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The Itô-Tanaka Trick : a non-semimartingale approach

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Abstract. In this paper we provide an Itô-Tanaka trick formula in a non semimartingale context, filling a gap in the theory of regularisation by noise. In a classical Brownian framework, the Itô-Tanaka trick links the time average of a function f along the solution to a Brownian SDE, with the solution of a Fokker-Planck PDE. Our main contribution is to provide such a link in a non-semimartingale framework, where the solution to the non-available PDE is replaced by a well-chosen random field. This allows us to improve well-posedness results for fractional SDEs with a singular drift coefficient.

1. Introduction

Consider the following ODE :

$$x_t = x_0 + \int_0^t b(s, x_s) ds, \quad t \in [0, T], \quad x_0 \in \mathbb{R}^d, \tag{1.1}$$

with $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ a given vector field. The well-posedness of this equation is obviously related to the smoothness of the coefficient b and in particular famous counter-examples to uniqueness can be derived even in dimension one. The so-called Peano example fits into that paradigm and consists of choosing :

$$d = 1, \quad x_0 = 0, \quad b(t, x) := \sqrt{2}\,\mathrm{sgn}(x)\sqrt{|x|},$$

for which any mapping of the form $t \mapsto \pm(t - t_0)^2$ (with t_0 in $[0, T]$) is solution to (1.1). However, the seminal works Veretennikov (1981); Zvonkin (1974) put in light the remarkable fact according

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to which the well-posedness of the ODE can be obtained under very week conditions on b by adding a random force to the system, which then becomes the following SDE :

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \sigma W_t, \quad t \in [0,T], \quad x_0 \in \mathbb{R}^d, \tag{1.2}$$

with $\sigma > 0$ and W a Brownian motion on $\mathbb{R}^d$ (we use the notation X to stress than the solution is not deterministic anymore). This phenomenon is usually referred to *regularization by noise effect* or *stochastic regularization*. To be more precise, pathwise uniqueness can be obtained for Equation (1.2) for any vector field b satisfying weak regularity conditions : a boundedness assumption (Veretennikov (1981)) or a Ladyzhenskaya-Prodi-Serrin (LPS) type condition (see Krylov and Röckner (2005)) $b \in L^q([0,T]; L^p(\mathbb{R}^d))$:

$$\frac{d}{p} + \frac{2}{q} < 1, \quad p, q \geq 2. \tag{1.3}$$

In addition, this result can be captured and quantified by the so-called *Itô-Tanaka trick* or *Zvonkin's tranform* Flandoli et al. (2010) which reads as follows :

$$\int_0^T b(t, X_t + x) dt = -F(0, X_0 + x) - \int_0^T \nabla F(t, X_t + x) \cdot dW_t, \tag{1.4}$$

and which relates the process X to the solution $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ of the parabolic system of PDEs

$$\begin{cases} \partial_t F(t,x) + \mathcal{L}^X F(t,x) = b(t,x), & \forall (t,x) \in [0,T) \times \mathbb{R}^d, \\ F(T,x) = 0, & \forall x \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

with $\mathcal{L}_s^X \Phi(x) := b(s,x) \cdot \nabla \Phi(x) + \frac{1}{2} \Delta \Phi(x)$. Indeed, one can prove (see for a precise statement Flandoli et al. (2010); Krylov and Röckner (2005)) that the solution F to the PDE admits two weak derivatives in space and one in the time variable which entails that for any positive time t, the mapping

$$x \mapsto \int_0^t b(s, X_s + x) ds$$

is more regular than the field b itself (recall Relation (1.4)).

Note that investigating such regularization effect for ODEs finds interest in fluid mechanics equations which take the form of (non-linear) transport PDEs (we refer to Beck et al. (2019) for a survey on that account). For that purpose, the LPS condition (1.3) provides a natural framework in which fits this paper. However determining if the counterpart of the previous paradigm for ODEs transfers to non-linear transport PDEs is valid or not is mainly an open question. Although, most references in the literature, where regularization effects for SDEs are obtained, are based on the Itô-Tanaka trick it does not constitute the only technique for that regard (see for instance Beck et al. (2019); Catellier and Gubinelli (2016)).

In this paper we investigate a general framework in which the Itô-Tanaka trick is valid. Indeed, at this stage, one can point out at least two limitations to Relation (1.4). First, the strong link to the PDE (1.5) seems to be bound to the semimartingale realm (where one relates an SDE as a probabilistic counterpart of a parabolic PDE using the Itô formula). Another limitation is to investigate if Relation (1.4) can be extended to random fields b. Note that this step seems somehow mandatory to study the (possible) regularization phenomenon for a class of fluid mechanics equations which takes the form of non-linear transport PDEs (we refer to the comment Flandoli et al. (2010, page 6) on that question). For instance, counter-examples can be derived in the case where b is random as this extra randomness can cancel the effect of the noise W. As an example, consider

$b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ a non-smooth deterministic field, and $\tilde{b} : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ defined as : $\tilde{b}(\omega, t, x) := b(t, x - \sigma W_t(\omega))$, then it is clear that SDE

$$dX_t = \tilde{b}(t, X_t)dt + \sigma dW_t,$$

is equivalent to the deterministic ODE (by setting $x_t := X_t - \sigma W_t$) :

$$dx_t = b(t, x_t)dt.$$

This example enlights the fact that somehow the randomness and space variables (ω, x) have to be decoupled for a relation of the form (1.4) to be in force. In Duboscq and Réveillac (2016), the authors have extended the Itô-Tanaka trick to that framework, for which the improvement of regularity is obtained if the field b is Malliavin differentiable. In particular, this extra randomness is harmless for the regularity in the space variable for b if (ω, x) are "decoupled".

In this paper we revisit the Itô-Tanaka trick for random fields b and a non-semimartingale driving noise. More specifically, we bound ourselves to the case of a fractional Brownian motion (fBm) noise which allows one to compare our results with for instance the work Catellier and Gubinelli (2016) (extended in Lê (2020) for deterministic drift coefficient) in which pathwise uniqueness is proved for SDEs of the form (with W replaced by a fBm) but without using the Itô-Tanaka trick. Our approach is based on the use of Malliavin calculus arguments allowing one to escape the semimartingale context and to consider random fields b. To illustrate our key argument, we provide informal computations in the following particular example : $d = 1$, $b : \mathbb{R} \to \mathbb{R}$ (so b is deterministic and does not depend on the time variable). We stress that our main result is valid in any finite dimension and for a time-dependent vector field b, which is random (more precisely adapted according to assumptions presented in Section 3). Consider once again the solution X to the SDE (1.2), and let $(P_t^X)_{t \geq 0}$ the transition operator associated to it. For any fixed time $t > 0$, assuming that the random variable $A_t = b(t, X_t + x)$ is square integrable, one can apply the Clark-Ocone formula (which will be recalled below as Relation (2.7)) to get

$$A_t = \mathbb{E}\left[A_t | \mathcal{F}_0\right] + \int_0^t \mathbb{E}\left[D_s A_t | \mathcal{F}_s\right] dW_s,$$

where D denotes the Malliavin derivative (which will also be recalled in the next section). Hence, very formally, integrating with respect to t, we obtain :

$$\begin{aligned}
\int_0^T b(t, X_t + x)dt &= \int_0^T P_t^X b(t, X_0 + x)dt + \int_0^T \int_0^t D_s P_{t-s}^X b(t, X_s + x) dW_s dt \\
&= \int_0^T P_t^X b(t, X_0 + x)dt + \int_0^T \int_0^t \frac{\partial}{\partial x} P_{t-s}^X b(t, X_s + x) dW_s dt \\
&= \int_0^T P_t^X b(t, X_0 + x)dt + \int_0^T \int_s^T \frac{\partial}{\partial x} P_{t-s}^X b(t, X_s + x) dt dW_s,
\end{aligned}$$

where we have used stochastic Fubini's theorem. This relation exactly matches with the Itô-Tanaka trick (1.4) as the mild solution F to the PDE (1.5) writes down as :

$$F(s, x) = -\int_s^T P_{t-s}^X b(t, x)dt, \quad t \in [0,T], \quad x \in \mathbb{R}^d. \tag{1.6}$$

Unfortunately, most of the work done in the proof of Theorem 3.3 consists in making this heuristic correct. In the mean time, from these simple and very formal computations, one can make several remarks. First, the regularization effect is contained in the form of the solution to the PDE (using the semigroup associated to X). Then, this approach seems restricted to the deterministic case, as a measurability issue would prevent one to define the stochastic Itô integral $\int_0^T \int_s^T \frac{\partial}{\partial x} P_{t-s}^X b(t, X_s + x)dt dW_s$, even in the case of an adapted random field b. This problem has been solved in Duboscq and Réveillac (2016) where the PDE has to be replaced by a Backward Stochastic PDE whose solution is explicitly given as the predictable projection of the solution to the PDE (1.5). However, BSPDEs can only be solved and studied in a semimartingale context. The main idea of this paper is to use the classical representation of a fBm as the Itô integral of a well-chosen kernel against a standard Brownian motion, and to the apply (several times) the Clark-Ocone formula to a functional of the form (1.6). This functional will not be a solution to a PDE (or a BSPDE) which fits with the well-known result according to which the fBm cannot be related to a Markov semi-group, but it somehow plays this role. The several use of the Clark-Ocone formula allows us to precisely take into account the randomness coming from the field b and from the noise. Hence we obtain a generalization of the Itô-Tanaka trick as Theorem 3.3. We apply this result to recover the well-posedness of the fractional SDE associated to b in Theorem 4.4.

Finally, we would like to make a comment on the reference Catellier and Gubinelli (2016) where the authors prove the well-posedness of the fractional SDE. The proof relies on two ingredients: the study of the Fourier transform of the occupation measure related to W (to be more specific, on the (ρ,γ)-irregular property of W) and the reformulation of the SDE as a Young-type ODE where the time-integral of the drift is reinterpreted as a Young integral. The (ρ,γ)-irregular property of W provides the regularization effects of W and the authors do not rely on the Itô-Tanaka trick but on a kind of discrete martingale decomposition and a Hoeffding lemma. We remark that this martingale decomposition possesses some similarities with the Clark-Ocone formula. In Section 4, we follow the same reformulation (and the argument to construct the Young integral) to prove the existence and uniqueness of a fractional SDE but we do not prove exactly the (ρ,γ)-irregular property since we rely on more straightforward strategy in Sobolev spaces (at the cost of an embedding to recover estimates in Hölder spaces).

We proceed as follows. In the next section we present the main notations. The main result (Theorems 3.3) is presented in Section 3. The application to uniqueness of fractional SDEs (with additive noise) with adapted coefficients is presented in Section 4. The proof of Theorem 3.3 is postponed to Section 5.

2. Notations and preliminaries

2.1. *General notations.* Throughout this paper T denotes a positive real number, λ stands for the Lebesgue measure and $\mathcal{B}(E)$ denotes the Borelian σ-field of a given measurable pace E. We set also $\mathbb{N}^*$ the set of integers n with $n \geq 1$. As usual, $a \lesssim b$ means that there exists a constant $C \geq 0$ (independent of a and b), such that $a \leq C\,b$.

For any x in $\mathbb{R}^d$, we denote by x_k the k-th coordinate of x that is $x = (x_1, \ldots, x_d)$.

For any $r, d \in \mathbb{N}^*$, we denote by $\mathcal{C}^r(\mathbb{R}^d)$ the set of r-times continuously differentiable (real-valued) mappings defined on $\mathbb{R}^d$. We also let $\mathcal{C}_c^\infty(\mathbb{R}^d)$ the set of infinitely differentiable mappings with compact support.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ belongs to $\mathcal{C}^r(\mathbb{R}^d)$, $n \leq r$, $n_1, \ldots, n_d, p_1, \ldots, p_d$ in $\mathbb{N}$ with $\sum_{i=1}^d n_i^{p_i} = n$, we denote by $\frac{\partial^n \varphi}{\prod_{i=1}^d \partial {x_i}^{p_i}}$ the nth partial derivative of φ with respect to the variables x_i with order p_i. $\nabla\varphi$ will refer to the gradient of φ. Finally for any x and h in $\mathbb{R}^d$, and $k \in \mathbb{N}^*$, we write $\nabla^k\varphi(x) \cdot h^k$ the action of the k-order differentiable of φ (noted $\nabla^k\varphi(x)$) on $h^k := (h, \ldots, h)$. Finally, we denote by Δ the Laplacian operator.

For $p \geq 0$ and $m \in \mathbb{R}$, we set

$$W^{m,p}(\mathbb{R}^d) = \overline{\left\{\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d); \mathcal{F}^{-1}\left([1+|\xi|^2]^{m/2}\hat{\varphi}\right) \in L^p(\mathbb{R}^d)\right\}},$$

the usual Sobolev spaces equipped with its natural norm

$$\|\varphi\|_{W^{m,p}(\mathbb{R}^d)} := \left\|\mathcal{F}^{-1}\left([1+|\xi|^2]^{m/2}\hat{\varphi}\right)\right\|_{L^p(\mathbb{R}^d)},$$

where $\hat{\varphi}(\xi) = \mathcal{F}(\varphi)(\xi)$ and $\mathcal{F}$ (resp. $\mathcal{F}^{-1}$) denotes the Fourier transform (resp. the inverse Fourier transform).

We also make use of the following notation : let $(\mathcal{E}, \mathcal{B}, \mu)$ be a mesured space and $(G, \|\cdot\|_G)$ be a Banach space, and $r \geq 0$. We denote by $L^r(\mathcal{E}; G)$ the space of measurable mappings $\varphi : \mathcal{E} \to G$ with

$$\|\varphi\|^r_{L^r(\mathcal{E};G)} := \int_{\mathcal{E}} \|\varphi(y)\|^r_G \, \mu(dy) < +\infty.$$

Depending on the context, the definition of the integral will be made precise.

2.2. *The fractional Brownian motion.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $d \in \mathbb{N}$ ($d \geq 1$) and $B := (B_1(s), \ldots, B_d(s))_{s\in(-\infty,T]}$ a standard $\mathbb{R}^d$-valued two-sided Brownian motion (with independent components). We set $(\mathcal{F}_t)_{t\in(-\infty,T]}$ the natural (completed and right-continuous) filtration of B. We assume for simplicity that $\mathcal{F} = \sigma\left(B(s),\ s \in (-\infty, T]\right)$.

More generally, for any $\mathbb{R}^d$-valued stochastic process $(X(t))_{t\in(-\infty,T]}$ we will denote by X^j the jth component of X.

The main object of our analysis will be d-dimensional fractional Brownian motion

$$W^H := (W_1^H(s), \ldots, W_d^H(s))_{s\in[0,T]},$$

defined as

$$W_j^H(s) = \int_{-\infty}^s \left((s-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right) dB_j(u), \quad s \in [0,T], \quad j \in \{1, \cdots, d\}$$

where H is a given parameter in $(0,1) \setminus \left\{\frac{1}{2}\right\}$. A crucial decomposition is on analysis relies on the following split of the fBm W^H as follows :

$$\begin{aligned} W_j^H(s) &= \int_{-\infty}^s \left[(s-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right] dB_j(u) \\ &= \int_t^s (s-u)^{H-1/2} dB_j(u) + \int_{-\infty}^t \left[(s-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right] dB_j(u) \\ &=: W_j^{1,H}(t,s) + W_j^{2,H}(t,s), \end{aligned} \tag{2.1}$$

Note that for a given (s,t) with $t < s$, the random variable $W_j^{1,H}(t,s)$ is independent of $\mathcal{F}_t$ whereas the process $W^{2,H} := (W^{2,H}(t,s))_{t\in[0,s]}$ is $(\mathcal{F}_t)_{t\in[0,s]}$-adapted. It is worth noting that this decomposition is somehow natural in the context of stochastic regularisation and was already used in Catellier (2016) as only the component $W^{1,H}$ contributes to the regularising effect we will describe in the next sections.

We now turn to the notion of (smooth) adapted random field.

Definition 2.1 ((smooth) adapted random field)**.**

(i) A random field is a $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}$.

(ii) An adapted random field is a $\mathcal{F}_T\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)$-measurable mapping $\varphi:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ such that for any x in $\mathbb{R}^d$, $\varphi(\cdot,x)$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted.

(iii) A smooth adapted random field is an adapted random field φ such that $x\mapsto\varphi(\omega,t)$ is infinitely continuously differentiable with bounded derivatives of any order for $\lambda\otimes\mathbb{P}$-a.e. (ω,t) in $[0,T]\times\Omega$.

We denote by $P:=(P_t)_{t\in[0,T]}$ the Heat semigroup. For simplicity, we will use throughout this paper, the following notation for the conditional expectation.

Notations 2.2. For t in $[0,T]$, we set $\mathbb{E}_t[\cdot]:=\mathbb{E}[\cdot|\mathcal{F}_t]$.

2.3. *Malliavin-Sobolev spaces.* In this section, we introduce the main notations about the Malliavin calculus for random fields.

Definition 2.3. (i) Consider $\mathcal{S}_{r.v.}$ be the set of cylindrical random variables, that is the set of random fields $F:\Omega\times\mathbb{R}^d\to\mathbb{R}$ such that there exist :

$$n\in\mathbb{N}^*,\ 0\le\gamma_1<\gamma_2<\cdots<\gamma_n\le T,\quad \varphi:\mathbb{R}^{nd}\times\mathbb{R}^d\to\mathbb{R}\in\mathcal{C}_c^\infty(\mathbb{R}^{d(n+1)})$$

such that

$$F(\omega,x)=\varphi(B(\gamma_1),\cdots,B(\gamma_n),x),\quad \omega\in\Omega,\ x\in\mathbb{R}^d. \tag{2.2}$$

(ii) The set of cylindrical random fields denoted by $\mathcal{S}$, consists of random fields $F:[0,T]\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ such that there exist :

$$n\in\mathbb{N}^*,\ 0\le\gamma_1<\gamma_2<\cdots<\gamma_n\le T,\quad \varphi:[0,T]\times(\mathbb{R}^d)^n\times\mathbb{R}^d\to\mathbb{R}$$

such that

$$F(\omega,t,x)=\varphi(t,B(\gamma_1),\cdots,B(\gamma_n),x),\quad \omega\in\Omega,\ x\in\mathbb{R}^d,\quad t\in[0,T], \tag{2.3}$$

where $\varphi(t,\cdot)\in\mathcal{C}_c^\infty\left((\mathbb{R}^d)^n\times\mathbb{R}^d\right)$, $\forall t\in[0,T]$, and

$$\sup_{t\in[0,T]}\left(\|\varphi(t,\cdot)\|_\infty+\|\mathcal{L}\varphi(t,\cdot)\|_\infty\right)<+\infty$$

with $\mathcal{L}$ any partial derivative of any order.

(iii) The set of adapted cylindrical random fields denoted by $\mathcal{S}_{ad}$, consists of adapted random field is a random field $F:[0,T]\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ such that there exist :

$$n\in\mathbb{N}^*,\ 0\le\gamma_1<\gamma_2<\cdots<\gamma_n\le T,\quad \varphi:[0,T]\times(\mathbb{R}^d)^n\times\mathbb{R}^d\to\mathbb{R}$$

such that

$$F(\omega,t,x)=\varphi(t,B(\gamma_1\wedge t),\cdots,B(\gamma_n\wedge t),x),\quad \omega\in\Omega,\ x\in\mathbb{R}^d,\quad t\in[0,T], \tag{2.4}$$

where $\varphi(t,\cdot)\in\mathcal{C}_c^\infty\left((\mathbb{R}^d)^n\times\mathbb{R}^d\right)$, $\forall t\in[0,T]$, and

$$\sup_{t\in[0,T]}\left(\|\varphi(t,\cdot)\|_\infty+\|\mathcal{L}\varphi(t,\cdot)\|_\infty\right)<+\infty$$

with $\mathcal{L}$ any partial derivative of any order.

Obviously, $\mathcal{S}_{r.v.}\subset\mathcal{S}$ and $\mathcal{S}_{ad}\subset\mathcal{S}$.

We now define the Malliavin derivative of any adapted random field F in $\mathcal{S}$.

Definition 2.4. Let F in $\mathcal{S}$ with representation (2.3). Then, we define the Malliavin gradient DF of F as follows :

$$DF:[0,T]\times\Omega\times\mathbb{R}^d\to L^p([0,T];\mathbb{R}^d)$$

with for any j in $\{1,\cdots,d\}$,

$$(D_jF(t,\omega,x))\,(u):=\sum_{i=1}^{n}\frac{\partial\varphi}{\partial y_i}(t,B(\gamma_1)(\omega),\cdots,B(\gamma_n)(\omega),x)\mathbf{1}_{[0,\gamma_i]}(u),\quad u\in[0,T],\ \omega\in\Omega,x\in\mathbb{R}^d.$$

In particular, for any i,j in $\{1,\cdots,d\}$ and u,γ in $[0,T]$,

$$(D_jB(\gamma_i)(\omega))(u):=\begin{cases}\mathbf{1}_{[0,\gamma_i]}(u), & \text{if } i=j,\\ 0, \text{ else}\end{cases}.$$

We can now define Malliavin-Sobolev spaces associated to the Malliavin and the spatial derivatives for random fields.

Definition 2.5. Set $m\in\mathbb{R}$.

(i) We set $\mathbb{D}^{1,m,p}$ the closure of $\mathcal{S}_{r.v.}$ with respect to the seminorm $\|\cdot\|_{\mathbb{D}^{1,m,p}}$ with

$$\|F\|^p_{\mathbb{D}^{1,m,p}}:=\mathbb{E}[\|F\|^p_{W^{m,p}(\mathbb{R}^d)}]+\int_0^T\mathbb{E}\left[\|D_\theta F\|^p_{W^{m,p}(\mathbb{R}^d)}\right]d\theta. \tag{2.5}$$

(ii) We set $\mathbb{D}^{1,m,p}_p$ the closure of $\mathcal{S}$ with respect to the seminorm $\|\cdot\|_{\mathbb{D}^{1,m,p}_p}$ with

$$\|F\|^p_{\mathbb{D}^{1,m,p}_p}:=\int_0^T\|F(t,\cdot)\|^p_{\mathbb{D}^{1,m,p}}dt. \tag{2.6}$$

This definition, requires some justifications. Indeed, note that $D\nabla^kF=\nabla^kDF$ for F in $\mathcal{S}_{r.v.}$. In addition, as proved in Duboscq and Réveillac (2016, Lemma Appendix A.1 and Lemma Appendix A.2), the operators $D\nabla^k$ (and so ∇^kD) are closable from $\mathcal{S}$ to $L^p([0,T]\times\Omega\times\mathbb{R}^d;\mathbb{R}^d)$.

Remark 2.6. By definition

$$\mathcal{S}_{ad}\subset\mathbb{D}^{1,m,p}_p,\quad\forall m\geq 0,\ p\geq 2$$

We conclude this section with two properties of the Malliavin derivative.

Lemma 2.7.

(i) *(Chain rule). Let F be in $\mathcal{S}$ and G be in $\mathcal{S}_{r.v.}$. Then, for any t in $[0,T]$, $F(t,G)$ belongs to $\mathcal{S}_{r.v.}$ and :*

$$(D_jF(t,G))(u)=(D_jF(t,x))(u))_{x=G}+\frac{\partial F}{\partial x_j}(t,G)\times(D_jG)(u),\quad j\in\{1,\cdots,d\},\ u\in[0,T].$$

(ii) *Let m a real number, $p\geq 2$, t in $[0,T]$ and G be in $\mathbb{D}^{1,m,p}$. If G is $\mathcal{F}_t$-measurable, then for any $j\in\{1,\cdots,d\}$:*

$$((D_jG)(s))_{s>t}=0,\quad\textit{where the equality is understood in } L^2(\Omega\times(t,T])).$$

2.4. *Clark-Ocone formula.* Let $\mathcal{S}_{r.v.}$ be the set of random variables of the form $F=\varphi(B(t_1),\cdots,B(t_n))$ in $\mathcal{S}$ (that is that do not depend on the x-variable). We start with the following lemma whose proof can be found for instance in Privault (2009, Lemma 3.2.5).

Lemma 2.8. *The operator*

$$\begin{cases}D^{\mathcal{P}}: & \mathcal{S}_{r.v.} & \to L^2([0,T]\times\Omega;\mathbb{R}^d)\\ & F & \mapsto(\mathbb{E}_s[(DF)(s)])_{s\in[0,T]},\end{cases}$$

is continuous with respect to the $L^2(\Omega)$-norm. In particular in extends to $L^2(\Omega)$.

Consider $F:\Omega\to\mathbb{R}$ a random variable with $\mathbb{E}[|F|^2]<+\infty$. Then for any t in $[0,T]$,

$$F=\mathbb{E}_t[F]+\sum_{j=1}^d\int_t^T\mathbb{E}_s[(D_jF)(s)]dB_j(s) \tag{2.7}$$

$$\left(D^{\mathcal{P}}F\right)(u):=\mathbb{E}_u[(DF)(u)]$$

Note that by Lemma 2.8, the operator $(\mathbb{E}_s[D_sF])_s$ is well-defined even though F is not Malliavin differentiable.

3. Main result

Assumption 3.1. Let $m\in\mathbb{R}$, $p\geq 2$ and α in $\mathbb{R}$. An adapted random field $f:[0,T]\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ is said to enjoy Assumption 3.1 if :

$$1/2-H\alpha-1/p>0 \quad \text{ and } \quad f\in\mathbb{D}_p^{1,m-\alpha,p}.$$

We set :

Notations 3.2. Given an adapted random field f, we set for $0\leq t\leq u\leq s\leq T$, $x\in\mathbb{R}^d$, $j\in\{1,\cdots,d\}$:

$$f^a(s,t,x)=\mathbb{E}_t\left[f(s,x)\right] \quad \text{ and } \quad g_j(s,u,x)=D^{\mathcal{P}}f_j(s,u,x)=\mathbb{E}_u\left[(D_jf(s,x))(u)\right] \tag{3.1}$$

With these notations at hand we can state a non-semimartingale counterpart of the Itô-Tanaka-Wentzell trick for as:

Theorem 3.3. *Let $f:[0,T]\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ be an adapted random field and (p,m,α) such that Assumption 3.1 is in force. Then, $\forall S\in[0,T]$, we have*

$$\begin{aligned}\int_0^S f(r,W_r^H+x)dr=&\int_0^S P_{\frac{1}{2H}r^{2H}}f(r,W^H(r)+x)dr\\&+\sum_{j=1}^d\int_0^S\int_u^S P_{\frac{1}{2H}(r-u)^{2H}}\frac{\partial}{\partial x_j}f^a(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}drdB_j(u)\\&+\sum_{j=1}^d\int_0^S\int_u^S P_{\frac{1}{2H}(r-u)^{2H}}\frac{\partial}{\partial x_j}g_j(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}drdu\\&-\sum_{j=1}^d\int_0^S\int_u^S P_{\frac{1}{2H}(r-u)^{2H}}g_j(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}drdB_j(u),\end{aligned} \tag{3.2}$$

where the equality holds in $L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d)))$.

Remark 3.4.

(i) Note that the second term in the right-hand side of Formula (3.2) rewrites as :

$$\int_0^S\int_u^S P_{\frac{1}{2H}(r-u)^{2H}}\nabla f^a(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}dr\cdot dB(u),$$

whereas the third term is some sort of divergence term with respect to both the Malliavin derivative and the usual spatial derivative. More precisely, if we define $\mathrm{div}^{(\omega,x)}$ this joint divergence operator (applied to a random field $F:\Omega\times\mathbb{R}^d$) as :

$$\left(\mathrm{div}^{(\omega,x)}F\right)(u):=\sum_{j=1}^d\frac{\partial}{\partial x_j}\mathbb{E}_u\left[D_j(F(\cdot,x))(u)\right],$$

then the third term rewrites as

$$\int_0^S \int_u^S (r-u)^{H-1/2} P_{\frac{1}{2H}(r-u)^{2H}} \left(\mathrm{div}^{(\omega,x)} f(r,y)\right)(u)_{|y=W^{2,H}(u,r)+x} drdu$$

(ii) As mentioned in the introduction, the Itô-Tanaka trick relates a Brownian SDE with its Fokker-Planck PDE (see Krylov and Röckner (2005)). However, there is no PDE associated to the fBm W^H. In our approach we replace this non-available PDE by a random field which plays the role of the solution to the Fokker-Planck PDE in the (standard) Brownian setting and which is defined as :

$$F(t) := \int_t^T P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s)+x) ds, \quad t \in [0,T].$$

Note that this random field F does not appear explicitly in the right-hand-side of (3.2) although it is at the core of our approach.

We postpone the proof of this result to Section 5.

4. Application to fractional SDEs

In this section, we use Theorem 3.3 to obtain new results concerning the existence and uniqueness of SDEs with singular drifts and additive fractional Brownian motions. Our result applies in fact to a reformulation of such SDEs as Young ODEs (see Galeati (2021) for a survey) and we state some key results around these equations.

4.1. *Main result.* We consider the following SDE

$$X_t = X_0 + \int_0^t b(s, X_s) ds + W_t^H, \tag{4.1}$$

where $b : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is an adapted (generalized) function and $(W_t^H)_{t \geq 0}$ a fractional Brownian motion of Hurst index $H \in (0,1)$. By making the following change of variable

$$Y_t := X_t - W_t^H,$$

and setting, $\forall (u,x) \in \mathbb{R}^+ \times \mathbb{R}^d$,

$$A_u(x) = \int_0^u b(s, x + W_s^H) ds, \tag{4.2}$$

we can relate (4.1) to the following Young type ODE

$$Y_t = Y_0 + \int_0^t A_{ds}(Y_s), \tag{4.3}$$

where the integral is understood as a nonlinear generalization of the Young integral, $\forall Z \in \mathcal{C}^\gamma([0,T];\mathbb{R}^d)$,

$$\int_0^t A_{ds}(Z_s) = \lim_{|\Pi_{[0,t]}| \to 0} \sum_{[u,v] \in \Pi_{[0,t]}} \delta A_{u,v}(Z_u),$$

with

$$\delta A_{u,v}(x) := A_v(x) - A_u(x),$$

and $\Pi_{[0,t]}$ denoting a discretization of $[0,t]$. Before stating our result, we need the following "chain rule" assumption on the Malliavin derivative of b.

Assumption 4.1. Let $\vartheta \geq 0$, $\ell, q \in (2,+\infty)$, $p \in [2,+\infty)$, $k \in \mathbb{R}$, $\iota \in [0,1]$ and $\sigma, \bar{\sigma} \in [\ell,+\infty]$ such that

$$\frac{1}{2}-\frac{1}{\ell}-\frac{1}{q}>0, \quad \frac{1}{H}\left(\frac{1}{2}-\frac{1}{\ell}-\frac{1}{q}\right)+k-\vartheta-\frac{d}{p}>0 \quad \text{and} \quad \frac{1}{\sigma}+\frac{1}{\bar{\sigma}}=\frac{1}{\ell}. \tag{4.4}$$

We assume that b is an adapted function which belongs to $L^\ell(\Omega; L^q([0,T]; W^{k,p}(\mathbb{R}^d)))$ and that:

i) there exist a function a function $b' \in L^\sigma(\Omega; L^q([0,T]; L^\infty([0,T]; W^{k-\iota,p}(\mathbb{R}^{d\times d}))))$ and a mapping $\upsilon \in L^{\bar{\sigma}}(\Omega; L^\infty([0,T]\times\mathbb{R}^{d\times d}))$ such that
$$D_\theta b(t,x) = b'(t,\theta,x)\upsilon(\theta,t), \quad \forall \theta \leq t \leq T,\ \mathbb{P}-a.s.,$$
where, $\forall t \in [0,T]$, $b'(t,\theta,x)$ is $\mathcal{F}_t$-adapted for any $x \in \mathbb{R}^d$ and $\upsilon(\theta,t)$ is a $\mathcal{F}_t$ adapted function for any $0 \leq \theta \leq t$,

ii) there exists $C_1 \in L^{\bar{\sigma}}(\Omega; \mathbb{R}^{+*})$ such that one of the following statement is in force
- for any $0 \leq \theta \leq s \leq t \leq T$,
$$|\upsilon(\theta,t)| \leq C_1|\theta-t|^{H\iota}, \tag{4.5}$$
- $\iota = 0$ and $\upsilon(\theta,t) = C_1 \mathbf{1}_{\{\theta\leq\tau_b\}}$ where τ_b is a random variable with values in $[0,t]$,

Example 4.2. One can consider, for instance, the Gaussian functional

$$b(t,x) = \int_0^T \varphi(t,s,x)dB_s,$$

with $\varphi \in L^q(\mathbb{R}^+; L^\infty([0,T]; W^{k,p}(\mathbb{R}^d)))$ with $q \geq p \geq 2$ and k satisfying (4.4) for a certain $\ell \in (2,+\infty)$ and such that $\varphi(s,t,\cdot) = 0$ in $W^{k,p}(\mathbb{R}^d)$ if $s \geq t$. Thanks to the BDG inequality (see Remark 4.3 below), it follows that, for any $\ell \in (2,+\infty)$,

$$\mathbb{E}\left[\|b\|^\ell_{L^q(\mathbb{R}^+;W^{k,p}(\mathbb{R}^d))}\right]^{1/\ell} \lesssim \sqrt{T}\sup_{s\in[0,T]}\|\varphi(s,\cdot,\cdot)\|_{L^q(\mathbb{R}^+;W^{k,p}(\mathbb{R}^d))} \leq \sqrt{T}\|\varphi\|_{L^q(\mathbb{R}^+;L^\infty([0,T];W^{k,p}(\mathbb{R}^d)))}.$$

Furthermore, we have

$$D_\theta b(t,x) = \varphi(\theta,t,x)\mathbf{1}_{\theta\leq T},$$

which satisfies the required hypothesis.

Remark 4.3. The Burkholder-Davis-Gundy inequality can be used in the context of Banach spaces if they are UMD Banach space of type 2 (see Hytönen et al. (2017)) which is the case of $L^q([0,T]; W^{m,p}(\mathbb{R}^d))$ or $W^{m,p}$ for $q \geq p$ and $p \in (2,+\infty)$.

We can now give our result.

Theorem 4.4. *Let $T > 0$. Under Assumption 4.1 with $\vartheta = 2$, there exists $\beta > 1/2$ such that Equation (4.3) admits a unique solution $Y \in \mathcal{C}^\beta([0,T]; \mathbb{R}^d)$.*

Remark 4.5. We note that the previous result is not optimal compared to the ones of Catellier and Gubinelli (2016); Lê (2020) since our assumptions are more restrictives. Indeed, we consider a completely deterministic point of view when dealing with the Cauchy problem whereas, in Catellier and Gubinelli (2016); Lê (2020), the authors rely on more stochastic approach based on a Girsanov transformation. Our result gives in fact a "path-by-path uniqueness" while the (less restrictives) ones from Catellier and Gubinelli (2016); Lê (2020) give a "pathwise uniqueness".

Remark 4.6. The equality obtained in Theorem 3.3 holds in $L^\infty([0,T]; L^p(\Omega; W^{m,p}(\mathbb{R}^d)))$. However, in the proof of Theorem 4.4, we bound an increment of each term in $L^\ell(\Omega; W^{m,p}(\mathbb{R}^d))$. That is why we need the stronger Assumption 4.1.

Remark 4.7. Even though b might be defined in the sense of generalized functions (or Schwarz distribution), the Young integral (4.3) can still be well-defined due to regularization effect of $(W_t^H)_{t\geq 0}$ whereas the integral of the drift in (4.1) does not make sense. Nevertheless, it is possible to define a notion of "controlled solution" for (4.1) (see Catellier and Gubinelli (2016)). See also Remark 4.12 below.

4.2. *The Cauchy problem for Young ODEs.* We recall here some results on the nonlinear Young integration procedure and the Cauchy problem related to the Young ODE. Here, we simply give the results from Catellier and Gubinelli (2016) but the reader might also be interested in Galeati (2021).

Definition 4.8. Let $T > 0$, $\beta, \gamma \in (0, 1]$, $I = [0, T]$ and V, W to Banach spaces. For all $n \in \mathbb{N}$, and any mapping $A : I \times V \to W$, we define the norm

$$\|A\|_{\beta,\gamma} = \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \sup_{\substack{x,y\in V\\ x\neq y}} \frac{|\delta A_{s,t}(x) - \delta A_{s,t}(y)|_W}{|t-s|^{\beta}|x-y|_V^{\gamma}},$$

and

$$\|A\|_{\beta,n+\gamma} = \|\mathfrak{D}^n A\|_{\beta,\gamma} + \sum_{k=0}^{n} \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \sup_{x\in\mathbb{R}^d} \frac{|\mathfrak{D}^n \delta A_{s,t}(x)|_{\mathcal{L}^k(V;W)}}{|t-s|^{\beta}},$$

where $\mathfrak{D}$ denotes the Fréchet derivative from V to W.

We can now proceed to state the results from Catellier and Gubinelli (2016). The first result concerns the existence of the nonlinear Young integral.

Theorem 4.9. *Let $\beta, \gamma, \rho > 0$ with $\beta + \gamma\rho > 1$, V, W two Banach spaces and I a finite interval of $\mathbb{R}$. We consider $A \in \mathcal{C}^{\beta,\gamma}(I, V; W)$ and $Y \in \mathcal{C}^{\rho}(I; V)$. For any $s, t \in I$ such that $s \leq t$, the following nonlinear Young integral exists and is independent of the partition*

$$\int_s^t A_{dr}(Y_r) := \lim_{\substack{\Pi \text{ partition of } [s,t]\\ |\Pi|\to 0}} \sum_{[u,v]\in\Pi} \delta A_{u,v}(Y_u).$$

Furthermore, we have

(1) *for all $u \in [s, t]$, the equality*

$$\int_s^t A_{dr}(Y_r) = \int_s^u A_{dr}(Y_r) + \int_u^t A_{dr}(Y_r),$$

(2) *the following bound*

$$\left|\int_s^t A_{dr}(Y_r) - \delta A_{s,t}(Y_s)\right|_W \lesssim_{\beta,\gamma,\rho} \|A\|_{\beta,\gamma}\|Y\|_{\mathcal{C}^{\rho}(I;V)}^{\gamma}(t-s)^{\beta+\gamma\rho}$$

(3) *for all $s, t \in I$ such that $s \leq t$ and $R > 0$, the map*

$$(Y, A) \to \int_s^t A_{dr}(Y_r)$$

is a continuous function from
$(\{Y \in \mathcal{C}^{\rho}(I; V);\ \|Y\|_{\mathcal{C}^{\rho}(I;V)} \leq R\}, \|\cdot\|_{L^\infty([s,t];V)}) \times (\mathcal{C}^{\beta,\gamma}(I, V; W), \|\cdot\|_{\beta,\gamma})$ *to W.*

The next result gives the existence of a solution to the Equation (4.3).

Theorem 4.10. *Let $\beta > 1/2$, $\gamma \in [0,1)$ such that*

$$\beta(1+\gamma) > 1.$$

We consider $A \in \mathcal{C}^{\beta,\gamma}([0,T];\mathbb{R}^d)$. There exists a solution $Y \in \mathcal{C}^{\beta}([0,T];\mathbb{R}^d)$ to the nonlinear Young differential equation (4.3). Furthermore, there exists a constant C depending on β, γ, T and $\|A\|_{\beta,\gamma}$ such that

$$\|Y\|_{\mathcal{C}^{\beta}([0,T])} \le C(|Y_0| + 1).$$

We finally state a uniqueness result which only relies on the regularity of A.

Theorem 4.11. *Let $\beta > 1/2$, $\gamma > 0$ such that $A \in \mathcal{C}^{\beta,\gamma+2}$. Then, there exists a unique solution $Y \in \mathcal{C}^{\beta}([0,T];\mathbb{R}^d)$ to the nonlinear Young differential equation (4.3).*

Remark 4.12. Let b is a continuous bounded function. A classical solution of the Cauchy problem (4.1) is a path α-Hölder continuous with $\alpha < H$, such that almost surely, $Y = X - W^H$ is absolutely continuous with respect to the Lebesgue measure and its derivative $\dot{Y}_s = b(X_s)$ almost surely in s. In general, there is no uniqueness of such solution without additional assumptions on b.

Following Bass and Chen (2001) Definition 1 and Corollary 2.18 of Catellier and Gubinelli (2016), a Young-Catellier-Gubinelli solution of (4.1) is a process X such

(i) that $Y = X - W^H$ is γ Hölder continuous for some $\gamma > 1/2$,

(ii) for all sequence $(b_n)_{n\in\mathbb{N}}$ of bounded, Lipschitz continuous functions such that $(A^n)_{n\in\mathbb{N}}$ converge to A in some space $\mathcal{C}^{\beta,\gamma+1}$, the sequence of the solution of (4.3) $(Y^n)_{n\in\mathbb{N}}$ converges to Y in the β-Hölder continuous seminorms.

Let b fulfilling Assumption 4.1 and be continuous bounded. Let $(b^n)_{n\in\mathbb{N}}$ be a mollification of b and

$$A^n_{s,t}(x) := \int_s^t b^n(x + w_u)du, \ \forall x \in \mathbb{R}^d, \ s,t \in [0,T].$$

For any $n \in \mathbb{N}$, the equations (4.1) with b^n instead of b and (4.3) with A^n instead of A have a unique solution, namely X^n and Y^n and

$$X^n = Y^n + W^H.$$

Using Corollary 2.18 of Catellier and Gubinelli (2016), one can prove that $(Y^n)_{n\in\mathbb{N}}$ converges in a space of γ-Hölder continuous for some $\gamma > 1/2$ to Y solution of (4.3) with $A_{s,t}(x) := \int_s^t b(y_s + W^H_s)ds$, $\forall x \in \mathbb{R}^d$, $t \in [0,T]$. The limiting process Y does not depend of the choosen sequence of mollifier. Then, since b is continuous, bounded, $X := Y + W^H$ is a solution of (4.1).

One can said, that X is the unique Young-Catellier-Gubinelli solution.

4.3. *Proof of Theorem 4.4.* To obtain such results in our context, we need Theorem 3.3 and, from there, we essentially have to derive the proper bounds on A in adequate Sobolev spaces. Before proceeding in this direction, we recall the smoothing properties of the heat semigroup.

Lemma 4.13. *Let $m \in \mathbb{R}$, $\gamma > 0$ and $p \in (1,\infty)$. For any $f \in W^{m,p}(\mathbb{R}^d)$ and $\tau \in \mathbb{R}^{+*}$, we have*

$$\|P_\tau f\|_{W^{m,p}(\mathbb{R}^d)} \lesssim \tau^{-\gamma/2}\|f\|_{W^{m-\gamma,p}(\mathbb{R}^d)}.$$

We are now in position to prove the following result.

Proposition 4.14. *Under Assumption 4.1 and for A defined by (4.2), there exists $\gamma > 0$ and $\beta > 1/2$ such that, up to a modification, $A \in \mathcal{C}^{\beta}([0,T];\mathcal{C}^{\vartheta+\gamma}_b(\mathbb{R}^d))$ where $\mathcal{C}^{\vartheta+\gamma}_b(\mathbb{R}^d)$ is the space of bounded and $\vartheta + \gamma$-Hölder functions.*

Proof: By Assumption 4.1, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\frac{1}{2H} - \frac{1}{H\ell} - \frac{1}{Hq} + k - \vartheta - \frac{d}{p} = \varepsilon_1 + \frac{\varepsilon_2}{H}.$$

We denote

$$m := \vartheta + \frac{d}{p} + \varepsilon_1 = k + \frac{1}{H}\left(\frac{1}{2} - \frac{1}{\ell} - \frac{1}{q} - \varepsilon_2\right).$$

Step 1: By Theorem 3.3 and (4.2), we have that, for any $x \in \mathbb{R}^d$, $\delta A_{s,t}(x)$ is given by

$$\begin{aligned}
\delta A_{s,t}(x) = &\int_s^t P_{\frac{1}{2H}r^{2H}} b(r, W^H(r) + x)dr \\
&+ \sum_{j=1}^d \left(\int_s^t \int_u^t + \int_0^s \int_s^t\right) P_{\frac{1}{2H}(r-u)^{2H}} \frac{\partial}{\partial x_j} b^a(r, u, W^{2,H}(u, r) + x)(r-u)^{H-1/2} dr dB_j(u) \\
&+ \sum_{j=1}^d \left(\int_s^t \int_u^t + \int_0^s \int_s^t\right) P_{\frac{1}{2H}(r-u)^{2H}} \frac{\partial}{\partial x_j} D^{\mathcal{P}} b_j(r, u, W^{2,H}(u, r) + x)(r-u)^{H-1/2} dr du \\
&- \sum_{j=1}^d \left(\int_s^t \int_u^t + \int_0^s \int_s^t\right) P_{\frac{1}{2H}(r-u)^{2H}} D^{\mathcal{P}} b_j(r, u, W^{2,H}(u, r) + x)(r-u)^{H-1/2} dr dB_j(u), \\
&=: \delta A^{(1)}_{s,t} + \delta A^{(2)}_{s,t} + \delta A^{(3)}_{s,t} + \delta A^{(4)}_{s,t},
\end{aligned}$$

where we denote

$$D^{\mathcal{P}} b_j(r, u, x) = (D^{\mathcal{P}} b(r, x))_j(u).$$

We now estimate each term from the right-hand-side in the $L^\ell(\Omega; W^{\vartheta+d/p+\varepsilon_1,p}(\mathbb{R}^d))$-norm. By a density argument, we can assume that b is a smooth random field. For the first term, we have, thanks to Hölder's inequality and Lemma 4.13,

$$\begin{aligned}
\left\|\delta A^{(1)}_{s,t}\right\|_{W^{m,p}(\mathbb{R}^d)} &\le \int_s^t \left\|P_{\frac{1}{2H}r^{2H}} b(r, \cdot)\right\|_{W^{m,p}(\mathbb{R}^d)} dr \lesssim \int_s^t r^{-1/2+1/\ell+1/q+\varepsilon_2} \|b(r, \cdot)\|_{W^{k,p}(\mathbb{R}^d)} dr \\
&\lesssim (t-s)^{1/2+1/\ell+\varepsilon_2} \|b\|_{L^q([0,T]; W^{k,p}(\mathbb{R}^d))}.
\end{aligned}$$

We now turn to $A^{(2)}_{s,t}$ and use the BDG inequality (see Remark 4.3 above) together with Lemma 4.13, to deduce that, for any $j \in \{1, \ldots, d\}$,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_s^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} \partial_{x_j} b^a(r, u, W^{2,H}(u, r) + \cdot)(r-u)^{H-1/2} dr dB_j(u)\right\|^\ell_{W^{m,p}(\mathbb{R}^d)}\right]^{1/\ell} \\
&\lesssim \mathbb{E}\left[\left(\int_s^t \left(\int_u^t (r-u)^{-H-1/2+1/\ell+1/q+\varepsilon_2} \|b^a(r, u, \cdot)\|_{W^{k,p}(\mathbb{R}^d)} (r-u)^{H-1/2} dr\right)^2 du\right)^{\ell/2}\right]^{1/\ell} \\
&\lesssim \left(\int_s^t (t-u)^{2/\ell+2\varepsilon_2} \|b\|^2_{L^\ell(\Omega; L^q([0,T]; W^{k,p}(\mathbb{R}^d)))} du\right)^{1/2} \\
&\lesssim (t-s)^{1/2+1/\ell+\varepsilon_2} \|b\|_{L^\ell(\Omega; L^q([0,T]; W^{k,p}(\mathbb{R}^d)))},
\end{aligned}$$

and also

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^s \int_s^t P_{\frac{1}{2H}(r-u)^{2H}} \partial_{x_j} b^a(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2} dr dB_j(u)\right\|_{W^{m,p}(\mathbb{R}^d)}^{\ell}\right]^{1/\ell} \\
&\lesssim \mathbb{E}\left[\left(\int_0^s \left(\int_s^t (r-u)^{-H-1/2+1/\ell+1/q+\varepsilon_2} \|b^a(r,u,\cdot)\|_{W^{k,p}(\mathbb{R}^d)} (r-u)^{H-1/2} dr\right)^2 du\right)^{\ell/2}\right]^{1/\ell} \\
&\lesssim \left(\int_s^t \left[(t-u)^{1/\ell+\varepsilon_2} - (s-u)^{1/\ell+\varepsilon_2}\right]^2 \|b\|_{L^\ell(\Omega;L^q([0,T];W^{k,p}(\mathbb{R}^d)))}^2 du\right)^{1/2} \\
&\lesssim (t-s)^{1/2+1/\ell+\varepsilon_3} \|b\|_{L^\ell(\Omega;L^q([0,T];W^{k,p}(\mathbb{R}^d)))},
\end{aligned}$$

for some $0<\varepsilon_3<\varepsilon_2$. Still by similar arguments, Jensen's inequality and ii) of Assumption 4.1, we can bound $\delta A^{(4)}_{s,t}$. We obtain, for any $j \in \{1,\dots,d\}$,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_s^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} D^{\mathcal{P}} b_j(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2} dr dB_j(u)\right\|_{W^{m,p}(\mathbb{R}^d)}^{\ell}\right]^{1/\ell} \\
&= \mathbb{E}\left[\left\|\int_s^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} \mathbb{E}_u[b_j'(r,u,W^{2,H}(u,r)+\cdot)\upsilon(u,r)](r-u)^{H-1/2} dr dB_j(u)\right\|_{W^{m,p}(\mathbb{R}^d)}^{\ell}\right]^{1/\ell} \\
&\lesssim \mathbb{E}\left[\left(\int_s^t\left(\int_u^t \mathbb{E}_u\left[\left\|P_{\frac{1}{2H}(r-u)^{2H}} b_j'(r,u,W^{2,H}(u,r)+\cdot)\upsilon(u,r)\right\|_{W^{m,p}(\mathbb{R}^d)}\right](r-u)^{H-1/2} dr\right)^2 du\right)^{\frac{\ell}{2}}\right]^{1/\ell} \\
&\lesssim \mathbb{E}\left[\left(\int_s^t\left(\int_u^t (r-u)^{-1/2+1/\ell+1/q+\varepsilon_2} \mathbb{E}_u\left[C_1 \|b'(r,\cdot,\cdot)\|_{L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))}\right](r-u)^{H-1/2} dr\right)^2 du\right)^{\frac{\ell}{2}}\right]^{1/\ell} \\
&\lesssim (t-s)^{1/2+H+1/\ell+\varepsilon_2} \|b'\|_{L^\sigma(\Omega;L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))))},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^s \int_s^t P_{\frac{1}{2H}(r-u)^{2H}} D^{\mathcal{P}} b_j(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2} dr dB_j(u)\right\|_{W^{m,p}(\mathbb{R}^d)}^{\ell}\right]^{1/\ell} \\
&\lesssim \mathbb{E}\left[\left(\int_0^s\left(\int_s^t (r-u)^{-1/2+1/\ell+1/q+\varepsilon_2} \mathbb{E}_u\left[C_1 \|b'(r,\cdot,\cdot)\|_{L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))}\right](r-u)^{H-1/2} dr\right)^2 du\right)^{\frac{\ell}{2}}\right]^{\frac{1}{\ell}} \\
&\lesssim (t-s)^{1/2+H+1/\ell+\varepsilon_3} \|b'\|_{L^\sigma(\Omega;L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))))},
\end{aligned}$$

We finally estimate $\delta A^{(3)}_{s,t}$. We have, for any $j \in \{1,\dots,d\}$,

$$\begin{aligned}
&\left\|\int_s^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} \partial_{x_j} D^{\mathcal{P}} b_j(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2} dr du\right\|_{W^{m,p}(\mathbb{R}^d)} \\
&\lesssim \int_s^t \int_u^t (r-u)^{-H-1/2+1/\ell+1/q+\varepsilon_2} \mathbb{E}_u\left[C_1 \|b'(r,\cdot,\cdot)\|_{L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))}\right](r-u)^{H-1/2} dr du \\
&\lesssim \int_s^t (t-u)^{1/\ell+\varepsilon_2} \mathbb{E}_u\left[C_1^2\right]^{1/2} \mathbb{E}_u\left[\|b'\|_{L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d)))}^2\right]^{1/2} du,
\end{aligned}$$

as well as

$$\left\|\int_0^s\int_s^t P_{\frac{1}{2H}(r-u)^{2H}}\partial_{x_j}D^{\mathcal{P}}b_j(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2}drdu\right\|_{W^{m,p}(\mathbb{R}^d)}$$
$$\lesssim \int_0^s\int_s^t (r-u)^{-H-1/2+1/\ell+1/q+\varepsilon_2}\mathbb{E}_u\left[C_1\|b'(r,\cdot,\cdot)\|_{L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))}\right](r-u)^{H-1/2}drdu$$
$$\lesssim \int_s^t\left[(t-u)^{1/\ell+\varepsilon_2}-(s-u)^{1/\ell+\varepsilon_2}\right]\mathbb{E}_u\left[C_1^2\right]^{1/2}\mathbb{E}_u\left[\|b'\|^2_{L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d)))}\right]^{1/2}du,$$

This leads to

$$\mathbb{E}\left[\left\|\int_s^t\int_u^t P_{\frac{1}{2H}(r-u)^{2H}}\partial_{x_j}D^{\mathcal{P}}b_j(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2}drdu\right\|^\ell_{W^{k,p}(\mathbb{R}^d)}\right]^{1/\ell}$$
$$\lesssim (t-s)^{1+\varepsilon_2+1/\ell}\left\|b'\right\|_{L^\sigma(\Omega;L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))))},$$

and

$$\mathbb{E}\left[\left\|\int_0^s\int_s^t P_{\frac{1}{2H}(r-u)^{2H}}\partial_{x_j}D^{\mathcal{P}}b_j(r,u,W^{2,H}(u,r)+\cdot)(r-u)^{H-1/2}drdu\right\|^\ell_{W^{k,p}(\mathbb{R}^d)}\right]^{1/\ell}$$
$$\lesssim (t-s)\left\|b'\right\|_{L^\sigma(\Omega;L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))))}.$$

Step 2: From the Sobolev embedding

$$W^{\vartheta+d/p+\varepsilon_1,p}(\mathbb{R}^d)\hookrightarrow\mathcal{C}_b^{\vartheta+\gamma}(\mathbb{R}^d),$$

for any $0<\gamma<\varepsilon_1$, we deduce that

$$\mathbb{E}\left[\|\delta A_{s,t}\|^\ell_{\mathcal{C}^{\vartheta+\gamma}(\mathbb{R}^d)}\right]^{1/\ell}$$
$$\leq C|t-s|^{1/2+\varepsilon_3+1/\ell}\left(\|b\|_{L^\ell(\Omega;L^q([0,T];L^\infty([0,T];W^{k,p}(\mathbb{R}^d)))}+\|b'\|_{L^\sigma(\Omega;L^q([0,T];L^\infty([0,T];W^{k-\iota,p}(\mathbb{R}^d))))}\right),$$

with $0<\varepsilon_3<\varepsilon_2$. It follows from Kolmogorov's continuity theorem that, up to a modification,

$$A\in\mathcal{C}^\beta([0,T];\mathcal{C}_b^{\vartheta+\gamma}(\mathbb{R}^d)),$$

with $\beta\in(0,1/2+\varepsilon_3)$. □

As a direct consequence from the previous proposition, it follows from Theorem 4.11, that Equation (4.3) admits a unique solution under Assumption 4.1 with $\vartheta=2$.

5. Proof of Theorem 3.3

Formula (3.2) is valid for any fixed x in $\mathbb{R}^d$ and any $S\in[0,T]$, however to avoid cumbersome notations we fix in this proof :

$$x=0\quad\text{and}\quad S=T.$$

Throughout this proof, C will denote a generic constant that may vary from line to line. The proof is divided into several steps. For any N in $\mathbb{N}^*$ and i in $\{0,\cdots,N\}$, we set $t_i^N:=i\frac{T}{N}$. To prevent notations to become cumbersome we will often write t_i instead of t_i^N.

In the following we make use of the following notation : For i in $\{0,\ldots,N-1\}$, and $s\geq t_{i+1}$, we set

$$\begin{cases}\delta_{i,s}(W^{2,H}):=\left(\delta_{1,i,s}(W^{2,H}),\cdots,\delta_{d,i,s}(W^{2,H})\right),\\[2mm] \delta_{k,i,s}(W^{2,H}):=\left(W^{2,H}(t_{i+1},s)-W^{2,H}(t_i,s)\right)_k,\quad k\in\{1,\ldots,d\}.\end{cases}\tag{5.1}$$

Step 1 : We first assume that f belongs to $\mathcal{S}_{ad}$, that is there exist

$$n \in \mathbb{N}^*,\ 0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_n \leq T, \quad \varphi : [0,T] \times (\mathbb{R}^d)^n \times \mathbb{R}^d \to \mathbb{R}$$

such that

$$f(t,y) = \varphi(t, B(\gamma_1 \wedge t), \cdots, B(\gamma_n \wedge t), y),\ y \in \mathbb{R}^d, \quad t \in [0,T], \tag{5.2}$$

and $\varphi(t\cdot)$ is bounded and admits bounded partial derivatives of any order which are uniformly bounded in t on $[0,T]$. Hence, for any $0 \leq r \leq u \leq s \leq T$, for any $\mathcal{F}_u$-measurable random variable G, and for any differential operator $\mathcal{L} = \partial_y^\alpha$ with $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq 4$,

$$\begin{aligned}\sup_{0\leq u\leq r\leq s\leq T} \left| \mathbb{E}_u [P_{\frac{1}{2H}(s-r)^{2H}} \mathcal{L} f(s,y)]_{y=G} \right| + \sum_{j=1}^n \sup_{0\leq u\leq r\leq s\leq T} \left| \mathbb{E}_u [(P_{\frac{1}{2H}(s-r)^{2H}} D_j \mathcal{L} f(s,y))(u)]_{y=G} \right| \\ \leq \sum_{j=1}^d \sup_{\substack{0\leq s\leq T \\ b\in\mathbb{R}^n \\ y\in\mathbb{R}^d}} \left| \frac{\partial}{\partial_{b_j}} \mathcal{L}\varphi(s,b,y)) \right| < \infty.\end{aligned}$$

Throughout this step, C will denote a generic constant which may differ from line to line and which depends on : T, H, d and on :

$$\sum_{j=1}^n \sup_{0\leq s\leq T} \left\| \frac{\partial}{\partial_{b_i}} \mathcal{L}\varphi(s,\cdot,\cdot) \right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^d)} < +\infty.$$

We set :

$$F(t) := \int_t^T P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s)) ds, \quad t \in [0,T], \tag{5.3}$$

$$F^a(t) := \mathbb{E}_t \left[\int_t^T P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s)) ds \right], \quad t \in [0,T] \tag{5.4}$$

First of all, the Clark-Ocone formula (2.7) applies to the random variable $F(t)$ (defined as (5.3)) allows one to decompose for any time t the random variable $F(t)$ as follows :

$$F(t) = \mathbb{E}_t [F(t)] + \sum_{j=1}^d \int_t^T \mathbb{E}_u [(D_j F(t))(u)] dB_j(u), \quad t \in [0,T]. \tag{5.5}$$

By defintion, $F^a(t) = \mathbb{E}_t [F(t)]$ and set $G(t) := \sum_{j=1}^d \int_t^T \mathbb{E}_u [(D_j F(t))(u)] dB_j(u)$ so that

$$F(t) = F^a(t) + G(t), \quad t \in [0,T]. \tag{5.6}$$

Using Definition (5.3) of F we have for any i in $\{0, \cdots, N-1\}$ that :

$$
\begin{aligned}
F(t_{i+1}) - F(t_i) &= \int_{t_{i+1}}^{T} P_{\frac{1}{2H}(s-t_{i+1})^{2H}} f(s, W^{2,H}(t_{i+1}, s))ds - \int_{t_i}^{T} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s))ds \\
&= - \int_{t_i}^{t_{i+1}} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s))ds \\
&\quad + \int_{t_{i+1}}^{T} \left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}} f(s, W^{2,H}(t_{i+1}, s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \right] ds \\
&= - \int_{t_i}^{t_{i+1}} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s))ds \\
&\quad + \int_{t_{i+1}}^{T} \left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}} - P_{\frac{1}{2H}(s-t_i)^{2H}} \right] f(s, W^{2,H}(t_{i+1}, s))ds \\
&\quad + \int_{t_{i+1}}^{T} P_{\frac{1}{2H}(s-t_i)^{2H}} \left[f(s, W^{2,H}(t_{i+1}, s)) - f(s, W^{2,H}(t_i, s)) \right] ds. \qquad (5.7)
\end{aligned}
$$

We aim here to use a Taylor expansion. To this end we set (using Notation (5.1)) :

$$
W^{2,H}(t_i, s, \theta) := W^{2,H}(t_i, s) + \theta \, \delta_{i,s}(W^{2,H}), \quad \theta \in [0,1]. \qquad (5.8)
$$

With this notation at hand, the last term in this expression writes as follows :

$$
\begin{aligned}
& P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_{i+1}, s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \\
&= \nabla P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \cdot \delta_{i,s}(W^{2,H}) \\
&+ \frac{1}{2} \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \left(\delta_{i,k,s}(W^{2,H}) \right)^2 \\
&+ \frac{1}{2} \sum_{k,\ell=1; k \neq \ell}^{d} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \delta_{i,k,s}(W^{2,H}) \delta_{i,\ell,s}(W^{2,H}) \\
&+ \frac{1}{6} \int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i, s, \theta)\right) d\theta \cdot \left(\delta_{i,s}(W^{2,H}) \right)^3.
\end{aligned}
$$

To proceed with our analysis we apply the Clark-Ocone formula (2.7) to each element

$$
\mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s))
$$

with $\mathcal{L} = \frac{\partial}{\partial y_k}$ (for k in $\{1, \cdots, d\}$) or $\mathcal{L} = \frac{\partial^2}{\partial y_k \partial y_\ell}$ for k, ℓ in $\{1, \cdots, d\}$ with $k \neq \ell$. We have

$$
\begin{aligned}
& \mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \\
&= \mathbb{E}_{t_i} \left[\mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \right] + \sum_{j=1}^{d} \int_{t_i}^{s} \mathbb{E}_u \left[D_j \left(\mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \right)(u) \right] dB_j(u).
\end{aligned}
$$

Since $W^{2,H}(t_i, s)$ is $\mathcal{F}_{t_i}$-measurable, the first term of the right hand side is :

$$
\mathbb{E}_{t_i} \left[\mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \right] = \mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s, t_i, W^{2,H}(t_i, s)),
$$

whereas (i) and (ii) of Lemma 2.7 implies that :

$$
\left(\mathbb{E}_u \left[D_j \left(\mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i, s)) \right)(u) \right] \right)_{t_i \le u \le s} = \left(\mathcal{L} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s, u, W^{2,H}(t_i, s)) \right)_{t_i \le u \le s},
$$

where the equality is understood as processes in $L^2(\Omega \times [0,T])$ and where we recall Notation (3.1). Hence

$$\begin{aligned}
&\mathcal{L}P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i,s)) \\
&= \mathcal{L}P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s, t_i, W^{2,H}(t_i,s)) + \sum_{j=1}^d \int_{t_i}^s \mathcal{L}P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dB_j(u).
\end{aligned} \tag{5.9}$$

Thus, using the segment addition postulate in (5.9) between t_i, t_{i+1} and s, we get

$$\begin{aligned}
&P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_{i+1},s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i,s)) \\
&= \sum_{k=1}^d \frac{\partial}{\partial x_k} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s, t_i, W^{2,H}(t_i,s))\ \delta_{k,i,s}(W^{2,H}) \\
&\quad + \sum_{k=1}^d \sum_{j=1}^d \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial x_k} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dB_j(u)\ \delta_{k,i,s}(W^{2,H}) \\
&\quad + \sum_{k=1}^d \sum_{j=1}^d \int_{t_{i+1}}^{s} \frac{\partial}{\partial x_k} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dB_j(u)\ \delta_{k,i,s}(W^{2,H}) \\
&\quad + \frac{1}{2} \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i,s)) dB_j(u)\ \left(\delta_{k,i,s}(W^{2,H})\right)^2 \\
&\quad + \frac{1}{2} \sum_{k,\ell=1;k\neq\ell}^d \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s, t_i, W^{2,H}(t_i,s)) dB_j(u)\ \delta_{k,i,s}(W^{2,H}) \delta_{\ell,i,s}(W^{2,H}) \\
&\quad + \frac{1}{2} \sum_{k,\ell=1;k\neq\ell}^d \sum_{j=1}^d \int_{t_i}^{t_{i+1}} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dB_j(u)\ \delta_{k,i,s}(W^{2,H}) \delta_{\ell,i,s}(W^{2,H}) \\
&\quad + \frac{1}{2} \sum_{k,\ell=1;k\neq\ell}^d \sum_{j=1}^d \int_{t_{i+1}}^{s} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dB_j(u)\ \delta_{k,i,s}(W^{2,H}) \delta_{\ell,i,s}(W^{2,H}) \\
&\quad + \frac{1}{6} \int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s,\theta)\right) d\theta \cdot \left(\delta_{i,s}(W^{2,H})\right)^3.
\end{aligned}$$

Coming back to the expression (5.7) of an increment of F we obtain

$$
\begin{aligned}
&F(t_{i+1}) - F(t_i)\\
&= -\int_{t_i}^{t_{i+1}} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i,s))ds\\
&\quad + \int_{t_{i+1}}^{T} \left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}} - P_{\frac{1}{2H}(s-t_i)^{2H}}\right] f(s, W^{2,H}(t_{i+1},s))ds\\
&\quad + \sum_{k=1}^{d} \int_{t_{i+1}}^{T} \frac{\partial}{\partial x_k} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s, t_i, W^{2,H}(t_i,s))\delta_{k,i,s}(W^{2,H})ds\\
&\quad + \sum_{k=1}^{d}\sum_{j=1}^{d} \int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}} \frac{\partial}{\partial x_k} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u, W^{2,H}(t_i,s))dB_j(u)\ \delta_{k,i,s}(W^{2,H})ds\\
&\quad + \sum_{k=1}^{d}\sum_{j=1}^{d} \int_{t_{i+1}}^{T}\int_{t_{i+1}}^{s} \frac{\partial}{\partial x_k} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u, W^{2,H}(t_i,s))dB_j(u)\ \delta_{k,i,s}(W^{2,H})ds\\
&\quad + \frac{1}{2}\sum_{k=1}^{d} \int_{t_{i+1}}^{T} \frac{\partial^2}{\partial x_k^2} P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_i,s))dB_j(u)\ \left(\delta_{k,i,s}(W^{2,H})\right)^2 ds\\
&\quad + \frac{1}{2}\sum_{k,\ell=1;k\neq\ell}^{d} \int_{t_{i+1}}^{T} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s, t_i, W^{2,H}(t_i,s))\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})ds\\
&\quad + \frac{1}{2}\sum_{\substack{k,\ell=1;\\ k\neq\ell}}^{d}\sum_{j=1}^{d} \int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u, W^{2,H}(t_i,s))dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})ds\\
&\quad + \frac{1}{2}\sum_{\substack{k,\ell=1;\\ k\neq\ell}}^{d}\sum_{j=1}^{d} \int_{t_{i+1}}^{T}\int_{t_{i+1}}^{s} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u, W^{2,H}(t_i,s))dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})ds\\
&\quad + \frac{1}{6}\int_{t_{i+1}}^{T}\int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s,\theta)\right)d\theta \cdot \left(\delta_{i,s}(W^{2,H})\right)^3 ds\\
&=: \sum_{k=1}^{10} I_{1,k}(t_i,t_{i+1}).
\end{aligned}
\tag{5.10}
$$

We now compute an increment of G. To this end we first remark that (recall Notation in (2.7))

$$
\begin{aligned}
G(t) &:= \sum_{j=1}^{d}\int_t^T (D_j^{\mathcal{P}} F(t))(u)dB_j(u)\\
&= \sum_{j=1}^{d}\int_t^T \mathbb{E}_u\left[D_j\left(\int_t^T P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s))ds\right)(u)\right]dB_j(u)\\
&= \sum_{j=1}^{d}\int_t^T\int_t^T \mathbb{E}_u\left[D_j\left(P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s))\right)(u)\mathbf{1}_{\{u\leq s\}}\right]dsdB_j(u)\\
&= \sum_{j=1}^{d}\int_t^T\int_u^T D_j^{\mathcal{P}}\left(P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s))\right)(u)dsdB_j(u),
\end{aligned}
\tag{5.11}
$$

where the first equality is a consequence of the stochastic Fubini theorem as for any j in $\{1, \dots, d\}$

$$\begin{aligned}
&\int_t^T \mathbb{E}\left[\left|\mathbb{E}_u\left[D_j\left(\int_u^T P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s))ds\right)(u)\right]\right|^2\right] du \\
&\int_t^T \int_u^T \mathbb{E}\left[\left|D_j\left(P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s))\right)(u)\right|^2\right] duds \\
&\leq C\int_t^T |T-u|^2 du < +\infty.
\end{aligned}$$

In addition, since for any t, $W^{2,H}(t,s)$ is $\mathcal{F}_t$-measurable, Lemma 2.7 implies that

$$\left(D^{\mathcal{P}}\left(P_{\frac{1}{2H}(s-t)^{2H}} f(s, W^{2,H}(t,s))\right)(u)\right)_u = \left(P_{\frac{1}{2H}(s-t)^{2H}} g(s, u, W^{2,H}(t,s))\right)_u.$$

Thus,

$$G(t) = \sum_{j=1}^d \int_t^T \int_u^T P_{\frac{1}{2H}(s-t)^{2H}} g_j(s, u, W^{2,H}(t,s)) ds dB_j(u).$$

This form allows us to proceed in the analysis of an increment of G. Indeed,

$$\begin{aligned}
&G(t_{i+1}) - G(t_i) \\
&= \sum_{j=1}^d \int_{t_{i+1}}^T \int_u^T \left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s))\right] dsdB_j(u) \\
&\qquad - \sum_{j=1}^d \int_{t_i}^{t_{i+1}} \int_u^T P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dsdB_j(u) \\
&= \sum_{j=1}^d \int_{t_{i+1}}^T \int_u^T \left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s))\right] dsdB_j(u) \\
&+ \sum_{j=1}^d \int_{t_{i+1}}^T \int_u^T \left[P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s))\right] dsdB_j(u) \\
&\qquad - \sum_{j=1}^d \int_{t_i}^{t_{i+1}} \int_u^T P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dsdB_j(u)
\end{aligned}$$

In a similar fashion to the computation of an increment of F, we expand using Taylor expansion the second term to obtain

$$\begin{aligned}
&P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) \\
&= \nabla P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) \cdot \delta_{i,s}(W^{2,H}) + \frac{1}{2}\nabla^2 P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) \cdot \left(\delta_{i,s}(W^{2,H})\right)^2 \\
&\qquad + \frac{1}{6}\int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s,\theta)) d\theta \cdot \left(\delta_{i,s}(W^{2,H})\right)^3,
\end{aligned}$$

where we recall Notation (5.8). Plugging this expansion in the expression above, we get

$$
\begin{aligned}
&G(t_{i+1}) - G(t_i) \\
&= \sum_{j=1}^{d} \int_{t_{i+1}}^{T} \int_{u}^{T} \Big[P_{\frac{1}{2H}(s-t_{i+1})^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s)) - P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_{i+1},s)) \Big] ds dB_j(u) \\
&+ \sum_{j=1}^{d} \int_{t_{i+1}}^{T} \int_{u}^{T} \left[\nabla P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) \cdot \delta_{i,s}(W^{2,H})) \right] ds dB_j(u) \\
&+ \frac{1}{2} \sum_{j=1}^{d} \int_{t_{i+1}}^{T} \int_{u}^{T} \left[\nabla^2 P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) \cdot \left(\delta_{i,s}(W^{2,H})\right)^2) \right] ds dB_j(u) \\
&+ \frac{1}{6} \sum_{j=1}^{d} \int_{t_{i+1}}^{T} \int_{u}^{T} \left[\int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s,\theta)) d\theta \cdot \left(\delta_{i,s}(W^{2,H})\right)^3) \right] ds dB_j(u) \\
&\quad - \sum_{j=1}^{d} \int_{t_i}^{t_{i+1}} \int_{u}^{T} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) ds dB_j(u) \\
&=: \sum_{k=1}^{5} I_{2,k}(t_i,t_{i+1}).
\end{aligned} \tag{5.12}
$$

As a consequence, using Relation (5.6) with $t=0$, we get :

$$
\begin{aligned}
F^a(0) &= F(0) - G(0) \\
&= - \lim_{N\to+\infty} \sum_{i=0}^{N-1} \left(F(t_{i+1}) - F(t_i) - (G(t_{i+1}) - G(t_i)) \right) \\
&= - \lim_{N\to+\infty} \sum_{i=0}^{N-1} \left(\sum_{k=1}^{8} I_{1,k}(t_i,t_{i+1}) - \sum_{k=1}^{5} I_{2,k}(t_i,t_{i+1}) \right) \\
&= - \lim_{N\to+\infty} \sum_{i=0}^{N-1} \left(I_{1,1}(t_i,t_{i+1}) + I_{1,3}(t_i,t_{i+1}) + I_{1,4}(t_i,t_{i+1})(t_i,t_{i+1}) - I_{2,5}(t_i,t_{i+1}) \right) \qquad (5.13) \\
&+ \lim_{N\to+\infty} \sum_{i=0}^{N-1} R(t_i,t_{i+1}),
\end{aligned}
$$

with

$$
R(t_i,t_{i+1}) := I_{1,2}(t_i,t_{i+1}) + I_{1,5}(t_i,t_{i+1}) + \sum_{k=6}^{10} I_{1,k}(t_i,t_{i+1}) - \sum_{k=1}^{4} I_{2,k}(t_i,t_{i+1}),
$$

where the terms involved in this expression are defined in (5.10) and in (5.12).

By Lemma 5.1 (postponed at the end of this section), we have that

$$
\lim_{N\to+\infty} \sum_{i=0}^{N-1} I_{1,1}(t_i,t_{i+1}) = - \int_0^T f(t, W_t^H) dt, \tag{5.14}
$$

$$
\lim_{N\to+\infty} \sum_{i=0}^{N-1} I_{1,3}(t_i,t_{i+1}) = \sum_{k=1}^{d} \int_0^T \int_t^T P_{\frac{1}{2H}(s-t)^{2H}} \nabla f_t^a(s, W^{2,H}(t,s))(s-t)^{H-1/2} ds \cdot dB(t), \tag{5.15}
$$

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} I_{1,4}(t_i,t_{i+1})(t_i,t_{i+1}) = \sum_{j=1}^d \int_0^T \int_t^T \frac{\partial}{\partial x_j} P_{\frac{1}{2H}(s-t)^{2H}} g_j(s,t,W^{2,H}(t,s))(s-t)^{H-1/2} ds dt, \tag{5.16}$$

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} I_{2,5}(t_i,t_{i+1}) = -\int_0^T \int_u^T P_{\frac{1}{2H}(s-t)^{2H}} g_j(s,t,W^{2,H}(t,s)) ds dB_j(t), \tag{5.17}$$

and that, thanks to Lemma 5.2 (postponed below after the proof of Lemma 5.1),

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} R(t_i,t_{i+1}) = 0. \tag{5.18}$$

Step 2 :

In a first step, we have proved Formula (3.2) for f in $\mathcal{S}_{ad}$ for any (s,t,x) in $[0,T]^2\times\mathbb{R}^d$ $(s\le t)$. We now extend it to any element f in $\mathbb{D}_p^{1,m-\alpha,p}$. To this end, we set the operators :

$$\begin{cases} \mathcal{A}_{LHS}: & \mathbb{D}_p^{1,m-\alpha,p} & \to L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d))) \\ & f & \mapsto (\mathcal{A}_{LHS}(t,x))_{t\in[0,T],x\in\mathbb{R}^d}, \end{cases}$$

$$\mathcal{A}_{LHS}(t,x) := \int_0^t f(r, W_r^H + x) dr;$$

and

$$\begin{cases} \mathcal{A}_{RHS}: & \mathcal{S}_{ad} & \to L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d))) \\ & f & \mapsto (\mathcal{A}_{RHS}(t,x))_{t\in[0,T],x\in\mathbb{R}^d}, \end{cases}$$

with

$$\begin{aligned} \mathcal{A}_{RHS}(f)(t,x) := &\int_0^t P_{\frac{1}{2H}r^{2H}} f(r, W^H(r)+x) dr \\ &+ \sum_{j=1}^d \int_0^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} \frac{\partial}{\partial x_j} f^a(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2} dr dB_j(u) \\ &+ \sum_{j=1}^d \int_0^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} \frac{\partial}{\partial x_j} g_j(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2} dr du \\ &- \sum_{j=1}^d \int_0^t \int_u^t P_{\frac{1}{2H}(r-u)^{2H}} g_j(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2} dr dB_j(u). \end{aligned} \tag{5.19}$$

In Step 1, we have proved that for any f in $\mathcal{S}_{ad}$

$$\mathcal{A}_{LHS} = \mathcal{A}_{RHS}, \text{ in } L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d))).$$

Note also that by definition,

$$\|\mathcal{A}_{LHS}(f)\| \le \|f\|_{L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d)))}.$$

So Formula (3.2) holds true for any adapted random field f in $\mathbb{D}_p^{1,m-\alpha,p}$ (that is the equality of the operators $\mathcal{A}_{LHS}$ and $\mathcal{A}_{RHS}$) if we prove that $\mathcal{A}_{RHS}$ is a well-defined bounded operator on $\mathbb{D}_p^{1,m-\alpha,p}$. We thus prove that for any adapted random field f in $\mathbb{D}_p^{1,m-\alpha,p}$ we have that :

$$\|\mathcal{A}_{RHS}(f)\|_{L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d)))} \lesssim \|f\|_{\mathbb{D}_p^{1,m-\alpha,p}}. \tag{5.20}$$

Proof of (5.20) :

We remark that the following estimates are different from the ones in the proof of Theorem 4.4 (see Remark 4.6). Let f be an adapted random field in $\mathbb{D}_p^{1,m-\alpha,p}$. We now estimate each term in the

$L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d)))$ space with $p\geq 2$ and $1/2-H\alpha-1/p>0$. For the first term, we have, by Hölder's inequality,

$$\begin{aligned}\left\|\int_0^t P_{\frac{1}{2H}r^{2H}}f(r,\cdot)dr\right\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d))} &\leq \int_0^t \left\|P_{\frac{1}{2H}r^{2H}}f(r,\cdot)\right\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d))}dr\\ &\lesssim \int_0^t r^{-H\alpha}\left\|f(r,\cdot)\right\|_{L^p(\Omega;W^{m-\alpha,p}(\mathbb{R}^d))}dr\\ &\lesssim t^{1-H\alpha-1/p}\|f\|_{L^p([0,T]\times\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))},\end{aligned}$$

which yields

$$\left\|\int_0^\cdot P_{\frac{1}{2H}r^{2H}}f(r,\cdot)dr\right\|_{L^\infty([0,T];L^p(\Omega;W^{m,p}(\mathbb{R}^d)))}\lesssim T^{1-H\alpha-1/p}\|f\|_{L^p([0,T]\times\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))}.$$

We now turn to the second term. It follows from the BDG, Minkowski and Hölder inequalities that, for any $j\in\{1,\ldots,d\}$,

$$\begin{aligned}&\left\|\int_0^t\int_u^t P_{\frac{1}{2H}(r-u)^{2H}}\frac{\partial}{\partial x_j}f^a(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}drdB_j(u)\right\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d))}\\ &\qquad\lesssim\left(\int_0^t\left\|\int_u^t(r-u)^{-1/2-H\alpha}\mathbb{E}_u\left[\|f(r,\cdot)\|_{W^{m-\alpha,p}(\mathbb{R}^d)}\right]dr\right\|^2_{L^p(\Omega)}du\right)^{1/2}\\ &\qquad\lesssim\left(\int_0^t(t-u)^{1-2H\alpha-2/p}\|f\|^2_{L^p([0,T]\times\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))}du\right)^{1/2}\\ &\qquad\lesssim T^{1-H\alpha-1/p}\|f\|_{L^p([0,T];L^p(\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))}.\end{aligned}$$

By rather similar arguments, we estimate the fourth term as

$$\begin{aligned}&\left\|\int_0^t\int_u^t P_{\frac{1}{2H}(r-u)^{2H}}g_j^a(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}drdB_j(u)\right\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d))}\\ &\quad\lesssim\left(\int_0^t\left\|\int_u^t(r-u)^{-1/2-H(\alpha-1)}\mathbb{E}_u\left[\|g_j(r,u,\cdot)\|_{W^{m-\alpha,p}(\mathbb{R}^d)}\right]dr\right\|^2_{L^p(\Omega)}du\right)^{1/2}\\ &\quad\lesssim\left(\int_0^t(t-u)^{1-2H(\alpha-1)-2/p}\|g_j(,\cdot,u,\cdot)\|^2_{L^p([0,T]\times\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))}du\right)^{1/2}\\ &\quad\lesssim T^{1-H(\alpha-1)-3/(2p)}\|g_j\|_{L^p([0,T]^2\times\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))}.\end{aligned}$$

Finally, we have, for the third term,

$$\begin{aligned}&\left\|\int_0^t\int_u^t P_{\frac{1}{2H}(r-u)^{2H}}\frac{\partial}{\partial x_j}g_j(r,u,W^{2,H}(u,r)+x)(r-u)^{H-1/2}drdu\right\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d))}\\ &\qquad\lesssim\int_0^t\int_u^t(r-u)^{-1/2-H\alpha}\left\|g_j(r,u,\cdot)\right\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d))}drdu\\ &\qquad\lesssim T^{3/2-H\alpha-2/p}\|g_j\|_{L^p([0,T]^2\times\Omega;W^{m-\alpha,p}(\mathbb{R}^d)))}.\end{aligned}$$

Since each term in (3.2) is linear with respect to f and from each of the previous estimates, we can deduce that Formula (3.2) is in force for any f in $\mathbb{D}_p^{1,m-\alpha,p}$. □

Lemma 5.1. *With the notations of the proof of Theorem 3.3, the convergences (5.14)-(5.17) hold true in $L^2(\Omega)$:*

(i)

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} I_{1,1}(t_i,t_{i+1}) = -\int_0^T f(t,W_t^H)dt.$$

(ii)

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} I_{1,3}(t_i,t_{i+1}) = \int_0^T\int_t^T P_{\frac{1}{2H}(s-t)^{2H}}\nabla f_t^a(s,W^{2,H}(t,s))(s-t)^{H-1/2}ds\cdot dB(t).$$

(iii)

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} I_{1,4}(t_i,t_{i+1})(t_i,t_{i+1}) = \sum_{j=1}^{d}\int_0^T\int_t^T \frac{\partial}{\partial x_j}P_{\frac{1}{2H}(s-t)^{2H}}g_j(s,t,W^{2,H}(t,s))(s-t)^{H-1/2}dsdt,$$

(also see Remark 3.4 for this term).

(iv)

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} I_{2,5}(t_i,t_{i+1}) = -\sum_{j=1}^{d}\int_0^T\int_u^T P_{\frac{1}{2H}(s-t)^{2H}}g_j(s,t,W^{2,H}(t,s))dsdB_j(t).$$

Proof: Throughout this proof, C denotes a positive constant (which can vary from line to line) and that represents the sup norm of f and its derivatives up to order 4.

Proof of (i) :

We set using Decomposition (2.1), $W^{2,H}(s,s) := W^H(s)$, for any s. We have that

$$\begin{aligned}
&I_{1,1}(t_i,t_{i+1}) - \int_{t_i}^{t_{i+1}} f(s,W^H(s))ds \\
&= -\int_{t_i}^{t_{i+1}}\left(P_{\frac{1}{2H}(s-t_i)^{2H}}f(s,W^{2,H}(t_i,s)) - f(s,W^{2,H}(s,s))ds\right)ds \\
&= -\int_{t_i}^{t_{i+1}}\left(P_{\frac{1}{2H}(s-t_i)^{2H}}f(s,W^{2,H}(t_i,s)) - P_0 f(s,W^{2,H}(t_i,s))\right)ds \\
&- \int_{t_i}^{t_{i+1}}\left(P_0 f(s,W^{2,H}(t_i,s)) - P_0 f(s,W^{2,H}(s,s))\right)ds.
\end{aligned}$$

Since the semigroup P is associated to the heat equation, the first term of the right-hand side can be re-written as :

$$\begin{aligned}
&= \left|\int_{t_i}^{t_{i+1}}\left(P_{\frac{1}{2H}(s-t_i)^{2H}}f(s,W^{2,H}(t_i,s)) - f(s,W^{2,H}(s,s))\right)ds\right| \\
&\le \frac{1}{2}\int_{t_i}^{t_{i+1}}\int_0^{\frac{1}{2H}(s-t_i)^{2H}} |\Delta P_r f(s,W^{2,H}(t_i,s))|drds \\
&\le \frac{C}{4H}\sup_{s\in[0,T]}\int_{t_i}^{t_{i+1}}(s-t_i)^{2H}ds.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(P_{\frac{1}{2H}(s-t_i)^{2H}}f(s,W^{2,H}(t_i,s))-f(s,W^{2,H}(s,s))\right)ds\right|^2\right]\\
&\leq C\left|\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}(s-t_i)^{2H}ds\right|^2\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

We now turn to the second term. Since

$$
\mathbb{E}\left[\left|W^{2,H}(u,s)-W^{2,H}(v,s)\right|^2\right]\leq|u-v|^{\min\{2H,1\}}\quad\forall u,v\leq s, \tag{5.21}
$$

we deduce that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(f(s,W^{2,H}(t_i,s))-f(s,W^{2,H}(s,s))\right)ds\right|^2\right]^{1/2}\\
&\leq C\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[\left|W^{2,H}(t_i,s)-W^{2,H}(s,s)\right|^2\right]^{1/2}ds\\
&\leq C\sum_{i=0}^{N-1}|t_{i+1}-t_i|^{1+\min\{H,1/2\}}\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

So Item (i) (or equivalently (5.14)) is proved.

Proof of (ii) :

Fix k in $\{1,\cdots,d\}$. First note that as f belongs to $\mathcal{S}_{ad}$, and since $W^{2,H}(t_i,s)$ is $\mathcal{F}_{t_i}$-measurable $(s\geq t_{i+1})$, we have that :

$$
\mathbb{E}_{t_i}\left[\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f(s,W^{2,H}(t_i,s))\right]=\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a_{t_i}(s,W^{2,H}(t_i,s)).
$$

Hence, (ii) will be proved if the following holds true for any k in $\{1,\ldots,d\}$:

$$
\begin{aligned}
&\lim_{N\to+\infty}\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a_{t_i}(s,W^{2,H}(t_i,s))\delta_{k,i,s}(W^{2,H})ds\\
&\overset{L^2(\Omega)}{=}\int_0^T\int_t^T\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t)^{2H}}f^a_t(s,W^{2,H}(t,s))(s-t)^{H-1/2}dsdB_k(t).
\end{aligned} \tag{5.22}
$$

By definition, (recall Definition (5.1) for the increments of $W^{2,H}$)

$$
\begin{aligned}
I_{1,3,k}(t_i,t_{i+1}) &:= \int_{t_{i+1}}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a_{t_i}(s,W^{2,H}(t_i,s))\delta_{k,i,s}(W^{2,H})ds\\
&=\int_{t_{i+1}}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a_{t_i}(s,W^{2,H}(t_i,s))\int_{t_i}^{t_{i+1}}(s-u)^{H-1/2}dB_k(u)ds\\
&=\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a_{t_i}(s,W^{2,H}(t_i,s))(s-u)^{H-1/2}dsdB_k(u),
\end{aligned}
$$

where the last equality is justified by the stochastic Fubini theorem. Indeed,

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\left|\frac{\partial}{\partial x_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f_{t_i}^a(s,W^{2,H}(t_i,s))\right|^2(s-u)^{2H-1}duds\right]\\
&=\mathbb{E}\left[\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\left|\mathbb{E}_{t_i}\left[P_{\frac{1}{2H}(s-t_i)^{2H}}\frac{\partial}{\partial y_k}f(s,W^{2,H}(t_i,s))\right]\right|^2(s-u)^{2H-1}duds\right]\\
&\leq C\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}(s-u)^{2H-1}duds<+\infty.
\end{aligned}$$

Using this expression, the Itô isometry and the independence of the disjoint increments of the Brownian motion, we get that

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\left(I_{1,3,k}(t_i,t_{i+1})-\int_{t_i}^{t_{i+1}}\int_{t_i}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f_{t_i}^a(s,W^{2,H}(t_i,s))(s-t_i)^{H-\frac{1}{2}}dsdB_k(u)\right)\right|^2\right]\\
&\leq 2\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\left(\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f_{t_i}^a(s,W^{2,H}(t_i,s))\Big((s-u)^{H-\frac{1}{2}}-(s-t_i)^{H-\frac{1}{2}}\Big)dsdB_k(u)\right)\right|^2\right]\\
&\quad+2\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f_{t_i}^a(s,W^{2,H}(t_i,s))(s-t_i)^{H-\frac{1}{2}}dsdB_k(u)\right)\right|^2\right]\\
&=2\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[\left|\int_{t_{i+1}}^{T}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f_{t_i}^a(s,W^{2,H}(t_i,s))\left((s-u)^{H-\frac{1}{2}}-(s-t_i)^{H-\frac{1}{2}}\right)ds\right|^2\right]du\\
&\quad+2\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\frac{\partial}{\partial y_k}P_{\frac{1}{2H}(s-t_i)^{2H}}f_{t_i}^a(s,W^{2,H}(t_i,s))(s-t_i)^{H-1/2}ds\right|^2\right]du\\
&\leq 2CS_N,
\end{aligned}$$

where

$$S_N:=\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(\left|\int_{t_{i+1}}^{T}\left|(s-u)^{H-1/2}-(s-t_i)^{H-1/2}\right|ds\right|^2+\left|\int_{t_i}^{t_{i+1}}(s-t_i)^{H-1/2}ds\right|^2\right)du.$$

Using the fact that for $t_i<u<t_{i+1}<s$ and $0<\varepsilon<H+1/2$

$$\begin{aligned}
0&\leq-(s-u)^{H-1/2}+(s-t_i)^{H-1/2}\\
&\leq\begin{cases}|u-t_i|^{H-1/2}\leq|t_{i+1}-t_i|^{H-1/2} & \text{for } H>1/2,\\ |u-t_i|^{\varepsilon}|s-u|^{H-1/2-\varepsilon}\leq|t_{i+1}-t_i|^{\varepsilon}|s-u|^{H-1/2-\varepsilon} & \text{for } H<1/2,\end{cases}
\end{aligned}$$

a direct computation gives that $\lim_{N\to+\infty}S_N=0$.

A direct computation gives that $\lim_{N\to+\infty}S_N=0$. It remains to prove that the process

$$t\to\int_t^T P_{\frac{1}{2H}(s-t)^{2H}}\frac{\partial}{\partial x_k}f_t^a(s,W^{2,H}(s,t))(s-t)^{H-1/2}ds, \tag{5.23}$$

is continuous in $L^2(\Omega \times [0,T])$ in order to verifies the assumptions of Jacod (1979, Theorem 2.74) in order to deduce that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{T} \frac{\partial}{\partial y_k} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a_{t_i}(s, W^{2,H}(t_i,s))(s-t_i)^{H-1/2} ds dB_k(u)$$
$$\underset{N\to\infty}{\to} \int_0^T \int_t^T \frac{\partial}{\partial y_k} P_{\frac{1}{2H}(s-t)^{2H}} f^a_t(s, W^{2,H}(t,s))(s-t)^{H-1/2} ds dB_k(t).$$

First, we prove the domination assumption. Using the change of variable $u = s-t$ and the fact that f is a smooth random field, we obtain the following estimate

$$\left| \int_t^T P_{\frac{1}{2H}(s-t)^{2H}} \frac{\partial}{\partial x_k} f^a_t(s, W^{2,H}(s,t))(s-t)^{H-1/2} ds \right|$$
$$= \left| \int_0^{T-t} P_{\frac{1}{2H}u^{2H}} \frac{\partial}{\partial x_k} f^a_t(u+t, W^{2,H}(u+t,t)) u^{H-1/2} du \right|$$
$$\lesssim \int_0^{T-t} u^{H-1/2} du \leq (T-t)^{H+1/2}.$$

We now turn to the continuity of the process (5.23) itself. By the change of variable $u = s-t$, we essentially have to prove that $f^a_t(u+t, W^{2,H}(u+t,t))$ is continuous with respect to t. The only difficulty is the continuity of $t \to f^a_t(u,y)$ for any $(u,y) \in [0,T] \times \mathbb{R}^d$. Clark-Ocone's formula gives

$$f(u,y) = \mathbb{E}[f(u,y)] + \sum_{j=1}^d \int_0^T \mathbb{E}_r[D_j(f(u,y))(r)] dB_j(r),$$

then, we derive

$$f^a_t(u,y) = \mathbb{E}[f(u,y)] + \sum_{j=1}^d \int_0^t \mathbb{E}_r[D_j(f(u,y))(r)] dB_j(r),$$

which is continuous with respect to t uniformly in (u,y). This ends the proof of (5.22).

Proof of (iii) :

For fixed $i \in \{0, \cdots, N-1\}$, j,k in $\{1, \cdots, d\}$, $s \in [t_{i+1}, T]$, we set

$$\alpha_{i,j,k,s}(u) := \frac{\partial}{\partial y_k} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)),$$

$$M_{i,j,k,s}(r) := \int_{t_i}^r \alpha_{i,j,k,s}(u) dB_j(u), \quad N_{i,k,s}(r) := \int_{t_i}^r (s-u)^{H-\frac{1}{2}} dB_k(u),\ r \in [t_i, t_{i+1}],$$

which are the integral form of the increments defined in (5.1). so that $M_{i,j,k,s}$ and $N_{i,k,s}$ are continuous martingales. Note once again that since f belongs to $\mathcal{S}_{ad}$, $\alpha_{i,j,k,s}(u)$ is uniformly (in i,j,k,s,u) bounded $\mathbb{P}$-a.s. Thus

$$I_{1,4}(t_i, t_{i+1}) = \sum_{k=1}^d \sum_{j=1}^d \int_{t_{i+1}}^T M_{i,j,k,s}(t_{i+1}) N_{i,k,s}(t_{i+1}) ds.$$

The integration by parts formula for semimartingales implies that

$$M_{i,j,k,s}(t_{i+1}) N_{i,k,s}(t_{i+1}) - \mathbf{1}_{j=k} \int_{t_i}^{t_{i+1}} \alpha_{i,j,k,s}(u)(s-u)^{H-\frac{1}{2}} du$$
$$= \int_{t_i}^{t_{i+1}} M_{i,j,k,s}(r) dN_{i,k,s}(r) + \int_{t_i}^{t_{i+1}} N_{i,k,s}(r) dM_{i,j,k,s}(r). \tag{5.24}$$

We show below that both terms in the right hand side do not contribute to the limit. Indeed, using the fact that the co-variation $[B_j(\cdot), B_{j'}(\cdot)] = 0$ for any $j \neq j'$, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}} M_{i,j,k,s}(r)dN_{i,k,s}(r)ds\right|^2\right]\\
&=\sum_{i,i'=0}^{N-1}\sum_{k,k'=1}^{d}\sum_{j,j'=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i'+1}}^{T}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} M_{i,j,k,s}(r)dN_{i,k,s}(r)\int_{t_i'}^{t_{i'+1}} M_{i',j',k',s'}(r')dN_{i',k',s'}(r')\right]dsds'\\
&=\sum_{i=0}^{N-1}\sum_{k=1}^{d}\sum_{j,j'=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[M_{i,j,k,s}(r)M_{i,j',k,s'}(r)\right](s-r)^{H-1/2}(s'-r)^{H-1/2}drdsds'\\
&=\sum_{i=0}^{N-1}\sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}\mathbb{E}[\alpha_{i,j,k,s}(u)\alpha_{i,j,k,s'}(u)]du(s-r)^{H-1/2}(s'-r)^{H-1/2}drdsds'\\
&\leq\frac{C}{N}\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{H-1/2}(s'-r)^{H-1/2}drdsds'\\
&=\frac{C}{N}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(\int_{t_{i+1}}^{T}(s-r)^{H-1/2}ds\right)^2dr\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}\tag{5.25}$$

Now we turn to the analysis of the the second term in the right hand side of (5.24). The first arguments follow the same line as for the term above (using mainly the independence of the components of the Brownian motion B). Indeed, we have :

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}} N_{i,k,s}(r)dM_{i,j,k,s}(r)ds\right|^2\right]\\
&=\sum_{i=0}^{N-1}\sum_{k,k'=1}^{d}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[N_{i,k,s}(r)N_{i,k',s'}(r)\alpha_{i,j,k,s}(r)\alpha_{i,j,k',s'}(r)\right]drdsds'\\
&\leq C\sum_{i=0}^{N-1}\sum_{k,k'=1}^{d}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[|N_{i,k,s}(r)N_{i,k',s'}(r)|\right]drdsds'\\
&=C\sum_{i=0}^{N-1}\sum_{k,k'=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{r}(s-v)^{2H-1}dv\int_{t_i}^{r}(s'-v)^{2H-1}dv\right)^{1/2}drdsds'.
\end{aligned}\tag{5.26}$$

So plugging this estimate in (5.26), we get

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}} N_{i,k,s}(r)dM_{i,j,k,s}(r)ds\right|^2\right]\\
&\leq C\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(\int_{t_{i+1}}^{T}\left(\int_{t_i}^{r}(s-v)^{2H-1}dv\right)^{1/2}ds\right)^2 dr\\
&\leq C\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}\int_{t_i}^{r}(s-v)^{2H-1}dvdsdr\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
\tag{5.27}
$$

So to summarize, Relations (5.24), (5.25) and (5.27) imply that :

$$
\lim_{N\to+\infty}\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{1,4}(t_i,t_{i+1})-\sum_{i=0}^{N-1}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\alpha_{i,j,j,s}(u)(s-u)^{H-\frac{1}{2}}duds\right|^2\right]=0.
$$

However we have that :

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\sum_{j=1}^{d}\Bigg(\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}\alpha_{i,j,j,s}(u)(s-u)^{H-\frac{1}{2}}duds\\
&\qquad\qquad -\int_0^T\int_t^T \partial_{y_j}P_{\frac{1}{2H}(s-t)^{2H}}g_j(s,t,W^{2,H}(t,s))(s-t)^{H-1/2}dsdt\Bigg)\Bigg|^2\Bigg]\\
&=\mathbb{E}\Bigg[\Bigg|\sum_{j=1}^{d}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\Bigg(\int_{t_{i+1}}^{T}\alpha_{i,j,j,s}(u)(s-u)^{H-\frac{1}{2}}ds\\
&\qquad\qquad -\int_u^T\partial_{y_j}P_{\frac{1}{2H}(s-u)^{2H}}g_j(s,u,W^{2,H}(u,s))(s-u)^{H-1/2}ds\Bigg)du\Bigg|^2\Bigg]\\
&=\mathbb{E}\left[\left|\sum_{j=1}^{d}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(\int_u^T\beta_{i,j,s}(u)(s-u)^{H-\frac{1}{2}}ds-\int_u^{t_{i+1}}\alpha_{i,j,j,s}(u)(s-u)^{H-\frac{1}{2}}ds\right)du\right|^2\right]\\
&\leq 2\mathbb{E}\left[\left|\sum_{j=1}^{d}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_u^T\beta_{i,j,s}(u)(s-u)^{H-\frac{1}{2}}dsdu\right|^2\right]\\
&+2\mathbb{E}\left[\left|\sum_{j=1}^{d}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_u^{t_{i+1}}\alpha_{i,j,j,s}(u)(s-u)^{H-\frac{1}{2}}dsdu\right|^2\right]\\
&=:2\left(I_{1,4,1}+I_{1,4,2}\right).
\end{aligned}
$$

with

$$
\beta_{i,j,s}(u):=\alpha_{i,j,j,s}(u)-\partial_{y_j}P_{\frac{1}{2H}(s-u)^{2H}}g_j(s,u,W^{2,H}(u,s)).
$$

The proof of (iii) is then established if we prove that

$$
\lim_{N\to+\infty} I_{1,4,1}+I_{1,4,2}=0. \tag{5.28}
$$

Note first that :

$$
\begin{aligned}
&\beta_{i,j,s}(u)\\
&= \partial_{x_j}\left(P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) - P_{\frac{1}{2H}(s-u)^{2H}} g_j(s,u,W^{2,H}(u,s))\right)\\
&= \left(P_{\frac{1}{2H}(s-t_i)^{2H}} - P_{\frac{1}{2H}(s-u)^{2H}}\right)\partial_{x_j} g_j(s,u,W^{2,H}(t_i,s))\\
&+ P_{\frac{1}{2H}(s-u)^{2H}}\left(\partial_{x_j} g_j(s,u,W^{2,H}(t_i,s)) - \partial_{x_j} g_j(s,u,W^{2,H}(u,s))\right)\\
&= -\frac{1}{2}\int_{t_i}^{u} \Delta P_{\frac{1}{2H}(s-r)^{2H}}\partial_{x_j} g_j(s,u,W^{2,H}(t_i,s))dr\\
&+ \int_0^1 P_{\frac{1}{2H}(s-u)^{2H}} \nabla\partial_{x_j} g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\cdot(W^{2,H}(t_i,s) - W^{2,H}(u,s)), \qquad (5.29)
\end{aligned}
$$

where we recall Notation (5.8). Using once again the fact that f belongs to $\mathcal{S}_{ad}$, we immediately obtain that

$$
|\beta_{i,j,s}(u)| \leq C\left((u-t_i) + \sum_{k=1}^{d}\left|W^{2,H}(t_i,s) - W^{2,H}(u,s)\right|\right), \qquad (5.30)
$$

from which we deduce that (using (5.21))

$$
\begin{aligned}
&(I_{1,4,1})^{1/2}\\
&\leq C\sum_{j=1}^{d}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_u^T\left((u-t_i) + \sum_{k=1}^{d}\mathbb{E}\left[\left|W^{2,H}(t_i,s) - W^{2,H}(u,s)\right|^2\right]^{1/2}\right)(s-u)^{H-\frac{1}{2}}dsdu\\
&\leq C\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_u^T\left((u-t_i) + |u-t_i|^{\min\{H,1/2\}}\right)(s-u)^{H-\frac{1}{2}}dsdu\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Thus

$$
\lim_{N\to+\infty} I_{1,4,1} = 0.
$$

The convergence of the term $I_{1,4,2}$ is easy to handle as :

$$
\begin{aligned}
(I_{1,4,2})^{1/2} &\leq \sum_{j=1}^{d}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_u^{t_{i+1}}(s-u)^{H-\frac{1}{2}}\mathbb{E}\left[|\alpha_{i,j,j,s}(u)|^2\right]^{1/2}dsdu\\
&\leq C\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_u^{t_{i+1}}(s-u)^{H-\frac{1}{2}}dsdu\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

So (5.28) is proved.

Proof of (iv) :

Recall that

$$
I_{2,5}(t_i,t_{i+1}) = -\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}}\int_u^T P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s))dsdB_j(u).
$$

Hence :

$$
\begin{aligned}
&\sum_{i=0}^{N-1} I_{2,5}(t_i,t_{i+1}) + \int_0^T \int_u^T P_{\frac{1}{2H}(s-u)^{2H}} g_j(s,u,W^{2,H}(t_i,s))dsdB_j(u) \\
&= -\sum_{j=1}^{d}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_u^T \left(P_{\frac{1}{2H}(s-t_i)^{2H}} - P_{\frac{1}{2H}(s-u)^{2H}}\right) g_j(s,u,W^{2,H}(t_i,s))dsdB_j(u) \\
&- \sum_{j=1}^{d}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_u^T P_{\frac{1}{2H}(s-u)^{2H}}(g_j(s,u,W^{2,H}(t_i,s)) - g_j(s,u,W^{2,H}(u,s)))dsdB_j(u) \\
&= \sum_{j=1}^{d}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_u^T \gamma_{s,t_i}(u)dsdB_j(u)
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_{s,t_i}(u) := &\left(P_{\frac{1}{2H}(s-t_i)^{2H}} - P_{\frac{1}{2H}(s-u)^{2H}}\right) g_j(s,u,W^{2,H}(t_i,s)) \\
&+ P_{\frac{1}{2H}(s-u)^{2H}}\left[g_j(s,u,W^{2,H}(t_i,s)) - g_j(s,u,W^{2,H}(u,s))\right].
\end{aligned}
$$

Hence using the Itô isometry,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{2,5}(t_i,t_{i+1}) + \int_0^T \int_u^T P_{\frac{1}{2H}(s-u)^{2H}} g_j(s,u,W^{2,H}(t_i,s))dsdB_j(u)\right|\right]^{1/2} \\
&\leq \sum_{j=1}^{d}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\left|\int_u^T \gamma_{s,t_i}(u)ds\right|^2\right]^{1/2} du \\
&\leq \sum_{j=1}^{d}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_u^T \mathbb{E}\left[|\gamma_{s,t_i}(u)|^2\right]^{1/2} dsdu.
\end{aligned}
$$

Up to the gradient, the quantity γ_{s,t_i} is very similar to $\beta_{i,j,s}$ defined in (5.29) and using (5.30) and (5.21), we get

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{2,5}(t_i,t_{i+1}) + \int_0^T \int_u^T P_{\frac{1}{2H}(s-u)^{2H}} g_j(s,u,W^{2,H}(t_i,s))dsdB_j(u)\right|\right]^{1/2} \\
&\leq C \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_u^T \left((u-t_i) + |t_{i+1}-t_i|^{\min\{H,1/2\}}\right) dsdu \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

□

Lemma 5.2. *We use notations introduced in the proof of Theorem 3.3, the following convergences hold true in $L^2(\Omega)$:*

(i)

$$
\lim_{N\to+\infty} \sum_{i=0}^{N-1} I_{1,2}(t_i,t_{i+1}) + \lim_{N\to+\infty} \sum_{i=0}^{N-1} I_{1,6}(t_i,t_{i+1}) = 0,
$$

(ii)

$$
\lim_{N\to\infty} \sum_{i=0}^{N-1} I_{2,1}(t_i,t_{i+1}) + \lim_{N\to+\infty} \sum_{i=0}^{N-1} I_{2,3}(t_i,t_{i+1}) + \lim_{N\to+\infty} \sum_{i=0}^{N-1} I_{2,4}(t_i,t_{i+1}) = 0,
$$

(iii)

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} I_{1,7}(t_i,t_{i+1})=0,$$

(iv)

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} I_{1,8}(t_i,t_{i+1})=0,$$

(v)

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} I_{1,9}(t_i,t_{i+1})=0,$$

(vi)

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} I_{1,10}(t_i,t_{i+1})=0,$$

(vii)

$$I_{1,5}(t_i,t_{i+1})=I_{2,5}(t_i,t_{i+1}),\quad \forall i\in\{0,\cdots,N\},\ \forall N\in\mathbb{N}^*.$$

Proof: **Proof of (i)**

As we will see some cancellations appear among the terms in the rest. We start with one of these cancellations, that is we first prove that

$$\lim_{N\to+\infty}\sum_{i=0}^{N} I_{1,2}(t_i,t_{i+1})+\lim_{N\to+\infty}\sum_{i=0}^{N} I_{1,6}(t_i,t_{i+1}) \overset{L^2(\Omega)}{=} 0. \tag{5.31}$$

Recall first that

$$\begin{aligned}
I_{1,2}(t_i,t_{i+1}) &= \int_{t_{i+1}}^{T}\left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}}-P_{\frac{1}{2H}(s-t_i)^{2H}}\right] f(s,W^{2,H}(t_{i+1},s))ds\\
&= -\frac{1}{2}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}(s-u)^{2H-1}\Delta P_{\frac{1}{2H}(s-u)^{2H}} f(s,W^{2,H}(t_{i+1},s))duds\\
&= -\frac{1}{2}\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}(s-u)^{2H-1}\Delta P_{\frac{1}{2H}(s-u)^{2H}} f(s,W^{2,H}(t_{i+1},s))dsdu.
\end{aligned}\tag{5.32}$$

Concerning the term $I_{1,6}(t_i,t_{i+1})$ we have

$$\begin{aligned}
&I_{1,6}(t_i,t_{i+1})\\
&=\frac{1}{2}\sum_{k=1}^{d}\int_{t_{i+1}}^{T}\frac{\partial^2}{\partial y_k^2}P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s,W^{2,H}(t_i,s)\right)\ \left(\delta_{k,i,s}(W^{2,H})\right)^2 ds.
\end{aligned}$$

So

$$\begin{aligned}
&I_{1,6}(t_i,t_{i+1})\\
&=\frac{1}{2}\sum_{k=1}^{d}\int_{t_{i+1}}^{T}\frac{\partial^2}{\partial y_k^2}P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s,W^{2,H}(t_i,s)\right)\ \left(\left|\delta_{k,i,s}(W^{2,H})\right|^2-\int_{t_i}^{t_{i+1}}(s-u)^{2H-1}du\right)ds\\
&+\frac{1}{2}\sum_{k=1}^{d}\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}(s-u)^{2H-1}\Delta P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s,W^{2,H}(t_i,s)\right)dsdu\\
&=: I_{1,6,1}(t_i,t_{i+1})+I_{1,6,2}(t_i,t_{i+1}).
\end{aligned}$$

As a consequence using (5.32), and letting :

$$\begin{aligned}
&A(t_i, t_{i+1}) \\
&:= \frac{1}{2}\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}(s-u)^{2H-1}\Big(\Delta P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s)\right) - \Delta P_{\frac{1}{2H}(s-u)^{2H}} f(s, W^{2,H}(t_{i+1},s))\Big) ds du,
\end{aligned} \tag{5.33}$$

$I_{1,2}(t_i, t_{i+1}) + I_{1,6}(t_i, t_{i+1})$ writes down as

$$I_{1,2}(t_i, t_{i+1}) + I_{1,6}(t_i, t_{i+1}) = I_{1,6,1}(t_i, t_{i+1}) + A(t_i, t_{i+1}).$$

Hence, (5.31) is proved if we prove

$$\lim_{N\to+\infty} \mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{1,6,1}(t_i, t_{i+1})\right|^2\right] = 0, \tag{5.34}$$

and

$$\lim_{N\to+\infty} \mathbb{E}\left[\left|\sum_{i=0}^{N-1} A(t_i, t_{i+1})\right|^2\right] = 0. \tag{5.35}$$

We start with an analysis of Term $I_{1,6,1}(t_i, t_{i+1})$, and we write $I_{1,6,1}(t_i, t_{i+1}) = \frac{1}{2}\sum_{k=1}^{d} I_{1,6,1,k}(t_i, t_{i+1})$, with

$$I_{1,6,1,k}(t_i, t_{i+1}) := \int_{t_{i+1}}^{T} \frac{\partial^2}{\partial y_k^2} P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s)\right) \left(\left|\delta_{k,i,s}(W^{2,H})\right|^2 - \int_{t_i}^{t_{i+1}}(s-u)^{2H-1}du\right) ds.$$

We have by letting $\rho_{i,s} := \frac{\partial^2}{\partial y_k^2} P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s)\right)$, and

$$\epsilon_{i,s,k} := \left|\delta_{k,i,s}(W^{2,H})\right|^2 - \int_{t_i}^{t_{i+1}}(s-u)^{2H-1}du. \tag{5.36}$$

We have

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{1,6,1,k}(t_i, t_{i+1})\right|^2\right] \\
&= 2\sum_{i,i'=0; i<i'}^{N-1} \int_{t_{i+1}}^{T}\int_{t_{i'+1}}^{T} \mathbb{E}\left[\rho_{i,s}\rho_{i',s'}\epsilon_{i,s,k}\underbrace{\mathbb{E}_{t_{i'}}\left[\epsilon_{i',s',k}\right]}_{=0}\right] ds ds' \\
&+ \sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T} \mathbb{E}\left[\rho_{i,s}\rho_{i,s'}\epsilon_{i,s,k}\epsilon_{i,s',k}\right] ds ds' \\
&\le C\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T} \mathbb{E}\left[\left|\epsilon_{i,s,k}\epsilon_{i,s',k}\right|\right] ds ds' \\
&\le C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}(s-v)^{2H-1} dv ds\right)^2 \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

So (5.34) is proved. Convergence (5.35) is obtained as follows. Note first that :

$$
\begin{aligned}
&\left|\Delta P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s)\right) - \Delta P_{\frac{1}{2H}(s-u)^{2H}} f(s, W^{2,H}(t_{i+1},s))\right| \\
&\leq \left|\Delta P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s)\right) - \Delta P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_{i+1},s))\right| \\
&+ \left|\Delta P_{\frac{1}{2H}(s-t_i)^{2H}} f(s, W^{2,H}(t_{i+1},s)) - \Delta P_{\frac{1}{2H}(s-u)^{2H}} f(s, W^{2,H}(t_{i+1},s))\right| \\
&\leq C \sum_{k=1}^{d} \left|\delta_{k,i,s}(W^{2,H})\right| + C \int_{t_i}^{u} (s-r)^{2H-1} dr,
\end{aligned}
$$

where C depends on the sup norms of partial derivatives of φ (recall (5.2)) up to order 4 and where we have used the definition of the Heat semigroup as in (5.32). Thus, since

$$
\mathbb{E}\left[\delta_{k,i,s}(W^{2,H})\delta_{k,i',s}(W^{2,H})\right] = 0, \forall i \neq i',
$$

we have (recalling (5.21))

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} A(t_i,t_{i+1})\right|^2\right] \\
&\leq C \sum_{k=1}^{d} \mathbb{E}\left[\left|\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_{i+1}}^{T} (s-u)^{2H-1} \left|\delta_{k,i,s}(W^{2,H})\right| ds du\right|^2\right] \\
&+ C \left|\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_{i+1}}^{T} (s-u)^{2H-1} \int_{t_i}^{u} (s-r)^{2H-1} dr ds du\right|^2 \\
&\leq C \sum_{k=1}^{d} \left(\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_{i+1}}^{T} (s-u)^{2H-1} |t_{i+1}-t_i|^{\min\{H,1/2\}} ds du\right)^2 \\
&\underset{N\to+\infty}{\longrightarrow} 0,
\end{aligned}
$$

which proves (5.35).

Proof of (ii)

The second cancellation is the following

$$
\lim_{N\to\infty} \sum_{i=0}^{N} I_{2,1}(t_i,t_{i+1}) + I_{2,3}(t_i,t_{i+1}) + I_{2,4}(t_i,t_{i+1}) = 0. \tag{5.37}
$$

Before getting into the computations, it is worth noting that $I_{2,1}(t_i,t_{i+1})$ (respectively $I_{2,3}(t_i,t_{i+1})$) has the same structure (up to the Brownian integral) than $I_{1,2}(t_i,t_{i+1})$ (respectively $I_{1,6}(t_i,t_{i+1})$ and $I_{1,7}(t_i,t_{i+1})$). So the proof will follow the same lines as in the one of (i). For the sake of completeness, we provide the main arguments. As we will notice a the end of the proof of this step, term $I_{2,4}(t_i,t_{i+1})$ is similar to $I_{2,3}(t_i,t_{i+1})$ but smaller in norm which allows one tu use a more straightforward way treatment. Recall that

$$
\begin{aligned}
&I_{2,1}(t_i,t_{i+1})\\
&=\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\left[P_{\frac{1}{2H}(s-t_{i+1})^{2H}}-P_{\frac{1}{2H}(s-t_i)^{2H}}\right]g_j(s,u,W^{2,H}(t_{i+1},s))dsdB_j(u)\\
&=-\frac{1}{2}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1}\Delta P_{\frac{1}{2H}(s-r)^{2H}}g_j(s,u,W^{2,H}(t_i,s))drdsdB_j(u)\\
&\quad-\frac{1}{2}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1}\Delta P_{\frac{1}{2H}(s-r)^{2H}}\\
&\qquad\qquad\qquad\cdot\left(g_j(s,u,W^{2,H}(t_{i+1},s))-g_j(s,u,W^{2,H}(t_i,s))\right)drdsdB_j(u)\\
&=-\frac{1}{2}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1}\Delta P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))drdsdB_j(u)\\
&\quad-\frac{1}{2}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1}\Big(\Delta P_{\frac{1}{2H}(s-r)^{2H}}-\Delta P_{\frac{1}{2H}(s-t_i)^{2H}}\Big)g_j(s,u,W^{2,H}(t_i,s))drdsdB_j(u)\\
&\quad-\frac{1}{2}\sum_{j=1}^{d}\sum_{k,\ell=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1}\\
&\qquad\qquad\int_0^1\frac{\partial^3}{\partial y_k^2\partial_{y_\ell}}P_{\frac{1}{2H}(s-r)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\;\delta_{\ell,i,s}(W^{2,H})drdsdB_j(u),
\end{aligned}
$$

where we recall Notation (5.8). In addition

$$
\begin{aligned}
&I_{2,3}(t_i,t_{i+1})\\
&=\frac{1}{2}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}P_{\frac{1}{2H}(s-t_i)^{2H}}\nabla^2 g_j\left(s,u,W^{2,H}(t_i,s)\right)\cdot\left(\delta_{i,s}(W^{2,H})\right)^2dsdB_j(u)\\
&=\frac{1}{2}\sum_{j=1}^{d}\sum_{k,\ell=1;k\neq\ell}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j\left(s,u,W^{2,H}(t_i,s)\right)\delta_{i,k,s}(W^{2,H})\delta_{i,\ell,s}(W^{2,H})dsdB_j(u)\\
&+\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\frac{\partial^2}{\partial y_k^2}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j\left(s,u,W^{2,H}(t_i,s)\right)\left(\delta_{i,k,s}(W^{2,H})\right)^2dsdB_j(u).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&I_{2,1}(t_i,t_{i+1}) + I_{2,3}(t_i,t_{i+1}) \\
&= \frac{1}{2}\sum_{j=1}^{d}\sum_{\substack{k,\ell=1;\\ k\neq\ell}}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j\left(s,u,W^{2,H}(t_i,s)\right)\delta_{i,k,s}(W^{2,H})\delta_{i,\ell,s}(W^{2,H})dsdB_j(u) \\
&\quad + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\frac{\partial^2}{\partial x_k^2}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s)) \\
&\qquad\qquad\qquad \cdot\left[\left(\delta_{i,k,s}(W^{2,H})\right)^2 - \int_{t_i}^{t_{i+1}}(s-r)^{2H-1}dr\right]dsdB_j(u) \\
&\quad - \frac{1}{2}\sum_{j=1}^{d}\sum_{k,\ell=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1} \\
&\qquad\qquad\qquad \cdot\int_0^1\frac{\partial^3}{\partial x_k^2\partial_{x_\ell}}P_{\frac{1}{2H}(s-r)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\;\delta_{\ell,i,s}(W^{2,H})drdsdB_j(u) \\
&\quad - \frac{1}{2}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\int_{t_i}^{t_{i+1}}(s-r)^{2H-1} \\
&\qquad\qquad\qquad \cdot\left(\Delta P_{\frac{1}{2H}(s-r)^{2H}} - \Delta P_{\frac{1}{2H}(s-t_i)^{2H}}\right)g_j(s,u,W^{2,H}(t_i,s))drdsdB_j(u) \\
&=: C_1(t_i,t_{i+1}) + C_2(t_i,t_{i+1}) + C_3(t_i,t_{i+1}) + C_4(t_i,t_{i+1}).
\end{aligned}
$$

So obviously, (5.37) is proved if we prove that

$$
\lim_{N\to+\infty}\mathbb{E}\left[\left|\sum_{i=0}^{N-1}C_r(t_i,t_{i+1})\right|^2\right] = 0, \quad \forall r\in\{1,2,3,4\}. \tag{5.38}
$$

These three terms are of similar form and their treatment will follow the similar scheme, so we give all the details for $C_1(t_i,t_{i+1})$ and present only the key ingredients for $C_2(t_i,t_{i+1})$ and $C_3(t_i,t_{i+1})$. Hence we start with $C_1(t_i,t_{i+1})$.
Set $\mu_{s,i,k,\ell,u} := \frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j\left(s,u,W^{2,H}(t_i,s)\right)$. We write $C_1(t_i,t_{i+1})$ as

$$
C_1(t_i,t_{i+1}) = \sum_{j=1}^{d}\sum_{k,\ell=1;k\neq\ell}^{d}C_{1,j,k,\ell}(t_i,t_{i+1})
$$

with obvious notations. We have for j,k,ℓ (with $k\neq\ell$),

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{i=0}^{N-1} C_{1,j,k,\ell}(t_i,t_{i+1})\Big|^2\Big]\\
&= 2\sum_{i,i'=0;i<i'}^{N-1}\int_{t_{i'+1}}^T\int_u^T\int_u^T\\
&\qquad\qquad \mathbb{E}\Big[\mu_{s,i,k,\ell,u}\mu_{s',i',k,\ell,u}\delta_{i,k,s}(W^{2,H})\delta_{i,\ell,s}(W^{2,H})\underbrace{\mathbb{E}_{t_{i'}}\left[\delta_{i',k,s'}(W^{2,H})\delta_{i',\ell,s'}(W^{2,H})\right]}_{=0}\Big]dsds'du\\
&+\sum_{i=0}^{N-1}\int_{t_{i+1}}^T\int_u^T\int_u^T \mathbb{E}\left[\mu_{s,i,u}\mu_{s',i,u}\delta_{i,k,s}(W^{2,H})\delta_{i,\ell,s}(W^{2,H})\delta_{i,k,s'}(W^{2,H})\delta_{i,\ell,s'}(W^{2,H})\right]dsds'du\\
&\leq C\sum_{i=0}^{N-1}\int_{t_{i+1}}^T\left(\int_u^T\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dvds\right)^2du\\
&\leq C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dvds\right)^2\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

With the previous notation and using Notation (5.36),

$$
C_2(t_i,t_{i+1})=\frac{1}{2}\sum_{j=1}^d\sum_{k=1}^d\int_{t_{i+1}}^T\int_u^T \mu_{s,i,k,k,u}\epsilon_{i,s,k}dsdB_j(u).
$$

So we have

$$
\begin{aligned}
\mathbb{E}\left[\Big|\sum_{i=0}^{N-1} C_2(t_i,t_{i+1})\Big|^2\right] &\leq C\sum_{j=1}^d\sum_{k=1}^d\sum_{i,i'=0}^{N-1}\int_{t_{i'+1}\vee t_{i+1}}^T\int_u^T\int_u^T \mathbb{E}\left[\epsilon_{i,s,k}\epsilon_{i',s,k'}\right]dsds'du\\
&\leq C\left(\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dvds\right)^2\right)^2\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

We now turn to Term $C_3(t_i,t_{i+1})$, for which we have :

$$
\begin{aligned}
&\mathbb{E}\left[\Big|\sum_{i=0}^{N-1} C_3(t_i,t_{i+1})\Big|^2\right] \\
&\leq C\sum_{i,i'=0}^{N-1}\int_{t_{i'+1}\vee t_{i+1}}^T\int_u^T\int_u^T\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \\
&\qquad\qquad (s-r)^{2H-1}(s'-r')^{2H-1}\mathbb{E}\left[|\delta_{\ell,i,s}(W^{2,H})\delta_{\ell,i',s'}(W^{2,H})|\right] drdr'dsds'du \\
&\leq C\left(\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T\int_{t_i}^{t_{i+1}}(s-r)^{2H-1}\left(\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dv\right)^{1/2} drds\right)^2\right)^2 \\
&\leq C\left(\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dv\right)^3 ds\right)^2 \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Following the same lines and using once again the uniform boundedness of derivatives (spatial and in the Malliavin sense) of f, we get immediately that

$$
\begin{aligned}
\mathbb{E}\left[\left|\sum_{i=0}^{N-1} C_4(t_i,t_{i+1})\right|^2\right] &\leq C\left(\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T\int_{t_i}^{t_{i+1}}|t_{i+1}-t_i|^{\min\{2H,1\}}dvds\right)^2\right)^2 \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Reproducing a similar treatment than the one of Term $I_{2,3}$, we can prove that Term $I_{2,4}$ also converges to 0. The fact that it contains an increment of order 3 allows one to use more straightforward estimates. Indeed, remind that

$$
I_{2,4} = \int_{t_{i+1}}^T\int_u^T\left[\int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\cdot\left(\delta_{i,s}(W^{2,H})\right)^3\right]dsdB_u,
$$

which contains an extra increment $\delta_{i,s}(W^{2,H})$ compared to $I_{2,4}$ which makes it smaller (in norm). More precisely,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{2,4}\right|^2\right]^{1/2} \\
&= C\mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_{i+1}}^T\int_u^T\left[\int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\cdot\left(\delta_{i,s}(W^{2,H})\right)^3\right]dsdB_u\right|^2\right]^{1/2} \\
&\leq C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T\mathbb{E}\left[\left|\int_u^T\left[\int_0^1 \nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\cdot\left(\delta_{i,s}(W^{2,H})\right)^3\right]ds\right|^2\right]du\right)^{1/2}.
\end{aligned}
$$

Using Jensen's inequality and recalling that C is a generic constant (depending on T, and on the dimension d of the fractional Brownian motion) we get that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{2,4}\right|^2\right]^{1/2} \\
&\leq C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^{T}\mathbb{E}\left[\left|\int_u^T\left[\int_0^1\nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\cdot\left(\delta_{i,s}(W^{2,H})\right)^3\right]ds\right|^2\right]du\right)^{1/2} \\
&\leq C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^{T}\int_u^T\mathbb{E}\left[\left|\int_0^1\nabla^3 P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s,\theta))d\theta\cdot\left(\delta_{i,s}(W^{2,H})\right)^3\right|^2\right]dsdu\right)^{1/2} \\
&\leq C\sum_{k,\ell,m=1}^{d}\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^{T}\int_u^T\mathbb{E}\left[\left|\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\delta_{m,i,s}(W^{2,H})\right|^2\right]dsdu\right)^{1/2} \\
&\leq C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^{T}\int_u^T\left(\int_{t_i}^{t_{i+1}}(s-u)^{2H-1}du\right)^3 dsdu\right)^{1/2} \\
&\leq C\sum_{i=0}^{N-1}(t_{i+1}-t_i)^{3/2}\underset{N\to+\infty}{\longrightarrow}0.
\end{aligned}
$$

Proof of (iii)

We have $I_{1,7}(t_i,t_{i+1})=\frac{1}{2}\sum_{k,\ell=1;k\neq\ell}^{d}I_{1,7,k,\ell}(t_i,t_{i+1})$ with

$$
I_{1,7,k,\ell}(t_i,t_{i+1}):=\int_{t_{i+1}}^{T}\frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a(s,t_i,W^{2,H}(t_i,s))\;\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})ds.
$$

Fix $k\neq\ell$ and set :

$$
\rho_{i,k,\ell,s}:=\frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}f^a(s,t_i,W^{2,H}(t_i,s)).
$$

We have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{i=0}^{N-1} I_{1,7,k,\ell}(t_i,t_{i+1})\Big|^2\Big] \\
&= \mathbb{E}\left[\left|\sum_{i=0}^{N-1} \int_{t_{i+1}}^T \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} f^a(s,t_i,W^{2,H}(t_i,s))\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H}) ds\right|^2\right] \\
&= 2\sum_{\substack{i,i'=0;\\ i<i'}}^{N-1} \int_{t_{i+1}}^T \int_{t_{i'+1}}^T \mathbb{E}\Bigg[\rho_{i,k,\ell,s}\rho_{i',k,\ell,s'}\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\underbrace{\mathbb{E}_{t_{i'}}\left[\delta_{k,i',s'}(W^{2,H})\delta_{\ell,i',s'}(W^{2,H})\right]}_{=0}\Bigg] ds ds' \\
&+ \sum_{i=0}^{N-1} \int_{t_{i+1}}^T \int_{t_{i+1}}^T \mathbb{E}\left[\rho_{i,k,\ell,s}\rho_{i,k,\ell,s'}\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\delta_{k,i',s'}(W^{2,H})\delta_{\ell,i',s'}(W^{2,H})\right] ds ds' \\
&\leq C\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^T \int_{t_i}^{t_{i+1}} (s-v)^{2H-1} dv ds\right)^2 \\
&\leq CN^{-1}\sum_{i=0}^{N-1}\int_{t_{i+1}}^T \int_{t_i}^{t_{i+1}} (s-v)^{4H-2} dv ds \\
&= CN^{-1}\sum_{i=0}^{N-1}\left[(T-t_i)^{4H} - (t_{i+1}-t_i)^{4H} - (T-t_{i+1})^{4H}\right] \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Proof of (iv)

Term $I_{1,8}(t_i,t_{i+1})$

$$
I_{1,8}(t_i,t_{i+1}) := \sum_{j=1}^d \sum_{k,\ell=1;k\neq\ell}^d I_{1,8,k,\ell}(t_i,t_{i+1}),
$$

with

$$
\begin{aligned}
&I_{1,8,j,k,\ell}(t_i,t_{i+1}) \\
&:= \frac{1}{2}\int_{t_{i+1}}^T \int_{t_i}^{t_{i+1}} \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)) dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H}) ds.
\end{aligned}
$$

Fix $k\neq\ell$, j. Set

$$
\gamma_{k,\ell,i,u,s} := \frac{\partial^2}{\partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} g_j(s,u,W^{2,H}(t_i,s)).
$$

Fix i, we have

$$
\begin{aligned}
&\int_{t_{i+1}}^T \int_{t_i}^{t_{i+1}} \gamma_{k,\ell,i,u,s} dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H}) ds \\
&\qquad\qquad = \int_{t_i}^{t_{i+1}} \int_{t_{i+1}}^T \gamma_{k,\ell,i,u,s}\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H}) ds dB_j(u).
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{1,8,j,k,\ell}(t_i,t_{i+1})\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_{t_{i+1}}^{T}\gamma_{k,\ell,i,u,s}\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})dsdB_j(u)\right|^2\right] \\
&= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[\left|\int_{t_{i+1}}^{T}\gamma_{k,\ell,i,u,s}\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})ds\right|^2\right]du \\
&= \sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\gamma_{k,\ell,i,u,s}\gamma_{k,\ell,i,u,s'}du\ \delta_{k,i,s}(W^{2,H})\delta_{k,i,s'}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\delta_{\ell,i,s'}(W^{2,H})\right]dsds' \\
&\le CN^{-1}\sum_{i=0}^{N-1}\left(\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dvds\right)^2 \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Proof of (v)

Term $I_{1,9}(t_i,t_{i+1})$

$$I_{1,9}(t_i,t_{i+1}) := \frac{1}{2}\sum_{j=1}^{d}\sum_{k,\ell=1;k\neq\ell}^{d} I_{1,9,k,\ell}(t_i,t_{i+1}),$$

with

$$
\begin{aligned}
&I_{1,9,j,k,\ell}(t_i,t_{i+1}) \\
&:= \frac{1}{2}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{s}\frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})ds.
\end{aligned}
$$

Fix $k\neq\ell$, j. Set

$$\gamma_{k,\ell,i,u,s} := \frac{\partial^2}{\partial x_k\partial x_\ell}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s)).$$

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{1,9,j,k,\ell}(t_i,t_{i+1})\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{i=0}^{N-1} \int_{t_{i+1}}^{T} \int_{t_{i+1}}^{s} \gamma_{k,\ell,i,u,s} dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H}) ds\right|^2\right] \\
&= 2 \sum_{i,i'=0; i<i'}^{N-1} \int_{t_{i+1}}^{T} \int_{t_{i'+1}}^{T} \mathbb{E}\left[\int_{t_{i+1}}^{s} \gamma_{k,\ell,i,u,s} dB_j(u)\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\delta_{k,i',s'}(W^{2,H})\delta_{\ell,i',s'}(W^{2,H}) \times \right. \\
&\qquad\qquad \left. \underbrace{\mathbb{E}_{t_{i'+1}}\left[\int_{t_{i'+1}}^{s'} \gamma_{k,\ell,i',u',s'} dB_j(u')\right]}_{=0}\right] ds ds' \\
&= \sum_{i=0}^{N-1} \int_{t_{i+1}}^{T} \int_{t_{i+1}}^{T} \mathbb{E}\Big[\int_{t_{i+1}}^{s\wedge s'} \gamma_{k,\ell,i,u,s}\gamma_{k,\ell,i,u,s'} du\ \delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\delta_{k,i,s'}(W^{2,H})\delta_{\ell,i,s'}(W^{2,H})\Big] ds ds' \\
&\leq C \sum_{i=0}^{N-1} \left(\int_{t_{i+1}}^{T} \int_{t_i}^{t_{i+1}} (s-v)^{2H-1} dv ds\right)^2 \\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Proof of (vi)

Term $I_{1,10}(t_i,t_{i+1})$

$$
I_{1,10}(t_i,t_{i+1}) := \sum_{j,k,\ell=1}^{d} I_{1,10,j,k,\ell}(t_i,t_{i+1}),
$$

with

$$
I_{1,10,j,k,\ell}(t_i,t_{i+1}) := \frac{1}{6}\int_{t_{i+1}}^{T} \mu_{i,s,j,k,\ell}\ \delta_{j,i,s}(W^{2,H})\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H}) ds,
$$

where

$$
\mu_{i,s,j,k,\ell} := \int_0^1 \frac{\partial^3}{\partial x_j \partial x_k \partial x_\ell} P_{\frac{1}{2H}(s-t_i)^{2H}} f\left(s, W^{2,H}(t_i,s,\theta)\right) d\theta
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} I_{1,10,j,k,\ell}(t_i,t_{i+1})\right|^2\right]^{1/2}\\
&\leq \sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\mathbb{E}\left[\left|\delta_{j,i,s}(W^{2,H})\delta_{k,i,s}(W^{2,H})\delta_{\ell,i,s}(W^{2,H})\right|^2\right]^{1/2} ds\\
&\leq C\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\left(\int_{t_i}^{t_{i+1}}(s-v)^{2H-1}dv\right)^{3/2} ds\\
&\leq CN^{-1/2}\sum_{i=0}^{N-1}\int_{t_{i+1}}^{T}\int_{t_i}^{t_{i+1}}(s-v)^{3H-3/2}dvds\\
&= C\left(N^{-1/2}\sum_{i=0}^{N-1}\left[(T-t_i)^{3H+1/2}-(T-t_{i+1})^{3H+1/2}\right]\right)-CN^{-3H}\\
&\underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Proof of (vii)

Finally, as for any k,j in $\{1,\cdots,d\}$,

$$
\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{s}\mathbb{E}\left[\left|\frac{\partial}{\partial x_k}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))\right|^2\right] duds<+\infty,
$$

the stochastic Fubini theorem implies that

$$
\begin{aligned}
I_{1,5} &= \sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{s}\frac{\partial}{\partial x_k}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))dB_j(u)\ \delta_{k,i,s}(W^{2,H})ds\\
&= \sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{t_{i+1}}^{s}\frac{\partial}{\partial x_k}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))\delta_{k,i,s}(W^{2,H})dB_j(u)\ ds\\
&= \sum_{k=1}^{d}\sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\frac{\partial}{\partial x_k}P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))\delta_{k,i,s}(W^{2,H})ds\ dB_j(u)\\
&= \sum_{j=1}^{d}\int_{t_{i+1}}^{T}\int_{u}^{T}\left[\nabla P_{\frac{1}{2H}(s-t_i)^{2H}}g_j(s,u,W^{2,H}(t_i,s))\cdot\delta_{i,s}(W^{2,H}))\right] dsdB_j(u)=I_{2,2}.
\end{aligned}
$$

□

Lemma 5.3. *Let f a smooth random field (that is $f\in\mathcal{S}_{ad}$). Then each term in this relation (3.2) admits a version which jointly measurable in (s,t,x,ω) in $[0,T]^2\times\mathbb{R}^d\times\Omega$ $(s\leq t)$. We will always consider this version.*

Proof: Recall that f (together with all its derivatives) is by definition bounded. The result is true for all the integrals in dt as a consequence of Lebegue's dominated convergence. Concerning the terms involving a stochastic integral, we refer to Protter (2005, Theorem IV.63). □

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