ALEA, Lat. Am. J. Probab. Math. Stat. 19, 1001–1024 (2022)
DOI: 10.30757/ALEA.v19-40

Functional central limit theorem for tagged particle dynamics in stochastic ranking process with space-time dependent intensities

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Abstract. In this paper, we consider a "parabolic" scaling limit of tagged particle dynamics and that of empirical measure of the position of particles for stochastic ranking process with space-time dependent intensities. A stochastic ranking process is driven according to an algorithm for a self-organizing linear list of a finite number of items. We regard this process as a particle system. We fasten a tag to a "particle" (item) and observe the (normalized) motion of the "tagged particle". We obtain a sum of diffusion processes between each two successive jump time for a "parabolic" scaling limit of tagged particle dynamics. In order to obtain the diffusion process, we have to observe a "parabolic" scaling limit of empirical measure of the position of particles. We also obtain a generalized Ornstein-Uhlenbeck process for a "parabolic" scaling limit of empirical measure of the position of particles.

1. Introduction

A stochastic ranking process is introduced in Hattori and Hattori (2009b) in order to explain the reasons why some typical curve is observed in time evolution of ranking of books in online bookstores. In this paper, we consider a stochastic ranking process (or Poisson embedding of the move-to-front rules) with space-time dependent intensities, which is driven according to an algorithm for a self-organizing linear list of a finite number of items. The list is updated in the following way. Each item has an independent Poisson clock, whose rate depends on type of the item, (normalized) position of the item and time. If the Poisson clock of the i-th item rings, then it jumps to the top of the list and each of the items located in front of the i-th item accordingly descend simultaneously by one rank; those behind do not move at all. In this paper, we regard this process as an "interacting particle system". We fasten a tag to a "particle" (item) and observe the (normalized) motion of the "tagged particle".

A "hyperbolic" scaling limit of (multi-)tagged particle dynamics and that of joint empirical measure of the rate and the position, as the number of the particles tends to infinity, is obtained in Hattori (2019). In this scaling limit, the limit process of the scaled tagged particle jumps to the top

Received by the editors April 16th, 2021; accepted May 1st, 2022.
2010 *Mathematics Subject Classification.* 60K35, 60F17.
Key words and phrases. central limit theorem, tagged particle.

of the list when its own Poisson clock rings and moves deterministically along a curve otherwise. The joint empirical measure of the rate and the position converges to some measure, whose density function is given by some system of integral partial differential equations. We can regard these results as a law of large numbers. In this paper, we consider a "parabolic" scaling limit of tagged particle dynamics and that of empirical measure of the position of particles. In Nagahata (2013a) the author obtained a "parabolic" scaling limit of (multi-)tagged particle dynamics whose rate function does not depend on the position of the item. Our main result (Theorem 2.4) is the extension of this result. In the limit of tagged particle under this scaling, we obtain a sum of diffusion processes between each two successive jump times (Theorem 2.4). In the limit of joint empirical measure under this scaling, we obtain a generalized Ornstein-Uhlenbeck process (Theorem 2.3).

The move-to-front rule is introduced in Cetlin (1963) and studied in many papers Burville and Kingman (1973); Hendricks (1972); Kingman et al. (1975); Letac (1974); McCabe (1965). It is also studied as least-recently-used cashing Barrera and Fontbona (2010); Bitner (1979); Blom and Holst (1991); Chung et al. (1988); Fagin (1977); Fill (1996a,b); Fill and Holst (1996); Jelenković (1999); Jelenković and Radovanović (2004); Rivest (1976); Rodrigues (1995). It is reintroduced and studied as a mathematical model of the ranking in the web page of online bookstores or in the posting web pages Hariya et al. (2011); Hattori and Hattori (2009b,a); Hattori (2019); Hattori and Kusuoka (2012); Nagahata (2013b,a).

2. Model and result

A stochastic ranking process defined in Hattori (2019) is the stochastic system of N particles on the interval $[0,1]$. We set

$$\overline{\mathcal{W}} = \left\{ w(t,x) \in C^1([0,T]\times[0,1]); w \geq 0, \sup \left| \frac{\partial}{\partial x} w(t,x) \right| \leq C \right\}$$

for some constant C. Let $\mathcal{W}$ be a finite subset of $\overline{\mathcal{W}}$ of cardinality $K \geq 2$ and denote the elements of $\mathcal{W}$ by $\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_K$. Let $w_1, w_2, \dots$ be a sequence in $\mathcal{W}$. Let $y_1^{(N)}, y_2^{(N)}, \dots, y_N^{(N)}$ be a permutation of $\{\frac{i}{N}; i = 0, 1, 2, \dots, N-1\}$. A stochastic ranking process with space-time dependent intensities Hattori (2019) is the system of stochastic processes $\{Y_i^{(N)}; i = 1, 2, \dots, N\}$ defined by

$$\begin{aligned} &Y_i^{(N)}(t) \\ &:= \quad y_i^{(N)} + \frac{1}{N}\sum_{j=1}^N \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}(Y_j^{(N)}(s-) > Y_i^{(N)}(s-)) \\ &\qquad \times \mathbf{1}(0 \leq \xi < w_j(s, Y_j^{(N)}(s-)))\nu_j(ds, d\xi) \\ &\qquad - \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} Y_i^{(N)}(s-)\mathbf{1}(0 \leq \xi < w_i(s, Y_i^{(N)}(s-)))\nu_i(ds, d\xi), \end{aligned} \tag{2.1}$$

where $\mathbf{1}(\mathcal{S})$ equals 1 or 0 according as a statement $\mathcal{S}$ is true or false, and ν_i $i = 1, 2, \dots$ are independent Poisson random measures on $[0,\infty)\times[0,\infty)$ with uniform unit intensity measures.

We regard $Y_i^{(N)}(t)$ and $y_i^{(N)}$ as (normalized) positions of the i-th particle at time t and 0 respectively. Each particle has an independent Poisson clock whose intensity is governed by the time and its position via w_i. If the i-th particle's Poisson clock rings, then i-th particle jumps to 0, the position of the top. If a Poisson clock of a particle located behind the i-th particle rings, then the i-th particle jumps backward by $1/N$, the unit step.

We consider an empirical measure of particles whose intensity are $\tilde{w}_k$ given by

$$\mu_{k,t}^{(N)}(dz) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}(w_i = \tilde{w}_k)\delta_{Y_i^{(N)}(t)}(dz),$$

where $\delta_\cdot$ is the Dirac measure.

We consider a Cauchy problem for a system of quasilinear integral partial differential equations

$$\begin{aligned}
&\frac{\partial}{\partial t}u_k(t,x)+\left(\sum_{r=1}^{K}\int_x^1 \tilde{w}_r(t,z)u_r(t,z)dz\right)\frac{\partial}{\partial x}u_k(t,x)\\
&=\left(\sum_{r=1}^{K}\tilde{w}_r(t,x)u_r(t,x)-\tilde{w}_k(t,x)\right)u_k(t,x)\\
&u_k(t,0)=\frac{\displaystyle\int_0^1 \tilde{w}_k(t,x)u_k(t,x)dx}{\displaystyle\sum_{r=1}^{K}\int_0^1 \tilde{w}_r(t,z)u_r(t,z)dz}\\
&u_k(0,x)=u_{k,0}(x),
\end{aligned} \tag{2.2}$$

for $1\le k\le K$ for some given initial data $\{u_{k,0}(x)\ge 0\}_k$, $u_{k,0}(x)\in C^1$ with $\sum_{r=1}^{K}u_{r,0}(x)=1$.

Lemma 2.1. *For each given initial data $\{u_{k,0}\}_k$, the Cauchy problem for the system of quasilinear integral partial differential equations (2.2) has the unique solution.*

We shall prove this lemma in Section 6.

By using the unique solution $\{u_k(t,x)\}_k$, for each given $0\le\tau\le T$, $0\le\xi<1$, we set $y_C(t)=y_C(t;\tau,\xi)$ by the unique solution to the ODE

$$\frac{dy_C}{dt}=Z(t,y_C),\ y_C(\tau)=\xi,\ Z(s,x)=\sum_{r=1}^{K}\int_x^1 \tilde{w}_r(s,z)u_r(s,z)dz.$$

We note that this y_C is the characteristic line for (2.2).

Given initial data $\{u_{k,0}\}_k$, we set

$$\mu_{k,t}(dx):=u_k(t,x)dx, \tag{2.3}$$

for $1\le k\le K$, where $\{u_k(t,x)\}_k$ is the unique solution to the Cauchy problem for the system of quasilinear integral partial differential equations (2.2).

We assume that there exist constants $0<\delta<1/2$ and $C>0$ such that for $N\in\mathbb{N}$, for any $1\le k\le K$, and for any $h\in C([0,1]\to\mathbb{R})$, we have

$$\left|\int_0^1 h(z)\mu_{k,0}^{(N)}(dz)-\int_0^1 h(z)\mu_{k,0}(dz)\right|\le\frac{C\|h\|_\infty}{N^\delta},$$

for some initial data. Here $\|h\|_\infty:=\sup_{x\in[0,1]}|h(x)|$.

We cite a result from Hattori (2019, Theorem 2.3), which we appropriately adapt to the present setting.

Proposition 2.2. *(Hattori (2019, Theorem 2.3)) If the above bound is satisfied for the initial data $\{u_{k,0}\}_k$, then with probability one $\mu_{k,t}^{(N)}\to\mu_{k,t}$ $(N\to\infty)$ weakly, uniformly in t, $1\le k\le K$. Furthermore $Y_1^{(N)}\to Y_1$ a.s. uniformly in t, where Y_1 is the unique solution of*

$$\begin{aligned}
Y_1(t)\ &=\ y_1+\int_{s=0}^{t}\sum_{k=1}^{K}\int_{z=Y_1(s-)}^{1}\tilde{w}_k(s,z)\mu_{k,s}(dz)\ ds\\
&\quad-\int_{s=0}^{t}\int_{\xi\in[0,\infty)}Y_1(s-)\mathbf{1}(0\le\xi<w_1(s,Y_1(s-)))\nu_1(ds,d\xi).
\end{aligned}$$

Proposition 2.2 states that the tagged particle moves deterministically on the corresponding characteristic line for (2.2), except for its successive Poisson epochs at each of which it jumps to the top.

We write $\tau_n^{(N)}$ for the n-th jump time of 1-st particle as is defined by

$$\tau_0^{(N)} := 0,$$
$$\tau_{n+1}^{(N)} = \inf\{t > \tau_n^{(N)} | \nu_1(\{(s,\xi) | \tau_n^{(N)} < s \le t, 0 \le \xi \le w_1(s, Y_1^{(N)}(s-))\}) > 0\}.$$

We also write τ_n for the n-th jump time of $Y_1(t)$ as is defined by

$$\tau_0 := 0,\ \tau_{n+1} = \inf\{t > \tau_n | \nu_1(\{(s,\xi) | \tau_n < s \le t, 0 \le \xi \le w_1(s, Y_1(s-))\}) > 0\}.$$

In Hattori (2019, (117)(118)) some constant $\mathcal{X}_1^{(N)}(T)$ is defined such that $P(\tau_n^{(N)} \neq \tau_n) \le \mathcal{X}_1^{(N)}(T)$. It is shown in Hattori (2019, below (125)) that $\mathcal{X}_1^{(N)}(T) \le C/N^{2p\delta'}$ for some constants C and p, δ'. Here we can take p and δ' such that $2p\delta' > 1$. By applying Borel-Cantelli lemma, we can deduce that $\tau_n^{(N)} = \tau_n$ for all sufficiently large N with probability one for each $n \ge 1$. Hence in this paper, we can regard $\tau_n^{(N)}$ as τ_n, which is independent of N. We set $\mathcal{F} := \sigma(\{\tau_n; n\})$. We note that stochastic processes defined below are conditioned on $\mathcal{F}$. In order to simplify our notation, we abbreviate the symbol of the conditional expectation.

In this paper, we are interested in the behavior of the following scaling limit of centered tagged particle system;

$$Z^{(N)}(t) := \sqrt{N}\{Y_1^{(N)}(t) - Y_1(t)\}. \tag{2.4}$$

Precisely, what we are interested in is the following: If we take N large enough, then both $Y_1^{(N)}$ and Y_1 jump to 0 at the same time τ_n for $n \ge 1$. Hence $Z^{(N)}(\tau_n) = 0$ for $n \ge 1$. In the interval $[\tau_n, \tau_{n+1})$, the centered tagged particle may converge to a Gaussian process with some covariance. In this paper, we show the convergence of this process conditioned on $\mathcal{F}$ to some process and give some SDE for the limiting process. Note that we do not have a closed SDE for limit of $Z^{(N)}$. But if we decompose $Z^{(N)}$ into K parts, then each part converges to a Gaussian process and the limit process solves some SDE. We recall that K is the number of different intensities.

The process $Y^{(N)}(t)$ may be considered as a "hyperbolic" scaling limit of some process $X^{(N)}(t)$, (which is not defined in this paper) and $Z^{(N)}(t)$ as a "parabolic" scaling limit of $X^{(N)}(t)$. We repeat the reasoning advanced in Nagahata (2013a) why we use the words "hyperbolic" and "parabolic" in spite of no scale change of time in both cases. First we consider

$$\tilde{W}^{(N)}(t) := \sum_{j=1}^{N} \int_0^t \tilde{\nu}_j(ds),\ W^{(N)}(t) := \frac{1}{N}\sum_{j=1}^{N} \int_0^t \tilde{\nu}_j(ds),$$

where $\{\tilde{\nu}_j\}$ are independent Poisson random measures with intensity ds. It is obvious that $\tilde{W}^{(N)}(t)$ is a time homogeneous continuous time random walk with mean waiting time $1/N$. It is standard to see that $W^{(N)}(t)$ converges to t and $\sqrt{N}(W^{(N)}(t) - t)$ converge to a standard Brownian motion as a consequence of a law of large numbers and an invariance principle respectively. In this case, we have scaled the processes in space, not in time. Nevertheless we have a law of large numbers and an invariance principle, since adding independent Poisson measures plays a speed up (time scaling) role. The situation for $Y^{(N)}(t)$ and $Z^{(N)}(t)$ is the same, so the expressions "hyperbolic" and "parabolic" scaling according as the space scaling factors $1/N$ and $1/\sqrt{N}$ respectively. (In the "parabolic" scaling case, we may also need some centering.)

We define $\Psi^{(N)}(t) = (\Psi_1^{(N)}(t), \Psi_2^{(N)}(t), \ldots, \Psi_K^{(N)}(t))$, whose k-th component is a centered normalized number of type k particles between $Y_1^{(N)}(t)$ and 1, by

$$\Psi_k^{(N)}(t) := \sqrt{N} \left\{ \frac{1}{N} \sum_{j=2}^{N} \mathbf{1}(w_j = \tilde{w}_k) \mathbf{1}(Y_j^{(N)}(t) > Y_1^{(N)}(t)) - \int_{y_C(t)}^{1} u_k(t,z) dz \right\},$$

where $y_C(t) = y_C(t; 0, y_1)$ if $0 \leq t < \tau_1$ and $y_C(t) = y_C(t; \tau_n, 0)$ if $\tau_n \leq t < \tau_{n+1}$ for $n \geq 1$. It is easy to see that

$$\begin{aligned}
&\sum_{k=1}^{K} \frac{1}{N} \sum_{j=2}^{N} \mathbf{1}(w_j = \tilde{w}_k) \mathbf{1}(Y_j^{(N)}(t) > Y_1^{(N)}(t)) \\
&= \frac{1}{N} \sum_{j=2}^{N} \mathbf{1}(Y_j^{(N)}(t) > Y_1^{(N)}(t)) = 1 - Y_1^{(N)}(t), \\
&\sum_{k=1}^{K} \int_{y_C(t)}^{1} u_k(t,z) dz = \int_{y_C(t)}^{1} dz = 1 - y_C(t) = 1 - Y_1(t).
\end{aligned}$$

Hence we have

$$Z^{(N)}(t) := \sqrt{N} \{ Y_1^{(N)}(t) - Y_1(t) \} = - \sum_{k=1}^{K} \Psi_k^{(N)}(t). \tag{2.5}$$

For our convenience, we define $\mathcal{A}_{k,t}$ by

$$\begin{aligned}
\mathcal{A}_{k,t} f(t,z) \quad = \quad & \frac{\partial}{\partial t} f(t,z) + \frac{\partial}{\partial z} f(t,z) \sum_{r=1}^{K} \int_{z}^{1} \tilde{w}_r(t,x) u_r(t,x) dx \\
& + \Big\{ f(t,0) - f(t,z) \Big\} \tilde{w}_k(t,z),
\end{aligned}$$

for $f \in C^1$, $1 \leq k \leq K$.

In order to give our SDEs, we give martingales $\mathcal{M}_k$, for $1 \le k \le K$ and $\mathcal{N}_k(\cdot; f)$ for $1 \le k \le K$ and $f \in C^1$ with quadratic variation

$$\begin{aligned}
&\mathcal{M}_k(\tau_n) = 0, \quad \text{for } n \ge 0,\ 1 \le k \le K,\\
&\langle \mathcal{M}_k \rangle_t = \int_{s=\tau_n}^t \int_{z=y_C(s)}^1 \tilde{w}_k(s,z) u_k(s,z) dz\ ds,\\
&\ \text{for } 1 \le k \le K,\ \text{for } \tau_n \le t < \tau_{n+1},\\
&\mathcal{N}_k(0; f) = 0, \quad \text{for } 1 \le k \le K, \quad \text{for all } f,\\
&\langle \mathcal{N}_k(\cdot; f) \rangle_t = \int_{s=0}^t \int_{x=0}^1 \sum_{r=1}^K \Bigg[\int_{z=0}^x \frac{\partial}{\partial x} f(s,z) u_k(s,z) dz\\
&+ \mathbf{1}(r = k) \Big\{ f(s,0) - f(s,x) \Big\} \Bigg]^2 u_r(s,x) \tilde{w}_r(s,x) dx ds,\\
&\ \text{for } 1 \le k \le K, \quad \text{for all } f,\\
&\langle \mathcal{M}_k, \mathcal{M}_{k'} \rangle_t = 0, \quad \text{for all } 1 \le k \ne k' \le K,\\
&\langle \mathcal{N}_k(\cdot; f), \mathcal{N}_{k'}(\cdot; g) \rangle_t\\
&= \int_{s=0}^t \int_{x=0}^1 \sum_{r=1}^K \Bigg[\int_{z=0}^x \frac{\partial}{\partial x} f(s,z) u_k(s,z) dz\\
&+ \mathbf{1}(r = k) \Big\{ f(s,0) - f(s,x) \Big\} \Bigg]\\
&\times \Bigg[\int_{z=0}^x \frac{\partial}{\partial x} g(s,z) u_{k'}(s,z) dz + \mathbf{1}(r = k') \Big\{ g(s,0) - g(s,x) \Big\} \Bigg]\\
&\times u_r(s,x) \tilde{w}_r(s,x) dx ds, \quad \text{for all } 1 \le k, k' \le K, \quad \text{for all } f, g,\\
&\langle \mathcal{M}_k, \mathcal{N}_{k'}(\cdot; f) \rangle_t\\
&= \int_{s=\tau_n}^t \int_{x=y_C(s)}^1 \Bigg[\int_{z=0}^x \frac{\partial}{\partial x} f(s,z) u_{k'}(s,z) dz + \mathbf{1}(k' = k) \Big\{ f(s,0) - f(s,x) \Big\} \Bigg]\\
&\times \tilde{w}_k(s,x) u_k(s,x) dx\ ds, \quad \text{for all } 1 \le k, k' \le K, \quad \text{for all } f, \quad \text{for } \tau_n \le t < \tau_{n+1}.
\end{aligned}$$

Note that each $\mathcal{M}_k$, $(1 \le k \le K)$ is a martingale in the time interval in $[\tau_n, \tau_{n+1})$ for each $n \ge 0$. By using these martingales, we give a system of SDEs by

$$\Psi_k(t) = -\mathcal{M}_k(t) - \int_{\tau_n}^t \tilde{w}_k(s, y_C(s)) \Psi_k(s) ds - \int_{\tau_n}^t F_k(s) ds, \tag{2.6}$$
$$\text{for } \tau_n \le t < \tau_{n+1},\ \text{for } n \ge 0,$$

$$\Xi_{k,t}(f(t,z)) - \Xi_{k,0}(f(0,z)) = \mathcal{N}_k(t; f) + \int_0^t \Xi_{k,s}(\mathcal{A}_{k,s} f(s,z)) ds, \tag{2.7}$$

$$F_k(s) = \Xi_{k,s}(\mathbf{1}(y_C(s) \le z < 1)\{\tilde{w}_k(s,z) - \tilde{w}_k(s, y_C(s))\}), \tag{2.8}$$

for $1 \le k \le K$, where $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_K) \in (D([\tau_n, \tau_{n+1}), \mathbb{R})^K$, $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_K) \in (D([\tau_n, \tau_{n+1}), \mathcal{H}_{-\alpha})^K$ for $\alpha \ge 3$, and f is a test function. Here $D(\mathcal{I}, \mathcal{S})$ denotes a $\mathcal{S}$ valued functions that are right continuous with left limits and $\mathcal{H}_{-\alpha} = \mathcal{H}_{-\alpha}([0,1))$ denotes a Sobolev space with index

$-\alpha$. We remark that we regard

$$\begin{aligned}\mathcal{A}_{k,t}\Big(\mathbf{1}(y_C(t)\le z<1)f(t,z)\Big) &= \mathbf{1}(y_C(t)\le z<1)\mathcal{A}_{k,t}f(t,z)\\ =\ \mathbf{1}(y_C(t)\le z<1)\Bigg[&\frac{\partial}{\partial t}f(t,z)\\ &+\frac{\partial}{\partial z}f(t,z)\sum_{r=1}^{K}\int_z^1 \tilde{w}_r(t,x)u_r(t,x)dx\Big\{f(t,0)-f(t,z)\Big\}\tilde{w}_k(t,z)\Bigg],\end{aligned}$$

for $f\in C^1$.

Before expressing our main result, we shall give some interpretation on measure-valued generalized Ornstein-Uhlenbeck process Ξ.

By referring to the result on generalized Ornstein-Uhlenbeck process due to Holley and Stroock (1978) (see also Kipnis and Landim (1999, Sec. 11.4. p.307)), Ξ has unique weak solution. Indeed, for each given f, $\Xi_{k,t}(f)$ conditioned on $\mathcal{G}_s$ (which is the filtration generated by $\Xi_{k,s}$) has a Gaussian distribution whose mean and variance is given as follows: For given t, and $f(t,z)$, we set $g(u,z)$ by the unique solution of

$$\mathcal{A}_{k,u}g(u,z)=0,\ \ g(t,z)=f(t,z),$$

(see Lemma 4.1 in Section 4.) Then the mean and variance of $\Xi_{k,t}(f)$ conditioned on $\mathcal{G}_s$ are given by $\Xi_{k,s}(g(s,z))$ and $\langle\mathcal{N}_k(\cdot;g)\rangle_t-\langle\mathcal{N}_k(\cdot;g)\rangle_s$, respectively.

We define a fluctuation of empirical measures by

$$\begin{aligned}\Xi^{(N)}_{k,s}(dz) &:= \sqrt{N}(\mu^{(N)}_{k,s}-\mu_{k,s})(dz),\\ \Xi^{(N)}_{k,s}(f) &:= \int_0^1 f(z)\Xi^{(N)}_{k,s}(dz),\\ F^{(N)}_k(s) &= \Xi^{(N)}_{k,s}(\mathbf{1}(y_C(s)\le z<1)\{\tilde{w}_k(s,z)-\tilde{w}_k(s,y_C(s))\})\\ &= \sqrt{N}\Big[\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y^{(N)}_j(s)>y_C(s))\\ &\quad\times\{\tilde{w}_k(s,Y^{(N)}_j(s))-\tilde{w}_k(s,y_C(s))\}\\ &\quad-\int_{y_C(s)}^1\{\tilde{w}_k(s,z)-\tilde{w}_k(s,y_C(s))\}u_k(s,z)dz\Big]\end{aligned}$$

for $1\le k\le K$, where $\Xi^{(N)}_s=(\Xi^{(N)}_{1,s},\Xi^{(N)}_{2,s},\dots,\Xi^{(N)}_{K,s})\in\mathcal{H}^K_{-\alpha}$,
$F^{(N)}(s)=(F^{(N)}_1(s),F^{(N)}_2(s),\dots,F^{(N)}_K(s))\in\mathbb{R}^K$ and $f\in C^1$ is a test function.

Denote by $Q^{(N)}$, $Q^{(N)}_f$ the probability measures on $D([0,T],\mathbb{R}^K\times\mathcal{H}^K_{-\alpha})$, $D([0,T],\mathcal{H}^K_{-\alpha})$ induced by $(\Psi^{(N)},\Xi^{(N)})$, and $\Xi^{(N)}$, respectively. We also denote by Q, Q_f the probability measures on $C([0,T],\mathbb{R}^K\times\mathcal{H}^K_{-\alpha})$, $C([0,T],\mathcal{H}^K_{-\alpha})$ induced by the system of SDEs (2.6), (2.7), (2.8), and SDEs (2.7), respectively. We note that all of them are conditioned on $\mathcal{F}$. Here $D(\mathcal{I},\mathcal{S})$ denotes a $\mathcal{S}$ valued functions that are right continuous with left limits and $C(\mathcal{I},\mathcal{S})$ denotes a $\mathcal{S}$ valued continuous functions.

Theorem 2.3. *We assume that the assumption for Proposition 2.2 holds. Furthermore we suppose that $\Xi^{(N)}_0$ converges in distribution to Ξ_0 and there is a constant C such that*

$$E\Big[\Big|\Xi^{(N)}_{k,0}(f)\Big|^2\Big]\le C\Big\{\sup_{0\le x<1}\Big|\frac{d}{dx}f(x)\Big|+\sup_{0\le x<1}|f(x)|\Big\}^2,$$

for all $f\in C^1$.

Then the sequence $Q_f^{(N)}$ *converges weakly to the probability measure* Q_f*.*

Theorem 2.4. *We assume that the assumption for Theorem 2.3 holds. Furthermore we suppose that* $y_1^{(N)} - y_1 = o(1/\sqrt{N})$ *with probability one. Then the sequence* $Q^{(N)}$ *converges weakly to* Q*.*

Theorem 2.3 expresses that a fluctuation of empirical measures converges to a generalized Ornstein-Uhlenbeck process. By using these measure-valued processes and using special test functions $\mathbf{1}(y_C(t) \le z < 1)\{\tilde{w}_k(s,z) - \tilde{w}_k(s, y_C(s))\}$, we have Gaussian processes F_k. Our target $Z^{(N)} = \sqrt{N}\{Y_1^{(N)} - Y_1\}$ is decomposed into K parts $-\sum_{k=1}^K \Psi_k^{(N)}$ and each $\Psi_k^{(N)}$, $(1 \le k \le K)$ converges to a solution of Ornstein-Uhlenbeck process perturbed by F_k.

We can easily understand that this result is an extension of that given in Nagahata (2013a, Theorem 2.2). Indeed, in Nagahata (2013a), we assumed that each of $\{\tilde{w}_k(t,x)\}$ $(1 \le k \le K)$ does not depend on x. Hence each of F_k, $(1 \le k \le K)$ vanishes and each component of Ψ becomes a solution of Ornstein-Uhlenbeck process which is given in Nagahata (2013a, below (4)).

One of the main significances of this paper is that we find out these F_k terms.

In Nagahata (2013a, Theorem 2.5), a scaling limit of centered multi-tagged particles system is discussed. The "hyperbolically" scaled multi-tagged particles divide $[0,1)$ interval into $L+1$ layers, where L is a number of tagged particles. In each layer, we give a system of Ornstein-Uhlenbeck processes. The scaling limit of a centered tagged particle is given by a sum of these Ornstein-Uhlenbeck processes which are given in layers behind the tagged particles.

We can extend this result to our situation. Namely Ornstein-Uhlenbeck processes given in Nagahata (2013a, Theorem 2.5) becomes Ornstein-Uhlenbeck process perturbed by a fluctuation of empirical measure, similar to our Theorem 2.4. Due to notational complexity, we omit the details.

Finally, we give a strategy of the proof of our main theorems, Theorem 2.3, and 2.4. In both cases, our strategy of the proof is simple and standard such that we show the tightness of the sequence of probability measures induced by our processes, and we derive the martingale problem related to the limiting measure. Precisely, in section 3 we derive the martingale problem related to (2.7) from $\Xi^{(N)}$ as $N \to \infty$. In section 4, we show that the sequence of probability measures $Q_f^{(N)}$, which is induced by $\{\Xi^{(N)}\}$, is tight. In section 5, we derive the martingale problem related to (2.6) from $\Psi^{(N)}$ as $N \to \infty$, and we show that the sequence of probability measures, which is induced by $\{\Psi^{(N)}\}$, is tight.

3. The martingale problem for density fluctuation

In this section, by using the Ito calculus, we derive the martingale problem related to (2.7) from $\Xi^{(N)}$.

We recall that

$$\begin{aligned}
\mu_{k,t}^{(N)}(dz) &= \frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k)\delta_{Y_j^{(N)}(t)}(dz), \\
\mu_{k,t}(dz) &= u_k(t,z)dz, \\
\Xi_{k,s}^{(N)} &:= \sqrt{N}(\mu_{k,s}^{(N)} - \mu_{k,s}), \\
\Xi_{k,s}^{(N)}(f) &:= \int_0^1 f(z)\Xi_{k,s}^{(N)}(dz)
\end{aligned}$$

for $1 \le k \le K$.

Let $f(s,z)$ be a test function. By the definition of $\mu_{k,s}(dz) = u_k(s,z)dz$ and (2.2), we have

$$\begin{aligned}
&\int_0^1 f(t,z)\mu_{k,t}(dz) - \int_0^1 f(0,z)\mu_{k,0}(dz) \\
&= \int_{s=0}^t \Bigg[\int_{z=0}^1 \frac{\partial}{\partial s} f(s,z) u_k(s,z)dz \\
&- \int_{z=0}^1 f(s,z) \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y)u_r(s,y)dy \frac{\partial}{\partial z} u_k(s,z)dz \\
&+ \int_{z=0}^1 f(s,z) \Big\{ \sum_{r=1}^K \tilde{w}_r(s,z)u_r(s,z) - \tilde{w}_k(s,z) \Big\} u_k(s,z)dz \Bigg] ds.
\end{aligned}$$

By using integration by parts, we can rewrite the second term of the right hand side above by

$$\begin{aligned}
&- \int_{z=0}^1 f(s,z) \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ \frac{\partial}{\partial z} u_k(s,z)dz \\
&= - \Bigg[f(s,z) \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ u_k(s,z) \Bigg]_{z=0}^1 \\
&+ \int_{z=0}^1 \frac{\partial}{\partial z} f(s,z) \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ u_k(s,z)dz \\
&- \int_{z=0}^1 f(s,z) \sum_{r=1}^K \tilde{w}_r(s,z)u_r(s,z)u_k(s,z)dz \\
&= f(s,0) \sum_{r=1}^K \int_{y=0}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ u_k(s,0) \\
&+ \int_{z=0}^1 \frac{\partial}{\partial z} f(s,z) \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ u_k(s,z)dz \\
&- \int_{z=0}^1 f(s,z) \sum_{r=1}^K \tilde{w}_r(s,z)u_r(s,z)u_k(s,z)dz.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_0^1 f(t,z)\mu_{k,t}(dz) - \int_0^1 f(0,z)\mu_{k,0}(dz) \\
&= \int_{s=0}^t \Bigg[\int_{z=0}^1 \frac{\partial}{\partial s} f(s,z) u_k(s,z)dz \\
&+ \int_{z=0}^1 \frac{\partial}{\partial z} f(s,z) \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ u_k(s,z)dz \\
&+ \int_{z=0}^1 \Big\{ f(s,0) - f(s,z) \Big\} \tilde{w}_k(s,z)u_k(s,z)dz \\
&+ f(s,0) \Big\{ \sum_{r=1}^K \int_{y=0}^1 \tilde{w}_r(s,y)u_r(s,y)dy \ u_k(s,0) - \int_{z=0}^1 \tilde{w}_k(s,z)u_k(s,z)dz \Big\} \Bigg] ds.
\end{aligned}$$

Thanks to the boundary condition of (2.2), the last term above vanishes. Hence we have

$$\begin{aligned}
&\int_0^1 f(t,z)\mu_{k,t}(dz) - \int_0^1 f(0,z)\mu_{k,0}(dz) \\
&= \int_{s=0}^t \int_{z=0}^1 \Bigg[\frac{\partial}{\partial s} f(s,z) + \sum_{r=1}^K \int_{y=z}^1 \tilde{w}_r(s,y) u_r(s,y) dy \frac{\partial}{\partial z} f(s,z) \\
&+\tilde{w}_k(s,z)\Big\{f(s,0) - f(s,z)\Big\}\Bigg]\mu_{k,s}(dz)\ ds,
\end{aligned} \tag{3.1}$$

for $1 \le k \le K$.

By using the Ito formula, we have

$$\begin{aligned}
&\sqrt{N}\Big\{\int_0^1 f(t,z)\mu_{k,t}^{(N)}(dz) - \int_0^1 f(0,z)\mu_{k,0}^{(N)}(dz)\Big\} \\
&= \sqrt{N}\frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k) f(t, Y_j^{(N)}(t)) - \sqrt{N}\frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k) f(0, Y_j^{(N)}(0)) \\
&= \sqrt{N}\int_0^t \frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k)\frac{\partial}{\partial s} f(s, Y_j^{(N)}(s-))ds \\
&+\sqrt{N}\int_{s=0}^t \frac{1}{N}\sum_{i=1}^N \Bigg[\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k)\mathbf{1}(Y_i^{(N)}(s-) > Y_j^{(N)}(s-))\nabla^N f(s, Y_j^{(N)}(s-)) \\
&+\mathbf{1}(w_i = \tilde{w}_k)\Big\{f(s,0) - f(s, Y_i^{(N)}(s-))\Big\}\Bigg] \\
&\times \int_\xi \mathbf{1}(0 \le \xi < w_i(s, Y_i^{(N)}(s-)))\nu_i(ds, d\xi),
\end{aligned} \tag{3.2}$$

where $\nabla^N f(t,z) = f(t, z+1/N) - f(t,z)$. Note that

$$\nabla^N f(t,z) = \frac{1}{N}\frac{\partial}{\partial z} f(t,z) + o(\frac{1}{N}),$$

if $f \in C^1$.

We define $\mathcal{N}_k^{(N)}$ by

$$\begin{aligned}
&\mathcal{N}_k^{(N)}(t) = \mathcal{N}_k^{(N)}(t; f) \\
&= \sqrt{N}\int_{s=0}^t \frac{1}{N}\sum_{i=1}^N \Bigg[\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k)\mathbf{1}(Y_i^{(N)}(s-) > Y_j^{(N)}(s-))\nabla^N f(s, Y_j^{(N)}(s-)) \\
&+\mathbf{1}(w_i = \tilde{w}_k)\Big\{f(s,0) - f(s, Y_i^{(N)}(s-))\Big\}\Bigg] \\
&\times\Big\{\int_\xi \mathbf{1}(0 \le \xi < w_i(s, Y_i^{(N)}(s-)))\nu_i(ds, d\xi) - w_i(s, Y_i^{(N)}(s-)))ds\Big\}
\end{aligned} \tag{3.3}$$

Then it is easy to see that $\mathcal{N}_k^{(N)}$ is a martingale with quadratic variation

$$\langle \mathcal{N}_k^{(N)}(\cdot; f)\rangle_t$$

$$= \int_0^t \frac{1}{N}\sum_{i=1}^N \Bigg[\sum_{j=1}^N \mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_i^{(N)}(s)>Y_j^{(N)}(s))\nabla^N f(s,Y_j^{(N)}(s))$$
$$+\mathbf{1}(w_i=\tilde{w}_k)\Big\{f(s,0)-f(s,Y_i^{(N)}(s))\Big\}\Bigg]^2 w_i(s,Y_i^{(N)}(s))ds$$

$$\langle \mathcal{N}_k^{(N)}(\cdot;f),\mathcal{N}_{k'}^{(N)}(\cdot;g)\rangle_t$$
$$= \int_0^t \frac{1}{N}\sum_{i=1}^N \Bigg[\sum_{j=1}^N \mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_i^{(N)}(s)>Y_j^{(N)}(s))\nabla^N f(s,Y_j^{(N)}(s))$$
$$+\mathbf{1}(w_i=\tilde{w}_k)\Big\{f(s,0)-f(s,Y_i^{(N)}(s))\Big\}\Bigg]$$
$$\times\Bigg[\sum_{j=1}^N \mathbf{1}(w_j=\tilde{w}_{k'})\mathbf{1}(Y_i^{(N)}(s)>Y_j^{(N)}(s))\nabla^N g(s,Y_j^{(N)}(s))$$
$$+\mathbf{1}(w_i=\tilde{w}_{k'})\Big\{g(s,0)-g(s,Y_i^{(N)}(s))\Big\}\Bigg] w_i(s,Y_i^{(N)}(s))ds$$

for $t\geq 0$, $1\leq k,k'\leq K$. Thanks to Proposition 2.2, we have

$$\langle \mathcal{N}_k^{(N)}(\cdot;f)\rangle_t$$
$$= \int_{s=0}^t\int_{x=0}^1\sum_{r=1}^K\Bigg[\int_{z=0}^x \frac{\partial}{\partial x}f(s,z)u_k(s,z)dz+\mathbf{1}(r=k)\Big\{f(s,0)-f(s,x)\Big\}\Bigg]^2$$
$$\times u_r(s,x)\tilde{w}_r(s,x)dxds(1+o(1)), \tag{3.4}$$
$$\langle \mathcal{N}_k^{(N)}(\cdot;f),\mathcal{N}_{k'}^{(N)}(\cdot;g)\rangle_t$$
$$= \int_{s=0}^t\int_{x=0}^1\sum_{r=1}^K\Bigg[\int_{z=0}^x \frac{\partial}{\partial x}f(s,z)u_k(s,z)dz+\mathbf{1}(r=k)\Big\{f(s,0)-f(s,x)\Big\}\Bigg]$$
$$\times\Bigg[\int_{z=0}^x \frac{\partial}{\partial x}g(s,z)u_{k'}(s,z)dz+\mathbf{1}(r=k')\Big\{g(s,0)-g(s,x)\Big\}\Bigg]$$
$$\times u_r(s,x)\tilde{w}_r(s,x)dxds(1+o(1)),$$

for $1\leq k,k'\leq K$.

We rewrite the drift term as follows;

$$\sqrt{N}\int_0^t \frac{1}{N}\sum_{i=1}^N \Bigg[\sum_{j=1}^N \mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_i^{(N)}(s-)>Y_j^{(N)}(s-))\nabla^N f(s,Y_j^{(N)}(s-))$$
$$+\mathbf{1}(w_i=\tilde{w}_k)\Big\{f(s,0)-f(s,Y_i^{(N)}(s-))\Big\}\Bigg] w_i(s,Y_i^{(N)}(s-)))ds$$
$$=\sqrt{N}\int_0^t \frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j=\tilde{w}_k)\frac{\partial}{\partial z}f(s,Y_j^{(N)}(s-))\frac{1}{N}\sum_{i=1}^N \mathbf{1}(Y_i^{(N)}>Y_j^{(N)}(s-))$$
$$\times w_i(s,Y_i^{(N)}(s-))ds(1+o(1))$$

$$+\sqrt{N}\int_0^t\sum_{i=1}^N \mathbf{1}(w_i=\tilde{w}_k)\Big\{f(s,0)-f(s,Y_i^{(N)}(s-))\Big\}\tilde{w}_k(s,Y_i^{(N)}(s-))ds.$$

Thanks to Proposition 2.2, the above expression is equal to

$$\sqrt{N}\int_0^t\int_0^1\Bigg[\frac{\partial}{\partial z}f(s,z)\sum_{r=1}^K\int_z^1\tilde{w}_r(s,x)u_r(s,x)dx(1+o(1)) \\ +\Big\{f(s,0)-f(s,z)\Big\}\tilde{w}_k(s,z)\Bigg]\mu_{k,s}^{(N)}(dz)ds. \tag{3.5}$$

We recall that

$$\begin{aligned}\mathcal{A}_{k,t}f(t,z) &= \frac{\partial}{\partial t}f(t,z)+\frac{\partial}{\partial z}f(t,z)\sum_{r=1}^K\int_z^1\tilde{w}_r(t,x)u_r(t,x)dx \\ &\quad +\Big\{f(t,0)-f(t,z)\Big\}\tilde{w}_k(t,z),\end{aligned}$$

By using (3.1), (3.2) and (3.5), we have

$$\Xi_{k,t}^{(N)}(f(t,z))-\Xi_{k,0}^{(N)}(f(0,z))=\int_0^t\Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,z))(1+o(1))ds+\mathcal{N}_k^{(N)}(t;f), \tag{3.6}$$

for $1\le k\le K$. Here we recall that $\mathcal{N}_k^{(N)}(t)$ is a martingale with quadratic variation (3.4).

Hence we have derived the martingale problem related to (2.7) from $\Xi^{(N)}$ as $N\to\infty$.

4. Tightness for density fluctuation

In this section, we set functions $h_0=1$, $h_n(x)=\sqrt{2}\cos(2\pi nx)$ for $n>0$ and $h_n(x)=\sqrt{2}\sin(2\pi nx)$ for $n<0$, which are the eigenvectors of $1-\frac{\partial^2}{\partial x^2}$ on $\mathbf{L}^2([0,1))$ with eigenvalue $\gamma_n=1+4\pi^2n^2$, for $n\in\mathbb{Z}$. Then we have

$$\|\mu\|_{-\alpha}^2=\sum_{n\in\mathbb{Z}}\left(\int_0^1 h_n(x)\mu(dx)\right)^2\gamma_n^{-\alpha}$$

for $\mu\in\mathcal{H}_{-\alpha}$.

In order to prove the tightness for $\Xi_{k,t}^{(N)}$, we give the following lemma;

Lemma 4.1. *For given a pair $(t,f(t,x))$, the Cauchy problem to the semilinear PDE*

$$\mathcal{A}_{k,s}g(s,x)=0,\ \ g(t,x)=f(t,x),$$

has a unique solution. Furthermore we have

$$\sup_{0\le s\le t,0\le x<1}\Big|g(s,x)\Big|\le C\sup_{0\le x<1}\Big|f(t,x)\Big|,$$

$$\sup_{0\le s\le t,0\le x<1}\Big|\frac{\partial}{\partial x}g(s,x)\Big|\le C\Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big|+\sup_{0\le x<1}\Big|f(t,x)\Big|\Big\}$$

for some constant C.

We shall prove this lemma in Section 6.

Thanks to Lemma 4.1, for each given a pair $(t,f(t,x))$, we set $g_k(s,x)$ by the unique solution of

$$\mathcal{A}_{k,s}g(s,x)=0,\ \ g(t,x)=f(t,x).$$

Thanks to (3.6), we have

$$\begin{aligned}\Xi_{k,t}^{(N)}(f(t,z)) &= \Xi_{k,t}^{(N)}(g(t,z))\\ &= \Xi_{k,0}^{(N)}(g(0,z)) + \int_0^t \Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}g(s,z))(1+o(1))ds + \mathcal{N}_k^{(N)}(t;g).\end{aligned}$$

Hence $\Xi_{k,t}^{(N)}(f(t,z))$ has mean $E[\Xi_{k,0}^{(N)}(g(0,z))] + o(1)$ and variance $E[\langle \mathcal{N}_k^{(N)}(\cdot;g)\rangle_t]$.

Thanks to the assumption on $\Xi_{k,0}^{(N)}$ and Lemma 4.1, we have

$$\begin{aligned}\Big|E[\Xi_{k,0}^{(N)}(g(0,z))]\Big|^2 &\le E\Big[\Big|\Xi_{k,0}^{(N)}(g(0,z))\Big|^2\Big]\\ &\le C\Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}g(0,x)\Big| + \sup_{0\le x<1}|g(0,x)|\Big\}^2\\ &\le C'\Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big| + \sup_{0\le x<1}|f(t,x)|\Big\}^2\end{aligned}$$

for some constants C, C'.

Thanks to (3.4) and Lemma 4.1, we have

$$\begin{aligned}E[\langle \mathcal{N}_k^{(N)}(\cdot;g)\rangle_t] &\le C\int_{s=0}^t\int_{z=0}^1\Big\{\Big(\frac{\partial}{\partial x}g(s,z)\Big)^2 + g(s,z)^2\Big\}dz\,ds\\ &\le C't\Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big|^2 + \sup_{0\le x<1}\Big|f(t,x)\Big|^2\Big\}\end{aligned}$$

for some constants C, C'.

Hence we have

$$E\Big[\Big|\Xi_{k,t}^{(N)}(f(t,z))\Big|^2\Big] \le C\Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big|^2 + \sup_{0\le x<1}\Big|f(t,x)\Big|^2\Big\} \tag{4.1}$$

for some constant C.

Due to (3.6), it is easy to see that

$$\begin{aligned}&\sup_{0\le t\le T}\Big|\Xi_{k,t}^{(N)}(f(t,x))\Big|^2\\ &\le 2\sup_{0\le t\le T}\Big|\Xi_{k,t}^{(N)}(f(t,x)) - \int_0^t \Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))ds\Big|^2\\ &\quad +2\sup_{0\le t\le T}\Big|\int_0^t \Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))ds\Big|^2\\ &\le 4\Big|\Xi_{k,0}^{(N)}(f(0,x))\Big|^2 + 4\sup_{0\le t\le T}\Big|\mathcal{N}_k^{(N)}(t;f)\Big|^2 + 2\sup_{0\le t\le T}\Big|\int_0^t \Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))ds\Big|^2.\end{aligned}$$

By using the Schwarz inequality the third term above has the following estimate:

$$\begin{aligned}&2\sup_{0\le t\le T}\Big|\int_0^t \Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))ds\Big|^2\\ &\le 2\sup_{0\le t\le T} t\int_0^t\Big|\Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))\Big|^2ds \le 2T\int_0^T\Big|\Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))\Big|^2ds.\end{aligned}$$

Since $\mathcal{N}_k^{(N)}$ is a martingale, by applying Doob's inequality, we have

$$E\Big[\sup_{0\le t\le T}\Big|\Xi_{k,t}^{(N)}(f(t,x))\Big|^2\Big] \tag{4.2}$$
$$\le 4E\Big[\Big|\Xi_{k,0}^{(N)}(f(0,x))\Big|^2\Big] + 16E\Big[\Big|\mathcal{N}_k^{(N)}(T;f)\Big|^2\Big]$$
$$+2T\int_0^T E\Big[\Big|\Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))\Big|^2\Big]ds.$$

By our assumption, the first term of (4.2) has the following estimate:

$$E\Big[\Big|\Xi_{k,0}^{(N)}(f(0,x))\Big|^2\Big] \le C\Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big| + \sup_{0\le x<1}\Big|f(t,x)\Big|\Big\}^2$$

for some constant C. Thanks to (3.4) and Lemma 4.1, the second term of (4.2) has the following estimate:

$$E\Big[\Big|\mathcal{N}_k^{(N)}(T;f)\Big|^2\Big] \le C\Big\{\sup_{0\le t\le T,0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big|^2 + \sup_{0\le t\le T,0\le x<1}\Big|f(t,x)\Big|^2\Big\}$$

for some constant C. Thanks to (4.1), the third term of (4.2) has the following estimate:

$$\int_0^T E\Big[\Big|\Xi_{k,s}^{(N)}(\mathcal{A}_{k,s}f(s,x))\Big|^2\Big]ds$$
$$\le CT\Big\{\sup_{0\le t\le T,0\le x<1}\Big|\frac{\partial}{\partial x}\mathcal{A}_{k,t}f(t,x)\Big|^2 + \sup_{0\le t\le T,0\le x<1}\Big|\mathcal{A}_{k,t}f(t,x)\Big|^2\Big\}$$

for some constant C. Hence we conclude that

$$E\Big[\sup_{0\le t\le T}\Big|\Xi_{k,t}^{(N)}(f(t,x))\Big|^2\Big]$$
$$\le C\Big\{\sup_{0\le t\le T,0\le x<1}\Big|\frac{\partial}{\partial x}f(t,x)\Big|^2 + \sup_{0\le t\le T,0\le x<1}\Big|f(t,x)\Big|^2 \tag{4.3}$$
$$+\sup_{0\le t\le T,0\le x<1}\Big|\frac{\partial}{\partial x}\mathcal{A}_{k,t}f(t,x)\Big|^2 + \sup_{0\le t\le T,0\le x<1}\Big|\mathcal{A}_{k,t}f(t,x)\Big|^2\Big\}.$$

for some constant C. Substituting $f(t,x)=h_n(x)$ for (4.3), we have

$$E\Big[\sup_{0\le t\le T}\Big\|\Xi_{k,t}^{(N)}\Big\|_{-\alpha}^2\Big]$$
$$\le \sum_{n\in\mathbb{Z}}E\Big[\sup_{0\le t\le T}\Big|\Xi_{k,t}^{(N)}(h_n(x))\Big|^2\Big]\gamma_n^{-\alpha}$$
$$\le C\sum_{n\in\mathbb{Z}}\Big\{\sup_{0\le t\le T,0\le x<1}\Big|\frac{\partial}{\partial x}h_n(x)\Big|^2 + \sup_{0\le t\le T,0\le x<1}\Big|h_n(x)\Big|^2$$
$$+\sup_{0\le t\le T,0\le x<1}\Big|\frac{\partial}{\partial x}\mathcal{A}_{k,t}h_n(x)\Big|^2 + \sup_{0\le t\le T,0\le x<1}\Big|\mathcal{A}_{k,t}h_n(x)\Big|^2\Big\}\gamma_n^{-\alpha}$$
$$\le C'\sum_{n\in\mathbb{Z}}n^4\gamma_n^{-\alpha},$$

for some constants C, C'. Hence if $\alpha\ge 3$, we have

$$E\Big[\sup_{0\le t\le T}\Big\|\Xi_{k,t}^{(N)}\Big\|_{-\alpha}^2\Big] \le C \tag{4.4}$$

for some constant C.

Similarly, if $0 \le s < t \le T$, then we have

$$E\left[\sup_{s\le u<v\le t}\left\|\Xi^{(N)}_{k,v}-\Xi^{(N)}_{k,u}\right\|^2_{-\alpha}\right] \le (t-s)C\sum_{n\in\mathbb{Z}} n^4\gamma_n^{-\alpha},$$

for some constant C. Here we allow for u, v to be stopping times. Hence if $\alpha \ge 3$, we have

$$E\left[\sup_{s\le u<v\le t}\left\|\Xi^{(N)}_{k,v}-\Xi^{(N)}_{k,u}\right\|^2_{-\alpha}\right] \le C(t-s) \tag{4.5}$$

for some constant C.

Thanks to the Aldous tightness criteria (see Billingsley (1999, p.176)), (4.4) and (4.5) imply the sequence of probability measures $Q^{(N)}_f$, which is induced by $\Xi^{(N)}$, is tight.

5. Convergence of Ψ

We recall that $\Psi^{(N)}(t) = (\Psi^{(N)}_1(t), \Psi^{(N)}_2(t), \dots, \Psi^{(N)}_K(t))$, whose k-th component is a centered normalized number of k-th particle between $Y^{(N)}_1(t)$ and 1, is given by

$$\Psi^{(N)}_k(t) := \sqrt{N}\left\{\frac{1}{N}\sum_{j=2}^N \mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y^{(N)}_j(t) > Y^{(N)}_1(t)) - \int_{y_C(t)}^1 u_k(t,z)dz\right\},$$

where $y_C(t) = y_C(t;0,y_1)$ if $0 \le t < \tau_1$ and $y_C(t) = y_C(t;\tau_n,0)$ if $\tau_n \le t < \tau_{n+1}$ for $n \ge 1$.

By the definition of the characteristic line y_C, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{y_C(t)}^1 u_k(t,z)dz &= -\frac{d}{dt}y_C(t)u_k(t,y_C(t)) + \int_{y_C(t)}^1 \frac{\partial}{\partial s}u_k(t,z)dz \\ &= -\int_{y_C(t)}^1 \tilde{w}_k(t,z)u_k(t,z)dz.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_{y_C(t)}^1 u_k(t,z)dz - \int_{y_C(\tau_n)}^1 u_k(\tau_n,z)dz \\ &= -\int_{s=\tau_n}^t\int_{z=y_C(s)}^1 \tilde{w}_k(s,z)u_k(s,z)dzds \\ &= -\int_{s=\tau_n}^t \tilde{w}_k(s,y_C(s))\int_{z=y_C(s)}^1 u_k(s,z)dzds \\ &\quad -\int_{s=\tau_n}^t\int_{z=y_C(s)}^1 \{\tilde{w}_k(s,z)-\tilde{w}_k(s,y_C(s))\}u_k(s,z)dzds,\end{aligned} \tag{5.1}$$

for $\tau_n \le t < \tau_{n+1}$, $1 \le k \le K$.

By using Ito formula, we have the following expression:

$$\begin{aligned}&\sqrt{N}\frac{1}{N}\sum_{j=2}^N \mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y^{(N)}_j(t) > Y^{(N)}_1(t)) \\ &-\sqrt{N}\frac{1}{N}\sum_{j=2}^N \mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y^{(N)}_j(\tau_n) > Y^{(N)}_1(\tau_n))\end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{N}\int_{s=\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_j^{(N)}(s-)>Y_1^{(N)}(s-))\\
&\times\int_{\xi}\mathbf{1}(0\le\xi<\tilde{w}_k(s,Y_j^{(N)}(s-)))\nu_j(ds,d\xi)\\
&= -\Big\{\sqrt{N}\int_{s=\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_j^{(N)}(s-)>Y_1^{(N)}(s-))\\
&\times\int_{\xi}\mathbf{1}(0\le\xi<\tilde{w}_k(s,Y_j^{(N)}(s-)))\nu_j(ds,d\xi)\\
&-\sqrt{N}\int_{s=\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_j^{(N)}(s-)>Y_1^{(N)}(s-))\tilde{w}_k(s,Y_j^{(N)}(s-))ds\Big\}\\
&-\sqrt{N}\int_{s=\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_j^{(N)}(s-)>Y_1^{(N)}(s-))\tilde{w}_k(s,Y_j^{(N)}(s-))ds\\
&=: -\mathcal{M}_k^{(N)}(t)+D_k^{(N)}(t)
\end{aligned}
$$

for $\tau_n\le t<\tau_{n+1}$, $1\le k\le K$.

It is standard to see that the first term $\mathcal{M}_k^{(N)}(t)$ is a martingale with quadratic variation

$$
\begin{aligned}
&\langle\mathcal{M}_k^{(N)}\rangle_t=\int_{\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j=\tilde{w}_k)\mathbf{1}(Y_j^{(N)}(s)>Y_1^{(N)}(s))\tilde{w}_k(s,Y_j^{(N)}(s))ds,\\
&\quad\text{for } 1\le k\le K,\\
&\langle\mathcal{M}_k^{(N)},\mathcal{M}_{k'}^{(N)}\rangle_t=0,\quad\text{if } 1\le k\ne k'\le K,
\end{aligned}
$$

for $\tau_n\le t<\tau_{n+1}$. Thanks to Proposition 2.2, we have

$$
\langle\mathcal{M}_k^{(N)}\rangle_t=\int_{s=\tau_n}^{t}\int_{z=y_C(s)}^{1}\tilde{w}_k(s,z)u_k(s,z)(1+o(1))dzds, \tag{5.2}
$$

for $\tau_n\le t<\tau_{n+1}$, $1\le k\le K$.

We recall that $\mathcal{N}_k^{(N)}(t)$ is a martingale defined by (3.3). We also have

$$
\begin{aligned}
&\langle\mathcal{M}_k^{(N)},\mathcal{N}_{k'}^{(N)}(\cdot;f)\rangle_t\\
&\quad=\int_{\tau_n}^{t}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}(w_i=\tilde{w}_k)\mathbf{1}(Y_i^{(N)}(s)>Y_1^{(N)}(s))\\
&\qquad\times\Bigg[\sum_{j=1}^{N}\mathbf{1}(w_j=\tilde{w}_{k'})\mathbf{1}(Y_i^{(N)}(s)>Y_j^{(N)}(s))\nabla^N f(s,Y_j^{(N)}(s))\\
&\qquad+\mathbf{1}(w_i=\tilde{w}_{k'})\Big\{f(s,0)-f(s,Y_i^{(N)}(s))\Big\}\Bigg]w_i(s,Y_i^{(N)}(s))ds,
\end{aligned}
$$

for $\tau_n \le t < \tau_{n+1}$, $1 \le k, k' \le K$ and f is a test function. Thanks to Proposition 2.2, we also have

$$\begin{aligned}
&\langle \mathcal{M}_k^{(N)}, \mathcal{N}_{k'}^{(N)}(\cdot; f)\rangle_t \\
&= \int_{s=\tau_n}^t \int_{x=y_C(s)}^1 \left[\int_{z=0}^x \frac{\partial}{\partial x} f(s,z) u_{k'}(s,z)dz + \mathbf{1}(k=k')\Big\{ f(s,0) - f(s,x) \Big\} \right] \\
&\times \tilde{w}_k(s,x)dxds(1+o(1)),
\end{aligned}$$

for $\tau_n \le t < \tau_{n+1}$, $1 \le k, k' \le K$ and f is a test function.

We shall decompose $D_k^{(N)}(t)$ into three terms as follows;

$$\begin{aligned}
&D_k^{(N)}(t) \\
&= -\sqrt{N} \int_{\tau_n}^t \frac{1}{N} \sum_{j=2}^N \mathbf{1}(w_j = \tilde{w}_k) \mathbf{1}(Y_j^{(N)}(s) > Y_1^{(N)}(s)) \tilde{w}_k(s, Y_j^{(N)}(s))ds \\
&= -\sqrt{N} \int_{\tau_n}^t \tilde{w}_k(s, y_C(s)) \frac{1}{N} \sum_{j=2}^N \mathbf{1}(w_j = \tilde{w}_k) \mathbf{1}(Y_j^{(N)}(s) > Y_1^{(N)}(s))ds \\
&\quad -\sqrt{N} \int_{\tau_n}^t \frac{1}{N} \sum_{j=2}^N \mathbf{1}(w_j = \tilde{w}_k) \mathbf{1}(Y_j^{(N)}(s) > y_C(s)) \\
&\quad \times \{\tilde{w}_k(s, Y_j^{(N)}(s)) - \tilde{w}_k(s, y_C(s))\}ds \\
&\quad +\sqrt{N} \int_{\tau_n}^t \frac{1}{N} \sum_{j=2}^N \Big\{ \tilde{w}_k(s, Y_j^{(N)}(s)) - \tilde{w}_k(s, y_C(s)) \Big\} \\
&\quad \times \mathbf{1}(w_j = \tilde{w}_k) \Big\{ \mathbf{1}(y_C(s) < Y_j^{(N)}(s) < Y_1^{(N)}(s)) \\
&\quad - \mathbf{1}(Y_1^{(N)}(s) < Y_j^{(N)}(s) < y_C(s)) \Big\} ds.
\end{aligned}$$

By our assumption, we have

$$\sqrt{N} \frac{1}{N} \sum_{j=2}^N \mathbf{1}(w_j = \tilde{w}_k) \mathbf{1}(Y_j^{(N)}(\tau_n) > Y_1^{(N)}(\tau_n)) - \int_{y_C(\tau_n)}^1 u_k(\tau_n, z)dz = o(1),$$

for $1 \le k \le K$. Hence we have $\Psi_k^{(N)}(\tau_n) = o(1)$ for $1 \le k \le K$ and

$$\begin{aligned}
&\Psi_k^{(N)}(t) - \Psi_k^{(N)}(\tau_n) \\
&= -\mathcal{M}_k^{(N)}(t) - \int_{\tau_n}^t \tilde{w}_k(s, y_C(s)) \Psi_k^{(N)}(s)ds - \int_{\tau_n}^t F_k^{(N)}(s)ds \qquad (5.3) \\
&+\sqrt{N} \int_{\tau_n}^t \frac{1}{N} \sum_{j=2}^N \Big\{ \tilde{w}_k(s, Y_j^{(N)}(s)) - \tilde{w}_k(s, y_C(s)) \Big\} \\
&\times \mathbf{1}(w_j = \tilde{w}_k) \Big\{ \mathbf{1}(y_C(s) < Y_j^{(N)}(s) < Y_1^{(N)}(s)) - \mathbf{1}(Y_1^{(N)}(s) < Y_j^{(N)}(s) < y_C(s)) \Big\} ds,
\end{aligned}$$

for $1 \le k \le K$.

Since $\tilde{w}_k \in C^1$, we have

$$|\tilde{w}_k(s, Y_j^{(N)}(s)) - \tilde{w}_k(s, y_C(s))| \le C |Y_j^{(N)}(s) - y_C(s)|,$$

for some constant C for all $1 \le j \le N$ and $1 \le k \le K$. It is easy to see that

$$\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}(y_C(s) < Y_j^{(N)}(s) < Y_1^{(N)}(s)) \le |Y_1^{(N)}(s) - y_C(s)|,$$
$$\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}(Y_1^{(N)}(s) < Y_j^{(N)}(s) < y_C(s)) \le |Y_1^{(N)}(s) - y_C(s)|. \tag{5.4}$$

Hence the absolute value of the last term in (5.3) has the following estimate:

$$\begin{aligned}
&\Big|\sqrt{N}\int_{\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\Big\{\tilde{w}_k(s, Y_j^{(N)}(s)) - \tilde{w}_k(s, y_C(s))\Big\}\mathbf{1}(w_j = \tilde{w}_k)\\
&\times\Big\{\mathbf{1}(y_C(s) < Y_j^{(N)}(s) < Y_1^{(N)}(s)) - \mathbf{1}(Y_1^{(N)}(s) < Y_j^{(N)}(s) < y_C(s))\Big\}ds\Big|\\
&\le C\int_{\tau_n}^{t}\sqrt{N}|Y_1^{(N)}(s) - y_C(s)|^2 ds
\end{aligned}$$

for some constant C.

If we show that $NE[|Y_1^{(N)}(s) - y_C(s)|^2] \le C$ for some constant C, then we have derived the martingale problem related to (2.6) from $\Psi_k^{(N)}$ as $N \to \infty$.

Thanks to (2.5), we have the following inequality:

$$\begin{aligned}
&\sqrt{N}|Y_1^{(N)}(t) - y_C(t)| - \sqrt{N}|Y_1^{(N)}(\tau_n) - y_C(\tau_n)|\\
=\ &\left|\sum_{k=1}^{K}\Psi_k^{(N)}(t)\right| - \left|\sum_{k=1}^{K}\Psi_k^{(N)}(\tau_n)\right|\\
\le\ &\sum_{k=1}^{K}|\mathcal{M}_k^{(N)}(t)| + C\int_{\tau_n}^{t}\sqrt{N}|Y_1^{(N)}(s) - y_C(s)|ds + \int_{\tau_n}^{t}\left|\sum_{k=1}^{K}F_k^{(N)}(s)\right|ds\\
&+\left|\sum_{k=1}^{K}\sqrt{N}\int_{\tau_n}^{t}\frac{1}{N}\sum_{j=2}^{N}\Big\{\tilde{w}_k(s, Y_j^{(N)}(s)) - \tilde{w}_k(s, y_C(s))\Big\}\mathbf{1}(w_j = \tilde{w}_k)\right.\\
&\left.\times\Big\{\mathbf{1}(y_C(s) < Y_j^{(N)}(s) < Y_1^{(N)}(s)) - \mathbf{1}(Y_1^{(N)}(s) < Y_j^{(N)}(s) < y_C(s))\Big\}ds\right|.
\end{aligned}$$

Thanks to (5.4), we have

$$\begin{aligned}
&\sqrt{N}|Y_1^{(N)}(t) - y_C(t)| - \sqrt{N}|Y_1^{(N)}(\tau_n) - y_C(\tau_n)|\\
&\le \sum_{k=1}^{K}|\mathcal{M}_k^{(N)}(t)| + C'\int_{\tau_n}^{t}\sqrt{N}|Y_1^{(N)}(s) - y_C(s)|ds + \int_{\tau_n}^{t}\left|\sum_{k=1}^{K}F_k^{(N)}(s)\right|ds
\end{aligned}$$

for some constant C'. By applying the Cauchy Schwarz inequality, we have

$$\begin{aligned}
&NE[|Y_1^{(N)}(t) - y_C(t)|^2]\\
&\le C\Big\{NE[|Y_1^{(N)}(\tau_n) - y_C(\tau_n)|^2] + \sum_{k=1}^{K}E[|\mathcal{M}_k^{(N)}(t)|^2] + \sum_{k=1}^{K}\int_{\tau_n}^{t}E[F_k^{(N)}(s)]ds\Big\}\\
&\quad +C\int_{\tau_n}^{T}NE[|Y_1^{(N)}(s) - y_C(s)|^2]ds.
\end{aligned}$$

By our assumption, we have $\sqrt{N}|Y_1^{(N)}(0) - y_C(0)| = o(1)$ and $|Y_1^{(N)}(\tau_n) - y_C(\tau_n)| = 0$ for $n \geq 1$. Thanks to (5.2) and (4.3), we have

$$E[\sup_{\tau_n \leq t < \tau_{n+1}} |\mathcal{M}_k^{(N)}(t)|^2] \leq C, \quad \text{and} \quad E[\sup_{0 \leq t \leq T} |F_k^{(N)}(s)|^2] \leq C$$

for some constant C.

By applying Gronwall's inequality, we conclude that

$$NE[|Y_1^{(N)}(s) - y_C(s)|^2] \leq C \tag{5.5}$$

for some constant C. Hence we have derived the martingale problem related to (2.6) from $\Psi_k^{(N)}$ as $N \to \infty$.

Thanks to (5.3) and (5.4), we have

$$\begin{aligned} |\Psi_k^{(N)}(t)| \quad &\leq \quad |\Psi_k^{(N)}(\tau_n)| + |\mathcal{M}_k^{(N)}(t)| + C\int_{\tau_n}^t |\Psi_k^{(N)}(s)|ds \\ &\quad + \int_{\tau_n}^t |F_k^{(N)}(s)|ds + C\int_{\tau_n}^t \sqrt{N}|Y_1^{(N)}(s) - y_C(s)|ds \end{aligned}$$

for $\tau_n \leq t < \tau_{n+1}$ for some constant C. Hence we have

$$\begin{aligned} &E[\sup_{\tau_n \leq t < \tau_{n+1}} |\Psi_k^{(N)}(t)|^2] \\ &\leq \; CE[|\Psi_k^{(N)}(\tau_n)|^2] + CE[\sup_{\tau_n \leq t < \tau_{n+1}} |\mathcal{M}_k^{(N)}(t)|^2] + C\int_{\tau_n}^{\tau_{n+1}} E[|F_k^{(N)}(s)|^2]ds \\ &\quad + C\int_{\tau_n}^{\tau_{n+1}} NE[|Y_1^{(N)}(s) - y_C(s)|^2]ds + C\int_{\tau_n}^{\tau_{n+1}} E[\sup_{\tau_n \leq u \leq s} |\Psi_k^{(N)}(u)|^2]ds, \end{aligned}$$

for some constant C.

By our assumption, we have $\sqrt{N}|Y_1^{(N)}(0) - y_C(0)| = o(1)$ and $|Y_1^{(N)}(\tau_n) - y_C(\tau_n)| = 0$ for $n \geq 1$. Thanks to (5.2), (4.3) and (5.5), we have

$$\begin{aligned} &E[\sup_{\tau_n \leq t < \tau_{n+1}} |\mathcal{M}_k^{(N)}(t)|^2] \leq C, \; E[\sup_{0 \leq t \leq T} |F_k^{(N)}(s)|^2] \leq C, \\ &\text{and} \;\; NE[|Y_1^{(N)}(s) - y_C(s)|^2] \leq C \end{aligned}$$

for some constant C.

By applying Gronwall's inequality we conclude that

$$E[\sup_{\tau_n \leq t < \tau_{n+1}} |\Psi_k^{(N)}(t)|^2] \leq C$$

for some constant C. Similarly, we have

$$E[\sup_{\tau_n \leq s \leq u < v \leq t < \tau_{n+1}} |\Psi_k^{(N)}(v) - \Psi_k^{(N)}(u)|^2] \leq C(t-s)$$

for some constant C. Here we allow for u, v to be stopping times. By applying Aldous tightness criteria (see Billingsley (1999, p.176)), these inequalities imply that the probability measures induced by $\Psi^{(N)}$ are tight.

6. On related PDE

In this section our proof is based on Bressan (2000, Sec. 3.4 p.46).

Proof of Lemma 2.1.

We shall show the existence and uniqueness of the solution to the system of quasilinear integral partial differential equation (2.2) by using standard fixed point theorem.

For each given $u(t,x) = (u_1(t,x), u_2(t,x), \dots, u_K(t,x))$, we set $a(t,x) = a(t,x;u)$ and $h_k(t,x) = h_k(t,x;u)$ for $1 \le k \le K$ by

$$\begin{aligned} a(t,x;u) &:= \sum_{r=1}^{K} \int_x^1 \tilde{w}_r(t,z) u_r(t,z) dz, \\ h_k(t,x;u) &:= \sum_{r=1}^{K} \tilde{w}_r(t,x) u_r(t,x) - \tilde{w}_k(t,x). \end{aligned}$$

It is easy to see that our integral partial differential equation is rewritten by

$$\frac{\partial}{\partial t} u_k(t,x) + a(t,x;u) \frac{\partial}{\partial x} u_k(t,x) = h_k(t,x;u) u_k(t,x),$$

for $1 \le k \le K$.

Suppose that $u(t,x)$ is nonnegative, i.e. $u_k(t,x) \ge 0$ for all $1 \le k \le K$, $0 \le t \le T$ and $0 \le x < 1$, then it is easy to see that $a(t,x) \ge 0$ for all t and $0 \le x < 1$. Furthermore $a(t,x) \to 0$ as $x \to 1$. It is also easy to see that if u is continuous, then $a(t,x;u)$ is a continuous w.r.t. (t,x) and Lipschitz continuous w.r.t. x.

By using standard ODE theorem, for given nonnegative u, we set $y(t;\tau,\xi) = y(t;\tau,\xi;u)$ for $\tau_0 \le t \le T$, $0 \le \tau \le T$, and $0 \le \xi < 1$, by the unique solution to the ODE

$$\frac{dy}{dt} = a(t,y), \ \ y(\tau) = \xi,$$

where $\tau_0 = \tau_0(\tau,\xi) = \max\{0, \tau_0'\}$ and $\tau_0' = \inf\{t; y(t) \ge 0\}$. We note that this y is the characteristic line for the semi linear PDE $\frac{\partial}{\partial t} v_k(t,x) + a(t,x) \frac{\partial}{\partial x} v_k(t,x) = h_k(t,x) v_k(t,x)$, for all $1 \le k \le K$.

We consider a linear ODE

$$\frac{d}{dt} f_k(t) = h_k(t, y(t;\tau,\xi)) f_k(t), \ \ f_k(\tau_0) = u_k(\tau_0, y(\tau_0))$$

for each $1 \le k \le K$, $0 \le \tau \le T$ and $0 \le \xi < 1$. Note that if $\tau_0 \ne 0$, then $y(\tau_0) = 0$. It is easy to see that the unique solution to this ODE is given by

$$f_k(t) = f_k(t;\tau,\xi) = u_k(\tau_0, y(\tau_0)) \exp\Big\{ \int_{\tau_0}^t h_k(s, y(s;\tau,\xi)) ds \Big\},$$

for $1 \le k \le K$. It is obvious that if u is nonnegative, then $f = (f_1, f_2, \dots, f_K)$ is also nonnegative.

It is easy to see that if we set $v_k(\tau,\xi) := f_k(\tau;\tau,\xi)$, then $v = (v_1, v_2, \dots, v_K)$ solves

$$\begin{aligned} &\frac{\partial}{\partial t} v_k(t,x) + a(t,x) \frac{\partial}{\partial x} v_k(t,x) = h_k(t,x) v_k(t,x), \\ &v_k(t,0) = u_k(t,0), \\ &v_k(0,x) = u_k(0,x), \end{aligned}$$

for $1 \le k \le K$, $0 \le t \le T$ and $0 \le x < 1$.

We suppose that $\sum_{r=1}^K u_r(t,x) = 1$ for all $0 \le t \le T$ and $0 \le x < 1$. Then it is easy to see that $\sum_{r=1}^K h_r(t,x;u) = 0$. Hence we have $F(t,x) = \sum_{r=1}^K f_r(t,x) = 1$ for all $0 \le t \le T$ and $0 \le x < 1$ and $\sum_{r=1}^K v_r(t,x) = 1$ for all $0 \le t \le T$ and $0 \le x < 1$.

For nonnegative u, we define a transformation $\mathcal{T}$ by

$$(\mathcal{T}u(t,x))_k = \begin{cases} v_k(t,x) & \text{for } 0<t<T,\ 0<x<1, \\ \dfrac{\int_0^1 \tilde{w}_k u_k(t,x)dx}{\sum_{r=1}^K \int_0^1 \tilde{w}_r u_r(t,x)dx} & \text{for } 0<t<T,\ x=0, \\ u_k(0,x) & \text{for } t=0,\ 0\le x<1, \end{cases}$$

for $1 \le k \le K$.

It is standard to see that the fixed point of $\mathcal{T}$ is the unique solution to the Cauchy problem for our quasilinear integral differential equation.

It is not difficult to see that there are constants $T_0 > 0$ and C such that

$$\begin{aligned} &\sup_{1\le k\le K, 0<t\le T_0, 0\le x<1} \Big| (\mathcal{T}u(t,x))_k - (\mathcal{T}\tilde{u}(t,x))_k \Big| \\ &\le CT_0 \sup_{1\le k\le K, 0\le t\le T_0, 0\le x<1} \Big| (u(t,x))_k - (\tilde{u}(t,x))_k \Big|. \end{aligned}$$

Furthermore, we can take T_0 small enough so that $CT_0 < 1$. Hence if we restrict our time interval to $[0, T_0]$, and our initial data to $u_{k,0}$, then the transformation $\mathcal{T}$ is a contraction transformation. By applying the fixed point theorem, for each given initial data $u_{k,0}$, we have $u = (u_1, u_2, \dots, u_K)$ such that

$$\begin{aligned} &u_k(t,x) = (\mathcal{T}u(t,x))_k, \quad \text{for } 0\le t\le T_0,\ 0\le x<1, \\ &u_k(0,x) = u_{k,0}(x), \quad \text{for } 0\le x<1. \end{aligned}$$

Namely, we have the unique solution to the Cauchy problem for our quasilinear integral partial differential equation in the time interval $[0, T_0]$. Similarly, we can extend our solution in the time intervals, $[T_0, 2T_0],\ [2T_0, 3T_0], \dots$. Hence we have the unique global solution to the Cauchy problem for our quasilinear integral partial differential equation. □

Proof of Lemma 4.1.

To simplify our notation, we set

$$a(s,x) := \sum_{r=1}^K \int_x^1 \tilde{w}_r(s,z) u_r(s,z)dz.$$

In our setting, $a(s,x)$ is continuous w.r.t. (s,x), is C^1 w.r.t. x, $a(s,x) \ge 0$, $a(s,x) \to 0$ as $x \to 1$ and $\sup_{0\le s\le T, 0\le x<1} |a(s,x)| < C$ for some constant C. For given $g(s,x)$, we set $h_k(s,x) = h_k(s,x;g)$ by

$$h_k(s,x;g) := \tilde{w}_k(s,x)\Big\{ g(s,x) - g(s,0) \Big\}.$$

We rewrite our PDE by

$$\begin{aligned} &\frac{\partial}{\partial s} g(s,x) + a(s,x) \frac{\partial}{\partial x} g(s,x) = h_k(s,x;g), \\ &g(t,x) = f(t,x) \end{aligned}$$

for some given pair of $(t, f(t,x))$ for each $1 \le k \le K$. We shall show the existence and uniqueness of Cauchy problem for this semilinear PDE by using standard fixed point theorem.

By using standard ODE theorem, we set characteristic line for our PDE $y(s) = y(s;\tau,\xi)$ for $0 \le \tau \le s \le t$, $0 \le \xi < 1$, by the unique solution to the ODE

$$\frac{dy}{ds} = a(s,y), \ \ y(\tau) = \xi.$$

Since $a(s,y) \ge 0$ and vanishes at $x = 1$, for each $0 \le \tau \le t$, $0 \le \xi < 1$, we have $0 \le y(t;\tau,\xi) < 1$.

For given $g(s,x)$ we consider a Cauchy problem for the ODE

$$\frac{d}{ds}v(s;\tau,\xi;g) = h_k(s,y(s;\tau,\xi);g),\ \ v(t;\tau,\xi;g) = f(t,y(t;\tau,\xi)),$$

for each $0 \le \tau \le t$, $0 \le \xi < 1$. It is easy to see that the solution v is rewritten by

$$v(s;\tau,\xi;g) = f(t,y(t;\tau,\xi)) - \int_s^t h_k(u,y(u;\tau,\xi);g)du.$$

If we set $\tilde{g}(s,x) := v(s;s,x;g)$, then $\tilde{g}(s,x)$ solves

$$\begin{aligned}&\frac{\partial}{\partial s}\tilde{g}(s,x) + a(s,x)\frac{\partial}{\partial x}\tilde{g}(s,x) = h_k(s,x;g),\\ &\tilde{g}(t,x) = f(t,x).\end{aligned}$$

Given a pair $(t, f(t,x))$, we define a transformation $\mathcal{T}_k$ by

$$\mathcal{T}_k g(s,x) = \begin{cases}\tilde{g}(s,x) & \text{for } s<t,\ 0<x<1,\\ f(t,x) & \text{for } s=t,\ 0\le x<1,\end{cases}$$

for $1 \le k \le K$.

It is not difficult to see that there are constants $t_0 > 0$ and C such that

$$\sup_{1\le k\le K, t-t_0<s\le t, 0\le x<1}\Big|\mathcal{T}_k g(s,x) - \mathcal{T}_k\bar{g}(s,x)\Big| \le Ct_0 \sup_{t-t_0\le s\le t, 0\le x<1}\Big|g(s,x)-\bar{g}(s,x)\Big|.$$

Furthermore we can take t_0 and C such that $Ct_0 < 1$ and they are independent of t. Hence we have a contraction transform $\mathcal{T}_k$. By applying standard fixed point theorem, we have the unique solution to the Cauchy problem for our semilinear PDE in the time interval $[t-t_0, t]$. We can extend our solution in the time interval $[t-2t_0, t-t_0], [t-3t_0, t-2t_0], \ldots$. Hence we have the unique solution to the Cauchy problem for our semilinear PDE in the time interval $[0,t]$.

Note that $a(s,x)$ only depends on $\tilde{w}_k$ and u_k for $1 \le k \le K$. We recall that $a(s,x)$ is continuous w.r.t. s and C^1 w.r.t. x. By standard ODE theorem, the characteristic line $y(s;\tau,\xi)$ is C^1 w.r.t. ξ. By using the change of variables, we rewrite $\mathcal{T}_k g$ by

$$\begin{aligned}&\mathcal{T}_k g(s,x) = f(t,y(t;s,x))\\ &+\int_0^{t-s}\tilde{w}_k(s+u,y(s+u;s,x))\Big\{g(s+u,y(s+u;s,x)) - g(s+u,0)\Big\}du.\end{aligned}$$

By applying Gronwall's inequality, for g the fixed point of $\mathcal{T}_k$, we have

$$\sup_{0\le s\le t, 0\le x<1}\Big|g(s,x)\Big| \le C\sup_{0\le x<1}\Big|f(t,x)\Big|,$$

for some C.

We also have

$$\begin{aligned}&\mathcal{T}_k g(s,x+h) - \mathcal{T}_k g(s,x)\\ &\quad= \Big\{f(t,y(t;s,x+h)) - f(t,y(t;s,x))\Big\}\\ &\qquad+\int_0^{t-s}\Big\{\tilde{w}_k(s+u,y(s+u;s,x+h))g(s+u,y(s+u;s,x+h))\\ &\qquad-\tilde{w}_k(s+u,y(s+u;s,x))g(s+u,y(s+u;s,x))\Big\}du\\ &\qquad+\int_0^{t-s}\Big\{\tilde{w}_k(s+u,y(s+u;s,x)) - \tilde{w}_k(s+u,y(s+u;s,x))\Big\}g(s+u,0)du.\end{aligned}$$

Hence if f and g are C^1 w.r.t. x, then $\mathcal{T}_k g$ is also C^1 w.r.t. x. Furthermore if f is C^1 w.r.t. x, then the fixed point of $\mathcal{T}_k$ is also C^1 w.r.t. x and by using Gronwall's inequality, g, the fixed point of $\mathcal{T}_k$, satisfies

$$\sup_{0\le s\le t, 0\le x<1}\Big|\frac{\partial}{\partial x}g(s,x)\Big| \le \Big\{\sup_{0\le x<1}\Big|\frac{\partial}{\partial x}f(s,x)\Big| + C\sup_{0\le x<1}\Big|f(t,x)\Big|\Big\}(1+Cte^{Ct}),$$

for some constant C. □

Acknowledgements

The author would like to thank Professor T. Hattori, Professor K. Uchiyama for their valuable comments. The author also would like to thank the anonymous referee for helpful comments.

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