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Asymptotic behaviour of the lattice Green function

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Abstract. The lattice Green function, i.e., the resolvent of the discrete Laplace operator, is fundamental in probability theory and mathematical physics. We derive its long-distance behaviour via a detailed analysis of an integral representation involving modified Bessel functions. Our emphasis is on the decay of the massive lattice Green function in the vicinity of the massless (critical) case, and the recovery of Euclidean isotropy in the massless limit. This provides a prototype for the expected but unproven long-distance behaviour of near-critical two-point functions in statistical mechanical models such as percolation, the Ising model, and the self-avoiding walk above their upper critical dimensions.

Dedicated to the memory of Dmitry Ioffe, 1963–2020.

1. The lattice Green function and its decay

1.1. *Introduction.* The *lattice Green function* is defined to be the Fourier integral

$$C_a(x) = \int_{[-\pi,\pi]^d} \frac{e^{ik\cdot x}}{a^2 + 1 - \hat{D}(k)} \frac{\mathrm{d}k}{(2\pi)^d} \tag{1.1}$$

with $x \in \mathbb{Z}^d$, $a \geq 0$, and

$$\hat{D}(k) = \frac{1}{d} \sum_{j=1}^{d} \cos k_j \qquad (k = (k_1, \ldots, k_d)). \tag{1.2}$$

The integral (1.1) converges for all $a > 0$ in all dimensions $d \geq 1$, but converges for $a = 0$ only for the transient case $d > 2$ since the denominator of the integrand is quadratic in small k when $a = 0$. The integral $C_0(0)$ in dimension $d = 3$ is a Watson integral that has been evaluated explicitly

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in Watson (1939); see Zucker (2011) and Guttmann (2010) for the interesting history and further developments.

The lattice Green function derives its name from the fact that it is equal to the $0, x$ matrix element of $(-\Delta + a^2)^{-1}$ (the inverse is as an operator on $\ell_2(\mathbb{Z}^d)$). Here Δ is the Laplacian on $\mathbb{Z}^d$, i.e.,

$$\Delta = D - I \tag{1.3}$$

where D is the $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix with $D_{xy} = \frac{1}{2d}$ if $\|x - y\|_1 = 1$, and otherwise $D_{xy} = 0$, and I denotes the identity matrix. Thus C_a is the resolvent of the lattice Laplacian. In the physics literature, C_a is often called the *Euclidean lattice scalar propagator*.

Our purpose is to study the precise asymptotic behaviour of $C_a(x)$ as x goes to infinity, with emphasis on how this behaviour depends on values of a close to the critical value $a = 0$, and on how Euclidean invariance is restored in the small a limit. Let m_a denote the unique nonnegative solution to $\cosh m_a = 1 + da^2$. For $a > 0$, an elementary proof that m_a is the exponential rate of decay of $C_a(x)$ when $x \to \infty$ along a coordinate axis, and that $C_a(x) \le C_a(0)e^{-m_a\|x\|_\infty}$ for all $x \in \mathbb{Z}^d$, is given in Madras and Slade (1993, Theorem A.2) (with a change of notation $2dz = \frac{1}{1+a^2}$). On the other hand, for the critical case $a = 0$ it is well-known that there is instead polynomial decay $\|x\|_2^{-(d-2)}$ Lawler and Limic (2010). We will prove, in a unified way and with precise constants for the amplitudes in the asymptotic formulas, that there are the following four regimes of decay for dimensions $d \ge 1$ (with the restriction $d > 2$ for regime (IV)):

(I)	fixed $a > 0$	anisotropic OZ	$m_a^{(d-3)/2}\lvert x\rvert_a^{-(d-1)/2}e^{-m_a\lvert x\rvert_a}$
(II)	$a\|x\|_2 \to \infty,\ a^3\|x\|_2 \to 0$	isotropic OZ	$a^{(d-3)/2}\|x\|_2^{-(d-1)/2}e^{-\sqrt{2d}a\|x\|_2}$
(III)	fixed $a\|x\|_2 > 0$	massive continuum	$\|x\|_2^{-(d-2)}$
(IV)	$a = 0\ \ (d > 2)$	massless continuum	$\|x\|_2^{-(d-2)}$.

The decay in regime (I) is called *Ornstein–Zernike decay*. The norm $|\cdot|_a$ is an explicitly defined a-dependent anisotropic norm on $\mathbb{R}^d$ (not an ℓ^p norm). Ornstein–Zernike decay is widely studied and has been proved for a variety of subcritical statistical mechanical models, e.g., Campanino et al. (2008); Chayes and Chayes (1986); Campanino et al. (1991). These proofs for much more difficult models than the lattice Green function show decay of the form $|x|_a^{-(d-1)/2}e^{-m_a|x|_a}$, but do not however reveal the factor $m_a^{(d-3)/2}$ for small $a > 0$. Indeed, a solution to the latter problem would be tantamount to a control of the critical behaviour of those models, a topic with difficult unsolved problems of great current interest.

The decay in regime (II) is also Ornstein–Zernike decay, but the mass in regime (I) is now replaced by its asymptotic form $\sqrt{2d}a$ as $a \to 0$, and Euclidean invariance is restored since the norm $|x|_a$ from regime (I) is replaced by the Euclidean norm. This is natural: we will prove that $\lim_{a\to 0}|x|_a = \|x\|_2$.

The decay in regimes (III) and (IV) is in fact expressed in terms of the continuum Green function for the Laplacian on $\mathbb{R}^d$: the massive Green function in regime (III) and massless Green function in regime (IV). These continuum Green functions appear explicitly in the full asymptotic formulas in these regimes. In both cases, the decay is Euclidean invariant and is expressed in terms of the ℓ_2 norm.

The transition from regime (II) to (III) can be anticipated by replacing a in (II) by $\|x\|_2^{-1}$, which corresponds to x on the order of the *correlation length* m_a^{-1}. This replacement causes the asymptotic formula in (II) to transform into the formula in (III).

More generally, for real numbers $q \in (0, \infty)$ we consider the decay of

$$C_a^{(q)}(x) = \int_{[-\pi,\pi]^d} \frac{e^{ik\cdot x}}{(a^2 + 1 - \hat{D}(k))^q} \frac{\mathrm{d}k}{(2\pi)^d} \tag{1.4}$$

in the above four regimes. When $q \geq 2$ is a positive integer, $C_a^{(q)}$ is the q-fold convolution of C_a with itself. For $q = 2, 3, 4$, $C_a^{(q)}(0)$ is known respectively as the bubble, triangle and square diagram. These diagrams play an important role in the study of various statistical mechanical models above their upper critical dimensions, especially when $a = 0$; see, e.g., Slade (2006). For integers $q \geq 2$, $C_0^{(q)}(x)$ is the critical lattice polyharmonic Green function. Polyharmonic functions have been widely studied, especially on $\mathbb{R}^d$ rather than on the lattice $\mathbb{Z}^d$ (e.g., Aronszajn et al. (1983)).

We note in passing that the lattice Green function has the following probabilistic interpretation. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with each X_i equally likely to be any one of the $2d$ unit vectors (positive or negative) in $\mathbb{Z}^d$, for any fixed integer $d \geq 1$. For $a \in [0, \infty)$, let N be a geometric random variable with

$$\mathbb{P}(N = n) = \Big(\frac{1}{1+a^2}\Big)^n \frac{a^2}{1+a^2} \qquad (n \geq 0), \tag{1.5}$$

with N independent of the X_i. Then $\mathbb{P}(N \geq n) = (\frac{1}{1+a^2})^n$. Let $S_0 = 0$, and consider the nearest-neighbour random walk $S_n = X_1 + \cdots + X_n$ on $\mathbb{Z}^d$ subjected to a-dependent killing, i.e., the walk takes N steps and then dies. Let $p_n(x)$ denote the probability that the random walk without killing makes a transition from 0 to x in n steps. The expected number of visits of the random walk to a point $x \in \mathbb{Z}^d$ is

$$\mathbb{E}\left(\sum_{n=0}^{N} \mathbb{1}_{S_n = x}\right) = \sum_{n=0}^{\infty} \Big(\frac{1}{1+a^2}\Big)^n p_n(x) \qquad (x \in \mathbb{Z}^d). \tag{1.6}$$

The expectation in (1.6) is equal to $(1+a^2)C_a^{(1)}(x)$, as a consequence of the fact that the Fourier transform of $p_n(x)$ is simply

$$\sum_{x \in \mathbb{Z}^d} p_n(x) e^{ik\cdot x} = \hat{D}(k)^n. \tag{1.7}$$

We do not consider more general random walks, which would correspond to operators other than the Laplacian. We expect that our results should extend to the Green function for random walks taking finite-range symmetric steps, but as can be seen in Aoun et al. (2021) the nature of the decay can change when arbitrarily long steps are permitted.

Our motivation to study the decay of the lattice Green function originates from statistical mechanics. The long-distance asymptotic behaviour of the two-point function is an essential feature in the analysis of critical phenomena in lattice statistical mechanical models such as percolation, the Ising model, or the self-avoiding walk. In high dimensions, $\|x\|_2^{-(d-2)}$ decay of the critical two-point function has been proved in several cases, including Hara et al. (2003); Hara (2008); Brydges et al. (2021); Sakai (2007). However, the near-critical behaviour, which merges the subcritical exponential decay and the power-law critical decay, has received scant attention despite the fact that it has the potential to reveal important and hitherto unstudied aspects of the critical behaviour, particularly for models defined on a torus. Recently progress has been made in this direction for weakly self-avoiding walk for dimensions $d > 4$ Michta and Slade (2021); Slade (2020) and percolation for $d > 6$ Hutchcroft et al. (2021). In high dimensions, where mean-field behaviour is known to occur, the near-critical two-point function is conjectured to have similar decay to that of the lattice Green function. It is therefore important to have a detailed understanding of the long-distance behaviour of the lattice Green function as a prototype. In this paper, we provide a comprehensive account of the decay of the lattice Green function.

1.2. *The anisotropic norm.* Lattice effects play a significant role in the asymptotic behaviour of $C_a^{(q)}(x)$ when $a > 0$ is fixed, and lead to anisotropy in the decay. The following definition enters into the description of the anisotropy.

Definition 1.1. Let $d \geq 1$ and $a \geq 0$. We define the *mass*, or *inverse correlation length*, to be the unique solution $m_a \geq 0$ of

$$\cosh m_a = 1 + da^2. \tag{1.8}$$

For nonzero $x \in \mathbb{R}^d$, let $u = u_a(x) \geq 0$ be the unique solution of

$$\frac{1}{d}\sum_{i=1}^{d} \sqrt{1 + x_i^2 u^2} = 1 + a^2, \tag{1.9}$$

which exists since the left-hand side of (1.9) is a strictly increasing function of $u \in [0, \infty)$ onto $[1, \infty)$. Finally, with the restriction now that $a > 0$, we define $|0|_a = 0$ and for nonzero $x \in \mathbb{R}^d$ define

$$|x|_a = \frac{1}{m_a}\sum_{i=1}^{d} x_i \operatorname{arcsinh}(x_i u_a(x)). \tag{1.10}$$

It follows from (1.8) and Taylor's theorem that, as $a \to 0$,

$$m_a = \sqrt{2d}\,a(1 + O(a^2)). \tag{1.11}$$

Equation (1.10) defines a norm on $\mathbb{R}^d$ whose properties are indicated in Proposition 1.2. In particular, the norm $|\cdot|_a$ interpolates between the ℓ_1 norm when $a = \infty$ and the ℓ_2 norm when $a = 0$. The norm's unit ball in dimensions $d = 2, 3$ is depicted in Figure 1.1.

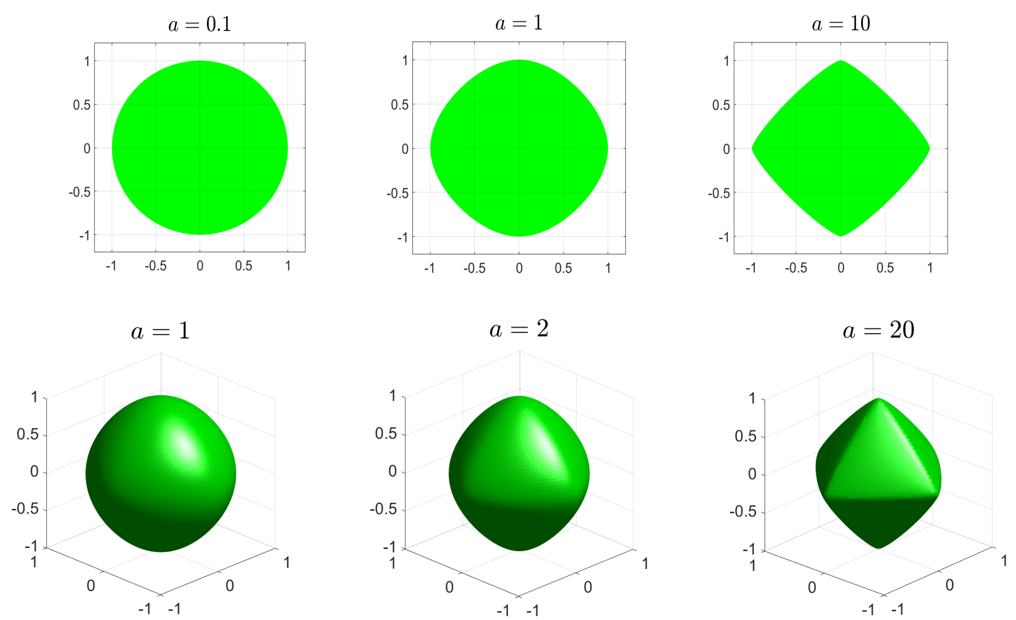

FIGURE 1.1. Unit ball for the norm $|\cdot|_a$ in dimensions $d = 2, 3$.

Proposition 1.2. *Let $d \geq 1$ and $a > 0$. The function $|\cdot|_a$ defines a norm on $\mathbb{R}^d$ which is monotone increasing in a and for all $x \in \mathbb{R}^d$ obeys*

$$\lim_{a\to 0} |x|_a = \|x\|_2, \qquad \lim_{a\to\infty} |x|_a = \|x\|_1, \tag{1.12}$$

in fact $|x|_a = \|x\|_2(1 + O(a^2))$ *uniformly in* $x \neq 0$. *In particular,*

$$\|x\|_2 \le |x|_a \le \|x\|_1. \tag{1.13}$$

By definition, $m_a(0) = 0$ and m_a is a strictly positive strictly increasing function of $a > 0$. To understand why the factor m_a^{-1} appears on the right-hand side of (1.10), we note that for any $a > 0$ and for any unit vector $e_j \in \mathbb{R}^d$,

$$u_a(e_j) = \sqrt{(1 + da^2)^2 - 1} = \sqrt{\cosh^2 m_a - 1} = \sinh m_a, \tag{1.14}$$

and hence, for all $a > 0$,

$$|e_j|_a = 1. \tag{1.15}$$

The $|\cdot|_a$ norm originated in the analysis of the 2-dimensional Ising model McCoy and Wu (1973, pp. 302–303), although it was not identified there as a norm. A proof that it defines a norm was given in Pfister (1991, Lemma 6.5); there the proof of the triangle inequality was based on the second Griffiths inequality applied to the 2-dimensional Ising model. We provide a simple alternate proof based on a random walk argument. We do not know of any direct proof of the triangle inequality based on the definition of the norm. Neither are we aware of any prior proof of the monotonicity of the norm.

1.3. *The continuum Green function.* In Appendix A we consider and interpret the integral

$$G_s^{(q)}(x) = \int_{\mathbb{R}^d} \frac{e^{ik\cdot x}}{(\frac{1}{2d}\|k\|_2^2 + s^2)^q} \frac{\mathrm{d}k}{(2\pi)^d} \qquad (x \in \mathbb{R}^d \setminus \{0\}), \tag{1.16}$$

which in the case $q = 1$ is the Green function for the (normalised) continuum Laplace operator on $\mathbb{R}^d$. It follows from (1.16) that there is a scaling relation

$$G_s^{(q)}(x) = s^{d-2q} G_1^{(q)}(sx) \qquad (s > 0). \tag{1.17}$$

In Appendix A, we recall the elementary proof that for $s > 0$, integers $d \ge 1$, and nonzero $q \in \mathbb{R}$, the massive and massless continuum Green functions are given explicitly (in the sense of tempered distributions) by (1.17) together with

$$G_1^{(q)}(x) = \frac{2d^q}{\Gamma(q)(2\pi)^{d/2}} \left(\frac{\sqrt{2d}}{\|x\|_2} \right)^{(d-2q)/2} K_{(d-2q)/2}(\sqrt{2d}\|x\|_2), \tag{1.18}$$

$$G_0^{(q)}(x) = \frac{d^q \Gamma(\frac{d-2q}{2})}{2^q \pi^{d/2} \Gamma(q)} \frac{1}{\|x\|_2^{d-2q}}, \tag{1.19}$$

where K_α is the modified Bessel function of the second kind and for (1.19) we restrict to $d > 2q$. For $\alpha > 0$ the asymptotic behaviour of K_α is known to be

$$K_\alpha(z) \sim \frac{\Gamma(\alpha)}{2}\left(\frac{2}{z}\right)^\alpha \quad (z \to 0), \qquad K_\alpha(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad (z \to \infty). \tag{1.20}$$

1.4. *Asymptotic behaviour.*

1.4.1. *Main result.* The following theorem gives a precise statement of the asymptotic decay of the lattice Green function for arbitrary dimension $d \ge 1$ and for $a \ge 0$ (possibly n-dependent). The norm $|\cdot|_a$ plays a key role in the anisotropic limit (1.21), for which lattice effects persist when a is independent of n. Recall that $m_a = \operatorname{arccosh}(1 + da^2)$ for $a \ge 0$. We write $f(n) \sim g(n)$ to mean $\lim_{n\to\infty} f(n)/g(n) = 1$.

Theorem 1.3. *Let $d \geq 1$ and $q \in (0, \infty)$ (not necessarily integer). Fix $x \in \mathbb{Z}^d \setminus \{0\}$.*
(i) (Ornstein–Zernike decay). *Let $a_n \in (0, \infty)$ and suppose that $a_n n \to \infty$ in such a manner that a_n remains bounded (this includes in particular the case of fixed $a_n = a \in (0, \infty)$). There exists $c_{a,q,\hat{x}} > 0$ depending on a, q, and the direction $\hat{x} = x/|x|_a$ (and on the dimension d), such that, as $n \to \infty$,*

$$C_{a_n}^{(q)}(nx) \sim c_{a_n,q,\hat{x}} \frac{m_{a_n}^{(d-1-2q)/2}}{(n|x|_{a_n})^{(d+1-2q)/2}} \, e^{-m_{a_n} n|x|_{a_n}}. \tag{1.21}$$

The ratio of the above left- and right-hand sides converges to 1 uniformly in nonzero x, and the constant $c_{a,q,\hat{x}}$ has the explicit $\hat{x}$-independent limit

$$c_{0,q} = \lim_{a \to 0} c_{a,q,\hat{x}} = \frac{d^q}{(2\pi)^{(d-1)/2}\Gamma(q)}. \tag{1.22}$$

(ii) (Critical decay). *Let $a_n = s/n$ with $s \in [0, \infty)$, with $d > 2q$ if $s = 0$. Then, as $n \to \infty$,*

$$C_{a_n}^{(q)}(nx) \sim \frac{1}{n^{d-2q}} G_s^{(q)}(x). \tag{1.23}$$

The asymptotic formula (1.21) encompasses both regimes (I) and (II) mentioned in Section 1.1. The anisotropic OZ regime (I) is the case of fixed $a_n = a > 0$, for which the anisotropic norm $|x|_a$ plays a role.

For the isotropic OZ regime (II), we are interested in the case where $a_n \to 0$ in such a way that $a_n n \to \infty$ and $a_n^3 n \to 0$. Recall from (1.11) and Proposition 1.2 that $m_a = \sqrt{2d}a(1 + O(a^2))$ and $|x|_a = \|x\|_2(1 + O(a^2))$. Consequently, as $a_n \to 0$ we have

$$m_{a_n} n|x|_{a_n} = \sqrt{2d} a_n n \|x\|_2 (1 + O(a_n^2)) \tag{1.24}$$

and hence it follows from (1.21) that if $a_n n \to \infty$ then

$$C_{a_n}^{(q)}(nx) \sim c_{0,q} \frac{(\sqrt{2d}a_n)^{(d-1-2q)/2}}{(n\|x\|_2)^{(d+1-2q)/2}} e^{-\sqrt{2d}a_n n \, \|x\|_2 [1+O(a_n^2)]}. \tag{1.25}$$

If we now assume additionally that $a_n^3 n \to 0$ then the error term in the exponential can be neglected and we obtain the result claimed for regime (II), namely

$$C_{a_n}^{(q)}(nx) \sim c_{0,q} \frac{(\sqrt{2d}a_n)^{(d-1-2q)/2}}{(n\|x\|_2)^{(d+1-2q)/2}} e^{-\sqrt{2d}a_n n \, \|x\|_2}. \tag{1.26}$$

If the condition $a_n^3 n \to 0$ is violated then we see from (1.25) that modifications to the exponential decay will occur from the error term in the exponent.

The massive critical regime (III) and the massless critical regime (IV) are respectively the $s > 0$ and $s = 0$ cases of (1.23). There is coherence between regimes (II) and (III) in the sense that if a_n in (II) is replaced by s/n then the exponential factor becomes a constant and the powers $a_n^{(d-1-2q)/2} n^{-(d+1-2q)/2}$ reduce to an s-dependent multiple of $n^{-(d-2q)}$. The continuum Green function $G_s^{(q)}(x)$ provides the amplitude for the asymptotic decay in the critical regimes. There is no statement of uniformity in x in (1.23) because uniformity is impossible for $s > 0$: e.g., if $x = n^2 y$ with y independent of n then $\frac{s}{n}x = sny \to \infty$ as $n \to \infty$ and we are actually in regime (I), not regime (III).

1.4.2. *Previous results.* The proof of Theorem 1.3 is based on the representation

$$C_a^{(q)}(x) = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-(a^2+1)t} \prod_{j=1}^d I_{x_j}(t/d) \mathrm{d}t \tag{1.27}$$

in terms of the modified Bessel function of the first kind. Much of Theorem 1.3 has been proved previously by other authors, and we now describe what was done previously and how our approach simplifies, extends and unifies earlier work.

For $q = 1$ and for fixed $a_n = a > 0$, the asymptotic formula (1.21) is proved in Molchanov and Yarovaya (2012, Theorem 3.2) for $d \geq 1$, and for $d = 2$ in Messikh (2006, Proposition 13). Neither of those references identified the role of the anisotropic norm in (1.21), and the norm makes the statement significantly more transparent. In Molchanov and Yarovaya (2012, Theorem 3.3), (1.26) is stated to hold in the limit in which $a_n \to 0$ with $a_n n \to \infty$; in fact the further restriction $a_n^3 n \to 0$ is necessary for the simplification of the exponential in (1.21) to yield the isotropic form (1.26). Our method of proof is based on the Laplace method as in Molchanov and Yarovaya (2012) but it is simplified by our appeal to well-established properties of the modified Bessel function rather than deriving them as part of the proof as in Molchanov and Yarovaya (2012). Also, unlike the separate proofs for the anisotropic and isotropic cases in Molchanov and Yarovaya (2012), we give one unified proof.

The massive critical regime was considered in Paladini and Sexton (1999) (indeed these authors computed higher-order terms as well), but the arguments used in Paladini and Sexton (1999) do not constitute a proof. The formula (1.23) for $q = 1$ and $s > 0$ can be inferred from the statement of Deng et al. (2022, Proposition 3.1), which is proved via the local central limit theorem. Our proof, which again uses known properties of the modified Bessel function, involves a straightforward application of the dominated convergence theorem and does not involve the Laplace method.

For the massless critical regime (IV), the asymptotic behaviour of the critical lattice polyharmonic Green function is given in Mangad (1967) as

$$C_0^{(q)}(x) \sim \frac{d^q \Gamma(\frac{d-2q}{2})}{2^q \pi^{d/2} \Gamma(q)} \frac{1}{\|x\|_2^{d-2q}} \qquad (q = 1, 2, 3, \ldots;\ d > 2q) \tag{1.28}$$

with explicit higher-order correction term. Since higher-order terms are known we make no effort here to compute them, as our focus in the proof is on simplicity. We prove (1.28) as the $s = 0$ case of (1.23) for arbitrary real $q \in (0, \infty)$ when $d > 2q$. This special case of our proof of (1.23) in the entire critical regime $s \geq 0$ does not require separate attention. When $q = 1$, (1.28) gives the well-known decay of the critical lattice Green function. In fact, the $\|x\|_2^{-(d-2)}$ decay in (1.28) for $q = 1$ holds more generally under a second-moment condition for D_{xy} (recall (1.3)), with error term of order $\|x\|_2^{-d}$ with known coefficient. For $q = 1$ see, e.g., Lawler and Limic (2010, p. 82) or Molchanov and Yarovaya (2012, Theorem 3.4), or Spitzer (1976, p. 308) for $d = 3$, and for further error terms see Uchiyama (1998). A version of (1.28) for $q = 1$ holds under certain conditions even when the transition matrix D_{xy} is permitted to assume negative values Hara (2008).

1.4.3. *Explicit calculation for $d = 1$.* For $d = 1$ and integers $q \geq 1$, the condition $d = 1 > 2q$ is violated and $C_0^{(q)} = \infty$, so regime (IV) does not apply. The computation of $C_a^{(q)}(x)$ for $d = 1$, integer $q \geq 1$, and $a > 0$ can be done explicitly with the result that

$$\int_{-\pi}^{\pi} \frac{e^{ikx}}{(1 + a^2 - \cos k)^q} \frac{dk}{2\pi} = \frac{e^{-m_a |x|}}{\sinh^q m_a} \sum_{l=0}^{q-1} \binom{|x| + q - 1}{q - 1 - l} \binom{q - 1 + l}{l} \Big(\frac{e^{-m_a}}{2 \sinh m_a} \Big)^l \tag{1.29}$$

with $m_a = \operatorname{arccosh}(1 + a^2)$. The above formula can be verified by residue calculus or by an appropriate rewriting of the formula Gradshteyn and Ryzhik (2007, (3.616.7)). In detail, the cases

$q = 1$ and $q = 2$ are

$$C_a^{(1)}(x) = \frac{e^{-m_a|x|}}{\sinh m_a} \qquad (d = 1), \tag{1.30}$$

$$C_a^{(2)}(x) = \frac{|x|e^{-m_a|x|}}{\sinh^2 m_a}\left[1 + \frac{1}{|x|}\Big(1 + \frac{e^{-m_a}}{\sinh m_a}\Big)\right] \qquad (d = 1). \tag{1.31}$$

Both of the formulas (1.30)–(1.31) refine and are consistent with (1.21) and (1.23) from Theorem 1.3. In particular, for $a = s/n$ with fixed $s > 0$, (1.30) gives

$$C_{s/n}^{(1)}(nx) \sim \frac{n}{\sqrt{2}s} e^{-\sqrt{2}s|x|}, \tag{1.32}$$

and since $K_{-1/2}(y) = K_{1/2}(y) = \pi^{1/2}(2y)^{-1/2}e^{-y}$ for $y > 0$ (see Gradshteyn and Ryzhik (2007, 8.432.8)) this agrees with (1.23).

1.4.4. *Ornstein–Zernike vs critical decay.* In the physics literature, the inverse mass $\xi_a = m_a^{-1}$ is known as the *correlation length*. With $a = s/n$ and $s > 0$, Theorem 1.3 can then be interpreted informally as identifying the following decay of the lattice Green function:

$$s > 0 \qquad n\|x\|_2 \asymp \xi_a \qquad \text{massive continuum limit}, \tag{1.33}$$

$$s \to \infty \qquad n\|x\|_2 \gg \xi_a \qquad \text{Ornstein–Zernike decay}. \tag{1.34}$$

For the latter case, we see the Euclidean invariance if $s = o(n)$ but not for $s = an$ with fixed $a > 0$.

The Ornstein–Zernike and critical regimes occur in general dimensions in lattice statistical mechanical models such as the self-avoiding walk, percolation, and the Ising model Campanino et al. (2008); Chayes and Chayes (1986); Campanino et al. (1991). This perspective is standard in the physics literature but a mathematical description of the near-critical behaviour which crosses over between the two regimes is lacking in most examples, even in high dimensions where the lace expansion applies. The asymptotic formula (1.21) provides a prototype for what can be expected for the near-critical two-point functions of the high-dimensional statistical mechanical models.

The bounds in regimes (I)–(II) in general do *not* hold uniformly in all $a > 0$, $n \ge 1$, and nonzero $x \in \mathbb{Z}^d$. This is evident in the explicit formula (1.31) for $d = 1$ and $q = 2$, where the first term $|x|\frac{e^{-m_a|x|}}{\sinh^2 m_a}$ dominates when $x \to \infty$ with fixed a, in agreement with (1.21), whereas with fixed x, in the limit $a \to 0$ we have $m_a \sim \sqrt{2}a \to 0$, the exponentials become insignificant, and (1.31) is dominated by the factor $\sinh^{-3} m_a \sim (\sqrt{2}a)^{-3}$ arising from its last term. This shows the impossibility for this case of an upper bound of the form $|x|a^{-2}e^{-m_a|x|}$ that is uniform in both x and a.

Similarly, for $d > 3$ and $q = 1$, there can be no upper bound on $C_a^{(1)}$ of the form

$$m_a^{(d-3)/2} \frac{1}{|x|_a^{(d-1)/2}} e^{-m_a|x|_a} \tag{1.35}$$

that is uniform in all $a > 0$ and nonzero $x \in \mathbb{Z}^d$, because (1.35) vanishes as $a \to 0$ with fixed x due to the factor $m_a^{(d-3)/2}$, whereas if $|x|_a$ grows like m_a^{-1} then $C_a^{(1)}$ is in regime (III) and decays as a multiple of $\|x\|_2^{-(d-2)}$.

It remains an open problem to determine for which values of d, q the formula (1.21) in fact gives a bound which is unform in $a > 0$ and nonzero x. On the other hand, for $q = 1$ and $d > 2$ an upper bound that *is* uniform in $a \ge 0$ and in x is given in Slade (2020, Proposition 2.1), which asserts that there are constants $\kappa_1 > 0$ and $\kappa \in (0, 1)$ such that for all $a \ge 0$ and all $x \ne 0$,

$$C_a^{(1)}(x) \le \kappa_1 \frac{1}{|x|_a^{d-2}} e^{-\kappa m_a|x|_a}. \tag{1.36}$$

(By changing the constants, another norm than $|x|_a$ could be used in the above.) As in Michta and Slade (2021, Lemma 3.3), the inequality (1.36) easily implies that for general integers $q \geq 1$ and dimensions $d > 2q$,

$$C_a^{(q)}(x) \leq \kappa_q \frac{1}{|x|_a^{d-2q}} e^{-\kappa m_a |x|_a}. \tag{1.37}$$

The uniform upper bound (1.36) combines the critical $|x|_a^{-(d-2)}$ decay with the exponential decay for $a > 0$. The relaxation of the exponential decay via $\kappa < 1$ compensates for the differing power laws in (1.37) and in regime (I). Bounds of the form (1.36) have been proved and applied to analyse the critical behaviour of weakly self-avoiding walk in dimensions $d > 4$ Michta and Slade (2021) and Slade (2020) and of percolation in dimensions $d > 6$ Hutchcroft et al. (2021).

1.5. *Organisation.* The remainder of the paper is organised as follows.

In Section 2, we give the elementary derivation of the representation (1.27) of $C_a^{(q)}(x)$ in terms of the modified Bessel function I_ν. This representation in terms of a 1-dimensional integral is the basis for all of our analysis. We then recall properties of I_ν which enable the asymptotic evaluation of the integral (1.27).

In Section 3, we prove Theorem 1.3(i), pertaining to the Ornstein–Zernike regime. In this regime, the Bessel integral (1.27) has an exponential factor in the integrand which makes it amenable to application of the Laplace method. The norm $|\cdot|_a$ emerges naturally from a computation involving the critical point which dominates the behaviour arising in the Laplace method.

In Section 4, we prove Theorem 1.3(ii), pertaining to the critical regime. In the critical regime, there is no longer any exponential behaviour in the integrand of the Bessel integral (1.27) and there is no need for the Laplace method. Given the well-known asymptotics for I_ν recalled in Section 2, the proof follows quickly from the dominated convergence theorem.

Finally, in Appendix A we provide an elementary proof that the formulas (1.18)–(1.19) for the continuum Green function are equal to the integral (1.16) over $\mathbb{R}^d$ in the sense of tempered distributions, and in Appendix B we discuss properties of the modified Bessel function.

2. Bessel representation

For any integer $\nu \geq 0$ and $t \in \mathbb{R}$ the modified Bessel function of the first kind $I_\nu(t)$ is given by

$$I_\nu(t) = \frac{1}{\pi} \int_0^\pi e^{t\cos\theta + i\nu\theta} \mathrm{d}\theta. \tag{2.1}$$

For our purposes it is more useful to consider

$$\bar{I}_\nu(t) = e^{-t} I_\nu(t) = \frac{1}{\pi} \int_0^\pi e^{-t(1-\cos\theta) + i\nu\theta} \mathrm{d}\theta \tag{2.2}$$

which has the exponential growth of $I_\nu(t)$ cancelled. The following lemma provides the well-known integral representation that is the foundation for the proof of Theorem 1.3.

Lemma 2.1. *For $d \geq 1$, $a \geq 0$, $q > 0$, $x \in \mathbb{Z}^d$, and with the restriction $d > 2q$ when $a = 0$,*

$$C_a^{(q)}(x) = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-a^2 t} \prod_{j=1}^d \bar{I}_{x_j}(t/d) \mathrm{d}t. \tag{2.3}$$

Proof: Let $\hat{F}(k) = a^2 + 1 - \hat{D}(k)$. We use the identity

$$\frac{1}{v^q} = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-tv} \mathrm{d}t \qquad (v > 0) \tag{2.4}$$

in the definition (1.4) to obtain

$$C_a^{(q)}(x) = \int_{[-\pi,\pi]^d} \frac{e^{ik\cdot x}}{\hat{F}(k)^q} \frac{\mathrm{d}k}{(2\pi)^d}$$
$$= \frac{1}{\Gamma(q)} \int_{[-\pi,\pi]^d} \int_0^\infty t^{q-1} e^{-t\hat{F}(k)} \mathrm{d}t \, e^{ikx} \frac{\mathrm{d}k}{(2\pi)^d}$$
$$= \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-a^2 t} \prod_{j=1}^d \bar{I}_{x_j}(t/d) \mathrm{d}t, \tag{2.5}$$

and the proof is complete. Note that there is no issue with convergence of this last integral at $t = 0$, and for large t convergence is guaranteed (assuming $d > 2q$ when $a = 0$) by the fact that $\bar{I}_\nu(z) \sim (2\pi z)^{-1/2}$ as $z \to \infty$. In particular, this justifies the above application of Fubini's Theorem. □

To study the Ornstein–Zernike regime we apply the change of variable $t = dnv$ to the integral representation (2.3) to obtain

$$C_a^{(q)}(nx) = \frac{d^q n^q}{\Gamma(q)} \int_0^\infty v^{q-1} e^{-dna^2 v} \prod_{j=1}^d \bar{I}_{nx_j}(nv) \mathrm{d}v. \tag{2.6}$$

For the continuum regime we will also make the replacement $v = nt/d$ in (2.6) and use

$$C_{s/n}^{(q)}(nx) = \frac{n^{2q}}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-s^2 t} \prod_{j=1}^d \bar{I}_{nx_j}(n^2 t/d) \, \mathrm{d}t. \tag{2.7}$$

To study the integrals (2.6)–(2.7) we will make use of well-established asymptotic properties for I_ν. To state these properties, for $\nu > 0$ and $t > 0$ we define

$$L_\nu(t) = \frac{1}{(2\pi\nu)^{1/2}} \frac{e^{\nu\psi(t)}}{(1+t^2)^{1/4}}, \qquad \psi(t) = -t + \sqrt{1+t^2} + \log\Big(\frac{t}{1+\sqrt{1+t^2}}\Big). \tag{2.8}$$

The identity $\log(\frac{t}{1+\sqrt{1+t^2}}) = -\operatorname{arcsinh}(t^{-1})$ gives a useful alternate representation for ψ. The first three derivatives of ψ are:

$$\psi'(t) = -1 + \sqrt{1+t^{-2}}, \tag{2.9}$$
$$\psi''(t) = -\frac{t^{-3}}{\sqrt{1+t^{-2}}}, \tag{2.10}$$
$$\psi'''(t) = \frac{2t^{-6} + 3t^{-4}}{(1+t^{-2})^{3/2}}. \tag{2.11}$$

The following lemma gives asymptotic representations of the Bessel function of large argument and large order. The proof of the lemma is deferred to Appendix B. We use (2.12) for the OZ regime and (2.13)–(2.14) for the continuum regime.

Lemma 2.2. *As $\nu \to \infty$,*

$$\bar{I}_\nu(\nu t) = L_\nu(t)(1 + o(1)) \tag{2.12}$$

where the $o(1)$ is uniform in $t > 0$. Also, as $\nu \to \infty$, for any $s > 0$,

$$\bar{I}_\nu(\nu^2 s) \sim \frac{e^{-1/2s}}{\nu(2\pi s)^{1/2}}, \tag{2.13}$$

with an error that is not uniform in s. Finally, there exist $C, \delta, \nu_0 > 0$ such that

$$\bar{I}_\nu(\nu^2 s) \le C\Big(e^{-\delta\nu} \mathbb{1}_{2\nu s < 1} + \nu^{-1} s^{-1/2} e^{-\delta/s} \mathbb{1}_{2\nu s \ge 1}\Big) \qquad (\nu \ge \nu_0, \ s > 0). \tag{2.14}$$

3. Ornstein–Zernike regime: Proof of Theorem 1.3(i)

In this section, we prove Theorem 1.3(i). Let x be a vector in $\mathbb{Z}^d \setminus \{0\}$, and without loss of generality assume that $x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$. We write r for the number of nonzero components of x. Throughout this section, we consider a bounded sequence $a_n \in (0, a_{\max}]$ with $a_n n \to \infty$. To lighten the notation, we write simply a in place of a_n. In particular, a can be independent of $n \to \infty$, or we can have $a \to 0$ as long as $an \to \infty$.

We start with (2.6), which states that

$$C_a^{(q)}(nx) = \frac{d^q n^q}{\Gamma(q)} \int_0^\infty v^{q-1} e^{-dna^2 v} (\bar{I}_0(nv))^{d-r} \prod_{j=1}^r \bar{I}_{nx_j}(nv) \mathrm{d}v. \tag{3.1}$$

With the asymptotic formula for $\bar{I}_\nu(\nu t)$ from (2.12) together with the definitions of L_ν and ψ from (2.8), after some algebra this leads to

$$C_a^{(q)}(nx) = (1+\delta_n)\alpha_q n^{q-r/2} \int_0^\infty h_{n,x}(v) e^{-ng_{a,x}(v)} \mathrm{d}v, \tag{3.2}$$

where $\delta_n \to 0$ (uniformly in nonzero x) and

$$\alpha_q = \frac{d^q}{(2\pi)^{d/2}\Gamma(q)}, \tag{3.3}$$

$$h_{n,x}(v) = v^{q-1}(\sqrt{2\pi}\bar{I}_0(nv))^{d-r} \prod_{j=1}^r \frac{1}{(x_j^2+v^2)^{1/4}}, \tag{3.4}$$

$$g_{a,x}(v) = da^2 v - \sum_{j=1}^r x_j \psi(v/x_j). \tag{3.5}$$

We first solve $g'_{a,x}(v) = 0$. By definition of $g_{a,x}$ and by (2.9),

$$g'_{a,x}(v) = d(1+a^2) - \sum_{j=1}^d \sqrt{1+v^{-2}x_j^2}. \tag{3.6}$$

By the definition of $u_a(x)$ in (1.9), we see that the unique solution of the equation $g'_{a,x}(v) = 0$ is $v_a(x) = u_a^{-1}$, where for notational convenience we write $u_a(x)$ simply as u_a. We will soon see that this solution is the location of the unique minimum of $g_{a,x}$. Since we are allowing the variable a to go to zero, which sends $v_a(x)$ to infinity, it is convenient to relocate this minimum to 1. We therefore rescale the representation (3.2) via $v = y/u_a$ and obtain

$$C_a^{(q)}(nx) = (1+\delta_n)\alpha_q \Big(\frac{n}{u_a}\Big)^{q-d/2} \int_0^\infty \bar{h}_{n,a,x}(y) e^{-n\bar{g}_{a,x}(y)} \mathrm{d}y \tag{3.7}$$

with

$$\bar{h}_{n,a,x}(y) = h_n(y/u_a) n^{(d-r)/2} u_a^{q-1-d/2}, \quad \bar{g}_{a,x}(y) = g_{a,x}(y/u_a). \tag{3.8}$$

The minimum of $\bar{g}_{a,x}$ is located exactly at 1, as is illustrated in Figure 3.2.

Recall the norm $|x|_a$ from Definition 1.1. We write $\hat{x} = x/|x|_a$ and $\hat{u}_a = u_a(\hat{x})$. The scaling relation $\lambda u_a(\lambda x) = u_a(x)$ for all $\lambda > 0$ follows from the definition of $u_a(x)$ in (1.9), and implies that $u_a x_j = \hat{u}_a \hat{x}_j$ and $\hat{u}_a = |x|_a u_a$. The definitions lead to

$$\bar{g}_{a,x}(y) = |x|_a \, \bar{g}_{a,\hat{x}}(y), \tag{3.9}$$

$$\bar{h}_{n,a,x}(y) = y^{q-1}\Big(\sqrt{2\pi n/u_a}\,\bar{I}_0(ny/u_a)\Big)^{d-r} \prod_{j=1}^r \frac{1}{(y^2+\hat{u}_a^2\hat{x}_j^2)^{1/4}}. \tag{3.10}$$

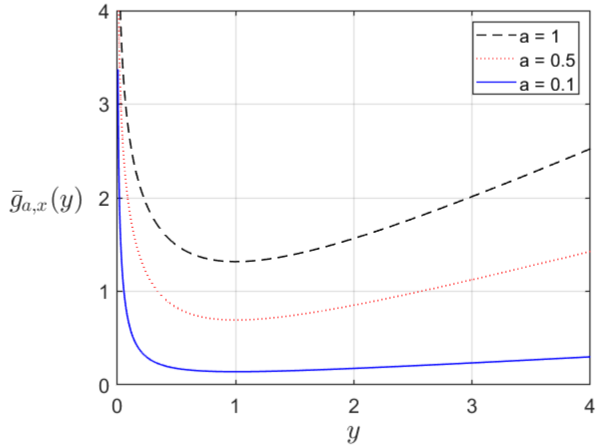

FIGURE 3.2. Plot of $\bar g_{a,x}$ for different values of a, for $d = 1$ and $x = 1$.

The a-dependence of $\bar g''_{a,x}(y)$ hinders an immediate application of a standard theorem for the Laplace method such as Olver (1997, Theorem 7.1, p. 127), so we prove Theorem 1.3(i) by analysing the integral in (3.7) directly. To do so, we require the detailed understanding of the $\bar g_{a,x}$ that we present next. As a preliminary, we note that it follows from the definition of u_a in (1.9) that $\hat u_a = O(a)$ as $a \to 0$ uniformly in $x \neq 0$, and moreover that

$$\hat u_a = \sqrt{2d}\frac{a}{\|\hat x\|_2}(1 + O(a^2)) \quad \text{uniformly in } x \neq 0. \tag{3.11}$$

Lemma 3.1. *Let $a > 0$. The function $\bar g_{a,x}$ is convex and attains its unique minimum on $(0,\infty)$ at 1, with $\bar g_{a,x}(1) = m_a|x|_a$. Also*

$$\bar g''_{a,x}(y) = |x|_a\, \bar g''_{a,\hat x}(y) = |x|_a \hat u_a \sum_{j=1}^{d} \frac{\hat x_j^2 y^{-3}}{\sqrt{1+\hat u_a^2 \hat x_j^2 y^{-2}}}, \tag{3.12}$$

$$\bar g'''_{a,x}(y) = |x|_a\, \bar g'''_{a,\hat x}(y) = -|x|_a \hat u_a \sum_{j=1}^{d} \frac{3\hat x_j^2 y^{-4} + 2\hat u_a^2 \hat x_j^4 y^{-6}}{(1+\hat u_a^2 \hat x_j^2 y^{-2})^{3/2}}. \tag{3.13}$$

In addition, for any $\alpha \in \mathbb{R}$ and any $n \geq 1$,

$$\lim_{y\to\infty} y^\alpha e^{-n\bar g_{a,x}(y)} = 0. \tag{3.14}$$

Finally, if $\alpha < 0$ then the maximum of $y \mapsto y^\alpha e^{-n\bar g_{a,x}(y)}$ for $y \in (0,1]$ is uniquely attained and lies in the interval $[\frac{1}{2}, 1]$ provided that a is bounded and an is sufficiently large (depending on α but not on nonzero x).

Proof: By definition, $\bar g_{a,x}(1) = g_{a,x}(1/u_a)$ and

$$\begin{aligned} g_{a,x}(1/u_a) &= u_a^{-1}\Big[d(a^2+1) - \sum_{j=1}^{d}\sqrt{1+u_a^2 x_j^2}\,\Big] + \sum_{j=1}^{d} x_j \operatorname{arcsinh}(u_a x_j) \\ &= \sum_{j=1}^{d} x_j \operatorname{arcsinh}(u_a x_j) = m_a|x|_a, \end{aligned} \tag{3.15}$$

with the last equality due to the definition of the norm in (1.10). For the derivatives we use (3.9) together with (3.5) and the expressions for ψ'' and ψ''' given in (2.10)–(2.11). In particular, it follows from (3.12) that $\bar{g}_{a,x}$ is convex and therefore the unique critical point at 1 is the location of the unique minimum.

For (3.14), it suffices to consider $g_{a,x}$ since there is no claim of uniformity in a. It can be seen from the definition of ψ in (2.8) that $\psi(t) \to 0$ as $t \to \infty$. With the definition of $g_{a,x}$ in (3.5), this implies that $g_{a,x}(y) \sim da^2 y$ as $y \to \infty$, so (3.14) holds for any $n \geq 1$.

Finally, and most substantially, we let $\alpha < 0$ and prove that the maximum of $y \mapsto y^{\alpha} e^{-n\bar{g}_{a,x}(y)}$ for $y \in [0,1]$ is uniquely attained and lies in the interval $[\frac{1}{2}, 1]$, provided that $a = a_n$ is bounded and an is sufficiently large (depending on α but not on nonzero x). We write

$$y^{-|\alpha|} e^{-n\bar{g}_{a,x}(y)} = \exp[-n\varphi_{a,x,\alpha}(y)] \quad \text{with} \quad \varphi_{a,x,\alpha,n}(y) = \bar{g}_{a,x}(y) + \frac{|\alpha|}{n} \log y. \tag{3.16}$$

To find a critical point of $\varphi = \varphi_{a,x,\alpha,n}(y)$ we first observe, as in (3.6), that

$$\varphi'(y) = \frac{1}{u_a} F(y/u_a) \quad \text{with} \quad F(t) = d(1+a^2) - \sum_{j=1}^{d} \sqrt{1 + t^{-2} x_j^2} + \frac{|\alpha|}{n} t^{-1}. \tag{3.17}$$

Note that $F(u_a^{-1}) = |\alpha| u_a / n > 0$, and that $F(t) \sim -t^{-1}(\|x\|_1 - |\alpha|/n)$ as $t \to 0$ so $F(t) \to -\infty$ uniformly in $n \geq |\alpha| + 1$ and $x \neq 0$. To prove that φ has a unique critical point in $[0,1]$, it therefore suffices to prove that $F(t)$ is increasing on $t \in [0, u_a^{-1}]$. The derivative of F is

$$F'(t) = t^{-2} G(t) \quad \text{with} \quad G(t) = \sum_{j=1}^{d} \frac{t^{-1} x_j^2}{\sqrt{1 + t^{-2} x_j^2}} - \frac{|\alpha|}{n}. \tag{3.18}$$

By multiplying by t in the numerator and denominator of the above sum, we see that G is decreasing. As $t \to 0$, $G(t) \to \|x\|_1 - |\alpha|/n \geq 1 - |\alpha|/n > 0$ uniformly in $x \neq 0$ and in $n \geq |\alpha| + 1$. Also, since $u_a x_j = \hat{u}_a \hat{x}_j$,

$$G(1/u_a) = \sum_{j=1}^{d} \frac{\hat{u}_a \hat{x}_j x_j}{\sqrt{1 + \hat{u}_a^2 \hat{x}_j^2}} - \frac{|\alpha|}{n}. \tag{3.19}$$

Recall (3.11) and (1.13). The square root on the right-hand side is bounded above since a is bounded, so there is a constant $c_0 > 0$ such that, uniformly in nonzero x,

$$G(1/u_a) \geq c_0 \hat{u}_a \frac{\|x\|_2^2}{|x|_a} - \frac{|\alpha|}{n} \geq \hat{u}_a \Big(\frac{c_0 \|x\|_2^2}{\|x\|_1} - \frac{|\alpha|}{n \hat{u}_a} \Big). \tag{3.20}$$

This proves that $G(1/u_a) > 0$ for an large enough (independent of $x \neq 0$). Therefore $G(t) > 0$ for all $t \in [0, u_a^{-1}]$, which completes the proof that F is increasing on $[0, u_a^{-1}]$. As noted previously, this proves that there exists a unique $t^*(a,n,x) \in [0, u_a^{-1}]$ such that $F(t^*) = 0$.

To conclude, we now verify that $t^* \in [(2u_a)^{-1}, u_a^{-1}]$. It suffices to show that $F(1/(2u_a)) < 0$ if an is large enough (independent of $x \neq 0$). By definition of u_a,

$$\begin{aligned} F(1/(2u_a)) &= d(1+a^2) - \sum_{j=1}^{d} \sqrt{1 + 4u_a^2 x_j^2} + 2u_a \frac{|\alpha|}{n} \\ &= -\sum_{j=1}^{d} \Big(\sqrt{1 + 4\hat{u}_a^2 \hat{x}_j^2} - \sqrt{1 + \hat{u}_a^2 \hat{x}_j^2} \Big) + 2\hat{u}_a \frac{|\alpha|}{n|x|_a}. \end{aligned} \tag{3.21}$$

If a is bounded below away from zero then the last term on the right-hand side is as small as desired by taking n large, whereas the difference in the first term is bounded below by a positive constant, so $F(1/(2u_a)) < 0$ in this case. It therefore suffices to consider small a, for which we see that

$$F(1/(2u_a)) \le -c\sum_{j=1}^d \hat{u}_a^2\hat{x}_j^2 + 2\hat{u}_a\frac{|\alpha|}{n|x|_a} \le -c'\hat{u}_a^2 + 2\hat{u}_a\frac{|\alpha|}{n|x|_a} \le -\hat{u}_a^2\big(c' - \frac{2|\alpha|}{\hat{u}_a n}\big). \tag{3.22}$$

For an sufficiently large (independent of $x \neq 0$) we conclude that $F(1/(2u_a)) < 0$. This completes the proof. □

Next, we establish further properties of the functions $\bar{g}_{a,x}$ and $\bar{h}_{n,a,x}$. Let $\varepsilon > 0$ and set $A_2 = [1-\varepsilon, 1+\varepsilon]$. In the following, we are interested in the limit $\varepsilon \to 0$ and we write $o(1)$ for error terms that go to zero in this limit.

Properties of $\bar{g}_{a,x} = |x|_a\bar{g}_{a,\hat{x}}$. By Lemma 3.1, $\bar{g}_{a,x}$ is convex and has unique minimum $\bar{g}_{a,x}(1) = m_a|x|_a$. In particular, $\bar{g}_{a,\hat{x}}(1) = m_a$. Taylor expansion of $\bar{g}_{a,\hat{x}}$ about 1 gives

$$\bar{g}_{a,\hat{x}}(y) = m_a + \frac{1}{2!}\bar{g}''_{a,\hat{x}}(1)(y-1)^2 + \frac{1}{3!}g'''_{a,\hat{x}}(y^*)(y-1)^3 \tag{3.23}$$

for some y^* between 1 and y. We see from (3.11) and Lemma 3.1 that as $\varepsilon \to 0$ we have

$$\bar{g}''_{a,\hat{x}}(y) = \bar{g}''_{a,\hat{x}}(1)(1 + O(\varepsilon)), \tag{3.24}$$

$$\bar{g}'''_{a,\hat{x}}(y) = O(a), \tag{3.25}$$

uniformly in $y \in A_2$, in $a \le a_{\max}$, and in $x \neq 0$. By (3.12), $\bar{g}''_{a,\hat{x}}(1) \asymp a$ uniformly in $a \le a_{\max}$ and in x. For the endpoints of A_2, the above implies that there exists a constant $\gamma > 0$ such that, for ε sufficiently small

$$\bar{g}_{a,\hat{x}}(1 \pm \varepsilon) = \bar{g}_{a,\hat{x}}(1) + \frac{1}{2}\bar{g}''_{a,\hat{x}}(1)\varepsilon^2 + O(a)\varepsilon^3 \ge m_a + \gamma a\varepsilon^2, \tag{3.26}$$

$$\bar{g}'_{a,\hat{x}}(1 \pm \varepsilon) = \pm\bar{g}''_{a,\hat{x}}(1)\varepsilon_n + O(a\varepsilon^2) = O(a\varepsilon), \tag{3.27}$$

uniformly in $a \le a_{\max}$ and in $x \neq 0$.

Properties of $\bar{h}_{n,a,x}$. We first prove that, as $\varepsilon \to 0$,

$$\bar{h}_{n,a,x}(y) = (1 + o(1))\prod_{j=1}^d(1 + \hat{u}_a^2\hat{x}_j^2)^{-1/4} \quad \text{uniformly in } y \in A_2, \text{ in } a \le a_{\max}, \text{ and in } x \neq 0. \tag{3.28}$$

When $y \in A_2$, the ratio $y/u_a = y|x|_a/\hat{u}_a$ is bounded away from zero uniformly in $a \le a_{\max}$ and $x \neq 0$. The estimate (3.28) then follows from (3.10) and the fact (see Olver (1997, p. 83)) that

$$\sqrt{2\pi nt}\,\bar{I}_0(nt) = 1 + o(1) \text{ uniformly in } t \text{ bounded away from zero.} \tag{3.29}$$

Next, we claim that there is a $C > 0$ such that

$$\bar{h}_{n,a,x}(y) \le Cy^{q-1-d/2} \quad \text{uniformly in } y > 0, \text{ in } a \le a_{\max}, \text{ and in } x \neq 0. \tag{3.30}$$

To obtain (3.30), we use (3.10) and the fact that $\bar{I}_0(t) \le O(t^{-1/2})$ which can also be seen from Olver (1997, p. 83). Finally, we use (3.12) and (3.28) to see, after some algebra, that as $\varepsilon \to 0$ we have

$$\frac{\bar{h}_{n,a,x}(1)}{\sqrt{\bar{g}''_{a,x}(1)}} = \frac{1 + o(1)}{(|x|_a\bar{g}''_{a,\hat{x}}(1)\prod_{j=1}^d(1 + \hat{u}_a^2\hat{x}_j^2)^{1/2})^{1/2}} = \frac{\kappa_a(\hat{x})}{\sqrt{|x|_a\hat{u}_a}}(1 + o(1)), \tag{3.31}$$

with

$$\kappa_a(\hat{x}) = \Big(\sum_{j=1}^d \hat{x}_j^2\prod_{i\neq j}(1 + \hat{u}_a^2\hat{x}_i^2)^{1/2}\Big)^{-1/2}, \tag{3.32}$$

and where the $o(1)$ term goes to zero as $\varepsilon \to 0$ uniformly in $a \le a_{\max}$ and $x \neq 0$.

Proof of Theorem 1.3(i): Recall from (3.7) that

$$C_a^{(q)}(nx) = (1+\delta_n)\Big(\frac{n}{u_a}\Big)^{q-d/2}\alpha_q \int_0^\infty \bar h_{n,a,x}(y)e^{-n\bar g_{a,x}(y)}\mathrm{d}y. \tag{3.33}$$

With $c_{0,q} = \sqrt{2\pi}\,\alpha_q$ as in (1.22) (recall (3.3)), we define

$$c_{a,q,\hat x} = c_{0,q}\kappa_a(\hat x)\Big(\frac{\hat u_a}{m_a}\Big)^{(d-1-2q)/2}. \tag{3.34}$$

By definition, $c_{a,q,\hat x}$ depends on x only via its direction $\hat x$. Also, as stated in (1.22), $\lim_{a\to 0} c_{a,q,\hat x} = c_{0,q}$ due to (3.11), (3.32), the relation $m_a \sim \sqrt{2d}\,a$, and the fact that $\|\hat x\|_2 = \|x\|_2/|x|_a \to 1$ by Proposition 1.2. Our goal is to prove that (1.21) holds, which by (3.33) will follow if we prove that, uniformly in $x \neq 0$ and in $a \le a_{\max}$, as $an\to\infty$ we have

$$\begin{aligned}\alpha_q \int_0^\infty \bar h_{n,a,x}(y)e^{-n\bar g_{a,x}(y)}\mathrm{d}y &\sim \Big(\frac{u_a}{n}\Big)^{q-d/2} c_{a,q,\hat x}\frac{m_a^{(d-1-2q)/2}}{(n|x|_a)^{(d+1-2q)/2}}\, e^{-m_a n|x|_a}\\ &= c_{0,q}\frac{1}{\sqrt{n|x|_a\hat u_a}}\kappa_a(\hat x)e^{-m_a n|x|_a}\end{aligned} \tag{3.35}$$

(the equality holds by definition—recall that $u_a = \hat u_a/|x|_a$).

We set $\varepsilon_n = (an)^{-1/4}$, which does obey $\varepsilon_n \to 0$ as we imposed below (3.11), and we divide the interval of integration in (3.35) into three subintervals:

$$A_1 = [0, 1-\varepsilon_n], \qquad A_2 = [1-\varepsilon_n, 1+\varepsilon_n] \qquad A_3 = [1-\varepsilon_n, \infty). \tag{3.36}$$

Then we set

$$J_i = \alpha_q \int_{A_i} \bar h_{n,a,x}(y)e^{-n\bar g_{a,x}(y)}dy \qquad (i=1,2,3). \tag{3.37}$$

We will prove that J_2 gives the main contribution to (3.35), with J_1 and J_3 relatively small.

The integral J_2. By (3.23), (3.24), and (3.28),

$$J_2 = (1+o(1))\alpha_q \bar h_{n,a,x}(1)e^{-nm_a|x|_a}\int_{-\varepsilon_n}^{\varepsilon_n}\exp\Big(-\frac{n}{2}\bar g''_{a,x}(1)(1+o(1))y^2\Big)\mathrm{d}y, \tag{3.38}$$

with the $o(1)$ (as $\varepsilon_n\to 0$) uniform in y, in $a\le a_{\max}$, and in $x\neq 0$. We make the change of variables $v = y(n\bar g''_{a,x}(1))^{1/2}$ and obtain, with $M_n = \varepsilon_n(n\bar g''_{a,x}(1))^{1/2}$,

$$J_2 = (1+o(1))\alpha_q \frac{\bar h_{n,a,x}(1)}{\sqrt{n\bar g''_{a,x}(1)}}e^{-nm_a|x|_a}\int_{-M_n}^{M_n}\exp\Big(-\frac12(1+o(1))v^2\Big)\mathrm{d}v. \tag{3.39}$$

By our choice of ε_n, and by the fact that $\bar g''_{a,x}(1) = |x|_a \bar g''_{a,\hat x}(1) \asymp a|x|_a$ (as noted above (3.26)), there exists a $c>0$ such that

$$M_n \ge c\varepsilon_n(n|x|_a a)^{1/2} \ge c(na)^{1/4}\to\infty. \tag{3.40}$$

Since $\alpha_q(2\pi)^{1/2} = c_{0,q}$, this gives

$$J_2 = (1+o(1))c_{0,q}\frac{\bar h_{n,a,x}(1)}{\sqrt{n\bar g''_{a,x}(1)}}e^{-nm_a|x|_a}. \tag{3.41}$$

To obtain the desired right-hand side of (3.35), we replace the ratio in the above using (3.31).

It remains to show that the contributions from the integrals J_1 and J_3 are relatively small.

The integral J_1. To show that J_1 is relatively small compared to J_2, it suffices to prove that as $an\to\infty$

$$\int_0^{1-\varepsilon_n}\bar h_{n,a,x}(y)e^{-n\bar g_{a,x}(y)}\mathrm{d}y \le \frac{o(1)}{\sqrt{na|x|_a}}e^{-n\bar g_{a,x}(1)} \tag{3.42}$$

uniformly in $x \neq 0$. Let $\alpha = q - 1 - d/2$. By the upper bound $\bar{h}_{n,a,x}(y) \le C y^\alpha$ of (3.30), the above integral is at most

$$C \int_0^{1-\varepsilon_n} y^\alpha e^{-n\bar{g}_{a,x}(y)} \mathrm{d}y. \tag{3.43}$$

If $\alpha > -1$ then we simply bound the exponential by its maximum value to obtain an upper bound proportional to $\exp[-n\bar{g}_{a,x}(1-\varepsilon_n)]$. By (3.26), this gives an upper bound (with $\gamma > 0$)

$$e^{-nm_a|x|_a} e^{-n\gamma a \varepsilon_n^2 |x|_a} = e^{-nm_a|x|_a} e^{-\gamma (na)^{1/2}|x|_a} = o((na|x|_a)^{-1/2}) e^{-nm_a|x|_a}, \tag{3.44}$$

which is sufficient.

If instead $\alpha \le -1$, then we apply Lemma 3.1 to bound $y^\alpha e^{-n\bar{g}_{a,x}(y)}$ by its maximum which is attained on $[\frac{1}{2}, 1]$ and hence is at most $2^{|\alpha|} \exp[-n\bar{g}_{a,x}(1-\varepsilon_n)]$, and this is again sufficient to obtain (3.42). This proves that J_1 is negligible compared to J_2.

The integral J_3. We bound $\bar{h}(y)$ by y^α with $\alpha = q - 1 - d/2$. If $\alpha < -1$, so that y^α is integrable, then we simply extract additional exponential decay (compared to J_2) using (3.26) again. Then we integrate y^α over $[1, \infty]$ and obtain an upper bound of the form

$$C e^{-\gamma (na)^{1/2}|x|_a} = o\Big(\frac{1}{\sqrt{na|x|_a}}\Big), \tag{3.45}$$

which is sufficient for the case $\alpha < -1$.

If instead $\alpha \ge -1$ then we use integration by parts to reduce the power. For example, if $\alpha \in (-1, 0]$ then, with $t = 1 + \varepsilon_n$ and $f(y) = n\bar{g}_{a,x}(y)$, we use the facts that by (3.27) $f'(t) \ge c(na)^{3/4}|x|_a$ which eventually exceeds 1, that f' is positive and increasing on $[t, \infty)$, and that $\lim_{y\to\infty} y^\alpha e^{-f(y)} = 0$ by (3.14), to obtain

$$\int_t^\infty y^\alpha e^{-f(y)} \mathrm{d}y \le \frac{1}{f'(t)} \int_t^\infty y^\alpha f'(y) e^{-f(y)} \mathrm{d}y \le t^\alpha e^{-f(t)} + |\alpha| \int_t^\infty y^{\alpha-1} e^{-f(y)} \mathrm{d}y. \tag{3.46}$$

The term $t^\alpha e^{-f(t)}$ is bounded above by a multiple of $e^{-f(1+\varepsilon_n)}$, which is bounded as in (3.45). This process can be iterated to reduce the power of y to below -1, which we have seen to be sufficient.

This completes the proof. □

4. Continuum regime: Proof of Theorem 1.3(ii)

In this section, we prove Theorem 1.3(ii). The method of proof is different from the proof of Theorem 1.3(i) and relies instead on a dominated convergence argument which applies simultaneously for both $s > 0$ and $s = 0$.

We define the *heat kernel* (for the normalised Laplacian $\frac{1}{2d}\Delta_{\mathbb{R}^d}$)

$$p_t(x) = \Big(\frac{d}{2\pi t}\Big)^{d/2} e^{-d\|x\|_2^2/2t} \qquad (x \in \mathbb{R}^d,\ t > 0). \tag{4.1}$$

Recall the definitions of $G_s(x)$ and $G_0(x)$ in (1.18) and (1.19). The following representations of the continuum Green function will be useful. For the case $a = 0$ (with $d > 2q$), we observe that the change of variables $s = d\|x\|_2^2/2t$ leads to

$$\frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} p_t(x) = \frac{d^q \Gamma(\frac{d-2q}{2})}{2^q \pi^{d/2} \Gamma(q)} \frac{1}{\|x\|_2^{d-2q}} = G_0^{(q)}(x). \tag{4.2}$$

For $a > 0$, we use

$$\frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} e^{-ta^2} p_t(x) = \Big(\frac{d}{2}\Big)^{d/2} \frac{1}{\Gamma(q)} \frac{a^{d-2q}}{2^d \pi^{d/2}} \int_0^\infty \mathrm{d}s \frac{1}{s^{(d-2q)/2}} e^{-s-d(a\|x\|_2)^2/2s} = G_a(x), \tag{4.3}$$

where we applied the formula

$$K_\alpha(z) = \frac{1}{2}\Big(\frac{z}{2}\Big)^\alpha \int_0^\infty \frac{1}{t^{\alpha+1}} e^{-t-z^2/4t} \mathrm{d}t \qquad (\alpha \in \mathbb{R},\ z > 0) \tag{4.4}$$

from Gradshteyn and Ryzhik (2007, 8.432.6) for the last equality.

Proof of Theorem 1.3(ii): Fix $x \neq 0$; without loss of generality we may assume that $x_j \geq 0$ for all j. We are interested in $a = s/n$ with some fixed $s \geq 0$. We rewrite (2.7) as

$$n^{d-2q} C^{(q)}_{s/n}(nx) = \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} e^{-s^2 t} f_n(t) \tag{4.5}$$

where we define (with x-dependence suppressed in the notation)

$$f_n(t) = n^d \prod_{j=1}^d \bar{I}_{nx_j}(n^2 t/d). \tag{4.6}$$

Our goal is to prove that (4.5) has limit $G^{(q)}_s(x)$ as $n \to \infty$.

We prove below in Lemmas 4.1–4.2 that

$$\lim_{n\to\infty} f_n(t) = p_t(x), \tag{4.7}$$

and that there are positive constants C, δ (depending on x, d) such that

$$f_n(t) \leq C\Big(\mathbb{1}_{t\leq 1} + t^{-d/2}\mathbb{1}_{t>1}\Big) \quad \text{uniformly in } n \geq 0 \text{ and } t \geq 0. \tag{4.8}$$

Once (4.7)–(4.8) are proved, since the upper bound in (4.8) is integrable after insertion in the integral on the right-hand side of (4.5) (assuming $d > 2q$ if $s = 0$), the dominated convergence theorem can be applied. For $s > 0$, this gives

$$\lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} e^{-ts^2} f_n(t) = \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} e^{-ts^2} p_t(x), \tag{4.9}$$

and we have seen in (4.3) that the right-hand side is equal to $G^{(q)}_s(x)$. Similarly, if $s = 0$ and $d > 2q$, we instead obtain

$$\lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} f_n(t) = \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} p_t(x), \tag{4.10}$$

which is the integral identified as $G_0(x)$ in (4.2). This completes the proof. □

It remains to prove (4.7)–(4.8). We do this in Lemmas 4.1–4.2, whose proofs rely on the asymptotic form of the modified Bessel function from Lemma 2.2.

Lemma 4.1. *For $d \geq 1$, for $x \in \mathbb{Z}^d$ with $x_j \geq 0$, and for $t > 0$,*

$$\lim_{n\to\infty} f_n(t) = p_t(x). \tag{4.11}$$

Proof: Recall the definition of $f_n(t)$ in (4.6). When $x_j > 0$, it follows from (2.13) that

$$\bar{I}_{nx_j}(n^2 t/d) = \bar{I}_{nx_j}((nx_j)^2 t/dx_j^2) \sim \frac{1}{n}\Big(\frac{d}{2\pi t}\Big)^{1/2} e^{-dx_j^2/2t}. \tag{4.12}$$

If $x_j = 0$ then, since $\bar{I}_0(z) \sim (2\pi z)^{-1/2}$ as $z \to \infty$,

$$\bar{I}_0(n^2 t/d) \sim \frac{1}{n}\Big(\frac{d}{2\pi t}\Big)^{1/2}, \tag{4.13}$$

which is the same formula as (4.12) but with x_j set equal to zero. Substitution of (4.12)–(4.13) into (4.6), together with the definition of $p_t(x)$ in (4.1), then gives

$$\lim_{n\to\infty} f_n(t) = p_t(x) \tag{4.14}$$

and the proof is complete. □

Lemma 4.2. *Let $d \geq 1$ and let $x \in \mathbb{Z}^d$ be nonzero with $x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$. There are constants $C, \delta, n_0 > 0$ (depending only on d) such that*

$$f_n(t) \leq C\Big(\mathbb{1}_{t \leq 1} + t^{-d/2}\mathbb{1}_{t>1}\Big) \quad \textit{uniformly in } n \geq n_0 \textit{ and } t \geq 0. \tag{4.15}$$

Proof: We use C to denote a constant that may depend on d and may change value from line to line. By hypothesis, $x_1 \geq 1$. Since $I_{\alpha'}(z) < I_\alpha(z)$ for any $z \geq 0$ and any $\alpha' > \alpha \geq 0$ Cochran (1967), we can bound each factor with $j \geq 2$ in (4.6) above by $\bar{I}_0(n^2 t/d)$ to obtain

$$f_n(t) \leq n^{d-1}\Big(\bar{I}_0(n^2t/d)\Big)^{d-1} n\bar{I}_n(n^2t/d). \tag{4.16}$$

Since $\bar{I}_0(z) \sim (2\pi z)^{-1/2}$ as $z \to \infty$, and since $\bar{I}_0(z) \leq 1$ for all $z \geq 0$, we see that

$$f_n(t) \leq C\min(n^{d-1}, t^{-(d-1)/2})\, n\bar{I}_n(n^2t/d). \tag{4.17}$$

By (2.14), there exist $C, \delta, n_0 > 0$ such that

$$n\bar{I}_n(n^2t/d) \leq C\Big(ne^{-\delta n}\mathbb{1}_{t<d/2n} + t^{-1/2}e^{-\delta/t}\mathbb{1}_{t \geq d/2n}\Big) \qquad (n \geq n_0,\ t > 0). \tag{4.18}$$

We insert (4.18) into (4.17), using n^{d-1} for the first term and $t^{-(d-1)/2}$ for the second, and obtain

$$f_n(t) \leq C\Big(n^d e^{-\delta n}\mathbb{1}_{t<d/2n} + t^{-d/2}e^{-\delta/t}\Big) \qquad (n \geq n_0,\ t > 0). \tag{4.19}$$

The second term on the right-hand side is bounded for $t \leq 1$ and is less than $t^{-d/2}$ for $t > 1$. Also, $n^d e^{-\delta n}$ is bounded as a function of n, and $\mathbb{1}_{t<d/2n} \leq \mathbb{1}_{t \leq 1}$ once $n \geq d/2$ so the first term is bounded by a multiple of $\mathbb{1}_{t \leq 1}$. This completes the proof. □

5. Properties of the norm: Proof of Proposition 1.2

In this section, we prove Proposition 1.2. We assume throughout that $d \geq 1$ and $a > 0$. Recall the definition

$$|x|_a = \frac{1}{m_a}\sum_{i=1}^d x_i \operatorname{arcsinh}(x_i u) \qquad (x \neq 0). \tag{5.1}$$

Proposition 1.2 asserts that $|\cdot|_a$ defines a norm on $\mathbb{R}^d$ which is monotone increasing in a and for all $x \in \mathbb{R}^d$ obeys

$$|x|_a = \|x\|_2(1 + O(a^2)), \qquad \lim_{a\to\infty}|x|_a = \|x\|_1, \tag{5.2}$$

with the error term in the first equality uniform in nonzero x as $a \to 0$. From this, we conclude immediately that $\|x\|_2 \leq |x|_a \leq \|x\|_1$.

For the limit $a \to 0$, it follows from the relation $u_a(x) = \frac{\sqrt{2d}a}{\|x\|_2}(1 + O(a^2))$ from (3.11), together with $m_a = \sqrt{2d}a(1 + O(a^2))$ from (1.11) and the definition (5.1), that

$$|x|_a = \frac{1}{m_a}\sum_{i=1}^d x_i^2 \frac{\sqrt{2d}a}{\|x\|_2}\big(1 + O(a^2) + O(a^2 x_i^2\|x\|_2^{-2})\big) = \|x\|_2(1 + O(a^2)). \tag{5.3}$$

To see that $\lim_{a\to\infty}|x|_a = \|x\|_1$, we first observe that $m_a = \operatorname{arccosh}(1 + da^2) \sim \log a^2$ as $a \to \infty$. Also, it follows from (1.9) that $u = u_a(x) \sim \|x\|_1^{-1} da^2$ as $a \to \infty$, and therefore

$$|x|_a \sim \frac{1}{\log a^2}\sum_{i=1}^d |x_i| \log a^2 = \|x\|_1. \tag{5.4}$$

Thus, to complete the proof of Proposition 1.2, it suffices to prove that $|\cdot|_a$ defines a norm on $\mathbb{R}^d$, and that $|x|_a$ is monotone increasing in a for each fixed x. We prove these two items in Lemmas 5.2–5.3. To lighten the notation, we will write $C_a(x)$ instead of $C_a^{(1)}(x)$.

The following elementary lemma is the basis for our proof of the triangle inequality for $|\cdot|_a$.

Lemma 5.1. *For $d \geq 1$, for $x, y \in \mathbb{Z}^d$ and for $a > 0$ (also for $a = 0$ if $d > 2$),*

$$C_a(0)C_a(x) \geq C_a(y)C_a(x-y). \tag{5.5}$$

Proof: Let $p_n(x)$ be the n-step transition probability for simple random walk (without killing) to travel from 0 to x in n steps, and let $P_\kappa(x) = \sum_{n=0}^\infty (1-\kappa)^n p_n(x)$. We have seen below (1.6) that $(1+a^2)C_a(x) = P_\kappa(x)$ with $\kappa = \frac{a^2}{1+a^2}$, so it suffices to prove (5.5) instead for P_κ.

Let $q_n(x)$ be the n-step transition probability for simple random walk to travel from 0 to x in n steps without revisiting 0, and let $Q_\kappa(x) = \sum_{n=0}^\infty (1-\kappa)^n q_n(x)$. By considering only walks from 0 to x which pass through a fixed $y \in \mathbb{Z}^d$ and visit y for the last time at the m^{th} step, we obtain

$$p_n(x) \geq \sum_{m=0}^n p_m(y) q_{n-m}(x-y). \tag{5.6}$$

This inequality gives

$$P_\kappa(x) \geq P_\kappa(y) Q_\kappa(x-y). \tag{5.7}$$

Also, with m the time of the last return to 0,

$$p_n(x) = \sum_{m=0}^n p_m(0) q_{n-m}(x), \tag{5.8}$$

and by replacing x with $x-y$ we similarly obtain

$$P_\kappa(x-y) = P_\kappa(0) Q_\kappa(x-y). \tag{5.9}$$

Therefore,

$$P_\kappa(0)P_\kappa(x) \geq P_\kappa(y)P_\kappa(x-y), \tag{5.10}$$

and the proof is complete. □

Lemma 5.2. *For $d \geq 1$ and $a > 0$, $|\cdot|_a$ is a norm on $\mathbb{R}^d$.*

Proof: By its definition in (5.1), $|\cdot|_a$ is non-negative and homogeneous (recall that $u_a(\lambda x) = |\lambda|^{-1} u_a(x)$), with $|x|_a = 0$ if and only if $x = 0$. It remains only to prove the triangle inequality.

To prove the triangle inequality first for points in $\mathbb{Z}^d$, we conclude from Lemma 5.1 that

$$C_a(0)C_a(nx) \geq C_a(ny)C_a(nx-ny) \qquad (x, y \in \mathbb{Z}^d). \tag{5.11}$$

The asymptotic formula (1.21) (whose proof did not use the triangle inequality we are now proving) implies that

$$-\lim_{n\to\infty} \frac{1}{n} \log C_a(nx) = m_a |x|_a. \tag{5.12}$$

From this, we obtain

$$m_a|x|_a \leq m_a|y|_a + m_a|x-y|_a, \tag{5.13}$$

and hence the triangle inequality does hold when the norm is evaluated at points in $\mathbb{Z}^d$.

For $x, y \in \mathbb{R}^d$, we write $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \cdots, \lfloor x_d \rfloor)$. The triangle inequality holds for $\lfloor 2^n x \rfloor, \lfloor 2^n y \rfloor$, for all $n \in \mathbb{N}$. By homogeneity, it also holds for $\frac{\lfloor 2^n x \rfloor}{2^n}, \frac{\lfloor 2^n y \rfloor}{2^n}$. Since $x \mapsto u(x)$ is a continuous function on $\mathbb{R}^d \setminus \{0\}$, $x \mapsto |x|_a$ is a continuous function on $\mathbb{R}^d$ which is extended continuously at 0 by $|0|_a = 0$. Thus by letting $n \to \infty$ we obtain the triangle inequality for all $x, y \in \mathbb{R}^d$. □

Lemma 5.3. *For $d \geq 1$, $a > 0$, and $x \in \mathbb{R}^d$, the norm $|x|_a$ is a monotone increasing function of a.*

Proof: We fix $x \in \mathbb{R}^d$ and prove that the function $f(a) = |x|_a$ is increasing in a. It is convenient to introduce the notation

$$\sigma_i = \sigma_i(x) = x_i^2(1 + x_i^2 u^2)^{-1/2}, \qquad \|\sigma\|_1 = \sum_{i=1}^d \sigma_i. \tag{5.14}$$

Implicit differentiation of $\cosh m_a = 1 + da^2$ with respect to a gives

$$m_a'(a) = \frac{2da}{\sinh m_a}, \tag{5.15}$$

and differentiation of (1.9) leads to

$$u_a' = \frac{2da}{u\|\sigma\|_1}. \tag{5.16}$$

Therefore, by (5.1),

$$f'(a) = -\frac{m_a'}{m_a^2} \sum_{i=1}^d x_i \operatorname{arcsinh}(x_i u) + \frac{u'}{m_a}\|\sigma\|_1 = \frac{2da}{um_a}\Big(1 - \frac{|x|_a u}{\sinh m_a}\Big). \tag{5.17}$$

Let

$$U(x) = |x|_a u_a(x) \quad (x \in \mathbb{R}^d \setminus \{0\}), \tag{5.18}$$

and note that when $x = e_i$ is a unit vector, it follows from (1.14)–(1.15) that

$$U(e_i) = u_a(e_i) = \sinh m_a. \tag{5.19}$$

Thus it suffices to show that U is maximal at e_i, as this implies $f'(a) \geq 0$. Since $U(\lambda x) = U(x)$ for all $\lambda > 0$, and since U is continuously differentiable and bounded on $\mathbb{R}^d \setminus \{0\}$, the maximum exists and will be attained along lines through the origin. There will be several lines since $U(x)$ is invariant under permutation or sign changes of the coordinates of x, so we may restrict attention to nonzero x with $x_1 \geq \cdots \geq x_d$.

We first argue that any critical point x^* of U must have all its nonzero coordinates equal. A critical point of U obeys

$$\frac{\partial U}{\partial x_i}(x^*) = \frac{u \operatorname{arcsinh}(x_i^* u)}{m_a} + |x^*|_a \frac{\partial u}{\partial x_i}(x^*) = 0 \qquad (i = 1, \ldots, d). \tag{5.20}$$

Differentiation of (1.9) with respect to x_i gives

$$x_i \frac{\partial u}{\partial x_i} = -u \frac{\sigma_i}{\|\sigma\|_1}. \tag{5.21}$$

Thus, with $\sigma_i^* = \sigma_i(x^*)$ and $u^* = u_a(x^*)$, (5.20) can be rephrased as

$$\frac{\sigma_i^*}{\|\sigma^*\|_1} = \frac{x_i^* \operatorname{arcsinh} u^* x_i^*}{m_a |x^*|_a} \qquad (i = 1, \ldots, d). \tag{5.22}$$

Let $k \geq 1$ denote the largest subscript i such that $x_i^* > 0$. From (5.22), we see that

$$\frac{\sigma_i^* u^{*2}}{u^* x_i^* \operatorname{arcsinh} u^* x_i^*} = \frac{\sigma_j^* u^{*2}}{u^* x_j^* \operatorname{arcsinh} u^* x_j^*} \qquad (i, j \leq k). \tag{5.23}$$

An elementary calculation shows that the function

$$t \mapsto \frac{t}{\sqrt{1+t^2} \operatorname{arcsinh} t} \tag{5.24}$$

is a bijection from $\mathbb{R}^+$ onto $[0,1]$. Thus (5.23) implies the equality $u^* x_i^* = u^* x_j^*$, so indeed all nonzero coordinates of any critical vector x^* must be equal.

Let $v_k = \sum_{i=1}^k e_i$, for $k = 1, \dots, d$. It remains only to determine which value of k maximises $U(v_k)$. The explicit values of $u_a(v_k)$ and $|v_k|_a$ can be computed from (1.9)–(1.10), with the result that

$$u_a(v_k) = \sqrt{(1 + da^2/k)^2 - 1} = \sinh(\operatorname{arccosh}(1 + da^2/k)), \tag{5.25}$$

$$|v_k| = \frac{1}{m_a} k \operatorname{arcsinh} u_a(v_k) = \frac{1}{m_a} k \operatorname{arccosh}(1 + da^2/k). \tag{5.26}$$

From this, we find that

$$U(v_k) = \frac{da^2}{m_a}\psi(1 + da^2/k), \qquad \psi(x) = \Big(\frac{x+1}{x-1}\Big)^{1/2} \operatorname{arccosh} x. \tag{5.27}$$

A computation gives

$$\psi'(x) = \frac{1}{x-1}\Big(1 - \frac{\operatorname{arccosh} x}{\sqrt{x^2-1}}\Big) \ge 0, \tag{5.28}$$

with the inequality due to the fact that $\operatorname{arccosh} x \le \sqrt{x^2-1}$ for all $x \ge 1$. Therefore $U(v_k)$ is decreasing in k and the maximum of $U(v_k)$ is attained at $v_1 = e_1$. We have noted previously that this suffices, so the proof is complete. □

Appendix A. Continuum Green function

Let $\Delta_{\mathbb{R}}$ denote the Laplace operator for functions on $\mathbb{R}^d$, normalised by $\frac{1}{2d}$, for dimensions $d \ge 1$. Let $s \ge 0$ and $q > 0$. The Green function (or *fundamental solution*) of the operator $-\Delta_{\mathbb{R}} + s^2$ corresponds to the $q = 1$ case of the integral

$$G_s^{(q)}(x) = \int_{\mathbb{R}^d} \frac{e^{ik\cdot x}}{(\frac{1}{2d}\|k\|_2^2 + s^2)^q} \frac{\mathrm{d}k}{(2\pi)^d} \qquad (x \in \mathbb{R}^d \setminus \{0\}). \tag{A.1}$$

However this integral requires some interpretation, as it is not absolutely convergent for large k unless $d < 2q$, and for $s = 0$ it is not convergent at $k = 0$ unless $d > 2q$. The interpretation is in terms of tempered distributions in the next proposition. For the case $s = 0$, see Grafakos (2014, Theorem 2.4.6) for an extension without the restriction that $d > 2q$.

Proposition A.1. *Let $d \ge 1$, $q > 0$, and $s \ge 0$ (with the restriction $d > 2q$ if $s = 0$). In the sense of tempered distributions, the Fourier transform of $G_s^{(q)}(x)$ defined by (1.17)–(1.19) is*

$$\hat{G}_s^{(q)}(k) = \frac{1}{(\frac{1}{2d}\|k\|_2^2 + s^2)^q}. \tag{A.2}$$

For $f \in L^1(\mathbb{R}^d)$, we define

$$\hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{ik\cdot x} \mathrm{d}x \qquad (k \in \mathbb{R}^d). \tag{A.3}$$

In particular, $\hat{p}_t(k) = e^{-t\|k\|_2^2/2d}$, where p_t is the heat kernel defined in (4.1).

Proof of Proposition A.1: Let $d \ge 1$, $q > 0$ and $s \ge 0$, with the additional assumption that $d > 2q$ if $s = 0$. Let φ be a Schwartz-class test function on $\mathbb{R}^d$. In the sense of tempered distributions, the statement that the Fourier transform is given by (A.2) is the statement that

$$\int_{\mathbb{R}^d} \frac{1}{(\frac{1}{2d}\|k\|_2^2 + s^2)^q} \hat{\varphi}(k) \frac{\mathrm{d}k}{(2\pi)^d} = \int_{\mathbb{R}^d} G_s^{(q)}(x)\varphi(x)\mathrm{d}x. \tag{A.4}$$

To prove (A.4), we use (2.4) and Fubini's Theorem to obtain

$$\int_{\mathbb{R}^d} \frac{1}{(\frac{1}{2d}\|k\|_2^2 + s^2)^q} \hat{\varphi}(k) \frac{\mathrm{d}k}{(2\pi)^d} = \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} e^{-ts^2} \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^d} e^{-t\|k\|_2^2/2d} \hat{\varphi}(k). \tag{A.5}$$

Fubini's Theorem indeed applies since the integral on the right-hand side is absolutely convergent, because the t-integral is bounded uniformly in k when $s > 0$, and is $O(\|k\|_2^{-2q})$ when $s = 0$ so there is no divergence at $k = 0$ when $d > 2q$ (of course there is no divergence as $k \to \infty$ because $\hat{\varphi}$ is a Schwartz function). By Parseval's relation, the last integral in (A.5) is equal to

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^d} \hat{p}_t(k)\hat{\varphi}(k) = \int_{\mathbb{R}^d} p_t(x)\varphi(x)\mathrm{d}x. \tag{A.6}$$

A second application of Fubini's Theorem (justified below) then gives

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{(\frac{1}{2d}\|k\|_2^2 + s^2)^q} \hat{\varphi}(k) \frac{\mathrm{d}k}{(2\pi)^d} &= \int_{\mathbb{R}^d} \mathrm{d}x\, \varphi(x) \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{d}t\, t^{q-1} e^{-ts^2} p_t(x) \\ &= \int_{\mathbb{R}^d} \mathrm{d}x\, \varphi(x) G_s(x), \end{aligned} \tag{A.7}$$

where we used (4.2)–(4.3) for the last equality. To justify the application of Fubini's Theorem in (A.7), it suffices to prove that the integral on its first right-hand side is absolutely convergent. We have just shown that this integral has integrand $\varphi(x)G_s^{(q)}(x)$. There is no issue for large x, since $G_s(x)$ decays as $x \to \infty$ and φ is a Schwartz function. As $x \to 0$, $G_s^{(q)}(x)$ is asymptotically a multiple of $\|x\|_2^{d-2q}$ for all $s \geq 0$ (recall (1.20) for asymptotics of $K_{(d-2q)/2}$ when $s > 0$), and this is integrable. This completes the proof. □

Appendix B. Bessel function with large order and large argument

We now prove Lemma 2.2, which we restate here for convenience as Lemma B.1.

Lemma B.1. *As $\nu \to \infty$,*

$$\bar{I}_\nu(\nu t) = L_\nu(t)(1 + o(1)) \tag{B.1}$$

where the $o(1)$ is uniform in $t > 0$. Also, as $\nu \to \infty$, for any $s > 0$,

$$\bar{I}_\nu(\nu^2 s) \sim \frac{e^{-1/2s}}{\nu(2\pi s)^{1/2}}, \tag{B.2}$$

with an error that is not uniform in s. Finally, there exist $C, \delta, \nu_0 > 0$ such that

$$\bar{I}_\nu(\nu^2 s) \leq C\Big(e^{-\delta\nu} \mathbb{1}_{2\nu s<1} + \nu^{-1} s^{-1/2} e^{-\delta/s} \mathbb{1}_{2\nu s\geq 1}\Big) \qquad (\nu \geq \nu_0,\ s > 0). \tag{B.3}$$

Proof: The uniform asymptotic formula (B.1) is given in Olver (1997, (7.18), p. 378).

To prove (B.2), we fix $s > 0$ and set $t = \nu s$ in (B.1). By the definitions of L, ψ in (2.8) we have

$$L_\nu(\nu s) \sim \frac{e^{\nu\psi(\nu s)}}{\nu(2\pi s)^{1/2}} \tag{B.4}$$

and

$$e^{\nu\psi(\nu s)} = e^{-\nu^2 s} e^{\nu\sqrt{1+\nu^2 s^2}} \Big(\frac{\nu s}{1 + \sqrt{1 + \nu^2 s^2}}\Big)^\nu \sim e^{-1/2s}, \tag{B.5}$$

so (B.2) then follows from the uniformity in (B.1) together with

$$L_\nu(\nu s) \sim \frac{e^{-1/2s}}{\nu(2\pi s)^{1/2}}. \tag{B.6}$$

To prove (B.3), we use (B.1) to see that there is a $\nu_0 > 0$ such that

$$\bar{I}_\nu(\nu s) \leq 2L_\nu(\nu s) \qquad (\nu \geq \nu_0,\ s > 0). \tag{B.7}$$

By the definition of L_ν in (2.8),

$$L_\nu(t) = \frac{e^{-\nu t+\nu\sqrt{1+t^2}}}{(2\pi\nu)^{1/2}(1+t^2)^{1/4}} h_\nu(t), \qquad h_\nu(t) = \Big(\frac{t}{1+\sqrt{1+t^2}}\Big)^\nu. \tag{B.8}$$

We need an estimate for (B.8) when $t = \nu s$. We use

$$\sqrt{1+t^2} \le t + \min(1, (2t)^{-1}), \tag{B.9}$$

$$(1+t^2)^{-1/4} \le \min(1, t^{-1/2}), \tag{B.10}$$

to obtain

$$L_\nu(\nu s) \le C\nu^{-1/2}\min(1,(\nu s)^{-1/2}) e^{\min(\nu,(2s)^{-1})} h_\nu(\nu s). \tag{B.11}$$

If $2\nu s \ge 1$ then we use (with C possibly changing from line to line)

$$L_\nu(\nu s) \le C\nu^{-1}s^{-1/2}e^{1/(2s)}h_\nu(\nu s) \qquad (2\nu s \ge 1), \tag{B.12}$$

while if $2\nu s < 1$ we use simply

$$L_\nu(\nu s) \le Ce^\nu h_\nu(\nu s) \qquad (2\nu s < 1). \tag{B.13}$$

Thus it remains to prove that there exists a $\delta > 0$ such that

$$h_\nu(\nu s) \le \begin{cases} e^{-(1+\delta)\nu} & (2\nu s < 1) \\ e^{-(\delta+1/2)/s} & (2\nu s \ge 1). \end{cases} \tag{B.14}$$

Suppose first that $2\nu s < 1$. The inequality (B.14) holds in this case because

$$h_\nu(\nu s) \le 4^{-\nu} \qquad (2\nu s < 1). \tag{B.15}$$

Finally, if $2\nu s \ge 1$ then, since the function $\tau(y) = (\frac{y}{1+\sqrt{1+y^2}})^y$ is decreasing in y,

$$h_\nu(\nu s) \le \tau(1/2)^{1/s} = e^{-(\log(2+\sqrt{5}))/2s} \qquad (2\nu s \ge 1), \tag{B.16}$$

and this suffices for (B.14) because $\log(2+\sqrt{5}) > 1$. □

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