



# On the local limit theorems for lower psi-mixing Markov chains

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**Abstract.** In this paper we investigate the local limit theorem for additive functionals of nonstationary Markov chains that converge in distribution. We consider both the lattice and the non-lattice cases. The results are also new in the stationary setting and lead to local limit theorems linked to convergence to stable distributions. The conditions are imposed to individual summands and are expressed in terms of lower psi-mixing coefficients.

## 1. Introduction

The local limit theorem for partial sums  $(S_n)_{n \geq 1}$  of a sequence of random variables deals with the rate of convergence of the probabilities of the type  $\mathbb{P}(c \leq S_n \leq d)$ . Local limit theorems have been studied for the case of lattice random variables and the case of non-lattice random variables. A random variable is said to have a lattice distribution if there exists  $h > 0$  and  $\ell \in \mathbb{R}$  such that its values are concentrated on the lattice  $\{\ell + kh : k \in \mathbb{Z}\}$ . The non-lattice distribution means that no such  $\ell$  and  $h$  exist.

This problem was intensively studied for sums of i.i.d. random variables. For i.i.d. random variables in the domain of attraction of a stable law the local limit theorem was solved by [Stone \(1965\)](#) and [Feller \(1967\)](#).

It should be mentioned that the local limit theorem is more delicate than its convergence in distribution counterpart and often requires additional conditions. An important counterexample

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is given by [Gamkrelidze \(1964\)](#), pointing out this phenomenon for independent summands, and a variety of sufficient conditions were developed over the years. We mention especially papers by [Roazanov \(1957\)](#), [Mineka and Silverman \(1970\)](#), [Maller \(1978\)](#), [Shore \(1978\)](#) and [Dolgopyat \(2016\)](#).

In the dependent case, we mention early works on Markov chains by [Kolmogorov \(1962\)](#). On the other hand, [Nagaev \(1961\)](#) gives rates of convergence in the CLT for stationary  $\psi$ -mixing Markov chains. In the lattice case, [Szewczak \(2010\)](#) established a local limit theorem for continued fractions, which is an example of  $\psi$ -mixing sequence. [Hafouta and Kifer \(2016\)](#) proved a local limit theorem for nonconventional sums of stationary  $\psi$ -mixing Markov chains, while the case of infinite variance is analyzed in [Aaronson and Denker \(2001a\)](#). Also in the stationary case we mention the local limit theorems for Markov chains in the papers by [Hervé and Pène \(2010\)](#), [Ferré et al. \(2012\)](#). Recently, there are two additional works by [Merlevède et al. \(2021\)](#) and [Dolgopyat and Sarig \(2020\)](#), which cover several aspects of local limit theorems associated with the central limit theorem for  $\psi$ -mixing nonstationary Markov chains.

These results raise the natural question if a local limit theorem is valid for more general Markov chains. In this paper we positively answer this question and consider a class larger than  $\psi$ -mixing Markov chains defined by using the so called one sided lower  $\psi$ -mixing coefficient. The key tool in proving these results is a delicate factorization of the characteristic function of partial sums.

We shall comment that all the results obtained by [Merlevède et al. \(2021\)](#) for  $\psi$ -mixing nonstationary Markov chains are also valid for this larger class. Furthermore, we shall also obtain local limit theorems associated with other limiting distributions than the normal attraction, which is actually the attraction to a stable distribution with index  $p = 2$ . More precisely we shall also obtain the local limit theorem associated to attraction to any stable distributions with index  $1 < p < 2$ . The key tools for obtaining these results are some general local limit theorems, which assume convergence to distributions with integrable characteristic functions.

Our paper is organized as follows. In [Section 2](#) we present some preliminary considerations about the types of local limit theorems we shall obtain. In [Section 3](#) we present the main results and provide some examples of classes of lower  $\psi$ -mixing Markov chains. [Section 4](#) contains relevant background material pertaining to mixing coefficients. [Section 5](#) contains the proofs. We present first two general local limit theorems, then a bound on the characteristic function of sums and the proofs of the main results.

In the paper, by  $\Rightarrow$  we denote the convergence in distribution.

## 2. Preliminary considerations

We formulate first the conclusions of the local limit theorems. Let  $(S_n)_n$  be a sequence of random variables and let  $L$  be a random variable with characteristic function  $f_L$ . The basic assumption will be an underlying convergence in distribution:

$$\frac{S_n}{B_n} \Rightarrow L, \text{ where } f_L \text{ is integrable and } B_n \rightarrow \infty. \quad (2.1)$$

Note that assuming that  $f_L$  is integrable implies that  $L$  has a continuous density we shall denote by  $h_L$  (see pages 370-371 in [Billingsley \(2012\)](#)).

In the case when the variables  $(S_n)$  do not have values in a fixed minimal lattice we shall say that the sequence satisfies a local limit theorem if for any function  $g$  on  $\mathbb{R}$  which is continuous and with compact support,

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left| B_n \mathbb{E}g(S_n + u) - h_L\left(-\frac{u}{B_n}\right) \int g(t) \lambda(dt) \right| = 0, \quad (2.2)$$

where  $\lambda$  is the Lebesgue measure.

If all the variables have values in the same fixed lattice

$$\mathcal{S} = \{kh, k \in \mathbb{Z}\},$$

where  $h$  is some fixed positive number, by the local limit theorem we understand that for any function  $g$  on  $\mathbb{R}$  which is continuous and with compact support,

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathcal{S}} \left| B_n \mathbb{E}g(S_n + u) - h_L \left( -\frac{u}{B_n} \right) \int g(v) \mathcal{L}_h(dv) \right| = 0, \tag{2.3}$$

where  $\mathcal{L}_h$  is the measure assigning  $h$  to each point  $kh$ .

By standard arguments, for any real numbers  $c$  and  $d$  such that  $c \leq d$ , we also have that (2.2) implies in the non-lattice case that

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left| B_n \mathbb{P}(c - u \leq S_n \leq d - u) - (d - c)h_L \left( -\frac{u}{B_n} \right) \right| = 0,$$

and in the lattice case

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathcal{S}} \left| B_n \mathbb{P}(c - u \leq S_n \leq d - u) - h \sum_k I(c - u \leq kh \leq d - u) h_L \left( -\frac{u}{B_n} \right) \right| = 0.$$

In particular, since  $B_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then for fixed  $A > 0$ , in the non-lattice case

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq A} |B_n \mathbb{P}(c - u \leq S_n \leq d - u) - (d - c)h_L(0)| = 0, \tag{2.4}$$

and in the lattice case

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq A, u \in \mathcal{S}} \left| B_n \mathbb{P}(c - u \leq S_n \leq d - u) - h \sum_k I(c - u \leq kh \leq d - u) h_L(0) \right| = 0. \tag{2.5}$$

If we further take  $u = 0$  in (2.4), then, in the nonlattice case we have

$$\lim_{n \rightarrow \infty} B_n \mathbb{P}(S_n \in [c, d]) = (d - c) h_L(0).$$

In other words, the sequence of measures  $B_n \mathbb{P}(S_n \in [c, d])$  of the interval  $[c, d]$  converges to a scalar multiple of the Lebesgue measure.

In the lattice case we can take  $u = 0$  in (2.5) and obtain,

$$\lim_{n \rightarrow \infty} B_n \mathbb{P}(S_n \in [c, d]) = h \sum_k I(c \leq kh \leq d) h_L(0).$$

We assume now that  $(\xi_k)_{k \geq 0}$  is a Markov chain defined on  $(\Omega, \mathcal{K}, \mathbb{P})$  with values in  $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ , where  $\mathcal{B}(\mathcal{X})$  is a  $\sigma$ -field on  $\mathcal{X}$ , with regular transition probabilities:

$$Q_k(x, A) = \mathbb{P}(\xi_k \in A | \xi_{k-1} = x) \tag{2.6}$$

and marginal distributions denoted by

$$\mathbb{P}_k(A) = \mathbb{P}(\xi_k \in A). \tag{2.7}$$

Throughout the paper we shall assume that there is a constant  $a > 0$  with the following property:

For all  $k \geq 1$  there is  $\mathcal{X}'_k \in \mathcal{B}(\mathcal{X})$  with  $\mathbb{P}_{k-1}(\mathcal{X}'_k) = 1$  such that for all  $A \in \mathcal{B}(\mathcal{X})$  and  $x \in \mathcal{X}'_k$  we have

$$Q_k(x, A) \geq a \mathbb{P}_k(A). \tag{2.8}$$

Triangular arrays  $(\xi_{kn})_{k \geq 0}$  can also be considered. In this case  $a$  should be replaced by  $a_n$  which may be different from line to line.

Let  $(g_j)_{j \geq 0}$  be real-valued measurable functions and define

$$X_j = g_j(\xi_j) \tag{2.9}$$

and set

$$S_n = \sum_{k=1}^n X_k.$$

In order to obtain our results we shall combine several techniques, specifically designed for obtaining local limit theorems, with a suitable bound on the characteristic function facilitated by condition (2.8).

### 3. Main Results

In the sequel, we shall denote by  $f_k(t)$  the characteristic function of  $X_k$ ,

$$f_k(t) = \mathbb{E}(\exp(itX_k)).$$

We shall introduce two useful conditions that will provide a version of the local limit theorem under conditions imposed to the characteristic function.

Let  $a$  be as in (2.8).

**Condition A.** There is  $\delta > 0$  and  $n_0 \in \mathbb{N}$  and a Borel function  $g : [1, \infty) \rightarrow (0, \infty)$  such that for  $1 \leq |u| \leq \delta B_n$  and  $n > n_0$

$$\frac{a^2}{2^4} \sum_{k=1}^n \left(1 - \left|f_k\left(\frac{u}{B_n}\right)\right|^2\right) > g(|u|) \text{ and } \exp(-g(|u|)) \text{ is integrable.} \quad (3.1)$$

**Condition B.** For  $u \neq 0$  there is an  $\varepsilon = \varepsilon(u)$ ,  $c(u)$  and a  $n_0 = n_0(u)$  such that for all  $t$  with  $|t - u| \leq \varepsilon$  and  $n > n_0$

$$\frac{a^4}{2^4(\ln B_n)} \sum_{k=1}^n (1 - |f_k(t)|^2) \geq c(u) > 1. \quad (3.2)$$

Our general local limit theorem is the following:

**Theorem 3.1.** *Let  $(X_j)_{j \geq 0}$  be defined by (2.9) and satisfying (2.8). Assume that conditions (2.1), A and B are satisfied. If not all the variables have values in a fixed lattice, then (2.2) holds. If all the variables have the values in a fixed lattice  $\mathcal{S} = \{kh, k \in \mathbb{Z}\}$ , under the same conditions with the exception that we assume now that Condition B holds for  $0 < |u| \leq \pi/h$ , then (2.3) holds.*

The conditions of Theorem 3.1 are verified in many situations of interest. Merlevède et al. (2021) provided sufficient conditions for Conditions A and B when second moment is finite and obtained the local limit theorems in the nonlattice case under a more restrictive condition than (2.8). Namely it was assumed there that there are two constants  $a > 0$  and  $b < \infty$  with the following property:

For all  $k \geq 1$  there is  $\mathcal{X}'_k \in \mathcal{B}(\mathcal{X})$  with  $\mathbb{P}_{k-1}(\mathcal{X}'_k) = 1$  such that for all  $A \in \mathcal{B}(\mathcal{X})$  and  $x \in \mathcal{X}'_k$  we have

$$a\mathbb{P}_k(A) \leq Q_k(x, A) \leq b\mathbb{P}_k(A). \quad (3.3)$$

By using our Proposition 5.5, all the results in Merlevède et al. (2021) also hold for a larger class of Markov chains satisfying only the one sided condition (2.8). Therefore in all that results in Merlevède et al. (2021) the condition  $b < \infty$  is superfluous. Furthermore, because of Theorem 5.3 below, all the results in Merlevède et al. (2021) can also be formulated for the lattice case. We shall not repeat all the results there and mention only that our results are also new in the stationary setting. In this particular case the results have a simple formulation.

The first theorem deals with local limit theorem in case of attraction to normal distribution, which is stable with index  $p = 2$ .

**Theorem 3.2.** *Assume that  $(\xi_k)_{k \in \mathbb{Z}}$  is a strictly stationary Markov chain satisfying (2.8). Define  $(X_k)_{k \in \mathbb{Z}}$  by  $X_k = g(\xi_k)$  and assume  $\mathbb{E}(X_0) = 0$  and  $H(x) = \mathbb{E}(X_0^2 I(|X_0| \leq x))$  is a slowly varying function as  $x \rightarrow \infty$ . Then, there is  $B_n \rightarrow \infty$  such that in the nonlattice case (2.2) holds and in the lattice case (2.3) holds with  $h_L(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ .*

*Remark 3.3.* In the case where  $\mathbb{E}(X_0^2) < \infty$ , we can take  $B_n^2 = \mathbb{E}(S_n^2)$ , and there are two constants  $C_1$  and  $C_2$  such that  $C_1\mathbb{E}(X_0^2)n \leq \mathbb{E}(S_n^2) \leq C_2\mathbb{E}(X_0^2)n$ . Furthermore, since in this case  $\mathbb{E}(S_n^2)/n \rightarrow c^2 > 0$ , we can also take  $B_n^2 = c^2n$ . In case  $\mathbb{E}(X_0^2) = \infty$  we can take  $B_n^2 = \sqrt{\pi/2}\mathbb{E}|S_n|$ .

We move now to other types of limiting distributions  $L$  in (2.1).

Let  $0 < p < 2$ . Assume that  $(\xi_k)_{k \in \mathbb{Z}}$  is a strictly stationary Markov chain and let  $(X_k)$  be defined by  $X_k = g(\xi_k)$ , with a nondegenerate marginal distribution satisfying the following condition:

$$\mathbb{P}(|X_0| > x) = x^{-p}\ell(x), \tag{3.4}$$

where  $\ell(x)$  is a slowly varying function at  $\infty$ ,

$$\frac{\mathbb{P}(X_0 > x)}{\mathbb{P}(|X_0| > x)} \rightarrow c^+ \quad \text{and} \quad \frac{\mathbb{P}(X_0 < -x)}{\mathbb{P}(|X_0| > x)} \rightarrow c^- \quad \text{as } x \rightarrow \infty \tag{3.5}$$

with  $0 \leq c^+ \leq 1$  and  $c^+ + c^- = 1$ .

These conditions practically mean that the distribution of  $X_0$  is in the domain of attraction of a stable law with index  $p$ . The convergence in distribution of a mixing sequence of random variables in the domain of attraction of a stable law is a delicate problem, which was often studied in the literature, starting with papers by [Davis \(1983\)](#), [Jakubowski \(1991\)](#), [Samur \(1987\)](#), [Denker and Jakubowski \(1989\)](#), [Kobus \(1995\)](#), [Tyran-Kamińska \(2010\)](#), [Cattiaux and Manou-Abi \(2014\)](#), [El Machkouri et al. \(2020\)](#), among many others. In general an additional "non clustering" assumption is relevant for this type of convergence. We shall assume that for every  $x > 0$  and for all  $k \geq 1$

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_k| > xB_n \mid |X_0| > xB_n) = 0, \tag{3.6}$$

where  $B_n$  is such that

$$\frac{n}{B_n^p} \ell(B_n) = n\mathbb{P}(|X_0| > B_n) \rightarrow 1. \tag{3.7}$$

In the context of  $\varphi$ -mixing sequences, for this type of condition we refer to Theorem 3.2 in [Samur \(1987\)](#), Corollary 5.9 of [Kobus \(1995\)](#) and Corollary 1.3 in [Tyran-Kamińska \(2010\)](#), where the necessity of the condition (3.6) for such a type of result is also discussed.

**Theorem 3.4.** *Assume  $0 < p < 2$  and (3.4) and (3.5) are satisfied. When  $1 < p < 2$  we assume  $\mathbb{E}(X_0) = 0$  and when  $p = 1$  we assume  $X_0$  has a symmetric distribution. Also assume (2.8) and (3.6). Then in the nonlattice case (2.2) holds and in the lattice case (2.3) holds with  $B_n$  defined by (3.7) and  $h_L$  the density of a strictly stable distribution  $L$  with index  $p$ .*

Note that condition (3.6) of Theorem 3.4 is satisfied if we assume that there is  $b < \infty$  such that the right hand side of (3.3) holds. Therefore we obtain:

**Corollary 3.5.** *Assume that  $(X_k)$  is as in Theorem 3.4, (3.3), (3.4) and (3.5) are satisfied. Then the conclusions of Theorem 3.4 hold.*

3.1. *Examples. Example 1.* The following is an example of a strictly stationary Markov chain satisfying (2.8) but not the right hand side of (3.3) (see [Bradley \(1997\)](#) Remark 1.5). The Markov chain  $(\xi_k)_{k \in \mathbb{Z}}$  is such that the marginal distribution of  $\xi_0$  is uniformly distributed on  $[0, 1]$  and the one step transition probabilities are as follows:

For each  $x \in [0, 1]$ , let

$$\mathbb{P}(\xi_1 = x | \xi_0 = x) = 1/2$$

and

$$\mathbb{P}(\xi_1 \in B | \xi_0 = x) = (1/2)\lambda(B),$$

where  $B \subset [0, 1] - x$  and  $\lambda$  is the Lebesgue measure on  $[0, 1]$ . Then (2.8) holds with  $a = 1/2$ . However no such  $b$  as in the right hand side of (3.3) exists.

**Example 2.** (Generalized Gibbs Markov chains). Let  $S$  be a countable set,  $p : S \times S \rightarrow [0, 1]$  be an aperiodic, irreducible stochastic matrix and  $(\pi_s)_{s \in S}$  an invariant distribution with  $\pi_s > 0$  for all  $s \in S$ . Let  $T : S^{\mathbb{N}} \rightarrow S^{\mathbb{N}}$  be the shift and define the Markov chain in a canonical way on  $S^{\mathbb{N}}$  by

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \pi_{x_1} p(x_1, x_2) \dots p(x_{n-1}, x_n).$$

Let  $\Omega = \{x \in S^{\mathbb{N}} : \mathbb{P}(X_1 = x_1, \dots, X_n = x_n) > 0\}$ . We assume that there is  $0 < m < 1$  such that for all  $s, t \in S$

$$p(s, t) \geq m\pi_t \tag{3.8}$$

Then our condition (2.8) is satisfied with  $a = m$ . This is a larger class than the so called Gibbs-Markov maps as defined in [Aaronson and Denker \(2001b\)](#)

**Example 3.** In the context of Example 2, a fairly large class of countable state Markov processes satisfying condition (3.8) can be constructed by defining for  $i, j \in \mathbb{N}^*$

$$p(i, j) = \pi_j + (\delta_{i,j} - \delta_{i+1,j})\varepsilon_i,$$

where for all  $i \in \mathbb{N}^*$ ,  $\delta_{i,i} = 1$  and for  $j \neq i$  we have  $\delta_{i,j} = 0$ . We take  $0 \leq \varepsilon_i \leq \min(1 - \pi_i, \pi_{i+1})$ . In addition we assume that there is  $0 < m < 1$  such that

$$\varepsilon_i \leq (1 - m)\pi_{i+1}.$$

For example, let  $m = 1/2$ ,  $\pi_j = 2^{-j}$  and set  $p(i, j) = 2^{-j} + (\delta_{i,j} - \delta_{i+1,j})2^{-(2+i)}$ .

#### 4. Relation to mixing coefficients

##### Relation to lower $\psi$ -mixing coefficient.

We introduce now a mixing condition which is comparable to condition (2.8).

Following [Bradley \(2007\)](#), for any two sigma algebras  $\mathcal{A}$  and  $\mathcal{B}$  define the lower  $\psi$ -mixing coefficient by

$$\psi'(\mathcal{A}, \mathcal{B}) = \inf \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)\mathbb{P}(B)}; \quad A \in \mathcal{A} \text{ and } B \in \mathcal{B}, \mathbb{P}(A)\mathbb{P}(B) > 0.$$

Obviously  $0 \leq \psi'(\mathcal{A}, \mathcal{B}) \leq 1$  and  $\psi'(\mathcal{A}, \mathcal{B}) = 1$  if and only if  $\mathcal{A}$  and  $\mathcal{B}$  are independent.

For a sequence  $(\xi_k)_{k \geq 0}$  of random variables we shall denote by  $\mathcal{F}_n^m = \sigma(\xi_i, m \leq i \leq n)$  and by

$$\psi'_k = \inf_{m \geq 0} \psi'(\mathcal{F}_0^m, \mathcal{F}_{k+m}^{k+m}).$$

In the Markov setting, by the Markov property,

$$\psi'_k = \inf_{m \geq 0} \psi'(\sigma(\xi_m), \sigma(\xi_{k+m})).$$

Notice that, in terms of conditional probabilities defined by (2.6), we also have the following equivalent definition:

$$\psi'_1 = \inf_{k \geq 1} \operatorname{ess\,inf}_x \inf_{A \in \mathcal{B}(\mathcal{X})} Q_k(x, A) / \mathbb{P}(\xi_k \in A).$$

Note that if (2.8) holds then  $\psi'_1 \geq a > 0$  and if  $\psi'_1 > 0$  then (2.8) holds with  $a = \psi'_1$ . Also note that condition (2.8) is equivalent to the existence of a constant  $a'$  such that  $\psi'_1 = a' > 0$ .

For Markov chains, by Theorem 7.4 (d) in [Bradley \(2007\)](#):

$$1 - \psi'_{k+m} \leq (1 - \psi'_k)(1 - \psi'_m). \tag{4.1}$$

So, by condition (2.8), we have  $1 - \psi'_n \leq (1 - a')^n \rightarrow 0$ , and in this case,  $(\xi_k)_{k \geq 0}$  is called lower  $\psi$ -mixing.

It should be mentioned that the condition on the coefficient  $\psi'_1$  is quite natural for obtaining large deviation results for stationary Markov chains (see [Bryc and Smoleński \(1993\)](#)).

##### Relation to $\rho$ -mixing coefficients.

Define the maximal coefficient of correlation

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{X \in \mathbb{L}_2(\mathcal{A}), Y \in \mathbb{L}_2(\mathcal{B})} |\text{corr}(X, Y)| .$$

By Lemma 10 in Merlevède et al. (2021) (which is actually due to R. Bradley), we know that

$$\rho(\mathcal{A}, \mathcal{B}) \leq 1 - \psi'(\mathcal{A}, \mathcal{B}). \tag{4.2}$$

For a sequence  $(\xi_k)_{k \geq 0}$  of random variables  $\rho_k = \sup_{m \geq 0} \rho(\mathcal{F}_0^m, \mathcal{F}_{k+m}^\infty)$ , and in the Markov case  $\rho_k = \sup_{m \geq 0} \rho(\sigma(\xi_m), \sigma(\xi_{k+m}))$ . In this latter case we also have the equivalent definition for  $\rho_1$ ,

$$\rho_1 = \sup_{k \geq 1} \sup_{\|f\|_2=1, \mathbb{E}f=0} \|Q_k f\|_2 ,$$

where

$$Q_k f(x) = \int f(y) Q_k(x, dy).$$

According to (4.2), condition (2.8) implies

$$\rho_1 \leq 1 - a < 1. \tag{4.3}$$

By Theorem 7.4 (a) in Bradley (2007), it follows that  $\rho_n \leq (1 - a)^n \rightarrow 0$ . Therefore, a Markov chain satisfying condition (2.8) is also  $\rho$ -mixing.

Define now  $(X_k)_{k \geq 0}$  by (2.9). Assume the variables  $(X_k)_{k \geq 0}$  are centered and have finite second moments. Denote  $\tau_n^2 = \sum_{j=1}^n \text{var}(X_j)$  and  $\sigma_n^2 = \mathbb{E}(S_n^2)$ . From Proposition 13 in Peligrad (2012) we know that

$$\frac{1 - \rho_1}{1 + \rho_1} \leq \frac{\sigma_n^2}{\tau_n^2} \leq \frac{1 + \rho_1}{1 - \rho_1}.$$

By combining this inequality with (4.3), we obtain

$$\frac{a}{2 - a} \leq \frac{\sigma_n^2}{\tau_n^2} \leq \frac{2 - a}{a}. \tag{4.4}$$

If the Markov chain is stationary define  $X_k = g(\xi_k)$  and assume that the variables  $X_k$  are nondegenerate, have finite second moment and are centered. Then, by the definition of  $\rho_k$  and (4.3), for all  $k > 0$ ,  $|\mathbb{E}(X_0 X_k)| \leq (1 - a)^k \mathbb{E}(X_k^2)$  and therefore it follows that there is  $c > 0$  such that

$$\frac{\sigma_n^2}{n} \rightarrow c^2. \tag{4.5}$$

**Relation to  $\varphi$ -mixing coefficients**

Relevant to our paper is also the relation with  $\varphi$ -mixing coefficient defined by

$$\varphi(\mathcal{A}, \mathcal{B}) = \sup \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)}; \quad A \in \mathcal{A} \text{ and } B \in \mathcal{B}, \mathbb{P}(A) > 0.$$

By Proposition 5.2 (III)(b) in Bradley (2007),

$$\varphi(\mathcal{A}, \mathcal{B}) \leq 1 - \psi'(\mathcal{A}, \mathcal{B}). \tag{4.6}$$

In the Markov case

$$\varphi_k = \sup_{m \geq 0} \varphi(\sigma(\xi_m), \sigma(\xi_{k+m})).$$

Then, if condition (2.8) is satisfied, by (4.6) and by (4.1), for any  $k \geq 1$

$$\varphi_k \leq 1 - \psi'_k \leq (1 - a)^k \rightarrow 0 \text{ as } k \rightarrow \infty, \tag{4.7}$$

and so, our Markov chain is also  $\varphi$ -mixing, at least exponentially fast.

**Relation to  $\psi$ -mixing coefficient.**

The class of  $\psi'$ -mixing Markov chains is strictly larger than the class of  $\psi$ -mixing. To see this we mention that the  $\psi$ -mixing coefficient is defined in the following way. For any two sigma algebras  $\mathcal{A}$  and  $\mathcal{B}$ , define the  $\psi$ -mixing coefficient by

$$\psi(\mathcal{A}, \mathcal{B}) = \sup \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)\mathbb{P}(B)}; \quad A \in \mathcal{A} \text{ and } B \in \mathcal{B}, \mathbb{P}(A)\mathbb{P}(B) > 0.$$

It is well-known that

$$\psi(\mathcal{A}, \mathcal{B}) = \max[\psi^*(\mathcal{A}, \mathcal{B}) - 1, 1 - \psi'(\mathcal{A}, \mathcal{B})],$$

where

$$\psi^* = \sup \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)\mathbb{P}(B)}; \quad A \in \mathcal{A} \text{ and } B \in \mathcal{B}, \mathbb{P}(A)\mathbb{P}(B) > 0.$$

For a sequence  $(\xi_k)_{k \geq 0}$  of random variables  $\psi_k = \sup_{m \geq 0} \psi(\mathcal{F}_0^m, \mathcal{F}_{k+m}^{k+m})$  and in the Markovian case  $\psi_k = \sup_{m \geq 0} \psi(\sigma(\xi_m), \sigma(\xi_{k+m}))$ . Note that  $\psi_1 < 1$  implies  $\psi'_1 > 0$ . Therefore all our results also hold for  $\psi$ -mixing Markov chains with  $\psi_1 < 1$ .

Furthermore, the class of  $\psi$ -mixing Markov chains is strictly smaller than the class of  $\psi'$ -mixing Markov chains (see for example Remark 1.5 in Bradley (1997) and Example 1 above). Therefore, clearly, the conclusions of our results also hold for the smaller class of  $\psi$ -mixing Markov chains with  $\psi_1 < 1$ .

Note that condition (3.3) is equivalent to the existence of two constants  $a'$  and  $b'$  such that  $\psi'_1 = a' > 0$  and  $\psi^*_1 = b' < \infty$ .

### 5. Proofs

5.1. *Preliminary general local CLT.* Here we give two general local limit theorems.

**Theorem 5.1.** *We require that convergence in distribution in (2.1) holds. In addition, suppose that for each  $D > 0$*

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T < |t| \leq DB_n} \left| \mathbb{E} \exp \left( it \frac{S_n}{B_n} \right) \right| dt = 0. \tag{5.1}$$

Then (2.2) holds.

*Remark 5.2.* By decomposing the integral in (5.1) into two parts, on  $\{T < |u| < \delta B_n\}$  and on  $\{\delta B_n \leq |u| \leq DB_n\}$ , and changing the variable in the second integral we easily argue that in order to prove this theorem it is enough to show that for each  $D$  fixed there is  $0 < \delta < D$  such that

$$(D_1) \quad \lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T < |t| < \delta B_n} \left| \mathbb{E} \exp \left( it \frac{S_n}{B_n} \right) \right| dt = 0$$

and

$$(D_2) \quad \lim_{n \rightarrow \infty} B_n \int_{\delta \leq |t| \leq D} |\mathbb{E} \exp(itS_n)| dt = 0.$$

If all the variables  $(S_n)$  have the values in a fixed lattice, we impose a different condition and the result we obtain is the following:

**Theorem 5.3.** *Assume now that all the variables  $(S_n)$  have values in a fixed lattice*

$$\mathcal{S} = \{kh, k \in \mathbb{Z}\}$$

and (2.1) holds. Suppose in addition that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T < |t| \leq \frac{\pi}{h} B_n} \left| \mathbb{E} \exp \left( it \frac{S_n}{B_n} \right) \right| dt = 0. \tag{5.2}$$

Then, the conclusion in (2.3) holds.

*Remark 5.4.* By decomposing the integral in (5.2) into two parts, on  $\{T < |u| < \delta B_n\}$  and on  $\{\delta B_n \leq |u| \leq \pi B_n/h\}$ , the conclusion of Theorem 5.3 holds if, for some  $0 < \delta < \pi/h$ ,

$$(D'_1) \quad \lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T < |t| < \delta B_n} \left| \mathbb{E} \exp \left( it \frac{S_n}{B_n} \right) \right| dt = 0$$

and

$$(D'_2) \quad \lim_{n \rightarrow \infty} B_n \int_{\delta \leq |t| \leq \pi/h} |\mathbb{E} \exp(itS_n)| dt = 0.$$

**Proof of Theorems 5.1 and 5.3.** The proofs of these theorems can be deduced from the corresponding arguments in Hafouta and Kifer (2016), who assumed convergence to normal distribution and normalization  $\sqrt{n}$ . Since the proof of Theorem 5.1 is given in Peligrad et al. (2022) we shall give here only the proof of Theorem 5.3.

The proof is based on the inverse Fourier formula for the sum  $S_n$ . According to Lemma 4.5 and arguments in Section VI.4 in Hennion and Hervé (2001) (see also Theorem 10.7 in Breiman (1992) and Section 10.4 there), it suffices to prove (2.3) for all continuous complex valued functions  $g$  defined on  $\mathbb{R}$  such that  $|g| \in L^1(\mathbb{R})$  and its Fourier transform

$$\hat{g}(t) = \int_{\mathbb{R}} e^{-itx} g(x) dx$$

has compact support contained in some finite interval  $[-D, D]$ .

The inversion formula gives:

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{isx} \hat{g}(s) ds.$$

Therefore, using a change of variable and taking the expected value, we obtain

$$\mathbb{E} [g(S_n + u)] = \frac{1}{2\pi B_n} \int_{\mathbb{R}} \hat{g}\left(\frac{t}{B_n}\right) f_{S_n}\left(\frac{t}{B_n}\right) \exp\left(\frac{itu}{B_n}\right) dt,$$

where we have used the notation

$$f_X(v) = \mathbb{E}(\exp(ivX)),$$

and  $u \in \mathcal{S}$ , where  $\mathcal{S}$  is as in the statement of Theorem 5.3. Denote

$$r(v) = \sum_{k=-\infty}^{\infty} \hat{g}\left(v + 2\pi \frac{k}{h}\right).$$

Since  $\hat{g}$  has compact support included in an interval  $[-D, D]$ , the sum is finite. So, with the notation  $M_v = h(D + |v|)/2\pi$ , clearly

$$|r(v)| \leq \sum_{|k| \leq M_v} \left| \hat{g}\left(v + 2\pi \frac{k}{h}\right) \right| \leq 2M_v \sup_{|w| \leq D} |\hat{g}(w)|. \tag{5.3}$$

Now,

$$\begin{aligned} & \int_{\mathbb{R}} \hat{g}\left(\frac{t}{B_n}\right) f_{S_n}\left(\frac{t}{B_n}\right) \exp\left(\frac{itu}{B_n}\right) dt \\ &= \sum_{k=-\infty}^{\infty} \int_{\frac{2k-1}{h}\pi B_n}^{\frac{2k+1}{h}\pi B_n} \hat{g}\left(\frac{t}{B_n}\right) f_{S_n}\left(\frac{t}{B_n}\right) \exp\left(\frac{itu}{B_n}\right) dt. \end{aligned}$$

We change the variable from  $t$  to  $v$  according to:

$$\frac{t}{B_n} = \frac{v}{B_n} + \frac{2\pi k}{h}.$$

Because  $u = k'h, k' \in \mathbb{Z}$ ,

$$\exp\left(\frac{itu}{B_n}\right) = \exp\left(iu \frac{v}{B_n} + ik'h \frac{2\pi k}{h}\right) = \exp\left(iu \frac{v}{B_n}\right).$$

Since  $S_n$  has values in  $\mathcal{S}$  we note that

$$f_{S_n}\left(\frac{t}{B_n} + \frac{2k}{h}\pi\right) = f_{S_n}\left(\frac{t}{B_n}\right) \text{ for all } k \in \mathbb{Z}.$$

Therefore, by using the definition of  $r(v)$ ,

$$\mathbb{E}[g(S_n + u)] = \frac{1}{2\pi B_n} \int_{-\frac{\pi}{h}B_n}^{\frac{\pi}{h}B_n} f_{S_n}\left(\frac{v}{B_n}\right) \exp\left(\frac{ivu}{B_n}\right) r\left(\frac{v}{B_n}\right) dv.$$

Note that, by the definition of the characteristic function, we have the identity  $f_L(u) = \hat{h}_L(-u)$ . So, by the Fourier inversion formula we also have

$$h_L\left(-\frac{u}{B_n}\right) = \frac{1}{2\pi} \int f_L(t) \exp\left(\frac{itu}{B_n}\right) dt.$$

We evaluate

$$\begin{aligned} & 2\pi \left| B_n \mathbb{E}[g(S_n + u)] - h_L\left(-\frac{u}{B_n}\right) \sum_{k=-\infty}^{\infty} hg(kh) \right| \\ &= \left| \int_{-\frac{\pi}{h}B_n}^{\frac{\pi}{h}B_n} f_{S_n}\left(\frac{v}{B_n}\right) \exp\left(\frac{ivu}{B_n}\right) r\left(\frac{v}{B_n}\right) dv - \int f_L(v) \exp\left(\frac{ivu}{B_n}\right) dv \int gd\mathcal{L}_h \right| \\ &= |I_n + II_n + III_n| \leq |I_n| + |II_n| + |III_n|, \end{aligned}$$

where the terms  $I_n, II_n$ , and  $III_n$  are given below and their moduli analyzed:

$$\begin{aligned} |I_n| &= \left| \int_{-T}^T \left( f_{S_n}\left(\frac{t}{B_n}\right) r\left(\frac{t}{B_n}\right) - f_L(t) \int gd\mathcal{L}_h \right) \exp\left(\frac{itu}{B_n}\right) dt \right| \\ &\leq \int_{-T}^T \left| f_{S_n}\left(\frac{t}{B_n}\right) r\left(\frac{t}{B_n}\right) - f_L(t) \int gd\mathcal{L}_h \right| dt, \\ |II_n| &\leq \left( \int_{|t|>T} |f_L(t)| dt \right) \left( \int gd\mathcal{L}_h \right), \end{aligned}$$

and

$$|III_n| \leq \int_{T < |v| \leq \frac{\pi}{h}B_n} \left| f_{S_n}\left(\frac{v}{B_n}\right) \right| \left| r\left(\frac{v}{B_n}\right) \right| dv.$$

To deal with  $|I_n|$ , we further decompose the sum and use the triangle inequality,

$$\begin{aligned} |I_n| &\leq \int_{-T}^T \left| f_{S_n}\left(\frac{v}{B_n}\right) r\left(\frac{v}{B_n}\right) - f_{S_n}\left(\frac{v}{B_n}\right) \int gd\mathcal{L}_h \right| dv \\ &\quad + \int_{-T}^T \left| f_{S_n}\left(\frac{v}{B_n}\right) \int gd\mathcal{L}_h - f_L(v) \int gd\mathcal{L}_h \right| dv. \end{aligned}$$

The first term in the right hand side of the above inequality tends to 0 because by Ch 10 in Breiman (1992) and since  $B_n \rightarrow \infty$ , we have

$$r\left(\frac{v}{B_n}\right) \rightarrow \int gd\mathcal{L}_h.$$

The second term in the right hand side converges to 0 because, by our hypotheses

$$f_{S_n}\left(\frac{v}{B_n}\right) \rightarrow f_L(v),$$

uniformly on compacts.

The term  $|II_n|$  converges to 0 as  $T \rightarrow \infty$ , because  $f_L$  is assumed to be integrable.

Next, for  $|III_n|$  we have

$$\begin{aligned} |III_n| &\leq \sup_{T < |w| \leq \frac{\pi}{h} B_n} \left| r\left(\frac{w}{B_n}\right) \right| \int_{T < |v| \leq \frac{\pi}{h} B_n} \left| f_{S_n}\left(\frac{v}{B_n}\right) \right| dv \\ &\leq \sup_{|w| \leq \frac{\pi}{h}} |r(w)| \int_{T < |v| \leq \frac{\pi}{h} B_n} \left| f_{S_n}\left(\frac{v}{B_n}\right) \right| dv. \end{aligned}$$

The result follows by condition (5.2) and the fact that by (5.3) we have

$$\sup_{|w| \leq \frac{\pi}{h}} |r(w)| \leq [(Dh + \pi)/\pi] \sup_{|w| \leq D} |\hat{g}(w)|.$$

□

5.2. *Bounds on the characteristic function.* The bound on the characteristic function of a Markov chain is inspired by Lemma 1.5 in Nagaev (1961). It is given in the following proposition. Recall the definition of  $a$  in (2.8).

**Proposition 5.5.** *Let  $(X_j)_{j \geq 0}$  be defined by (2.9). Then, for all  $n \in \mathbb{N}$ , and  $u \in \mathbb{R}$ ,*

$$|\mathbb{E}(\exp(iuS_n))| \leq \exp \left[ -\frac{a^4}{16} \sum_{j=1}^n (1 - |f_j(u)|^2) \right].$$

For proving this proposition we need some preliminary considerations.

For a complex, finite measure  $\mu$ , defined on a sigma algebra  $\mathcal{F}$ , denote by  $\text{Var}\mu$  its total variation. This is, for all measurable  $A$  we have

$$\text{Var}\mu(A) = \sup \sum_{k=1}^n |\mu(A_k)|,$$

where the supremum is taken over all  $n \in \mathbb{N}$ , and all  $A_1, A_2, \dots, A_n$ , disjoint sets in  $\mathcal{F}$ , with  $A = \cup_{k=1}^n A_k$ .

The property we shall use below is that for complex measurable functions integrable with respect to  $\mu$  we have

$$\left| \int h(y) d\mu \right| \leq \int |h(y)| d(\text{Var}\mu).$$

Denote by  $\mathbb{L}_\infty(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}')$  the space of complex valued, measurable, essentially bounded functions on the probability space  $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}')$ . For an operator  $T : \mathbb{L}_\infty(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}') \rightarrow \mathbb{L}_\infty(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}'')$  we denote by

$$\|T\| := \sup_{\|f\|_1=1} \|T(f)\|_1,$$

where  $\|T(f)\|_1$  is the norm of  $T(f)$  in the space of complex valued, integrable functions on  $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}'')$  and  $\|f\|_1$  is the norm of  $f$  in the space of complex valued integrable functions on  $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}')$ .

We introduce now a family of operators which are relevant to our proofs:

For  $u$  a fixed real number let us introduce the operator

$$T_{u,k} : \mathbb{L}_\infty(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}_k) \rightarrow \mathbb{L}_\infty(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P}_{k-1})$$

by

$$T_{u,k}(h)(x) := \int h(y) \exp(iug_k(y)) Q_k(x, dy).$$

So, for  $k \geq 1$ ,

$$T_{u,k}(h)(\xi_{k-1}) = \mathbb{E}([h(\xi_k) \exp(iuX_k)] \mid \xi_{k-1}).$$

**Lemma 5.6.** *For any  $k \in \mathbb{N}^*$  and  $u \in \mathbb{R}$  we have,*

$$\|T_{u,k-1} \circ T_{u,k}\|_1 \leq 1 - \frac{a^4}{8}(1 - |\mathbb{E}(\exp(iuX_{k-1}))|^2).$$

**Proof.** In the sequel, to simplify the notation we drop the index  $u$ , and write  $T_k = T_{u,k}$ . Let  $x \in \mathcal{X}'$ , where  $\mathcal{X}' \in \mathcal{B}(\mathcal{X})$  such that  $\mathbb{P}_{k-1}(\mathcal{X}') = 1$ , for which condition (3.3) holds. By the definition of  $T_k$ 's,

$$\begin{aligned} & T_{k-1} \circ T_k(h)(x) \\ &= \int \exp(iug_{k-1}(y)) \int h(z) \exp(iug_k(z)) Q_k(y, dz) Q_{k-1}(x, dy). \end{aligned}$$

Changing the order of integration

$$\begin{aligned} & T_{k-1} \circ T_k(h)(x) \\ &= \int h(z) \exp(iug_k(z)) \int \exp(iug_{k-1}(y)) Q_{k-1}(x, dy) Q_k(y, dz) \\ &= \int h(z) \exp(iug_k(z)) m_x(dz), \end{aligned}$$

where, for  $x$  fixed in  $\mathcal{X}'$ ,  $m_x$  is the complex finite measure defined on  $\mathcal{B}(\mathcal{X})$  by the formula

$$m_x(A) = \int \exp(iug_{k-1}(y)) Q_k(y, A) Q_{k-1}(x, dy).$$

Note that

$$\begin{aligned} \int |T_{k-1} \circ T_k(h)(x)| \mathbb{P}_{k-2}(dx) &= \int \left| \int h(z) \exp(iug_k(z)) m_x(dz) \right| \mathbb{P}_{k-2}(dx) \\ &\leq \int \int |h(z)| \text{Var}(m_x(dz)) \mathbb{P}_{k-2}(dx) := \int |h(z)| m(dz), \end{aligned} \tag{5.4}$$

where

$$m(A) = \int \text{Var}(m_x(A)) \mathbb{P}_{k-2}(dx).$$

To analyze  $m(A)$ , we start by noting that for an increasing sequence of measurable partitions  $\mathcal{P}_n$  of  $A$ , by the monotone convergence theorem,

$$m(A) = \lim_{n \rightarrow \infty} \sum_{A_i \in \mathcal{P}_n} \int |m_x(A_i)| \mathbb{P}_{k-2}(dx). \tag{5.5}$$

In order to find an upper bound for  $|m_x(A)|$ , we start from the following estimate, where we have used at the end condition (2.8):

$$\begin{aligned} & \left( \int Q_k(y, A) Q_{k-1}(x, dy) \right)^2 - \left( \left| \int \exp(iug_{k-1}(y)) Q_k(y, A) Q_{k-1}(x, dy) \right| \right)^2 \\ &= \iint (1 - \cos(u(g_{k-1}(y) - g_{k-1}(y')))) Q_k(y, A) Q_{k-1}(x, dy) Q_k(y', A) Q_{k-1}(x, dy') \\ &= \iint 2 \sin^2 \left( \frac{u}{2} (g_{k-1}(y) - g_{k-1}(y')) \right) Q_k(y, A) Q_{k-1}(x, dy) Q_k(y', A) Q_{k-1}(x, dy') \\ &\geq a^4 \mathbb{P}_k^2(A) M, \end{aligned}$$

where

$$M = 2 \iint \sin^2 \left( \frac{u}{2} (g_{k-1}(y) - g_{k-1}(y')) \right) \mathbb{P}_{k-1}(dy) \mathbb{P}_{k-1}(dy').$$

Assume for the moment that  $\mathbb{P}_k(A) > 0$ . By condition (2.8) this implies that, for all  $x \in \mathcal{X}'$

$$\int Q_k(y, A) Q_{k-1}(x, dy) \geq a \mathbb{P}_k(A) > 0.$$

Because

$$|m_x(A)| = \int Q_k(y, A)Q_{k-1}(x, dy) - \int Q_k(y, A)Q_{k-1}(x, dy) + \left| \int \exp(iug_{k-1}(y))Q_k(y, A)Q_{k-1}(x, dy) \right|,$$

we have

$$|m_x(A)| = \int Q_k(y, A)Q_{k-1}(x, dy) - \frac{(\int Q_k(y, A)Q_{k-1}(x, dy))^2 - (\int \exp(iug_{k-1}(y))Q_k(y, A)Q_{k-1}(x, dy))^2}{\int Q_k(y, A)Q_{k-1}(x, dy) + |\int \exp(iug_{k-1}(y))Q_k(y, A)Q_{k-1}(x, dy)|}.$$

But

$$\int Q_k(y, A)Q_{k-1}(x, dy) + \left| \int \exp(iug_{k-1}(y))Q_k(y, A)Q_{k-1}(x, dy) \right| \leq 2 \int Q_k(y, A)Q_{k-1}(x, dy).$$

Therefore, by the above considerations,

$$|m_x(A)| \leq \int Q_k(y, A)Q_{k-1}(x, dy) - \frac{a^4\mathbb{P}_k^2(A)M}{2 \int Q_k(y, A)Q_{k-1}(x, dy)}.$$

With the notation

$$Y_{k-2}(x) = \int Q_k(y, A)Q_{k-1}(x, dy) = \mathbb{P}(\xi_k \in A | \xi_{k-2} = x),$$

by integrating the above inequality with respect to  $\mathbb{P}_{k-2}(dx)$  we obtain

$$\int |m_x(A)|\mathbb{P}_{k-2}(dx) \leq \int Y_{k-2}(x)\mathbb{P}_{k-2}(dx) - a^4\mathbb{P}_k(A)M \int \frac{\mathbb{P}_k(A)}{2Y_{k-2}(x)}\mathbb{P}_{k-2}(dx). \tag{5.6}$$

By the Markov inequality,

$$\mathbb{P}_{k-2}(Y_{k-2} > 2\mathbb{P}_k(A)) \leq \frac{\int Y_{k-2}(x)\mathbb{P}_{k-2}(dx)}{2\mathbb{P}_k(A)} = \frac{\mathbb{P}_k(A)}{2\mathbb{P}_k(A)} = \frac{1}{2}.$$

So

$$\mathbb{P}_{k-2}(Y_{k-2} \leq 2\mathbb{P}_k(A)) \geq \frac{1}{2}.$$

Therefore

$$\int \frac{\mathbb{P}_k(A)}{Y_{k-2}(x)}\mathbb{P}_{k-2}(dx) \geq \int_{Y_{k-2} \leq 2\mathbb{P}_k(A)} \frac{\mathbb{P}_k(A)}{Y_{k-2}(x)}\mathbb{P}_{k-2}(dx) \geq \frac{1}{2}\mathbb{P}_{k-2}(Y_{k-2} \leq 2\mathbb{P}_k(A)) \geq \frac{1}{4}.$$

Combining the last two inequalities with (5.6) we obtain

$$\int |m_x(A)|\mathbb{P}_{k-2}(dx) \leq \mathbb{P}_k(A) - \frac{1}{8}a^4\mathbb{P}_k(A)M = \left(1 - \frac{1}{8}a^4M\right) \mathbb{P}_k(A). \tag{5.7}$$

Let us note now that inequality (5.7) is also true for  $\mathbb{P}_k(A) = 0$ . Indeed, since

$$\mathbb{P}_k(A) = \int Q_k(y, A)\mathbb{P}_{k-1}(dy),$$

we get  $Q_k(y, A) = 0$ ,  $\mathbb{P}_{k-1} - a.s.$ , and by its definition,  $m_x(A) = 0$  for all  $x \in \mathcal{X}'$ .

At this moment, inequality (5.7) allows to estimate  $m(A)$  defined in (5.5). Hence, we get

$$\begin{aligned}
 m(A) &= \lim_n \sum_{A_i \in \mathcal{P}_n} \int |m_x(A_i)| \mathbb{P}_{k-2}(dx) \\
 &\leq \lim_n \sum_{A_i \in \mathcal{P}_n} \left(1 - \frac{1}{8} a^4 M\right) \mathbb{P}_k(A_i) = \left(1 - \frac{1}{8} a^4 M\right) \mathbb{P}_k(A).
 \end{aligned}$$

Whence, by (5.4), we obtain

$$\begin{aligned}
 \int |T_{k-1} \circ T_k(h)(x)| \mathbb{P}_{k-2}(dx) &\leq \int |h(z)| m(dz) \\
 &\leq \left(1 - \frac{1}{8} a^4 M\right) \int |h(z)| \mathbb{P}_k(dz),
 \end{aligned}$$

and Lemma 5.6 follows by noting that

$$M = 1 - |f_{k-1}(u)|^2.$$

□

**Lemma 5.7.** *For any  $k \in \mathbb{N}$  and  $u \in \mathbb{R}$ , we have*

$$\|T_{u,k}(1)\|_\infty^2 \leq 1 - a^2(1 - |\mathbb{E}(\exp(iuX_k))|^2).$$

**Proof:** We start by noticing that

$$\begin{aligned}
 |T_{u,k}(1)(x)|^2 &= \iint \exp(iu(g_k(y) - g_k(y'))) Q_k(x, dy) Q_k(x, dy') \\
 &= \iint \cos(u(g_k(y) - g_k(y'))) Q_k(x, dy) Q_k(x, dy') \\
 &= 1 - 2 \iint \sin^2\left(\frac{u}{2}(g_k(y) - g_k(y'))\right) Q_k(x, dy) Q_k(x, dy').
 \end{aligned}$$

By (2.8) we have

$$\begin{aligned}
 \iint \sin^2\left(\frac{u}{2}(g_k(y) - g_k(y'))\right) Q_k(x, dy) Q_k(x, dy') \\
 \geq a^2 \iint \sin^2\left(\frac{u}{2}(g_k(y) - g_k(y'))\right) \mathbb{P}_k(dy) \mathbb{P}_k(dy').
 \end{aligned}$$

and the result follows. □

We are now in the position to prove Proposition 5.5.

**Proof of Proposition 5.5**

Note that, for  $k \geq 1$

$$\mathbb{E}(\exp(iuS_{2k})|\xi_0) = T_1 \circ T_2 \circ \dots \circ T_{2k}(1)(\xi_0).$$

So

$$\mathbb{E}(\exp(iuS_{2k})) = \int T_1 \circ T_2 \circ \dots \circ T_{2k}(1)(x) \mathbb{P}_0(dx),$$

and then

$$|\mathbb{E}(\exp(iuS_{2k}))| \leq \|T_1 \circ T_2\| \cdots \|T_{2k-1} \circ T_{2k}\| .$$

Hence, by Lemma 5.6 we have that, for  $k \geq 1$ ,

$$|\mathbb{E}(\exp(iuS_{2k}))| \leq \prod_{j=1}^k [1 - a^4(1 - |\mathbb{E}(\exp(iuX_{2j-1})|^2)/8] .$$

Also, by Lemma 5.6 and Lemma 5.7

$$\begin{aligned} |\mathbb{E}(\exp(iuS_{2k}))| &\leq \|T_2 \circ T_3\| \cdots \|T_{2k-2} \circ T_{2k-1}\| \|T_{2k}(1)\| \\ &\leq \prod_{j=1}^k [1 - a^4(1 - |\mathbb{E}(\exp(iuX_{2j})|^2)/8] , \end{aligned}$$

and so, by multiplying these two inequalities we get

$$|\mathbb{E}(\exp(iuS_{2k}))|^2 \leq \prod_{j=1}^{2k} [1 - a^4(1 - |\mathbb{E}(\exp(iuX_j)|^2)/8]$$

A similar result can be obtained for  $|\mathbb{E}(\exp(iuS_{2k+1}))|^2$ , and therefore

$$|\mathbb{E}(\exp(iuS_n))|^2 \leq \prod_{j=1}^n \left[ 1 - \frac{a^4}{8} (1 - |f_j(u)|^2) \right].$$

Now, for any real  $x$ ,  $1 + x \leq \exp(x)$ , and the result in Proposition 5.5 follows.  $\square$

5.3. *Proof of Theorem 3.1.* We consider first the non-lattice case. According to Remark 5.2, it remains to verify conditions  $(D_1)$  and  $(D_2)$ . Suppose  $D > 0$ . Choose any  $\delta$  such that  $0 < \delta < D$ . By Proposition 5.5 combined with Condition **A**, for any  $1 \leq |u| < \delta B_n$ ,

$$\begin{aligned} \left| \mathbb{E} \exp \left( iu \frac{S_n}{B_n} \right) \right| &\leq \exp \left[ - \frac{a^4}{16} \sum_{j=1}^n (1 - |f_j(\frac{u}{B_n})|^2) \right] \\ &\leq \exp(-g(|u|)). \end{aligned}$$

Integrating both sides of this inequality on the intervals  $T \leq |u| \leq \delta B_n$ , with the restriction  $T \geq 1$ , we obtain

$$\begin{aligned} \int_{T \leq |u| \leq \delta B_n} \left| \mathbb{E} \exp \left( iu \frac{S_n}{B_n} \right) \right| du &\leq \int_{T \leq |u| \leq \delta B_n} \exp(-g(|u|)) du \\ &\leq \int_{|u| \geq T} \exp(-g(|u|)) du. \end{aligned}$$

Whence, taking first  $\limsup_n$  and then  $T \rightarrow \infty$ , condition  $(D_1)$  is verified.

We move now to verify  $(D_2)$ . Because the interval  $[-D, -\delta] \cup [\delta, D]$  is compact,  $(D_2)$  is verified if we can show that for any  $u$  fixed in this interval we can find an open interval  $O_u$  containing  $u$  such that

$$B_n \sup_{t \in O_u} |\mathbb{E} \exp(itS_n)| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{5.8}$$

By Proposition 5.5, for any  $t$ ,

$$\begin{aligned} B_n |\mathbb{E} \exp(itS_n)| &\leq B_n \exp \left[ - \frac{a^4}{16} \sum_{j=1}^n (1 - |f_j(t)|^2) \right] \\ &= \exp \left[ \ln B_n - \frac{a^4}{16} \sum_{k=1}^n (1 - |f_k(t)|^2) \right]. \end{aligned}$$

Now (5.8) is satisfied provided that

$$\ln B_n \left( 1 - \inf_{t \in O_u} \frac{1}{\ln B_n} \frac{a^4}{16} \sum_{k=1}^n (1 - |f_k(t)|^2) \right) \rightarrow -\infty.$$

Since  $B_n \rightarrow \infty$ , we obtain in this case that  $(D_2)$  follows from Condition **B**.

The proof of the lattice case is similar.

□

5.4. *Proof of Theorem 3.2 and Remark 3.3.* The proof of Theorem 3.2 is similar to the proof of the corresponding result in Merlevède et al. (2021) replacing Proposition 8 there by Proposition 5.5 here. In case  $\mathbb{E}(X_0^2) < \infty$ , the identification of normalizer  $B_n$  in Remark 3.3 follows by (4.4) and (4.5).

When  $\mathbb{E}(X_0^2) = \infty$ , we shall invoke Theorem 2.1 in Peligrad (1990) to decide that  $B_n$  can be taken  $\sqrt{\pi/2}\mathbb{E}|S_n|$ . To apply this theorem, note that by (4.7) we have  $\varphi_1 < 1$ , and also the fact that  $X_0$  is in the domain of attraction of the normal law with  $\mathbb{E}(X_0^2) = \infty$ , is equivalent to (1.2) in Peligrad (1990). Moreover, by Lemma 5.2 and Corollary 5.1 in Peligrad (1990) it follows that  $B_n \rightarrow \infty$ .

5.5. *Proof of Theorem 3.4.* We shall verify first condition (2.1). To derive this convergence in distribution, we use the existing results, inspired by Theorem 3.2 in Samur (1987) and further developed by Kobus (1995) and Tyran-Kamińska (2010). As a matter of fact, we shall apply Corollary (5.9) in Kobus (1995).

Note that, by relations between the mixing coefficients in Section 4, our Markov chain is  $\varphi$ -mixing with exponential rate of convergence to 0 of the  $\varphi$ -mixing coefficients. Therefore, also the sequence  $(X_k)_{k \in \mathbb{Z}}$  is  $\varphi$ -mixing with exponential rate of convergence to 0, and also  $\rho$ -mixing with exponential rate of convergence to 0. Whence the two mixing conditions (i) and (ii) in Corollary (5.9) in Kobus (1995) are satisfied. Since we assumed that the distribution of  $X_0$  satisfies conditions (3.4) and (3.5), and in addition  $\mathbb{E}(X_0) = 0$  for  $1 < p < 2$  and  $X_0$  has a symmetric distribution for  $p = 1$ , by classical results, it follows that  $X_0$  is in the domain of attraction of a strictly stable distribution. This gives that condition (5.12) in Kobus (1995) holds without centering. By the conclusion of Corollary 5.9 in Kobus (1995) in order to prove

$$\frac{S_n}{B_n} \Rightarrow L,$$

with  $L$  strictly stable with exponent  $p$ , it is enough to verify the following condition: for any  $x > 0$  and any  $k \in \mathbb{N}$  we have

$$\lim_{n \rightarrow \infty} n\mathbb{P}(|X_0| \geq xB_n, |X_k| \geq xB_n) = 0.$$

Clearly, by conditions (3.4) we can write

$$x^p n\mathbb{P}(|X_0| \geq xB_n) = \frac{n}{B_n^p} \ell(B_n) \frac{\ell(xB_n)}{\ell(B_n)}.$$

From the properties of slowly varying functions and from (3.7) we deduce that for any  $x > 0$ ,

$$\lim_{n \rightarrow \infty} n\mathbb{P}(|X_0| \geq xB_n) = \frac{1}{x^p}.$$

Therefore we can find a constant  $C_x$  such that

$$\begin{aligned} & n\mathbb{P}(|X_0| \geq xB_n, |X_j| \geq xB_n) \\ &= n\mathbb{P}(|X_0| \geq B_n x) (\mathbb{P}|X_j| \geq xB_n \mid |X_0| \geq xB_n) \\ &\leq C_x \mathbb{P}(|X_j| \geq xB_n \mid |X_0| \geq xB_n), \end{aligned}$$

and the result follows by condition (3.6).

Also note that a stable distribution with index  $p \neq 1$  and symmetric for  $p = 1$ , has an integrable characteristic function. This can be seen by the form of the modulus of the characteristic function for these situations, namely for a constant  $c_p$ :

$$|f_L(t)| = \exp(-|c_p t|^p).$$

(Lévy, 1954). Therefore condition (2.1) holds.

To prove the results, according to Theorems 5.1 and 5.3, it remains to verify conditions **A** and **B**.

We shall invoke relation (7.9) in Feller (1967), which states that under (3.4) and (3.5), there are constants  $\delta > 0$ ,  $n_0 \in \mathbb{N}$  and  $c > 0$  such that for  $0 < |u| \leq \delta B_n$  and  $n > n_0$

$$n \left( 1 - |f_0\left(\frac{u}{B_n}\right)|^2 \right) > c|u|^p.$$

Clearly Condition **A** is satisfied.

In the stationary setting Condition **B** reads: For  $u \neq 0$  there is an  $\varepsilon = \varepsilon(u)$ ,  $c(u)$  and a  $n_0 = n_0(u)$  such that for all  $t$  with  $|t - u| \leq \varepsilon$  and  $n > n_0$ ,

$$\frac{na^4}{2^4(\ln B_n)}(1 - |f_0(t)|^2) \geq c(u) > 1.$$

On another hand, by the definition of  $B_n$ , and the properties of slowly varying functions, it is well known that  $B_n < n^{p'}$  for  $p' > 1/p$  and  $n$  large enough. So  $\ln B_n < p' \ln n$  for  $p' > 1/p$  and  $n$  sufficiently large.

Therefore, if  $X$  does not have a lattice distribution, then for every  $u \neq 0$ ,  $|f_0(u)| < 1$ , and by the continuity of  $f_0$ , for any  $u \neq 0$  we can certainly find an  $\varepsilon > 0$  and  $d < 1$ , such that for all  $t$  with  $|t - u| \leq \varepsilon$  we have  $|f_0(t)| < d$ . Therefore, in this case, for such  $t$ ,

$$\frac{na^4}{(\ln B_n)}(1 - |f_0(t)|^2) \geq \frac{na^4}{p' \ln n}(1 - d^2) \rightarrow \infty$$

and Condition **B** is satisfied.

In the lattice case we use a similar argument on the interval  $0 < u < \frac{\pi}{h}$ .  $\square$

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## References

- Aaronson, J. and Denker, M. A local limit theorem for stationary processes in the domain of attraction of a normal distribution. In *Asymptotic methods in probability and statistics with applications (St. Petersburg, 1998)*, Stat. Ind. Technol., pp. 215–223. Birkhäuser Boston, Boston, MA (2001a). [MR1890328](#).
- Aaronson, J. and Denker, M. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.*, **1** (2), 193–237 (2001b). [MR1840194](#).
- Billingsley, P. *Probability and measure*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, NJ (2012). ISBN 978-1-118-12237-2. [MR2893652](#).
- Bradley, R. C. Every “lower psi-mixing” Markov chain is “interlaced rho-mixing”. *Stochastic Process. Appl.*, **72** (2), 221–239 (1997). [MR1486554](#).
- Bradley, R. C. *Introduction to strong mixing conditions. Vol. 1*. Kendrick Press, Heber City, UT (2007). [MR2325294](#).

- Breiman, L. *Probability*, volume 7 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1992). ISBN 0-89871-296-3. [MR1163370](#).
- Bryc, W. and Smoleński, W. On the convergence of averages of mixing sequences. *J. Theoret. Probab.*, **6** (3), 473–483 (1993). [MR1230342](#).
- Cattiaux, P. and Manou-Abi, M. Limit theorems for some functionals with heavy tails of a discrete time Markov chain. *ESAIM Probab. Stat.*, **18**, 468–482 (2014). [MR3333999](#).
- Davis, R. A. Stable limits for partial sums of dependent random variables. *Ann. Probab.*, **11** (2), 262–269 (1983). [MR690127](#).
- Denker, M. and Jakubowski, A. Stable limit distributions for strongly mixing sequences. *Statist. Probab. Lett.*, **8** (5), 477–483 (1989). [MR1040810](#).
- Dolgopyat, D. A local limit theorem for sums of independent random vectors. *Electron. J. Probab.*, **21**, Paper No. 39, 15 (2016). [MR3515569](#).
- Dolgopyat, D. and Sarig, O. Local limit theorems for inhomogeneous Markov chains. *ArXiv Mathematics e-prints* (2020). [arXiv: 2109.05560](#).
- El Machkouri, M., Jakubowski, A., and Volný, D. Stable limits for Markov chains via the principle of conditioning. *Stochastic Process. Appl.*, **130** (4), 1853–1878 (2020). [MR4074686](#).
- Feller, W. On regular variation and local limit theorems. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1*, pp. 373–388. Univ. California Press, Berkeley, Calif. (1967). [MR0219117](#).
- Ferré, D., Hervé, L., and Ledoux, J. Limit theorems for stationary Markov processes with  $L^2$ -spectral gap. *Ann. Inst. Henri Poincaré Probab. Stat.*, **48** (2), 396–423 (2012). [MR2954261](#).
- Gamkrelidze, N. G. On a local limit theorem for lattice random variables. *Teor. Veroyatnost. i Primenen.*, **9**, 733–736 (1964). [MR0177433](#).
- Hafouta, Y. and Kifer, Y. A nonconventional local limit theorem. *J. Theoret. Probab.*, **29** (4), 1524–1553 (2016). [MR3571253](#).
- Hennion, H. and Hervé, L. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2001). ISBN 3-540-42415-6. [MR1862393](#).
- Hervé, L. and Pène, F. The Nagaev-Guivarc’h method via the Keller-Liverani theorem. *Bull. Soc. Math. France*, **138** (3), 415–489 (2010). [MR2729019](#).
- Jakubowski, A. Asymptotic independent representations for sums and order statistics of stationary sequences (1991). Research’s report. <https://www-users.mat.uni.torun.pl/~adjakubo/hab.pdf>.
- Kobus, M. Generalized Poisson distributions as limits of sums for arrays of dependent random vectors. *J. Multivariate Anal.*, **52** (2), 199–244 (1995). [MR1323331](#).
- Kolmogorov, A. N. A local limit theorem for Markov chains. In *Select. Transl. Math. Statist. and Probability, Vol. 2*, pp. 109–129. American Mathematical Society, Providence, R.I. (1962). [MR0150810](#).
- Maller, R. A. A local limit theorem for independent random variables. *Stochastic Process. Appl.*, **7** (1), 101–111 (1978). [MR492695](#).
- Merlevède, F., Peligrad, M., and Peligrad, C. On the local limit theorems for psi-mixing Markov chains. *ALEA Lat. Am. J. Probab. Math. Stat.*, **18** (2), 1221–1239 (2021). [MR4282187](#).
- Mineka, J. and Silverman, S. A local limit theorem and recurrence conditions for sums of independent non-lattice random variables. *Ann. Math. Statist.*, **41**, 592–600 (1970). [MR256447](#).
- Nagaev, S. V. More exact limit theorems for homogeneous Markov chains. *Teor. Veroyatnost. i Primenen.*, **6**, 67–86 (1961). [MR0131291](#).
- Peligrad, M. On Ibragimov-Iosifescu conjecture for  $\phi$ -mixing sequences. *Stochastic Process. Appl.*, **35** (2), 293–308 (1990). [MR1067114](#).
- Peligrad, M. Central limit theorem for triangular arrays of non-homogeneous Markov chains. *Probab. Theory Related Fields*, **154** (3-4), 409–428 (2012). [MR3000551](#).

- Peligrad, M., Sang, H., Xiao, Y., and Yang, G. Limit theorems for linear random fields with innovations in the domain of attraction of a stable law. *Stochastic Process. Appl.*, **150**, 596–621 (2022). [MR4433071](#).
- Rozanov, Y. A. On a local limit theorem for lattice distributions. *Teor. Veroyatnost. i Primenen.*, **2**, 275–281 (1957). [MR0092264](#).
- Samur, J. D. On the invariance principle for stationary  $\phi$ -mixing triangular arrays with infinitely divisible limits. *Probab. Theory Related Fields*, **75** (2), 245–259 (1987). [MR885465](#).
- Shore, T. R. On the local limit theorem for independent nonlattice random variables. *Ann. Probab.*, **6** (4), 563–573 (1978). [MR496406](#).
- Stone, C. A local limit theorem for nonlattice multi-dimensional distribution functions. *Ann. Math. Statist.*, **36**, 546–551 (1965). [MR175166](#).
- Szewczak, Z. S. A local limit theorem for continued fractions. *Stoch. Dyn.*, **10** (3), 429–439 (2010). [MR2671385](#).
- Tyran-Kamińska, M. Convergence to Lévy stable processes under some weak dependence conditions. *Stochastic Process. Appl.*, **120** (9), 1629–1650 (2010). [MR2673968](#).