



# Long-time asymptotic behaviour of the value function in nonlinear stopping problems

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**Abstract.** We provide general conditions ensuring that the value functions of some nonlinear stopping problems with finite horizon converge to the value functions of the corresponding problems with infinite horizon. Our result can be formulated as result on stability, with respect to time horizon, of nonlinear  $f$ -expectations. We also study the rate of convergence. Many examples are given to illustrate our results. They include the analysis of time asymptotics of the fair prices of American options in a multidimensional exponential Lévy model.

## 1. Introduction

Let  $\mathbb{M} = \{(X, P_x), x \in E\}$  be a Borel right Markov process with state space  $E$  and life time  $\zeta$ , and  $D$  be an open subset of  $E$ . For  $T > 0$  and measurable functions  $g : E \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $h, \varphi : E \rightarrow \mathbb{R}$  and  $\psi : E \setminus D \rightarrow \mathbb{R}$  we consider the value functions

$$V_T(s, x) = \sup_{\sigma \leq T_s \wedge \tau_D} \mathcal{E}_{0, \sigma}^{x, f} [h(X_\sigma) \mathbf{1}_{\{\sigma < T_s \wedge \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma = \tau_D, \tau_D \leq T_s\}} + \varphi(X_{T_s}) \mathbf{1}_{\{\sigma = T_s, \tau_D > T_s\}}], \quad (s, x) \in [0, T] \times D, \quad (1.1)$$

and

$$V(x) = \sup_{\sigma \leq \tau_D} \mathcal{E}_{0, \sigma}^{x, f} [h(X_\sigma) \mathbf{1}_{\{\sigma < \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma = \tau_D\}}], \quad x \in D. \quad (1.2)$$

Here

$$f(t, y) = g(X_t, y), \quad t \geq 0, y \in \mathbb{R},$$

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$\mathcal{E}^{x,f}$  is the nonlinear  $f$ -expectation (defined under the measure  $P_x$ ) introduced by Peng (1997) (see Section 2),  $\sigma$  are stopping times with respect to the minimum admissible filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  generated by  $\mathbb{M}$ , and

$$\tau_D = \inf\{t > 0 : X_t \notin D\}, \quad T_s = T - s.$$

Perhaps it is appropriate at this point to note that under some assumptions,  $V_T$  and  $V$  are solutions of some "usual" nonlinear stopping problems, i.e. can be defined without recourse to the notion of nonlinear expectation. Specifically, from our results it follows that under natural assumptions on the data  $g, h, \psi$  and  $\varphi$  the function  $V_T$  is a solution of the nonlinear equation

$$V_T(s, x) = \sup_{0 \leq \sigma \leq T_s \wedge \tau_D} \mathbb{E}_x \left[ \int_0^\sigma g(X_t, V_T(s+t, X_t)) dt + h(X_\sigma) \mathbf{1}_{\{\sigma < T_s \wedge \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma = \tau_D, \tau_D \leq T_s\}} + \varphi(X_{T_s}) \mathbf{1}_{\{\sigma = T_s, \tau_D > T_s\}} \right], \tag{1.3}$$

and  $V$  is a solution of the equation

$$V(x) = \sup_{0 \leq \sigma \leq \tau_D} \mathbb{E}_x \left[ \int_0^\sigma g(X_t, V(X_t)) dt + h(X_\sigma) \mathbf{1}_{\{\sigma < \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma = \tau_D\}} \right], \tag{1.4}$$

where  $\mathbb{E}_x$  denotes the expectation with respect to  $P_x$  (for details and a generalization see Section 4). We stress, however, that in some of our results the generator  $f$  is merely continuous and nonincreasing with respect to the  $y$ -variable. Therefore, in general, the integrals in (1.3), (1.4) involving  $g$  need not be well defined. Nevertheless, one can still define and study  $V_T, V$  defined by (1.1), (1.2).

In this paper, we give general conditions on the data  $g, h, \varphi, \psi$  guaranteeing that

$$\lim_{T_s \rightarrow \infty} V_T(s, x) = V(x), \quad x \in D. \tag{1.5}$$

We also provide some estimates on the rate of convergence. In many cases it is very important to have some information about the dynamic of the value functions, i.e. about the processes  $t \mapsto V_T(s+t, \cdot, X_t)$  and  $t \mapsto V(X_t)$ . The second main result of the paper is the dynamic version of (1.5). We first prove that if  $D$  is Dirichlet regular (i.e.  $P_x(\tau_D > 0) = 0$  for  $x \in \partial D$ ), then for every stopping time  $\alpha \leq \tau_D \wedge T_s$ ,

$$V_T(s + \alpha, X_\alpha) = \operatorname{ess\,sup}_{\alpha \leq \sigma \leq T_s \wedge \tau_D} \mathcal{E}_{\alpha, \sigma}^{x,f} [h(X_\sigma) \mathbf{1}_{\{\sigma < T_s \wedge \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma = \tau_D, \tau_D \leq T_s\}} + \varphi(X_{T_s}) \mathbf{1}_{\{\sigma = T_s, \tau_D > T_s\}}],$$

and for every stopping time  $\alpha \leq \tau_D$  we have

$$V(X_\alpha) = \operatorname{ess\,sup}_{\alpha \leq \sigma \leq \tau_D} \mathcal{E}_{\alpha, \sigma}^{x,f} [h(X_\sigma) \mathbf{1}_{\{\sigma < \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma = \tau_D\}}], \quad x \in D.$$

We show that these two formulas together with the integrability condition

$$\mathbb{E}_x \int_0^{T_s \wedge \tau_D} |g(X_t, V_T(s+t, X_t))| dt + \mathbb{E}_x \int_0^{\tau_D} |g(X_t, V(t, X_t))| dr < \infty$$

imply (1.3), (1.4). Then we prove that

$$V_{T_s}^*(x) := \sup_{\alpha \leq \tau_D \wedge T_s} \mathbb{E}_x |V_T(s + \alpha, X_\alpha) - V(X_\alpha)| \leq \mathbb{E}_x |\gamma - \varphi|(X_{T_s}) \mathbf{1}_{\{\tau_D \geq T_s\}} + \mathbb{E}_x \int_{T_s \wedge \tau_D}^{\tau_D} |g(X_t, \gamma(X_t))| dt + \sup_{T_s \wedge \tau_D \leq \tau \leq \tau_D} \mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)|, \tag{1.6}$$

where  $\gamma(X_t) = \mathbb{E}_x(\psi(X_{\tau_D})|\mathcal{F}_t)$ , and that the right-hand side of (1.6) converges to zero as  $T_s \rightarrow \infty$ . From (1.6) and the elementary inequality

$$\mathbb{E}_x \sup_{t \leq T_s \wedge \tau_D} |V_T(s+t, X_t) - V(X_t)|^q \leq \frac{1}{1-q} (V_{T_s}^*(x))^q, \quad q \in (0, 1), \tag{1.7}$$

we get uniform (in  $t$ ) convergence of the value processes.

The key idea of our proofs of (1.5) and (1.6) is to look at the processes  $V(X)$  and  $V_T(s + \cdot, X)$  as solutions to certain reflected backward stochastic differential equations (RBSDEs). In the non-dynamic case this link is immediate. Some work is required to prove it in the dynamic case. After establishing these links, we get the desired results by applying new stability results for RBSDEs obtained recently in Klimsiak (2021a) (see Section 3).

In the second part of the paper (Sections 5 and 6), we study the rate of convergence in (1.5). To this end, in Section 5 we describe several methods of estimating the right-hand side of (1.6). In general, these methods are incomparable. Its applicability depends on the underlying process  $\mathbb{M}$  (Lévy type process, (intrinsic) ultracontractive process or symmetric Hunt process related to a symmetric Dirichlet form) and assumptions on the data  $\varphi, \psi, h$  (bounded or in  $L^q$  with some  $q \geq 1$ ) and  $D$  (bounded, unbounded or of finite reference measure, with regular boundary or not). To illustrate the utility of these methods, let us mention here that in Section 5, among other things, we get several results of type

$$V_{T_s}^*(x) \sim e^{-T_s \lambda(x)}, \quad x \in D, \quad \text{as } T_s \rightarrow \infty.$$

Depending on the structure of the problem, the function  $\lambda(\cdot)$  may be positive and vanishing near the boundary of  $D$ , may be constant and equal to a number  $\lambda > 0$  or be equal to the principal eigenvalue  $\lambda_1$  associated with the semigroup  $(P_t^D)$  of the process  $\mathbb{M}$  killed upon leaving  $D$ .

The problem of controlling the rate of decay of the right-hand side of (1.6) is by no means obvious. Let  $R^D$  denote the potential operator associated with the process  $\mathbb{M}$  killed upon leaving  $D$ . We prove that for general Markov process  $\mathbb{M}$  we have

$$V_{T_s}^*(x) \leq P_{T_s}^D w(x) + P_{T_s}^D \hat{h}_\infty(x), \quad x \in D, \quad 0 \leq s \leq T,$$

where

$$w(x) = |\gamma - \varphi|(x) + R^D(|g(\cdot, \gamma)|)(x), \quad \hat{h}_\infty(x) = \sup_{\tau \leq \tau_D} \hat{h}(x, \tau).$$

and

$$\hat{h}(x, \tau) = \mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)|.$$

One easily checks that  $P_{T_s}^D w(x)$  is equal to the sum of the first two terms on the right-hand side of (1.6). Thus, the study of the rate of convergence of these two terms reduces to the study of the rate of decay, as  $t \rightarrow \infty$ , of the semigroup  $(P_t^D)$ . For this one can use some known results from the semigroup theory and probabilistic potential theory. In Section 5, we indicate some of them. The third term on the right-hand side of (1.6) causes specific problems due to the compound term  $\hat{h}_\infty$ . We show how to deal with this term in some typical situations, for instance, when  $h(X)$  is a submartingale or supermartingale under  $P_x$  for  $x \in E$  or  $\gamma, h \in C_b^2(\mathbb{R}^d)$  and the generator of  $\mathbb{M}$  is a Lévy-type operator. Another interesting situation we consider is when  $\mathbb{M}$  is associated with a symmetric regular Dirichlet form and  $\gamma, h$  belong to its extended domain. In this case, in general, the process  $h(X)$  need not be a semimartingale.

In Section 6, we provide a detailed analysis of (1.5) in a stopping problem arising in the study of American options. We consider dividend paying options in an exponential Lévy model. We show that if the payoff function is continuous and satisfies the linear growth condition, then under natural assumptions on the model the fair price of the option with maturity  $T$  converges as  $T \rightarrow \infty$  to the fair price of the corresponding perpetual American option, and we provide the rate of convergence.

In the present paper, we deal exclusively with stopping problems and RBSDEs with one (lower) barrier. RBSDEs with two continuous barriers satisfying Mokobodzki's separability condition and some integrability conditions were introduced in Cvitanić and Karatzas (1996) in case the underlying filtration is Brownian. In Cvitanić and Karatzas (1996) also a connection of such equations with a pair of some coupled optimal stopping problems (so-called Dynkin's games) is given. At present, the theory initiated in Cvitanić and Karatzas (1996) is quite well developed. The existence and uniqueness of solutions of RBSDEs with possibly infinite terminal time is known to hold under fairly general assumptions on the data and for general filtration (see Klimsiak (2021a); Klimsiak and Rzymowski (2021) and the references therein). It is also known that under fairly general assumptions the value functions in Dynkin's games can be represented by solutions of RBSDEs (see Klimsiak (2021a); Klimsiak and Rzymowski (2021)). It would be interesting to use these results to give, in the case of two barriers, some asymptotic results similar to those given in the present paper.

## 2. RBSDEs and nonlinear optimal stopping problem

For the sake of completeness, in this section we recall some known results on the existence and uniqueness of solutions to backward stochastic differential equations (BSDEs) and reflected BSDEs (RBSDEs) with one barrier.

In what follows  $(\Omega, \mathcal{F}, P)$  is a complete probability space and  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is a right-continuous filtration satisfying the usual conditions. We denote by  $\mathcal{T}$  the set of all  $\mathbb{F}$ -stopping times, and for given  $\mathbb{F}$ -stopping times  $\tau, \sigma$  such that  $\tau \leq \sigma$  we denote by  $\mathcal{T}_\tau^\sigma$  the set of all  $\mathbb{F}$ -stopping times  $\alpha$  such that  $\tau \leq \alpha \leq \sigma$ . To simplify notation, we write  $\mathcal{T}^\sigma$  for  $\mathcal{F}_0^\sigma$  and  $\mathcal{T}_\tau$  for  $\mathcal{T}_\tau^\infty$ .

Below we assume as given an  $\mathbb{F}$ -stopping time  $\vartheta$  (not necessarily finite), an  $\mathcal{F}_\vartheta$ -measurable random variable  $\xi$ , an  $\mathbb{F}$ -adapted càdlàg process  $L$  of class (D) and a function  $f : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is  $\mathbb{F}$ -adapted with respect to  $(\omega, t) \in \Omega \times [0, \infty)$  for any fixed  $y \in \mathbb{R}$ . As usual, in the sequel in our notation we omit the dependence of  $f$  on  $\omega \in \Omega$ .

Recall that a càdlàg  $\mathbb{F}$ -adapted process  $Y$  is said to be of class (D) under the measure  $P$  if the collection of random variables  $\{Y_\tau : \tau \in \mathcal{T}, \tau < \infty\}$  is uniformly integrable under  $P$ .

**Definition 2.1.** We say that a pair  $(Y, M)$  of  $\mathbb{F}$ -adapted càdlàg processes is a solution, on the interval  $[0, \vartheta]$ , of the BSDE with terminal condition  $\xi$  and coefficient  $f$  (BSDE $^\vartheta(\xi, f)$  for short) if  $Y$  is a process of class (D),  $M$  is a local martingale such that  $M_0 = 0$ ,  $P$ -a.s., and the following conditions are satisfied  $P$ -a.s.:

- (a)  $\int_0^{a \wedge \vartheta} |f(t, Y_t)| dt < \infty$  for every  $a \geq 0$ ,
- (b) For every  $a \geq 0$ ,

$$Y_t = Y_{a \wedge \vartheta} + \int_{t \wedge \vartheta}^{a \wedge \vartheta} f(r, Y_r) dr - \int_{t \wedge \vartheta}^{a \wedge \vartheta} dM_r, \quad t \in [0, a], \quad (2.1)$$

- (c)  $Y_{a \wedge \vartheta} \rightarrow \xi$   $P$ -a.s. as  $a \rightarrow \infty$ .

We will need the following assumptions.

- (A1)  $E|\xi| < \infty$  and there exists a càdlàg process  $S$  such that  $S$  is a difference of two supermartingales of class (D) and  $E \int_0^\vartheta |f(t, S_t)| dt < \infty$ .
- (A2) For a.e.  $t \in [0, \vartheta]$  the function  $y \mapsto f(t, y)$  is non-increasing  $P$ -a.s.
- (A3) For a.e.  $t \in [0, \vartheta]$  the function  $y \mapsto f(t, y)$  is continuous  $P$ -a.s.
- (A4) For every  $y \in \mathbb{R}$ ,  $\int_0^\vartheta |f(t, y)| dt < \infty$   $P$ -a.s.
- (A5)  $L$  is a càdlàg adapted process of class (D) such that  $\limsup_{a \rightarrow \infty} L_{a \wedge \vartheta} \leq \xi$ ,
- (A6) There exists a process  $U$  such that  $L \leq U$ ,  $U$  is a difference of two supermartingales of class (D) and  $E \int_0^\vartheta f^-(t, U_t) dt < \infty$ .

**Theorem 2.2.** (i) Under (A2) there exists at most one solution of BSDE $^\vartheta(\xi, f)$ .

- (ii) If (A1)–(A4) are satisfied, then there exists a solution to  $\text{BSDE}^\vartheta(\xi, f)$ . Moreover,  $M$  is a uniformly integrable martingale and

$$E \int_0^\vartheta |f(t, Y_t)| dt < \infty. \tag{2.2}$$

*Proof:* Part (i) is a direct consequence of Klimsiak (2019, Proposition 2.4). For (ii) see Klimsiak (2019, Theorem 2.9, Remark 2.2).  $\square$

We now recall the notion of the nonlinear  $f$ -expectation introduced by Peng (1997) (see also Peng (2004)). For  $\alpha, \beta \in \mathcal{T}$  such that  $\alpha \leq \beta$  and  $f$  satisfying (A1)–(A4) we define the operator

$$\mathcal{E}_{\alpha, \beta}^f : L^1(\Omega, \mathcal{F}_\beta, P) \rightarrow L^1(\Omega, \mathcal{F}_\alpha, P)$$

by

$$\mathcal{E}_{\alpha, \beta}^f(\xi) = Y_\alpha, \quad \xi \in L^1(\Omega, \mathcal{F}_\beta, P),$$

where  $(Y, M)$  is the unique solution of  $\text{BSDE}^\beta(\xi, f)$ . Note that in general  $\mathcal{E}^f$  is a nonlinear operator. Applying Itô’s formula shows that if  $f$  is linear of the form  $f(t, y) = -c(t)y + b(t)$  for some  $\mathbb{F}$ -adapted processes  $b$  and  $c \geq 0$ , then

$$\mathcal{E}_{\alpha, \beta}^f(\xi) = E\left(e^{-\int_0^\beta c(r) dr} \xi + \int_0^\beta e^{-\int_0^t c(r) dr} b(t) dt \mid \mathcal{F}_\alpha\right).$$

We say that a càdlàg process  $X$  of class (D) is an  $\mathcal{E}^f$ -supermartingale (resp.  $\mathcal{E}^f$ -submartingale) on  $[\alpha, \beta]$  if  $\mathcal{E}_{\sigma, \tau}^f(X_\tau) \leq X_\sigma$  (resp.  $\mathcal{E}_{\sigma, \tau}^f(X_\tau) \geq X_\sigma$ ) for all  $\tau, \sigma \in \mathcal{T}$  such that  $\alpha \leq \sigma \leq \tau \leq \beta$ . Of course,  $X$  is called an  $\mathcal{E}^f$ -martingale on  $[\alpha, \beta]$  if it is both  $\mathcal{E}^f$ -supermartingale and  $\mathcal{E}^f$ -submartingale on  $[\alpha, \beta]$ . For a given càdlàg process  $V$  and stopping times  $\alpha, \beta$  ( $\alpha \leq \beta$ ) we denote by  $|V|_{\alpha, \beta}$  the total variation of the process  $V$  on  $[\alpha, \beta]$ .

**Proposition 2.3.** *Assume that  $f$  satisfies (A1)–(A4) and let  $\alpha, \beta \in \mathcal{T}^\vartheta$  be such that  $\alpha \leq \beta$ .*

- (i) *Let  $\xi \in L^1(\Omega, \mathcal{F}_\beta; P)$  and  $G$  be a càdlàg  $\mathbb{F}$ -adapted finite variation process such that  $G_\alpha = 0$  and  $E|G|_{\alpha, \beta} < \infty$ . Then there exists a unique solution  $(X, N)$  of  $\text{BSDE}^{\alpha, \beta}(\xi, f + dG)$ . Moreover, if  $G$  (resp.  $-G$ ) is an increasing process, then  $X$  is an  $\mathcal{E}^f$ -supermartingale (resp.  $\mathcal{E}^f$ -submartingale) on  $[\alpha, \beta]$ .*
- (ii) *If  $\xi_1, \xi_2 \in L^1(\Omega, \mathcal{F}_\beta, P)$  and  $\xi_1 \leq \xi_2$ , then  $\mathcal{E}_{\alpha, \beta}^f(\xi_1) \leq \mathcal{E}_{\alpha, \beta}^f(\xi_2)$ .*
- (iii) *Let  $f_1, f_2$  satisfy (A1)–(A4) and  $\alpha, \beta_1, \beta_2 \in \mathcal{T}^\vartheta$  be such that  $\alpha \leq \beta_1 \leq \beta_2$ . Then for any  $\xi_1 \in L^1(\Omega, \mathcal{F}_{\beta_1}, P)$  and  $\xi_2 \in L^1(\Omega, \mathcal{F}_{\beta_2}, P)$ ,*

$$\begin{aligned} |\mathcal{E}_{\alpha, \beta_1}^{f_1}(\xi_1) - \mathcal{E}_{\alpha, \beta_2}^{f_2}(\xi_2)| &\leq E\left(|\xi_1 - \xi_2| + \int_\alpha^{\beta_1} |f^1(t, Y_t^1) - f^2(t, Y_t^1)| dt \right. \\ &\quad \left. + \int_{\beta_1}^{\beta_2} |f^2(t, Y_t^2)| dt \mid \mathcal{F}_\alpha\right), \end{aligned}$$

where  $Y_t^1 = \mathcal{E}_{t \wedge \beta_1, \beta_1}^{f_1}(\xi_1)$ ,  $Y_t^2 = \mathcal{E}_{t \wedge \beta_2, \beta_2}^{f_2}(\xi_2)$ .

*Proof:* Assertion (iii) follows from Klimsiak (2019, Theorem 2.9) and (ii) follows from Klimsiak (2019, Proposition 2.4). The existence part in (i) follows from Klimsiak (2019, Theorem 2.9). Now assume that  $X$  is as in (i) and  $G$  is an increasing process. Let  $\sigma, \tau \in \mathcal{T}$  be such that  $\alpha \leq \sigma \leq \tau \leq \beta$ , and let  $(X^\tau, N^\tau)$  be a solution of  $\text{BSDE}^{\alpha, \tau}(X_\tau, f)$ . It is clear that  $(X, N)$  is a solution of  $\text{BSDE}^{\alpha, \tau}(X_\tau, f + dG)$ . Therefore, by Klimsiak (2019, Proposition 2.4),  $X \geq X^\tau$  on  $[\alpha, \tau]$ . In particular,  $X_\sigma \geq X_\sigma^\tau$ . By the definition of the nonlinear expectation,  $\mathcal{E}_{\sigma, \tau}^f(X_\tau) = X_\sigma^\tau$ , so  $\mathcal{E}_{\sigma, \tau}^f(X_\tau) \leq X_\sigma$ . A similar reasoning in the case where  $-V$  is increasing gives the result.  $\square$

**Definition 2.4.** We say that a triple  $(Y, M, K)$  of adapted càdlàg processes is a solution, on the interval  $[0, \vartheta]$ , of the RBSDE with terminal condition  $\xi$ , coefficient  $f$  and barrier  $L$  (RBSDE $^\vartheta(\xi, f, L)$  for short) if  $Y$  is a of class (D),  $M$  is a local martingale such that  $M_0 = 0$ ,  $K$  is an increasing process with  $K_0 = 0$  and the following conditions are satisfied  $P$ -a.s.:

- (a)  $\int_0^{a \wedge \vartheta} |f(t, Y_t)| dt < \infty$  for every  $a \geq 0$ .
- (b) For every  $a \geq 0$ ,

$$Y_t = Y_{a \wedge \vartheta} + \int_{t \wedge \vartheta}^{a \wedge \vartheta} f(r, Y_r) dr + \int_{t \wedge \vartheta}^{a \wedge \vartheta} dK_r - \int_{t \wedge \vartheta}^{a \wedge \vartheta} dM_r, \quad t \in [0, a], \tag{2.3}$$

- (c)  $Y_t \geq L_t, t \in [0, a \wedge \vartheta]$  and  $\int_0^{a \wedge \vartheta} (Y_{t-} - L_{t-}) dK_t = 0$  for every  $a \geq 0$ ,
- (d)  $Y_{a \wedge \vartheta} \rightarrow \xi$  as  $a \rightarrow \infty$ .

Let  $\eta$  be a strictly positive  $\mathbb{F}$ -progressively measurable process such that  $\eta \leq 1$  and

$$E \int_0^\vartheta \eta_t (S_t - L_t)^- dt < \infty, \tag{2.4}$$

where  $S$  is the process appearing in (A1). As an example of such  $\eta$  can serve any deterministic strictly positive bounded by 1 process such that  $\int_0^\infty \eta_t dt < \infty$ . Since  $S$  and  $L$  are assumed to be of class (D),  $\eta$  satisfies (2.4). We let

$$f_n(t, y) = f(t, y) + n\eta_t(y - L_t)^-, \quad t \geq 0, y \in \mathbb{R}. \tag{2.5}$$

**Theorem 2.5.** Assume that (A1)–(A5) are satisfied. Then

- (i) There exists a unique solution  $(Y, M, K)$  to RBSDE $^\vartheta(\xi, f, L)$ .
- (ii) Let  $\xi$  be an  $\mathcal{F}_\vartheta$ -measurable integrable random variable. Then for every  $n \geq 1$  there exists a unique solution  $(Y^n, M^n)$  of BSDE $^\vartheta(\xi, f_n)$  with  $f_n$  defined by (2.5), and  $Y^n \nearrow Y$   $P$ -a.s. as  $n \rightarrow \infty$ .
- (iii) If we assume additionally that (A6) is satisfied, then  $M$  is a uniformly integrable martingale,  $EK_\vartheta < \infty$  and (2.2) is satisfied.

*Proof:* See Klimsiak (2021a, Proposition A.16) and Klimsiak (2019, Theorem 3.9). □

For  $\varepsilon > 0$ , we set

$$\sigma_\varepsilon = \inf\{t \geq \alpha : Y_t \leq L_t + \varepsilon\} \wedge \vartheta. \tag{2.6}$$

**Theorem 2.6.** Assume that assumptions (A1)–(A5) are satisfied. Then  $(Y, M, K)$  is a solution of RBSDE $^\vartheta(\xi, f, L)$  if and only if for every  $\alpha \in \mathcal{T}^\vartheta$ ,

$$Y_\alpha = \text{ess sup}_{\sigma \in \mathcal{T}^\vartheta} \mathcal{E}_{\alpha, \sigma}^f(L_\sigma \mathbf{1}_{\{\sigma < \vartheta\}} + \xi \mathbf{1}_{\{\sigma = \vartheta\}}). \tag{2.7}$$

*Proof:* Let  $\sigma \in \mathcal{T}^\vartheta$  and  $\{\delta_n\}$  be a fundamental sequence for the local martingale  $M$  on  $[\alpha, \infty)$ . From the minimality condition we deduce that  $(Y, M)$  is a solution of BSDE $^{\alpha, \sigma_\varepsilon}(Y_{\sigma_\varepsilon}, f)$ . By Proposition 2.3(i),  $Y$  is an  $\mathcal{E}^f$ -martingale on  $[\alpha, \sigma_\varepsilon]$ . Hence

$$Y_\alpha = \mathcal{E}_{\alpha, \sigma_\varepsilon}^f(Y_{\sigma_\varepsilon}).$$

On the other hand, by the definition of  $\sigma_\varepsilon$  and Proposition 2.3(ii), it follows from the above equality that

$$Y_\alpha \leq \mathcal{E}_{\alpha, \sigma_\varepsilon}^f(L_{\sigma_\varepsilon} \mathbf{1}_{\{\sigma_\varepsilon < \vartheta\}} + \varepsilon + \xi \mathbf{1}_{\{\sigma_\varepsilon = \vartheta\}}).$$

From this and Proposition 2.3(iii) we get

$$Y_\alpha \leq \mathcal{E}_{\alpha, \sigma_\varepsilon}^f(L_{\sigma_\varepsilon} \mathbf{1}_{\{\sigma_\varepsilon < \vartheta\}} + \xi \mathbf{1}_{\{\sigma_\varepsilon = \vartheta\}}) + \varepsilon, \tag{2.8}$$

from which one can easily deduce that (2.7) is satisfied. To prove the sufficiency part, we denote by  $\bar{Y}_\alpha$  the right-hand side of (2.7). By Klimsiak (2019, Theorem 3.9), there exists a unique solution

$(Y, M, K)$  of  $\text{RBSDE}^\vartheta(\xi, f, L)$ . By the necessity part in (i),  $\bar{Y}_\alpha = Y_\alpha$ ,  $\alpha \in \mathcal{T}^\vartheta$ , so  $(\bar{Y}, M, K)$  is a solution of  $\text{RBSDE}^\vartheta(\xi, f, L)$ .  $\square$

### 3. Stability results for solutions of RBSDEs

In this section, we prove stability results for solutions of RBSDEs. For given  $\alpha, \beta \in \mathcal{T}^\vartheta$  such that  $\alpha \leq \beta$  we set

$$\|Y\|_{1,\alpha,\beta} = \sup_{\alpha \leq \tau \leq \beta, \tau < \infty} E|Y_\tau|, \quad \|Y\|_{1,\beta} = \|Y\|_{1,0,\beta}. \tag{3.1}$$

In what follows,  $L^1$  and  $L^2$  are càdlàg adapted processes of class (D).

**Proposition 3.1.** *Assume that  $\xi^1, \xi^2$  are  $\mathcal{F}_\beta$ -measurable and  $E|\xi^1| + E|\xi^2| < \infty$ . Let  $(Y^i, M^i, K^i)$  be a solution of  $\text{RBSDE}^\vartheta(\xi^i, f^i, L^i)$ ,  $i = 1, 2$ , and  $f^1$  satisfy (A2). Then*

$$\|Y^1 - Y^2\|_{1;\alpha} \leq E|\xi^1 - \xi^2| + E \int_0^\vartheta |f^1(t, Y_t^2) - f^2(t, Y_t^2)| dt + \|L^1 - L^2\|_{1;\alpha}.$$

*Proof:* See [Klimsiak \(2021a, Corollary 3.15, Remark 4.2\)](#).  $\square$

*Remark 3.2.* One can get the above result by using the representation (2.7) and properties of nonlinear expectation. However, to apply this second method one has to impose much stronger conditions on  $f^1, f^2$ . This is due to the fact that the nonlinear expectations  $\mathcal{E}^{f^1}$  and  $\mathcal{E}^{f^2}$  are well defined under (A1)–(A4).

For a finite variation process  $C$ , we denote by  $|C|_t$  its total variation on the interval  $[0, t]$ .

**Theorem 3.3.** *Let  $\alpha, \beta \in \mathcal{T}^\vartheta$  be such that  $\alpha \leq \beta$  and let  $\xi^1 \in \mathcal{F}_\alpha, \xi^2 \in \mathcal{F}_\beta$  satisfy  $E|\xi^1| + E|\xi^2| < \infty$ . Suppose that  $(Y^1, M^1, K^1)$  is a solution of  $\text{RBSDE}^\alpha(\xi^1, f, L^1)$  and  $(Y^2, M^2, K^2)$  is a solution of  $\text{RBSDE}^\beta(\xi^2, f, L^2)$  with some  $f$  satisfying (A2). Then*

- (i)  $\|Y^1 - Y^2\|_{1;\alpha} \leq E|\xi^1 - Y_\alpha^2| + \|L^1 - L^2\|_{1;\alpha}$ .
- (ii) *Set  $\tilde{Y}_t^1 = Y_t^1 \mathbf{1}_{[0,\alpha)}(t) + H_t \mathbf{1}_{[\alpha,\beta]}(t) \mathbf{1}_{\{\alpha < \infty\}}$ , where  $H$  is a càdlàg process which is a difference of two supermartingales of class (D) on  $[0, \beta]$  with  $\lim_{a \rightarrow \infty} H_{\beta \wedge a} = H_\beta$ . Let  $H_t = H_0 + C_t + N_t$  be the Doob–Meyer decomposition of  $H$  ( $C$  is a predictable finite variation process with  $C_0 = 0$ , and  $N$  is a martingale with  $N_0 = 0$ ). Then*

$$\begin{aligned} \|\tilde{Y}^1 - Y^2\|_{1;\beta} &\leq E|\xi^1 - \xi^2| \mathbf{1}_{\{\alpha = \infty\}} + E|\xi^1 - H_\alpha| + E|\xi^2 - H_\beta| \\ &\quad + E \int_\alpha^\beta |f(t, H_t)| dt + E \int_\alpha^\beta d|C|_t + \|H - L^2\|_{1;\alpha,\beta} + \|L^1 - L^2\|_{1;\alpha}. \end{aligned} \tag{3.2}$$

*Proof:* Observe that  $(Y^2, M^2, K^2)$  is a solution of  $\text{RBSDE}^\alpha(Y_\alpha^2, f, L^2)$ . Therefore (i) follows immediately from Proposition 3.1. To prove (ii), set

$$\begin{aligned} N_t^\alpha &= N_{t \vee \alpha} - N_\alpha, & C_t^\alpha &= C_{t \vee \alpha} - C_\alpha, \\ \tilde{K}_t^1 &= K_{t \wedge \alpha}^1, & \tilde{M}_t^1 &= M_{t \wedge \alpha}^1 + N_t^\alpha, & V_t^1 &= (\xi^1 - H_\alpha) \mathbf{1}_{[\alpha,\beta]}(t) \mathbf{1}_{\{\alpha < \infty\}}, \quad t \in [0, \beta], \end{aligned}$$

and

$$\tilde{f}(t, y) := f(t, y) \mathbf{1}_{[0,\alpha]}(t), \quad \tilde{L}_t^1 = L_t^1 \mathbf{1}_{[0,\alpha)}(t) + H_t \mathbf{1}_{[\alpha,\beta]}(t) \mathbf{1}_{\{\alpha < \infty\}}. \quad t \in [0, \beta], y \in \mathbb{R}.$$

Then  $(\tilde{Y}^1, \tilde{M}^1, \tilde{K}^1)$  is a solution of  $\text{RBSDE}^\beta(\xi^1 \mathbf{1}_{\{\alpha = \infty\}} + H_\beta \mathbf{1}_{\{\alpha < \infty\}}, \tilde{f} + dV^1 + dC^\alpha, \tilde{L}^1)$ . Hence, by Proposition 3.1,

$$\begin{aligned} \|\tilde{Y}^1 - Y^2\|_{1;\beta} &\leq E|\xi^1 \mathbf{1}_{\{\alpha = \infty\}} + H_\beta \mathbf{1}_{\{\alpha < \infty\}} - \xi^2| + E \int_0^\beta d|V^1|_t + E \int_0^\beta d|C^\alpha|_t \\ &\quad + E \int_0^\beta |\tilde{f}(t, \tilde{Y}_t^1) - f(t, \tilde{Y}_t^1)| dt + \|\tilde{L}^1 - L^2\|_{1;\beta}, \end{aligned}$$

which leads to (3.2). □

*Remark 3.4.* (i) The basic difference between assertions (i) and (ii) is that unlike (i), the right-hand side of the estimate in (ii) does not depend on the solution. This allows one to provide some results on the rate of convergence.

(ii) At first glance the presence of a process  $H$  on the right-hand side of the inequality in (ii) is puzzling. We shall see later on that in order to get the rate of convergence in (1.5) it is necessary to use (ii) with a suitable process  $H$  depending on the model. In most cases  $H$  cannot be taken to be zero.

#### 4. RBSDEs and value functions of stopping problems

In what follows,  $\mathbb{M} = (X = (X_t)_{t \geq 0}, (\theta_t)_{t \geq 0}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, (P_x)_{x \in E})$  is a Borel right Markov process with state space  $E$  (augmented by a cemetery state  $\partial$ ), shift operators  $(\theta_t)_{t \geq 0}$  and life time  $\zeta$ , defined on some measurable space  $(\Omega, \mathcal{F})$  (see, e.g., [Sharpe \(1988\)](#)). We shall use the symbol  $\mathbb{E}_x$  to denote the expectation with respect to  $P_x$ . We adopt the convention that  $X_\infty = \partial$ . We also adopt the convention that every function  $g$  on  $E$  (resp.  $E \times \mathbb{R}$ ) is extended to  $E \cup \{\partial\}$  (resp.  $(E \cup \{\partial\}) \times \mathbb{R}$ ) by setting  $g(\partial) = 0$  (resp.  $g(\partial, y) = 0, y \in \mathbb{R}$ ).

Let  $D$  be an open subset of  $E$  and  $(P_t^D)_{t > 0}$  be the semigroup associated with the process  $\mathbb{M}$  killed when exiting  $D$ , that is

$$P_t^D \rho(x) = \mathbb{E}_x(\rho(X_t) \mathbf{1}_{\{t < \tau_D\}}), \quad t \geq 0, \quad x \in D,$$

and let  $R^D$  denote the potential operator, that is

$$R^D \rho(x) = \mathbb{E}_x \int_0^{\tau_D} \rho(X_t) dt, \quad x \in D$$

for any positive Borel  $\rho : E \rightarrow \mathbb{R}$ . We assume that  $(P_t^D)$  is transient, i.e. there exists a strictly positive function  $\bar{\rho}$  such that  $R^D \bar{\rho}$  is finite. By the strong Markov property, for all positive Borel functions  $\rho : E \rightarrow \mathbb{R}$  and  $t > 0$  we have

$$\mathbb{E}_x \int_{t \wedge \tau_D}^{\tau_D} \rho(X_s) ds = P_t^D R^D \rho(x), \quad x \in D. \tag{4.1}$$

For given Borel measurable functions  $\varphi : E \rightarrow \mathbb{R}$ ,  $\psi : D^c \rightarrow \mathbb{R}$  and  $g : E \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $h : E \rightarrow \mathbb{R}$  we set

$$\xi^{T,s} = \varphi(X_{T_s}) \mathbf{1}_{\{\tau_D > T_s\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\tau_D \leq T_s\}}, \quad \xi = \psi(X_{\tau_D})$$

and

$$f(\omega, t, y) = g(X_t(\omega), y), \quad t \geq 0, y \in \mathbb{R}, \quad L_t = h(X_t), \quad t \geq 0.$$

We will need the following assumptions:

(H) For any  $x \in D$  the function  $y \mapsto g(x, y)$  is continuous and nonincreasing

and, in the stationary case,

(S1)  $\mathbb{E}_x |\psi(X_{\tau_D})| + \mathbb{E}_x \int_0^{\tau_D} |g(X_t, 0)| dt < \infty$ ,

(S2) For every  $y \in \mathbb{R}$ ,  $\int_0^{\tau_D} |g(X_t, y)| dt < \infty$   $P_x$ -a.s.

(S3)  $L$  is a càdlàg process of class (D) under  $P_x$  such that  $\lim_{a \rightarrow \infty} h(X_{a \wedge \tau_D}) = \psi(X_{\tau_D})$   $P_x$ -a.s.

The counterparts to (S1)–(S3) in the evolutionary case are as follows:

(E1)  $\mathbb{E}_x |\xi^{T,s}| + \mathbb{E}_x \int_0^{T_s \wedge \tau_D} |g(X_t, 0)| dt < \infty$ ,

(E2) For every  $y \in \mathbb{R}$ ,  $\int_0^{T_s \wedge \tau_D} |g(X_t, y)| dt < \infty$   $P_x$ -a.s.,

(E3)  $L$  is a càdlàg process on  $[0, T_s \wedge \tau_D]$  of class (D) under the measure  $P_x$  and  $h(X_{T_s \wedge \tau_D}) \leq \xi^{T,s}$   $P_x$ -a.s.

If (H) and (S1)–(S3) are satisfied, then by Theorem 2.5 there exists a unique solution  $(Y^x, M^x, K^x)$  of RBSDE $^{\tau_D}(\xi, f, L)$  under the measure  $P_x$ . Moreover, from Theorem 2.5 we conclude that if (H) and (E1)–(E3) are satisfied, then there exists a unique solution  $(Y^{T,s,x}, M^{T,s,x}, K^{T,s,x})$  of RBSDE $^{T_s \wedge \tau_D}(\xi^{T,s}, f, L)$ . We are going to show that

$$V(X) = Y^x, \quad V_T(s + \cdot, X) = Y^{T,s,x} \quad P_x\text{-a.s.}, \tag{4.2}$$

where  $V_T$  is defined by (1.1) and  $V$  is defined by (1.2). This together with the stability results for RBSDEs proved in Theorem 3.3 yields the main result of the paper. First, however, we shall prove a weaker result.

**Proposition 4.1.** *Let  $x \in D$  and  $s \geq 0$ . Assume that (H) and (S1)–(S3) are satisfied, and for every  $T > 0$ , (E3) is satisfied and  $\mathbb{E}_x|\varphi(X_T)| < \infty$ .*

(i) *If*

$$\lim_{a \rightarrow \infty} \mathbb{E}_x(|\varphi(X_a)|\mathbf{1}_{\{a < \tau_D\}}) = 0, \tag{4.3}$$

*then  $\lim_{T_s \rightarrow \infty} V_T(s, x) = V(x)$ .*

(ii) *Let  $\gamma$  be a Borel function such that  $\gamma(X)$  is a difference of two supermartingales of class (D) on  $[0, \tau_D]$  and  $\gamma(X_{\tau_D}) = \psi(X_{\tau_D})$ . Then for any  $0 \leq s \leq T$ ,*

$$\begin{aligned} |V_T(s, x) - V(x)| \leq & \mathbb{E}_x(|\gamma - \varphi|(X_{T_s})\mathbf{1}_{\{\tau_D > T_s\}}) + \mathbb{E}_x \int_{T_s \wedge \tau_D}^{\tau_D} |g(X_t, \gamma(X_t))| dt \\ & + \mathbb{E}_x \int_{T_s \wedge \tau_D}^{\tau_D} d|C^x|_t + \sup_{T_s \wedge \tau_D \leq \tau \leq \tau_D} \mathbb{E}_x|\gamma(X_\tau) - h(X_\tau)|, \end{aligned} \tag{4.4}$$

*where  $C^x$  is the predictable finite variation part of the Doob-Meyer decomposition of  $\gamma(X)$  with  $C_0^x = 0$ .*

*Proof:* By Theorem 2.6,  $V(x) = \mathbb{E}_x Y_0^x$  and  $V_T(s, x) = \mathbb{E}_x Y_0^{T,s,x}$ . Therefore, by Theorem 3.3(i) applied to  $Y^1 = Y^{T,s,x}$ ,  $Y^2 = Y^x$  and  $\alpha = T_s \wedge \tau_D$ ,  $\beta = \tau_D$ , we have

$$\begin{aligned} |V_T(s, x) - V(x)| & \leq \mathbb{E}_x|\xi^{T,s} - Y_{T_s \wedge \tau_D}^x| \\ & = \mathbb{E}_x|\varphi(X_{T_s})\mathbf{1}_{\{\tau_D > T_s\}} + \psi(X_{\tau_D})\mathbf{1}_{\{\tau_D \leq T_s\}} - Y_{T_s \wedge \tau_D}^x|. \end{aligned}$$

Since  $Y^x$  is of class (D) under the measure  $P_x$ , and  $Y_{T_s \wedge \tau_D}^x \rightarrow \xi = \psi(X_{\tau_D}) = \psi(X_{\tau_D})\mathbf{1}_{\{\tau_D < \infty\}}$   $P_x$ -a.s. as  $T_s \rightarrow \infty$ , then  $\mathbb{E}_x|\psi(X_{\tau_D})\mathbf{1}_{\{\tau_D \leq T_s\}} - Y_{T_s \wedge \tau_D}^x| \rightarrow 0$  as  $T_s \rightarrow \infty$ . From this and (4.3) we obtain at once that the right-hand side of the above inequality converges to zero as  $T_s \rightarrow \infty$ . This proves (i). Part (ii) follows from Theorem 3.3(ii) with  $H = \gamma(X)$  and  $Y^1, Y^2, \alpha, \beta$  as above.  $\square$

Recall that  $D$  is called Dirichlet regular if  $P_x(\tau_D > 0) = 0$  for all  $x \in \partial D$ .

**Lemma 4.2.** *Let  $\gamma$  be a positive Borel function on  $E$  and  $v(x) = \mathbb{E}_x \gamma(X_{\tau_D})$ ,  $x \in E$ . Then for every  $\alpha \in \mathcal{T}^{\tau_D}$ ,*

$$\mathbf{1}_{\{\alpha < \tau_D\}}v(X_\alpha) = \mathbf{1}_{\{\alpha < \tau_D\}}\mathbb{E}_x(\gamma(X_{\tau_D})|\mathcal{F}_\alpha), \quad P_x\text{-a.s.}, \quad x \in E.$$

*Moreover, if  $D$  is Dirichlet regular, then for every  $\alpha \in \mathcal{T}^{\tau_D}$  we have*

$$v(X_\alpha) = \mathbb{E}_x(\gamma(X_{\tau_D})|\mathcal{F}_\alpha), \quad P_x\text{-a.s.}, \quad x \in E.$$

*Proof:* Let  $A = \{\alpha < \tau_D\}$ ,  $B = \{\alpha = \tau_D\}$ . All the following equations hold  $P_x$ -a.s. for  $x \in E$ . By the strong Markov property,

$$\begin{aligned} v(X_\alpha) & = E_{X_\alpha} \gamma(X_{\tau_D}) = \mathbb{E}_x(\gamma(X_{\tau_D} \circ \theta_\alpha)|\mathcal{F}_\alpha) \\ & = \mathbb{E}_x(\mathbf{1}_A \gamma(X_{\tau_D} \circ \theta_\alpha)|\mathcal{F}_\alpha) + \mathbb{E}_x(\mathbf{1}_B \gamma(X_{\tau_D} \circ \theta_\alpha)|\mathcal{F}_\alpha). \end{aligned} \tag{4.5}$$

On the set  $A$  we have  $\tau_D \circ \theta_\alpha = \tau_D - \theta_\alpha$ , so  $\mathbf{1}_A \gamma(X_{\tau_D} \circ \theta_\alpha) = \mathbf{1}_A \gamma(X_{\tau_D})$ . Therefore (4.5) implies the first assertion. To prove the second one, it suffices now to observe that by the Dirichlet regularity of  $D$ ,  $\mathbf{1}_B \gamma(X_{\tau_D} \circ \theta_\alpha) = \mathbf{1}_B \gamma(X_{\tau_D})$ .  $\square$

**Corollary 4.3.** *Let assumption (H) hold and (S1)–(S2) be satisfied for every  $x \in D$ . Then for every  $x \in D$  there exists a unique solution  $(Y^x, M^x)$  of  $\text{BSDE}^{\tau_D}(\xi, f)$ . Furthermore, the function  $u(x) := \mathbb{E}_x Y_0^x$  is Borel measurable, and*

$$Y_t^x = u(X_t), \quad t < \tau_D, \quad x \in D. \tag{4.6}$$

If  $D$  is Dirichlet regular, then the above equation holds for all  $t \leq \tau_D$  and  $x \in D$ .

*Proof:* Let  $v$  be as in Lemma 4.2. By Lemma 4.2, and a simple calculation, we have  $Y^x = \bar{Y}^x + v(X)$ , where  $(\bar{Y}^x, \bar{M}^x)$  is a solution of  $\text{BSDE}^{\tau_D}(0, f_v)$  with

$$f_v(t, y) = f(t, y + v(X_t)).$$

By Klimsiak and Rozkosz (2013, Theorem 4.7), there exists a Borel function  $\bar{u}$  such that  $\bar{Y}^x = \bar{u}(X)$ ,  $x \in E$ . Thus, we have (4.6) with  $u = \bar{u} + v$ .  $\square$

In the next theorem, we give a precise meaning of (4.2) and give conditions ensuring that it is satisfied.

**Theorem 4.4.** *Let assumption (H) hold.*

- (i) *If (S1)–(S3) are satisfied for all  $x \in D$ , then  $V(X_t) = Y_t^x$ ,  $t \in [0, \tau_D)$ ,  $P_x$ -a.s. for  $x \in D$ .*
- (ii) *If (E1)–(E3) are satisfied for all  $x \in D$  and  $s \in [0, T)$ , then  $V_T(s + t, X_t) = Y_t^{T, s, x}$ ,  $t \in [0, T_s \wedge \tau_D)$ ,  $P_x$ -a.s. for  $(s, x) \in [0, T) \times D$ .*

Moreover, if  $D$  is Dirichlet regular, then the assertions of (i) and (ii) hold on the random intervals  $[0, \tau_D]$  and  $[0, T_s \wedge \tau_D]$ , respectively.

*Proof:* (i) Let  $f_n(t, y) = f(t, y) + n\rho(X_t)(y - L_t)^-$ ,  $n \geq 1$  with  $\rho$  being a strictly positive bounded Borel function such that  $R^D \rho$  is bounded (it exists since we assumed that  $(P_t^D)$  is transient). By Theorem 2.5, for every  $x \in D$ ,

$$Y_t^{n, x} \nearrow Y_t^x, \quad t \in [0, \tau_D], \quad P_x\text{-a.s.},$$

where  $(Y^{n, x}, M^{n, x}, K^{n, x})$  is the unique solution of  $\text{BSDE}^{\tau_D}(\xi, f_n)$  under the measure  $P_x$ . By Corollary 4.3, there exists a Borel function  $u_n$  such that  $u_n(X_t) = Y_t^{n, x}$ ,  $t \in [0, \tau_D]$  (for  $t \in [0, \tau_D]$  in case  $D$  is Dirichlet regular)  $P_x$ -a.s. Clearly  $u_n(x) = E_x Y_0^{n, x} \nearrow E_x Y_0^x =: u(x)$ ,  $x \in D$ . Thus  $u(X_t) = Y_t^x$ ,  $t \in [0, \tau_D]$  (for  $t \in [0, \tau_D]$  in case  $D$  is Dirichlet regular)  $P_x$ -a.s. By Theorem 2.6,  $u = V$  on  $D$ .

(ii) Let  $v$  be the uniform motion to the right, that is  $v(0) = s$  and  $v(s) = s + t$ ,  $t \geq 0$ , under the measure  $P_{s, x}$ . Set  $\mathcal{X}_t = (v(t), X_{v(t)})$ . Then  $\{(\mathcal{X}, P_{s, x}), (s, x) \in \mathbb{R}_+ \times D\}$ , where  $P_{s, x}(\mathcal{X}_t \in A) = P_x((t + s, X_t) \in A)$  for any Borel subset of  $\mathbb{R}_+ \times D$ , is a Markov process with state space  $\mathbb{R}_+ \times D$  (see, e.g., Wentzell (1981, Section 8.5.5)). Set  $\hat{D} = [0, T) \times D$ ,  $\tau_{\hat{D}} = \inf\{t > 0 : \mathcal{X}_t \notin \hat{D}\}$  and  $\hat{\xi}^T = \hat{\psi}(\mathcal{X}_{\tau_{\hat{D}}})$ , where

$$\hat{\psi}(t, x) = \varphi(x)\mathbf{1}_{\{t=T, x \in D\}} + \psi(x)\mathbf{1}_{\{t < T, x \notin D\}}.$$

We also set  $\hat{f}(t, y) = g(\Pi(\mathcal{X}_t), y)$ ,  $\hat{L}_t = h(\Pi(\mathcal{X}_t))$ , where  $\Pi$  denotes the canonical projection on  $E$ . By Theorem 2.5, for every  $(s, x) \in \hat{D}$  there exists a unique solution  $(\hat{Y}^{s, x}, \hat{M}^{s, x}, \hat{K}^{s, x})$  of  $\text{RBSDE}^{\tau_{\hat{D}}}(\hat{\xi}^T, \hat{f}, \hat{L})$  under  $P_{s, x}$ . Moreover,

$$\hat{Y}_t^{n, s, x} \nearrow \hat{Y}_t^{s, x}, \quad t \in [0, \tau_{\hat{D}}],$$

where  $(\hat{Y}^{n, s, x}, \hat{M}^{n, s, x})$  is the unique solution to  $\text{BSDE}^{\tau_{\hat{D}}}(\hat{\xi}^T, \hat{f}_n)$  with  $\hat{f}_n(t, y) = \hat{f}(t, y) + n(y - \hat{L}_t)^-$ . By Corollary 4.3, there exists a Borel function  $u_n$  on  $\hat{D}$  such that  $u_n(\mathcal{X}_t) = \hat{Y}_t^{n, s, x}$ ,  $t \in [0, \tau_{\hat{D}}]$ ,  $P_{s, x}$ -a.s. (for  $t \in [0, \tau_{\hat{D}}]$  in case  $D$  is Dirichlet regular). Clearly  $u_n(s, x) = \mathbb{E}_{s, x} \hat{Y}_0^{n, s, x}$ . However, by

the relation between  $P_x$  and  $P_{s,x}$ , we have

$$\begin{aligned} \mathbb{E}_x u_n(s, X_0) = u_n(s, x) &= \mathbb{E}_{s,x} \hat{Y}_0^{n,s,x} = \mathbb{E}_x (\varphi(X_{T-s}) \mathbf{1}_{\{T_s \wedge \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\tau_D < T_s\}}) \\ &+ \mathbb{E}_x \int_0^{T_s \wedge \tau_D} g(X_t, u_n(s+t, X_t)) dt \\ &+ n \mathbb{E}_x \int_0^{T_s \wedge \tau_D} (u_n(s+t, X_t) - h(X_t))^- dt. \end{aligned}$$

Set  $f_n(t, y) = g(X_t, y) + n(y - h(X_t))^-$ . Using the strong Markov property of  $X$  we deduce that  $u_n(s+t, X_t) = Y_t^{n,T,s,x}$ ,  $t \in [0, T_s \wedge \tau_D]$  (for  $t \in [0, T_s \wedge \tau_D]$  in case of Dirichlet regular  $D$ ), where  $(Y^{n,T,s,x}, M^{n,T,s,x})$  is the unique solution of BSDE $^{T_s \wedge \tau_D}(\xi^{T,s}, f_n)$ . By Theorem 2.5,  $Y^{n,T,s,x} \nearrow Y^{T,s,x}$   $P_x$ -a.s. In particular,  $u_n(s, x) = \mathbb{E}_x Y_0^{n,T,s,x} \nearrow \mathbb{E}_x Y_0^{T,s,x} =: u(s, x)$ . Therefore  $u(s+t, X_t) = Y_t^{T,s,x}$ ,  $t \in [0, T_s \wedge \tau_D]$  (for  $t \in [0, T_s \wedge \tau_D]$  in case of Dirichlet regular  $D$ ),  $P_x$ -a.s. On the other hand, by Theorem 2.6,  $u = V_T$  on  $\hat{D}$ .  $\square$

*Remark 4.5.* Let (H) hold and (S1)–(S3) be satisfied for every  $x \in E$ . Let  $Y^x$  be the first component of the solution of RBSDE $^{\tau_D}(\xi, f, L)$ . By Theorem 4.4(i),  $Y^x = V(X)$ , so from Klimsiak (2019, Remark 3.6) it follows that for any stopping time  $\alpha \leq \tau_D$ ,

$$V(X_\alpha) = \text{ess sup}_{\sigma \geq \alpha} \mathbb{E}_x \left( \int_\alpha^{\sigma \wedge \tau_D} g(X_r, V(X_r)) dr + h(X_\sigma) \mathbf{1}_{\{\sigma < \tau_D\}} + \psi(X_{\tau_D}) \mathbf{1}_{\{\sigma \wedge \tau_D = \tau_D\}} \mid \mathcal{F}_\alpha \right).$$

Letting  $\alpha = 0$  we get (1.4). Similarly, if (H), (S1)–(S3) are satisfied, then from Theorem 4.4(ii) we get (1.3). Note also that from Klimsiak and Rzymowski (2021, Proposition 4.3, Proposition 5.1) it follows that  $V : D \rightarrow \mathbb{R}$  satisfying (1.4) is unique and  $V_T : [0, T] \times D \rightarrow \mathbb{R}$  satisfying (1.3) is unique.

From Theorem 3.3 and Theorem 4.4 we get the dynamic version of (4.4). It is one of the main results of our paper.

**Theorem 4.6.** *Assume that (H) hold and for every  $x \in D$  assumptions (S1)–(S3) are satisfied and  $\mathbb{E}_x |\varphi(X_T)| < \infty$ ,  $T > 0$ . Moreover, assume that (E3) is satisfied for every  $(s, x) \in [0, T] \times D$ .*

(i) *If (4.3) is satisfied, then*

$$\lim_{T_s \rightarrow \infty} \sup_{\alpha \in \mathcal{T}^{T_s \wedge \tau_D}} \mathbb{E}_x [\mathbf{1}_{\{\alpha < T_s \wedge \tau_D\}} |V_T(s + \alpha, X_\alpha) - V(X_\alpha)|] = 0.$$

*If  $D$  is Dirichlet regular, then the above convergence holds true without the factor  $\mathbf{1}_{\{\alpha < T_s \wedge \tau_D\}}$ .*

(ii) *Let  $\gamma, C^x$  be as in Proposition 4.1. For every  $(s, x) \in [0, T] \times D$ ,*

$$\begin{aligned} &\sup_{\alpha \in \mathcal{T}^{T_s \wedge \tau_D}} \mathbb{E}_x [\mathbf{1}_{\{\alpha < T_s \wedge \tau_D\}} |V_T(s + \alpha, X_\alpha) - V(X_\alpha)|] \\ &\leq \mathbb{E}_x |\gamma - \varphi|(X_{T_s}) \mathbf{1}_{\{\tau_D > T_s\}} + \mathbb{E}_x \int_{T_s \wedge \tau_D}^{\tau_D} |g(X_t, \gamma(X_t))| dt \\ &+ \mathbb{E}_x \int_{T_s \wedge \tau_D}^{\tau_D} d|C^x|_t + \sup_{\tau \in \mathcal{T}_{T_s \wedge \tau_D}^{\tau_D}} \mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)|. \end{aligned} \tag{4.7}$$

*If  $D$  is Dirichlet regular, then the above estimate holds true without the factor  $\mathbf{1}_{\{\alpha < T_s \wedge \tau_D\}}$  on the left-hand side of (4.7).*

*Proof:* Follows from Theorem 4.4 and Theorem 3.3 applied to  $Y^1 = Y^{T,s,x}$ ,  $Y^2 = Y^x$ ,  $H = \gamma(X)$  and  $\alpha = T_s \wedge \tau_D$ ,  $\beta = \tau_D$  (see the proof of Proposition 4.1).  $\square$

*Remark 4.7.* It is well known (see, e.g., Klimsiak et al. (2020, Remark 2.1)) that for every  $q \in (0, 1)$ ,

$$\mathbb{E}_x \sup_{t \leq T_s \wedge \tau_D} |V_T(s+t, X_t) - V(X_t)|^q \leq \frac{1}{1-q} \left( \sup_{\alpha \in \mathcal{T}^{T_s \wedge \tau_D}} \mathbb{E}_x |V_T(s+\alpha, X_\alpha) - V(X_\alpha)| \right)^q.$$

This together with (4.7) yields the rate of convergence of the value function in the supremum norm.

### 5. Rate of convergence

Let

$$V_{T_s}^*(x) = \sup_{\alpha \in \mathcal{T}^{T_s \wedge \tau_D}} \mathbb{E}_x |V_T(s + \alpha, X_\alpha) - V(X_\alpha)|.$$

The aim of this section is to provide the rate of convergence of  $V_{T_s}^*$  as  $T_s \rightarrow \infty$ . To this end, we shall estimate the right-hand side of (4.7). We begin with some general remarks. Then we discuss in more detail some specific situations.

Throughout this section, we assume that (S1)–(S3) and (E1)–(E3) are satisfied for all  $x \in E$  and  $s \in [0, T]$ ,  $T \geq 0$ . Moreover, we assume that  $D$  is Dirichlet regular and  $\gamma$  is of the form

$$\gamma(x) = \mathbb{E}_x \psi(X_{\tau_D}), \quad x \in D.$$

Under the measure  $P_x$  the process  $\gamma(X)$  is a martingale on  $[0, \tau_D]$  (see the argument in the proof of Lemma 4.2), so  $C^x = 0$ . Therefore, by Theorem 4.6,

$$V_{T_s}^*(x) \leq \mathbb{E}_x |\gamma - \varphi|(X_{T_s}) \mathbf{1}_{\{\tau_D \geq T_s\}} + \mathbb{E}_x \int_{T_s \wedge \tau_D}^{\tau_D} |g(X_t, \gamma(X_t))| dt + \sup_{\tau \in \mathcal{T}_{T_s \wedge \tau_D}^{\tau_D}} \mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)|.$$

This together with (4.1) gives

$$V_{T_s}^*(x) \leq P_{T_s}^D |\gamma - \varphi|(x) + P_{T_s}^D R^D(|g(\cdot, \gamma)|)(x) + \sup_{\tau \in \mathcal{T}_{T_s \wedge \tau_D}^{\tau_D}} \mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)|. \tag{5.1}$$

Define  $w : D \rightarrow \mathbb{R}$  by

$$w(x) := |\gamma - \varphi|(x) + R^D(|g(\cdot, \gamma)|)(x) \tag{5.2}$$

and  $\hat{h} : E \times \mathcal{T} \rightarrow \mathbb{R}$  by

$$\hat{h}(x, \tau) := \mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)| = \mathbb{E}_x |\mathbb{E}_x(h(X_{\tau_D}) | \mathcal{F}_\tau) - h(X_\tau)|. \tag{5.3}$$

With this notation (5.1) can be rewritten in the form

$$V_{T_s}^*(x) \leq P_{T_s}^D w(x) + \sup_{\tau \in \mathcal{T}_{\tau_D \wedge T_s}^{\tau_D}} \hat{h}(x, \tau). \tag{5.4}$$

The rate of convergence of the first term on the right-hand side of (5.4) depends on the rate of decay of the semigroup  $(P_t^D)$  as  $t \rightarrow \infty$ . There are various results in the literature concerning this issue and we shall indicate a few of them. However, it is by no means clear how to control the rate of decay of the second term on the right-hand side of (5.4).

5.1. *General Markov processes.* Set

$$\hat{h}_\infty(x) := \sup_{\tau \in \mathcal{T}^{\tau_D}} \hat{h}(x, \tau), \quad x \in D.$$

**Theorem 5.1.** *For all  $0 \leq s \leq T$  and  $x \in D$ ,*

$$V_{T_s}^*(x) \leq P_{T_s}^D w(x) + P_{T_s}^D \hat{h}_\infty(x). \tag{5.5}$$

*Proof:* Let  $Y^x$  be the first component of the solution of RBSDE $^{\tau_D}(0, 0, |\gamma - h|(X))$  under the measure  $P_x^D$ . By Theorem 4.4 and Theorem 2.6,

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{T_s \wedge \tau_D}^{\tau_D}} \mathbb{E}_x^D (|\gamma(X_\tau) - h(X_\tau)| | \mathcal{F}_{T_s \wedge \tau_D}) = Y_{T_s \wedge \tau_D}^x = u(X_{T_s \wedge \tau_D}) \quad P_x^D\text{-a.s.}, \tag{5.6}$$

where  $\mathbb{E}_x^D$  denotes the expectation with respect to  $P_x^D$  and

$$u(x) = \sup_{\tau \in \mathcal{T}^{\tau_D}} \mathbb{E}_x^D |\gamma(X_\tau) - h(X_\tau)|, \quad x \in D. \tag{5.7}$$

Taking the expectation of both sides of (5.6) we get

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_{T_s \wedge \tau_D}^{\tau_D}} \mathbb{E}_x^D |\gamma(X_\tau) - h(X_\tau)| &= \mathbb{E}_x^D \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{T_s \wedge \tau_D}^{\tau_D}} \mathbb{E}_x^D (|\gamma(X_\tau) - h(X_\tau)| | \mathcal{F}_{T_s \wedge \tau_D}) \\ &= \mathbb{E}_x^D u(X_{T_s \wedge \tau_D}) = P_{T_s}^D u(x). \end{aligned}$$

On the other hand, by (5.7),  $u(x) = \sup_{\tau \in \mathcal{T}^{\tau_D}} \hat{h}(x, \tau) = \hat{h}_\infty(x)$ . This together with (5.4) proves (5.5).  $\square$

By Theorem 5.1 the problem of the rate of convergence of  $V_{T_s}^*$  as  $T_s \rightarrow \infty$  is reduced to the problem of the rate of decay of the semigroup  $(P_t^D)$  as  $t \rightarrow \infty$ . Some simple consequences of this fact are given below.

**Corollary 5.2.** *Assume that  $w$  and  $|h - \gamma|$  are bounded. Then for all  $0 \leq s \leq T$  and  $x \in D$ ,*

$$V_{T_s}^*(x) \leq (\|w\|_\infty + \|h - \gamma\|_\infty) P_{T_s}^D 1(x) = (\|w\|_\infty + \|h - \gamma\|_\infty) P_x(\tau_D \geq T_s).$$

*Proof:* Follows immediately from Theorem 5.1 since  $\|\hat{h}_\infty\|_\infty \leq \|\gamma - h\|_\infty$ .  $\square$

In applications very often it is known that  $h(X)$  is a supermartingale or submartingale. In such a case the following lemma is useful.

**Lemma 5.3.** *Assume that  $x \in D$  and  $h(X)$  is a submartingale (or supermartingale) under the measure  $P_x$ . Then*

$$\hat{h}_\infty(x) = |\gamma - h|(x).$$

*Proof:* Clearly  $\hat{h}_\infty(x) \geq |\gamma - h|(x)$ ,  $x \in D$ . By the assumption that  $h(X)$  is a submartingale, (S3) and the definition of  $\gamma$ , for  $\tau \leq \tau_D$  we have

$$\gamma(X_\tau) = \mathbb{E}_x(\psi(X_{\tau_D}) | \mathcal{F}_\tau) = \mathbb{E}_x(h(X_{\tau_D}) | \mathcal{F}_\tau) \geq h(X_\tau) \quad P_x\text{-a.s.}$$

From this and the fact that  $\gamma(X)$  is a martingale, we infer that for every  $\tau \in \mathcal{T}^{\tau_D}$ ,

$$\mathbb{E}_x |\gamma(X_\tau) - h(X_\tau)| = \mathbb{E}_x \gamma(X_\tau) - \mathbb{E}_x h(X_\tau) \leq \mathbb{E}_x \gamma(X_0) - \mathbb{E}_x h(X_0) = |\gamma - h|(x).$$

An analogous reasoning applies to the case where  $h(X)$  is a supermartingale.  $\square$

**Corollary 5.4.** *Assume that  $h(X)$  is a submartingale (or supermartingale) under the measure  $P_x$ . Then for all  $0 \leq s \leq T$  and  $x \in D$ ,*

$$V_{T_s}^*(x) \leq P_{T_s}^D (w + |\gamma - h|)(x).$$

*Proof:* Follows immediately from Theorem 5.1 and Lemma 5.3.  $\square$

Let  $m$  be a positive Borel measure on  $E$  with full support. In the rest of this subsection, we assume that  $\mathbb{M}^D$  has the transition density  $p_D(t, x, y)$  with respect to  $m$ , that is

$$P_t^D f(x) = \int_D f(y) p_D(t, x, y) m(dy), \quad x \in D, f \in \mathcal{B}_b(D).$$

For  $q \in [1, \infty]$ , we let

$$r_q(t, x) := \|p_D(t, x, \cdot)\|_{L^q(D; m)}, \quad x \in D, t > 0. \tag{5.8}$$

Note that  $r_1(t, x) = P_t^D 1(x)$ . For  $q \in [1, \infty]$  we let  $q^* = \frac{q}{q-1}$  if  $q \in (1, \infty)$ ,  $q^* = \infty$  if  $q = 1$ , and  $q^* = 1$  if  $q = \infty$ .

**Proposition 5.5.** *For all  $0 \leq s \leq T$ ,  $q \in [1, \infty]$  and  $x \in D$ ,*

$$V_{T_s}^*(x) \leq r_{q^*}(T_s, x) (\|w\|_{L^q(D; m)} + \|\hat{h}_\infty\|_{L^q(D; m)}).$$

*Proof:* Follows easily from Theorem 5.1 by applying Hölder's inequality.  $\square$

**Corollary 5.6.** *Assume that  $h(X)$  is a submartingale (or supermartingale) under the measure  $P_x$ . Then for all  $0 \leq s \leq T$ ,  $q \in [1, \infty]$  and  $x \in D$ ,*

$$V_{T_s}^*(x) \leq r_{q^*}(T_s, x)(\|w\|_{L^q(D;m)} + \|h - \gamma\|_{L^q(D;m)}).$$

*Proof:* Follows from Proposition 5.5 and Lemma 5.3. □

5.2. *Lévy-type operators.* Let  $L$  be an integro-differential operator defined for  $u \in C^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$  by

$$\begin{aligned} Lu(x) &= \text{Tr}(Q(x)\nabla^2u(x)) + b(x) \cdot \nabla u(x) - c(x)u(x) \\ &+ \int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x)\mathbf{1}_{\{|y|\leq 1\}}) N(x, dy). \end{aligned} \tag{5.9}$$

We assume that its coefficients  $q_{ij}$ ,  $b_i$ ,  $c$ ,  $i, j = 1, \dots, d$  are bounded Borel measurable functions on  $\mathbb{R}^d$ ,  $c$  is nonnegative and the matrix  $Q(x) = [q_{ij}(x)]_{i,j=1}^d$  is symmetric and positive definite for every  $x \in \mathbb{R}^d$ . As for  $N(x, dy)$ , we assume that it is a Lévy kernel, that is  $N(x, dy)$  is a  $\sigma$ -finite positive Borel measure on  $\mathbb{R}^d \setminus \{0\}$  for each  $x \in \mathbb{R}^d$ , and

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |y|^2) N(x, dy) < \infty, \quad x \in \mathbb{R}^d.$$

Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ . Recall that a probability measure  $P_\mu$  on the Skorokhod space  $\mathcal{D}$  of càdlàg functions on  $[0, \infty)$  is called a solution of the martingale problem, in the sense of Stroock and Varadhan, associated with the operator  $L$  and initial measure  $\mu$ , if for every  $f \in C_b^2(\mathbb{R}^d)$ ,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_r) dr, \quad t \geq 0, \tag{5.10}$$

is a martingale under the measure  $P_\mu$ , and  $P_\mu(X_0 \in B) = \mu(B)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ . In what follows we assume that there exists a strong Markov process  $\mathbb{M} = \{(X, P_x), x \in \mathbb{R}^d\}$  with the property that for every probability measure  $\mu$  on  $\mathbb{R}^d$  the measure  $P_\mu(\cdot) := \int_{\mathbb{R}^d} P_x(\cdot) \mu(dx)$  is a solution of the martingale problem associated with the operator  $L$  and initial measure  $\mu$ . Any Markov process  $\mathbb{M}$  enjoying the above properties is called a strong Markov solution of the martingale problem associated with  $L$ .

By Kühn (2020, Theorem 4.1) (see also Klimsiak and Komorowski (2021, Section 6.1)) there exists a strong Markov solution of the martingale problem associated with the operator  $L$  provided that for every probability measure  $\mu$  on  $\mathbb{R}^d$  there exists a solution of the martingale problem for  $L$  and initial measure  $\mu$ .

*Remark 5.7.* Consider the following hypotheses:

- (M1) The matrix  $Q(x)$  is *uniformly strictly positive definite* on compact sets, i.e. for any compact set  $K \subset \mathbb{R}^d$  there exists  $\lambda_K > 0$  such that

$$\lambda_K |\xi|^2 \leq \sum_{i,j=1}^d q_{ij}(x) \xi_i \xi_j, \quad x \in K, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$

- (M2) The mapping  $\mathbb{R}^d \ni x \mapsto Q(x) \in \mathbb{R}^{d \times d}$  is continuous and for every Borel set  $B \subset B(0, 1) = \{y \in \mathbb{R}^d : |y| < 1\}$  the mapping

$$\mathbb{R}^d \ni x \mapsto N_B(x) := \int_B (1 \wedge |y|^2) N(x, dy)$$

is continuous.

Let the assumptions on the coefficients of  $L$  made after (5.9) be satisfied. Then there exists a strong Markov solution of the martingale problem associated with  $L$  if (M1) is satisfied (see Anulova and Pragarauskas (1977); Lepeltier and Marchal (1976)) or (M2) is satisfied and  $Q(x)$  is invertible for every  $x \in \mathbb{R}^d$  (see Jacod and Shiryaev (2003, Theorem III.2.34, p. 159) and also Stroock (1975)) or (M2) is satisfied and the mapping  $x \mapsto b(x)$  is continuous (see Hoh (1998) and also Böttcher et al. (2013, Theorem 3.24)).

Recall here that a martingale problem is said to be *well posed* if for every probability measure  $\mu$  on  $\mathbb{R}^d$  there exists a unique solution  $P_\mu$  of the martingale problem associated with  $L$  and initial measure  $\mu$ .

The following proposition can be useful for estimating  $\hat{h}_\infty$ .

**Proposition 5.8.** *Assume that  $h, \gamma \in C_b^2(\mathbb{R}^d)$ . Then*

$$\hat{h}_\infty(x) \leq R^D |Lh|(x) \leq \left( \mathbb{E}_x \sup_{t \leq \tau_D} |Lh(X_t)|^p \right)^{1/p} \left( \mathbb{E}_x (\tau_D)^{p^*} \right)^{1/p^*}, \quad x \in D, p \in [1, \infty].$$

*Proof:* By (5.10) applied to  $h$  and  $\gamma$ ,

$$M_t^\gamma - M_t^h = (\gamma - h)(X_t) - (\gamma - h)(X_0) + \int_0^t Lh(X_r) dr, \quad t \geq 0.$$

By (S3), for all  $a \geq 0$  and  $t \in [0, a]$  we have

$$(\gamma - h)(X_t) = (\gamma - h)(X_{a \wedge \tau_D}) + \int_{t \wedge \tau_D}^{a \wedge \tau_D} Lh(X_r) dr - \int_{t \wedge \tau_D}^{a \wedge \tau_D} d(M_r^\gamma - M_r^h).$$

Thus the pair  $((\gamma - h)(X), M^\gamma - M^h)$  is a solution of  $\text{BSDE}^{\tau_D}(0, F)$  with the coefficient  $F(t, \omega) := Lh(X_t(\omega))$ ,  $t \geq 0$ ,  $\omega \in \Omega$ . Therefore applying Proposition 3.1 we get the desired result.  $\square$

In the rest of this subsection we assume additionally that  $\mathbb{M} = \{(X, P_x), x \in \mathbb{R}^d\}$  is a Feller process, i.e.  $P_t(C_\infty(\mathbb{R}^d)) \subset C_\infty(\mathbb{R}^d)$ , where  $C_\infty(\mathbb{R}^d)$  is the space of continuous functions on  $\mathbb{R}^d$  vanishing at infinity, with the symbol

$$\begin{aligned} p(x, \xi) &= c(x) - i\langle b(x), \xi \rangle + \frac{1}{2} \langle Q(x)\xi, \xi \rangle \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} \left( 1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) N(x, dz). \end{aligned} \tag{5.11}$$

For an overview of sufficient conditions on  $p$  or the coefficients of  $L$  guaranteeing that  $L$  generates a Feller process  $\mathbb{M}$  see Böttcher et al. (2013, Chapter 3). Here we recall one general criterion (see Böttcher et al. (2013, Theorem 3.25, Lemma 3.26)). It says that if

- (a)  $\lim_{|x| \rightarrow \infty} N(x, B(-x, r)) = 0$  for any  $r > 0$ ,
- (b)  $x \mapsto p(x, \xi)$  is continuous for any  $\xi \in \mathbb{R}^d$ ,
- (c) the martingale problem for  $L$  is well posed,

then  $\mathbb{M}$  is Feller. Using this criterion and the results of Stroock (1975) we get the following example of a Feller process.

*Example 5.9.* Assume that

$$\lambda^{-1}I \leq Q(x) \leq \lambda I, \quad |b(x)| \leq \Lambda, \quad 0 \leq c(x) \leq \Lambda, \quad x \in \mathbb{R}^d, \tag{5.12}$$

for some  $\lambda \geq 1$ ,  $\Lambda > 0$  ( $I$  is the  $d$ -dimensional identity matrix),  $q_{ij}, b_i, c$  are continuous, and  $N_B$  (cf. condition (M2)) is continuous for any  $B \in \mathcal{B}(\mathbb{R}^d)$ . Furthermore, suppose that

$$\lim_{|x| \rightarrow \infty} \int_{\mathbb{R}^d} \min\{|y|^2/|x|^2, 1\} N(x, dy) = 0. \tag{5.13}$$

(for instance, the last condition is satisfied if  $N$  is independent of  $x$  or big jumps are integrable, that is  $\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \geq 1\}} N(x, dy) < \infty$ ). Then  $\mathbb{M}$  solving the martingale problem for  $L$  is a Feller process. Indeed, by Stroock (1975, Theorem 2.2, Theorem 4.3) the martingale problem for  $L$  is well posed, so we have (c). By the assumptions made on the coefficients of  $L$  and function  $N_B$ , (b) is satisfied. What is left is to show that (a) is satisfied. By Böttcher et al. (2013, Lemma 3.26) condition (i) is satisfied if

$$\lim_{|x| \rightarrow \infty} \sup_{|\xi| \leq 1/|x|} (\operatorname{Re} p(x, \xi) - p(x, 0) - \frac{1}{2} \xi \cdot Q(x) \xi) = 0. \tag{5.14}$$

The quantity in the bracket equals to

$$\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) N(x, dy) \leq \int_{\mathbb{R}^d} \min\{|y|^2 |\xi|^2, 1\} N(x, dy).$$

From this and (5.13) we easily get (5.14).

Consider the following condition

$$\lim_{|\xi| \rightarrow \infty} \frac{\inf_{z \in \mathbb{R}^d} \operatorname{Re} p(z, \xi)}{\log(1 + |\xi|)} = \infty. \tag{5.15}$$

Clearly, it is satisfied for the operator from Example 5.9. By Schilling and Wang (2013, Theorem 1.2), if (5.15) holds, then  $\mathbb{M}$  has a transition density  $p(t, x, y)$ . Consequently, the part  $\mathbb{M}^D$  of  $\mathbb{M}$  on  $D$  has a transition density, which we denote by  $p_D(t, x, y)$ . Set

$$r(t) = (4\pi)^{-d} \int_{\mathbb{R}^d} \exp\left(-\frac{t}{16} \inf_{z \in \mathbb{R}^d} \operatorname{Re} p(z, \xi)\right) d\xi. \tag{5.16}$$

By Schilling and Wang (2013, Theorem 1.2), under (5.15) we have

$$r_{q^*}(t, x) \leq \begin{cases} r(t), & \text{if } q^* = \infty, \\ r(t)[m(D)]^{1/q^*}, & \text{if } q^* \in [1, \infty). \end{cases} \tag{5.17}$$

*Example 5.10.* Let  $\alpha \in C_b^1(\mathbb{R}^d)$ , and

$$0 < \underline{\alpha} := \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \bar{\alpha} := \sup_{x \in \mathbb{R}^d} \alpha(x) < 2.$$

Furthermore, suppose that either  $d \geq 2$ , or  $d = 1$  and there exists  $K > 0$  such that  $\sup_{|x| \geq K} \alpha(x) \in (0, 1)$ . Let

$$p(x, \xi) = |\xi|^{\alpha(x)}, \quad x, \xi \in \mathbb{R}^d.$$

Then, by Schilling and Wang (2013), there exists a Feller process  $\mathbb{M}$  with symbol  $p$ . One can easily check that there is  $C > 0$  such that for every  $t \geq 1$ ,

$$r(t) \leq Ct^{-d/\bar{\alpha}}.$$

The semigroup  $(P_t)$  associated with  $\mathbb{M}$  is generated by the operator  $\Delta^{\alpha(\cdot)}$ .

In case  $D$  is bounded one can estimate  $r_1$  without imposing condition (5.15). Let  $\delta = \operatorname{diam} D$ . If  $\delta < \infty$ , then by Böttcher et al. (2013, Theorem 5.9),

$$r_1(t, x) \leq 3e^{-tc(x, \delta)/16}, \quad x \in D, \tag{5.18}$$

where

$$c(x, \delta) := \sup_{|\xi| \leq 1/(2\delta k^*(x, \delta))} \inf_{|y-x| \leq 3\delta} \operatorname{Re} p(y, \xi),$$

and

$$k^*(x, \delta) := \inf \left\{ k \geq (\arcsin \sqrt{2/3})^{-1} : \sup_{|\xi| \leq 1/(2k\delta)} \sup_{|y-x| \leq \delta} \frac{\operatorname{Re} p(y, \xi)}{|\xi| |\operatorname{Im} p(y, \xi)|} \geq 4\delta \right\}.$$

*Remark 5.11.* Observe that if  $p$  satisfies the following *sector condition*: there exists  $\kappa > 0$  such that

$$|\operatorname{Im} p(x, \xi)| \leq \kappa \operatorname{Re} p(x, \xi), \quad x, \xi \in \mathbb{R}^d,$$

then  $k^*(x, \delta) = (\arcsin \sqrt{2/3})^{-1}$ ,  $x \in \mathbb{R}^d$ ,  $\delta > 0$ . The sector condition is trivially satisfied when  $\operatorname{Im} p(\cdot, \cdot) = 0$ . In particular, the sector condition is satisfied by the symbol of the operator  $\Delta^{\alpha(\cdot)}$  from Example 5.10.

**Proposition 5.12.**

(i) If (5.15) is satisfied and  $w, \hat{h}_\infty \in L^1(D; m)$ , then

$$V_{T_s}^*(x) \leq r(T_s)(\|w\|_{L^1(D; m)} + \|\hat{h}_\infty\|_{L^1(D; m)}).$$

(ii) Let  $q \in (1, \infty]$ . If (5.15) is satisfied,  $m(D) < \infty$  and  $w, \hat{h}_\infty \in L^q(D; m)$ , then

$$V_{T_s}^*(x) \leq r(T_s)[m(D)]^{1/q^*} (\|w\|_{L^q(D; m)} + \|\hat{h}_\infty\|_{L^q(D; m)}).$$

(iii) If  $\delta := \operatorname{diam} D < \infty$  and  $w, \hat{h}_\infty \in L^\infty(D; m)$ , then

$$V_{T_s}^*(x) \leq 3e^{-(T_s)c(x, \delta)/16} (\|w\|_{L^\infty(D; m)} + \|\hat{h}_\infty\|_{L^\infty(D; m)}).$$

*Proof:* Assertions (i), (ii) follow from Proposition 5.5 and (5.17), and (iii) follows from Proposition 5.5 and (5.18). □

5.3. *Ultracontractivity and intrinsic ultracontractivity.* Let  $m$  be a  $\sigma$ -finite measure on  $\mathbb{R}^d$  with full support and  $D \subset \mathbb{R}^d$  be an open subset such that  $m(D) < \infty$ . We assume that  $\mathbb{M}^D$  has the transition density  $p_D(t, x, y)$  with respect to  $m$ . Let  $(\hat{P}_t^D)$  be the dual semigroup to  $(P_t^D)$  relative to  $m$  in the sense that for all  $t > 0$  and nonnegative  $f, g \in \mathcal{B}(D)$ ,

$$\int_D f(x) P_t^D g(x) m(dx) = \int_D g(x) \hat{P}_t^D f(x) m(dx).$$

Clearly,  $\hat{P}_t f(y) = \int_D p_D(t, x, y) f(x) m(dx)$ ,  $y \in D$ ,  $f \in \mathcal{B}^+(D)$ . The semigroup  $(P_t^D)$  is said to be *ultracontractive* if for every  $t > 0$  there exists  $b(t) > 0$  such that

$$p_D(t, x, y) \leq b(t), \quad x, y \in D. \tag{5.19}$$

It is known that if  $(\hat{P}_t^D)$  is Markov, then  $(P_t^D)$  is ultracontractive if and only if the operators  $P_t^D : L^2(D; m) \rightarrow L^\infty(D; m)$  and  $\hat{P}_t^D : L^2(D; m) \rightarrow L^\infty(D; m)$  are bounded. Moreover, if the semigroup  $(P_t^D)$  is ultracontractive and  $(\hat{P}_t^D)$  is Markov, then (5.19) holds with  $b(t) = \max\{\|P_t^D\|_{L^2(D; m) \rightarrow L^\infty(D; m)}, \|\hat{P}_t^D\|_{L^2(D; m) \rightarrow L^\infty(D; m)}\}$  and  $b(t)$  is nonincreasing as  $t \rightarrow \infty$  (see, e.g., Kim and Song (2008, Proposition 2.2)).

*Remark 5.13.* Let  $\mathbb{M}$  be a Feller process with symbol (5.11). By Schilling and Wang (2013, Theorem 1.2), if (5.15) is satisfied, then  $(P_t^D)$  is ultracontractive and

$$p_D(t, x, y) \leq r(t), \quad t > 0, x, y \in D, \tag{5.20}$$

where  $r$  is defined by (5.16).

**Proposition 5.14.** Let  $q \in [1, \infty)$ . If  $w, \hat{h}_\infty \in L^q(D; m)$  and  $(P_t^D)$  is ultracontractive, then

$$V_{T_s}^*(x) \leq (r_1(T_s, x))^{1/q^*} (b(T_s))^{1/q} (\|w\|_{L^q(D; m)} + \|\hat{h}_\infty\|_{L^q(D; m)}). \tag{5.21}$$

*Proof:* Let  $\rho \in L^q(D; m)$ . Then

$$\begin{aligned} P_t^D \rho &= \mathbb{E}_x \mathbf{1}_{\{t < \tau_D\}} \rho(X_t) = \mathbb{E}_x \mathbf{1}_{\{t < \tau_D\}} (\rho(X_t) \mathbf{1}_{\{t < \tau_D\}}) \\ &\leq (P_x(t < \tau_D))^{1/q^*} (\mathbb{E}_x \mathbf{1}_{\{t < \tau_D\}} \rho^q(X_t))^{1/q} \\ &= (r_1(t, x))^{1/q^*} \left( \int_D p_D(t, x, y) \rho^q(y) m(dy) \right)^{1/q} \\ &\leq (r_1(T_s, x))^{1/q^*} (b(T_s))^{1/q} \|\rho\|_{L^q(D; m)}. \end{aligned}$$

From this and Theorem 5.1 we get the desired estimate. □

*Remark 5.15.* (i) It is worth noting that the difference between (5.21) and the inequality formulated in Proposition 5.12(ii) is that in (5.21) the factor  $r(T_s)[m(D)]^{1/q^*}$  appearing in Proposition 5.12(ii) has been replaced by  $(r_1(T_s, x))^{1/q^*} (b(T_s))^{1/q}$ . This is an important refinement. Recall that

$$r_1(T_s, x) = P_x(\tau_D > T_s) = \int_D p_D(T_s, x, y) m(dy),$$

while  $r(T_s) = \sup_{x, y \in \mathbb{R}^d} p(T_s, x, y)$ . The asymptotics of the quantity  $P_x(\tau_D > T_s)$  as  $T_s \rightarrow \infty$  is well studied in the literature. Moreover, the term  $r_1(T_s, x)$  tends to zero when  $x$  approaches the boundary of  $D$ , while  $r(T_s)$  is independent of  $x \in D$ .

(ii) The decay of  $V_{T_s}^*$  similar to (5.21) is asserted in Proposition 5.12(iii). However, in Proposition 5.12(iii) boundedness of  $w$  and  $\hat{h}_\infty$  is required. Thanks to the ultracontractivity of  $(P_t^D)$  we may dispense with this restriction.

**Corollary 5.16.** *Let  $q \in [1, \infty)$ . Assume that  $\delta = \text{diam } D < \infty$ ,  $w, \hat{h}_\infty \in L^q(D; m)$ , and  $\mathbb{M} = \{(X, P_x), x \in \mathbb{R}^d\}$  is a Feller process with the symbol  $p$  (cf. (5.11)). If  $(P_t^D)$  is ultracontractive, then*

$$V_{T_s}^*(x) \leq 3^{1/q^*} e^{-T_s c(x, \delta)/(16q^*)} (b(T_s))^{1/q} (\|w\|_{L^q(D; m)} + \|\hat{h}_\infty\|_{L^q(D; m)}).$$

*Proof:* Follows from Proposition 5.14 and (5.18). □

*Example 5.17* (divergence form operators). Let  $Q, b$  and  $c$  be as in (5.9) and satisfy (5.12). By Stroock (1988) there exists a diffusion process  $\mathbb{M}$  with transition density  $p(\cdot, \cdot, \cdot)$  being a fundamental solution for the operator

$$L = \sum_{i, j=1}^d \partial_{x_i} (q_{ij}(x) \partial_{x_j}) + \sum_{i=1}^d b_i \partial_{x_i} + c. \tag{5.22}$$

By Aronson’s estimates, there exists  $M > 0$  depending only on  $\Lambda, d$  and  $T > 0$  such that

$$p(t, x, y) \leq Mt^{-d/2} e^{-|x-y|^2/(Mt)}$$

for all  $t \in (0, T]$  and  $x, y \in \mathbb{R}^d$ . By Proposition 5.14, (5.21) holds with  $b$  defined by the right-hand side of the above inequality. Better rate of convergence we get in case  $c = 0$ ,  $b^i = 0$ ,  $i = 1, \dots, d$ , and  $D$  is Lipschitz (see Example 5.20).

In the rest of this subsection we assume that for each  $t > 0$ ,  $p_D(t, \cdot, \cdot)$  is bounded and strictly positive. Moreover, we assume that  $(\hat{P}_t^D)$  is Markov. Let

$$-\lambda_1 = \sup\{\text{Re } \lambda : \lambda \in \sigma(L^D)\}, \tag{5.23}$$

where  $\sigma(L^D)$  is the spectrum of the infinitesimal generator of the semigroup  $(P_t^D)$  on  $L^2(D; m)$ . By Jentzsch’s theorem (see Schaefer (1974, Theorem V.6.6, page 337)) there exist unique, up to a multiplicity constant, strictly positive functions  $\phi_1, \hat{\phi}_1 \in L^2(D; m)$  such that

$$P_t^D \phi_1(x) = e^{-t\lambda_1} \phi_1(x), \quad \hat{P}_t^D \hat{\phi}_1(x) = e^{-t\lambda_1} \hat{\phi}_1(x), \quad x \in D, t > 0,$$

The semigroup  $(P_t^D)$  is called *intrinsically ultracontractive* (see [Davies and Simon \(1984\)](#); [Kim and Song \(2008\)](#)) if for every  $t > 0$  there exists a constant  $c_t > 0$  such that

$$p_D(t, x, y) \leq c_t \phi_1(x) \hat{\phi}_1(y), \quad x, y \in D.$$

Equivalently,  $(P_t^D)$  is intrinsically ultracontractive if  $(Q_t^D)$  is ultracontractive on  $L^2(D; \mu)$ , where

$$Q_t^D f(x) = e^{\lambda_1 t} P_t^D(f \phi_1) \phi_1^{-1}, \quad \mu(dy) = \phi_1(y) \hat{\phi}_1(y) m(dy).$$

From the last statement and the fact that  $(Q_t^D)$  ( $\hat{Q}_t^D$ ) are Markov, we infer, in particular, that for every  $t_0 > 0$  there exists  $M(t_0) > 0$  such that

$$p_D(t, x, y) \leq M(t_0) e^{-\lambda_1 t} \phi_1(x) \hat{\phi}_1(y), \quad x, y \in D, t \geq t_0, \tag{5.24}$$

with  $M(t_0) = \max\{\|Q_{t_0}^D\|_{L^2(D; \mu) \rightarrow L^\infty(D; \mu)}, \|\hat{Q}_{t_0}^D\|_{L^2(D; \mu) \rightarrow L^\infty(D; \mu)}\}$ .

**Proposition 5.18.** *Let  $q \in [1, \infty]$  and  $t_0 > 0$ . Assume that  $(P_t^D)$  is intrinsically ultracontractive and  $w, \hat{h}_\infty \in L^q(D; m)$ . Then for all  $T_s \geq t_0$  and  $x \in D$ ,*

$$V_{T_s}^*(x) \leq M(t_0) e^{-\lambda_1 T_s} \phi_1(x) \|\hat{\phi}_1\|_{L^{q^*}(D; m)} (\|w\|_{L^q(D; m)} + \|\hat{h}_\infty\|_{L^q(D; m)}).$$

*Proof:* Let  $t \geq t_0$ . By (5.24),

$$r_{q^*}(t, x) \leq M(t_0) e^{-\lambda_1 t} \phi_1(x) \|\hat{\phi}_1\|_{L^{q^*}(D; m)},$$

so the desired estimate follows from Proposition 5.5. □

*Example 5.19* (nondivergence form operator). Let  $m$  be the Lebesgue measure on  $\mathbb{R}^d$  and  $D \subset \mathbb{R}^d$  be a bounded domain. Consider the operator (5.9) with  $N = 0$ . Assume that  $q_{ij}, b_i, i, j = 1, \dots, d$ , and  $c$  are bounded  $C^\infty$  functions on  $\mathbb{R}^d$ . Furthermore, we assume that (5.12) is satisfied,  $\partial b_i / \partial x_i, i = 1, \dots, d$ , are bounded and

$$c(x) - \sum_{i=1}^d \partial_{x_i} b_i(x) \geq 0, \quad x \in \mathbb{R}^d.$$

Then the assumptions formulated in the first paragraph of this subsection are satisfied (see [Kim and Song \(2008, p. 538\)](#)). If  $D$  is Lipschitz, then by [Kim and Song \(2008, Theorem 3.9\)](#) the semigroup  $(P_t^D)$  is intrinsically ultracontractive.

*Example 5.20* (divergence form operator). Consider the operator  $L$  from Example 5.17 but assume that  $b^i = 0, i = 1, \dots, d$ , and  $c = 0$ . In [Davies and Simon \(1984\)](#) it is proved that if  $D$  is bounded and Lipschitz, then  $(P_t^D)$  is intrinsically ultracontractive. Thus, Proposition 5.18 is applicable. This proposition requires  $\hat{h}_\infty$  to be in  $L^q(D; m)$ . In Subsection 5.4 (see Example 5.29) we provide different results on the rate of convergence requiring assumptions on  $h$  and not on  $\hat{h}_\infty$ .

*Example 5.21.* Let  $\mathbb{M} = (X, P_x)$  be a symmetric Lévy process in  $\mathbb{R}^d$  with Lévy measure  $\nu$  such that

$$\nu(B(x, r)) > 0 \tag{5.25}$$

for all  $x \in \mathbb{R}^d$  and  $r > 0$ , where  $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ . Assume also that  $\mathbb{M}$  has the transition density  $p(t, x, y) = p(t, x - y)$  (with respect to the Lebesgue measure) such that  $p(t, \cdot, \cdot)$  is continuous for every  $t > 0$  and moreover, for every  $\delta > 0$  there exists a constant  $c(\delta) > 0$  such that  $p(t, x) \leq c(\delta)$  for all  $t > 0$  and  $|x| \geq \delta$ . In [Grzywny \(2008, Theorem 3.1\)](#) (see also [Kulczycki \(1998\)](#) for the special case of rotationally symmetric  $\alpha$ -stable Lévy process) it is proved that for any bounded open set  $D \subset \mathbb{R}^d$  the semigroup  $(P_t^D)$  associated with the process  $\mathbb{M}$  killed upon exiting  $D$  is intrinsically ultracontractive. Note also that (5.25) can be weakened if we additionally assume that  $D$  is a connected Lipschitz set.

5.4. *Dirichlet forms.* Let  $E$  be a locally compact separable metric space and  $m$  be a Radon measure on  $E$  with full support. Let  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  be a regular symmetric Dirichlet form on  $L^2(E; m)$ . For an open  $U \subset E$  we define the capacity of  $U$  by

$$\text{Cap}_E(U) = \inf\{\mathcal{E}_1(u, u) : u \geq \mathbf{1}_U \text{ } m\text{-a.e.}, u \in \mathcal{D}(\mathcal{E})\},$$

where  $\mathcal{E}_1(u, u) = \mathcal{E}(u, u) + \|u\|_{L^2(E; m)}^2$ . For an arbitrary  $B \subset D$  we set

$$\text{Cap}_E(B) = \inf_{B \subset U} \text{Cap}_E(U).$$

Recall that a function  $u$  on  $E$  is called quasi-continuous if for every  $\varepsilon > 0$  there exists an open set  $U_\varepsilon \subset E$  such that  $\text{Cap}_E(U_\varepsilon) < \varepsilon$  and  $u|_{E \setminus U_\varepsilon}$  is continuous. By Fukushima et al. (2011, Theorem 2.1.3) every function  $u \in \mathcal{D}(\mathcal{E})$  has an  $m$ -version  $\tilde{u}$  which is quasi-continuous. Let  $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$  be the part of  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  on  $D$ , i.e.

$$\mathcal{E}^D(u, v) = \mathcal{E}(u, v), \quad u, v \in \mathcal{D}(\mathcal{E}^D) := \{u \in \mathcal{D}(\mathcal{E}) : \tilde{u} = 0 \text{ q.e. on } E \setminus D\}$$

(here  $\tilde{u} = 0$  q.e. means that  $\tilde{u} = 0$  except for a set of capacity  $\text{Cap}_E$  equal to zero). By Fukushima et al. (2011, Section 7) there exists a Hunt process  $\mathbb{M}$  associated with  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ . Moreover,  $\mathbb{M}^D$  is associated with  $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$  which is again a regular Dirichlet form (see Fukushima et al. (2011, Theorem 4.4.3)). Analogously to  $\text{Cap}_E$  we define  $\text{Cap}_D$ . By Fukushima et al. (2011, Theorem 2.1.3), any function  $u \in \mathcal{D}(\mathcal{E}^D)$  has an  $m$ -version  $\tilde{u}$  which is quasi-continuous. Formally we should write "quasi-continuous with respect to  $\text{Cap}_E$  or  $\text{Cap}_D$ " and not just "quasi-continuous". However both capacities are equivalent on  $D$  (see Fukushima et al. (2011, Theorem 4.4.3)), so the above terminology does not lead to ambiguity.

Throughout the subsection, we assume that  $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$  is transient, i.e. there exists a strictly positive  $\chi \in L^2(D; m)$  such that

$$\int_D |u| \chi \, dm \leq (\mathcal{E}^D(u, u))^{1/2}, \quad u \in \mathcal{D}(\mathcal{E}^D). \tag{5.26}$$

By Fukushima et al. (2011, Theorems 1.5.2, 1.5.3), there exists an extension  $\mathcal{D}_e(\mathcal{E}^D) \subset L^1(D; \nu \cdot m)$  of  $\mathcal{D}(\mathcal{E}^D)$ , called the extended Dirichlet space, such that  $(\mathcal{E}^D, \mathcal{D}_e(\mathcal{E}^D))$  is a Hilbert space. Furthermore,  $\mathcal{D}_e(\mathcal{E}^D) \cap L^2(D; m) = \mathcal{D}(\mathcal{E}^D)$  is dense in  $\mathcal{D}_e(\mathcal{E}^D)$ . By Fukushima et al. (2011, Theorem 2.1.3) (see also the comments following Fukushima et al. (2011, (2.1.14))) every function  $u \in \mathcal{D}_e(\mathcal{E}^D)$  has an  $m$ -version  $\tilde{u}$  which is quasi-continuous. From now on, we consider quasi-continuous versions of functions in  $\mathcal{D}(\mathcal{E}^D)$  or  $\mathcal{D}_e(\mathcal{E}^D)$ .

Below we first give estimates of  $V_{T_s}^*$  in the space  $L^1(D; \rho \cdot m)$  with some weight  $\rho$ . Then we give pointwise estimates under additional assumptions on  $\mathbb{M}^D$ .

**Lemma 5.22.** *Assume that  $u \in \mathcal{D}_e(\mathcal{E}^D)$ . Then for every positive  $\rho \in L^2(D; m)$  such that  $R^D \rho \in L^2(D; m)$ , and every  $t \geq 0$ ,*

$$\int_D \left( \sup_{\tau \in T^{\tau D}} \mathbb{E}_x |u(X_\tau)| \right) \rho(x) \, m(dx) \leq (\mathcal{E}^D(u, u))^{1/2} \|\rho\|_{L^2(D; m)}^{1/2} \|R^D \rho\|_{L^2(D; m)}^{1/2}. \tag{5.27}$$

*Proof:* We let

$$\Gamma = \{\eta \in \mathcal{D}_e(\mathcal{E}) : \eta \geq |u| \text{ } m\text{-a.e.}\}. \tag{5.28}$$

By Oshima (2013, Theorem 1.1.1) applied to  $\Gamma$  and  $J = 0$  there exists a unique  $v \in \Gamma$  such that  $\mathcal{E}(v, w - v) \geq 0$  for all  $w \in \Gamma$ . Moreover,  $v$  has the property that

$$\mathcal{E}^D(v, v) = \inf\{\mathcal{E}^D(\phi, \phi) : \phi \in \mathcal{D}_e(\mathcal{E}^D) \text{ and } \phi \geq |u| \text{ } m\text{-a.e.}\}.$$

Let  $\bar{w}$  be a positive element of  $\mathcal{D}(\mathcal{E}^D)$ . Then  $\bar{w} + v \in \Gamma$ , so  $\mathcal{E}(v, \bar{w}) = \mathcal{E}(v, \bar{w} + v - v) \geq 0$ . Consequently,  $v$  is excessive by Oshima (2013, Theorem 1.4.1). From this and Fukushima et al. (2011, Theorem A.2.5) it follows that  $v(X)$  is a càdlàg process. Furthermore, from the fact that  $v$

is excessive and the Markov property it follows that  $v(X)$  is a supermartingale under the measure  $P_x$  for  $m$ -a.e.  $x \in D$ . Hence

$$\sup_{\tau \in \mathcal{T}^{\tau D}} \mathbb{E}_x |u(X_\tau)| \leq \sup_{\tau \in \mathcal{T}^{\tau D}} \mathbb{E}_x v(X_\tau) \leq \mathbb{E}_x v(X_0) = v(x)$$

for  $m$ -a.e.  $x \in D$ . Multiplying both sides of the above inequality by  $\rho$  and then integrating with respect to  $m$  yields

$$\int_D \left( \sup_{\tau \in \mathcal{T}^{\tau D}} \mathbb{E}_x |u(X_\tau)| \right) \rho(x) m(dx) \leq \int_D v \cdot \rho dm = \mathcal{E}^D(v, R^D \rho). \tag{5.29}$$

On the other hand,

$$\begin{aligned} \mathcal{E}^D(v, R^D \rho) &\leq (\mathcal{E}^D(v, v))^{1/2} (\mathcal{E}^D(R^D \rho, R^D \rho))^{1/2} \\ &= (\mathcal{E}^D(v, v))^{1/2} (\rho, R^D \rho)_{L^2(D; m)}^{1/2} \\ &\leq (\mathcal{E}^D(|u|, |u|))^{1/2} \|\rho\|_{L^2(D; m)}^{1/2} \|R^D \rho\|_{L^2(D; m)}^{1/2} \end{aligned} \tag{5.30}$$

Since  $\mathcal{E}^D$  is a transient Dirichlet form, every normal contraction operates on  $\mathcal{E}^D$ . Hence  $|u| \in \mathcal{D}(\mathcal{E}^D)$  and  $\mathcal{E}^D(|u|, |u|) \leq \mathcal{E}^D(u, u)$ . Therefore (5.29) and (5.30) imply (5.27).  $\square$

By Fukushima et al. (2011, Theorem 4.3.2), if  $\psi \in \mathcal{D}_e(\mathcal{E})$ , then  $\gamma \in \mathcal{D}_e(\mathcal{E})$ . Below we shall freely use this fact without explicit mention.

**Lemma 5.23.** *Assume that  $h, \psi \in \mathcal{D}_e(\mathcal{E})$ . Then for all  $t > 0$  and positive  $\rho \in L^2(D; m)$  such that  $R^D \rho \in L^2(D; m)$ ,*

$$\int_D P_t^D \hat{h}_\infty(x) \rho(x) m(dx) \leq (\mathcal{E}^D(\gamma - h, \gamma - h))^{1/2} \|P_t^D \rho\|_{L^2(D; m)}^{1/2} \|R^D \rho\|_{L^2(D; m)}^{1/2}.$$

*Proof:* Since  $\gamma \in \mathcal{D}_e(\mathcal{E})$  and  $h = \gamma$  on  $E \setminus D$ , we have  $\gamma - h \in \mathcal{D}_e(\mathcal{E}^D)$ . Therefore the result follows from the definition of  $\hat{h}_\infty$  and Lemma 5.22 applied to  $u = \gamma - h$ .  $\square$

We are now ready to give an estimate for  $V_{T_s}^*$  in case  $h, \psi \in \mathcal{D}_e(\mathcal{E})$ . It is worth stressing that in general, under the assumption  $h \in \mathcal{D}_e(\mathcal{E})$  the process  $h(X)$  is not a semimartingale under  $P_x$ . Therefore Lemma 5.23 and Proposition 5.24 below apply in situations quite different from those considered earlier, for instance in Corollary 5.4 or Proposition 5.8.

**Proposition 5.24.** *Assume that  $w \in L^2(D; m)$ ,  $\psi, h \in \mathcal{D}_e(\mathcal{E})$  and  $\rho \in L^2(D; m)$  is a positive function such that  $R^D \rho \in L^2(D; m)$ . Then*

$$\begin{aligned} &\int_D V_{T_s}^*(x) \rho(x) m(dx) \\ &\leq \left( \|w\|_{L^2(D; m)} + (\mathcal{E}^D(\gamma - h, \gamma - h))^{1/2} \|R^D \rho\|_{L^2(D; m)}^{1/2} \right) \|P_{T_s}^D \rho\|_{L^2(D; m)}^{1/2}. \end{aligned}$$

*Proof:* Follows from Lemma 5.23 and Theorem 5.1.  $\square$

Let  $\alpha : (0, \infty) \rightarrow (0, \infty)$  be a nonincreasing function and  $\Phi : L^2(D; m) \rightarrow [0, \infty]$  satisfy the following conditions:

- (a)  $\Phi(cu) = c^2 \Phi(u)$  for all  $u \in L^2(D; m)$  and  $c \in \mathbb{R}$ ,
- (b)  $\Phi(P_t^D \rho) \leq \Phi(\rho)$  for all  $\rho \in L^2(D; m)$ .

Suppose that the following Sobolev-type inequality is satisfied:

$$\|u\|_{L^2(D; m)}^2 \leq \alpha(r) \mathcal{E}^D(u, u) + r \Phi(u), \quad r > 0, u \in \mathcal{D}(\mathcal{E}^D). \tag{5.31}$$

By Wang (2003), if (5.31) is satisfied, then

$$\|P_t^D \rho\|_{L^2(D; m)}^2 \leq \beta(t) (\Phi(\rho) + \|\rho\|_{L^2(D; m)}^2), \quad t > 0, \rho \in L^2(D; m), \tag{5.32}$$

where

$$\beta(t) = \inf\{r > 0 : -\alpha(r) \ln r \leq 2t\}, \quad t > 0.$$

*Remark 5.25.* Let  $\lambda_2$  be the bottom of the spectrum, that is

$$\lambda_2 = \inf\{\mathcal{E}^D(u, u) : u \in \mathcal{D}(\mathcal{E}^D), \|u\|_{L^2(D;m)} = 1\}. \tag{5.33}$$

If  $\lambda_2 > 0$ , then by (5.32),

$$\|P_t^D \rho\|_{L^2(D;m)} \leq e^{-2\lambda_2 t} \|\rho\|_{L^2(D;m)}, \quad \rho \in L^2(D; m).$$

Indeed, for  $u \in \mathcal{D}_e(\mathcal{E}^D)$  we have  $\mathcal{E}^D(u, u) \leq \lambda_2^{-1} \|u\|_{L^2(D;m)}^2$ , so (5.31) is satisfied with  $\alpha(r) = \lambda_2^{-1}$ ,  $r > 0$ ,  $\Phi = 0$ . Clearly, we then have  $\beta(t) = e^{-2t\lambda_2}$ , and we apply (5.32).

**Corollary 5.26.** *Let  $w, \psi, h$  and  $\rho$  satisfy the assumptions of Proposition 5.24. If (5.31) is satisfied, then*

$$\int_D V_{T_s}^*(x) \rho(x) m(dx) \leq C(\beta(T_s))^{1/4},$$

where

$$C = \left( \|w\|_{L^2(D;m)} + (\mathcal{E}^D(\gamma - h, \gamma - h))^{1/2} \|R^D \rho\|_{L^2(D;m)}^{1/2} \right) (\Phi(\rho) + \|\rho\|_{L^2(D;m)}^2)^{1/4}.$$

*Proof:* Follows from Proposition 5.24 and (5.32). □

One can improve slightly the rate of convergence given above provided we know that  $\Phi(\hat{h}_\infty) < \infty$ .

**Proposition 5.27.** *Assume that  $w \in L^2(D; m)$ ,  $\psi, h \in \mathcal{D}_e(\mathcal{E})$ , and (5.31) is satisfied. Then for every positive  $\rho \in L^2(D; m)$ , and any  $r > 0$ ,*

$$\int_D V_{T_s}^*(x) \rho(x) m(dx) \leq C_r(\beta(T_s))^{1/2},$$

where

$$C_r = \left( \|w\|_{L^2(D;m)} + (\alpha(r)\mathcal{E}^D(\gamma - h, \gamma - h) + r\Phi(\hat{h}_\infty))^{1/2} \right) (\|\rho\|_{L^2(D;m)}^2 + \Phi(\rho))^{1/2}.$$

*Proof:* First observe that if we knew that inequality (5.27) holds with the right-hand side replaced by  $(\alpha(r)\mathcal{E}^D(u, u) + r\Phi(\hat{h}_\infty))^{1/2} \|\rho\|_{L^2(D;m)}$ , then using this modified inequality and repeating step by step the reasoning of the proof of Lemma 5.23 and Proposition 5.24, we would get

$$\int_D V_{T_s}^*(x) \rho(x) m(dx) \leq C_r \|P_{T_s}^D \rho\|_{L^2(D;m)}$$

with  $C_r$  as in the proposition. This together with (5.32) implies the desired estimate. Therefore, what is left is to show that for every  $u \in \mathcal{D}_e(\mathcal{E}^D)$ ,

$$\int_D \left( \sup_{\tau \in \mathcal{T}^{\tau D}} \mathbb{E}_x |u(X_\tau)| \right) \rho(x) m(dx) \leq (\alpha(r)\mathcal{E}^D(u, u) + r\Phi(\hat{h}_\infty))^{1/2} \|\rho\|_{L^2(D;m)}. \tag{5.34}$$

Let  $v$  be defined as in the proof of Lemma 5.22. The proof of (5.34) differs from the proof of (5.27) only in the estimate of the term  $\int_D v \rho dm$  appearing in (5.29). In the present situation, by Schwartz's inequality and (5.31) we have

$$\begin{aligned} \int_D v \rho dm &\leq \|v\|_{L^2(D;m)} \|\rho\|_{L^2(D;m)} \leq (\alpha(r)\mathcal{E}^D(v, v) + r\Phi(v))^{1/2} \|\rho\|_{L^2(D;m)} \\ &\leq (\alpha(r)\mathcal{E}^D(|u|, |u|) + r\Phi(v))^{1/2} \|\rho\|_{L^2(D;m)} \\ &\leq (\alpha(r)\mathcal{E}^D(u, u) + r\Phi(v))^{1/2} \|\rho\|_{L^2(D;m)}. \end{aligned}$$

Take  $u = \gamma - h$  in (5.28). Then, by Klimsiak (2021b, Proposition 3.16, Lemma 3.7), the process  $v(X)$  is the first component of the unique solution of RBSDE $^{\tau D}(0, 0, |\gamma - h|(X))$ . Hence, by Theorem 2.6,  $v = \hat{h}_\infty$ , and the proof is complete. □

**Corollary 5.28.** *Suppose that the assumptions of Proposition 5.27 hold, and moreover there exists  $\varepsilon > 0$  such that  $\Phi(\eta) \leq \varepsilon \|\eta\|_{L^2(D;m)}^2$ ,  $\eta \in L^2(D; m)$ . Then the conclusion of Proposition 5.27 holds with  $C_r$  replaced by*

$$\hat{C} := (1 + \varepsilon)^{1/2} (\|w\|_{L^2(D;m)} + (2\alpha(1/(2\varepsilon))\mathcal{E}^D(\gamma - h, \gamma - h))^{1/2}) \|\rho\|_{L^2(D;m)}.$$

*Proof:* Let  $u = \gamma - h \in \mathcal{D}_e(\mathcal{E}^D)$  and  $v$  be defined as in the proof of Lemma 5.22. In the proof of Proposition 5.27 it is shown that then  $v = \hat{h}_\infty$ . Hence

$$\mathcal{E}^D(\hat{h}_\infty, \hat{h}_\infty) \leq \mathcal{E}^D(|\gamma - h|, |\gamma - h|) \leq \mathcal{E}^D(\gamma - h, \gamma - h).$$

The desired result follows from Proposition 5.27, the assumption on  $\Phi$ , and the above inequality.  $\square$

*Example 5.29.* Let  $D$  be a domain in  $\mathbb{R}^d$  bounded at least in one direction, and  $m$  be the Lebesgue measure on  $\mathbb{R}^d$ . Let  $q_{ij} : D \rightarrow \mathbb{R}$ ,  $i, j = 1, \dots, d$ , be measurable functions such for each  $x \in D$  the matrix  $Q(x) = [q_{ij}(x)]_{i,j=1}^d$  is symmetric and satisfies the first condition in (5.12). Consider the form

$$\mathcal{E}^D(u, v) = \int_D Q(x) \nabla u(x) \nabla v(x) dx, \quad u, v \in \mathcal{D}(\mathcal{E}^D) = H_0^1(D).$$

It is well known (see, e.g., Fukushima et al. (2011, Section 3.1)) that  $(\mathcal{E}^D, H_0^1(D))$  is a symmetric regular Dirichlet form on  $L^2(D; m)$ . It is transient by (5.12) and Poincaré’s inequality (see Fukushima et al. (2011, Example 1.5.1)). Its generator is the divergence form operator (cf. (5.22))

$$L = \sum_{i,j=1}^d \partial_{x_i} (q_{ij}(x) \partial_{x_j}).$$

Define  $\lambda_2$  by (5.33). By Poincaré’s inequality  $\lambda_2 > 0$ , so by Remark 5.25, Corollary 5.28 applies with  $\beta(t) = e^{-2t\lambda_2}$ ,  $t > 0$ . Note that in general, even for regular  $h$ , the process  $h(X)$  is not a semimartingale under  $P_x$ . It is known that if  $h \in H_0^1(D)$ , then for  $m$ -a.e.  $x \in D$ , under the measure  $P_x$ , it is a Dirichlet process in the sense of Föllmer (for details and refinements see Rozkosz (2001)).

To get pointwise estimates for  $V_{T_s}^*$ , in the rest of this subsection we assume that  $D$  is connected and  $\mathbb{M}^D$  is strongly Feller, that is  $R_1^D(\mathcal{B}_b(D)) \subset C_b(D)$ , and  $R_1^D 1 \in C_\infty(D)$ . Therefore assumptions I–III of Fukushima et al. (2011, Section 6.4) are satisfied. Recall that  $\lambda_2$  is defined by (5.33). By Fukushima et al. (2011, Theorem 6.4.4), for every  $\lambda < \lambda_2$  we have

$$c_\lambda := \sup_{x \in D} \mathbb{E}_x e^{\lambda \tau_D} < \infty.$$

**Proposition 5.30.** *Assume that  $w, \hat{h}_\infty \in L^\infty(D; m)$ . Then for all  $\lambda < \lambda_2$  and  $x \in D$ ,*

$$\begin{aligned} V_{T_s}^*(x) &\leq c_\lambda e^{-\lambda T_s} (\|w\|_{L^\infty(D;m)} + \|\hat{h}_\infty\|_{L^\infty(D;m)}) \\ &\leq c_\lambda e^{-\lambda T_s} (\|w\|_{L^\infty(D;m)} + \|h - \gamma\|_{L^\infty(D;m)}). \end{aligned}$$

*Proof:* Let  $\rho$  be a positive function in  $L^\infty(D; m)$ . Then for any  $t > 0$ ,

$$\begin{aligned} P_t^D \rho(x) &= \mathbb{E}_x \mathbf{1}_{\{t < \tau_D\}} \rho(X_t) \leq P_x(t < \tau_D) \|\rho\|_{L^\infty(D;m)} = P_x(e^t < e^{\tau_D}) \|\rho\|_{L^\infty(D;m)} \\ &\leq e^{-\lambda t} \mathbb{E}_x e^{\lambda \tau_D} \|\rho\|_{L^\infty(D;m)} \leq e^{-\lambda t} c_\lambda \|\rho\|_{L^\infty(D;m)}. \end{aligned}$$

The desired result now follows from Theorem 5.1.  $\square$

5.5. *The semigroup theory approach.* Let  $m$  be a  $\sigma$ -finite positive Borel measure on  $E$ . In what follows  $Y$  denotes the Banach space  $C_\infty(D)$  or  $L^p(D; m)$  with  $p \geq 1$ . By  $\|\cdot\|_Y$  we denote the natural norm on  $Y$ . Assume that  $(P_t^D)$  is a  $C_0$ -semigroup on  $Y$ . By Pazy (1983, Theorem 4.4.1), if for some  $q \in [1, \infty)$ ,

$$\int_0^\infty \|P_t^D f\|_Y^q dt < \infty, \quad f \in Y,$$

then there are constants  $b \geq 1$  and  $\lambda > 0$  such that

$$\|P_t^D f\|_Y \leq b e^{-\lambda t} \|f\|_Y, \quad t > 0. \tag{5.35}$$

This estimate and Theorem 5.1 can be used to estimate  $V_{T_s}^*$  for some specific  $w$  and  $\hat{h}_\infty$ .

The assumption that  $(P_t^D)$  is a  $C_0$ -semigroup on  $L^p(D; m)$  with  $p \geq 1$  is satisfied for instance if  $m$  is excessive, that is

$$\int_E P_t f dm \leq \int_E f dm, \quad f \in \mathcal{B}^+(D).$$

On the other hand, if  $\mathbb{M}$  is Feller, then  $(P_t^D)$  is a  $C_0$ -semigroup of contractions on  $C_\infty(D)$  (since  $D$  is assumed to be Dirichlet regular).

The other useful result which can be applied in our context says (see, e.g., Pazy (1983, Theorem 4.4.3)) that if  $(P_t^D)$  is analytic and  $-\lambda_1 < 0$ , where  $-\lambda_1$  is defined by (5.23), then there are constants  $b \geq 1$  and  $\lambda > 0$  such that (5.35) is satisfied. Note also that if  $\mathbb{M}^D$  is  $m$ -symmetric, i.e.  $\int_E P_t^D f \cdot g dm = \int_E f \cdot P_t^D g dm$  for  $f, g \in \mathcal{B}^+(E)$ , then  $(P_t^D)$  is analytic on  $L^p(D; m)$  for any  $p > 1$  (see Liskevich and Perelmuter (1995)).

For analyticity of  $(P_t^D)$  on  $C_\infty(D)$  see, e.g., Taira (2020, Section 1.2.2) and Cerrai (2001, Section 3). If  $(P_t^D)$  is ultracontractive and is associated with a symmetric Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$ , then by Davies and Simon (1984, Theorem 2.1.5),  $(P_t^D)$  is analytic on  $L^\infty(D)$ . Finally, note that in Gomilko and Tomilov (2015) it is proved that if  $(P_t^D)$  is analytic on a Banach space  $Y$ , then for any Bernstein function  $\psi : (0, \infty) \rightarrow [0, \infty)$  the subordinated semigroup

$$P_t^{D, \psi} f := \int_0^\infty P_s^D f \mu_t(ds), \quad t > 0, f \in Y,$$

is again analytic. Here  $(\mu_t)_{t \geq 0}$  is a vaguely continuous semigroup of positive Borel measures on  $[0, \infty)$  with  $\mu_t([0, \infty)) \leq 1$ , which represents the Bernstein function  $\psi$ , i.e.

$$e^{-t\psi(\lambda)} = \int_0^\infty e^{-\lambda s} \mu_t(ds), \quad t > 0, \lambda > 0.$$

### 6. Valuation of American options

We consider  $d$ -dimensional dividend paying exponential Lévy models. In these models, under a risk-neutral measure (generally nonunique), the evolution of prices, on the time interval  $[0, \infty)$ , of financial assets with initial prices  $x_1 > 0, \dots, x_d > 0$  at time 0 is modeled by a Markov process  $\mathbb{M} = (X, P_x)$  (with  $x = (x_1, \dots, x_d)$ ) of the form

$$X_t^i = X_0^i e^{(r-\delta_i)t + \xi_t^i}, \quad t \geq 0, \tag{6.1}$$

where  $r \geq 0$  is the interest rate,  $\delta_i \geq 0$   $i = 1, \dots, d$ , are dividend rates and  $\xi = (\xi^1, \dots, \xi^d)$  is some Lévy process with  $\xi_0 = 0$  and the characteristic triple chosen so that if  $\delta_i = 0$ ,  $i = 1, \dots, d$ , then the discounted price processes  $t \mapsto e^{-rt} X_t^i = e^{\xi_t^i}$  are martingales under  $P_x$ . The state space of  $\mathbb{M}$  is  $E = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i > 0, i = 1, \dots, d\}$  and its life time is  $\zeta = \infty$ . The generator of  $\mathbb{M}$  has the form

$$Lf(x) = L_{BS}f(x) + L_I f(x), \quad f \in C_c(\mathbb{R}^d), x \in \mathbb{R}^d, \tag{6.2}$$

where  $L_{BS}$  is the Black–Scholes operator

$$L_{BS} = \frac{1}{2} \sum_{i,j=1}^d a_{ij} x_i x_j \partial_{x_i x_j}^2 + \sum_{i=1}^d (r - \delta_i) x_i \partial_{x_i}$$

with some positive definite symmetric matrix  $a = \{a_{ij}\}$  (volatility matrix). The operator  $L_I$  is defined by

$$L_I f(x) = \int_{\mathbb{R}^d} \left( f(x_1 e^{y_1}, \dots, x_d e^{y_d}) - f(x) - \sum_{i=1}^d x_i (e^{y_i} - 1) \partial_{x_i} f(x) \right) \nu(dy),$$

where  $\nu$  is a Lévy measure satisfying some additional integrability conditions (see below). For more details on the model see [Cont and Tankov \(2004\)](#); [Klimsiak and Rozkosz \(2018\)](#). When  $\nu = 0$ , the above model reduces to the multidimensional Black–Scholes model analysed carefully in [Klimsiak and Rozkosz \(2016\)](#); [Rozkosz \(2022\)](#). From the assumptions on the model it follows that for  $i = 1, \dots, d$  we have

$$X_t^i = x_i + \int_0^t (r - \delta_i) X_s^i ds + M_t^i, \quad t \geq 0, \quad P_x\text{-a.s.}, \tag{6.3}$$

for some martingales  $M^i$ ,  $i = 1, \dots, d$  (see [Klimsiak and Rozkosz \(2018, \(2.5\)\)](#)).

By the definition, the value function  $V_T$  of the American option with payoff function  $h : \mathbb{R}^d \rightarrow [0, \infty)$  and exercise time  $T > 0$  is given by the formula

$$V_T(x) = \sup_{\sigma \in \mathcal{T}^T} \mathbb{E}_x e^{-r\sigma} h(X_\sigma) = \sup_{\sigma \in \mathcal{T}^T} \mathbb{E}_x^r h(X_\sigma), \quad x \in E,$$

and the value function of the perpetual American option by the formula

$$V(x) = \sup_{\sigma \in \mathcal{T}} \mathbb{E}_x e^{-r\sigma} h(X_\sigma) = \sup_{\sigma \in \mathcal{T}} \mathbb{E}_x^r h(X_\sigma), \quad x \in E,$$

where  $\mathbb{M}^r = (X, P_x^r)$  is a Markov process which is the transformation of the process  $\mathbb{M}$  by the multiplicative functional  $A_t = e^{-rt}$ ,  $t \geq 0$ , and  $\mathbb{E}_x^r$  denotes the expectation with respect  $P_x^r$ .

In what follows, we assume that the payoff function  $h$  is positive, continuous and there is  $K > 0$  such that

$$|h(x)| \leq K(1 + |x|), \quad x \in \mathbb{R}^d.$$

As for  $\nu$ , we will assume that

$$\int_{\{|y|>1\}} |y|^2 e^{\beta|y|} \nu(dy) < \infty \tag{6.4}$$

for some  $\beta > 1$  if  $h$  is bounded, and  $\beta > 2$  in the general case. Note that (6.4) implies that  $\mathbb{E}_x |X_t|^\beta < \infty$ ,  $t \geq 0$  (see [Sato \(1999, Theorem 25.3\)](#)). We will also assume that

$$\det a > 0.$$

In the proof of the next theorem we apply our general results to the above model. In the notation of Section 5, in this model  $D = E$  and  $\mathbb{M}^r$  is the driving process. As for the data, we have  $\varphi = h$  and  $g = \psi = 0$ . Consequently,  $\gamma = 0$  and  $w = |h|$ .

**Theorem 6.1.** *Let  $\mathbb{M}$  and  $h$  satisfy the assumptions described above. Assume additionally that  $\delta_i > 0$  for  $i = 1, \dots, d$  or  $\|h\|_\infty < \infty$ . Then  $V_T(s, x) \nearrow V(x)$  as  $T \rightarrow \infty$  for all  $s \geq 0$  and  $x \in E$ . In fact, for all  $T > 0$  and  $x \in E$ ,*

$$V(x) - V_T(x) \leq 2e^{-rT} \|h\|_\infty$$

if  $h$  is bounded, and in the general case,

$$V(x) - V_T(x) \leq 2K(e^{-rT} + |x| \sum_{i=1}^d e^{-\delta_i T}). \tag{6.5}$$

*Proof:* The first inequality is immediate from Corollary 5.2. To prove (6.5), we first observe that it follows from (6.1) and the fact that  $t \mapsto e^{\xi_t^i}$  is a martingale that

$$\mathbb{E}_x X_t^i = x_i e^{(r-\delta_i)t}, \quad t \geq 0. \quad (6.6)$$

Let  $\eta_t^i = e^{-rt} X_t^i$ . By (6.3) and the integration by parts formula,

$$\eta_t = x_i e^{-rt} - \delta_i \int_0^t e^{-rs} X_s^i ds + \int_0^t e^{-rs} dM_s^i, \quad t \geq 0, \quad (6.7)$$

so  $\eta$  is a positive supermartingale with respect to  $P_x$ , and hence  $X$  is a supermartingale with respect to  $P_x^r$ . It follows that  $h(X)$  is a supermartingale with respect to  $P_x^r$ . We shall show that  $h(X)$  is of class (D) with respect to  $P_x^r$ . Observe that  $\mathbb{E}_x \eta_t^i \leq x_i e^{-\delta_i t} \leq x_i$ . Therefore, by Dellacherie and Meyer (1982, Theorem VI.6),  $\{\eta_t\}$  converges  $P_x$ -a.s. as  $t \rightarrow \infty$  to some integrable random variable. Moreover, under (6.4),  $\lim_{p \rightarrow 1^+} \mathbb{E}_x |X_1^i|^p = \mathbb{E}_x |X_1^i| = x_i e^{(r-\delta_i)}$ . Hence  $\mathbb{E}_x |\eta_1^i|^p \leq x_i e^{-\delta_1/2}$  for some  $p > 1$ . As a result, since  $\eta^i$  is stationary and with independent increments,  $\mathbb{E}_x |\eta_t^i| \leq x_i e^{-\delta_i t/2}$ ,  $t \geq 0$ . Therefore, if  $\delta_i > 0$  for  $i = 1, \dots, d$ , then  $\sup_{t \geq 0} \mathbb{E}_x |\eta_t^i|^p < \infty$  for some  $p > 1$ , so  $\{\eta_t\}$  converges in  $L^1(dP_x)$ . Furthermore, by (6.6),  $\mathbb{E}_x \int_0^\infty e^{-rs} X_s^i ds < \infty$ , so by monotone convergence,  $\int_0^t e^{-r\theta} X_\theta^i d\theta \rightarrow \int_0^\infty e^{-r\theta} X_\theta^i d\theta$  in  $L^1(dP_x)$  as  $t \rightarrow \infty$ . Therefore the martingale  $t \mapsto \int_0^t e^{-r\theta} dM_\theta$  is convergent in  $L^1(dP_x)$ . Consequently, it is uniformly integrable with respect to  $P_x$  and hence of class (D) (with respect to  $P_x$ ) by Dellacherie and Meyer (1982, Theorem VI.23). Since  $t \mapsto \int_0^t e^{-rs} X_s^i ds$  is also of class (D) with respect to  $P_x$ , it follows from (6.7) that  $\eta$  is of class (D) with respect to  $P_x$ . Clearly, this implies that  $t \mapsto e^{-rt} h(X_t)$  is of class (D) with respect to  $P_x$ . Therefore,  $t \mapsto h(X_t)$  is of class (D) with respect to  $P_x^r$ . Applying now Corollary 5.4 gives the second assertion of the theorem.  $\square$

In the language of the option pricing theory Theorem 6.1 says that in exponential Lévy models satisfying the assumptions given at the beginning of this section the fair price of an American option with payoff function  $h$  and maturity  $T$  converges to the fair price of the corresponding perpetual American option. This result generalizes the corresponding result from Rozkosz (2022) proved (by a different method) for multidimensional Black–Scholes models, i.e. when  $\nu = 0$ , and with the additional assumption that  $h$  is convex and Lipschitz. Note, however, that the method of Rozkosz (2022) together with the results from Klimsiak and Rozkosz (2016) also provides the early exercise formula for perpetual options.

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