

A de Finetti-type representation of joint hierarchically exchangeable arrays on DAGs

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Abstract. We define joint exchangeability on arrays indexed by a vector of natural numbers with coordinates being the vertices of directed acyclic graphs (DAGs) using local isomorphisms. The notion provides a new version of exchangeability, which is a joint version of hierarchical exchangeability defined in Jung, L., Staton, Yang (2020). We also prove the existence of a generic representation by independent uniform random variables.

1. Introduction

DAG exchangeability is a notion of exchangeability on a family of indexed random elements

$$\mathbf{X} = (X_\alpha : \alpha \in \mathbb{N}^V)$$

on a Borel space \mathcal{X} , where $G = (V, E)$ is a directed acyclic graphs (DAGs). DAG exchangeability was introduced by Jung et al. (2021) as a generalization of *hierarchical exchangeability* in Austin and Panchenko (2014). The main purpose of this paper is to extend Jung et al. (2021) to a wider class of exchangeable structures including *jointly exchangeable arrays* using probabilistic methods. These methods were first deployed by David Aldous in Aldous (1981). Later, Olav Kallenberg applied this method in a systematic way for more general results (see Kallenberg (1989) or Kallenberg (1992) for example). All of these results are organized in his textbook Kallenberg (2005) which is a standard reference for fundamental results in exchangeability.

Our work is motivated by studies on Bayesian inference modeling, probabilistic programming, and neural networks as discussed in the introduction to Jung et al. (2021). In fact, the original motivation and hope in that work was to obtain a representation for *jointly* DAG exchangeable arrays as opposed to the representation obtained there for *separately* DAG exchangeable arrays (precise definitions are given later). In this work, we close this gap by providing such a representation. Briefly, the idea is that de Finetti-type representations of hierarchically exchangeable structures can identify when a hierarchical generative model can be replaced by an equivalent one but with more explicit independence structure (see Staton et al. (2017)). For

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general applications of exchangeability theory, one can also find in [Orbanz and Roy \(2015\)](#) a recent survey on various applications of exchangeability theory to Bayesian inference models including [Hoff \(2007\)](#), [Fortini and Petrone \(2012\)](#), and [Lloyd et al. \(2012\)](#). Structure theorems on exchangeable processes also provide canonical representations of neural networks with hierarchical symmetries. Readers can consult, for example, [Bloem-Reddy and Teh \(2020\)](#), [Cohen and Welling \(2016\)](#), and [Bruna et al. \(2013\)](#) for applications in this direction.

Let $G = (V, E)$ be a DAG. We assume for the rest of the paper that G is finite and simple. Also, when we write G as a set, we refer to the set of vertices V . We write $v \prec w$ if there exists a directed nonempty path from v to w . Note that \prec defines a partial order in G . Conversely, given a finite partially ordered set (G, \prec) , we can build a corresponding set of directed edges E by adding the edge \overrightarrow{vw} if and only if $v \prec w$ and there is no $v' \in G$ such that $v \prec v' \prec w$. To make this correspondence bijective, we assume that G always have the minimal set of edges under its induced partial order: that is, whenever there is a directed path from v to w that passes other vertices than v and w , we have no edge from v to w .

We say that a subgraph C of a DAG G is **downward-closed** (or just closed) if it is downward-closed under the induced partial order, that is, $v \in C$ whenever there exists $w \in C$ such that $v \prec w$. We write \mathcal{A}_G for the collection of all closed (possibly empty) subgraphs of G . For $\alpha \in \mathbb{N}^G$ and $C \in \mathcal{A}_G$, let $\alpha|_C$ denote the restriction of α to C when viewing α as a function from G to \mathbb{N} .

Definition 1.1. Let G be a DAG. Then, a G -**permutation**¹ is a bijection $\tau : \mathbb{N}^G \rightarrow \mathbb{N}^G$ such that

$$\alpha|_C = \beta|_C \iff \tau(\alpha)|_C = \tau(\beta)|_C \quad (1.1)$$

for all $\alpha, \beta \in \mathbb{N}^G$, $C \in \mathcal{A}_G$. We write $S_{\mathbb{N}}^{\times G}$ for the collection of all G -permutations. A random array $\mathbf{X} = (X_\alpha : \alpha \in \mathbb{N}^G)$ is **DAG-exchangeable** if for all $\tau \in S_{\mathbb{N}}^{\times G}$, we have

$$\left(X_\alpha : \alpha \in \mathbb{N}^G \right) \stackrel{d}{=} \left(X_{\tau(\alpha)} : \alpha \in \mathbb{N}^G \right). \quad (1.2)$$

For $C \in \mathcal{A}_G$, let $I_C := \bigcup_{C' \in \mathcal{A}_G} \mathbb{N}^{C'}$. Given a G -permutation τ and $C \in \mathcal{A}_G$, one can always define the action of τ on \mathbb{N}^C by $\tau(\alpha)(v) = \tau(\beta)(v)$ for any $\beta \in \mathbb{N}^G$ such that $\beta|_C = \alpha$, since by the definition of G -permutations the choice of β is irrelevant. Since this induced map on \mathbb{N}^C is also a C -permutation, we can regard τ as a bijection from I_C into itself, and identify it as a G -permutation of the index set I_C . Therefore, we can define DAG exchangeability on a random array indexed by I_G instead of \mathbb{N}^G . For $\alpha \in I_G$, let $Dom(\alpha)$ be the domain of the function α , and $Restr(\alpha) := \{\alpha|_C : C \in \mathcal{A}_{Dom(\alpha)}\}$.

For any DAG-exchangeable array, we have a canonical representation using independent uniform random variables, as long as the underlying probability space is rich enough. We will assume this condition for the rest of the paper.

Theorem 1.2 ([Jung et al. \(2021\)](#)). *Let G be a DAG. Let $\mathbf{X} = (X_\alpha : \alpha \in \mathbb{N}^G)$ be a DAG-exchangeable array taking values in a Borel space \mathcal{X} . Then, there exist a measurable function $f : [0, 1]^{\mathcal{A}_G} \rightarrow \mathcal{X}$ and an i.i.d. array $\mathbf{U} = (U_\beta : \beta \in I_G)$ of uniform random variables such that*

$$X_\alpha = f\left(U_\beta : \beta \in Restr(\alpha)\right) \quad (1.3)$$

almost surely for all $\alpha \in \mathbb{N}^G$.

Example 1.3. This setup covers the following past results on the representations of exchangeable structures by independent uniform random variables.

- (a) *Exchangeable sequences:* Let G be a graph with a single vertex. Then, $S_{\mathbb{N}}^{\times G}$ is simply the group of all bijections from \mathbb{N} to itself. So, a DAG-exchangeable array is merely an exchangeable sequence.

Theorem 1.2 implies that for an exchangeable sequence $\mathbf{X} = (X_n : n \in \mathbb{N})$, there exist an i.i.d.

¹Although in [Jung et al. \(2021\)](#) we used the word ‘‘automorphism,’’ we change the terminology in order to distinguish them with automorphisms of the DAG itself which appear in Section 2.

sequence of uniform random variables (U_0, U_1, U_2, \dots) and a measurable function $f : [0, 1]^2 \rightarrow \mathcal{X}$ such that

$$X_n = f(U_0, U_n)$$

almost surely for all $n \in \mathbb{N}$. This is a variant of de Finetti’s theorem (de Finetti (1929), de Finetti (1937), Hewitt and Savage (1955)) proposed by Aldous (1981).

- (b) *Separately exchangeable arrays:* Let $G = (\{1, 2\}, \emptyset)$. Then, $\mathbb{N}^G = \mathbb{N}^2$ and $S_{\mathbb{N}}^{\times G}$ is isomorphic to $(S_{\mathbb{N}})^2$, acting naturally on \mathbb{N}^2 . Thus, a DAG-exchangeable array is a separately exchangeable array of dimension 2, that is, it satisfies the distributional equation

$$(X_{ij} : i, j \in \mathbb{N}) \stackrel{d}{=} (X_{\tau(i)\rho(j)} : i, j \in \mathbb{N}). \tag{1.4}$$

It is guaranteed by either Theorem 1.2 or the Aldous-Hoover theorem (Aldous (1981), Hoover (1979)) that there exist an i.i.d. array of uniform random variables $\mathbf{U} = (U_{ij} : i, j \geq 0)$ and a measurable function $f : [0, 1]^4 \rightarrow \mathcal{X}$ such that

$$X_{ij} = f(U_{00}, U_{i0}, U_{0j}, U_{ij}) \tag{1.5}$$

almost surely for all $i, j \in \mathbb{N}$. The result can be extended to arrays of higher dimensions. (See Kallenberg (2005) for a deep analysis on exchangeable arrays of high dimensions.)

- (c) *Hierarchically exchangeable arrays:* Let $G = (\{v_1, v_2, u_1, u_2\}, E)$ where $E = \{\overrightarrow{v_1 v_2}, \overrightarrow{u_1 u_2}\}$. Then, a DAG-exchangeable array is an example of hierarchical exchangeability introduced by Austin and Panchenko (2014), which can be written in the form $(X_{ij,k\ell} : i, j, k, \ell \in \mathbb{N})$ where i, j, k, ℓ are the coordinates on v_1, v_2, u_1, u_2 , respectively. If an array \mathbf{X} indexed by \mathbb{N}^G is DAG-exchangeable, then for $\tau, \rho \in S_{\mathbb{N}}$ and $\tau_i, \rho_k \in S_{\mathbb{N}}$ for each $i, k \in \mathbb{N}$, we have

$$(X_{ij,k\ell} : i, j, k, \ell \in \mathbb{N}) \stackrel{d}{=} (X_{\tau(i)\tau_i(j), \rho(k)\rho_k(\ell)} : i, j, k, \ell \in \mathbb{N}).$$

The representation theorem by Austin and Panchenko (2014) allows us to have

$$X_{ij,k\ell} = f(U_{00,00}, U_{i0,00}, U_{ij,00}, U_{00,k0}, U_{i0,k0}, U_{ij,k0}, U_{00,k\ell}, U_{i0,k\ell}, U_{ij,k\ell}).$$

almost surely for some measurable function f and some i.i.d. array \mathbf{U} of uniform random variables.

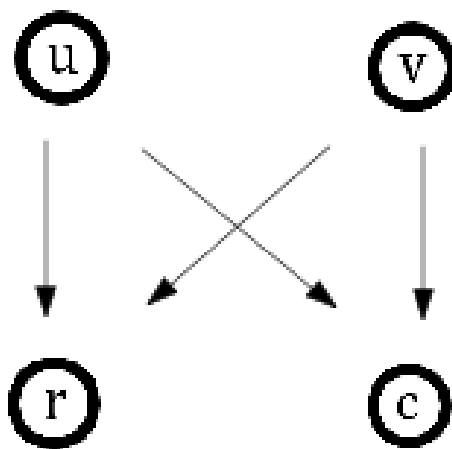


FIGURE 1.1. The DAG for Example 1.4.

Example 1.4. We introduce *random block matrices* from Example 2.2 of Jung et al. (2021), which is a new example covered by Theorem 1.2. Let $V = \{u, v, r, c\}$, $E = \{\overrightarrow{ur}, \overrightarrow{uc}, \overrightarrow{vr}, \overrightarrow{vc}\}$. (See Figure 1.1.) An array

$(X_{ij,k\ell} : i, j, k, \ell \in \mathbb{N})$ is DAG-exchangeable (regarding i, j, k, ℓ as coordinates on u, v, r, c , respectively) if for all $\tau, \rho \in S_{\mathbb{N}}$ and $\tau_{ij}, \rho_{ij} \in S_{\mathbb{N}}$ with $i, j \in \mathbb{N}$, we have

$$(X_{ij,k\ell} : i, j, k, \ell \in \mathbb{N}) \stackrel{d}{=} (X_{\tau(i)\rho(j), \tau_{ij}(k)\rho_{ij}(\ell)} : i, j, k, \ell \in \mathbb{N}). \tag{1.6}$$

By Theorem 1.2, there exist a measurable function f and an i.i.d. array of uniform random variables \mathbf{U} such that for all $i, j, k, \ell \in \mathbb{N}$, we have

$$X_{ij,k\ell} = f(U_{00,00}, U_{i0,00}, U_{0j,00}, U_{ij,00}, U_{ij,k0}, U_{ij,0\ell}, U_{ij,k\ell}) \tag{1.7}$$

almost surely.

To motivate the main objective of this paper, let us revisit (b) of Example 1.3. Let us consider the case where the array is jointly exchangeable, that is,

$$(X_{ij} : i, j \in \mathbb{N}) \stackrel{d}{=} (X_{\tau(i)\tau(j)} : i, j \in \mathbb{N}) \tag{1.8}$$

for all $\tau \in S_{\mathbb{N}}$. This is weaker than separate exchangeability, where we can choose permutations on the two coordinates separately. Jointly exchangeable arrays of dimension two are, in particular, closely connected to random graph theory. We recommend Diaconis and Janson (2008) or Austin (2008) as an introduction towards this direction.

For jointly exchangeable arrays, we have a representation of the form

$$X_{ij} = f(U_0, U_i, U_j, U_{\{i,j\}}) \tag{1.9}$$

almost surely for $i \neq j$ (Hoover (1979)). One can see that, compared to (1.5), the indices on the rows are merged with those on the columns. We can naturally ask if the similar merging occurs on joint versions of DAG-exchangeable arrays. That is, if a random array $\mathbf{X} = (X_{ij,k\ell} : i, j, k, \ell \in \mathbb{N})$, for instance, satisfies the distributional equation

$$(X_{ij,k\ell} : i, j, k, \ell \in \mathbb{N}) \stackrel{d}{=} (X_{\tau(i)\tau(j), \tau_{\{i,j\}}(k)\rho_{\{i,j\}}(\ell)} : i, j, k, \ell \in \mathbb{N}) \tag{1.10}$$

for all $\tau, \tau_{\{i,j\}}, \rho_{\{i,j\}} \in S_{\mathbb{N}}$, we can ask whether we have a representation of the form

$$X_{ij,k\ell} = f(U_{0,00}, U_{i,00}, U_{j,00}, U_{\{i,j\},00}, U_{\{i,j\},k0}, U_{\{i,j\},0\ell}, U_{\{i,j\},k\ell}) \tag{1.11}$$

almost surely for $i \neq j$.

The main objective of this paper is to extend the representation given by Theorem 1.2 to a wider class of exchangeable structures. This new model includes Hoover’s joint exchangeable arrays, the representation (1.11), and exchangeable arrays associated to arbitrary DAGs with merging of the vertices in the sense described above. We will rigorously define the model in the next section with more examples.

2. Settings and Main Results

2.1. *Main Results.* Let $Aut(G)$ denote the directed graph automorphism group of G , and let K be a subgroup of $Aut(G)$. Define a left group action of $Aut(G)$ acting on I_G by

$$\kappa\beta(v) = \beta(\kappa^{-1}(v)), \kappa \in Aut(G), \beta \in I_G.$$

Since elements in both $Aut(G)$ and $S_{\mathbb{N}}^{\times G}$ act as bijections from I_G to itself, we can composite them as functions. For $\kappa \in Aut(G)$ and $\tau \in S_{\mathbb{N}}^{\times G}$, we write $\kappa\tau, \tau\kappa$ to denote composite functions $\kappa \circ \tau, \tau \circ \kappa : I_G \rightarrow I_G$, respectively.

As we can see in Example 1.3, in the case of a separately exchangeable array of dimension d , we can regard G as a graph of order d with no edges. An array $\mathbf{X} = (X_{\alpha} : \alpha \in \mathbb{N}^d)$ is separately exchangeable if and only if

$$(X_{\alpha} : \alpha \in \mathbb{N}^d) \stackrel{d}{=} (X_{\tau\alpha} : \alpha \in \mathbb{N}^d) \tag{2.1}$$

for all $\tau \in S_{\mathbb{N}}^d$.

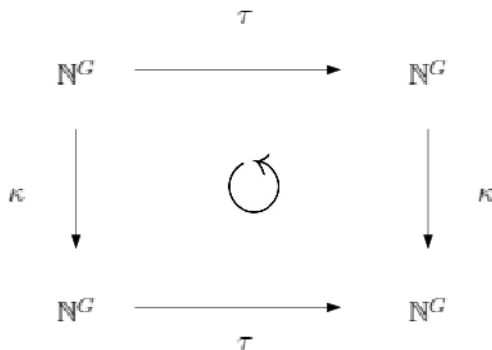


FIGURE 2.2. A diagram associated to permutations commuting with graph automorphisms. $\tau \in S_{\mathbb{N}}^{\times G}$ commutes with all $\kappa \in K \subseteq \text{Aut}(G)$.

On the other hand, the joint exchangeability can be described in terms of commutativity with the automorphism group $\text{Aut}(G)$, which is isomorphic to S_d . The array \mathbf{X} is jointly exchangeable, or equivalently,

$$(X_{n_1 n_2 \dots n_d} : n_1, \dots, n_d \in \mathbb{N}) \stackrel{d}{=} (X_{\sigma(n_1)\sigma(n_2)\dots\sigma(n_d)} : n_1, \dots, n_d \in \mathbb{N}) \tag{2.2}$$

for all $\sigma \in S_{\mathbb{N}}$ if and only if (2.1) holds for all $\tau \in S_{\mathbb{N}}^d$ satisfying $\tau\kappa = \kappa\tau$ for any $\kappa \in \text{Aut}(G)$.

Furthermore, if we require that τ commutes with $\kappa \in K$ for subgroups K of S_d instead of the whole S_d , we obtain a different notion of exchangeability lying between separate and joint exchangeability. For example, if we let $d = 3$ and K be the subgroup of S_3 generated by the transposition $(2\ 3)$, we have

$$(X_{ijk} : i, j, k \in \mathbb{N}^3) \stackrel{d}{=} (X_{\tau(i)\sigma(j)\sigma(k)} : i, j, k \in \mathbb{N}^3)$$

for all $\tau, \sigma \in S_{\mathbb{N}}$.

Considering the above observations, it is tempting to define joint exchangeability on random arrays defined on DAGs by assigning a subgroup K of $\text{Aut}(G)$ and allowing law-invariance for permutations which commute with K (see Figure 2.2). However, there are a few issues we have to handle. One is that the exchangeability structure does not uniquely determine the group K .

Example 2.1. let $G = \{v_1, v_2, v_3\}$ be a graph with no edges, and let $K = \mathbb{Z}/3\mathbb{Z}$ acting naturally on G . Then, since K acts transitively on G , a permutation $\tau = (\tau_1, \tau_2, \tau_3) \in S_{\mathbb{N}}^3$ of \mathbb{N}^G commutes with K if and only if $\tau_1 = \tau_2 = \tau_3$. So we obtain the same exchangeability structure in this setting if we choose either $K = \mathbb{Z}/3\mathbb{Z}$ or $K = S_3$.

The other issue is more serious. Many of the proofs of representation theorems on exchangeable arrays use induction on the dimension of the arrays, and we will also follow this strategy. However, by restricting K to be a subgroup of $\text{Aut}(G)$, we encounter an issue when deploying this type of induction, as the following example shows.

Example 2.2. Let $V = \{u_1, u_2, v_1, v_2\}$, $E = \{\overrightarrow{u_1 u_2}, \overrightarrow{v_1 v_2}\}$ (Example 1.3, (c)). Then, $\text{Aut}(G)$ is a group of order two, where the nonidentity element exchanges u_i and v_i for $i = 1, 2$ respectively. However, the closed subgraph $C = \{u_1, u_2, v_1\}$ has a trivial automorphism group. If we assign joint exchangeability on a random array $\mathbf{X} = (X_\alpha : \alpha \in \mathbb{N}^G)$ associated to $K = \text{Aut}(G)$, the permutations in consideration should act identically on vertices u_1 and v_1 . However, there is no way to assign such a class of exchangeability on random arrays defined on the subgraph C via its automorphism group, since it has no nontrivial graph automorphism at all.

Both of these issues arise from the nature that the class of permutations that commute with K is determined only by the *local* behavior of K in the following sense. Let $Z_K(S_{\mathbb{N}}^{\times G})$ denote the group of G -permutations that commute with K . Let C_v denote the closure of $\{v\}$, i.e. the smallest closed subgraph of

G containing v . Then, $Z_K(S_{\mathbb{N}}^{\times G})$ consists of all $\tau \in S_{\mathbb{N}}^{\times G}$ such that $\tau\kappa(\alpha) = \kappa\tau(\alpha)$ for all $v \in G, \kappa \in K, \alpha \in \mathbb{N}^{C_v}$. In other words, the only relevant information from K is its action on “local” indices, that is, elements in I_G whose domain is of the form C_v .

Now, in order to handle the above issues from the examples, instead of a subgroup of $Aut(G)$, we will use a collection of mappings which takes into account the local nature required by joint exchangeability. We will continue to use K to denote such a collection. These mappings are not necessarily defined on the whole of G , but only on a specific vertex and its closure. We use the word **isomorphism** to describe a bijective function from a closed subgraph of G to another closed subgraph that preserves directed edges.

Definition 2.3. Let G be a DAG. A **local isomorphism** of G is a sub-DAG isomorphism of the form

$$\kappa : C_v \rightarrow C_w$$

for some $v, w \in G$.

A collection K of local isomorphisms is called a **consistent local isomorphism class (CLIC)** of G if

- K contains all the identity mappings and is closed under inversion, composition, and restrictions to subgraphs of the form C_u .
- If κ is a local isomorphism such that for each $v \in Dom(\kappa)$ we have $\kappa' \in K$ such that $\kappa'(v) = \kappa(v)$, then $\kappa \in K$.

Example 2.4. Let us go back to the random block matrices in Example 1.4. The following are the list of all the local isomorphisms of G :

- $\kappa_1 : \{u\} \rightarrow \{v\}$.
- $\kappa_{01} : \{u, v, r\} \rightarrow \{u, v, c\}$ where $\kappa_{01}(u) = u, \kappa_{01}(v) = v$, and $\kappa_{01}(r) = c$.
- $\kappa_{11} : \{u, v, r\} \rightarrow \{u, v, c\}$ where $\kappa_{11}(u) = v, \kappa_{11}(v) = u$, and $\kappa_{11}(r) = c$.
- $\kappa_{1r} : \{u, v, r\} \rightarrow \{u, v, r\}$ where $\kappa_{1r}(u) = v, \kappa_{1r}(v) = u$, and $\kappa_{1r}(r) = r$.
- $\kappa_{1c} : \{u, v, c\} \rightarrow \{u, v, c\}$ where $\kappa_{1c}(u) = v, \kappa_{1c}(v) = u$, and $\kappa_{1c}(c) = c$.
- Inverses of the above maps
- Identities

The following are all the possible lists of members a CLIC can have, where the identities and the inverses are omitted:

- (1) $\kappa_1, \kappa_{1r}, \kappa_{1c}$
- (2) κ_{01}
- (3) $\kappa_1, \kappa_{1r}, \kappa_{1c}, \kappa_{01}, \kappa_{11}$

For another example, let us consider $G = (V, E)$ with $V = \{u_1, u_2, v_1, v_2, v_3\}$ with edges $E = \{\overrightarrow{u_1u_2}, \overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}\}$. It corresponds to Austin and Panchenko’s setting with $r = 2, d_1 = 2, d_2 = 3$ (See Austin and Panchenko (2014)). Although there is no nontrivial automorphism of G , we have the following nontrivial local isomorphisms along with their inverses:

- $\rho_1 : \{u_1\} \rightarrow \{v_1\}$.
- $\rho_2 : \{u_1, u_2\} \rightarrow \{v_1, v_2\}$ where $\rho_2(u_i) = v_i$.

The following are all the possible lists of members a CLIC can have, where again the identities and the inverses are omitted:

4. ρ_1
5. ρ_1, ρ_2

Remark 2.5. A local isomorphism need not be extendable to an automorphism. For instance, in the case of Example 2.2, the cause of the second issue is that the local isomorphism $\kappa : \{u_1\} \rightarrow \{v_1\}$ cannot be extended to an automorphism of C . Let $K = \{id, \kappa, \kappa^{-1}, \rho, \rho^{-1}\}$, where $\rho : \{u_1, u_2\} \rightarrow \{v_1, v_2\}$ with $\rho(u_i) = v_i$ for $i = 1, 2$. Then, the (jointly) exchangeable random array associated to the automorphism group of G is law-invariant under the permutations that commute “locally” with K . Unlike the case using automorphisms, the induced symmetry on the subgraph C is well-described by just taking the elements in K which are defined inside C , which are id, κ , and κ^{-1} .

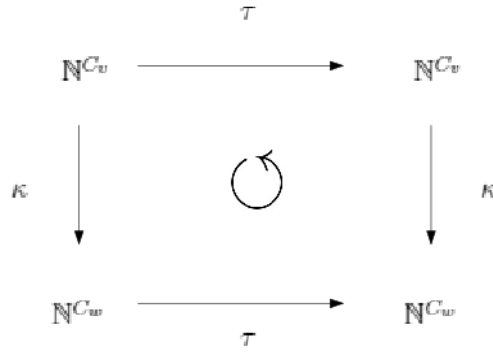


FIGURE 2.3. A commutative diagram associated to Definition 2.6.

Let K be a CLIC. For $v \in G$, let K_v be the collection of $\kappa \in K$ such that $C_v \subseteq \text{Dom}(\kappa)$. We say that two vertices $v, w \in G$ are **equivalent** under K if there exists $\kappa \in K_v$ such that $\kappa(v) = w$, and denote this relation by $v \stackrel{K}{\sim} w$.

We can define a similar equivalence in I_G as well. Given $v \in G$, $\kappa \in K_v$ and $\alpha \in \mathbb{N}^{C_v}$, define $\kappa(\alpha) \in \mathbb{N}^{C_{\kappa(v)}}$ as

$$\kappa(\alpha)(u) := \alpha(\kappa^{-1}u)$$

for $u \in C_{\kappa(v)}$. We say that two indices $\alpha, \beta \in I_G$ are **equivalent** under K if there exists a bijection $\phi : \text{Dom}(\alpha) \rightarrow \text{Dom}(\beta)$ such that for each $v \in \text{Dom}(\alpha)$, there exists $\kappa \in K_v$ such that $\kappa(\alpha|_{C_v}) = \beta|_{C_{\phi(v)}}$. We also write $\alpha \stackrel{K}{\sim} \beta$ for this relation. It is easy to check that both the relations on G and I_G denoted by the symbol $\stackrel{K}{\sim}$ are equivalence relations.

It is convenient to have our index set to be transitive under the group action. Thus, instead of \mathbb{N}^G , we restrict our index set to

$$\mathbb{N}_K^G := \{\alpha \in \mathbb{N}^G : \kappa(\alpha|_{C_v}) \neq \alpha|_{C_{\kappa(v)}} \text{ for all } v \in G, \kappa \in K_v \text{ such that } \kappa|_{C_v} \neq \text{id}_{C_v}\}.$$

We will see that this index set is transitive under the action of the group we will introduce soon. Let us also write

$$I_K^G := \bigcup_{C \in \mathcal{A}_G} \mathbb{N}_K^C.$$

Roughly speaking, a consistent isomorphism class K is an indicator that restricts the permutations of interest to act identically on vertices that are equivalent under $\stackrel{K}{\sim}$. The inclusion of identities and taking closure under inversion, composition and restriction, has ensured that $\stackrel{K}{\sim}$ is an equivalence relation in both V and I_K^G .

Now we are ready to define joint DAG-exchangeability and state the main theorem.

Definition 2.6. Let G be a DAG and K a CLIC of G . A permutation $\tau \in S_{\mathbb{N}}^{\times G}$ is said to be **K -commuting** if for all $\beta \in \mathbb{N}_K^{C_v}$ with $v \in G$ and $\kappa \in K_v$, we have

$$\tau \circ \kappa(\beta) = \kappa \circ \tau(\beta). \tag{2.3}$$

An array $\mathbf{X} = (X_\alpha : \alpha \in \mathbb{N}_K^G)$ (or I_K^G) is **(G, K) -exchangeable** if

$$(X_\alpha : \alpha \in \mathbb{N}_K^G) \stackrel{d}{=} (X_{\tau\alpha} : \alpha \in \mathbb{N}_K^G) \tag{2.4}$$

for all K -commuting τ .

We will keep using the notation $Z_K(S_{\mathbb{N}}^{\times G})$ for the collection of all K -commuting permutations. Note that Definition 1.1 is a special case of Definition 2.6 where K consists only of identity mappings. Also, thanks

to the second condition in Definition 2.3, for any two CLIC's K_1 and K_2 on a DAG G , we have $K_1 = K_2$ whenever $Z_{K_1}(S_{\mathbb{N}}^{\times G}) = Z_{K_2}(S_{\mathbb{N}}^{\times G})$.

Furthermore, one can show that $Z_K(S_{\mathbb{N}}^{\times G})$ indeed acts transitively on \mathbb{N}_K^G . It suffices to show that for every $\beta \in \mathbb{N}_K^G$, there exists $\tau \in Z_K(S_{\mathbb{N}}^{\times G})$ such that $\tau(\beta)(v)$'s are distinct for all $v \in G$, since every $\alpha \in \mathbb{N}_K^G$ with distinct $\alpha(v)$'s over $v \in G$ are in the same orbit.

For $\beta \in \mathbb{N}_K^G$, choose $\sigma_v \in S_{\mathbb{N}}$ for each $v \in G$ so that $\sigma_v = \sigma_u$ whenever $v \stackrel{K}{\sim} u$ and $\sigma_v(\beta(v))$'s are all distinct. This is possible because $\beta|_{C_v}$'s are never equivalent under K . Define $\tau \in Z_K(S_{\mathbb{N}}^{\times G})$ as

$$\tau(\gamma)(v) = \begin{cases} \sigma_u(\gamma(v)), & \gamma|_{C_v \setminus \{v\}} \stackrel{K}{\sim} \beta|_{C_u \setminus \{u\}} \\ \gamma(v), & \text{otherwise.} \end{cases}$$

Then, $\tau(\beta)(v) = \sigma_v(\beta(v))$ and hence $\tau\beta$ has distinct indices, proving the claim.

Now we are ready to state the main result.

Theorem 2.7. *Let G be a finite DAG, K a CLIC of G . Let $\mathbf{X} = (X_\alpha : \alpha \in \mathbb{N}_K^G)$ be an array whose entries take values in a Borel space. Then, \mathbf{X} is (G, K) -exchangeable if and only if there exists a measurable function $f : [0, 1]^{\mathcal{A}_G} \rightarrow \mathcal{X}$ such that for all $\alpha \in \mathbb{N}_K^G$,*

$$X_\alpha \stackrel{a.s.}{=} f(U_{[\alpha|_C]_K} : C \in \mathcal{A}_G) \tag{2.5}$$

for some array \mathbf{U} of i.i.d. uniform random variables indexed by $([\beta]_K : \beta \in I_K^G)$, where $[\beta]_K$ denotes the equivalence class of β with respect to $\stackrel{K}{\sim}$.

Example 2.8. Let us inspect the classes of permutations associated to the CLIC's introduced in Example 2.4 and the representations of the according exchangeable arrays. For the first case (random block matrices), each of the describe CLIC's represents the law-invariance under the following permutations of indices, respectively, where i, j, k, ℓ are the index values at u, v, r, c , respectively:

- (1) $X_{ij,kl} \rightarrow X_{\tau(i)\tau(j),\rho_{\{i,j\}}(k)\lambda_{\{i,j\}}(\ell)}$
- (2) $X_{ij,kl} \rightarrow X_{\tau(i)\theta(j),\rho_{ij}(k)\rho_{ij}(\ell)}$
- (3) $X_{ij,kl} \rightarrow X_{\tau(i)\tau(j),\rho_{\{i,j\}}(k)\rho_{\{i,j\}}(\ell)}$

For the three cases, Theorem 2.7 provides representations of the following forms, respectively:

- (1) $X_{ij,kl} = f(U_{0,00}, U_{i,00}, U_{j,00}, U_{\{i,j\},00}, U_{\{i,j\},k0}, U_{\{i,j\},0\ell}, U_{\{i,j\},kl}), i \neq j.$
- (2) $X_{ij,kl} = f(U_{00,0}, U_{i0,0}, U_{0j,0}, U_{ij,00}, U_{ij,k}, U_{ij,\ell}, U_{ij,\{k,\ell\}}), k \neq \ell.$
- (3) $X_{ij,kl} = f(U_{0,0}, U_{i,0}, U_{j,0}, U_{\{i,j\},0}, U_{\{i,j\},k}, U_{\{i,j\},\ell}, U_{\{i,j\},\{k,\ell\}}), i \neq j, k \neq \ell.$

For the second example, each of the cases allows permutations of the following forms, respectively, where i, j, k, ℓ, m are the index values at u_1, v_1, u_2, v_2, v_3 , respectively:

4. $X_{ij,kl,m} \rightarrow X_{\tau(i)\tau(j),\rho_i(k)\lambda_j(\ell)\theta_{j\ell}(m)}$
5. $X_{ij,kl,m} \rightarrow X_{\tau(i)\tau(j),\rho_i(k)\rho_j(\ell)\theta_{j\ell}(m)}$

For each of the two cases, Theorem 2.7 provides a representation of the following forms:

4. $X_{ij,kl,m} = f(U_{0,00,0}, U_{i,00,0}, U_{i,k0,0}, U_{j,00,0}, U_{\{i,j\},00,0}, U_{\{i,j\},k0,0}, U_{j,0\ell,0}, U_{\{i,j\},0\ell,0}, U_{\{i,j\},kl,0}, U_{j,0\ell,m}, U_{\{i,j\},0\ell,m}, U_{\{i,j\},kl,m}), i \neq j.$
5. $X_{ij,kl,m} = f(U_{0,0,0}, U_{i,0,0}, U_{i,k,0}, U_{j,0,0}, U_{\{i,j\},0,0}, U_{\{i,j\},k,0}, U_{j,\ell,0}, U_{\{i,j\},\ell,0}, U_{\{i,j\},\{k,\ell\},0}, U_{j,\ell,m}, U_{\{i,j\},\ell,m}, U_{\{i,j\},\{k,\ell\},m}), i \neq j, k \neq \ell.$

2.2. *Symmetry random variables associated to jointly DAG-exchangeable arrays.* The overall plan of the proof of Theorem 2.7 is similar to that of Theorem 1.2. We deploy induction on the number of vertices of G . To do this, we have to construct random variables which encode the intermediate information associated to \mathbf{X} , which we call a *symmetry random variables* associated to \mathbf{X} . We will see that the randomness of the uniform random variables affect \mathbf{X} only through symmetry random variables. A typical example of this phenomenon is the role of the empirical distribution in an exchangeable sequence (see Lemma 7.1 of Kallenberg (2005)).

The key property we need to show in this strategy is conditional independence among the involved σ -fields, and that is Proposition 2.12 in our case. It is a parallel of Proposition 4.1 of Jung et al. (2021), of which the proof is based on results of Hoover (1979). One aspect of Hoover's proof is that it depends heavily on nonstandard analysis and symbolic logic. In this paper, we provide a probabilistic proof of Proposition 2.12 independent of Hoover's. As mentioned at the beginning of the paper, our strategy resembles that of Kallenberg (2005) (especially Chapter 7) in the proof of the Aldous-Hoover representation theorem in a sense that we use systematic tools to prove conditional independence between involved random variables to deploy coding lemmas that provides representations using independent uniform random variables. (See the appendix for the lemmas that we use in the proof of the main result.)

Given a Borel space \mathcal{X} and a countable index set I endowed with a left group action of a group H on I , the action of $\tau \in H$ on the space \mathcal{X}^I is defined as

$$\left(\tau(x_i : i \in I)\right)_j := x_{\tau^{-1}(j)}.$$

Assuming that $\mathbf{X} = (X_i : i \in I)$ is exchangeable in a sense that

$$\mathbf{X} \stackrel{d}{=} \tau \mathbf{X}$$

for all $\tau \in H$, the action of H on a random variable $Y \in \sigma(\mathbf{X})$ is defined as

$$\tau Y := f(\tau \mathbf{X}),$$

where f is a measurable function such that $Y = f(\mathbf{X})$ almost surely.² The action on measurable sets $A \in \sigma(\mathbf{X})$ is defined by identifying the sets with indicator functions.

Let \mathbf{X} be a (G, K) -exchangeable array. For $\alpha \in I_K^G$, let us write $C_{K,\alpha}(S_{\mathbb{N}}^{\times G})$ for the collection of K -commuting permutations τ such that $\tau\alpha = \alpha$.

The following are basic properties of $C_{K,\alpha}(S_{\mathbb{N}}^{\times G})$.

- (a) If $\alpha \in \text{Restr}(\beta)$, then $C_{K,\beta}(S_{\mathbb{N}}^{\times G}) \subseteq C_{K,\alpha}(S_{\mathbb{N}}^{\times G})$.
- (b) If $\alpha \stackrel{K}{\sim} \beta$, then $C_{K,\alpha}(S_{\mathbb{N}}^{\times G}) = C_{K,\beta}(S_{\mathbb{N}}^{\times G})$.

The property (a) is obvious. The new property (b) follows from the fact that $\tau \in Z_K(S_{\mathbb{N}}^{\times G})$ commutes with the elements of K .

Let \mathcal{F}_α denote the invariant σ -field of $C_{K,\alpha}(S_{\mathbb{N}}^{\times G})$, that is,

$$\mathcal{F}_\alpha := \{A \in \sigma(\mathbf{X}) : \tau A = A \text{ for all } \tau \in C_{K,\alpha}(S_{\mathbb{N}}^{\times G})\}$$

where (we have $\mathcal{F}_\alpha = \mathcal{F}_\beta$ for $\alpha \stackrel{K}{\sim} \beta$ by (b).)

We want to construct a Borel-valued random array $\mathbf{S} := (S_\alpha : \alpha \in I_K^G)$ satisfying the following properties, and call it a **random symmetry array** associated to \mathbf{X} :

- (1) $\mathcal{F}_\alpha = \sigma(S_\alpha)$.
- (2) The array (\mathbf{X}, \mathbf{S}) is (G, K) -exchangeable.
- (3) For $\alpha \stackrel{K}{\sim} \beta$, $S_\alpha = S_\beta$.

The existence of symmetry arrays is a straightforward exercise. We give the proof in Appendix B.

²The choice of f is irrelevant due to exchangeability: If g is another measurable function such that $Y = g(\mathbf{X})$ almost surely, then we have $\mathbf{P}[f(\tau \mathbf{X}) = g(\tau \mathbf{X})] = \mathbf{P}[f(\mathbf{X}) = g(\mathbf{X})] = 1$.

Proposition 2.9. *For any (G, K) -exchangeable array \mathbf{X} taking values in a Borel space, a random symmetry array exists.*

Once we have an associated symmetry array, we can improve Theorem 2.7 so that the dependence structure of intermediate σ -fields is more explicit. For a generic array $\mathbf{Y} = (Y_i : i \in I)$ and $J \subseteq I$, we write $\mathbf{Y}_J := (Y_i : i \in J)$.

Assign a total ordering on the finite set \mathcal{A}_G . For each equivalence class of I_K^G under $\overset{K}{\sim}$, choose a representative whose domain is the smallest under this ordering. From now on, let us assume that we have a fixed collection of such representatives, and denote this collection by Γ_K^G . For $\alpha \in I_K^G$, let us write γ_α for the representative of $[\alpha]_K$.

Also, we let $Restr'(\alpha) := Restr(\alpha) \setminus \{\alpha\}$.

Theorem 2.10. *Let G, K, \mathbf{X} be as in Theorem 2.7, and let \mathbf{S} be a symmetry array of \mathbf{X} . Then, there exist measurable functions $f_C : [0, 1]^{\mathcal{A}_C} \rightarrow \mathcal{X}$ such that for all $\alpha \in \Gamma_K^G$,*

$$S_\alpha \stackrel{a.s.}{=} f_{Dom(\alpha)}(\mathbf{S}_{Restr'(\alpha)}, U_\alpha) \tag{2.6}$$

for some array \mathbf{U} of i.i.d. uniform random variables indexed by Γ_K^G .

The proof will come after introducing the required tools.

Note that by recursively replacing S_β with $f_{Dom(\gamma_\beta)}(\mathbf{S}_{Restr'(\gamma_\beta)}, U_{\gamma_\beta})$, for each $\alpha \in I_K^G$ we obtain the alternate representation of (2.6) of the form

$$S_\alpha = g_{Dom(\gamma_\alpha)}(U_\beta : \beta \in Restr(\gamma_\alpha)) \tag{2.7}$$

for some measurable functions g_C , where $U_\beta = U_{\gamma_\beta}$ for $\beta \in I_K^G$.

As mentioned earlier, the basic strategy of our proof is using induction on $|G|$, the number of vertices of G . We first build representations on the proper subgraphs of G , and tie them all together into a representation in the whole G . Proposition 2.12 is a key result which makes this ‘‘tying’’ possible.

Let $\alpha, \beta \in I_K^G$, and define $Restr(\alpha, K)$ to be the collection of $\alpha' \in I_K^G$ such that $\alpha' \overset{K}{\sim} \alpha|_C$ for some $C \in \mathcal{A}_{Dom(\alpha)}$. Let

$$D_{\alpha, \beta} := \{v \in Dom(\alpha) : \alpha|_{C_v} \in Restr(\beta, K)\},$$

and define

$$\alpha \wedge \beta := \alpha|_{D_{\alpha, \beta}}.$$

These are the joint-exchangeability counterparts of restrictions and intersections of two indices for separate DAG-exchangeability.

Remark 2.11. We have $\alpha \wedge \beta \overset{K}{\sim} \beta \wedge \alpha$ for all $\alpha, \beta \in I_K^G$. For each $v \in D_{\alpha, \beta}$, there exists $u \in Dom(\beta)$ such that $\alpha|_{C_v} = \kappa(\beta|_{C_u})$ for some $\kappa \in K$. By the definition of I_K^G , such a vertex u is unique, so we can define a mapping $\phi : D_{\alpha, \beta} \rightarrow D_{\beta, \alpha}$ via this relation. One can easily see that ϕ is the desired bijective correspondence to guarantee that $\alpha \wedge \beta \overset{K}{\sim} \beta \wedge \alpha$. One can also easily check that $\alpha \wedge \beta, \beta \wedge \alpha \in Restr(\alpha, K)$. Similarly, it is straightforward to show that $(\alpha \wedge \beta) \wedge \gamma \overset{K}{\sim} \alpha \wedge (\beta \wedge \gamma)$, and $\alpha \wedge \beta \overset{K}{\sim} \gamma \wedge \beta$ whenever $\alpha \overset{K}{\sim} \gamma$. Thus, I_K^G modulo $\overset{K}{\sim}$ is a semigroup under the operation \wedge .

Proposition 2.12. *Let $\alpha_1, \dots, \alpha_n \in I_K^G$. Then, $(S_{\alpha_k} : k \leq n)$ are independent given $(S_{\alpha_k \wedge \alpha_j} : k \neq j)$.*

The proof of Proposition 2.12 will be presented in the next section.

The next corollary follows from Proposition 2.12. Let us first define some notation.

- $Restr'(\alpha, K) := Restr(\alpha, K) \setminus [\alpha]_K$,
- $J_k := \{\alpha \in \Gamma_K^G : |Dom(\alpha)| \leq k\}$,

Corollary 2.13. *Let $\mathbf{S}_k := (S_\alpha : \alpha \in J_k)$. Then, for $1 \leq k \leq |G|$, \mathbf{S}_k is a conditionally independent family given \mathbf{S}_{k-1} . In particular, for $\alpha \in \mathbb{N}_K^G$, we have*

$$S_\alpha \perp\!\!\!\perp_{\mathbf{S}_{\text{Restr}'(\alpha, K)}} \mathbf{S} \setminus \mathbf{S}_{[\alpha]_K}.$$

Proof: Fix $\alpha \in J_k$, and let A be a finite subset of $J_k \setminus \{\alpha\}$. We claim that

$$S_\alpha \perp\!\!\!\perp_{(S_{\alpha \wedge \beta} : \beta \in A)} (S_\beta : \beta \in A). \tag{2.8}$$

Let us first Assume that the claim is true. Then, since $\sigma(S_{\alpha \wedge \beta} : \beta \in A) \subseteq \sigma(\mathbf{S}_{\text{Restr}'(\alpha, K)}) \subseteq \sigma(S_\alpha)$, by Lemma A.1 we have

$$S_\alpha \perp\!\!\!\perp_{\mathbf{S}_{\text{Restr}'(\alpha, K)}} (S_\beta : \beta \in A).$$

Since A is arbitrary, we have

$$S_\alpha \perp\!\!\!\perp_{\mathbf{S}_{\text{Restr}'(\alpha, K)}} \mathbf{S} \setminus \mathbf{S}_{[\alpha]_K}, \tag{2.9}$$

and since $\mathbf{S}_{\text{Restr}'(\alpha, K)} \in \sigma(\mathbf{S}_{k-1}) \subseteq \sigma(\mathbf{S} \setminus \mathbf{S}_{[\alpha]_K})$, again by Lemma A.1 we have

$$S_\alpha \perp\!\!\!\perp_{\mathbf{S}_{k-1}} \mathbf{S} \setminus \mathbf{S}_{[\alpha]_K}. \tag{2.10}$$

Since α is arbitrary, the proof is complete. The remaining task is to prove (2.8).

For $j = 1, \dots, |A|$, let $A_j := \{\beta_1 \wedge \dots \wedge \beta_j : \beta_1, \dots, \beta_j \in A, \text{ distinct}\}$. Letting $\mathcal{B}_j := \sigma(S_\beta : \beta \in A_j)$, by the assumption on A we have that

$$\sigma(S_{\beta_1 \wedge \dots \wedge \beta_{|A|}}) = \mathcal{B}_{|A|} \subseteq \mathcal{B}_{|A|-1} \subseteq \dots \subseteq \mathcal{B}_1 = \sigma(S_\beta : \beta \in A).$$

In particular, we have $\mathcal{B}_{|A|} \subseteq \sigma(S_{\alpha \wedge \beta} : \beta \in A)$.

By Proposition 2.12, we have

$$S_\alpha \perp\!\!\!\perp_{(S_{\alpha \wedge \beta} : \beta \in A_j), \mathcal{B}_{j+1}} \mathcal{B}_j,$$

for all j . Since $\sigma(S_{\alpha \wedge \beta} : \beta \in A_j) \subseteq \sigma(S_{\alpha \wedge \beta} : \beta \in A) \subseteq \sigma(S_\alpha)$, Lemma A.1 implies that

$$S_\alpha \perp\!\!\!\perp_{(S_{\alpha \wedge \beta} : \beta \in A), \mathcal{B}_{j+1}} \mathcal{B}_j.$$

Therefore, successively using the chain rule for conditional independence, we obtain

$$S_\alpha \perp\!\!\!\perp_{(S_{\alpha \wedge \beta} : \beta \in A), \mathcal{B}_{|A|}} \mathcal{B}_1.$$

Finally, we have

$$S_\alpha \perp\!\!\!\perp_{(S_{\alpha \wedge \beta} : \beta \in A)} S_{\beta_1 \wedge \dots \wedge \beta_{|A|}}.$$

Combining the last two relations, the proof of the claim is obtained. □

Proof of Theorem 2.10: We build an induction to show that for all $k \leq |G|$, there exists an i.i.d. array of uniform random variables $(U_\alpha : \alpha \in J_k)$ such that (2.6) holds for all $\alpha \in J_k$. (The case $k = 0$ is obvious.) Let us assume that there exists an i.i.d. array $\mathbf{W}_{k-1} := (W_\alpha : \alpha \in J_{k-1})$ of uniform random variables and a family of measurable functions $(f_C : C \in \mathcal{A}_G, |C| \leq k - 1)$ such that almost surely,

$$S_\alpha = f_{\text{Dom}(\alpha)}(\mathbf{S}_{\text{Restr}'(\alpha)}, W_\alpha), \alpha \in J_{k-1}. \tag{2.11}$$

Fix $\alpha \in N_k$, where $N_k := J_k \setminus J_{k-1}$. By (2.9) from Corollary 2.13, we have

$$S_\alpha \perp\!\!\!\perp_{\mathbf{S}_{\text{Restr}'(\alpha, K)}} \mathbf{S}_{k-1}.$$

Thus, by Lemma A.5, there exists a uniform random variable V_α independent of \mathbf{S}_{k-1} such that

$$S_\alpha = f_\alpha(\mathbf{S}_{\text{Restr}'(\alpha)}, V_\alpha) \tag{2.12}$$

almost surely. By exchangeability and transitivity, we have that the distribution of $(S_\alpha, \mathbf{S}_{\text{Restr}'(\alpha)})$ is identical for all α with the same domain. Thus, there exists an array of uniform random variables $\partial \mathbf{V}_k := (V_\alpha : \alpha \in N_k)$, which are not necessarily independent, such that (2.12) holds for every $\alpha \in N_k$, with the choice of f_α identical for all α defined on the same domain, which we denote by $f_{\text{Dom}(\alpha)}$.

Now consider an array $\partial \mathbf{W}_k := (W_\alpha : \alpha \in N_k)$ of i.i.d. uniform random variables, which are also independent of \mathbf{W}_{k-1} , and for $\alpha \in N_k$ define

$$S'_\alpha := f_\alpha(\mathbf{S}_{\text{Restr}'(\alpha)}, W_\alpha). \tag{2.13}$$

Since we can replace (2.11) and (2.13) into equations of the form (2.7), we can combine them into a one-line expression of the form

$$\mathbf{S}'_k = F(\mathbf{W}_k), \tag{2.14}$$

for some function F where $\mathbf{S}'_k = (\mathbf{S}_{k-1}, (S'_\alpha : \alpha \in N_k))$ and $\mathbf{W}_k = (W_\alpha : \alpha \in J_k)$. Note that $\mathbf{W}_k = (\mathbf{W}_{k-1}, \partial \mathbf{W}_k)$ is an i.i.d. array. On the other hand, we have the following properties:

- $\mathbf{P}[S_\alpha \in \cdot | \mathbf{S}_{k-1}] = \mathbf{P}[S'_\alpha \in \cdot | \mathbf{S}_{k-1}]$ almost surely for all $\alpha \in N_k$ since both W_α and V_α are independent of \mathbf{S}_{k-1} .
- $(S_\alpha : \alpha \in N_k)$ is a conditionally independent family given \mathbf{S}_{k-1} by Corollary 2.13.
- $(S'_\alpha : \alpha \in N_k)$ is a conditionally independent family given \mathbf{S}_{k-1} by construction.

By Lemma A.7 we have

$$\mathbf{S}_k = (\mathbf{S}_{k-1}, (S_\alpha : \alpha \in N_k)) \stackrel{d}{=} (\mathbf{S}_{k-1}, (S'_\alpha : \alpha \in N_k)).$$

Thus, with (2.14), we can apply Lemma A.3 to obtain an array of i.i.d. uniform random variables $\mathbf{U}_k := (U_\alpha : \alpha \in J_k)$ such that

$$\mathbf{S}_k = F(\mathbf{U}_k) \tag{2.15}$$

almost surely. Splitting (2.15) back to individual equations of the form (2.6), we obtain the desired representation for dimension k . Since $k \leq |G|$ is arbitrary, we have (2.6) for all $\alpha \in \Gamma_K^G$ by induction. \square

Proof of Theorem 2.7: Let $\alpha \in \mathbb{N}_K^G \cap \Gamma_K^G$. Since X_α is $\sigma(\mathbf{X})$ -measurable and is invariant under permutations fixing α , we have $X_\alpha \in \mathcal{F}_\alpha$ and hence $X_\alpha = h(S_\alpha)$ for some measurable function h , where the choice of h being independent of α due to exchangeability and transitivity. By inserting (2.7), we obtain (2.5) by identifying I_K^G modulo $\overset{K}{\sim}$ with Γ_K^G .

Let $U_\alpha := U_{\gamma_\alpha}$ for $\alpha \in \mathbb{N}_K^G$ (not necessarily in Γ_K^G) and $\mathbf{U} := (U_\alpha : \alpha \in I_K^G)$. To obtain (2.5) for all $\alpha \in \mathbb{N}_K^G$, it suffices to show that we can choose \mathbf{U} in a way such that (\mathbf{X}, \mathbf{U}) , or equivalently (\mathbf{S}, \mathbf{U}) , is (G, K) -exchangeable. Indeed, if (\mathbf{X}, \mathbf{U}) is exchangeable, since for $\beta \in \mathbb{N}_K^G$ there exists $\alpha \in \mathbb{N}_K^G \cap \Gamma_K^G$ such that $\tau\alpha = \beta$ with $\tau \in Z_K(S_{\mathbb{N}}^{\times G})$ by transitivity of the group action, (2.5) holds if we replace α with $\beta = \tau\alpha$ by exchangeability.

By Transfer Lemma A.2, there exists a family of measurable functions $(\phi_C : C \in \mathcal{A}_G)$ such that for any $\alpha \in \Gamma_K^G$ and any uniform random variable V independent of S_α , we have

$$(S_\alpha, U_\alpha) \stackrel{d}{=} (S_\alpha, \phi_{\text{Dom}(\alpha)}(S_\alpha, V)).^3 \tag{2.16}$$

Let $\mathbf{V} = (V_\alpha : \alpha \in I_K^G)$ be an array of uniform random variables independent of \mathbf{S} , where $V_\alpha = V_{\gamma_\alpha}$ for all $\alpha \in I_K^G$ and different components are all independent. Then, by (2.15) and (2.16), we have

$$S_\alpha = f_G(\mathbf{S}_{\text{Restr}'(\alpha)}, U'_\alpha)$$

almost surely for all $\alpha \in \Gamma_K^G$ where $U'_\alpha = \phi_{\text{Dom}(\alpha)}(S_\alpha, V_\alpha)$. (Note that $\mathbf{S}_{\text{Restr}'(\alpha)} \in \sigma(S_\alpha)$.)

³Since (2.6) implies that the conditional distribution of S_α given U_α is identical for all α with the same domain, the joint distribution of (S_α, U_α) depends only on $\text{Dom}(\alpha)$. Thus, we can choose the coding function ϕ_C to depend only on the domain.

Since \mathbf{V} is an i.i.d. array independent of \mathbf{S} , we have

$$U'_\alpha \perp_{\mathbf{S}} (U'_\beta : \beta \in \Gamma_K^G \setminus \{\alpha\}) \quad (2.17)$$

for all $\alpha \in \Gamma_K^G$. Also, since $(U_\beta : \beta \in \Gamma_K^G)$ is an independent family and $\mathbf{S} \setminus \mathbf{S}_{[\alpha]_K} \in \sigma(U_\beta : \beta \in \Gamma_K^G \setminus \{\alpha\})$, we have

$$U_\alpha \perp_{\mathbf{S} \setminus \mathbf{S}_{[\alpha]_K}} (U_\beta : \beta \in \Gamma_K^G \setminus \{\alpha\}). \quad (2.18)$$

Since $S_\alpha \in \sigma(\mathbf{S} \setminus S_\alpha, U_\alpha)$, by Lemma A.1 we obtain

$$U_\alpha \perp_{\mathbf{S}} (U_\beta : \beta \in \Gamma_K^G \setminus \{\alpha\}). \quad (2.19)$$

Therefore by (2.17) and (2.19), both $((S_\alpha, U_\alpha) : \alpha \in \Gamma_K^G)$ and $((S_\alpha, U'_\alpha) : \alpha \in \Gamma_K^G)$ are conditionally independent family given \mathbf{S} . Since $(S_\alpha, U_\alpha) \stackrel{d}{=} (S_\alpha, U'_\alpha)$ and both are conditionally independent of \mathbf{S} given S_α , by Lemma A.7 we have

$$(\mathbf{S}, U_\alpha) \stackrel{d}{=} (\mathbf{S}, U'_\alpha) \quad (2.20)$$

for all $\alpha \in \Gamma_K^G$. Therefore, by Lemma A.7, we have

$$(\mathbf{S}, (U_\alpha : \alpha \in \Gamma_K^G)) \stackrel{d}{=} (\mathbf{S}, (U'_\alpha : \alpha \in \Gamma_K^G)) \quad (2.21)$$

where $U'_\alpha := U'_{\gamma_\alpha}$. Thus, the relations (2.6) still hold even if we replace \mathbf{U} by $\mathbf{U}' := (U'_\alpha : \alpha \in \Gamma_K^G)$. Since \mathbf{S} and \mathbf{V} are exchangeable and independent of each other, (\mathbf{S}, \mathbf{V}) is exchangeable. Thus, by Lemma A.6, $(\mathbf{S}, \mathbf{U}')$ is exchangeable. \square

3. Proof of Proposition 2.12

Let $\mathcal{F}_\alpha = \sigma(S_\alpha)$, as in Section 2.2.

Lemma 3.1. For $\alpha \in I_K^G$, $n \in \mathbb{N}$, let

$\mathcal{F}_\alpha^n := \sigma(X_\beta : \beta \in \mathbb{N}_K^G)$, there exists $C \in \mathcal{A}_G$ such that $\beta|_C \in \text{Restr}(\alpha, K)$ and $\beta(v) \geq n$ for all $v \notin C$.

Then, $\mathcal{F}_\alpha = \bigcap_{n \geq 1} \mathcal{F}_\alpha^n$.

Let us introduce some notations to be used in the proof. For $\alpha, \beta \in I_K^G$ and $v \in G$, let

$$A_\alpha(v) := \left\{ \alpha(w) : C_w \setminus \{w\} = C_v \setminus \{v\}, w \stackrel{K}{\sim} v \right\},$$

and for $B \subseteq \mathbb{N}$ we define the injection $\tau_B : \mathbb{N} \rightarrow \mathbb{N}$ as

$$\tau_B(n) := \begin{cases} \min\{k > n : k \notin B\}, & n \notin B, \\ n, & n \in B. \end{cases}$$

In other words, τ_B fixes the numbers in B and shifts the rest to the increasing direction.

We define ρ_α to be an injective map of I_K^G to itself as

$$\rho_\alpha(\beta)(v) = \begin{cases} \tau_{A_\alpha(w)}(\beta(v)), & \beta|_{C_v \setminus \{v\}} \stackrel{K}{\sim} \alpha|_{C_w \setminus \{w\}}, \\ \beta(v) + 1, & \text{otherwise.} \end{cases}$$

The choice of w is irrelevant because we will always have $C_w \setminus \{w\} = C_u \setminus \{u\}$ for any $u \in \text{Dom}(\alpha)$ such that $\beta|_{C_v \setminus \{v\}} \stackrel{K}{\sim} \alpha|_{C_u \setminus \{u\}}$ (otherwise we do not have $\alpha \in \mathbb{N}_K^G$). The evaluation divides into three cases:

- (1) If $\kappa(\beta|_{C_v}) = \alpha|_{C_w}$ for some $w \in \text{Dom}(\alpha)$ and $\kappa \in K$ (or equivalently, $\beta|_{C_v} \in \text{Restr}(\alpha, K)$), then $\rho_\alpha(\beta)(v) = \beta(v)$.

- (2) If for some $w \in \text{Dom}(\alpha)$ and $\kappa \in K_v$ we have $\kappa(\beta|_{C_v})(u) = \alpha|_{C_w}(u)$ for all $u \in C_w \setminus \{w\}$ but $\kappa(\beta|_{C_v})(w) \neq \alpha(w)$, we let $\rho_\alpha(\beta)(v) = \beta(v) + \ell$, where ℓ is the smallest positive integer such that $\beta_{\ell,v}|_{C_v} \notin \text{Restr}(\alpha, K)$ and

$$\beta_{\ell,v}(u) := \beta(u) + \ell \mathbb{1}_{\{u=v\}}.$$

- (3) Otherwise, we have $\rho_\alpha(\beta)(v) = \beta(v) + 1$.

It is straightforward (but tedious) to check that ρ_α is an injection satisfying (1.2) in Definition 1.1, and that it commutes with K . Thus, any restriction of ρ_α to a finite subset of \mathbb{N}_K^G can be extended to an element in $Z_K(S_{\mathbb{N}}^{\times G})$. Thus, by Kolmogorov extension theorem, the array \mathbf{X} is law-invariant under the action of ρ_α .

Also note that ρ_α fixes α .

Proof of Lemma 3.1: For any $\beta \in \mathbb{N}_K^G$, letting $C := D_{\beta,\alpha}$, we have $\beta|_C \in \text{Restr}(\alpha, K)$ (see the remark right before Proposition 2.12). For $v \in C$, we have $\rho_\alpha(\beta)(v) = \beta(v)$. On the other hand, for $v \notin C$, one can see that $\rho_\alpha(\beta)(v) \geq \beta(v) + 1$. Thus, we have $\rho_\alpha(\mathbf{X}) \in \mathcal{F}_\alpha^2$. Since $D_{\rho_\alpha(\beta),\alpha} = D_{\beta,\alpha}$ for all $\beta \in \mathbb{N}_K^G$, we can similarly show that $\rho_\alpha^n(\beta)(\mathbf{X}) \in \mathcal{F}_\alpha^{n+1}$.

So, for arbitrary $E \in \mathcal{F}_\alpha$, acting ρ_α^n on both sides of the inclusion $E \in \sigma(\mathbf{X})$ we obtain $E \in \sigma(\rho_\alpha^n(\mathbf{X})) \subseteq \mathcal{F}_\alpha^{n+1}$. This shows that $\mathcal{F}_\alpha \subseteq \bigcap_{n \leq 1} \mathcal{F}_\alpha^n$.

To prove the converse, consider the collection \mathcal{T}_n of all finite permutations $\tau \in Z_K(S_{\mathbb{N}}^{\times G})$ such that

- (1) τ fixes α .
- (2) for all β , $\tau(\beta)(v) = \beta(v)$ whenever there exists $u \preceq v$ such that $\beta(u) > n$.

Then, as $n \rightarrow \infty$, the collection \mathcal{T}_n eventually contains all finite permutations in $Z_K(S_{\mathbb{N}}^{\times G})$ fixing α , and \mathcal{F}_α^n is invariant under the action of \mathcal{T}_n . Therefore, we have $\bigcap_{n \leq 1} \mathcal{F}_\alpha^n \subseteq \mathcal{F}_\alpha$. □

Proof of Proposition 2.12: We use induction on the number of indices n . The case $n = 1$ is obvious.

Consider $\rho := \rho_{\alpha_1}$ which is defined as in the proof of Lemma 3.1. As we have seen in the proof of Lemma 3.1, acting ρ on β fixes the values on D_{β,α_1} , which is by definition the values in the domain of $\beta \wedge \alpha_1$, and shifts all the values outside D_{β,α_1} by at least +1. By exchangeability, we have

$$(S_{\alpha_1}, \dots, S_{\alpha_n}) \stackrel{d}{=} (S_{\alpha_1}, \rho^k(S_{\alpha_2}), \dots, \rho^k(S_{\alpha_n})).$$

By Lemma A.4, we have

$$S_{\alpha_1} \perp_{\rho^k(S_{\alpha_2}), \dots, \rho^k(S_{\alpha_n})} S_{\alpha_2}, \dots, S_{\alpha_n} \tag{3.1}$$

for all $k \in \mathbb{N}$. Since the indices generating $\rho^k(\mathcal{F}_{\alpha_j})$ are contained in the collection of the indices generating $\mathcal{F}_{\alpha_1 \wedge \alpha_j}^k$, we have $\rho^k(S_{\alpha_j}) \in \mathcal{F}_{\alpha_1 \wedge \alpha_j}^k$. Thus, for all $k \geq 2$ we have

$$S_{\alpha_1} \perp_{\mathcal{F}_{\alpha_1 \wedge \alpha_j}^k : 2 \leq j \leq n} S_{\alpha_2}, \dots, S_{\alpha_n} \tag{3.2}$$

due to Lemma A.4. By the inductive hypothesis, $S_{\alpha_2}, \dots, S_{\alpha_n}$ are conditionally independent given $(S_{\alpha_i \wedge \alpha_j} : 2 \leq i \neq j \leq n)$. Thus, by Lemma 3.1 and Lemma A.8, the σ -field generated by $\mathcal{F}_{\alpha_1 \wedge \alpha_2}^k, \dots, \mathcal{F}_{\alpha_1 \wedge \alpha_n}^k$ decreases to some $\mathcal{G} \subseteq \sigma(\mathcal{F}_{\alpha_i \wedge \alpha_j} : 1 \leq i \neq j \leq n)$. Applying the backward martingale convergence to (3.2) as $k \rightarrow \infty$, we obtain

$$S_{\alpha_1} \perp_{\mathcal{G}} S_{\alpha_2}, \dots, S_{\alpha_n},$$

and since each $\mathcal{F}_{\alpha_i \wedge \alpha_j}$ is a sub- σ -field of S_{α_i} , by Lemma A.1 we obtain

$$S_{\alpha_1} \perp_{(\mathcal{F}_{\alpha_i \wedge \alpha_j} : 1 \leq i \neq j \leq n)} S_{\alpha_2}, \dots, S_{\alpha_n}. \tag{3.3}$$

Since $S_{\alpha_2}, \dots, S_{\alpha_n}$ are conditionally independent given $(\mathcal{F}_{\alpha_i \wedge \alpha_j} : 2 \leq i \neq j \leq n)$ and for each $j \geq 2$, $\mathcal{F}_{\alpha_1 \wedge \alpha_j}$ is a sub- σ -field of \mathcal{F}_{α_j} , again by Lemma A.1 we have that $S_{\alpha_2}, \dots, S_{\alpha_n}$ are conditionally independent given $(\mathcal{F}_{\alpha_i \wedge \alpha_j} : 1 \leq i \neq j \leq n)$. Combining this with (3.3), we obtain the desired result. □

Appendix A. Supplementary Lemmas

Elementary results that we use in the main text are introduced in this section. All are standard results and frequently used in exchangeability theory. For those results without proofs we have added references where one can find the proofs. We note again that the richness of the probability space is always assumed.

Lemma A.1. *Let \mathcal{G} , $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be σ -fields satisfying $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_1 \vee \mathcal{F}_3$ and*

$$\mathcal{F}_3 \perp\!\!\!\perp_{\mathcal{F}_1} \mathcal{G}$$

for some σ -field \mathcal{G} . Then, we have

$$\mathcal{F}_3 \perp\!\!\!\perp_{\mathcal{F}_2} \mathcal{G}.$$

Proof: The proof is straightforward from the towering property of conditional expectations. \square

Lemma A.2 (Transfer Lemma: Theorem 6.10, [Kallenberg \(2002\)](#)). *Let X, Y be random elements in a Borel space. Then,*

- (1) *For all $X' \stackrel{d}{=} X$, there exists a measurable function f such that whenever W is a uniform random variable independent of X' , then $Y' := f(X', W)$ satisfies $(X, Y) \stackrel{d}{=} (X', Y')$.*
- (2) *There exist measurable functions h and g such that whenever W is a uniform random variable independent of X and Y , $V := h(X, Y, W)$ is a uniform random variable independent of X satisfying $Y = g(X, V)$ almost surely.*

Lemma A.3 (Corollary 6.11, [Kallenberg \(2002\)](#)). *Let X, Y be Borel-valued random variables such that $X \stackrel{d}{=} f(Y)$ for some measurable function f . Then, there exists a random variable $Y' \stackrel{d}{=} Y$ such that $X = f(Y')$ almost surely.*

Lemma A.4 (Lemma 1.3, [Kallenberg \(2005\)](#)). *Let X, Y, Z be random variables such that $(X, Y) \stackrel{d}{=} (X, Z)$ and $\sigma(Y) \subseteq \sigma(Z)$. Then, $X \perp\!\!\!\perp_Y Z$.*

Lemma A.5 (Proposition 6.13, [Kallenberg \(2002\)](#)). *Let X, Y, Z be random elements, where X lies in a Borel space. Then, X is conditionally independent of Z given Y if and only if there exists a measurable function f and a uniform random variable U independent of Y, Z such that $X = f(Y, U)$ almost surely.*

Lemma A.6. *Let H be a group acting measurably on Borel spaces \mathcal{X} and \mathcal{Y} , and let μ be an H -invariant probability measure on \mathcal{X} , that is, x is H -exchangeable under μ . Let $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ be a measurable function. If $\phi(\tau x) = \tau\phi(x)$ μ -almost surely for all $\tau \in H$, then $(x, \phi(x))$ is H -exchangeable under μ .*

Proof: $\mu[x \in A, \phi(x) \in B] = \mu[\tau x \in A, \phi(\tau x) \in B] = \mu[\tau x \in A, \tau\phi(x) \in B]$. \square

Lemma A.7. *Let $(X_i : i \in I), (Y_i : i \in I)$ be a family of random variables with a countable index set I . For a random variable S , assume that the following are true:*

- $(S, X_i) \stackrel{d}{=} (S, Y_i)$. Equivalently, $\mathbf{P}[X_i \in \cdot | S] = \mathbf{P}[Y_i \in \cdot | S]$ almost surely.
- Given S , Both $(X_i : i \in I)$ and $(Y_i : i \in I)$ are conditionally independent families.

Then, we have $(S, X_i : i \in I) \stackrel{d}{=} (S, Y_i : i \in I)$.

Proof: Without loss of generality, let $I = \mathbb{N}$. Then for $n \in \mathbb{N}$ and bounded measurable functions f_1, \dots, f_n ,

$$\begin{aligned} \mathbf{E}[f_1(X_1) \cdots f_n(X_n) | S] &= \mathbf{E}[f_1(X_1) | S] \cdots \mathbf{E}[f_n(X_n) | S] \\ &= \mathbf{E}[f_1(Y_1) | S] \cdots \mathbf{E}[f_n(Y_n) | S] = \mathbf{E}[f_1(Y_1) \cdots f_n(Y_n) | S]. \end{aligned}$$

\square

Lemma A.8. For each $n \in \mathbb{N}$, let $(\mathcal{F}_k^n : k \in \mathbb{N})$ be a sequence of decreasing σ -fields with $\mathcal{F}^n := \bigcap_{k \geq 1} \mathcal{F}_k^n$.

Assume that given \mathcal{G} , the family $(\mathcal{F}_1^n : n \in \mathbb{N})$ is independent.

Then, $\mathcal{F} := \bigcap_{k \in \mathbb{N}} (\bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n)$ is a sub- σ -field of $\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}^n$.⁴ In particular, if $(\mathcal{F}_1^n : n \in \mathbb{N})$ are unconditionally independent, then $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \mathcal{F}^n$.

Proof: For each $n \in \mathbb{N}$ choose an event $A_n \in \mathcal{F}_1^n$, and let $A \in \mathcal{G}$. Then,

$$\mathbf{P} \left[A \bigcap_{j \leq N} A_j \mid \mathcal{G}, \mathcal{F}_k^n : n \in \mathbb{N} \right] = \mathbb{1}_A \prod_{j \leq N} \mathbf{P}[A_j \mid \mathcal{F}_k^j, \mathcal{G}]$$

by conditional independence and Lemma A.1. By backward martingale convergence the right hand side converges to a $\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}^n$ -measurable random variable as $k \rightarrow \infty$. Since the collection of all the events of the form $A \bigcap_{j \leq N} A_j$ is a π -system generating $\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}_1^n$, we can use π - λ arguments to show that for any $\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}_1^n$ -measurable event B , we have

$$\lim_{k \rightarrow \infty} \mathbf{P}[B \mid \mathcal{G}, \mathcal{F}_k^n : n \in \mathbb{N}] \in \mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}^n.$$

This shows that $\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n$ converges to $\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}^n$ as $k \rightarrow \infty$. Since $\bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n \subseteq \mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n$, we can conclude that

$$\bigcap_{k \in \mathbb{N}} (\bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n) \subseteq \bigcap_{k \in \mathbb{N}} (\mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n) = \mathcal{G} \vee \bigvee_{n \in \mathbb{N}} \mathcal{F}^n.$$

The last statement is obvious since we always have $\bigvee_{n \in \mathbb{N}} \mathcal{F}^n \subseteq \bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n$ for each $k \in \mathbb{N}$, which implies that

$$\bigvee_{n \in \mathbb{N}} \mathcal{F}^n \subseteq \bigcap_{k \in \mathbb{N}} (\bigvee_{n \in \mathbb{N}} \mathcal{F}_k^n) = \mathcal{F}. \quad \square$$

Appendix B. Proof of Proposition 2.9

For each $C \in \mathcal{A}_G$, choose $\gamma_C \in \Gamma_K^G$ with $Dom(\gamma_C) = C$, if there is any. Since \mathcal{F}_{γ_C} is countably generated by Lemma 3.1, there exists a Borel-valued random variable S_{γ_C} such that $\sigma(S_{\gamma_C}) = \mathcal{F}_{\gamma_C}$.⁵ Since S_{γ_C} is $\sigma(\mathbf{X})$ -measurable, there exists a measurable function f_C such that $f_C(\mathbf{X}) = S_{\gamma_C}$ almost surely. For any other $\gamma \in \Gamma_K^G$ with $Dom(\gamma) = C$, choose $\tau \in Z_K(S_{\mathbb{N}}^{\times G})$ such that $\tau\gamma = \gamma_C$ and let

$$S_\gamma := f_C(\tau\mathbf{X}).$$

The choice of τ is irrelevant because for all τ' such that $\tau'\gamma = \gamma_C$, we have

$$\mathbf{P}[S_\gamma = f_C(\tau\mathbf{X})] = \mathbf{P}[\tau'\tau^{-1}S_\gamma = \tau'\tau^{-1}f_C(\tau\mathbf{X})] = \mathbf{P}[S_\gamma = f_C(\tau'\mathbf{X})].$$

For $\alpha \in I_K^G$, we let

$$S_\alpha := S_{\gamma_\alpha}.$$

Let $\alpha, \beta \in I_K^G, \tau \in Z_K(S_{\mathbb{N}}^{\times G})$ with $\alpha = \tau\beta$. We claim that

$$\phi(\tau\mathbf{X}) = S_\beta$$

⁴Without the conditional independence, we cannot guarantee the result. Consider two sequences of random variables $\mathbf{X} = (X_n : n \in \mathbb{N})$ and $\mathbf{Y} = (Y_n : n \in \mathbb{N})$, and let P be a uniform random variable. Suppose that given P , \mathbf{X} and \mathbf{Y} are independent i.i.d. sequences, where $\mathbf{P}[X_1 = 1|P] = 1 - \mathbf{P}[X_1 = -1|P] = P$ and $\mathbf{P}[Y_1 = 1|P] = 1 - \mathbf{P}[Y_1 = -1|P] = 1/2$. Let $Z_n := X_n Y_n$. Then, \mathbf{Y} and \mathbf{Z} are i.i.d. sequences of random variables independent of P , and hence their tail σ -fields are trivial. However, since the tail σ -field of the joint sequence (\mathbf{Y}, \mathbf{Z}) recovers P , and hence it is not equal to the join of the tail σ -fields of the components.

⁵Any countably generated σ -field can be generated by a random variable taking values on a Borel space. The obvious choice of the generator for $\mathcal{F} = \sigma(A_k : k \in \mathbb{N})$ would be $X = \sum_{k \in \mathbb{N}} 3^{-k} \mathbb{1}_{A_k}$.

whenever $\phi(\mathbf{X}) = S_\alpha$. The left hand side represents the action of τ as we regard S_α as an $\sigma(\mathbf{X})$ -measurable random element, while on the right hand side τ acts on $\sigma(\mathbf{S})$ -measurable random elements. Once we have that these actions are identical, Lemma A.6 implies that the array \mathbf{S} constructed this way satisfies the desired properties.

Let $\text{Dom}(\gamma_\beta) = D$. Then, we have $S_{\gamma_\beta} = f_D(\lambda\mathbf{X})$, for $\lambda \in Z_K(S_{\mathbb{N}}^{\times G})$ such that $\gamma_D = \lambda\gamma_\beta$. Considering the way we have chosen the representatives, γ_α and γ_β are defined on the same domain, and hence for some K -commuting ρ we have $\gamma_D = \rho\gamma_\alpha$ and thus $S_{\gamma_\alpha} = f_D(\rho\mathbf{X})$.

For $v \in C$, let $\kappa_1, \kappa_2 \in K$ be local isomorphisms such that

$$\kappa_1(\beta|_{C_{\kappa_1(v)}}) = \gamma_\beta|_{C_v}, \kappa_2(\gamma_\alpha|_{C_{\kappa_2\kappa_1(v)}}) = \alpha|_{C_{\kappa_1(v)}}.$$

Then, $\rho\tau\lambda^{-1}$ fixes γ_D at $v \in D$ since

$$\begin{aligned} \rho\tau\lambda^{-1}(\gamma_D|_{C_v}) &= \rho\tau(\gamma_\beta|_{C_v}) = \rho\tau\kappa_1(\beta|_{C_{\kappa_1(v)}}) \\ &= \rho\kappa_1\tau(\beta|_{C_{\kappa_1(v)}}) = \rho\kappa_1(\alpha|_{C_{\kappa_1(v)}}) \\ &= \rho\kappa_1\kappa_2(\gamma_\alpha|_{C_{\kappa_2\kappa_1(v)}}) = \kappa_1\kappa_2\rho(\gamma_\alpha|_{C_{\kappa_2\kappa_1(v)}}) \\ &= \kappa_1\kappa_2(\gamma_D|_{C_{\kappa_2\kappa_1(v)}}) = \gamma_D|_{C_v}. \end{aligned}$$

Since $v \in D$ is arbitrary, $\rho\tau\lambda^{-1}$ fixes γ_D . Thus, by definition of \mathbf{S} , we have

$$f_D(\mathbf{X}) = f_D(\rho\tau\lambda^{-1}\mathbf{X})$$

almost surely. By exchangeability the equation holds almost surely if we replace \mathbf{X} by $\lambda\mathbf{X}$, and hence

$$S_{\gamma_\beta} = f_D(\lambda\mathbf{X}) = f_D(\rho\tau\mathbf{X})$$

almost surely.

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