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Convex minorants and the fluctuation theory of Lévy processes

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Abstract. We establish a novel characterisation of the law of the convex minorant of any Lévy process. Our self-contained elementary proof is based on the analysis of piecewise linear convex functions and requires only very basic properties of Lévy processes. Our main result provides a new simple and self-contained approach to the fluctuation theory of Lévy processes, circumventing local time and excursion theory. Easy corollaries include classical theorems, such as Rogozin's regularity criterion, Spitzer's identities and the Wiener-Hopf factorisation, as well as a novel factorisation identity.

1. Introduction

This paper provides elementary, self-contained proofs of some of the main results of the fluctuation theory for Lévy processes, a subject of classical interest in probability (see e.g. monographs Bertoin (1996, Ch. VI), Kyprianou (2006, Ch. 6), Sato (2013, Ch. 9) and the references therein). Our approach is based on a novel characterisation of the law of the convex minorant of a path of *any* Lévy process, given in our main result (Theorem 3.1 in Section 3) below. Its direct consequence is a characterisation in Theorem 2.1 below of the law of the triplet of the supremum, the time the supremum was attained and the position at T of the Lévy process X on the time interval $[0, T]$ of fixed length T. In Section 2 we use Theorem 2.1 to provide short proofs of the Rogozin's criterion for the regularity of X at its starting point, Spitzer's formula for the supremum of X, the Wiener-Hopf identities and the continuity of the law of the triplet. All these results are easy corollaries of Theorem 2.1 and basic properties of X. In particular, our approach circumvents local

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times and excursion theory used in other probabilistic proofs of fluctuation identities, see Bertoin (1996); Kyprianou (2006), and the continuity of the law of the triplet, see Chaumont (2013). To demonstrate further the usefulness of Theorem 3.1, we conclude Section 2 by showing that the vertex process of a Lévy process (introduced in Groeneboom (1983) for Brownian motion and later studied in Nagasawa (2000, Ch.XI.1), both on infinite time horizon) has independent increments. This result characterises the joint law of an n-tuple of the suprema of the Lévy process X perturbed by n different drifts, see Corollary 2.10 below, yielding what appears to be a novel generalisation of the Wiener-Hopf factorisation.

In Section 3 we provide an elementary proof of Theorem 3.1, characterising the law of the convex minorant C_T^X of the path of the Lévy process X on the time interval $[0,T]$. Theorem 3.1 provides an explicit construction of a random piecewise linear convex function, whose law equals that of C_T^X. Our main result can be viewed as a generalisation of Pitman and Uribe Bravo (2012, Thm 1), where such a characterisation was obtained for Lévy processes with diffuse marginals. The main technical step in Pitman and Uribe Bravo (2012) is a limit result on the Skorkhod space establishing that the faces of the convex minorant of the random-walk skeleton of X, described by Abramson and Pitman (2011, Thm 1), converge to the faces of C_T^X. This is a highly non-trivial result because it requires the control of the convergence of the excursions of the approximating random-walk skeletons to the excursions of X away from the faces of C_T^X. The less activity X has, the harder the convergence argument in Pitman and Uribe Bravo (2012) becomes, with the conclusion of Pitman and Uribe Bravo (2012, Thm 1) not being true (by Theorem 3.1) when the marginals of X have atoms.

In contrast, the proof of Theorem 3.1 in Section 3 below focuses on the convergence of the entire convex minorant of the skeleton of X, rather than each face separately. This approach circumvents the delicate face-by-face convergence in the Skorkhod space, which does not hold in general. Our proof relies entirely on elementary geometrical arguments in Section 3.2 to control the convergence of the piecewise linear convex functions of the approximating random walks to the convex function whose law is equal to that of C_T^X for *all* Lévy processes. Considering the convex minorant as a whole, rather than face-by-face, was crucial both in finding the correct formulation of Theorem 3.1, which generalises Pitman and Uribe Bravo (2012, Thm 1), and in its eventual proof. In this paper we give a complete self-contained account of the proof of Theorem 3.1, which perhaps surprisingly requires only very basic properties of Lévy processes (see Section 3.1 and a YouTube presentation González Cázares and Mijatović (2021) for an overview of the proof).

2. Fluctuations of Lévy processes: the stick-breaking approach

Let $x:[0,T]\to\mathbb{R}$ be a càdlàg path (i.e. right-continuous with left limits), $\overline{x}:[0,T]\to\mathbb{R}$ be its running supremum and $\overline{\tau}(x):[0,T]\to[0,T]$ the first times at which $\overline{x}$ is attained:

$$\overline{x}_t := \sup_{s\le t} x_s \quad \text{and} \quad \overline{\tau}_t(x) := \inf\{s\in[0,t] : \max\{x_s, x_{s-}\} = \overline{x}_t\}, \quad t\in[0,T],$$

where $x_{t-} := \lim_{s\uparrow t} x_s$ for all $t\in(0,T]$ and $x_{0-} := x_0$. The running infimum $\underline{x}$ and its time of attainment process $\underline{\tau}(x)$ are defined analogously. The vectors $\overline{\chi}_t(x) = (x_t, \overline{x}_t, \overline{\tau}_t(x))$ and $\underline{\chi}_t(x) = (x_t, \underline{x}_t, \underline{\tau}_t(x))$, $t\in[0,T]$, are referred to as *extremal vectors* of x. Note that $\underline{\chi}_t(x)$ may be recovered from $\overline{\chi}_t(-x) = (-x_t, -\underline{x}_t, \underline{\tau}_t(x))$.

2.1. *Projection of the convex minorant of a Lévy process.* Let $X=(X_t)_{t\ge0}$ be a Lévy process started at zero (but not identically equal to zero) with càdlàg paths. Since $-X$ is a Lévy process if and only if X is, we may (and do) mostly focus our attention on one of the two extremal vectors. At the core of all fluctuation theory results in this paper is the following theorem characterising explicitly the law of $\overline{\chi}_T(X)$ in terms of the increments of X and an independent stick-breaking process ℓ on

$[0,T]$ (recall that $\ell=(\ell_n)_{n\in\mathbb{N}}$ and its remainder process $L=(L_n)_{n\in\mathbb{N}\cup\{0\}}$ are given by the recursion $L_0:=T$, $\ell_n:=V_nL_{n-1}$ and $L_n:=L_{n-1}-\ell_n$ for $n\in\mathbb{N}$, where $(V_n)_{n\in\mathbb{N}}$ are iid $\mathrm{U}(0,1)$).

Theorem 2.1 (Stick-breaking representation of extrema). *Let X be any Lévy process and $T>0$ a fixed time horizon. A stick-breaking process ℓ on $[0,T]$ and its remainder L, independent of X, satisfy*

$$\overline{\chi}_T(X)\overset{d}{=}\sum_{n=1}^{\infty}\big(X_{L_{n-1}}-X_{L_n},\max\{X_{L_{n-1}}-X_{L_n},0\},\ell_n\cdot\mathbb{1}_{\{X_{L_{n-1}}-X_{L_n}>0\}}\big). \tag{2.1}$$

We refer to (2.1) as the *SB representation* of the extremal vector $\overline{\chi}_T(X)$. The SB representation in (2.1) is an easy direct consequence of our main result, Theorem 3.1 in Section 3 below, because $\overline{\chi}_T(X)$ equals the extremal vector of the concave majorant of X on $[0,T]$. We stress that the power of the SB representation in Theorem 2.1 for the extremal vector $\overline{\chi}_T(X)$ for any fixed time horizon T lies in the fact that (2.1) essentially reduce the properties of the path functional $\overline{\chi}_T(X)$ to the properties of the marginals of X. We now illustrate this by deriving many of the classical highlights of the fluctuation theory of Lévy processes from Theorem 2.1. Note first that, since $-\log(\ell_n/T)$ is gamma distributed with density $s\mapsto s^{n-1}\mathrm{e}^{-s}/(n-1)!$ for $s>0$, for a measurable $f:[0,T]\to\mathbb{R}_+$ we have

$$\mathbb{E}\sum_{n\in\mathbb{N}}f(\ell_n)=\int_0^T s^{-1}f(s)\mathrm{d}s. \tag{2.2}$$

In the definition of $\overline{\tau}_t(X)$ (resp. $\underline{\tau}_t(X)$), we take the first rather than last time the maximum (resp. minimum) is attained. Our first corollary shows that this choice makes little difference.

Corollary 2.2. *A Lévy process X attains its maximum at a unique time a.s. if and only if X is not a driftless compound Poisson process; then $\overline{\chi}_t(X)\overset{d}{=}(X_t,X_t-\underline{X}_t,t-\underline{\tau}_t(X))$ for all $t>0$.*

Proof: If X is a driftless compound Poisson process process, it has piecewise constant paths, making the time of the maximum not unique. Assume X is not a driftless compound Poisson, then we have $\mathbb{P}(X_t=0)>0$ for at most countably many $t>0$. Indeed, either the law of X_t is diffuse or, by Doeblin's lemma, see Kallenberg (2002, Lem. 15.22), X is compound Poisson with drift $\mu\neq0$. In the latter case, $\mathbb{P}(X_t=0)>0$ if and only if $-\mu t$ is in a countable set generated by the atoms of the Lévy measure of X, implying the claim.

The time-reversal process $X'=(X'_s)_{s\in[0,t]}$, defined as $X'_s:=X_{(t-s)-}-X_t$, $s\in[0,t]$, has the same law as $(-X_s)_{s\in[0,t]}$, implying $\overline{\tau}_t(X')\overset{d}{=}\underline{\tau}_t(X)$. The gap $(t-\overline{\tau}_t(X'))-\overline{\tau}_t(X)\geq0$ between the time of the first and last maximum of X has expectation equal to zero and is hence zero a.s. Indeed, since $\mathbb{P}(X_t=0)>0$ for at most countably many $t>0$, Theorem 2.1 and (2.2) yield

$$t-\mathbb{E}[\underline{\tau}_t(X)]-\mathbb{E}[\overline{\tau}_t(X)]=\mathbb{E}\sum_{n=1}^{\infty}\ell_n\mathbb{1}_{\{X_{L_{n-1}}=X_{L_n}\}}=\mathbb{E}\sum_{n=1}^{\infty}\ell_n\mathbb{P}(X_{\ell_n}=0|\ell_n)=\int_0^t\mathbb{P}(X_s=0)\mathrm{d}s=0.$$

The identity in law follows from $\overline{\chi}_t(X')\overset{d}{=}\overline{\chi}_t(-X)$. □

Corollary 2.3. *For any Lévy process X, the following formulae hold for any $t>0$:*

$$\mathbb{E}[\overline{\tau}_t(X)]=\int_0^t\mathbb{P}(X_s>0)\mathrm{d}s\qquad\textit{and}\qquad\mathbb{E}[\overline{X}_t]=\int_0^t(\mathbb{E}\max\{X_s,0\}/s)\mathrm{d}s. \tag{2.3}$$

Proof: Let $\rho(s):=\mathbb{P}(X_s>0)$ and take expectations in the third coordinate of SB-representation (2.1). Fubini's theorem and the formula in (2.2) imply

$$\mathbb{E}\overline{\tau}_t(X)=\sum_{n=1}^{\infty}\mathbb{E}[\ell_n\rho(\ell_n)]=\int_0^t\rho(s)\mathrm{d}s\quad\text{for any }t>0.$$

The proof of the formula for the supremum, based on (2.1) and (2.2), is analogous. □

2.2. *Wiener–Hopf factorisation and Rogozin's criteria.* Consider an exponential time horizon $T_\theta \sim \mathrm{Exp}(\theta)$ with parameter $\theta \in (0,\infty)$ (i.e. $\mathbb{E}T_\theta = 1/\theta$), independent of the Lévy process X. Let $\ell^{(\theta)} = (\ell_n^{(\theta)})_{n\in\mathbb{N}}$ be a stick-breaking process with a random time horizon T_θ. The random measure $\sum_{n=1}^\infty \delta_{\ell_n^{(\theta)}}$ on $(0,\infty)$ is easily seen to be a Poisson point process (PPP) (see Appendix A below). By (2.2) its mean measure satisfies $\mathbb{E}\sum_{n\in\mathbb{N}} \delta_{\ell_n^{(\theta)}}(A) = \int_A t^{-1}\mathrm{e}^{-\theta t}\mathrm{d}t$ for a measurable A (δ_z denotes the Dirac delta at the point z). Let $F(t,\mathrm{d}x) := \mathbb{P}(X_t \in \mathrm{d}x)$ denote the law of X_t for any $t>0$. Marking each point $\ell_n^{(\theta)}$ by a random real number sampled from the law $F(\ell_n^{(\theta)},\cdot)$, by the Marking Theorem (see Kingman, 1993, p. 55), produces a PPP on $(0,\infty)\times\mathbb{R}$.

Proposition 2.4. *Let the time horizon $T_\theta \sim \mathrm{Exp}(\theta)$ and the stick-breaking process $\ell^{(\theta)}$ be independent of the Lévy process X. Define $\xi_n^{(\theta)} := X_{L_{n-1}^{(\theta)}} - X_{L_n^{(\theta)}}$, where $L^{(\theta)} = (L_k^{(\theta)})_{k\in\mathbb{N}\cup\{0\}}$ is the remainder process associated to $\ell^{(\theta)}$. Then $\Xi_\theta := \sum_{n=1}^\infty \delta_{(\ell_n^{(\theta)},\xi_n^{(\theta)})}$ is a Poisson point process with mean measure*

$$\mu_\theta(\mathrm{d}t,\mathrm{d}x) := t^{-1}\mathrm{e}^{-\theta t}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t, \qquad (t,x)\in(0,\infty)\times\mathbb{R}. \tag{2.4}$$

An immediate corollary of Theorem 2.1 and Proposition 2.4 characterises the laws of the supremum and infimum of X on the exponential time horizon T_θ.

Corollary 2.5. *Let $T_\theta\sim\mathrm{Exp}(\theta)$ be independent of the Lévy process X. Then the moment generating functions of $\overline{X}_{T_\theta}$ and $-\underline{X}_{T_\theta}$ are given by the following formulae for any $u\ge 0$:*

$$\mathbb{E}\big[\mathrm{e}^{-u\overline{X}_{T_\theta}}\big] = \exp\left(\int_0^\infty\int_{(0,\infty)}(\mathrm{e}^{-ux}-1)\mathrm{e}^{-\theta t}t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\right), \tag{2.5}$$

$$\mathbb{E}\big[\mathrm{e}^{u\underline{X}_{T_\theta}}\big] = \exp\left(\int_0^\infty\int_{(-\infty,0)}(\mathrm{e}^{ux}-1)\mathrm{e}^{-\theta t}t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\right). \tag{2.6}$$

Proof: By Theorem 2.1, $\overline{X}_{T_\theta} \overset{d}{=} \int_{(0,\infty)^2} x\Xi_\theta(\mathrm{d}t,\mathrm{d}x)$, where Ξ_θ is a PPP with mean measure μ_θ. Campbell's formula in Kingman (1993, p. 28) implies (2.5). Applying (2.5) to $-X$ yields (2.6). □

Recall that 0 is *regular* for the half-line $(0,\infty)$ if X visits $(0,\infty)$ almost surely immediately after time 0, i.e. $\mathbb{P}(\bigcap_{t>0}\bigcup_{s\le t}\{X_s>0\})=1$.

Theorem 2.6 (Rogozin's criterion)**.** *The starting point 0 of X is regular for $(0,\infty)$ if and only if*

$$\int_0^1 t^{-1}\mathbb{P}(X_t>0)\mathrm{d}t=\infty. \tag{2.7}$$

Proof: Let the time horizon $T_\theta\sim\mathrm{Exp}(\theta)$ and random sequences $\ell^{(\theta)}$ and $\xi^{(\theta)}$ be as in Proposition 2.4 above. As $t\mapsto\overline{X}_t$ is non-decreasing a.s., 0 is not regular for $(0,\infty)$ if and only if $\mathbb{P}(\overline{X}_{T_\theta}=0)>0$. Since $\overline{X}_{T_\theta}\overset{d}{=}\int_A x\Xi_\theta(\mathrm{d}t,\mathrm{d}x)$, where $A:=(0,\infty)\times(0,\infty)$, the event $\{\overline{X}_{T_\theta}=0\}$ is equal to the event $\{\Xi_\theta(A)=0\}$ that the PPP $\Xi_\theta=\sum_{n=1}^\infty\delta_{(\ell_n^{(\theta)},\xi_n^{(\theta)})}$ has no points in A. Thus, 0 is not regular for $(0,\infty)$ if and only if

$$\mathbb{P}\left(\overline{X}_{T_\theta}=0\right)=\mathbb{P}\big(\Xi_\theta(A)=0\big)=\exp\big(-\mathbb{E}\Xi_\theta(A)\big)=\exp\left(-\int_0^\infty t^{-1}\mathrm{e}^{-\theta t}\mathbb{P}(X_t>0)\mathrm{d}t\right)>0$$

for some positive θ, which is equivalent to (2.7). □

We can now characterise the behaviour of X as $t\to\infty$.

Theorem 2.7 (Rogozin)**.** *The possibly degenerate variables* $\overline{X}_\infty := \sup_{t\ge0} X_t$ *and* $\underline{X}_\infty := \inf_{t\ge0} X_t$ *satisfy*

$$\mathbb{E}\big[\mathrm{e}^{-u\overline{X}_\infty}\big] = \exp\bigg(\int_0^\infty \int_{(0,\infty)} (\mathrm{e}^{-ux}-1)t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\bigg), \tag{2.8}$$

$$\mathbb{E}\big[\mathrm{e}^{u\underline{X}_\infty}\big] = \exp\bigg(\int_0^\infty \int_{(-\infty,0)} (\mathrm{e}^{ux}-1)t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\bigg), \tag{2.9}$$

for any $u\ge 0$*. Define the integrals*

$$I_+ := \int_1^\infty t^{-1}\mathbb{P}(X_t>0)\mathrm{d}t \qquad \& \qquad I_- := \int_1^\infty t^{-1}\mathbb{P}(X_t<0)\mathrm{d}t.$$

Then the following statements hold for any non-constant Lévy process X*:*
(a) *if* $I_+<\infty$*, then* $\overline{X}_\infty$ *is non-degenerate, infinitely divisible and finite a.s. and* $\lim_{t\to\infty} X_t = -\infty$*;*
(b) *if* $I_-<\infty$*, then* $\underline{X}_\infty$ *is non-degenerate, infinitely divisible and finite a.s. and* $\lim_{t\to\infty} X_t = \infty$*;*
(c) *if* $I_+ = I_- = \infty$*, then* $\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty$*.*

Proof: Let $T_1\sim \mathrm{Exp}(1)$ be independent of X and note $T_1/\theta \sim \mathrm{Exp}(\theta)$ for any $\theta>0$. Since $\overline{X}_{T_1/\theta}\to\overline{X}_\infty$ as $\theta\to0$ a.s., the corresponding Laplace transforms converge pointwise. Thus the monotone convergence theorem applied to the right-hand sides of (2.5)–(2.6) implies (2.8)–(2.9). Identity (2.8) (resp. (2.9)) implies that $I_+<\infty$ (resp. $I_-<\infty$) if and only if $\mathbb{E}\exp(-u\overline{X}_\infty)>0$ (resp. $\mathbb{E}\exp(u\underline{X}_\infty)>0$) for all $u\ge0$. This implies part (c) and all but the limits in parts (a) & (b).

It remains to prove the limit in (b), as the proof of the limit in (a) is analogous. First we show that $I_+ + I_- = \infty$. Since X is not constant, by $I_+ + I_- = \int_1^\infty t^{-1}(1-\mathbb{P}(X_t=0))\mathrm{d}t$, it suffices to prove $\int_1^\infty t^{-1}\mathbb{P}(X_t=0)\mathrm{d}t<\infty$. As shown in the proof of Corollary 2.2 above, if X is not a driftless compound Poisson process, the function $t\mapsto\mathbb{P}(X_t=0)$ is zero for Lebesgue a.e. $t>0$. If X is driftless compound Poisson, then the function $t\mapsto\sqrt{t}\mathbb{P}(X_t=0)$ is bounded. Indeed, suppose without loss of generality that X has positive jumps and consider the a decomposition $X_t = Y_t + S_{N_t}$, where the compound Poisson process Y, the random walk S with strictly positive increments and the Poisson process N with intensity $\lambda>0$ are independent. Then $\mathbb{P}(X_t=0) = \int_{\mathbb{R}}\mathbb{P}(S_{N_t}=-x)\mathbb{P}(Y_t\in\mathrm{d}x)\le\sup_{x\in\mathbb{R}}\mathbb{P}(S_{N_t}=-x)$. For $x\in\mathbb{R}$ we have $\sum_{n=0}^\infty \mathbb{1}_{\{S_n=-x\}}\le 1$ (since $n\mapsto S_n$ is strictly increasing) and

$$\mathbb{P}(S_{N_t}=-x)\le \mathrm{e}^{-\lambda t}\sup_{m\in\mathbb{N}\cup\{0\}}\{(\lambda t)^m/m!\}\mathbb{E}\sum_{n=0}^\infty \mathbb{1}_{\{S_n=-x\}}\le \mathrm{e}^{-\lambda t}\sup_{m\in\mathbb{N}\cup\{0\}}\{(\lambda t)^m/m!\}, \tag{2.10}$$

Since $m\mapsto \mathrm{e}^{-\lambda t}(\lambda t)^m/m!$ is maximised at an integer m approximately equal to λt, by Stirling's formula and (2.10), a multiple of $t^{-1/2}$ is an upper bound for $\mathbb{P}(X_t=0)$ for all sufficiently large t.

Assume $I_-<\infty$ and pick a sequence $a_n\uparrow\infty$ such that $\mathbb{P}(\underline{X}_\infty<-a_n)\le 1/n^2$ for $n\in\mathbb{N}$. Since $I_-<\infty$, we must have $I_+=\infty$ and thus $\overline{X}_\infty=\infty$ a.s., implying that the hitting time S_n of X of the level $2a_n$ is a.s. finite for every $n\in\mathbb{N}$. Define the sets

$$B_n = \{X_t<a_n \text{ for some } t>S_n\}\subset\{X_t - X_{S_n}<-a_n \text{ for some } t>S_n\},$$

and note that $\mathbb{P}(B_n)\le\mathbb{P}(\underline{X}_\infty<-a_n)\le 1/n^2$ by the strong Markov property. The Borel–Cantelli lemma then shows that $\mathbb{P}(B_n \text{ i.o.})=0$ and hence $\mathbb{P}(\liminf_{t\to\infty}X_t<\infty)=0$. In other words, we have $\lim_{t\to\infty}X_t=\infty$ a.s., completing the proof. □

Another easy corollary of Theorem 2.1 is the Wiener-Hopf factorisation.

Theorem 2.8 (Wiener-Hopf factorisation)**.** *Let the time horizon* $T_\theta\sim\mathrm{Exp}(\theta)$ *be independent of* X*. The random vectors* $(\overline{\tau}_{T_\theta}(X), \overline{X}_{T_\theta})$ *and* $(T_\theta - \overline{\tau}_{T_\theta}(X), X_{T_\theta}-\overline{X}_{T_\theta})$ *are independent, infinitely* *divisible with respective Fourier-Laplace transforms given by*

$$\Psi_\theta^+(u,v) := \mathbb{E}\big[\mathrm{e}^{u\overline{\tau}_{T_\theta}(X)+v\overline{X}_{T_\theta}}\big] = \frac{\varphi_+(-\theta,0)}{\varphi_+(u-\theta,v)}, \tag{2.11}$$

$$\Psi_\theta^-(u,-v) := \mathbb{E}\big[\mathrm{e}^{u(T_\theta-\overline{\tau}_{T_\theta}(X))-v(X_{T_\theta}-\overline{X}_{T_\theta})}\big] = \frac{\varphi_-(-\theta,0)}{\varphi_-(u-\theta,v)}, \tag{2.12}$$

for any $u,v\in\mathbb{C}$ with $\Re u,\Re v\le 0$. Here $\varphi_\pm$ is defined as follows: set $A_+ := (0,\infty)$, $A_- := (-\infty,0]$,

$$\varphi_\pm(a,b) := \exp\left(\int_0^\infty\int_{A_\pm}\big(\mathrm{e}^{-t}-\mathrm{e}^{at+b|x|}\big)t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\right), \tag{2.13}$$

for any $a,b\in\mathbb{C}$ such that the integrals in (2.13) exist, including the $\Re a<0$, $\Re b\le 0$. The characteristic exponent Ψ of X_1 (i.e. $\mathbb{E}\exp(vX_1)=\exp\Psi(v)$ for $v\in\mathbb{C}$ with $\Re v=0$) satisfies

$$\theta/(\theta-u-\Psi(v)) = \Psi_\theta^+(u,v)\Psi_\theta^-(u,v), \qquad u,v\in\mathbb{C} \text{ with } \Re v=\Re u=0. \tag{2.14}$$

Proof: Let $\ell^{(\theta)}$, $\xi^{(\theta)}$ and $\Xi_\theta=\sum_{n=1}^\infty \delta_{(\ell_n^{(\theta)},\xi_n^{(\theta)})}$ be as in Proposition 2.4. By Theorem 2.1, we have

$$(\overline{\tau}_{T_\theta}(X),\overline{X}_{T_\theta}) \overset{d}{=} \int_{B_+}(t,x)\Xi_\theta(\mathrm{d}t,\mathrm{d}x) \quad \& \quad (T_\theta-\overline{\tau}_{T_\theta}(X),X_{T_\theta}-\overline{X}_{T_\theta}) \overset{d}{=} \int_{B_-}(t,x)\Xi_\theta(\mathrm{d}t,\mathrm{d}x),$$

where $B_\pm := (0,\infty)\times A_\pm$. Moreover, the joint law of $(\overline{\tau}_{T_\theta}(X),\overline{X}_{T_\theta})$ and $(T_\theta-\overline{\tau}_{T_\theta}(X),X_{T_\theta}-\overline{X}_{T_\theta})$ equals that of the two integrals in the display above, making the vectors independent because $B_+\cap B_-=\emptyset$. By Proposition 2.4, the mean measure of Ξ_θ equals $\mu_\theta(\mathrm{d}t,\mathrm{d}x)=t^{-1}\mathrm{e}^{-\theta t}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t$. Hence we have

$$\Psi_\theta^\pm(u,v) = \exp\left(\int_{(0,\infty)\times A_\pm}\big(\mathrm{e}^{ut+vx}-1\big)\frac{\mathrm{e}^{-\theta t}}{t}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\right) \quad \text{for all } u,v\in\mathbb{C} \text{ with } \Re u,\Re v\le 0,$$

by Campbell's Theorem, see Kingman (1993, p. 28). This representation of $\Psi_\theta^\pm(u,v)$ and (2.13) imply (2.11)–(2.12). The independence and the formula $\mathbb{E}\exp(uT_\theta+vX_{T_\theta})=\theta/(\theta-u-\Psi(v))$ imply identity (2.14). □

A circular argument would arise if one attempted to develop fluctuation theory for Lévy processes with diffuse transition laws using Pitman and Uribe Bravo (2012, Thm 1), because of its reliance on Rogozin's result for Lévy process of infinite variation, $\limsup_{t\downarrow 0}X_t/t=-\liminf_{t\downarrow 0}X_t/t=\infty$ a.s., which relies on the Wiener-Hopf factorisation in an essential way. In contrast, Theorem 3.1, applicable to all Lévy processes, has an elementary proof that does not use fluctuation theory. In fact, Theorem 3.1 can be used as a short-cut to Rogozin's result as it implies the necessary fluctuation theory as described in this paper.

The question of the absolute continuity of the law of $\overline{\chi}_t(X)=(X_t,\overline{X}_t,\overline{\tau}_t(X))$ was the main topic in Chaumont (2013), investigated using excursion theory. Again, Theorem 2.1 provides an easy approach. In fact, in Pitman and Uribe Bravo (2012, Thm 2) the authors use a version of Theorem 2.1 for diffuse Lévy processes (Theorem 1 in Pitman and Uribe Bravo, 2012) to show that the laws of $(\overline{\tau}_t(X),\overline{X}_t)$ and $(\overline{X}_t-X_t,\overline{X}_t)$ are equivalent to Lebesgue measure on $(0,t]\times(0,\infty)$ and $(0,\infty)^2$, respectively, for any Lévy process with absolutely continuous marginals.

2.3. *Vertex process of a Lévy process.* The final application of Theorem 3.1 describes the law of the vertex process of the convex minorant of X. Intuitively, the vertex process is naturally paremetrised by the slope of the minorant and its range coinciding with the extremal points of the graph of the convex minorant. In the infinite time horizon case, Groeneboom (1983) described the law of the vertex process of Brownian motion as a time inhomogeneous additive process (i.e. a process with independent but non-stationary increments). This description was later extended by Nagasawa (2000, Ch. XI.1) to Lévy processes with infinite activity, again over an infinite time horizon. We now show, as a simple corollary of Theorem 3.1, that the vertex process has independent increments for all Lévy processes and independent exponential (possibly infinite) time horizons. Before defining the vertex process, note that the convex minorant C^X_∞ of X on the infinite time interval $[0,\infty)$ exists and is finite if and only if $l := \liminf_{t\to\infty} X_t/t$, which is a.s. constant, lies in $(-\infty,\infty]$. Otherwise, we have $C^X_\infty \equiv -\infty$ on $(0,\infty)$ (see Section 3 below for a characterisation of $l=-\infty$ in terms of the Lévy measure of X).

For $\theta\in[0,\infty)$, let the exponential random variable $T_\theta\sim\mathrm{Exp}(\theta)$ (with $T_0:=\infty$ a.s.) be independent of X. The right-derivative of $C^X_{T_\theta}$, given by $(C^X_{T_\theta})'(t) := \lim_{h\downarrow 0}(C^X_{T_\theta}(t+h)-C^X_{T_\theta}(t))/h$, exists for any $t\in[0,T_\theta)$ and is a non-decreasing function of t with a possibly infinite limit at T_θ. Define the *vertex process* (σ,η) of X to be the càdlàg process parametrised by slope $s\in\mathbb{R}$, where $\sigma=(\sigma_s)_{s\in\mathbb{R}}$ is the right-inverse of $(C^X_{T_\theta})'$ and $\eta=(\eta_s)_{s\in\mathbb{R}}$ the value of the minorant at that time,

$$\sigma_s := T_\theta \wedge \inf\big\{t\in[0,T_\theta) \,:\, (C^X_{T_\theta})'(t) > s\big\} \quad\text{and}\quad \eta_s := C^X_{T_\theta}(\sigma_s) = \min\{X_{\sigma_s-}, X_{\sigma_s}\}, \tag{2.15}$$

respectively (here $a\wedge b := \min\{a,b\}$ and $\inf\emptyset := \infty$). Its construction is somewhat reminiscent of a ladder subordinator, indexed by the local time of X at its running minimum, appearing in the classical approach to the fluctuation theory of Lévy processes, see e.g. Bertoin (1996, Ch. VI). It is clear that σ has non-decreasing paths and, almost surely, we have

$$\lim_{s\to-\infty}\sigma_s = 0,\quad \lim_{s\to-\infty}\eta_s = X_0 = 0,\quad \eta_0 = \underline{X}_{T_\theta},\quad \lim_{s\to\infty}\sigma_s = T_\theta,\quad \lim_{s\to\infty}\eta_s = C^X_{T_\theta}(T_\theta), \tag{2.16}$$

where $C^X_{T_\theta}(T_\theta)$ equals X_{T_θ} (if $\theta>0$) or $\mathrm{sgn}(l)\cdot\infty$ (if $\theta=0$), where $\mathrm{sgn}(l)=1$ if $l=\liminf_{t\to\infty}X_t/t>0$ and -1 otherwise. The following result is an elementary consequence of the Poissonian structure of $C^X_{T_\theta}$, described in Corollary 3.2 below, which extends Proposition 2.4 above.

Theorem 2.9. *Pick $\theta\in[0,\infty)$. Extend the definition of the mean measure μ_θ in (2.4) (for $\theta>0$) by $\mu_0(\mathrm{d}t,\mathrm{d}x) := \mathbb{1}_{\{x/t<l\}}t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t$ on $(t,x)\in(0,\infty)\times\mathbb{R}$. Then the vertex process (σ,η) has independent increments and its Fourier-Laplace transform is given by*

$$\mathbb{E}\big[\mathrm{e}^{u\sigma_s+v\eta_s}\big] = \exp\bigg(\int_{(0,\infty)\times\mathbb{R}}(\mathrm{e}^{ut+vx}-1)\mathbb{1}_{\{x/t\le s\}}\mu_\theta(\mathrm{d}t,\mathrm{d}x)\bigg), \tag{2.17}$$

for any $s\in\mathbb{R}$, $u,v\in\mathbb{C}$ with $\Re u\le 0$ and $\Re v=0$. In particular, the Laplace transform of σ_s equals

$$\mathbb{E}[\mathrm{e}^{-u\sigma_s}] = \exp\bigg(\int_0^\infty(e^{-ut}-1)e^{-\theta t}\mathbb{P}(X_t\le st)\frac{\mathrm{d}t}{t}\bigg),$$

for all $u\ge 0$, and either $s\in\mathbb{R}$ (if $\theta>0$) or $s\in(-\infty,l)$ (if $\theta=0$).

Proof: Let $\Xi_\theta := \sum_{n=1}^\infty \delta_{(\ell_n^{(\theta)},\xi_n^{(\theta)})}$ be a PPP on $A:=(0,\infty)\times\mathbb{R}$ with mean measure μ_θ. By Corollary 3.2 below and definition (2.15) of the vertex process (σ,η), for any sequence of slopes $s_1<\ldots<s_n$ in $\mathbb{R}$ we have

$$\big((\sigma_{s_1},\eta_{s_1}),\ldots,(\sigma_{s_n},\eta_{s_n})\big) \overset{d}{=} \int_A \big(\mathbb{1}_{\{x/t\le s_1\}}(t,x),\ldots,\mathbb{1}_{\{x/t\le s_n\}}(t,x)\big)\Xi_\theta(\mathrm{d}t,\mathrm{d}x).$$

Intuitively, since the convex minorant $C^X_{T_\theta}$ is piecewise linear, σ_{s_1} is the sum of all the horizontal lengths of all the linear faces with slope strictly smaller than s_1, see Section 3.2 for a precise description. The increments $(\sigma_{s_1},\eta_{s_1})$, $(\sigma_{s_2}-\sigma_{s_1},\eta_{s_2}-\eta_{s_1})$, …, $(\sigma_{s_n}-\sigma_{s_{n-1}},\eta_{s_n}-\eta_{s_{n-1}})$ are equal in law to the integrals with respect to the PPP Ξ_θ over disjoint "pizza slices" $\{(t,x)\in A: x/t\le s_1\}$, $\{(t,x)\in A: s_1<x/t\le s_2\}$, …, $\{(t,x)\in A: s_{n-1}<x/t\le s_n\}$ and are thus independent. The Fourier-Laplace transform of the marginal (σ_s,η_s) follows from Campbell's formula in Kingman (1993, p. 28). □

The vertex process can be constructed from the path of X without a reference to the convex minorant $C^X_{T_\theta}$ as follows. Given $s\in\mathbb{R}$ (with $s<l$ if $\theta=0$), define the Lévy process $X^{(s)}=(X^{(s)}_t)_{t\ge0}$ by $X^{(s)}_t := X_t - st$. Then $\sigma_s = \underline{\tau}_{T_\theta}(X^{(s)})$ and $\eta_s - s\sigma_s = \underline{X}^{(s)}_{T_\theta}$. This description and Theorem 2.9 yield the following novel generalisation of the classical Wiener-Hopf factorisation.

Corollary 2.10. *Pick $\theta > 0$ and let $T_\theta \sim \mathrm{Exp}(\theta)$ be independent of X. Then, for any real numbers $s_1 < \cdots < s_n$, the following $n+1$ vectors are independent:*

$$(\underline{\tau}_{T_\theta}(X^{(s_1)}), \underline{X}_{T_\theta}^{(s_1)} + s_1\underline{\tau}_{T_\theta}(X^{(s_1)})), \quad (T_\theta - \underline{\tau}_{T_\theta}(X^{(s_n)}), X_{T_\theta} - \underline{X}_{T_\theta}^{(s_n)} - s_n\underline{\tau}_{T_\theta}(X^{(s_n)})), \quad \text{and}$$
$$(\underline{\tau}_{T_\theta}(X^{(s_{i+1})}) - \underline{\tau}_{T_\theta}(X^{(s_i)}), \underline{X}_{T_\theta}^{(s_{i+1})} - \underline{X}_{T_\theta}^{(s_i)} + s_{i+1}\underline{\tau}_{T_\theta}(X^{(s_{i+1})}) - s_i\underline{\tau}_{T_\theta}(X^{(s_i)})), \quad i = 1, \ldots, n-1.$$

3. Stick-breaking representations for convex minorants

3.1. *The convex minorant of a Lévy process (the main theorem).* Given any càdlàg function $x : [0,T] \to \mathbb{R}$, its convex minorant, denoted by C_T^x, is the largest convex function that is pointwise smaller than x. The goal of this section is to prove our main result, Theorem 3.1, which (when applied to $-X$) clearly yields Theorem 2.1 above.

Theorem 3.1. *Let X be a Lévy process and fix $T > 0$. Let $(\ell_n)_{n\in\mathbb{N}}$ be a uniform stick-breaking process on $[0,T]$ independent of X. Then the convex minorant C_T^X of X has the same law (in the space of continuous functions on $[0,T]$) as the piecewise linear convex function on $[0,T]$ given by the formula*

$$\begin{aligned} t \mapsto \sum_{n=1}^{\infty} \xi_n \min\{\max\{t-a_n, 0\}/\ell_n, 1\}, \quad &\text{where} \quad \xi_n := X_{L_{n-1}} - X_{L_n} \quad \text{and} \\ a_n := \sum_{k=1}^{\infty} \ell_k \cdot \mathbb{1}_{\{\xi_k/\ell_k < \xi_n/\ell_n\}} + \sum_{k=1}^{n-1} \ell_k \cdot \mathbb{1}_{\{\xi_k/\ell_k = \xi_n/\ell_n\}}, &\quad n \in \mathbb{N}. \end{aligned} \tag{3.1}$$

In particular, the face of the piecewise linear function with horizontal length ℓ_n has vertical height ξ_n.

We now present a simple but useful consequence of Theorem 3.1, first established in Pitman and Uribe Bravo (2012, Cor. 2 & 3) for diffuse Lévy processes. Recall the a.s. constant $l = \liminf_{t\to\infty} X_t/t$ lies in $[-\infty, \infty]$. Whenever the expectation $\mathbb{E}X_1$ is well defined, i.e., when we have $\min\{\mathbb{E}\max\{X_1, 0\}, \mathbb{E}\max\{-X_1, 0\}\} < \infty$, the strong law of large numbers (see Sato, 2013, Thms 36.4 & 36.5) implies that $l = \mathbb{E}X_1 = \lim_{t\to\infty} X_t/t$ a.s. Otherwise, we have $\mathbb{E}X_1^+ = \mathbb{E}X_1^- = \infty$ and Doney (2007, Thm 15) implies that $l \in \{-\infty, \infty\}$ and

$$l = -\infty \qquad \text{if and only if} \qquad \int_{(-\infty,-1)} \frac{|x|}{1 + \int_1^{|x|} \nu([y,\infty))\mathrm{d}y} \nu(\mathrm{d}x) = \infty. \tag{3.2}$$

The proof of Doney (2007, Thm 15) is an easy corollary of the analogous result for random walks (see also Erickson, 1973), proven using renewal theory. Indeed, the small-jump and Brownian components of X converge by the strong law of large numbers and the big-jump component is a random walk, time-changed by a Poisson process.

Corollary 3.2. *Let $\theta \in [0,\infty)$ and $T_\theta \sim \mathrm{Exp}(\theta)$ (with $T_0 = \infty$) be independent of X. When $\theta = 0$ we assume that $l = \liminf_{t\to\infty} X_t/t > -\infty$. Define a σ-finite measure on $(0,\infty) \times \mathbb{R}$:*

$$\mu_\theta(\mathrm{d}t, \mathrm{d}x) := \begin{cases} t^{-1}\mathrm{e}^{-\theta t}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t, & \theta > 0, \\ \mathbb{1}_{\{x/t < l\}} t^{-1}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t, & \theta = 0. \end{cases}$$

Let $\Xi_\theta = \sum_{n\in\mathbb{N}} \delta_{(\ell_n^{(\theta)}, \xi_n^{(\theta)})}$ be a Poisson point process with mean measure μ_θ. Then $C_{T_\theta}^X$ has the same law as the piecewise linear function given by (3.1). In particular, the face of the piecewise linear function with horizontal length $\ell_n^{(\theta)}$ has vertical height $\xi_n^{(\theta)}$ and, when $\theta = 0$, the corresponding slope $\xi_n^{(\theta)}/\ell_n^{(\theta)}$ lies on the interval $(-\infty, l)$.

Proof: The result for $\theta > 0$ follows from Proposition 2.4 and Theorem 3.1. We now describe the case $\theta = 0$ following the proof of Pitman and Uribe Bravo (2012, Cor. 3). First note that the definition of l implies that the right-derivative of C_∞^X is upper bounded by l. On the other hand, the derivative has no smaller upper bound. Indeed, suppose the derivative is bounded by some $a < l$. Then there exists some time $T > 0$ after which $X_t/t \ge (a+l)/2$ for all $t \ge T$. Thus $t \mapsto C_\infty^X(\min\{t,T\}) + \max\{t-T,0\}(a+l)/2$ is convex, dominates C_∞^X and is dominated by X, contradicting the maximality of C_∞^X. Moreover, the right-derivative of C_∞^X is never equal to l. If $l = \infty$ this is clear. If $l < \infty$, by the discussion preceding the corollary we have $\mathbb{E}|X_1| < \infty$ and thus $\mathbb{E}X_1 = l$. Since the martingale $(X_t - lt)_{t\ge 0}$ is recurrent (see Sato, 2013, Thm 36.7), then $\liminf_{t\to\infty} C_\infty^X(t) - lt \le \liminf_{t\to\infty} X_t - lt = -\infty$. Thus, the right-derivative of C_∞^X is strictly smaller than l on $[0,\infty)$ and its limit as $t \to \infty$ equals l.

For any $a < l$ let S_a be the first time the derivative of C_∞^X is greater than a. Note that for any $T > S_a$, the convex minorant C_T^X agrees with C_∞^X on $[0, S_a]$. Since $S_a \to \infty$ as $a \to l$, this establishes the piecewise linearity of C_∞^X. Let E be an independent exponential time with unit mean and let $T_\lambda := E/\lambda$ for $\lambda > 0$. Note that $C_{T_\lambda}^X$ converges to C_∞^X as $\lambda \to 0$ since, for every $a < l$, both convex minorants agree on $[0, S_a]$ for all sufficiently small λ. Thus, for every $a < l$, the associated PPP encoding the lengths and heights of the faces of $C_{T_\lambda}^X$ (as in Proposition 2.4 and Theorem 3.1) converges to the corresponding point process for the faces of C_∞^X on the set $Z_a := \{(t,x) \in [0,\infty) \times \mathbb{R} : x/t < a\}$ as $\lambda \to 0$, implying that the PPP representation of the faces of C_∞^X mean measure μ_0 holds on Z_a. Since $a < l$ is arbitrary, $S_a \to \infty$ as $a \to l$ and $Z_l = \bigcup_{a<l} Z_a$, the PPP representation extends to $[0,\infty)$, completing the proof. □

Overview of the proof of Theorem 3.1. Theorem 3.1 connects two worlds: (**W1**) X and its convex minorant C_T^X on the interval $[0,T]$ and (**W2**) a random piecewise linear convex function. We first establish a convergence result within (**W2**) for a sequence of piecewise linear convex functions, see Section 3.2. This crucial step in the proof requires only elementary geometric manipulations of piecewise linear convex functions. In Section 3.3, using Abramson and Pitman (2011, Thm 1), we establish a bridge between (**W1**) and (**W2**) for random walks. We recall the 3214 path transformation from Abramson and Pitman (2011) for random walks and provide a short proof, based on the convergence results in Section 3.2, of the connection between (**W1**) and (**W2**) for random walks with general increments, see Theorem 3.7 below. In Section 3.4, we establish Theorem 3.1 by taking the limit of the convex minorant of the random-walk skeleton of X in (**W1**) and, using the convergence results of Section 3.2, the corresponding limit in (**W2**).

We stress that the proof of Theorem 3.1 given in this section is self-contained, requiring only rudimentary real analysis and the fact that X has stationary, independent increments and right-continuous paths with left limits. In particular, we make no use of the Lévy measure, the Lévy-Khintchine formula for X or weak convergence in the J_1-topology on the Skorokhod space.

3.2. *Convex minorants and piecewise linear functions.* Let $[\![n]\!] := \{1,\ldots,n\}$ for $n \in \mathbb{N}$ and $[\![\infty]\!] := \mathbb{N}$. We say that a function $f : [0,T] \to \mathbb{R}$ is piecewise linear if there exists a set consisting of $N \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ pairwise disjoint non-degenerate subintervals $\{(a_n,b_n) : n \in [\![N]\!]\}$ of $[0,T]$ such that $\sum_{n=1}^N (b_n - a_n) = T$ and f is linear on each (a_n,b_n). A face of f, corresponding to a subinterval (a_n,b_n), has length $l_n = b_n - a_n > 0$, height $h_n = f(b_n) - f(a_n) \in \mathbb{R}$ and slope h_n/l_n. Note that, if f is continuous and of finite variation $\sum_{n=1}^N |f(b_n) - f(a_n)| < \infty$, the following representation holds:

$$f(t) = f(0) + \sum_{n=1}^N h_n \min\{\max\{t-a_n,0\}/l_n, 1\}, \qquad t \in [0,T]. \tag{3.3}$$

The number N in representation (3.3) is not unique in general as any face may be subdivided into two faces with the same slope. Moreover, for a fixed f and N, the set of intervals $\{(a_n,b_n) : n \in [\![N]\!]\}$

need not be unique. Furthermore we stress that the sequence of faces in (3.3) does not necessarily respect the chronological order. Put differently, the sequence $(a_n)_{n\in[\![N]\!]}$ need not be increasing. We use the convention $\sum_{k=n}^m=0$ when $n>m$ and denote $x^+:=\max\{x,0\}$ for all $x\in\mathbb{R}$, throughout.

Lemma 3.3. *Fix $T>0$, $N\in\overline{\mathbb{N}}$ and let $l=(l_n)_{n=1}^N$ be a sequence of positive lengths with $\sum_{n=1}^N l_n=T$.*
(a) *For any sequence of heights $h=(h_n)_{n=1}^N$ with $\sum_{n=1}^N|h_n|<\infty$, the function*

$$\begin{aligned} F_{l,h}(t) &:= \sum_{n=1}^N h_n \min\{(t-a_n)^+/l_n, 1\}, \qquad t\in[0,T], \quad \textit{where} \\ a_n &:= \sum_{k=1}^N l_k\cdot \mathbb{1}_{\{h_k/l_k<h_n/l_n\}} + \sum_{k=1}^{n-1} l_k\cdot \mathbb{1}_{\{h_k/l_k=h_n/l_n\}}, \qquad n\in[\![N]\!], \end{aligned} \tag{3.4}$$

is piecewise linear and convex with $F_{l,h}(0)=0$. Differently put, $F_{l,h}$ is linear on each interval (a_n,a_n+l_n) with length l_n and height h_n. Moreover, any piecewise linear convex function started at zero whose faces have lengths l and heights h must equal $F_{l,h}$.
(b) *Suppose $N<\infty$. Given two sequences of heights $h=(h_n)_{n=1}^N$ and $h'=(h'_n)_{n=1}^N$, denote the corresponding functions in (3.4) by $F_{l,h}$ and $F_{l,h'}$ with sequences $(a_n)_{n=1}^N$ and $(a'_n)_{n=1}^N$ of the left endpoints of the intervals on which these functions are linear, respectively. Define the function*

$$G_{l,h,h'}(t) := \sum_{n=1}^N h_n\min\{(t-a'_n)^+/l_n,1\}, \qquad t\in[0,T].$$

Then, we have

$$\max\{\|F_{l,h}-F_{l,h'}\|_\infty, \|F_{l,h'}-G_{l,h,h'}\|_\infty\} \le \max\left\{\sum_{n=1}^N (h_n-h'_n)^+, \sum_{n=1}^N (h'_n-h_n)^+\right\}, \tag{3.5}$$

where $\|f\|_\infty:=\sup_{t\in[0,T]}|f(t)|$ denotes the supremum norm.

The piecewise linear function $G_{l,h,h'}$ need not be convex. However, it can be easily compared (in all cases, including $N=\infty$) with $F_{l,h'}$, because the intervals of linearity for $F_{l,h'}$ and $G_{l,h,h'}$ coincide. The function $G_{l,h,h'}$ will play a key bridging role in the proof of Proposition 3.5 below.

Proof of Lemma 3.3: (a) The lengths of the subintervals (a_n,a_n+l_n), $n\le N$, of $[0,T]$ sum up to $\sum_{n=1}^N l_n=T$. By comparing the respective slopes in the definition of a_n, it follows that these intervals are pairwise disjoint. Moreover, $F_{l,h}$ is convex on $[0,T]$ and linear on every (a_n,a_n+l_n). Indeed, since a function is convex if and only if it has a non-decreasing right-derivative a.e., $F_{l,h}$ is convex. Any other piecewise linear convex function with the same faces must have the same derivative as $F_{l,h}$. Furthermore, if such a function also starts at 0, it must equal $F_{l,h}$.

(b) A termwise comparison shows that

$$-\sum_{n=1}^N (h_n-h'_n)^+ \le F_{l,h'}-G_{l,h,h'} \le \sum_{n=1}^N (h'_n-h_n)^+,$$

pointwise. Thus, it remains to show the inequality for $\|F_{l,h}-F_{l,h'}\|_\infty$, which requires two steps.

Step 1. First assume there exists $m\in[\![N]\!]$ such that $h'_m\ne h_m$ and $h_n=h'_n$ for $n\in[\![N]\!]\setminus\{m\}$. By symmetry we may assume $h'_m>h_m$. For all $n\in[\![N]\!]$, define the slopes $s_n:=h_n/l_n$ and $s'_n:=h'_n/l_n$. Thus $s'_m>s_m$ and, if $n\ne m$, we have $s_n=s'_n$. Since

$$a_n=\sum_{k=1}^N l_k\cdot\mathbb{1}_{\{s_k<s_n\}}+\sum_{k=1}^{n-1} l_k\cdot\mathbb{1}_{\{s_k=s_n\}}, \qquad a'_n=\sum_{k=1}^N l_k\cdot\mathbb{1}_{\{s'_k<s'_n\}}+\sum_{k=1}^{n-1} l_k\cdot\mathbb{1}_{\{s'_k=s'_n\}},$$

the right-derivatives $f_{l,h}$ and $f_{l,h'}$ of $F_{l,h}$ and $F_{l,h'}$, respectively, are piecewise constant non-decreasing functions satisfying $f_{l,h} \le f_{l,h'}$ on $[0,T]$. Since $F_{l,h}(0) = 0 = F_{l,h'}(0)$, we deduce that $F_{l,h'} - F_{l,h} \ge 0$.

By construction, $F_{l,h'} \ge G_{l,h,h'}$ pointwise (in fact, termwise) and $\|F_{l,h'} - G_{l,h,h'}\|_\infty = h'_m - h_m$. Put $b_n := a_n + l_n$ and $b'_n := a'_n + l_n$ for $n \in [\![N]\!]$ and note that, since $s_m \le s'_m$, we have $a_m \le a'_m$ and

$$\begin{aligned} &F_{l,h}(t) = G_{l,h,h'}(t) = F_{l,h'}(t), && \text{for } t \in [0, a_m],\\ &F_{l,h}(t) = G_{l,h,h'}(t) \le F_{l,h'}(t) \le G_{l,h,h'}(t) + (h'_m - h_m), && \text{for } t \in [b'_m, T]. \end{aligned}$$

Thus, to establish (3.5) in this case, it suffices to prove that $F_{l,h'}(t) - F_{l,h}(t) \le h'_m - h_m$ on $t \in [a_m, b'_m]$.

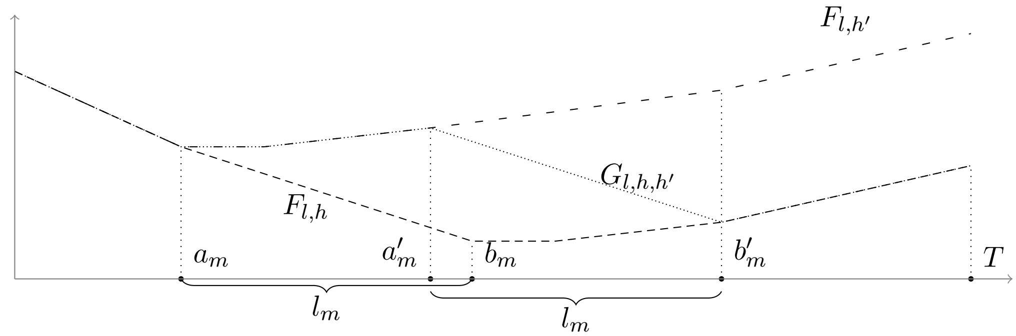

FIGURE 3.1. Comparison between $F_{l,h}$, $F_{l,h'}$ and $G_{l,h,h'}$.

By construction of a'_m, the right-derivative $f_{l,h'}$ is smaller or equal to $s'_m = h'_m/l_m$ on (a_m, b'_m). Since $F_{l,h'}(a_m) = F_{l,h}(a_m)$, for $t \in [a_m, b_m]$ we have

$$F_{l,h'}(t) - F_{l,h}(t) = \int_{a_m}^t f_{l,h'}(u)du - s_m(t - a_m) \le (s'_m - s_m)(t - a_m) \le (s'_m - s_m)l_m = h'_m - h_m.$$

For $t \in [b_m, b'_m]$ we have $t - l_m \in [a_m, a'_m]$ and thus $F_{l,h}(t) - h_m = G(t - l_m) = F_{l,h'}(t - l_m)$. Hence

$$F_{l,h'}(t) - F_{l,h}(t) = F_{l,h'}(t) - F_{l,h'}(t - l_m) - h_m = \int_{t-l_m}^t f_{l,h'}(u)du - h_m \le \int_{t-l_m}^t s'_m du - h_m = h'_m - h_m.$$

Thus, $F_{l,h'} - F_{l,h} \le h'_m - h_m$ on $[a_m, b'_m]$, proving (3.5) in this case.

Step 2. Consider the general case. For $k \in \{0, \ldots, N\}$, let $h^{(k)} = (h_n^{(k)})_{n \in [\![N]\!]}$ be given by $h_n^{(k)} := h_n \cdot \mathbb{1}_{\{n>k\}} + h'_n \cdot \mathbb{1}_{\{n \le k\}}$ for $n \in [\![N]\!]$. Note that $h' = h^{(0)}$ and $h = h^{(N)}$. Since the sequences $h^{(k)}$ and $h^{(k-1)}$ only differ in the coordinate $h_k^{(k)} \ne h_k^{(k-1)}$, the identity $F_{l,h'} - F_{l,h} = \sum_{k=1}^N (F_{l,h^{(k-1)}} - F_{l,h^{(k)}})$ and **Step 1** imply (3.5), completing the proof. □

Lemma 3.4. *Let $(N_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with a limit $N_k \to N_\infty \in \mathbb{N}$. For each $j \in \overline{\mathbb{N}}$, let $(l_{j,n})_{n\in[\![N_j]\!]}$ be positive numbers satisfying $\sum_{n=1}^{N_j} l_{j,n} = T$, $(h_{j,n})_{n\in[\![N_j]\!]}$ real numbers and C_j the piecewise linear convex function defined in (3.4) with lengths $(l_{j,n})_{j\in[\![N_j]\!]}$ and heights $(h_{j,n})_{j\in[\![N_j]\!]}$. Suppose $l_{k,n} \to l_{\infty,n}$ and $h_{k,n} \to h_{\infty,n}$ as $k \to \infty$ for all $n \in [\![N_\infty]\!]$. Then $\|C_\infty - C_k\|_\infty \to 0$ as $k \to \infty$.*

Proof: The convergence $N_k \to N_\infty$ as $k \to \infty$ implies $N_k = N_\infty = N$ for all sufficiently large k. Thus, we assume without loss of generality that $N_j = N$ for all $j \in \overline{\mathbb{N}}$. Define $s_{j,n} := h_{j,n}/l_{j,n}$ for $j \in \overline{\mathbb{N}}$ and $n \in [\![N]\!]$ and note that $s_{k,n} \to s_{\infty,n}$ as $k \to \infty$ for all $n \in [\![N]\!]$. Thus, for all sufficiently large k, if the inequality $s_{\infty,n} < s_{\infty,m}$ holds, then $s_{k,n} < s_{k,m}$. Thus, we assume this property holds for all $k \in \mathbb{N}$. Moreover, we assume without loss of generality, by relabeling if necessary, that $s_{\infty,1} \le \cdots \le s_{\infty,N}$.

We will next introduce a sequence of convex functions F_k satisfying the limits $\|C_\infty - F_k\|_\infty \to 0$ and $\|F_k - C_k\|_\infty \to 0$ as $k \to \infty$. These convex functions will replace each "block" of faces of C_k with a given common *limiting* slope, with a single face with the mean slope.

Let $M \le N$ be the number of distinct slopes in $\{s_{\infty,n} : n \in [\![N]\!]\}$ and note that $s_{\infty,i_1} < \cdots < s_{\infty,i_M}$, where we set $i_1 \coloneqq 1$ and $i_{n+1} \coloneqq \min\{m \in \{i_n+1,\dots,N\} : s_{\infty,m} > s_{\infty,i_n}\}$ for $n \in [\![M-1]\!]$. Note that $s_{k,m} \to s_{\infty,i_n}$ as $k \to \infty$ for all $m \in \{i_n,\dots,i_{n+1}-1\}$. Let $L_{j,n} \coloneqq \sum_{m=i_n}^{i_{n+1}-1} l_{j,m}$ and $H_{j,n} \coloneqq \sum_{m=i_n}^{i_{n+1}-1} h_{j,m}$ for $n \in [\![M]\!]$ and $j \in \overline{\mathbb{N}}$, where $i_{M+1} \coloneqq N+1$. Furthermore, for $j \in \overline{\mathbb{N}}$, let $(a_{j,n})_{n\in[\![N]\!]}$ be the left endpoints of the intervals in (3.4) on which C_j is linear. Note that C_∞ admits the representation $C_\infty(t) = \sum_{n=1}^M H_{\infty,n} \min\{(t-a_{\infty,i_n})^+/L_{\infty,n}, 1\}$ for $t \in [0,T]$ and define the convex functions $F_k(t) \coloneqq \sum_{n=1}^M H_{k,n} \min\{(t-a_{k,i_n})^+/L_{k,n}, 1\}$ for $k \in \mathbb{N}$. The limits $l_{k,n} \to l_{\infty,n}$ and $h_{k,n} \to h_{\infty,n}$ imply $a_{k,i_n} \to a_{\infty,i_n}$, $L_{k,n} \to L_{\infty,n}$ and $H_{k,n} \to H_{\infty,n}$ as $k \to \infty$ for $n \in [\![M]\!]$. Thus, we have the pointwise (in fact, termwise) convergence $F_k \to C_\infty$. Since the functions are convex, the pointwise convergence implies $\|C_\infty - F_k\|_\infty \to 0$ as $k \to \infty$.

To prove that $\|F_k - C_k\|_\infty \to 0$ as $k \to \infty$, note that for $a, c \in \mathbb{R}$ and $b, d > 0$ satisfying $a/b \le c/d$, we have $a/b \le (a+c)/(b+d) \le c/d$. Thus, $H_{k,n}/L_{k,n}$ lies between the smallest and largest values of $S_{k,n} \coloneqq \{h_{k,i_n}/l_{k,i_n},\dots,h_{k,i_{n+1}-1}/l_{k,i_{n+1}-1}\}$. Since all the slopes in $S_{k,n}$ converge to s_{∞,i_n}, by the triangle inequality, we have $\max_{s\in S_{k,n}} |H_{k,n}/L_{k,n} - s| \le \max_{s,s'\in S_{k,n}} |s'-s| \le b_k \coloneqq 2\max_{m\in[\![N]\!]} |s_{k,m} - s_{\infty,m}| \to 0$ as $k \to \infty$. Hence, the right-derivative of F_k is at most b_k away from the right-derivative of C_k, implying $\|F_k - C_k\|_\infty \le b_k T \to 0$ as $k \to \infty$, completing the proof. □

Proposition 3.5. *Let N_k and N_∞ be $\overline{\mathbb{N}}$-valued random variables with $N_k \to N_\infty$ a.s. as $k \to \infty$. Let $(l_{j,n})_{n=1}^{N_j}$, $j \in \overline{\mathbb{N}}$, be random sequences of positive numbers satisfying $\sum_{n=1}^{N_j} l_{j,n} = T$ and $(h_{j,n})_{n=1}^{N_j}$, $j \in \overline{\mathbb{N}}$, sequences of random variables with $\sum_{n=1}^{N_j} |h_{j,n}| < \infty$ a.s. Let C_j be the piecewise linear convex function in (3.4) with sequences of lengths $(l_{j,n})_{n=1}^{N_j}$ and heights $(h_{j,n})_{n=1}^{N_j}$ for $j \in \overline{\mathbb{N}}$. Suppose $l_{k,n} \to l_{\infty,n}$ a.s. and $h_{k,n} \to h_{\infty,n}$ a.s. as $k \to \infty$ for all $n < N_\infty + 1$ and*

$$\lim_{M\to\infty} \limsup_{k\to\infty} \mathbb{E} \min\left\{1, \sum_{n=M}^{N_k} |h_{k,n}|\right\} = 0. \tag{3.6}$$

Then $\|C_\infty - C_k\|_\infty \xrightarrow{\mathbb{P}} 0$ as $k \to \infty$.

Proof: On the event $\{N_\infty < \infty\}$, by Lemma 3.4 we have $\|C_\infty - C_k\|_\infty \to 0$ a.s. as $k \to \infty$ (and (3.6) holds by our summing convention). Assume we are on the event $\{N_\infty = \infty\}$. For each $M \in \mathbb{N}$ and $j \in \overline{\mathbb{N}}$, let $C_{j,M}$ be the piecewise linear convex function in (3.4) with lengths $(l_{j,n})_{n=1}^{N_j}$ and heights $(h_{j,n}\mathbb{1}_{\{n<M\}})_{n=1}^{N_j}$. Recall $a \wedge b = \min\{a,b\}$ for any $a, b \in \mathbb{R}$. For each $j \in \overline{\mathbb{N}}$, define

$$a_{j,n} \coloneqq \sum_{m=1}^{N_j} l_{j,m} \cdot \mathbb{1}_{\{h_{j,m}/l_{j,m} < h_{j,n}/l_{j,n}\}} + \sum_{m=1}^{n-1} l_{j,m} \cdot \mathbb{1}_{\{h_{j,m}/l_{j,m} = h_{j,n}/l_{j,n}\}}, \qquad n \in [\![N_j]\!],$$

end the function $G_{j,M}(t) \coloneqq \sum_{m=1}^{N_j\wedge(M-1)} h_{j,n} \min\{(t-a_{j,n})^+/l_{j,n}, 1\}$, $t \in [0,T]$. Note that C_j and $G_{j,M}$ are linear on every interval $(a_{j,n}, a_{j,n}+l_{j,n})$, $n \in [\![N_j]\!]$, but $C_{j,M}$ may have different intervals of linearity. Since $1 \wedge (x+y) \le 1 \wedge x + 1 \wedge y$ for all $x, y \ge 0$, the triangle inequality implies

$$1 \wedge \|C_\infty - C_k\|_\infty \le A_{\mathbf{(I)}} + A_{\mathbf{(II)}} + A_{\mathbf{(III)}} + A_{\mathbf{(IV)}} + A_{\mathbf{(V)}}, \tag{3.7}$$

where $A_{\mathbf{(I)}} \coloneqq 1 \wedge \|C_\infty - G_{\infty,M}\|_\infty$, $A_{\mathbf{(II)}} \coloneqq 1 \wedge \|G_{\infty,M} - C_{\infty,M}\|_\infty$, $A_{\mathbf{(III)}} \coloneqq 1 \wedge \|C_{\infty,M} - C_{k,M}\|_\infty$, $A_{\mathbf{(IV)}} \coloneqq 1 \wedge \|C_{k,M} - G_{k,M}\|_\infty$ and $A_{\mathbf{(V)}} \coloneqq 1 \wedge \|G_{k,M} - C_k\|_\infty$. As $\zeta_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ if and only if

$\mathbb{E}[1 \wedge |\zeta_n|] \to 0$, it suffices to prove that the expectation of each of the terms in (3.7) converges to 0 as we take $\limsup_{k\to\infty}$ and then $M \to \infty$.

(I)&**(V)**. By construction of C_j and $G_{j,M}$ we have $\|C_j - G_{j,M}\|_\infty \le \sum_{n=M}^{N_j} |h_{j,n}|$ for all $j \in \overline{\mathbb{N}}$. Thus, $\|C_\infty - G_{\infty,M}\|_\infty \to 0$ a.s. and hence $\mathbb{E}A_{\textbf{(I)}} = \mathbb{E}[1 \wedge \|C_\infty - G_{\infty,M}\|_\infty] \to 0$ as $M \to \infty$. Moreover, by assumption in (3.6), we have

$$\limsup_{k\to\infty} \mathbb{E}A_{\textbf{(V)}} = \limsup_{k\to\infty} \mathbb{E}[1 \wedge \|C_k - G_{k,M}\|_\infty] \le \limsup_{k\to\infty} \mathbb{E}\min\left\{1, \sum_{n=M}^{N_k} |h_{k,n}|\right\} \xrightarrow[M\to\infty]{} 0.$$

(III). For all $j \in \overline{\mathbb{N}}$, the faces of $C_{j,M}$ corresponding to $n \in [\![N_j]\!] \setminus [\![M-1]\!]$ are horizontal. By convexity, we may assume they lie next to each other in the graph of $C_{j,M}$. Merging all the lengths $l_{j,n}$, $n \in [\![N_j]\!] \setminus [\![M-1]\!]$, yields a representation of $C_{j,M}$ with $N_j \wedge M$ faces. Fix $M \in \mathbb{N}$. Lemma 3.4 yields $\|C_{\infty,M} - C_{k,M}\|_\infty \to 0$ a.s. and thus $\mathbb{E}A_{\textbf{(III)}} = \mathbb{E}[1 \wedge \|C_{\infty,M} - C_{k,M}\|_\infty] \to 0$ as $k \to \infty$.

(II)&**(IV)**. The idea is to apply (3.5) in Lemma 3.3(b) to bound $\|C_{j,M} - G_{j,M}\|_\infty$, with $F_{l,h}$, $G_{l,h,h'}$ and $F_{l,h'}$ in Lemma 3.3(b) given by $C_{j,M}$, $G_{j,M}$ and $F_{j,M}$, respectively. The piecewise linear convex function $F_{j,M}$, which shares the intervals of linearity with those of $G_{j,M}$, is yet to be defined.

Note that $G_{j,M}$ possesses a piecewise linear representation with at most $2M$ faces. Indeed, $G_{j,M}$ is linear on $(a_{j,n}, a_{j,n}+l_n)$, $n \in [\![N_j \wedge (M-1)]\!]$, and the complement $(0,T)\setminus\bigcup_{n=1}^{N_j\wedge(M-1)}[a_{j,n}, a_{j,n}+l_{j,n}]$ is a disjoint union of $M_j \le M+1$ open intervals, say $(a'_{j,n}, a'_{j,n}+l'_{j,n})$, $n \in [\![M_j]\!]$. For each $n \in [\![M_j]\!]$, define the height $h'_{j,n} \coloneqq \sum_{m\in S_{j,n}} h_{j,m}$, where $S_{j,n} \coloneqq \{m \in [\![N_j]\!]\setminus[\![M-1]\!] : a_{j,m} \in (a'_{j,n}, a'_{j,n}+l'_{j,n})\}$. Put differently, the height $h'_{j,n}$ equals the sum of all the heights of the faces of C_j that lie above the interval $[a'_{j,n}, a'_{j,n} + l'_{j,n}]$. For any $j \in \overline{\mathbb{N}}$, define the function

$$F_{j,M}(t) \coloneqq \sum_{n=1}^{N_j\wedge(M-1)} h_{j,n} \min\{(t-a_{j,n})^+/l_{j,n}, 1\} + \sum_{n=1}^{M_j} h'_{j,n} \min\{(t-a'_{j,n})^+/l'_{j,n}, 1\}, \quad t \in [0,T]. \tag{3.8}$$

We will show that $F_{j,M}$ is convex. It suffices to prove that the consecutive slopes of $F_{j,M}$ on adjacent intervals of linearity increase. If the consecutive intervals are $(a_{j,m}, a_{j,m} + l_{j,m})$ and $(a_{j,n}, a_{j,n} + l_{j,n})$ (i.e. they come from the first sum in (3.8)), then by construction the intervals must be adjacent with the same slopes in the convex function C_j, implying the corresponding slopes satisfy the correct ordering. Assume the consecutive intervals are $(a_{j,m}, a_{j,m} + l_{j,m})$ and $(a'_{j,n}, a'_{j,n} + l'_{j,n})$ (i.e. the first interval comes from first sum and the second interval comes from the second sum in (3.8)). Suppose $a_{j,m} = a'_{j,n} + l'_{j,n}$ and note that, for $a, c \in \mathbb{R}$ and $b, d > 0$ with $a/b \le c/d$ we have $a/b \le (a+c)/(b+d) \le c/d$. Thus, by definition of $h'_{j,n}$, we have $h'_{j,n}/l'_{j,n} \le \sup_{i\in S_{j,n}} h_{j,i}/l_{j,i} \le h_{j,m}/l_{j,m}$, where the last inequality holds because $a_{j,m} = a'_{j,n} + l'_{j,n}$ and C_j is convex. The case $a'_{j,n} = a_{j,m} + l_{j,n}$ is analogous since the slope $h'_{j,n}/l'_{j,n}$ is a mean of slopes at least as large as $h_{j,m}/l_{j,m}$, implying the convexity of $F_{j,M}$.

Define $l = (l_{j,1}, \dots, l_{j,N_j\wedge(M-1)}, l'_{j,1}, \dots, l'_{j,M_j})$, $h = (h_{j,1}, \dots, h_{j,N_j\wedge(M-1)}, h'_{j,1}, \dots, h'_{j,M_j})$ and $h' = (h_{j,1}, \dots, h_{j,N_j\wedge(M-1)}, 0, \dots, 0)$. Note that the corresponding functions $F_{l,h}$, $F_{l,h'}$ and $G_{l,h,h'}$ in (3.4) equal $F_{j,M}$, $C_{j,M}$ and $G_{j,M}$, so (3.5) implies the inequality

$$\|G_{j,M} - C_{j,M}\|_\infty \le \|G_{j,M} - F_{j,M}\|_\infty + \|F_{j,M} - C_{j,M}\|_\infty \le 2\sum_{m=1}^{M_j} |h'_{j,m}| \le 2\sum_{n=M}^{N_j} |h_{j,n}|.$$

Thus, $\|G_{\infty,M} - C_{\infty,M}\|_\infty \to 0$ a.s. and hence $\mathbb{E}A_{\textbf{(II)}} = \mathbb{E}[1 \wedge \|C_\infty - G_{\infty,M}\|_\infty] \to 0$ as $M \to \infty$. Moreover, by assumption in (3.6), we have

$$\limsup_{k\to\infty} \mathbb{E}A_{\textbf{(IV)}} = \limsup_{k\to\infty} \mathbb{E}[1 \wedge \|G_{k,M} - C_{k,M}\|_\infty] \le 2\limsup_{k\to\infty} \mathbb{E}\min\left\{1, \sum_{n=M}^{N_k} |h_{k,n}|\right\} \xrightarrow[M\to\infty]{} 0. \quad \square$$

3.3. *The convex minorant of random walks.* Let a function $f:[0,T]\to\mathbb{R}$ satisfy $f(0)=0$. Given parameters $0\le g\le u\le d\le T$, the *3214 transformation*, introduced in Abramson and Pitman (2011), is defined by

$$\Theta_{g,u,d}f(t)=\begin{cases} f(u+t)-f(u), & 0\le t\le d-u,\\ f(d)-f(u)+f(g+t-(d-u))-f(g), & d-u<t\le d-g,\\ f(d)-f(t-(d-g)), & d-g<t\le d,\\ f(t), & d<t.\end{cases}$$

The 3214 transformation reorders the segments of the graph of f as follows: the segments (I) $[0,g]$, (II) $[g,u]$, (III) $[u,d]$ and (IV) $[d,T]$ are moved to (III) $[0,d-u]$, (II) $[d-u,d-g]$, (I) $[d-g,d]$ and (IV) $[d,T]$, respectively (see also Figure 3.2 below). This transformation possesses the following remarkable property when applied to continuous piecewise linear functions with a given set of increments.

Proposition 3.6 (Abramson and Pitman, 2011, Thm 1)**.** *Fix $n\in\mathbb{N}$ and let $x_1,\ldots,x_n$ be real numbers, such that no two subsets have the same mean. Let $\lfloor y\rfloor:=\max\{m\in\mathbb{Z}:m\le y\}$, $y\in\mathbb{R}$, and $\pi:[\![n]\!]\to[\![n]\!]$ be a uniform random permutation. Define the piecewise linear random function $R=(R(t))_{t\in[0,T]}$ by $R(T):=\sum_{k=1}^n x_k$ and*

$$R(t):=(nt/T-\lfloor nt/T\rfloor)\,x_{\pi(\lfloor nt/T\rfloor+1)}+\sum_{k=1}^{\lfloor nt/T\rfloor}x_{\pi(k)},\qquad t\in[0,T). \tag{3.9}$$

Let C_T^R denote the convex minorant of R and let $W\sim\mathrm{U}(0,T)$ be independent of R. Let $0=V_0<\cdots<V_N=T$ be the sequence of contact points between the piecewise linear functions R and C_T^R and $j\in[\![N]\!]$ the unique index such that $W\in(V_{j-1},V_j]$. Define $U:=\lceil Wn/T\rceil\,T/n$, $G:=V_{j-1}$ and $D:=V_j$. Then the 3214 transform with parameters (G,U,D) satisfies the identity in law

$$(U,R)\overset{d}{=}(D-G,\Theta_{G,U,D}R).$$

We recall below a proof of Proposition 3.6 based on a simple argument from Abramson et al. (2011).

Theorem 3.7. *Let $x_1,\ldots,x_n$ be arbitrary real numbers and $\pi:[\![n]\!]\to[\![n]\!]$ a uniform random permutation. Define R by (3.9) and let $(V_k)_{k\in\mathbb{N}}$ be an iid sequence of $\mathrm{U}(0,1)$ random variables independent of π. Define recursively $L_{n,0}:=T$, $L_{n,k}:=\lfloor L_{n,k-1}V_kn/T\rfloor T/n$, $\ell_{n,k}:=L_{n,k-1}-L_{n,k}$ for $k\in\mathbb{N}$ and let $N\le n$ be the largest integer for which $\ell_{n,N}>0$. Then the convex minorant C_T^R has the same law as the piecewise linear convex function defined in (3.4) with sequences of lengths $(\ell_{n,k})_{k=1}^N$ and heights $(R(L_{n,k-1})-R(L_{n,k}))_{k=1}^N$.*

We stress that in Theorem 3.7, the reals $x_1,\ldots,x_n$ may have multiple subsets with the same mean. Our proof approximates a general sequence by one satisfying the "no ties" assumption of Proposition 3.6 and applies a convergence result for piecewise linear convex functions from Section 3.2. The proof of Theorem 3.7 in Abramson and Pitman (2011) sub-samples the ties, resulting in a more involved statement of the theorem.

Proof of Theorem 3.7: First assume that no two subsets of the numbers $x_1,\ldots,x_n$ have the same mean. Let π and (G,U,D) be as in Proposition 3.6. By Proposition 3.6, the face decomposition of C_T^R contains the face with length-height pair $(D-G,C_T^R(D)-C_T^R(G))$, which has the same law as $(U,R(U))$, and the faces of a copy of $C_{T-(D-G)}^R$ independent of the first face. Indeed, this copy is in fact the convex minorant of $\Theta_{G,U,D}R$ on $[T-(D-G),T]$ and we may apply the same procedure to this copy. Iterating this procedure, we obtain a (finite) sequence of lengths of the faces of C_T^R, which has the same law as the sequence $(\ell_{k,n})_{k=1}^N$, and the corresponding heights, which have the same law as $(R(L_{k-1,n}) - R(L_{k,n}))_{k=1}^N$, completing the proof in this case.

To prove the general case, we recall that $\|C_T^x - C_T^y\|_\infty \le \|x-y\|_\infty$ for any bounded functions $x, y : [0,T] \to \mathbb{R}$. Indeed, this follows from the fact that $C_T^x - \|x-y\|_\infty$ is convex and $C_T^x - \|x-y\|_\infty \le x - \|x-y\|_\infty \le y$ pointwise. For any $\varepsilon > 0$ consider real numbers $x_{1,\varepsilon}, \ldots, x_{n,\varepsilon}$ such that no two subsets have the same mean and $\sum_{k=1}^n |x_k - x_{k,\varepsilon}| \le \varepsilon$. Let R_ε be the corresponding random function in (3.9) (with the same permutation π). Note that $\|C_T^R - C_T^{R_\varepsilon}\|_\infty \le \|R - R_\varepsilon\|_\infty \le \varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, by the argument in the previous paragraph, $C_T^{R_\varepsilon}$ has the same law as the piecewise linear convex function C_ε given by (3.4) with lengths $(\ell_{n,k})_{k=1}^N$ and heights $(R_\varepsilon(L_{n,k-1}) - R_\varepsilon(L_{n,k}))_{k=1}^N$. Let C be the piecewise linear convex function given by (3.4) with lengths $(\ell_{n,k})_{k=1}^N$ and heights $(R(L_{n,k-1}) - R(L_{n,k}))_{k=1}^N$. Lemma 3.4 yields $\|C - C_\varepsilon\|_\infty \to 0$ a.s. as $\varepsilon \to 0$, implying $C \overset{d}{=} C_T^R$ and completing the proof. □

The proof of Proposition 3.6 requires the following lemma.

Lemma 3.8. *Let $x_1, \ldots, x_n$ be real numbers such that no two subsets have the same mean. Then there is a unique $k^* \in [\![n]\!]$ such that $\sum_{i=1}^k x_{(k^*+i) \bmod n} \ge \frac{k}{n} \sum_{i=1}^n x_i$ for all $k \in [\![n]\!]$, i.e. the walk with increments $x_{(k^*+1) \bmod n}, \ldots, x_{(k^*+n) \bmod n}$ is above the line connecting zero with the endpoint $\sum_{i=1}^n x_i$.*

Proof: Define $s \coloneqq \sum_{i=1}^n x_i/n$. If the walk $k \mapsto \sum_{i=1}^k (x_i - s)$, $k \in [\![n]\!]$, attained its minimum at two times $k_1 < k_2$, then $\sum_{i=k_1+1}^{k_2} x_i/(k_2 - k_1) = s$, contradicting the assumption. It is easily seen that the k^* in the statement of the lemma is the time at which this walk attains its minimum on $[\![n]\!]$. □

Proof of Proposition 3.6 (Abramson et al., 2011): Note that, if a random element ζ is uniformly distributed in some finite set $\mathcal{Z}$ and if the map $\varphi : \mathcal{Z} \to \mathcal{Z}$ is injective (and thus bijective), then $\varphi(\zeta)$ is also uniformly distributed on $\mathcal{Z}$. Thus, since π and U are uniform and independent, it is sufficient to show that the transformation $(u, f) \mapsto (d - g, \Theta_{g,u,d} f)$ is injective.

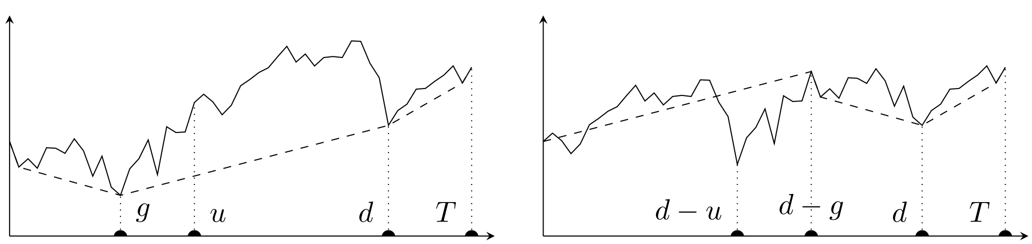

FIGURE 3.2. The pictures show a path of a random walk R (*solid*) and its convex minorant C_T^R (*dashed*) on $[0,T]$ on the left and their 3214 transforms on the right. The transform is associated to some $u \in (0,T)$ and the endpoints $\{g, d\}$ of the maximal face of C_T^R containing u.

Assume without loss of generality that $T = n$. To prove the injectivity, it suffices to describe the inverse transformation. Given $d - g$, and $\tilde{f} \coloneqq \Theta_{g,u,d} f$, note that $d - u$ is the unique time in Lemma 3.8 for the increments of $\tilde{f}$ over the set $[\![d-g]\!]$, see Figure 3.2. Consider the convex minorant of $\tilde{f}$ on the interval $[d-g, T]$ and note that d is the right end of the last face whose slope is less than $\tilde{f}(d-g)/(d-g)$. Thus we may identify d, u and g and then invert the 3214 transform to recover f. This shows that $(u, f) \mapsto (d - g, \Theta_{g,u,d} f)$ is injective, completing the proof. □

3.4. *Proof of Theorem 3.1.* **Step 1.** Let $\widetilde{C}_k$ be the largest convex function on $[0,T]$ that is smaller than X pointwise on the set $D_k := \{Tn/2^k : n \in \{0,1,\ldots,2^k\}\}$. Since $D_k \subset D_{k+1}$, we have $\widetilde{C}_k(t) \geq \widetilde{C}_{k+1}(t)$ for all $t \in [0,T]$. Moreover, the limit $\widetilde{C}_\infty := \lim_{k\to\infty} \widetilde{C}_k$ is clearly convex and smaller than X pointwise on the dense set $\bigcup_{k\in\mathbb{N}} D_k$ in $[0,T]$. As X is càdlàg, $\widetilde{C}_\infty$ is pointwise smaller than X on $[0,T]$, implying $\widetilde{C}_\infty \leq C_T^X$. Since C_T^X is convex and smaller than X on D_k, the maximality of $\widetilde{C}_k$ yields $\widetilde{C}_k \geq C_T^X$ for all $k \in \mathbb{N}$, implying $\widetilde{C}_\infty \geq C_T^X$ and thus $\widetilde{C}_\infty = C_T^X$.

Step 2. Let $U_1, U_2, \ldots$ be an iid sequence of $\mathrm{U}(0,1)$ random variables independent of X. Let $L_0 := T$, $L_n := U_n L_{n-1}$, $\ell_n := L_{n-1} - L_n$ and $\xi_n := X_{L_{n-1}} - X_{L_n}$ for $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, define $L_{k,0} := T$, $L_{k,n} := \lfloor L_{k,n-1} U_n 2^k/T \rfloor T/2^k$, $\ell_{k,n} := L_{k,n-1} - L_{k,n}$ and $\xi_{k,n} := X_{L_{k,n-1}} - X_{L_{k,n}}$ for $n \in \mathbb{N}$. Let N_k be the largest natural number for which $\ell_{k,N_k} > 0$, so that $\ell_{k,n}$, $L_{k,n}$ and $\xi_{k,n}$ are all zero for all $n > N_k$. For each $k \in \mathbb{N}$, let C_k (resp. C_∞) be the piecewise linear convex function given in (3.4) with lengths $(\ell_{k,n})_{n=1}^{N_k}$ (resp. $(\ell_n)_{n=1}^\infty$) and heights $(\xi_{k,n})_{n=1}^{N_k}$ (resp. $(\xi_n)_{n=1}^\infty$). Next we show that $\|C_k - C_\infty\|_\infty \xrightarrow{\mathbb{P}} 0$ as $k \to \infty$. Since X has càdlàg paths with countably many jumps, L_n has a density for every $n \in \mathbb{N}$ and $L_{k,n} \to L_n$ a.s. as $k \to \infty$, we have $\xi_{k,n} \to \xi_n$ a.s. as $k \to \infty$ for all $n \in \mathbb{N}$. Thus, by Proposition 3.5, it suffices to prove that $\lim_{M\to\infty} \limsup_{k\to\infty} \mathbb{E}[1 \wedge P_{k,M}] = 0$, where $P_{k,M} := \sum_{n=M}^{N_k} |\xi_{k_m,n}|$.

Theorem 3.7 implies that $C_k \stackrel{d}{=} \widetilde{C}_k$. Let R_k be the continuous piecewise linear function connecting the skeleton of X on D_k with line segments. Since the minimum and the final value of the convex minorant $C_T^{R_k} = \widetilde{C}_k$ agree with the corresponding functionals of R_k, the total variation $\sum_{n=1}^{N_k} |\xi_{k,n}|$ of C_k has the same distribution as $X_T - 2\min_{t\in D_k} X_t$. Moreover, by the independence and the definition of $(L_{k,n})_{n\in\mathbb{N}}$, it is easily seen that $P_{k,M} = \sum_{n=M+1}^{N_k} |\xi_{k,n}| \stackrel{d}{=} X_{L_{k,M}} - 2\min_{t\in D_k\cap[0,L_{k,M}]} X_t$. By the inequality $L_{k,M} \leq L_M$, we have

$$X_{L_{k,M}} - 2 \min_{t\in D_k\cap[0,L_{k,M}]} X_t \leq X_{L_{k,M}} - 2\underline{X}_{L_{k,M}} \leq \overline{X}_{L_M} - 2\underline{X}_{L_M}.$$

Since $L_M \to 0$ a.s. as $M \to \infty$ and $\overline{X}_t - 2\underline{X}_t \to 0$ a.s. as $t \to 0$, we have $\overline{X}_{L_M} - 2\underline{X}_{L_M} \to 0$ a.s. as $M \to \infty$, implying

$$\limsup_{k\to\infty} \mathbb{E}[1 \wedge P_{k,M}] \leq \mathbb{E}[1 \wedge (\overline{X}_{L_M} - 2\underline{X}_{L_M})] \xrightarrow[M\to\infty]{} 0.$$

Step 3. Recall that, by Theorem 3.7, we have $C_k \stackrel{d}{=} \widetilde{C}_k$. Since the limits $\|C_k - C_\infty\|_\infty \xrightarrow{\mathbb{P}} 0$ and $\|\widetilde{C}_k - C_T^X\|_\infty \to 0$ hold a.s. as $k \to \infty$, we conclude that $C_\infty \stackrel{d}{=} C_T^X$, implying Theorem 3.1.

Appendix A. Sticks on an exponential interval are a Poisson point processes

For $n \geq 2$, the Dirichlet law on the simplex $\{(x_1,\ldots,x_n) \in (0,1]^n : \sum_{i=1}^n x_i = 1\}$ with parameters $\theta_i > 0$ has a density proportional to $(x_1,\ldots,x_n) \mapsto \prod_{i=1}^n x_i^{\theta_i - 1}$. D is a Dirichlet random measure on $(0,1]$ if for any $0 = t_0 < t_1 < \ldots < t_n = 1$, the random vector $(D((t_0,t_1]),\ldots,D((t_{n-1},t_n]))$ follows the Dirichlet law with parameters $(t_i - t_{i-1})$. Let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ be independent iid $\mathrm{U}(0,1)$ sequences, independent of a Dirichlet random measure D_0 on $(0,1]$. Elementary calculations imply that $D_1 := (1-V_1)\delta_{U_1} + V_1 D_0 \stackrel{d}{=} D_0$ and hence $D_n := (1-V_n)\delta_{U_n} + V_n D_{n-1} \stackrel{d}{=} D_0$ for all $n \in \mathbb{N}$. Since D_n converges to $D_\infty := \sum_{n\in\mathbb{N}} \ell_n \delta_{U_n}$ in total variation, where $\ell_n := (1-V_n)\prod_{k=1}^{n-1} V_k$ is a uniform stick-breaking process on $[0,1]$, we have $D_0 \stackrel{d}{=} D_\infty$. Moreover, by construction we have $\sum_{n\in\mathbb{N}} (\ell_n^{(\theta)}/T_\theta)\delta_{U_n} \stackrel{d}{=} D_\infty$, where $(\ell_n^{(\theta)})_{n\in\mathbb{N}}$ is a stick-breaking process on an independent exponential time horizon $T_\theta \sim \mathrm{Exp}(\theta)$.

Let G be a gamma subordinator (i.e. G_t has density proportional to $s \mapsto s^{t-1}e^{-\theta s}$). The jump of G at $t > 0$, $\Delta G_t := G_t - \lim_{s\uparrow t} G_s$, is zero for all but countably many t, making $D' :=$

$\sum_{t\in(0,1]}((\Delta G_t)/G_1)\,\delta_t$ a Dirichlet random measure on $(0,1]$, independent of $G_1 \sim \mathrm{Exp}(\theta)$. Indeed, for any $0 = t_0 < t_1 < \ldots < t_n = 1$, the Jacobian change-of-variable formula shows that the vector

$$\big(D'((t_0,t_1]),\ldots,D'((t_{n-1},t_n]),G_1\big) = ((G_{t_1}-G_{t_0})/G_1,\ldots,(G_{t_n}-G_{t_{n-1}})/G_1,G_1)$$

has the desired law. Thus $(D',G_1) \overset{d}{=} (\sum_{n\in\mathbb{N}}(\ell_n^{(\theta)}/T_\theta)\delta_{U_n},T_\theta)$, implying that the law of the Poisson point process $\sum_{t\in(0,1]}\mathbb{1}_{\Delta G_t>0}\delta_{\Delta G_t}$ coincides with that of the random measure $\sum_{n\in\mathbb{N}}\delta_{\ell_n^{(\theta)}}$.

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