{2 - \sqrt{2}\alpha}{6 - 2\sqrt{2}} \right)^2 - \frac{2(\alpha + 2\rho)^2}{(6 - 2\sqrt{2})^2} \right), \tag{3.31}$$

which is nonnegative when

$$\frac{2 - 2\sqrt{2}(\alpha + \rho)}{6 - 2\sqrt{2}} \leq \gamma \leq 1, \tag{3.32}$$

satisfied by the corresponding γ range as $\frac{2 - 2\sqrt{2}(\alpha + \rho)}{6 - 2\sqrt{2}} < \frac{1 - \alpha}{2\sqrt{2} - 1}$ for $\alpha > -2\rho$, and the equality in (3.28) holds by (3.27) and

$$e^{-tI_{11}\left(\frac{1-\alpha}{\sqrt{2}}; 1\right) + o(t)} \geq e^{-tI_{11}(\gamma; 1) + o(t)}. \tag{3.33}$$

Then, (3.1) follows directly from (3.26)-(3.28).

Case (ii). Since $\alpha \in (-2\rho, -\rho)$, we have

$$0 < -\frac{\alpha + \rho}{\rho} < \frac{1 - \alpha}{2\sqrt{2} - 1} < 1 < \frac{1 - \alpha}{\sqrt{2}} < 1 - \alpha. \tag{3.34}$$

Thus (3.5) is equal to the sum of (3.8), (3.18), and (3.24),

$$\begin{cases} e^{-tI_{11}(\gamma; 1) + o(t)} + e^{-tI_{21}(\gamma; -\rho) + o(t)} + e^{-tI_{32}(\gamma) + o(t)} = e^{-tI_{32}(\gamma) + o(t)}, & \text{if } 0 < \gamma < -\frac{\alpha + \rho}{\rho}, & (3.35) \\ e^{-tI_{11}(\gamma; 1) + o(t)} + e^{-tI_{22}(\gamma) + o(t)} + e^{-tI_{31}(\gamma; -\rho) + o(t)} = e^{-tI_{22}(\gamma) + o(t)}, & \text{if } -\frac{\alpha + \rho}{\rho} \leq \gamma \leq \frac{1 - \alpha}{2\sqrt{2} - 1}, & (3.36) \\ 2e^{-tI_{11}(\gamma; 1) + o(t)} + e^{-tI_{21}(\gamma; -\rho) + o(t)} = e^{-tI_{11}(\gamma; 1) + o(t)}, & \text{if } \frac{1 - \alpha}{2\sqrt{2} - 1} < \gamma \leq 1, & (3.37) \end{cases}$$

where the equality in (3.36) follows from (3.26), the equality in (3.37) follows from (3.27), and the equality in (3.35) is because of the following two facts. The third term in (3.35) is of larger order than the second term, since

$$-I_{32}(\gamma) - (-I_{21}(\gamma; -\rho)) = \frac{1 - \gamma}{\gamma(1 + \gamma)} (\rho\gamma + (\alpha + \rho))^2 \geq 0 \text{ for all } 0 < \gamma \leq 1. \tag{3.38}$$

In addition, the second term in (3.35) dominates the first term, since $I_{21}(\cdot; -\rho) = I_{31}(\cdot; -\rho)$ and we learn from (3.31) that

$$-I_{21}(\gamma; -\rho) - (-I_{11}(\gamma; 1)) \leq 0 \iff \frac{2 - 2\sqrt{2}(\alpha + \rho)}{6 - 2\sqrt{2}} \leq \gamma \leq 1, \tag{3.39}$$

and $-\frac{\alpha + \rho}{\rho} < \frac{2 - 2\sqrt{2}(\alpha + \rho)}{6 - 2\sqrt{2}}$ when $\alpha > -2\rho$. Combining (3.35)-(3.37), (3.2) follows.

Case (iii). Since $\alpha \in (-\infty, -2\rho]$, we have

$$0 < 1 \leq \frac{1 - \alpha}{2\sqrt{2} - 1} \leq -\frac{\alpha + \rho}{\rho}. \tag{3.40}$$

Thus for all $0 < \gamma \leq 1$, (3.5) is equal to the sum of (3.8), (3.18), and (3.24),

$$e^{-tI_{11}(\gamma; 1) + o(t)} + e^{-tI_{21}(\gamma; -\rho) + o(t)} + e^{-tI_{32}(\gamma) + o(t)} = e^{-tI_{32}(\gamma) + o(t)}, \tag{3.41}$$

where the equality holds by the following two facts. Firstly, the first and the second exponents in (3.41) have appeared in (3.27) with different α ranges. The difference between these two exponents is recorded in (3.31), which, under the condition that $\alpha < -2\rho$, is not positive as that would require

$$1 < \gamma < \frac{2 - 2\sqrt{2}(\alpha + \rho)}{6 - 2\sqrt{2}}. \tag{3.42}$$

Thus the second term in (3.41) is of larger order than the first term. Secondly, from (3.38), we know that the third term in (3.41) dominates the second one for all $0 < \gamma \leq 1$. Hence, (3.41) holds and thus also (3.3) follows. This concludes the proof of Theorem 1.1. \square

Next, we prove Theorem 1.2, which focuses on the $\alpha \in (-\rho, 1)$ and restricts the first branching time τ to be later than the optimal one $\frac{1-\alpha}{\sqrt{2}}t$.

Proof of Theorem 1.2: In the same way of obtaining (3.4)-(3.5) in the proof of Theorem 1.1, we apply Lemma 2.2 with $A = [\gamma t, t]$ and $B = (-\infty, \sqrt{2}\alpha t + \sqrt{2}K(t - \tau)]$ and the change of variables $\lambda_\tau = \tau/t$ to rewrite $\mathbb{P}(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[\gamma t, t]})$ as

$$\begin{aligned} & \int_\gamma^1 tY_1 \left(\left(-\infty, \sqrt{2}\alpha t - \sqrt{2}(1 - \lambda_\tau)t \right] \right) d\lambda_\tau \\ & + \int_\gamma^1 tY_2 \left(\left[\sqrt{2}\alpha t - \sqrt{2}(1 - \lambda_\tau)t, \sqrt{2}\alpha t + \sqrt{2}\rho(1 - \lambda_\tau)t \right] \right) d\lambda_\tau \\ & + \int_\gamma^1 tY_3 \left(\left[\sqrt{2}\alpha t + \sqrt{2}\rho(1 - \lambda_\tau)t, \sqrt{2}\alpha t + \sqrt{2}K(1 - \lambda_\tau)t \right] \right) d\lambda_\tau. \end{aligned} \tag{3.43}$$

Notice that we restrict $-\rho < \alpha < 1$ and $\frac{1-\alpha}{\sqrt{2}} < \gamma \leq 1$ in the theorem. For the first summand in (3.43), we apply Lemma 2.4 with $\lambda_u = 1$ to rewrite it as

$$\int_\gamma^{\gamma\sqrt{1-\alpha}} e^{-tI_{11}(\lambda_\tau; 1)+o(t)} d\lambda_\tau + \int_{\gamma\sqrt{1-\alpha}}^1 e^{-tI_{12}(\lambda_\tau)+o(t)} d\lambda_\tau. \tag{3.44}$$

By Laplace’s method, (3.44) is equal to

$$\begin{cases} e^{-tI_{11}(\gamma; 1)+o(t)} + e^{-tI_{12}(1-\alpha)+o(t)}, & \text{if } 0 < \gamma < 1 - \alpha, \\ e^{-tI_{12}(\gamma)+o(t)}, & \text{if } 1 - \alpha \leq \gamma \leq 1. \end{cases} \tag{3.45}$$

Observe that

$$-I_{11}(\gamma; 1) - (-I_{12}(1 - \alpha)) = -\frac{2}{\gamma} \left(\left(\gamma - \frac{3(1 - \alpha)}{4} \right)^2 - \left(\frac{1 - \alpha}{4} \right)^2 \right), \tag{3.46}$$

which is positive when $\frac{1-\alpha}{2} < \gamma < 1 - \alpha$, satisfied by our γ range. Thus (3.44) is equal to

$$\begin{cases} e^{-tI_{11}(\gamma; 1)+o(t)}, & \text{if } 0 < \gamma < 1 - \alpha, \\ e^{-tI_{12}(\gamma)+o(t)}, & \text{if } 1 - \alpha \leq \gamma \leq 1. \end{cases} \tag{3.47}$$

For the second summand in (3.43), since $\lambda_\tau \geq \gamma > \frac{1-\alpha}{2\sqrt{2-1}}$, by Lemma 2.5 with $\lambda_u = 1, \lambda_v = -\rho$ and Laplace’s method, this summand is equal to

$$\int_\gamma^1 e^{-tI_{21}(\lambda_\tau; 1)+o(t)} d\lambda_\tau = e^{-tI_{21}(\gamma; 1)+o(t)}. \tag{3.48}$$

For the third summand in (3.43), since $\lambda_\tau \geq \gamma > -\frac{\alpha+\rho}{\rho}$, by Lemma 2.6 with $\lambda_v = -\rho$ and Laplace’s method, this summand is equal to

$$\int_\gamma^1 e^{-tI_{31}(\lambda_\tau; -\rho)+o(t)} d\lambda_\tau = e^{-tI_{31}(1; -\rho)+o(t)}. \tag{3.49}$$

To conclude the proof, we show that (3.47) dominates (3.48) and (3.49). Since $I_{11}(\cdot; 1) = I_{21}(\cdot; 1)$, it suffices to show that $-I_{12}(\gamma) \geq -I_{21}(\gamma; 1)$ and $-I_{21}(\gamma; 1) \geq -I_{31}(1; -\rho)$. The former

is obviously true for all $0 < \gamma \leq 1$. For the latter,

$$-I_{21}(\gamma; 1) - (-I_{31}(1; -\rho)) = -\frac{2}{\gamma} \left(\left(\gamma - \frac{\alpha^2 - 2\alpha + 3}{4} \right)^2 - \left(\frac{\alpha^2 - 2\alpha + 3}{4} \right)^2 + \frac{(1 - \alpha)^2}{2} \right), \tag{3.50}$$

which is nonnegative when

$$\frac{(1 - \alpha)^2}{2} \leq \gamma \leq 1. \tag{3.51}$$

As our range of γ is a subset of this, our claim is verified. Hence (3.47) indeed dominates over (3.48) and (3.49) and the proof is done. \square

3.2. Proofs for the location-constrained probabilities. In this subsection, we prove Theorem 1.3 and Theorem 1.4. We first prove Theorem (1.3), which restricts the first branching location to be below $\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau)$, for some $\beta \geq 1$.

Proof of Theorem 1.3: Let $A = [(\gamma - \epsilon)t, \gamma t]$ and $B = (-\infty, \sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau)]$. Since $\beta \geq 1$, $\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau) \leq \sqrt{2}\alpha t - \sqrt{2}(t - \tau)$ and thus with the notation from Lemma 2.2

$$B \cap I_1 = B, \quad B \cap I_2 = B \cap I_{3,K} = B \cap I_{0,K} = \emptyset. \tag{3.52}$$

Using Lemma 2.2, we rewrite $\mathbb{P} \left(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[(\gamma-\epsilon)t, \gamma t]} \cap L_{(-\infty, \sqrt{2}\alpha t - \sqrt{2}\beta(t-\tau))} \right)$ as

$$\int_{(\gamma-\epsilon)t}^{\gamma t} Y_1 \left(\left(-\infty, \sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau) \right) \right) d\tau = \int_{\gamma-\epsilon}^{\gamma} t Y_1 \left(\left(-\infty, \sqrt{2}\alpha t - \sqrt{2}\beta(1 - \lambda_\tau)t \right) \right) d\lambda_\tau, \tag{3.53}$$

after the change of variables $\lambda_\tau = \tau/t$. Applying Lemma 2.4 with $\lambda_u = \beta$, we obtain that

$$Y_1 \left(\left(-\infty, \sqrt{2}\alpha t - \sqrt{2}\beta(1 - \lambda_\tau)t \right) \right) = \begin{cases} e^{-tI_{11}(\lambda_\tau; \beta) + o(t)}, & \text{if } \beta > \beta_1^\alpha(\lambda_\tau), \\ e^{-tI_{12}(\lambda_\tau) + o(t)}, & \text{if } \beta \leq \beta_1^\alpha(\lambda_\tau). \end{cases} \tag{3.54}$$

Thus, by Laplace’s method, (3.53) is equal to

$$\begin{cases} e^{-tI_{11}(\gamma-\epsilon; \beta) + o(t)}, & \text{if } \beta > \beta_1^\alpha(\gamma), \\ e^{-tI_{12}(\gamma-\epsilon) + o(t)}, & \text{if } \beta \leq \beta_1^\alpha(\gamma). \end{cases} \tag{3.55}$$

Notice that the exponents in the above two cases are equal to each other at $\beta = \beta_1^\alpha(\gamma)$. The desired results follow if we take $t \rightarrow \infty$ and then $\epsilon \rightarrow 0$ in (3.55). \square

Next, we prove Theorem 1.4, where the first branching position is constrained to be above $\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau)$, for some $\beta \leq 1$.

Proof of Theorem 1.4: Let $A = [(\gamma - \epsilon)t, \gamma t]$ and $B = [\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau), \infty)$. For the set $B \cap I_{0,K}$, we choose K large enough so that $B \setminus I_{0,K}$ is not empty and the $B \cap I_{0,K}$ part is negligible by Lemma 2.3. From $\beta \leq 1$, we know that $B \cap I_1 = \emptyset$. To obtain the values of $B \cap I_2$ and $B \cap I_{3,K}$, we need to divide the β range into two and discuss two cases.

Case 1: $\beta \in (-\infty, -\rho]$. In this β range, we have that

$$B \cap I_1 = \emptyset, \quad B \cap I_2 = \emptyset. \tag{3.56}$$

Then by Lemma 2.2, $\mathbb{P} \left(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[(\gamma-\epsilon)t, \gamma t]} \cap L_{[\sqrt{2}\alpha t - \sqrt{2}\beta(t-\tau), \infty)} \right)$ can be rewritten as

$$\begin{aligned} & \int_{(\gamma-\epsilon)t}^{\gamma t} Y_3 \left(\left[\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau), \sqrt{2}\alpha t + \sqrt{2}K(t - \tau) \right] \right) d\tau \\ &= \int_{\gamma-\epsilon}^{\gamma} t Y_3 \left(\left[\sqrt{2}\alpha t - \sqrt{2}\beta(1 - \lambda_\tau)t, \sqrt{2}\alpha t + \sqrt{2}K(1 - \lambda_\tau)t \right] \right) d\lambda_\tau, \end{aligned} \tag{3.57}$$

after changing variables $\lambda_\tau = \tau/t$. Applying Lemma 2.6 with $\lambda_v = \beta$ and Laplace’s method, (3.57) is equal to

$$\begin{cases} e^{-tI_{31}(\gamma; \beta)+o(t)}, & \text{if } \beta < \frac{\alpha}{1+\gamma}, \\ e^{-tI_{32}(\gamma)+o(t)}, & \text{if } \beta \geq \frac{\alpha}{1+\gamma}. \end{cases} \tag{3.58}$$

As $\frac{\alpha}{1+\gamma} < -\rho \iff \gamma < -\frac{\alpha+\rho}{\rho}$, we summarize the results in Case 1 as follows.

- If $0 < \gamma < -\frac{\alpha+\rho}{\rho}$, then the estimate for the desired probability is of order

$$\begin{cases} e^{-tI_{31}(\gamma; \beta)+o(t)}, & \text{if } \beta < \frac{\alpha}{1+\gamma}, \\ e^{-tI_{32}(\gamma)+o(t)}, & \text{if } \frac{\alpha}{1+\gamma} \leq \beta \leq 1. \end{cases} \tag{3.59}$$

- If $-\frac{\alpha+\rho}{\rho} \leq \gamma < 1$, then the estimate is of order

$$e^{-tI_{31}(\gamma; \beta)+o(t)}, \text{ for all } \beta \leq 1. \tag{3.60}$$

Case 2: $\beta \in (-\rho, 1]$. In this β range,

$$B \cap I_2 = \left[\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau), \sqrt{2}\alpha t + \sqrt{2}\rho(t - \tau) \right], \quad B \cap I_{3,K} = I_{3,K}. \tag{3.61}$$

Thus by Lemma 2.2, $\mathbb{P} \left(X_{max}(t) \leq \sqrt{2}\alpha t, X \in T_{[(\gamma-\epsilon)t, \gamma t]} \cap L_{[\sqrt{2}\alpha t - \sqrt{2}\beta(t-\tau), \infty)} \right)$ can be rewritten as

$$\begin{aligned} & \int_{(\gamma-\epsilon)t}^{\gamma t} Y_2 \left(\left[\sqrt{2}\alpha t - \sqrt{2}\beta(t - \tau), \sqrt{2}\alpha t + \sqrt{2}\rho(t - \tau) \right] \right) d\tau \\ & + \int_{(\gamma-\epsilon)t}^{\gamma t} Y_3 \left(\left[\sqrt{2}\alpha t + \sqrt{2}\rho(t - \tau), \sqrt{2}\alpha t + \sqrt{2}K(t - \tau) \right] \right) d\tau, \end{aligned} \tag{3.62}$$

which, after the change of variables, becomes

$$\begin{aligned} & \int_{\gamma-\epsilon}^{\gamma} tY_2 \left(\left[\sqrt{2}\alpha t - \sqrt{2}\beta(1 - \lambda_\tau)t, \sqrt{2}\alpha t + \sqrt{2}\rho(1 + \lambda_\tau)t \right] \right) d\lambda_\tau \\ & + \int_{\gamma-\epsilon}^{\gamma} tY_3 \left(\left[\sqrt{2}\alpha t + \sqrt{2}\rho(1 - \lambda_\tau)t, \sqrt{2}\alpha t + \sqrt{2}K(1 - \lambda_\tau)t \right] \right) d\lambda_\tau. \end{aligned} \tag{3.63}$$

Applying Lemma 2.5 with $\lambda_u = \beta$ and $\lambda_v = -\rho$ and Laplace’s method, the first summand in (3.63) is equal to

$$\begin{cases} e^{-tI_{21}(\gamma; -\rho)+o(t)}, & \text{if } 0 < \gamma < -\frac{\alpha+\rho}{\rho}, \\ e^{-tI_{22}(\gamma)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} \leq \gamma \leq \frac{\beta-\alpha}{\beta+2\rho}, \\ e^{-tI_{21}(\gamma; \beta)+o(t)}, & \text{if } \frac{\beta-\alpha}{\beta+2\rho} < \gamma < 1. \end{cases} \tag{3.64}$$

By Lemma 2.6 with $\lambda_v = -\rho$ and Laplace’s method, the second summand in (3.63) is equal to

$$\begin{cases} e^{-tI_{32}(\gamma)+o(t)}, & \text{if } 0 < \gamma < -\frac{\alpha+\rho}{\rho}, \\ e^{-tI_{31}(\gamma; -\rho)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} \leq \gamma < 1. \end{cases} \tag{3.65}$$

Note that we put the equality sign in the different conditions compared to Lemma 2.6, since the two cases are equal when $\lambda_\tau = -\frac{\alpha+\rho}{\rho}$.

Adding together (3.64) and (3.65), we obtain that (3.63) is equal to

$$\begin{cases} e^{-tI_{21}(\gamma; -\rho)+o(t)} + e^{-tI_{32}(\gamma)+o(t)} = e^{-tI_{32}(\gamma)+o(t)}, & \text{if } 0 < \gamma < -\frac{\alpha+\rho}{\rho}, \end{cases} \tag{3.66}$$

$$\begin{cases} e^{-tI_{22}(\gamma)+o(t)} + e^{-tI_{31}(\gamma; -\rho)+o(t)} = e^{-tI_{22}(\gamma)+o(t)}, & \text{if } -\frac{\alpha+\rho}{\rho} \leq \gamma \leq \frac{\beta-\alpha}{\beta+2\rho}, \end{cases} \tag{3.67}$$

$$\begin{cases} e^{-tI_{21}(\gamma; \beta)+o(t)} + e^{-tI_{31}(\gamma; -\rho)+o(t)} = e^{-tI_{21}(\gamma; \beta)+o(t)}, & \text{if } \frac{\beta-\alpha}{\beta+2\rho} < \gamma < 1, \end{cases} \tag{3.68}$$

where the equalities in (3.66) and (3.67) follow from (3.35) and (3.26), respectively. The equality in (3.68) holds since

$$\begin{aligned} & -I_{21}(\gamma; \beta) - (-I_{31}(\gamma; -\rho)) \\ &= -\frac{(1-\gamma)^2}{\gamma} \left(\beta - \frac{2\rho\gamma + \alpha}{1-\gamma} \right)^2 + (3-2\sqrt{2})\gamma + \frac{(\alpha+\rho)^2}{\gamma} + 2\rho\alpha - 4\rho + 2 \\ &\geq -I_{21}(\gamma; -\rho) - (-I_{31}(\gamma; -\rho)) = 0, \end{aligned} \quad (3.69)$$

where the inequality is due to the facts that $-\rho < \beta \leq 1$ and, if $\gamma \neq 1$,

$$\gamma > \frac{\beta - \alpha}{\beta + 2\rho} \iff \beta < \frac{\alpha + 2\rho\gamma}{1 - \gamma}. \quad (3.70)$$

Hence, (3.66)-(3.68) follow and we obtain the desired estimate for Case 2. Grouping the estimates in (3.59)-(3.60) for Case 1 and (3.66)-(3.68) for Case 2 using the fact that

$$\gamma < -\frac{\alpha + \rho}{\rho} \iff \frac{\alpha}{1 + \gamma} < -\rho \quad (3.71)$$

and, when $\beta > -\rho$,

$$\alpha > -2\rho \iff -\frac{\alpha + \rho}{\rho} < \frac{\beta - \alpha}{\beta + 2\rho} < 1, \quad (3.72)$$

we obtain all estimates as stated in the theorem. \square

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