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Truncation of long-range percolation models with square non-summable interactions

Alberto M. Campos and Bernardo N. B. de Lima

Departamento de Matemática, Universidade Federal de Minas Gerais,
Av. Antônio Carlos 6627 C.P. 702 CEP 30123-970 Belo Horizonte-MG, Brazil
E-mail address: albertomizhahy@hotmail.com; bnblima@mat.ufmg.br

Abstract. We consider some problems related to the truncation question in long-range percolation. Probabilities are given that certain long-range oriented bonds are open; assuming that these probabilities are not summable, we ask if the probability of percolation is positive when we truncate the graph, disallowing bonds of range above a possibly large but finite threshold. This question is still open if the set of vertices is $\mathbb{Z}^2$. We give some conditions under which the answer is affirmative. One of these results generalizes a previous result in [Alves, Hilário, de Lima, Valesin, Journ. Stat. Phys. **122**, 972 (2017)].

1. Introduction

Long-range statistical mechanics models are an old topic that have been studied for a long time, e.g., Aizenman et al. (1988); Dyson (1969a,b) and Fröhlich and Spencer (1982) for Ising models or Aizenman et al. (1987); Aizenman and Newman (1986) and Newman and Schulman (1986) for percolation models.

One of the more intriguing questions in long-range percolation is the so-called *truncation question*. In words (we will become more formal later), this question can be stated as follows: consider a translation-invariant long-range percolation model that percolates with positive probability. Is the infinity of range indeed crucial for the occurrence of percolation?

More precisely, let $G = (\mathbb{V}, \mathbb{E})$ be a transitive graph, where the set of edges $\mathbb{E}$ can be partitioned as $\mathbb{E} = \cup_{n=1}^{\infty} \mathbb{E}_n$, where $\mathbb{E}_n$ is the set of edges of length n. Let $(p_n)_n \in [0,1]$ be a sequence of parameters. Consider on this graph an independent bond percolation model where bonds are open independently, in which each bond e is open with probability $p_{\|e\|}$, where $\|e\|$ is the length of e.

Thus, the probability space that describes this model is $(\Omega, \mathcal{F}, P)$, where $\Omega = \{0,1\}^{\mathbb{E}}$, $\mathcal{F}$ is the canonical product σ-algebra, and $P = \prod_{e\in\mathbb{E}} \mu_e$, where $\mu_e(\omega_e = 1) = p_{\|e\|} = 1 - \mu_e(\omega_e = 0)$. An element $\omega \in \Omega$ is called a percolation configuration.

Given a positive integer K, define the truncated sequence $(p_n^K)_n$ as

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$$p_n^K = \begin{cases} p_n, & \text{if } n \leqslant K, \\ 0, & \text{if } n > K, \end{cases} \tag{1.1}$$

and the truncated measure $P^K = \prod_{e\in\mathbb{E}} \mu_e^K$, where $\mu_e^K(\omega_e = 1) = p^K_{\|e\|} = 1 - \mu_e^K(\omega_e = 0)$.

Then, the truncation question can be restated as: fix a vertex $0 \in \mathbb{V}$ (remind that we consider transitive graphs), given a sequence $(p_n)_n$ where $P(0 \leftrightarrow \infty) > 0$, does there exist a large enough truncation constant K such that $P^K(0 \leftrightarrow \infty) > 0$? (Here we are using the standard notation in percolation where $(0 \leftrightarrow \infty)$ means the set of configurations $\omega \in \Omega$ such that there exists an infinite open path starting from the origin.)

Whenever $G = (\mathbb{V}, \mathbb{E})$ is the d-dimensional hypercubic lattice with long range bonds parallel to the coordinate axes, i.e., $\mathbb{V} = \mathbb{Z}^d$ and $\mathbb{E}_n = \{\langle x, x+n.\vec{e}_i\rangle; x \in \mathbb{Z}^d, i \in \{1, \dots, d\}\}$, where $\vec{e}_i$ is the i-th vector in the canonical basis of $\mathbb{Z}^d$; the truncation question can be placed for summable sequences $(p_n)_n$ as well as for non-summable sequences. In the latter case, if $\sum_n p_n = \infty$ by the Borel-Cantelli Lemma, it follows that $P(0 \leftrightarrow \infty) = 1$.

If $d = 1$, it is an exercise to see that the truncation question has a negative answer; when $d \geqslant 3$, it was shown in Friedli and de Lima (2006) that the truncation question has an affirmative answer. The case $d = 2$ is still an open problem and several works tackled this question adding some extra hypotheses upon the sequence $(p_n)_n$ like Berger (2002); Friedli and de Lima (2006); Friedli et al. (2004); de Lima and Sapozhnikov (2008); Menshikov et al. (2001) and Sidoravicius et al. (1999).

In some of these results, it is shown that $\lim_{K\to\infty} P^K(0 \leftrightarrow \infty) = 1$, which is a little stronger than the truncation question. Indeed in any situation, we have the weak convergence $P^K \Rightarrow P$ when $K \to \infty$, but the Portmanteau Theorem cannot be applied because the boundary (with respect the product topology) of the event $(0 \leftrightarrow \infty)$ has positive probability concerning the measure P.

In Section 2, we will give an affirmative answer, for the case $d = 2$, with some extra hypotheses that are not included in the papers cited above.

An analogous truncation question can be stated for the q-state ferromagnetic Potts model (see Proposition 2 of Friedli and de Lima (2006)) and rephrased as a percolation question, due to the Fortuin-Kastelyn random-cluster representation. It was shown in Fortuin (1972a) and Fortuin (1972b) that the magnetization of the truncated q-states long-range Potts model and the probability of percolation on the long-range processes are related by inequality

$$\mu^{\beta,s}_{\phi_n^K} \geqslant \frac{1}{q} + \frac{q-1}{q} P^K(0 \leftrightarrow \infty),$$

if the n-range potential function $(\phi_n)_n$ and the long-range percolation parameters are related by $p_n = \frac{1-\exp(-2\beta\phi_n)}{1+(q-1)\exp(-2\beta\phi_n)}$.

When the sequence $(p_n)_n$ is summable and there is percolation with positive probability, the truncation question can also be stated. The papers Meester and Steif (1996) and Berger (2002) are examples where affirmative answers are given. However in Biskup et al. (2006), a negative answer was given in the context of the Potts model with $q = 3$.

In each section, we will consider the truncation question on a different type of graph. In Section 3, we study the truncation question on a special oriented graph, generalizing the result of Theorem 1 of Alves et al. (2017).

2. Truncation question on a long-range square lattice

In this section, consider an anisotropic version of the graph G. Let $G^{an} = (\mathbb{Z}^2, \mathbb{E}^{an})$ be the graph whose set of bonds is $\mathbb{E}^{an} = \mathbb{E}^v \cup (\cup_n \mathbb{E}_n^h)$, where $\mathbb{E}^v = \{\langle x, x + (0, 1)\rangle; x \in \mathbb{Z}^2\}$ is the set of nearest neighbor vertical bonds and $\mathbb{E}_n^h = \{\langle x, x + (n, 0)\rangle; x \in \mathbb{Z}^2\}$ is the set of horizontal bonds with length n. Given the parameters δ and $(p_n)_n$, each bond e is open, independently, with probability δ or p_n, if e belongs to $\mathbb{E}^v$ or $\mathbb{E}^h_n$, respectively. We continue denoting by P and P^K the non-truncated and truncated at K measures.

Let us remember the H. Kesten result that $p_v + p_h = 1$ is the critical curve for independent anisotropic percolation on the ordinary square lattice $\mathbb{L}^2$ (see Kesten (1982) or Grimmett (1999)), where vertical and horizontal bonds are open with probabilities p_v and p_h, respectively. Indeed, in the next theorem, we will use the following lemma:

Lemma 2.1. *Consider an independent and anisotropic percolation model on the square lattice $\mathbb{L}^2$ with parameters p_v and p_h. Given any $p_v > 0$, it holds that $\lim_{p_h\to 1^-} P_{p_v,p_h}((0,0)\leftrightarrow\infty) = 1$.*

Theorem 2.2. *Consider the anisotropic percolation model on the graph G^{an} defined above. Given any $\delta > 0$, if the sequence $(p_n)_n$ satisfies $\sum_n p_n p_{n+N} = \infty$ for some $N > 0$, it holds that*

$$\lim_{K\to\infty} P^K\{(0,0)\leftrightarrow\infty\} = 1.$$

Proof: Fix $N > 0$ such that $\sum_n p_n p_{n+N} = \infty$, given any $\epsilon > 0$ we can choose integers M_1 and M_2 satisfying

$$\exp\left[-\sum_{n=1}^{M_1} p_n p_{n+N}\right] < \epsilon \quad \text{and} \quad \exp\left[-\sum_{n=M_1+1}^{M_2} p_n p_{n+N}\right] < \epsilon.$$

Given a vertex $(x,y)\in\mathbb{Z}^2$ and $n\in\mathbb{Z}_+$, let us define the following events:

$$E_{(x,y)}(n) = \{\langle (x,y);(x+n,y)\rangle \text{ and } \langle (x+n,y);(x-N,y)\rangle \text{ are open}\},$$
$$H^-_{(x,y)} = \cup_{n=1}^{M_1} E_{(x,y)}(n) \quad \text{and} \quad H^+_{(x,y)} = \cup_{n=M_1+1}^{M_2} E_{(x,y)}(n).$$

Observe that the events $H^\pm_\cdot$ use bonds with length at most M_2+N; therefore taking $K = M_2+N$ and by definition of M_1 and M_2, we have that

$$P^K(H^-_{(x,y)}) = 1 - P^K(\cap_{n=1}^{M_1}(E_{(x,y)}(n))^c) = 1 - \prod_{n=1}^{M_1}(1-p_n p_{n+N})$$
$$\geqslant 1 - \exp\left[-\sum_{n=1}^{M_1} p_n p_{n+N}\right] > 1-\epsilon.$$

Analogously, the same bound holds for the probability of $H^+_{(x,y)}$.

Now, we will couple a percolation process on the ordinary square lattice $\mathbb{L}^2$ (with only nearest neighbors non-oriented bonds) in the following manner: given $e = \langle (v_1,v_2);(u_1,u_2)\rangle$ a bond of $\mathbb{L}^2$, define the sequence of events $(X_e)_e$ as follows

$$X_e = \begin{cases} H^-_{(Nv_1,Nv_2)}, & \text{if } v_2 = u_2,\ u_1 - v_1 = 1 \text{ and } v_1 \text{ is even;} \\ H^+_{(Nv_1,Nv_2)}, & \text{if } v_2 = u_2,\ u_1 - v_1 = 1 \text{ and } v_1 \text{ is odd;} \\ \{\langle (Nv_1,Nv_2);(Nv_1,Nv_2+1)\rangle \text{ is open }\}, & \text{if } v_1 = u_1 \text{ and } \ u_2 - v_2 = 1. \end{cases}$$

We declare each bond e of $\mathbb{L}^2$ as *red* if and only if the event X_e occurs. The appropriate choice of the events $H^-_\cdot$ and $H^+_\cdot$ ensures that the events $(X_e)_e$ are independent. Thus, bonds in $\mathbb{L}^2$ are red following an independent anisotropic bond percolation, where each vertical bond is open with probability δ and horizontal bonds are open with probability at least $1-\epsilon$. It follows from the definition of $(X_e)_e$ that an infinite red path starting from the origin in $\mathbb{L}^2$ implies an infinite open path starting from the origin in the graph G^{an}. By Lemma 2.1, we can conclude that $\lim_{K\to\infty} P^K\{(0,0)\leftrightarrow\infty\} = 1$. □

The next lemma, due Kalikow and Weiss (see Theorem 2 of Kalikow and Weiss (1988)), is an important fact in the proof of our next result. We state it as we will need later.

Lemma 2.3. *Consider an independent long-range bond percolation model on the one-dimension graph* $(\mathbb{Z}^+, \{\langle i,j\rangle; i,j \in \mathbb{Z}^+\})$ *with parameters* $(p_n)_n$. *If* $\sum_n p_n = \infty$ *and* $\gcd\{n; p_n > 0\} = 1$, *then the random graph on* $\mathbb{Z}^+$ *formed by open bonds is connected a.s.. Moreover, for all* $l \in \mathbb{Z}^+$ *it holds that* $\lim_{L\to\infty} P(\{0,1,\dots,l\}$ *are connected in* $\{0,1,\dots,L\}) = 1$.

Theorem 2.4. *Consider the anisotropic percolation model on the graph* G^{an}. *Given any* $\delta > 0$, *if the sequence* $(p_n)_n$ *satisfies* $\limsup_{N\to\infty}\sum_n p_n p_{n+N} > 0$, *it holds that*

$$\lim_{K\to\infty} P^K\{(0,0) \leftrightarrow \infty\} = 1.$$

Proof: Suppose that $\gcd\{n; p_n > 0\} = 1$ and let $\eta > 0$ be such that $\limsup_{N\to\infty}\sum_n p_n p_{n+N} = 2\eta$. Given any $\epsilon > 0$, choose a large integer ℓ satisfying

$$\left[1 - [1 - \delta^2(1 - e^{-\eta})]^\ell\right] > 1 - \frac{\epsilon}{3}. \tag{2.1}$$

Given $x \in \mathbb{Z}^2$ and an integer $L > 2\ell$, define the following event

$$A_x(L) = \{x + \{0,1,\dots,2\ell\} \times \{0\} \text{ are connected in } x + \{0,1,\dots,L\} \times \{0\}\}.$$

The hypothesis $\limsup_{N\to\infty}\sum_n p_n p_{n+N} > 0$ implies that $\sum_n p_n = \infty$, then by Lemma 2.3 we can find a large L such that $P(A_x(L)) > 1 - \epsilon/3$.

Now, choose integers $k > 2L$ and $M > L$ such that

$$\sum_{n=1}^{M} p_n p_{n+k} > \eta. \tag{2.2}$$

Define the events

$$\begin{aligned} R_x^+ &= \{\langle x; x+(0,1)\rangle \text{ and } \langle x+(k,1); x+(k,2)\rangle \text{ are open}\} \\ &\quad \cap \left(\cup_{n=1}^{M}\{\langle x+(0,1); x+(n+k,1)\rangle \text{ and } \langle x+(n+k,1); x+(k,1)\rangle \text{ are open}\}\right) \end{aligned}$$

and

$$\begin{aligned} R_x^- &= \{\langle x; x+(0,1)\rangle \text{ and } \langle x+(-k,1); x+(-k,2)\rangle \text{ are open}\} \\ &\quad \cap \left(\cup_{n=1}^{M}\{\langle x+(0,1); x+(n,1)\rangle \text{ and } \langle x+(n,1); x+(-k,1)\rangle \text{ are open}\}\right). \end{aligned}$$

Observe that the events $A_\cdot(L)$ and $R_\cdot^\pm$ use only bonds whose length is at most $k+M$, then taking $K = k+M$, it follows that

$$\begin{aligned} P^K(R_x^\pm) &= \delta^2[1 - \prod_{n=1}^{M}(1 - p_n p_{n+k})] \\ &\geqslant \delta^2[1 - \exp(-\sum_{n=1}^{M} p_n p_{n+k})] \geqslant \delta^2(1 - e^{-\eta}) \end{aligned} \tag{2.3}$$

where in the last inequality we use (2.2).

Finally, we define the event T_x (see Figure 2.1) as follows

$$T_x = A_x(L) \cap (\cup_{i=0}^{\ell-1} R_{x+(i,0)}^+) \cap (\cup_{i=\ell+1}^{2\ell} R_{x+(i,0)}^-),$$

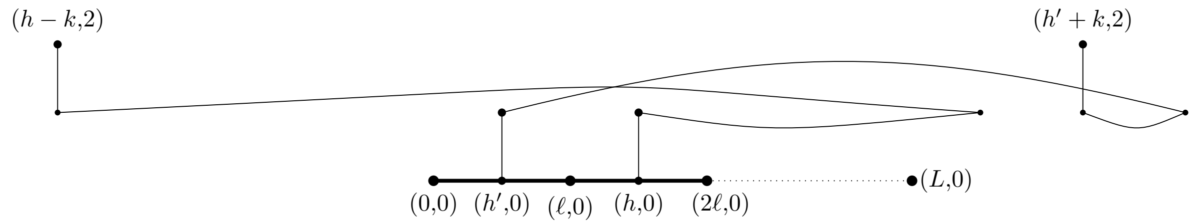

FIGURE 2.1. The event $T_{(0,0)}$ in the graph G^{an}. The thick segment indicates that the vertices therein are connected in the interval $\{0,\dots,L\}\times\{0\}$.

then

$$\begin{aligned}P^K(T_x) &\geqslant P^K(A_x(L))\cdot P^K(\cup_{i=0}^{\ell-1}R^+_{x+(i,0)})\cdot P^K(\cup_{i=\ell+1}^{2\ell}R^-_{x+(i,0)})\\ &\geqslant (1-\frac{\epsilon}{3})\cdot[1-P^K(\cap_{i=0}^{\ell-1}(R^+_{x+(i,0)})^c)]\cdot[1-P^K(\cap_{i=\ell+1}^{2\ell}(R^-_{x+(i,0)})^c)]\\ &\geqslant (1-\frac{\epsilon}{3})\cdot[1-(1-P^K(R^+_x))^\ell]^2\\ &\geqslant (1-\frac{\epsilon}{3})\cdot\Big[1-[1-\delta^2(1-e^{-\eta})]^\ell\Big]^2>1-\epsilon \end{aligned} \tag{2.4}$$

where in the expression above we are using FKG inequality, the independence of $(R^{\pm}_{x+(i,0)})_i$ and (2.1), respectively.

We will construct a site percolation model on the first quadrant of the square lattice $\mathbb{L}^2$. For each site $(v_1,v_2)\in\mathbb{Z}^2_+$, we declare the vertex (v_1,v_2) as *red* if and only if the event $T_{(k(v_1-v_2),2(v_1+v_2))}$ occurs. The choice of $k>2L$ and the definition of T_x ensures that all sites are red independently; observe that the path in the event R^+_x (R^-_x) starts in the left (respectively right) half of the segment $x+\{0,\dots,2\ell\}\times\{0\}$. By construction, an infinite path of red sites starting from the origin in $\mathbb{L}^2$ implies in an infinite path of open bonds starting from the origin in G^{an}. By (2.4), each site is red with probability at least $1-\epsilon$; thus $\lim_{K\to\infty}P^K\{(0,0)\leftrightarrow\infty\}=1$.

If $\gcd\{n;p_n>0\}=d>1$, the same proof can be done, with minor modifications, replacing the vertex set $\mathbb{Z}^2$ by $d\mathbb{Z}\times\mathbb{Z}$. □

Remark 2.5. The hypotheses of Theorems 2.2 and 2.4 can look strange at first glance. It is an exercise to see that any of these hypotheses are implied by $\sum_n p_n^2=\infty$, but it is not true the reciprocal affirmation. In the next section, we will give an affirmative answer for the truncation question in an oriented graph under the stronger hypothesis $\sum_n p_n^2=\infty$.

Remark 2.6. We finish this section giving examples of sequences where the hypothesis of Theorem 2.2 holds but not that of Theorem 2.4 and vice-versa. Consider the sequences:

$$p_n=\begin{cases}k^{-\frac{1}{2}}, & \text{if } n=3^k,3^k+1 \text{ for some } k,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$q_n=\begin{cases}\frac{1}{2\sqrt{k-1}}, & \text{if } n\in\{100^k+t3^k;t=1,\dots,k\} \text{ for some } k,\\ 0, & \text{otherwise.}\end{cases}$$

The sequence $(p_n)_n$ satisfies the hypothesis of Theorem 2.2 but not that of Theorem 2.4, whilst the opposite situation occurs for the sequence $(q_n)_n$.

3. Truncation question on an oriented graph

Let us consider the oriented graph $\mathcal{G} = (\mathbb{V}(\mathcal{G}), \mathbb{E}(\mathcal{G}))$. The vertex set is $\mathbb{V}(\mathcal{G}) = \mathbb{Z}^d \times \mathbb{Z}_+$, elements of $\mathbb{V}(\mathcal{G})$ will be denoted (x, m), where $x \in \mathbb{Z}^d$ and $m \in \mathbb{Z}_+$. The set $\mathbb{E}(\mathcal{G})$ of oriented bonds is

$$\{\langle (x,m), (x + n \cdot \vec{e}_i, m+1)\rangle : x \in \mathbb{Z}^d,\ m \in \mathbb{Z}_+,\ i \in \{1, \dots, d\},\ n \in \mathbb{Z}\}. \tag{3.1}$$

Again, given a sequence $(p_n)_n$ satisfying $\sum_n p_n = \infty$, assume each bond $\langle (x,m), (x+n\cdot\vec{e}_i, m+1)\rangle$ is open with probability $p_{|n|}$ independently of each other and let P and P^K be the non-truncated and truncated at K probability measures. The event $\{(0,0) \rightsquigarrow \infty\}$ means that there exists an infinite open oriented path starting from $(0,0)$.

It was proven in van Enter et al. (2016), under the hypothesis $\sum_n p_n = \infty$, that

$$\lim_{K\to\infty} P^K\{(0,0) \rightsquigarrow \infty\} = 1$$

for all $d \geqslant 2$. The case $d = 1$ is an open question and a partial answer was given in Alves et al. (2017), more precisely $\lim_{K\to\infty} P^K\{(0,0) \rightsquigarrow \infty\} = 1$ holds in $d = 1$ if $\limsup_{n\to\infty} p_n > 0$. The next theorem improves the result of Theorem 1 of Alves et al. (2017) replacing the hypothesis $\limsup_{n\to\infty} p_n > 0$ by $\sum_n p_n^2 = \infty$ (that is, some sequences $(p_n)_n$ decaying to zero are allowed like $p_n = 1/\sqrt{n}$).

Theorem 3.1. *For the graph $\mathcal{G}$ with $d = 1$, if the sequence $(p_n)_n$ satisfies $\sum_n p_n^2 = \infty$, the truncation question has an affirmative answer. Moreover,*

$$\lim_{K\to\infty} P^K\{(0,0) \rightsquigarrow \infty\} = 1.$$

Proof: This proof is similar to the proof of Theorem 1 of Alves et al. (2017). It consists in to define a family of special events, showing that they induce a supercritical oriented percolation process on an appropriate renormalized lattice, isomorphic to a subset of $\mathbb{Z}_+^2$.

Our first goal is to define the family of events $T^+_{(x,m)}$ and $T^-_{(x,m)}$ for all $(x, m) \in \mathbb{Z} \times \mathbb{Z}_+$. Define $k = \min\{n \in \mathbb{N}; p_n > 0\}$; given any $\epsilon > 0$ define large enough integers M and K such that

$$(1 - p_k^2)^M < \epsilon/3\ , \tag{3.2}$$

$$1 - \exp\left[-\sum_{i=k+1}^{K} p_i^2\right] \geqslant \left(1 - \frac{\epsilon}{3}\right)^{\frac{1}{M+1}}. \tag{3.3}$$

Given a vertex $(x, m) \in \mathbb{Z} \times \mathbb{Z}_+$ and $i \in \mathbb{Z}_+$, we define the following events:

$$R^+_{(x,m)}(i) = \{\langle (x,m); (x+i, m+1)\rangle \text{ and } \langle (x+i, m+1); (x, m+2)\rangle \text{ are open}\},$$

$$R^-_{(x,m)}(i) = \{\langle (x,m); (x-i, m+1)\rangle \text{ and } \langle (x-i, m+1); (x, m+2)\rangle \text{ are open}\},$$

$$S^+_{(x,m)} = \cup_{i=k+1}^{K} R^+_{(x,m)}(i), \quad S^-_{(x,m)} = \cup_{i=k+1}^{K} R^-_{(x,m)}(i)$$

and

$$L_{(x,m)} = \{\langle (x,m); (x+k, m+1)\rangle \text{ and } \langle (x+k, m+1); (x+2k, m+2)\rangle \text{ are open}\}.$$

Observe that $P(L_{(x,m)}) = p_k^2$; since $(R^{\pm}_{(x,m)}(i))_i$ are independent events, we have that

$$P(S^{\pm}_{(x,m)}) = 1 - P\left(\cap_{i=k+1}^{K} (R^{\pm}_{(x,m)}(i))^c\right)$$

$$= 1 - \prod_{i=k+1}^{K} (1 - p_i^2) \geqslant 1 - \exp\left[-\sum_{i=k+1}^{K} p_i^2\right] \geqslant \left(1 - \frac{\epsilon}{3}\right)^{\frac{1}{M+1}}, \tag{3.4}$$

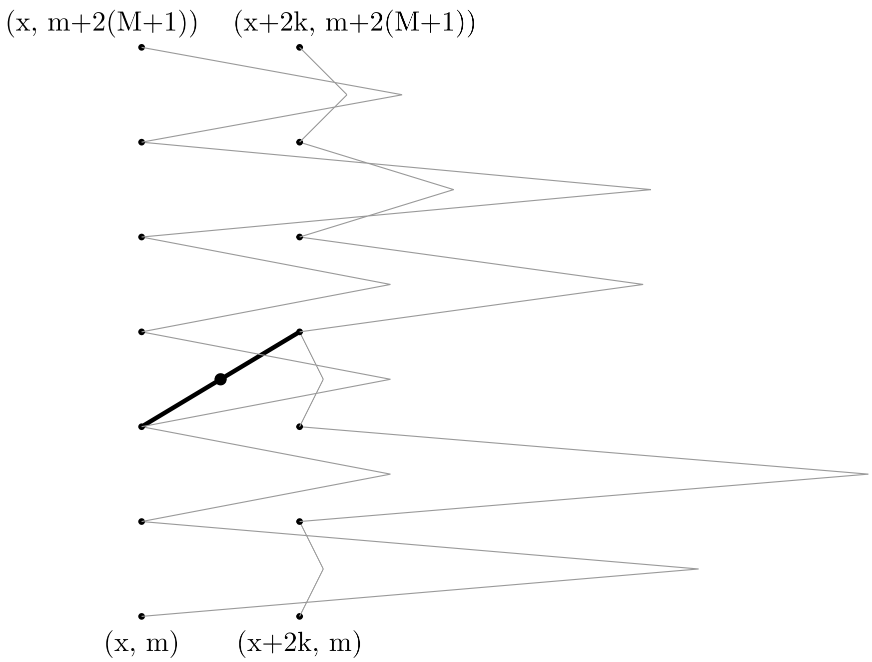

FIGURE 3.2. The event $T^+_{(x,m)}$ in the graph $\mathcal{G}$. The edges in bold correspond to the event $L_\cdot$.

where in the last inequality we used (3.3). Now, define our key events $T^+_{(x,m)}$ and $T^-_{(x,m)}$ as

$$T^\pm_{(x,m)} = \left(\cap_{i=0}^M S^\pm_{(x,m+2i)}\right) \bigcap \left(\cap_{i=0}^M S^\pm_{(x+2k,m+2i)}\right) \bigcap \left(\cup_{i=0}^{M-1} L_{(x,m+2i)}\right).$$

Observing that the events $(S^\pm_{(x,m+2i)})_i$, $(S^\pm_{(x+2k,m+2i)})_i$ and $(L_{(x,m+2i)})_i$ are independent, we have by (3.2) and (3.4) that

$$\begin{aligned} P(T^\pm_{(x,m)}) &= \left(\prod_{i=0}^M P(S^\pm_{(x,m+2i)})\right) \cdot \left(\prod_{i=0}^M P(S^\pm_{(x+2k,m+2i)})\right) \cdot \left(1 - P(\cap_{i=0}^{M-1}(L_{(x,m+2i)})^c)\right) \\ &\geqslant \left(1-\frac{\epsilon}{3}\right)^2 \left(1-(1-p_k^2)^M\right) \geqslant \left(1-\frac{\epsilon}{3}\right)^3 \geqslant 1-\epsilon. \end{aligned}$$

It is important to see that all bonds used to define the events $T^\pm_{(x,m)}$ have length at most K, then $P^K(T^\pm_{(x,m)}) \geqslant 1-\epsilon$ also for the truncated measure P^K and furthermore

$$T^\pm_{(x,m)} \subset \{(x,m) \rightsquigarrow (x, m+2(M+1))\} \cap \{(x,m) \rightsquigarrow (x+2k, m+2(M+1))\}. \tag{3.5}$$

See Figure 3.2 for an illustration of the event $T^+_{(x,m)}$.

Now, define a renormalized graph $G^* = (\mathbb{V}^*, \mathbb{E}^*)$ (an oriented graph), where $\mathbb{V}^* = \{(v,u) \in \mathbb{Z}^2_+; v \leqslant u\}$ and $\mathbb{E}^*$ is the set of oriented edges $\mathbb{E}^* = \{\langle (v,u),(w,u+1)\rangle; w = v \text{ or } w = v+1\}$. We define each vertex $(v,u) \in \mathbb{V}^*$ of the renormalized as *open* following the rule

$$\{(v,u) \text{ is open}\} = \begin{cases} T^+_{(2kv,2(M+1)u)}, & \text{if } v \text{ is even,} \\ T^-_{(2kv,2(M+1)u)}, & \text{if } v \text{ is odd,} \end{cases}$$

and $\{(v,u)$ is closed$\}$ otherwise. This appropriate choice of $T^+_\cdot$ or $T^-_\cdot$ holds that the events $(\{(v,u) \text{ is open}\})_{(v,u)}$ are independent, since the set of edges checked for each of these events are disjoint.

Hence,

$$P^K\left((v,u) \text{ is open}\right) = P^K(T^{\pm}_{(2kv,2(M+1)u)}) > 1-\epsilon. \tag{3.6}$$

Furthermore, by (3.5)

$$\begin{aligned}((0,0) \rightsquigarrow (v,u)) \subset \{(0,0) \rightsquigarrow (2kv, 2(M+1)(u+1))\}\\ \cap \{(0,0) \rightsquigarrow (2k(v+1), 2(M+1)(u+1))\}.\end{aligned}$$

Thus, the cluster of the origin in $\mathcal{G}$ dominates the oriented site percolation on G^* with parameter $1-\epsilon$.

Then, we can conclude that

$$\lim_{K\to\infty} P^K\{(0,0) \rightsquigarrow \infty\} = 1.$$

□

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