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Exponentially slow mixing and hitting times of rare events for a reaction–diffusion model

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Abstract. We consider the superposition of symmetric simple exclusion dynamics speeded-up in time, with spin-flip dynamics in a one-dimensional interval with periodic boundary conditions. We show that the mixing time has an exponential lower bound in the system size if the potential of the hydrodynamic equation has two or more local minima. We also apply our estimates to show that the normalized hitting times of rare events converge to a mean one exponential random variable if the potential has a unique minimum.

1. Introduction

In this paper, we study the superposition of symmetric simple exclusion dynamics speeded-up in time, with spin-flip dynamics in a one-dimensional interval with periodic boundary conditions. We call this model the *reaction–diffusion model*[1]. De Masi, Ferrari, and Lebowitz in De Masi et al. (1986) have introduced this model to study a reaction–diffusion equation of the form

$$\partial_t \rho = (1/2)\Delta\rho - V'(\rho), \tag{1.1}$$

where V is a potential from a stochastic microscopic systems viewpoint. They showed the *hydrodynamic limit*, that is, the macroscopic density of the reaction–diffusion model evolves according to the reaction–diffusion equation (1.1), under diffusive scaling. We refer to De Masi and Presutti (1991) and Bertini et al. (2019, Subsection 3.1) and the references therein for the recent development of the reaction–diffusion model.

This paper is a continuation of our studies Landim and Tsunoda (2018); Farfán et al. (2019); Tanaka and Tsunoda (2020) and we use several results established in these papers. We have studied the hydrostatic limit and the dynamical large deviation principle in Landim and Tsunoda (2018), the static large deviation principle in Farfán et al. (2019), and rapid mixing in Tanaka and Tsunoda (2020). More precisely, in Tanaka and Tsunoda (2020), we have shown that the total variation mixing time is of the order $\log N$ (N is the system size) if the reaction–diffusion model is *attractive*

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[1]This model is also known as *Glauber+Kawasaki model*.

and the hydrodynamic equation (1.1) has a strictly convex potential. Therefore, it is natural to ask what happens when the potential V has two or more local minima.

We first consider the case where the potential V has two or more local minima and show that the total variation mixing time is bounded below by e^{cN} for some constant c for any N sufficiently large, where N is the system size. In particular, for the case of the original model introduced in De Masi et al. (1986), our result and rapid mixing established in Tanaka and Tsunoda (2020) imply a phase transition for the mixing time. Namely, if an inverse temperature of the system is larger than some critical temperature, the reaction–diffusion model exhibits exponentially slow mixing, otherwise rapid mixing. Note that this type of phase transition cannot be observed for the Glauber dynamics related to the Ising model on the one-dimensional periodic domain Levin and Peres (2017, Theorem 15.5). On the other hand, due to the fast stirring mechanism of the exclusion process, this model may be regarded as a "local mean-filed" system. Therefore its structure should be similar to the one of the mean-field model Levin and Peres (2017, Theorem 15.3).

Using hitting time estimates, which will be established in this paper, we study the hitting times of rare events for the reaction–diffusion model when the potential has a unique minimum. As the second main result, we show that the hitting time of an open set rescaled by the mean, which does not contain a unique minimum of the potential, converges to a mean one exponential random variable. We note that the techniques developed in this paper are robust enough to apply to other models, including boundary–driven exclusion processes Bertini et al. (2003); Bodineau and Giacomin (2004); Farfan (2009); Farfan et al. (2011).

We mention several papers related to this work. Our motivation to study this problem originates from two recently developed theories. One is the *macroscopic fluctuation theory* and the other is the *martingale approach to metastability*. For details of each theory, see survey papers Bertini et al. (2015) and Landim (2019), respectively. This paper combines these two theories following the Freidlin–Wentzell theory Freidlin and Wentzell (1998). The mixing time for the exclusion process or Glauber dynamics is related to the problem we consider. Lacoin et al. have extensively studied the mixing time for the exclusion process Lacoin and Leblond (2011); Lacoin (2016b,a, 2017); Labbé and Lacoin (2019, 2020). The mixing time for the Glauber dynamics has been classically studied. We only refer to Levin and Peres (2017) and the sophisticated work by Lubetzky and Sly Lubetzky and Sly (2013). The convergence to a mean one exponential random variable also has a long history in probability theory Keilson (1979). As clarified later, we use a general criterion established in Benois et al. (2013). Therefore, we also refer to the references in this paper. We finally mention Hinojosa's study. He has studied the convergence to a mean one exponential random variable of an exit time for the reaction–diffusion model on the entire domain in a double-well case Hinojosa (2004) and a one-well case Hinojosa (2018).

This paper is organized as follows. We introduce our model and results in Section 2. We prove in Section 3 one of our main results, which is Theorem 2.2. Section 4 is devoted to proving Lemma 3.4, which is critical in proving Theorem 2.2. In Section 5, we prove our second main result, which is Theorem 5.3. Since our argument strongly relies on the results in Landim and Tsunoda (2018); Farfán et al. (2019), we summarize several results for reader's convenience in the appendix. Appendix A collects miscellaneous properties about the reaction–diffusion equation and Appendix B discusses the rate function of the dynamical large deviation principle.

2. Notation and Results

Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$, $N \geq 1$, be a one-dimensional discrete torus with N points. Denote the set $\{0,1\}^{\mathbb{T}_N}$ by X_N and the elements of X_N by η, called *configurations*. For each $x \in \mathbb{T}_N$ and $\eta \in X_N$, $\eta(x)$ represents the occupation variable at site x so that $\eta(x) = 1$ if site x is occupied, and $\eta(x) = 0$ if site x is vacant. For each $x \neq y \in \mathbb{T}_N$, denote by $\eta^{x,y}, \eta^x$, the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$, and the configuration given by flipping the occupation variable $\eta(x)$, respectively.

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise,} \end{cases} \qquad \eta^{x}(z) = \begin{cases} 1-\eta(x) & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x. \end{cases}$$

Consider a superposition of the speeded-up symmetric simple exclusion process with spin-flip dynamics. The generator of this X_N-valued, continuous-time Markov process acts on functions $f : X_N \to \mathbb{R}$ as

$$L_N f = L_G f + N^2 L_K f,$$

where L_G is the generator of spin-flip dynamics (Glauber dynamics)

$$L_G f(\eta) = \sum_{x \in \mathbb{T}_N} c(x,\eta)[f(\eta^x) - f(\eta)],$$

and L_K is the generator of a symmetric simple exclusion process (Kawasaki dynamics)

$$L_K f(\eta) = (1/2) \sum_{x \in \mathbb{T}_N} [f(\eta^{x,x+1}) - f(\eta)].$$

In defining L_G, the jump rate $\{c(x,\eta) : x \in \mathbb{T}_N, \eta \in X_N\}$ is chosen as $c(x,\eta) = c(\eta(\cdot + x))$ for a given function $c : \{0,1\}^{\mathbb{Z}} \to [0,\infty)$, where modulo N carries the sum. We also assume that c is *local* in the sense that c depends only on finitely many occupation variables $\eta(x)$. Then, c is identified with a function on X_N for N sufficiently large.

In this paper, we always assume that c is strictly positive, assuring that the Markov process generated by L_N is irreducible. Therefore, the process admits a unique probability distribution, which is invariant under the dynamics. We denote by μ_N its unique stationary probability measure.

Fix a topological space X. For $I = [0,T]$, $T > 0$, or $I = \mathbb{R}_+ = [0,\infty)$, let $C(I,X)$ be the space of continuous trajectories from I to X, endowed with the uniform topology. Similarly, let $D(I,X)$ be the space of right continuous trajectories from I to X with left limits, endowed with the Skorokhod topology. For each N, let $\{\eta_t^N : t \geq 0\}$ be the continuous-time Markov process on X_N whose generator is given by L_N. For a probability measure ν on X_N, denote by $\mathbb{P}_\nu$ the probability measure on $D(\mathbb{R}_+, X_N)$ induced by the process η_t^N starting from ν. Denote the measure $\mathbb{P}_\nu$ by $\mathbb{P}_\eta$ when the probability measure ν is the Dirac measure concentrated on the configuration η. The expectation with respect to $\mathbb{P}_\eta$ is represented by $\mathbb{E}_\eta$.

Let $\nu_\rho = \nu_\rho^N$, $0 \leq \rho \leq 1$, be the Bernoulli product measure on X_N with a density ρ. Define the polynomial functions $B, D : [0,1] \to \mathbb{R}$ by

$$B(\rho) = \int [1-\eta(0)]c(0,\eta)d\nu_\rho, \quad D(\rho) = \int \eta(0)c(0,\eta)d\nu_\rho.$$

B and D are the average birth and death rates under ν_ρ, respectively. We also set $F(\rho) = B(\rho) - D(\rho)$ and denote a primitive function of $-F$ by V. We call V a *potential*. Note that V has at least one local minimum on $(0,1)$ since $F(0) > 0, F(1) < 0$, and $V(\rho)$ is a polynomial in ρ.

As examined in the introduction, De Masi, Ferrari, and Lebowitz in De Masi et al. (1986) have shown that under an appropriate convergence of the initial distribution, the macroscopic density

$$\pi_t^N = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t^N(x)\delta_{x/N},$$

converges in probability to a unique weak solution to the reaction–diffusion equation

$$\partial_t \rho = (1/2)\Delta\rho + F(\rho).$$

As clarified later, our proof strongly relies on the corresponding large deviation principle (Theorem 3.2).

We here give an example of the jump rate c. The following example has been given in De Masi et al. (1986).

Example 2.1. For $0 \le \gamma < 1$, define

$$c(\eta) = 1 + \gamma(1 - 2\eta(0))(\eta(1) + \eta(-1) - 1) + \gamma^2(2\eta(-1) - 1)(2\eta(1) - 1).$$

Letting $\gamma = \tanh \beta, \beta \ge 0$, the Glauber dynamics generated by L_G is reversible with respect to a Gibbs measure of the one-dimensional nearest neighbor Ising model at the inverse temperature β. However, our stationary measure μ_N is neither Bernoulli nor Gibbs, except $\gamma = 0$ Gabrielli et al. (1996).

An elementary calculation shows

$$\begin{aligned} B(\rho) &= (1-\rho)\left\{1 - 2\gamma(1-2\rho) + \gamma^2(1-2\rho)^2\right\}, \\ D(\rho) &= \rho\left\{1 + 2\gamma(1-2\rho) + \gamma^2(1-2\rho)^2\right\}, \end{aligned}$$

and

$$F(\rho) = -2(\rho - 1/2)\left\{1 - 2\gamma + 4\gamma^2(\rho - 1/2)^2\right\},$$

for each $\rho \in [0,1]$, and V defined by

$$V(\rho) = (1-2\gamma)(\rho - 1/2)^2 + 2\gamma^2(\rho - 1/2)^4$$

is a potential. V has two local minima if, and only if, $\gamma > 1/2$, otherwise, a unique minimum.

Let us recall the notion of the *total variation mixing time*. For any probability measures μ, ν on X_N, define

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_{A \subset X_N} |\mu(A) - \nu(A)| = \frac{1}{2}\sum_{\eta \in X_N} |\mu(\eta) - \nu(\eta)|.$$

Then, for each $0 < \varepsilon < 1$, we define the mixing time $t^N_{\mathrm{mix}}(\varepsilon)$ by

$$t^N_{\mathrm{mix}}(\varepsilon) = \inf\left\{t \ge 0 : \max_{\eta \in X_N} \|\mathbb{P}_\eta(\eta^N_t \in \cdot) - \mu_N(\cdot)\|_{\mathrm{TV}} \le \varepsilon\right\}.$$

In this paper, we first study the mixing time of the reaction–diffusion model when the potential V has two or more local minima. In this setting, we show that the mixing time has an exponential lower bound in N. To introduce this result, let $h_0 > 0$ be the constant given in (3.5). Its definition is postponed since we need some notation. The precise statement is as follows.

Theorem 2.2. *Assume that the potential V has ℓ local minima with $\ell \ge 2$. Then, for any $0 < \varepsilon < \ell^{-1}$ and any N sufficiently large, we have*

$$t^N_{\mathrm{mix}}(\varepsilon) \ge e^{Nh_0}.$$

Remark 2.3. The jump rate provided in Example 2.1 is *attractive* in the sense that, for any configurations η, ξ such that $\eta(x) \ge \xi(x)$ for any x, it holds that

$$\begin{cases} c(\eta) \le c(\xi), & \text{if } \eta(0) = \xi(0) = 1, \\ c(\eta) \ge c(\xi), & \text{if } \eta(0) = \xi(0) = 0. \end{cases}$$

Note that V is strictly convex if, and only if, $0 \le \gamma < 1/2$. In Tanaka and Tsunoda (2020), we have established that for any attractive reaction–diffusion model with a strictly convex potential, the mixing time is in the order of $\log N$. Therefore, the reaction–diffusion model exhibits a phase transition regarding the mixing time with the critical parameter $\gamma = 1/2$.

Remark 2.4. It is natural to expect that

$$\lim_{N\to\infty} \frac{1}{N} \log t^N_{\mathrm{mix}}(\varepsilon) = \widetilde{h}_0,$$

for some constant $\widetilde{h}_0$. To capture this asymptotic behavior, the transition from a metastable well to another metastable well, so-called metastability, must be studied. The metastable behavior of this model has been longstanding as an open problem (Kipnis and Landim (1999, Chapter 10)). We leave this problem as future work.

We also study the hitting times of rare events by applying large deviation estimates and some mixing time estimates in the case where the potential has a unique minimum. Roughly speaking, we show that when the process starts from a small neighborhood of the unique minimum of the potential, the normalized hitting time of an open set that does not contain a unique minimum of the potential converges to a mean one exponential random variable. Since we need some notations to state this result, its precise statement is postponed to Theorem 5.3.

3. Proof of Theorem 2.2

We prove in this section Theorem 2.2. The proof of Theorem 2.2 mainly consists of Lemmata 3.1 and 3.4. Lemma 3.1 provides some concentration results for the stationary states established in Landim and Tsunoda (2018). Lemma 3.4, which will be proved in Section 4, provides an asymptotic estimate of the escape time from a small neighborhood of metastable states. We start from introducing these results and then prove Theorem 2.2. In Sections 3 and 4, we always assume that the potential V has ℓ local minima with $\ell \geq 2$.

Let $\mathbb{T}$ be the one-dimensional continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$ and $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T})$ be the space of all nonnegative measures on $\mathbb{T}$ with the total mass bounded by 1, endowed with the weak topology. Note that $\mathcal{M}_+$ is compact under the weak topology. For a measure ϱ in $\mathcal{M}_+$ and a continuous function $G : \mathbb{T} \to \mathbb{R}$, denote the integral of G with respect to ϱ by $\langle \varrho, G\rangle$

$$\langle \varrho, G\rangle = \int_{\mathbb{T}} G(\theta)\varrho(d\theta).$$

For a measurable function $\rho : \mathbb{T} \to \mathbb{R}$, let $\|\rho\|_2$ denote the L^2-norm with respect to the Lebesgue measure on $\mathbb{T}$

$$\|\rho\|_2^2 = \int_{\mathbb{T}} \rho(\theta)^2 d\theta.$$

We also denote by $\langle \rho_1, \rho_2\rangle$ the L^2-inner product for measurable functions $\rho_1, \rho_2 : \mathbb{T} \to [0, 1]$

$$\langle \rho_1, \rho_2\rangle = \int_{\mathbb{T}} \rho_1(\theta)\rho_2(\theta)d\theta.$$

The space $\mathcal{M}_+$ is metrizable. By letting $e_0(\theta) = 1$, $e_k(\theta) = \sqrt{2}\cos(2\pi k\theta)$, and $e_{-k}(\theta) = \sqrt{2}\sin(2\pi k\theta)$, $k \in \mathbb{N}$, one can define the distance d on $\mathcal{M}_+$ using

$$d(\varrho_1, \varrho_2) = \sum_{k\in\mathbb{Z}} \frac{1}{2^{|k|}} |\langle \varrho_1, e_k\rangle - \langle \varrho_2, e_k\rangle|, \tag{3.1}$$

and one can show that the topology induced by this distance corresponds to the weak topology. When $\varrho(d\theta) = \rho(\theta)d\theta$ for a measurable function $\rho : \mathbb{T} \to [0, 1]$, we sometimes write these notions with ρ instead of ϱ. For instance, we denote $d(\varrho_1, \varrho_2)$ as $d(\rho_1, \rho_2)$ for $\varrho_i(d\theta) = \rho_i(\theta)d\theta, i = 1, 2$. Note that for any measurable functions $\rho, \rho' : \mathbb{T} \to [0, 1]$, we have

$$d(\rho, \rho') \leq 3\|\rho - \rho'\|_2. \tag{3.2}$$

For each $\varrho \in \mathcal{M}_+$ and each $\alpha > 0$, let $\mathcal{B}(\alpha;\varrho)$, $\mathcal{B}[\alpha;\varrho]$ be the α-open, α-closed neighborhood of ϱ in $\mathcal{M}_+$, respectively.

Let $\pi_N : X_N \to \mathcal{M}_+$ be the empirical measure defined by

$$\pi_N(\eta) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta(x)\delta_{x/N}, \quad \eta \in X_N,$$

where δ_θ is the Dirac measure that has a point mass at $\theta \in \mathbb{T}$. We also let $\mathcal{P}_N = \mu_N \circ (\pi_N)^{-1}$, which is a probability measure on $\mathcal{M}_+$.

Let S be the set of all classical solutions to the semi-linear elliptic equation

$$(1/2)\Delta\rho + F(\rho) = 0, \quad \text{on } \mathbb{T}, \tag{3.3}$$

and $\mathcal{M}_{\mathrm{sol}}$ be the set of all measures whose density is a classical solution to (3.3)

$$\mathcal{M}_{\mathrm{sol}} = \{\bar\varrho \in \mathcal{M}_+ : \bar\varrho(d\theta) = \bar\rho(\theta)d\theta, \bar\rho \in S\}.$$

The following result has been established in Landim and Tsunoda (2018).

Lemma 3.1 (Landim and Tsunoda (2018, Theorem 2.2)). *For any $\alpha > 0$, we have*

$$\lim_{N\to\infty} \mathcal{P}_N\left(\varrho \in \mathcal{M}_+ : \min_{\bar\varrho\in\mathcal{M}_{\mathrm{sol}}} d(\varrho,\bar\varrho) \geq \alpha\right) = 0.$$

To describe a metastable well, we recall the dynamical large deviation principle from Jona-Lasinio et al. (1993); Landim and Tsunoda (2018). Let $\mathcal{M}_{+,1}$ be the closed subset of $\mathcal{M}_+$ consisting of all absolutely continuous measures with a density bounded by 1

$$\mathcal{M}_{+,1} = \{\varrho \in \mathcal{M}_+ : \varrho(d\theta) = \rho(\theta)d\theta,\ 0 \leq \rho(\theta) \leq 1 \text{ a.e. } \theta \in \mathbb{T}\}.$$

Fix $T > 0$. Denote by $C^{m,n}([0,T]\times\mathbb{T})$, m, n in $\mathbb{N}_0$, the set of all real functions defined on $[0,T]\times\mathbb{T}$ which are m times differentiable in the first variable and n times in the second one, and whose derivatives are continuous.

For each trajectory $\pi(t,d\theta) = \rho(t,\theta)d\theta$ in $D([0,T],\mathcal{M}_{+,1})$, define the energy $\mathcal{E}_T(\pi)$ as

$$\mathcal{E}_T(\pi) = \sup_{G\in C^{0,1}([0,T]\times\mathbb{T})}\left\{2\int_0^T dt\ \langle\rho_t, \nabla G_t\rangle - \int_0^T dt \int_{\mathbb{T}} d\theta\ G(t,\theta)^2\right\}.$$

Note that the energy $\mathcal{E}_T(\pi)$ is finite if, and only if, ρ has a generalized derivative denoted by $\nabla\rho$, and this generalized derivative is square–integrable on $[0,T]\times\mathbb{T}$

$$\int_0^T dt \int_{\mathbb{T}} d\theta\ |\nabla\rho(t,\theta)|^2 < \infty.$$

Here, we have

$$\mathcal{E}_T(\pi) = \int_0^T dt \int_{\mathbb{T}} d\theta\ |\nabla\rho(t,\theta)|^2.$$

For each function G in $C^{1,2}([0,T]\times\mathbb{T})$, define the functional $\bar J_{T,G} : D([0,T],\mathcal{M}_{+,1}) \to \mathbb{R}$ by

$$\begin{aligned}\bar J_{T,G}(\pi) &= \langle\pi_T, G_T\rangle - \langle\pi_0, G_0\rangle - \int_0^T dt\ \langle\pi_t, \partial_t G_t + \frac{1}{2}\Delta G_t\rangle \\ &\quad - \frac{1}{2}\int_0^T dt\ \langle\chi(\rho_t), (\nabla G_t)^2\rangle - \int_0^T dt\ \left\{\langle B(\rho_t), e^{G_t} - 1\rangle + \langle D(\rho_t), e^{-G_t} - 1\rangle\right\},\end{aligned}$$

where $\chi(r) = r(1-r)$ is the mobility.

Let $J_{T,G} : D([0,T],\mathcal{M}_+) \to [0,\infty]$ be the functional defined by

$$J_{T,G}(\pi) = \begin{cases} \bar J_{T,G}(\pi) & \text{if } \pi \in D([0,T],\mathcal{M}_{+,1}), \\ \infty & \text{otherwise,}\end{cases}$$

and let $I_T : D([0,T],\mathcal{M}_+) \to [0,\infty]$ be the functional defined by

$$I_T(\pi) = \begin{cases} \sup J_{T,G}(\pi) & \text{if } \mathcal{E}_T(\pi) < \infty, \\ \infty & \text{otherwise,} \end{cases}$$

where the supremum is carried over all functions G in $C^{1,2}([0,T]\times\mathbb{T})$. It has been established in Landim and Tsunoda (2018, Theorem 4.7) that $I_T(\cdot)$ is lower semicontinuous and has compact level sets.

For any $T > 0$ and any measurable function $\rho : \mathbb{T} \to [0,1]$, define the dynamical large deviation rate function $I_T(\cdot|\rho) : D([0,T],\mathcal{M}_+) \to [0,\infty]$ as

$$I_T(\pi|\rho) = \begin{cases} I_T(\pi) & \text{if } \pi(0,d\theta) = \rho(\theta)d\theta, \\ \infty & \text{otherwise.} \end{cases}$$

We say that a sequence of initial configurations $\{\eta^N\}_N$ is *associated with* a measurable function $\rho : \mathbb{T} \to [0,1]$ if, for any continuous function $G : \mathbb{T} \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{T}_N} G(x/N)\eta^N(x) = \int_{\mathbb{T}} G(\theta)\rho(\theta)d\theta.$$

Fix $T > 0$ and let Q_{η^N} be the distribution on the path space $D([0,T],\mathcal{M}_+)$ of the $\mathcal{M}_+$-valued process $\pi_N(\eta^N_\cdot)$ starting from a deterministic configuration $\pi_N(\eta^N)$. The large deviation principle for the reaction–diffusion model was first established in Jona-Lasinio et al. (1993, Theorem 2.2) for the case where product measures provide the initial distribution. When the process starts from a deterministic configuration, Landim and Tsunoda (2018) established the same result.

Theorem 3.2 (Landim and Tsunoda (2018, Theorem 2.5))**.** *Assume that a sequence of initial configurations $\{\eta^N\}_N$ is associated with a measurable function $\rho : \mathbb{T} \to [0,1]$. Then, for any closed set $\mathfrak{C} \subset D([0,T],\mathcal{M}_+)$, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_{\eta^N}(\mathfrak{C}) \le -\inf_{\pi\in\mathfrak{C}} I_T(\pi|\rho).$$

Remark 3.3. Besides the assumption of Theorem 3.2, under the additional condition

(C) the functions B and D are concave,

the large deviation lower bound was also established in Jona-Lasinio et al. (1993); Landim and Tsunoda (2018). Namely, for any open set $\mathfrak{O} \subset D([0,T],\mathcal{M}_+)$, we have

$$\liminf_{N\to\infty} \frac{1}{N} \log Q_{\eta^N}(\mathfrak{O}) \ge -\inf_{\pi\in\mathfrak{O}} I_T(\pi|\rho).$$

Note that we do not use the large deviation lower bound in Sections 2 and 4, so we do not have to assume the condition (C) in these sections. Moreover, several results, which we cited in this paper from Landim and Tsunoda (2018); Farfán et al. (2019), remain in force because they all hold without the condition (C) (the lower bound is not involved everywhere).

Denote the positions at which the local minima of V are attained by $\rho_1,\dots,\rho_\ell$, $\ell \ge 2$. For each $i = 1,\dots,\ell$, let $\bar\rho_i$ be the constant function defined by $\bar\rho_i(\theta) \equiv \rho_i, \theta \in \mathbb{T}$ and $\bar\varrho_i$ the measure $\bar\varrho_i(d\theta) = \bar\rho_i(\theta)d\theta$. Clearly, $\bar\varrho_i \in \mathcal{M}_{\rm sol}$ for each $i = 1,\dots,\ell$.

Fix $i = 1,\dots,\ell$. We construct small regions around $\bar\varrho_i$ denoted by $\mathcal{A}_i, \mathcal{B}_i$, and $\mathcal{C}_i$. We shall consider the exit problem from $\mathcal{C}_i$ when the process starts in $\mathcal{A}_i$. This problem will be examined in Section 4 precisely. To analyze this problem, we need several auxiliary properties concerning the dynamical large deviation functional, the quasi-potential and solutions to the reaction–diffusion equation.

For this purpose, let α_i, β_i, and γ_i be positive numbers chosen in the following way. We denote the following sets by $\mathcal{A}_i, \mathcal{B}_i$, and $\mathcal{C}_i$, respectively

$$\mathcal{A}_i = \mathcal{B}(\alpha_i; \bar{\varrho}_i), \quad \mathcal{B}_i = \mathcal{B}[2\beta_i; \bar{\varrho}_i] \setminus \mathcal{B}(\beta_i; \bar{\varrho}_i), \quad \mathcal{C}_i = \mathcal{B}(\gamma_i; \bar{\varrho}_i).$$

(γ) First, choose $\gamma_i > 0$ so that $\mathcal{C}_i$ does not intersect with the γ_i-closed neighborhood of $\mathcal{M}_{\rm sol} \setminus \{\bar{\varrho}_i\}$. This choice is possible by Corollary A.5. Moreover, by Lemma B.5, we have

$$2h_i := \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} > 0,$$

where the function $V_i : \mathcal{M}_+ \to [0, \infty]$ is the quasi-potential

$$V_i(\varrho) = \inf\left\{I_T(\pi|\bar{\rho}_i) : T > 0, \pi \in D([0,T], \mathcal{M}_+), \pi_T = \varrho\right\}, \quad \varrho \in \mathcal{M}_+.$$

(β) Second, we choose $\beta_i > 0$ so that the following conditions are in force:

(β-1) $2\beta_i < \gamma_i$.

(β-2) For any $\rho(\theta)d\theta \in \mathcal{B}_i$, denote by ρ_t the unique weak solution to the Cauchy problem

$$\begin{cases} \partial_t \rho = (1/2)\Delta\rho + F(\rho), \\ \rho(0, \cdot) = \rho(\cdot). \end{cases} \tag{3.4}$$

Then, it holds that ρ_t converges to $\bar{\rho}_i$ in the supremum norm as $t \to \infty$. This choice is possible by Lemma A.4.

(β-3) There exists $T_{1,i} > 0$ such that for any $\beta' \le \beta_i, T' \ge T_{1,i}$ and $\widetilde{\rho}(\theta)d\theta \in \mathcal{B}[\beta'; \bar{\varrho}_i]$,

$$\inf_{\pi \in \mathfrak{C}_{T'}(\mathcal{C}_i^c)} I_{T'}(\pi|\widetilde{\rho}) \ge (3/2)h_i,$$

where for each $T > 0$ and each set $\mathcal{D} \subset \mathcal{M}_+$, $\mathfrak{C}_T(\mathcal{D})$ stands for the subset of $D([0,T], \mathcal{M}_+)$ consisting of all trajectories π for which there exists some time $t \in [0,T]$ such that $\pi(t)$ belongs to $\mathcal{D}$ or $\pi(t-)$ belongs to $\mathcal{D}$. This choice is possible by Lemma 4.2.

(α) Finally, we choose $\alpha_i > 0$ so that $\alpha_i < \beta_i$.

We also set

$$h_0 = \min\{h_i : i = 1, \dots, \ell\}. \tag{3.5}$$

Since h_i is positive for each $i = 1, \dots, \ell$, so is h_0.

For a general set $\mathcal{D} \subset \mathcal{M}_+$, we define the set $\mathcal{D}^N \subset X_N$ by $\mathcal{D}^N = \pi_N^{-1}(\mathcal{D})$.[2] Let $\mathcal{A}$ and $\mathcal{C}$ be the open sets given by

$$\mathcal{A} = \bigcup_{i=1}^{\ell} \mathcal{A}_i, \quad \mathcal{C} = \bigcup_{i=1}^{\ell} \mathcal{C}_i,$$

respectively, and H_N be the hitting time of $[\mathcal{C}^N]^c$:

$$H_N = \inf\left\{t \ge 0 : \eta_t^N \in [\mathcal{C}^N]^c\right\}. \tag{3.6}$$

In the previous display, and in what follows, for a general set A, A^c denotes the complement of A. The following lemma is critical in proving Theorem 2.2. Its proof is postponed to Section 4.

Lemma 3.4. *We have*

$$\lim_{N\to\infty} \max_{\eta^N \in \mathcal{A}^N} \mathbb{P}_{\eta^N}\left(H_N \le e^{Nh_0}\right) = 0,$$

where h_0 is the constant defined in (3.5).

[2] We denote a subset of $\mathcal{M}_+$ by a calligraphic letter. The corresponding subset with superscript N is a subset of X_N. For instance, for $\mathcal{D} \subset \mathcal{M}_+$, $\mathcal{D}^N$ is the subset of X^N defined as above.

We can now prove Theorem 2.2. Note that Lemma 3.1 only shows that the stationary state $\mathcal{P}_N$ asymptotically concentrates on $\mathcal{M}_{\rm sol}$, not on $\mathcal{A}$. Therefore we need to partition $\mathcal{M}_{\rm sol}$ in a suitable way. By this reason, although the proof of Theorem 2.2 is a standard consequence of Lemmata 3.1 and 3.4, we give a complete proof.

Proof of Theorem 2.2: Fix $0 < \varepsilon < \ell^{-1}$ and $\varepsilon_0 = (1 - \ell\varepsilon)/3$. We also let $\alpha_0 = \min_i \alpha_i$. By applying Lemma 3.1 for $\alpha = \alpha_0$, there exists $N_0 \geq 1$ such that for all $N \geq N_0$, we have

$$\mathcal{P}_N \left(\varrho \in \mathcal{M}_+ : \min_{\bar{\varrho} \in \mathcal{M}_{\rm sol}} d(\varrho, \bar{\varrho}) \leq \alpha_0 \right) \geq 1 - \varepsilon_0. \tag{3.7}$$

For each $i = 1, \ldots, \ell$, let $\widetilde{\mathcal{M}}_i[\alpha_0]$ be the α_0-closed neighborhood of $\widetilde{\mathcal{M}}_i = \mathcal{M}_{\rm sol} \setminus \{\bar{\varrho}_i\}$. Note that

$$\left\{ \varrho \in \mathcal{M}_+ : \min_{\bar{\varrho} \in \mathcal{M}_{\rm sol}} d(\varrho, \bar{\varrho}) \leq \alpha_0 \right\} = \bigcup_{i=1}^{\ell} \widetilde{\mathcal{M}}_i[\alpha_0].$$

Therefore, the union bound shows that

$$\mathcal{P}_N \left(\varrho \in \mathcal{M}_+ : \min_{\bar{\varrho} \in \mathcal{M}_{\rm sol}} d(\varrho, \bar{\varrho}) \leq \alpha_0 \right) \leq \sum_{i=1}^{\ell} \mathcal{P}_N \left(\widetilde{\mathcal{M}}_i[\alpha_0] \right). \tag{3.8}$$

It follows from (3.7) and (3.8) that for each $N \geq N_0$ there exists an integer $i_N \in \{1, \ldots, \ell\}$ such that

$$\mathcal{P}_N \left(\widetilde{\mathcal{M}}_{i_N}[\alpha_0] \right) \geq (1 - \varepsilon_0)/\ell. \tag{3.9}$$

Let $j_N = i_N + 1$ if $i_N = 1, \ldots, \ell - 1$, otherwise $j_N = 1$. Let $\{\eta^N\}_N$ be a sequence in $\mathcal{A}^N_{j_N}$. Then, from Lemma 3.4, there exists $N_1 \geq 1$ such that for all $N \geq N_1$, we have

$$\mathbb{P}_{\eta^N} \left(H_N \leq e^{N h_0} \right) \leq \varepsilon_0/\ell. \tag{3.10}$$

We claim that for any $N \geq \max(N_0, N_1)$ and $t \leq e^{N h_0}$, we have

$$\max_{\eta \in X_N} \| \mathbb{P}_\eta(\eta^N_t \in \cdot) - \mu_N(\cdot) \|_{\rm TV} > \varepsilon. \tag{3.11}$$

Once (3.11) is proven, then the conclusion of Theorem 2.2 is immediate by the definition of $t^N_{\rm mix}(\varepsilon)$.

Let us prove (3.11). Let $N \geq \max(N_0, N_1)$ and $t \leq e^{N h_0}$. We have

$$\begin{aligned} \max_{\eta \in X_N} \| \mathbb{P}_\eta(\eta^N_t \in \cdot) - \mu_N(\cdot) \|_{\rm TV} &\geq \| \mathbb{P}_{\eta^N}(\eta^N_t \in \cdot) - \mu_N(\cdot) \|_{\rm TV} \\ &\geq \left| \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c \right) - \mu_N([\mathcal{C}^N_{i_N}]^c) \right| \\ &\geq \mu_N([\mathcal{C}^N_{i_N}]^c) - \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c \right). \end{aligned} \tag{3.12}$$

Since $[\mathcal{C}^N_{i_N}]^c$ contains the set $\pi_N^{-1}(\widetilde{\mathcal{M}}_{i_N}[\alpha_0])$, we have from (3.9) and (3.12)

$$\begin{aligned} \max_{\eta \in X_N} \| \mathbb{P}_\eta(\eta^N_t \in \cdot) - \mu_N(\cdot) \|_{\rm TV} &\geq \mu_N \left(\pi_N^{-1}(\widetilde{\mathcal{M}}_{i_N}[\alpha_0]) \right) - \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c \right) \\ &= \mathcal{P}_N \left(\widetilde{\mathcal{M}}_{i_N}[\alpha_0] \right) - \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c \right) \\ &\geq (1 - \varepsilon_0)/\ell - \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c \right). \end{aligned} \tag{3.13}$$

Note that the event $\{\eta^N_t \in [\mathcal{C}^N_{i_N}]^c, H_N > e^{N h_0}\}$ is empty because $\eta^N_0 = \eta^N \in \mathcal{A}^N_{j_N}$ and $t \leq e^{N h_0}$. Therefore, we have from (3.10)

$$\begin{aligned} \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c \right) &= \mathbb{P}_{\eta^N} \left(\eta^N_t \in [\mathcal{C}^N_{i_N}]^c, H_N \leq e^{N h_0} \right) \\ &\leq \mathbb{P}_{\eta^N} \left(H_N \leq e^{N h_0} \right) \leq \varepsilon_0/\ell. \end{aligned} \tag{3.14}$$

Note that by the choice of ε_0 and $\varepsilon < \ell^{-1}$, we have

$$(1-2\varepsilon_0)/\ell = (1/3\ell) + (2\varepsilon/3) > \varepsilon. \tag{3.15}$$

Therefore, (3.11) follows from (3.13), (3.14), and (3.15), which completes proving Theorem 2.2. □

4. Proof of Lemma 3.4

In this section, we prove Lemma 3.4. Fix $i = 1, \dots, \ell$. We first observe that, starting from a configuration η^N belonging to $\mathcal{A}_i^N$, the process can not reach $\mathcal{A}_j^N, j \neq i$ before exiting from $\mathcal{C}_i^N$. Therefore, if we can show

$$\lim_{N\to\infty} \max_{\eta^N \in \mathcal{A}_i^N} \mathbb{P}_{\eta^N}\left(H_N \le e^{Nh_i}\right) = 0, \tag{4.1}$$

Lemma 3.4 immediately follows from the definition of h_0 and (4.1).

In what follows, we fix $i = 1, \dots, \ell$ and prove (4.1). Since i is kept fixed, we sometimes omit dependence on i for some notation. Note that when the process starts from $\mathcal{A}_i^N$,

$$H_N = \inf\left\{t \ge 0 : \eta_t^N \in [\mathcal{C}_i^N]^c\right\}. \tag{4.2}$$

Recall the definitions of $\mathcal{A}_i^N, \mathcal{B}_i^N$, and $\mathcal{C}_i^N$ from Section 2. We inductively define the sequence of stopping times, denoted by $\tau_0 \le \sigma_0 \le \tau_1 \le \sigma_2 \le \tau_2 \le \cdots$, as $\tau_0 = 0$,

$$\sigma_k = \inf\{t > \tau_k : \eta_t^N \in \mathcal{B}_i^N\} \quad \text{and} \quad \tau_k = \inf\{t > \sigma_{k-1} : \eta_t^N \in \mathcal{A}_i^N \cup [\mathcal{C}_i^N]^c\}.$$

We avoid heavy notation by omitting dependence on N and i for σ_k and τ_k. Note that we consider the process starting from $\mathcal{A}_i^N$ and only up to exiting from $\mathcal{C}_i^N$. We also consider the discrete-time Markov chain Z_k^N defined by $Z_k^N = \eta_{\tau_k}^N$. Note that Z_k^N is a Markov chain on $\mathcal{A}_i^N \cup [\mathcal{C}_i^N]^c$. Let ν_N be the exit time of Z_k^N from $\mathcal{C}_i^N$

$$\nu_N = \inf\{k \in \mathbb{N} : Z_k^N \in [\mathcal{C}_i^N]^c\}.$$

First, we estimate the one-step transition probability of Z_k^N from $\mathcal{A}_i^N$ to $[\mathcal{C}_i^N]^c$. A similar estimate was given in Farfán et al. (2019, Lemma 24). We emphasize that no lower bounds of the dynamical large deviation principle are needed in what follows.

Lemma 4.1. *There exists N_2, such that for any $N \ge N_2$ and any sequence $\eta^N \in \mathcal{A}_i^N$, we have*

$$\mathbb{P}_{\eta^N}(\nu_N = 1) \le e^{-Nh_i}.$$

In particular,

$$\max_{\eta^N \in \mathcal{A}_i^N} \mathbb{P}_{\eta^N}(\nu_N = 1) \le e^{-Nh_i}.$$

Proof: Recall the formula (4.2) and fix any sequence $\eta^N \in \mathcal{A}_i^N$. When the process starts from η^N,

$$\{\nu_N = 1\} = \left\{\eta_{\sigma_0}^N \in \mathcal{B}_i^N, \eta_{\tau_1}^N \in [\mathcal{C}_i^N]^c\right\}.$$

Therefore, from the strong Markov property, we have

$$\mathbb{P}_{\eta^N}(\nu_N = 1) \le \sup_{\xi \in \mathcal{B}_i^N} \mathbb{P}_\xi(\tau_1 = H_N).$$

Let $\{\xi_0^N\}_N$ be a sequence in $\mathcal{B}_i^N$ satisfying

$$\mathbb{P}_{\xi_0^N}(\tau_1 = H_N) = \sup_{\xi \in \mathcal{B}_i^N} \mathbb{P}_\xi(\tau_1 = H_N).$$

Let

$$h_* = -\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N}(\tau_1 = H_N).$$

By the definition of h_*, once we show $h_* > h_i$, there exists some $N_2 > 0$ such that for any $N \geq N_2$ and any sequence $\eta^N \in \mathcal{A}_i^N$, we have

$$\mathbb{P}_{\eta^N}\left(\nu_N = 1\right) \leq \mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N\right) \leq e^{-Nh_i},$$

and the conclusion follows.

To see $h_* > h_i$, we decompose the probability $\mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N\right)$ into

$$\mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N, H_N > T\right) + \mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N, H_N \leq T\right)$$

for each $T > 0$. Then, we have

$$\begin{aligned} -h_* &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N\right) \\ &\leq \max\left\{\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N, H_N > T\right), \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N}\left(H_N \leq T\right)\right\} \end{aligned} \tag{4.3}$$

for any $T > 0$.

The probability

$$\frac{1}{N} \log \mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N, H_N > T\right)$$

can be handled by Lemma B.4. To see this, for each $\alpha > 0$, let $\mathcal{M}_{\text{sol}}(\alpha)$ be the α-open neighborhood of $\mathcal{M}_{\text{sol}}$ and $\widetilde{H}_N(\alpha)$ be the hitting time of $\mathcal{M}_{\text{sol}}^N(\alpha)$

$$\widetilde{H}_N(\alpha) = \inf\left\{t \geq 0 : \eta_t^N \in \mathcal{M}_{\text{sol}}^N(\alpha)\right\}.$$

On the event $\{\tau_1 = H_N, H_N > T\}$, the process starting from $\mathcal{B}_i^N$ does not hit $\mathcal{M}_{\text{sol}}^N(\alpha_i)$ during $[0, T]$ because $\mathcal{C}_i$ does not intersect with the γ_i-closed neighborhood of $[\mathcal{M}_{\text{sol}} \setminus \{\bar{\varrho}_i\}]$ by the conditions (γ) and $\alpha_i < \gamma_i$. Therefore, on the event $\{\tau_1 = H_N, H_N > T\}$, and started from $\mathcal{B}_i^N$, we have

$$\widetilde{H}_N(\alpha_i) \geq H_N \geq T.$$

Moreover, by Lemma B.4, there exist constants T_0, C_0, and N_0, depending only on α_i, such that for all $N \geq N_0$ and all $k \geq 1$,

$$\sup_{\eta \in X_N} \mathbb{P}_\eta\left[\widetilde{H}_N(\alpha_i) \geq kT_0\right] \leq e^{-kC_0N}.$$

By letting $k = (2h_i)/C_0, \widetilde{T}_0 = (2h_iT_0)/C_0$ in the previous display, we have

$$\mathbb{P}_{\xi_0^N}\left(\tau_1 = H_N, H_N > \widetilde{T}_0\right) \leq \sup_{\eta \in X_N} \mathbb{P}_\eta\left[\widetilde{H}_N(\alpha_i) \geq \widetilde{T}_0\right] \leq e^{-2h_iN} \tag{4.4}$$

for all $N \geq N_0$.

Let us turn to the probability

$$\mathbb{P}_{\xi_0^N}\left(H_N \leq T\right).$$

Because $\mathcal{B}_i$ is compact in $\mathcal{M}_+$, there exists a subsequence $\{\xi_0^{N_k}\}_k$ of $\{\xi_0^N\}$ such that $\pi_N(\xi_0^{N_k})$ converges to some $\varrho \in \mathcal{B}_i$ as $k \to \infty$. Moreover, ϱ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$ because each configuration in X_N has at most one particle per site. Let $\rho : \mathbb{T} \to [0, 1]$ be the density of ϱ: $\varrho(d\theta) = \rho(\theta)d\theta$. Taking a further subsequence if necessary, we can also assume without loss of generality that the sequence $\{\xi_0^{N_k}\}_k$ satisfies

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N}\left(H_N \leq T\right) = \lim_{k\to\infty} \frac{1}{N_k} \log \mathbb{P}_{\xi_0^{N_k}}\left(H_{N_k} \leq T\right)$$

For each $T > 0$ and each set $\mathcal{D} \subset \mathcal{M}_+$, recall the definition of $\mathfrak{C}_T(\mathcal{D})$, which was introduced in the condition (β-3). Note that if $\mathcal{D}$ is a closed subset of $\mathcal{M}_+$, $\mathfrak{C}_T(\mathcal{D})$ is a closed subset of $D([0,T],\mathcal{M}_+)$. When the process starts from $\mathcal{B}_i^N$, we have

$$\{H_N \le T\} \subset \{\pi_N(\eta_\cdot^N) \in \mathfrak{C}_T(\mathcal{C}_i^c)\}.$$

Therefore, by Theorem 3.2, we have

$$\lim_{k\to\infty} \frac{1}{N_k} \log \mathbb{P}_{\xi_0^{N_k}} (H_{N_k} \le T) \le - \inf_{\pi \in \mathfrak{C}_T(\mathcal{C}_i^c)} I_T(\pi|\rho).$$

Recall the definition of $T_{1,i}$ and the condition (β-3). For $\widetilde{T}_i = \max(\widetilde{T}_0, T_{1,i})$, by the condition ($\beta$-3) the last expression is bounded by $-(3/2)h_i$. Therefore, we have

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N} \left(H_N \le \widetilde{T}_i \right) \le -(3/2)h_i. \tag{4.5}$$

Taking $T = \widetilde{T}_0$ in (4.3), by (4.4) and (4.5), we have

$$\begin{aligned}
-h_* &\le \max \left\{ \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N} \left(\tau_1^N = H_N, H_N > \widetilde{T}_0 \right), \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N} \left(H_N \le \widetilde{T}_0 \right) \right\} \\
&\le \max \left\{ -2h_i, \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\xi_0^N} \left(H_N \le \widetilde{T}_i \right) \right\} \\
&\le \max \{-2h_i, -(3/2)h_i\} < -h_i,
\end{aligned}$$

concluding the proof. □

The proof of the following lemma is close to that of Farfán et al. (2019, Lemma 23), which shows a similar bound in the case where $\mathcal{C}_i^c$ is replaced with $\mathcal{B}[\alpha_j; \bar{\varrho}_j]$, for $j \ne i$. Since its proof relies on several arguments performed in Farfán et al. (2019) and some parts of the proof is omitted, we give a complete proof for reader's convenience.

Lemma 4.2. *Fix $i = 1, \dots, \ell$ and any $\varepsilon > 0$. There exist $\beta_i = \beta_i(\varepsilon) > 0$ and $T_{1,i} = T_{1,i}(\varepsilon) > 0$ such that for any $\beta' \le \beta_i, T' \ge T_{1,i}$ and $\widetilde{\rho}(\theta)d\theta \in \mathcal{B}[\beta'; \bar{\varrho}_i]$,*

$$\inf_{\pi \in \mathfrak{C}_{T'}(\mathcal{C}_i^c)} I_{T'}(\pi|\widetilde{\rho}) \ge \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} - \varepsilon. \tag{4.6}$$

Proof: Assume that the conclusion of the lemma fails. Here, for any $\beta > 0$ and $T > 0$, there exist $\beta' \le \beta, T' \ge T, \widetilde{\rho}(\theta)d\theta \in \mathcal{B}[\beta'; \bar{\varrho}_i]$ and $\pi(t, d\theta) \in \mathfrak{C}_{T'}(\mathcal{C}_i^c)$ such that

$$I_{T'}(\pi|\widetilde{\rho}) < \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} - \varepsilon.$$

In particular, by letting $\beta = 1/n$ and $T = 1$ for any $n \in \mathbb{N}$, there exist $\beta_n' \le 1/n, T_n' \ge 1, \widetilde{\rho}_n(\theta)d\theta \in \mathcal{B}[\beta_n'; \bar{\varrho}_i]$ and $\pi_n \in \mathfrak{C}_{T_n'}(\mathcal{C}_i^c)$ such that

$$I_{T_n'}(\pi_n|\widetilde{\rho}_n) < \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} - \varepsilon. \tag{4.7}$$

By Lemma B.1, $\pi_n(t, d\theta)$ has a density $\rho_n(t,\theta)$ for each $t \in [0, T_n']$ and the trajectory $\pi_n \mapsto \pi_n(t, d\theta)$ is continuous in $\mathcal{M}_+$. By the definition of $\mathfrak{C}_{T_n'}(\mathcal{C}_i^c)$ and the continuity of π_n, there exists $0 < T_n'' \le T_n'$ such that $\pi_n(T_n'') \in \mathcal{C}_i^c$.

Let $\mathbb{D}$ be the space of measurable functions $\rho : \mathbb{T} \to [0,1]$ endowed with the L^2-topology and define the function $\mathbb{V}_i : \mathbb{D} \to \mathbb{R}$ by

$$\mathbb{V}_i(\rho) = V_i(\rho(\theta)d\theta), \quad \rho \in \mathbb{D}.$$

Note that $\mathbb{V}_i(\bar{\rho}_i) = 0$ and, by Farfán et al. (2019, Theorem 6), $\mathbb{V}_i$ is continuous at $\bar{\rho}_i$ in $\mathbb{D}$. Therefore, there exists $\beta_\dagger > 0$ such that $6\beta_\dagger < \gamma_i$ and $\mathbb{V}_i(\rho) \le \varepsilon/2$ for any $\rho \in \mathbb{D}$ satisfying $\|\rho - \bar{\rho}_i\|_2 \le 2\beta_\dagger$. Note that the set $\{\rho(\theta)d\theta : \rho \in \mathbb{D}, \|\rho - \bar{\rho}_i\|_2 \le 2\beta_\dagger\}$ is a closed subset of $\mathcal{M}_+$.

Define $T_n''' \geq 0$ by

$$T_n''' = \sup\{0 \leq t \leq T_n'' : \|\rho_n(t,\cdot) - \bar{\rho}_i(\cdot)\|_2 \leq 2\beta_\dagger\}.$$

When the set inside the supremum is empty, let $T_n''' = 0$. Note that $T_n''' < T_n''$ for any $n \in \mathbb{N}$. To see this, we can assume that $T_n''' > 0$ and take a sequence $0 < t_k \uparrow T_n''$ such that

$$\|\rho_n(t_k,\cdot) - \bar{\rho}_i(\cdot)\|_2 \leq 2\beta_\dagger$$

for any $k \in \mathbb{N}$. Because this condition is closed under the weak topology, letting $k \to \infty$ provides

$$\|\rho_n(T_n'',\cdot) - \bar{\rho}_i(\cdot)\|_2 \leq 2\beta_\dagger \tag{4.8}$$

By (3.2) and $6\beta_\dagger < \gamma_i$, we have

$$d(\rho_n(T_n''), \bar{\rho}_i) \leq 3\|\rho_n(T_n'',\cdot) - \bar{\rho}_i(\cdot)\|_2 \leq 6\beta_\dagger < \gamma_i.$$

Therefore, the fact $\pi_n(T_n'') \in \mathcal{C}_i^c$ yields $T_n''' < T_n''$. Let $\widetilde{T}_n = T_n'' - T_n'''$ and define the trajectory $\widetilde{\pi}_n(t,d\theta) = \widetilde{\rho}_n(t,\theta)d\theta \in D([0,\widetilde{T}_n], \mathcal{M}_+)$ by $\widetilde{\pi}_n(t,d\theta) = \pi_n(t + T_n''', d\theta)$ for $t \in [0,\widetilde{T}_n]$.

For each $\beta > 0$ and $T > 0$, let $\mathbf{D}_{T,\beta}$ be the set of trajectories $\pi(t,d\theta) = \rho(t,\theta)d\theta \in D([0,T], \mathcal{M}_{+,1})$ such that $\|\rho(t,\cdot) - \bar{\rho}(\cdot)\|_2 > \beta$ for all $0 \leq t \leq T$ and all $\bar{\rho} \in S$. By Lemma B.3, there exists $T_\dagger$ such that

$$\inf_{\pi \in \mathbf{D}_{T_\dagger,\beta_\dagger}} I_T(\pi) \geq \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\}. \tag{4.9}$$

On the other hand, by the construction of $\widetilde{\pi}_n$, we have

$$I_{T_n'}(\pi_n) \geq I_{\widetilde{T}_n}(\widetilde{\pi}_n), \tag{4.10}$$

and $\widetilde{\pi}_n \in \mathbf{D}_{\widetilde{T}_n,\beta_\dagger}$ for any $n \in \mathbb{N}$.

Let us consider the case that there exists n such that $\widetilde{T}_n \geq T_\dagger$. Here, by (4.9) we obtain

$$I_{\widetilde{T}_n}(\widetilde{\pi}_n) \geq I_{T_\dagger}(\widetilde{\pi}_n) \geq \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\},$$

contradicting (4.7) and (4.10). Therefore, it remains to consider the case that $\widetilde{T}_n \leq T_\dagger$ for any $n \in \mathbb{N}$.

Assume that $\widetilde{T}_n \leq T_\dagger$ for any $n \in \mathbb{N}$. Here, we extend $\widetilde{\pi}_n$ as a trajectory in $D([0,T_\dagger], \mathcal{M}_+)$ in the following way

$$\widetilde{\pi}_n(t,d\theta) = \begin{cases} \widetilde{\rho}(t,\theta)d\theta, & 0 \leq t \leq \widetilde{T}_n, \\ \bar{\rho}_n(t - \widetilde{T}_n, \theta)d\theta, & \widetilde{T}_n \leq t \leq T_\dagger, \end{cases}$$

where $\bar{\rho}_n$ is the solution to the Cauchy problem (3.4) with the initial condition $\widetilde{\rho}_n(\widetilde{T}_n)$. By Lemma B.2, we have

$$I_{T_\dagger}(\widetilde{\pi}_n) = I_{\widetilde{T}_n}(\widetilde{\pi}_n).$$

Therefore, by (4.7) and (4.10), we have

$$I_{T_\dagger}(\widetilde{\pi}_n) < \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} - \varepsilon.$$

Since $I_{T_\dagger}$ has compact level sets, there exists a subsequence of $\widetilde{\pi}_n$ converging to some $\widetilde{\pi}(t,d\theta) = \widetilde{\rho}(t,\theta)d\theta \in D([0,T_\dagger], \mathcal{M}_{+,1})$ such that

(i) $\|\widetilde{\rho}(0,\cdot) - \bar{\rho}_i(\cdot)\|_2 \leq 2\beta_\dagger$.

(ii) There exists some $0 \leq \widetilde{T} \leq T_\dagger$ such that $\widetilde{\pi}(\widetilde{T}) \in \mathcal{C}_i^c$.

To see (i), we must consider cases $T_n''' > 0$ and $T_n''' = 0$. In the former case, (i) follows from (4.8). In the latter case, (i) follows from $\widetilde{\pi}_n(0) = \pi_n(T_n''') = \pi_n(0) \in \mathcal{B}[1/n; \bar{\rho}_i]$, thereby, $\widetilde{\rho}(0) = \bar{\rho}_i$. (ii) follows from $0 \le \widetilde{T}_n \le T_\dagger$, $\widetilde{\pi}_n(\widetilde{T}_n) = \pi_n(T_n'') \in \mathcal{C}_i^c$ and $\widetilde{\pi}_n$ converges to $\widetilde{\pi}$ in the uniform topology. Moreover, because $I_{T_\dagger}$ is lower semicontinuous, we have

$$I_{T_\dagger}(\widetilde{\pi}) \le \liminf_{n\to\infty} I_{T_\dagger}(\widetilde{\pi}_n) < \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} - \varepsilon.$$

By (i) and the choice of $\beta_\dagger$, there exist some $T^{(0)} > 0$ and a trajectory $\pi^{(0)} \in D([0, T^{(0)}], \mathcal{M}_+)$ such that $\pi^{(0)}(0) = \bar{\varrho}_i, \pi^{(0)}(T^{(0)}) = \widetilde{\pi}(0)$ and $I_{T^{(0)}}(\pi^{(0)}) \le \varepsilon/2$. Let $T = T^{(0)} + \widetilde{T}$ and define the trajectory $\pi \in D([0, T], \mathcal{M}_+)$ by

$$\pi(t, d\theta) = \begin{cases} \pi^{(0)}(t, d\theta), & 0 \le t \le T^{(0)}, \\ \widetilde{\pi}(t - T^{(0)}, d\theta), & T^{(0)} \le t \le T. \end{cases}$$

Then we have

$$I_T(\pi) = I_{T^{(0)}}(\pi^{(0)}) + I_{T_\dagger}(\widetilde{\pi}) < \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\} - \varepsilon/2.$$

Finally, since the trajectory π satisfies $\pi(0) = \pi^{(0)}(0) = \bar{\varrho}_i$ and $\pi(T) = \widetilde{\pi}(\widetilde{T}) \in \mathcal{C}_i^c$, by the definition of V_i, we have

$$I_T(\pi) \ge \inf\{V_i(\varrho) : \varrho \notin \mathcal{C}_i\}.$$

This contradicts the penultimate display, which completes proving the lemma. □

Secondly, we show that the process does not escape quickly from $\mathcal{C}_i^N$ with probability less than one half. Moreover, this occurs uniformly in the starting points in $\mathcal{A}_i^N$.

Lemma 4.3. *There exists $T_2 > 0$, such that*

$$\lim_{N\to\infty} \max_{\eta^N \in \mathcal{A}_i^N} \mathbb{P}_{\eta^N}(\tau_1 \le T_2) = 0.$$

In particular, there exist $T_2 > 0$ and $N_3 > 0$ such that for any $N \ge N_3$ and sequence $\eta^N \in \mathcal{A}_i^N$, we have

$$\mathbb{P}_{\eta^N}(\tau_1 > T_2) \ge 1/2.$$

Proof: Let $T_2 > 0$. The probability in the lemma can be decomposed into

$$\mathbb{P}_{\eta^N}\left(\tau_1 \le T_2, \eta_{\tau_1}^N \in \mathcal{A}_i^N\right) + \mathbb{P}_{\eta^N}\left(\tau_1 \le T_2, \eta_{\tau_1}^N \in [\mathcal{C}_i^N]^c\right). \tag{4.11}$$

By the definition of ν_N, $\mathbb{P}_{\eta^N}(\nu_N = 1)$ bounds the second probability in (4.11). From Lemma 4.1, this probability vanishes as $N \to \infty$. However, by the strong Markov property the first probability in (4.11) is bounded above by

$$\mathbb{E}_{\eta^N}\left[\mathbb{P}_{\eta_{\sigma_0}^N}\left(\tau_1 \le T_2, \eta_{\tau_1}^N \in \mathcal{A}_i^N\right)\right]. \tag{4.12}$$

Recall the definition of $\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i; \bar{\varrho}_i])$, which is introduced in the condition (β-3). Note that (4.12) is bounded by

$$\max_{\xi^N \in \mathcal{B}_i^N} \mathbb{P}_{\xi^N}\left(\eta_\cdot^N : \pi_N(\eta_\cdot^N) \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i; \bar{\varrho}_i])\right) = \max_{\xi^N \in \mathcal{B}_i^N} Q_{\xi^N}\left(\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i; \bar{\varrho}_i])\right).$$

Let $\{\xi_0^N\}_N$ be a sequence satisfying

$$Q_{\xi_0^N}\left(\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i; \bar{\varrho}_i])\right) = \max_{\xi^N \in \mathcal{B}_i^N} Q_{\xi^N}\left(\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i; \bar{\varrho}_i])\right).$$

Performing a similar argument, as we did in the proof of Lemma 4.1 (see the paragraph after (4.4)), there exists a subsequence $\{\xi_0^{N_k}\}_k$ of $\{\xi_0^N\}_N$ such that $\{\xi_0^{N_k}\}_k$ is associated with some $\rho : \mathbb{T} \to [0,1]$ with $\rho(\theta)d\theta \in \mathcal{B}_i$ and

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_{\xi_0^N} \left(\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])\right) = \lim_{k\to\infty} \frac{1}{N_k} \log Q_{\xi_0^{N_k}} \left(\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])\right).$$

Then, by Theorem 3.2, the right-hand side of the last display is bounded above by

$$-\inf_{\pi\in\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])} I_{T_2}(\pi|\rho).$$

It remains to show that there exists $T_2 > 0$ such that

$$\inf_{\pi\in\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])} I_{T_2}(\pi|\rho) > 0. \tag{4.13}$$

To see this, for each $\widetilde{\rho}(\theta)d\theta \in \mathcal{B}_i$, let $\widetilde{\rho}_t$ be the solution to the Cauchy problem (3.4), with the initial condition $\widetilde{\rho}$. Let $\tau^{(i)}(\widetilde{\rho})$ be the first entrance time of $\widetilde{\rho}_t$ into $\mathcal{B}[\alpha_i;\bar{\varrho}_i]$, that is,

$$\tau^{(i)}(\widetilde{\rho}) = \inf\{t \ge 0 : \widetilde{\rho}_t \in \mathcal{B}[\alpha_i;\bar{\varrho}_i]\}.$$

Note that $\tau^{(i)}(\widetilde{\rho})$ is finite because by (β-2), $\widetilde{\rho}_t$ converges to $\bar{\rho}_i$ as $t \to \infty$ for any $\widetilde{\rho}(\theta)d\theta \in \mathcal{B}_i$. Moreover, from Corollary A.3, the application $\widetilde{\rho}(\theta)d\theta \in \mathcal{B}_i \cap \mathcal{M}_{+,1} \mapsto \tau^{(i)}(\widetilde{\rho})$ is lower semicontinuous with respect to the weak topology. Let $\widetilde{T}_2$ be the constant defined by

$$\widetilde{T}_2 = \inf\left\{\tau^{(i)}(\widetilde{\rho}) : \widetilde{\rho}(\theta)d\theta \in \mathcal{B}_i\right\} = \min\left\{\tau^{(i)}(\widetilde{\rho}) : \widetilde{\rho}(\theta)d\theta \in \mathcal{B}_i\right\}.$$

The second equality follows from the compactness of $\mathcal{B}_i$ and the mentioned lower semi-continuity of $\tau^{(i)}$. Note that it is not difficult to see that $\widetilde{T}_2 > 0$. Let $T_2 = \widetilde{T}_2/2$.

Before turning to show (4.13), we see that some $\widetilde{\pi} \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])$ attains the infimum in (4.13). Indeed, let us take any $\pi' \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])$ satisfying $I_{T_2}(\pi'|\rho) < \infty$. Because $I_{T_2}(\cdot|\rho)$ has a compact level set and $\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])$ is closed, the subset of $D([0,T_2],\mathcal{M}_+)$ defined by

$$\mathfrak{C}' := \left\{\pi \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i]) : I_{T_2}(\pi|\rho) \le I_{T_2}(\pi'|\rho)\right\},$$

is a compact subset of $D([0,T_2],\mathcal{M}_+)$. Because $I_{T_2}(\cdot|\rho)$ is lower semicontinuous, there exists some $\widetilde{\pi} \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])$ such that

$$I_{T_2}(\widetilde{\pi}|\rho) = \inf_{\pi\in\mathfrak{C}'} I_{T_2}(\pi|\rho) = \inf_{\pi\in\mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])} I_{T_2}(\pi|\rho).$$

Therefore, $\widetilde{\pi} \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])$ can attain the infimum in (4.13).

Now, assume that $I_{T_2}(\widetilde{\pi}|\rho) = 0$. Here, from Lemma B.2, the density of $\widetilde{\pi}(t,d\theta)$, denoted by $\widetilde{\rho}(t,\theta), 0 \le t \le T_2$, is the weak solution to the Cauchy problem (3.4) with the initial condition ρ. However, this contradicts $\widetilde{\pi} \in \mathfrak{C}_{T_2}(\mathcal{B}[\alpha_i;\bar{\varrho}_i])$ and $\widetilde{\rho}_t(\theta)d\theta \notin \mathcal{B}[\alpha_i;\bar{\varrho}_i]$ for any $0 \le t \le T_2$. Therefore, (4.13) has been shown, completing the proof of Lemma 4.3. □

Invoking Lemmata 4.1 and 4.3, proving Lemma 3.4 is similar to the one of Freidlin and Wentzell (1998, Chapter 4, Theorem 4.2).

Proof of Lemma 3.4: As mentioned in the first paragraph of this section, to prove Lemma 3.4 it is enough to show (4.1). Fix any sequence $\eta^N \in \mathcal{A}_i^N$, such that

$$\mathbb{P}_{\eta^N}\left(H_N \le e^{Nh_i}\right) = \max_{\zeta^N\in\mathcal{A}_i^N} \mathbb{P}_{\zeta^N}\left(H_N \le e^{Nh_i}\right).$$

First, note that

$$\mathbb{P}_{\eta^N}\left(H_N \le e^{Nh_i}\right) \le \mathbb{P}_{\eta^N}\left(\tau_1 = H_N\right) + \mathbb{P}_{\eta^N}\left(\tau_1 < H_N, H_N \le e^{Nh_i}\right).$$

The first probability of the right-hand side equals

$$\mathbb{P}_{\eta^N}\left(\nu_N = 1\right),$$

and vanishes as $N \to \infty$ by Lemma 4.1. On the other hand, by the strong Markov property, the last probability of the penultimate display is bounded by

$$\sum_{k=1}^{\infty} \mathbb{E}_{\eta^N}\left[\mathbf{1}\left\{\tau_1 < H_N\right\} \mathbb{P}_{\eta^N_{\tau_1}}\left(\nu_N = k, H_N \leq e^{Nh_i}\right)\right].$$

Note that, on the event $\{\tau_1 < H_N\}$, we have $\eta^N_{\tau_1} \in \mathcal{A}^N_i$. Let T_2 be the constant chosen according to Lemma 4.3. We also let $m_N = \lceil (3/T_2) e^{N^{1/2} h_i} \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. For any configuration $\xi^N \in \mathcal{A}^N_i$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}_{\xi^N}\left(\nu_N = k, H_N \leq e^{Nh_i}\right) \\
&\quad \leq \mathbb{P}_{\xi^N}\left(\nu_N \leq m_N\right) + \sum_{k=m_N}^{\infty} \mathbb{P}_{\xi^N}\left(\nu_N = k, \tau_k \leq e^{Nh_i}\right) \\
&\quad \leq \mathbb{P}_{\xi^N}\left(\nu_N \leq m_N\right) + \mathbb{P}_{\xi^N}\left(\nu_N \geq m_N, \tau_{m_N} \leq e^{Nh_i}\right). \qquad (4.14)
\end{aligned}$$

From Lemma 4.1,

$$\mathbb{P}_{\xi^N}\left(\nu_N = 1\right) \leq e^{-Nh_i},$$

for any N sufficiently large and any $\xi^N \in \mathcal{A}^N_i$. Therefore, by the strong Markov property, we have

$$\mathbb{P}_{\xi^N}\left(\nu_N > m_N\right) \geq \left(1 - e^{-Nh_i}\right)^{m_N}.$$

As the right-hand side of the last expression converges to 1 as $N \to \infty$, we have

$$\lim_{N\to\infty} \max_{\xi^N \in \mathcal{A}^N_i} \mathbb{P}_{\xi^N}\left(\nu_N \leq m_N\right) = 0.$$

Let us address the second probability in (4.14). From the trivial decomposition

$$\tau_{m_N} = \left(\tau_{m_N} - \tau_{m_N - 1}\right) + \left(\tau_{m_N - 1} - \tau_{m_N - 2}\right) + \cdots + \left(\tau_1 - \tau_0\right),$$

τ_{m_N} can be bounded below by

$$T_2 \sum_{k=1}^{m_N} \mathbf{1}\left\{\tau_k - \tau_{k-1} > T_2\right\} =: R_N.$$

Therefore,

$$\begin{aligned}
\mathbb{P}_{\xi^N}\left(\nu_N \geq m_N, \tau_{m_N} \leq e^{Nh_i}\right) &\leq \mathbb{P}_{\xi^N}\left(\nu_N \geq m_N, R_N \leq e^{Nh_i}\right) \\
&\leq \mathbb{P}_{\xi^N}\left(\nu_N \geq m_N, \frac{R_N}{T_2 m_N} \leq 1/3\right).
\end{aligned}$$

Note that R_N/T_2 is the sum of independent Bernoulli random variables. Moreover, by Lemma 4.3, there exists N_4 such that, for any $N \geq N_4$ and any configuration $\xi^N \in \mathcal{A}^N_i$, under $\mathbb{P}_{\xi^N}$ and on the event $\{\nu_N \geq m_N\}$ the mean of each increment of R_N/T_2 is larger than 1/2. Therefore, the last probability vanishes as $N \to \infty$ uniformly in $\xi^N \in \mathcal{A}^N_i$. Therefore,

$$\lim_{N\to\infty} \max_{\xi^N \in \mathcal{A}^N_i} \mathbb{P}_{\xi^N}\left(\nu_N \geq m_N, \tau_{m_N} \leq e^{Nh_i}\right) = 0,$$

completing the proof of (4.1) and, hence, Lemma 3.4. □

5. Hitting times of rare events

In this section, we study the hitting times of rare events in the case where;

(UM) the potential V has a unique minimum.

Denote by ρ_* the position at which the minimum of V is attained. As before, let $\bar{\rho}_*(\theta) \equiv \rho_*, \theta \in \mathbb{T}$ and $\bar{\varrho}_*(d\theta) = \bar{\rho}_*(\theta)d\theta$, respectively.

Fix an open subset $\mathcal{O}$ of $\mathcal{M}_+$ such that $\mathcal{O} \cap \mathcal{M}_{+,1} \neq \emptyset$ and

$$d(\bar{\varrho}_*, \mathcal{O}) := \inf_{\varrho \in \mathcal{O}} d(\bar{\varrho}_*, \varrho) > 0.$$

Note that

$$\mu_N(\mathcal{O}^N) = \mathcal{P}_N(\mathcal{O}) \leq \mathcal{P}_N\left(\varrho \in \mathcal{M}_+ : d\left(\bar{\varrho}_*, \varrho\right) \geq d(\bar{\varrho}_*, \mathcal{O})\right).$$

Under condition (UM), the semi-linear elliptic equation (3.3) admits a unique classical solution given by $\bar{\rho}_*$. Therefore, by Lemma 3.1 the last expression vanishes as $N \to \infty$. Thus, we have

$$\lim_{N \to \infty} \mu_N(\mathcal{O}^N) = 0. \tag{5.1}$$

We apply our large deviation estimates and some mixing time estimates to show the convergence of hitting times of rare events. Therefore, let $H_N^{\mathcal{O}}$ be the hitting time of $\mathcal{O}^N$

$$H_N^{\mathcal{O}} = \inf\{t \geq 0 : \eta_t^N \in \mathcal{O}^N\}.$$

To establish the convergence of $H_N^{\mathcal{O}}$, we need the following result, which is Lemma 3.4 in this setting.

Lemma 5.1. *Let $\gamma_* = d(\bar{\varrho}_*, \mathcal{O})$. There exists $0 < \alpha_* < \gamma_*$, such that*

$$\lim_{N \to \infty} \max_{\eta^N \in \mathcal{A}_i^N} \mathbb{P}_{\eta^N}\left(H_N^{\mathcal{O}} \leq e^{Nh_*}\right) = 0,$$

where

$$\begin{aligned} \mathcal{A}_* &:= \mathcal{B}(\alpha_*, \bar{\varrho}_*) \\ 2h_* &:= \inf\{V_*(\varrho) : \varrho \notin \mathcal{C}_* = \mathcal{B}(\gamma_*, \bar{\varrho}_*)\} > 0, \\ V_*(\varrho) &:= \inf\left\{I_T(\pi|\bar{\rho}_*) : T > 0, \pi \in D([0,T], \mathcal{M}_+), \pi_T = \varrho\right\}, \quad \varrho \in \mathcal{M}_+. \end{aligned}$$

For two real-valued sequences a_N, and b_N, we denote $a_N \ll b_N$ if $a_N/b_N \to 0$ as $N \to \infty$. Recall the definition of the mixing time $t_{\text{mix}}^N(\varepsilon)$. Assume that for some $a > 0$;

(MT) $t_{\text{mix}}^N(1/4) \ll N^a$.

As mentioned before, in Tanaka and Tsunoda (2020), it has been shown that, for any attractive reaction–diffusion model with a strictly convex potential we have $t_{\text{mix}}^N(1/4) = O(\log N)$, where O denotes the Bachmann–Landau notation. Therefore in this case (MT) is satisfied.

We also need the static large deviation principle for empirical measures. More precisely, we assume the following static large deviation principle (SLDP).

(SLDP) The sequence of probability measures $\{\mathcal{P}_N\}_N$ on $\mathcal{M}_+$ satisfies a large deviation principle with speed N and rate function V_*. Namely, for each closed set $\mathcal{K} \subset \mathcal{M}_+$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathcal{P}_N(\mathcal{K}) \leq -\inf_{\varrho \in \mathcal{K}} V_*(\varrho),$$

and for each open set $\mathcal{U} \subset \mathcal{M}_+$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathcal{P}_N(\mathcal{U}) \geq -\inf_{\varrho \in \mathcal{U}} V_*(\varrho).$$

Under condition (C) in Remark 3.3 and (UM), (SLDP) has been established in Farfán et al. (2019). Note that we use the upper bound of (SLDP) only in the following argument.

Before stating the next result, let us return to Example 2.1 to see an example that satisfies the conditions (UM), (MT), and (SLDP).

Example 5.2. Recall the jump rate given in Example 2.1 for $0 \le \gamma < 1$, and define

$$c(\eta) = 1 + \gamma(1 - 2\eta(0))(\eta(1) + \eta(-1) - 1) + \gamma^2(2\eta(-1) - 1)(2\eta(1) - 1).$$

(UM) holds if, and only if, $0 \le \gamma \le 1/2$, (C) holds if, and only if, $0 \le \gamma \le 1/2$ and V is strictly convex if $0 \le \gamma < 1/2$. Thus, the jump rate c satisfies (UM), (MT), and (SLDP) for any $0 \le \gamma < 1/2$.

Our second main result is as follows.

Theorem 5.3. *Assume the conditions (UM), (MT) and (SLDP). Then, for any sequence $\eta^N \in \mathcal{A}_*^N$, $H_N^{\mathcal{O}}/\mathbb{E}_{\mu_N}[H_N^{\mathcal{O}}]$ under $\mathbb{P}_{\eta^N}$ converges in distribution to a mean one exponential random variable.*

Theorem 5.3 follows from a general result established in Benois et al. (2013). We first state their result alongside our setting and then prove Theorem 5.3.

Recall the definition of the generator L_N. Define the jump rates $R_N(\eta, \xi), \eta, \xi \in X_N$ using the formula

$$L_N f(\eta) = \sum_{\xi \in X_N} R_N(\eta, \xi)[f(\xi) - f(\eta)], \quad f : X_N \to \mathbb{R}.$$

Let A_N be a sequence of subsets of X_N and H_{A_N} be the hitting time of A_N

$$H_{A_N} = \inf\{t \ge 0 : \eta_t^N \in A_N\}.$$

Denote the average rate at which the process jumps from A_N^c to A_N by $r_N(A_N^c, A_N)$

$$r_N(A_N^c, A_N) = \frac{1}{\mu_N(A_N^c)} \sum_{\xi \in A_N^c} \mu_N(\xi) R_N(\xi, A_N),$$

where $R_N(\xi, A_N) = \sum_{\zeta \in A_N} R_N(\xi, \zeta)$.

The following result has been established in Benois et al. (2013).

Theorem 5.4 (Benois et al. (2013, Corollary 1.2))**.** *Let A_N be a sequence of subsets of X_N such that*

$$\lim_{N \to \infty} \mu_N(A_N) = 0, \tag{5.2}$$

$$t_{\text{mix}}^N(1/4) \ll r_N(A_N^c, A_N)^{-1}, \tag{5.3}$$

and there exists a sequence S_N such that

$$t_{\text{mix}}^N(1/4) \ll S_N \ll \mathbb{E}_{\mu_N}[H_{A_N}]. \tag{5.4}$$

Further, let $\{\nu_N\}_N$ be a sequence of probability measures on X_N such that

$$\lim_{N \to \infty} \mathbb{P}_{\nu_N}[H_{A_N} < S_N] = 0. \tag{5.5}$$

Then, $H_{A_N}/\mathbb{E}_{\mu_N}[H_{A_N}]$ under $\mathbb{P}_{\nu_N}$ converges in distribution to a mean one exponential random variable.

We can now prove Theorem 5.3.

Proof of Theorem 5.3: For Theorem 5.4, to prove Theorem 5.3, it is enough to show (5.2)-(5.5) in the case $A_N = \mathcal{O}^N$ and $\nu_N = \delta_{\eta^N}$ for a given sequence $\eta^N \in \mathcal{A}_*^N$.

Fix any sequence $\eta^N \in \mathcal{A}_*^N$. Note that (5.2) is nothing but (5.1), and (5.5) holds with $S_N = e^{(h_*/2)N}$ by Lemma 5.1. Moreover, the lower bound of (5.4) with $S_N = e^{(h_*/2)N}$ is clear by (MT). The upper bound of (5.4) can be computed as

$$\begin{aligned}
\mathbb{E}_{\mu_N}\left[H_N^{\mathcal{O}}\right] &= \sum_{\xi \in X_N} \mathbb{E}_\xi\left[H_N^{\mathcal{O}}\right]\mu_N(\xi) \\
&\geq \sum_{\xi \in \mathcal{A}_*^N} \mathbb{E}_\xi\left[H_N^{\mathcal{O}}\mathbf{1}\left\{H_N^{\mathcal{O}} > e^{Nh_*}\right\}\right]\mu_N(\xi) \\
&\geq e^{Nh_*}\sum_{\xi \in \mathcal{A}_*^N} \mathbb{P}_\xi\left(H_N^{\mathcal{O}} > e^{Nh_*}\right)\mu_N(\xi) \geq \frac{1}{2}e^{Nh_*}\mu_N\left(\mathcal{A}_*^N\right).
\end{aligned}$$

In the last inequality, we have used

$$\mathbb{P}_\xi\left(H_N^{\mathcal{O}} > e^{Nh_*}\right) \geq \frac{1}{2},$$

for N sufficiently large and any $\xi \in \mathcal{A}_*^N$ (this bound follows from Lemma 5.1). By Lemma 3.1, we have $\mu_N(\mathcal{A}_*^N) \geq 1/2$ for N sufficiently large. Thus, we have shown the upper bound of (5.4). By (MT), to conclude the proof, it is enough to show that there exists $b > 0$ such that

$$r_N([\mathcal{O}^N]^c, \mathcal{O}^N) \leq e^{-bN}, \tag{5.6}$$

for N sufficiently large.

Let us prove (5.6) for some $b > 0$. Let $\partial\mathcal{O}^N$ be the outer boundary of $\mathcal{O}^N$:

$$\partial\mathcal{O}^N = \left[\bigcup_{x \in \mathbb{T}_N}\left\{\eta \notin \mathcal{O}^N : \eta^{x,x+1} \in \mathcal{O}^N\right\}\right] \cup \left[\bigcup_{x \in \mathbb{T}_N}\left\{\eta \notin \mathcal{O}^N : \eta^{x} \in \mathcal{O}^N\right\}\right].$$

Note that for each $\xi \in [\mathcal{O}^N]^c$, $R_N(\xi, \mathcal{O}^N) = 0$ unless $\xi \in \partial\mathcal{O}^N$ and that

$$R_N(\xi, \mathcal{O}^N) \leq \sum_{\zeta \in X_N} R_N(\xi, \zeta) \leq N^3/2 + N\|c\|_\infty.$$

Then we have

$$r_N([\mathcal{O}^N]^c, \mathcal{O}^N) = \frac{1}{\mu_N([\mathcal{O}^N]^c)}\sum_{\xi \in \partial\mathcal{O}^N}\mu_N(\xi)R_N(\xi, \mathcal{O}^N) \leq N^4\frac{\mu_N(\partial\mathcal{O}^N)}{\mu_N([\mathcal{O}^N]^c)},$$

for N sufficiently large.

By (5.1), we have

$$\lim_{N\to\infty}\mu_N([\mathcal{O}^N]^c) = 1.$$

To estimate $\mu_N(\partial\mathcal{O}^N)$, let $\mathcal{K}$ be the closed set $\mathcal{B}(\alpha_*, \bar{\varrho}_*)^c$. Then we have $\partial\mathcal{O}^N \subset \mathcal{K}^N$ for N sufficiently large and by Lemma B.5, we have

$$\inf_{\varrho \in \mathcal{K}} V_*(\varrho) > 0.$$

Since

$$\mu_N(\partial\mathcal{O}^N) \leq \mu_N(\mathcal{K}^N) = \mathcal{P}_N(\mathcal{K}),$$

it follows from the upper bound of (SLDP) together with the previous bound that

$$\limsup_{N\to\infty}\frac{1}{N}\log\mu_N(\partial\mathcal{O}^N) \leq -\inf_{\varrho \in \mathcal{K}} V_*(\varrho).$$

Summarizing the previous arguments, we obtain

$$r_N([\mathcal{O}^N]^c, \mathcal{O}^N) \le \exp\left\{-\frac{N}{2} \inf_{\varrho \in \mathcal{K}} V_*(\varrho)\right\},$$

for N sufficiently large. Thus (5.6) is proven, completing the proof of Theorem 5.3. □

Appendix A. Reaction-diffusion equation

For reader's convenience, we collect miscellaneous lemmata from Landim and Tsunoda (2018); Farfán et al. (2019) which are used in this paper. When we need a generalization of existing results, we give a proof for the sake of completeness.

The following standard result is used for proving Proposition A.2.

Lemma A.1 (Farfán et al. (2019, Lemma 7))**.** *There exists a constant $C_0 > 0$ such that for any weak solutions ρ^j, $j = 1, 2$, to the Cauchy problem (3.4) with the initial condition ρ_0^j and for any $t > 0$,*

$$\|\rho_t^1 - \rho_t^2\|_2 \le e^{C_0 t}\|\rho_0^1 - \rho_0^2\|_2.$$

The following proposition is a generalization of a part of Farfán et al. (2019, Lemma 8). If we take ρ_0 as a stationary solution to the Cauchy problem (3.4), Farfán et al. (2019, Lemma 8) can be recovered.

Proposition A.2. *Let $\rho_0 : \mathbb{T} \to [0, 1]$ be a measurable function and $\rho_t(\theta) = \rho(t, \theta)$ be the unique weak solution to the Cauchy problem (3.4) with the initial condition ρ_0. For any $\beta > 0$ and $T > 0$, there exists $0 < \beta_0 < \beta$, depending only on β and T, such that for any measurable function $\widetilde{\rho}_0 : \mathbb{T} \to [0, 1]$ with $\widetilde{\rho}_0(\theta)d\theta \in \mathcal{B}[\beta_0; \rho_0]$, we have $\widetilde{\rho}(t, \theta)d\theta \in \mathcal{B}[\beta; \rho_t]$ for all $0 \le t \le T$, where $\widetilde{\rho}(t, \theta)$ is a unique weak solution to the Cauchy problem (3.4) with the initial condition $\widetilde{\rho}_0$.*

Proof: Fix $\beta > 0$ and $T > 0$. Let $\rho_0 : \mathbb{T} \to [0, 1]$ be a measurable function and $\rho_t(\theta) = \rho(t, \theta)$ be the unique weak solution of the Cauchy problem (3.4) with the initial condition ρ_0. Recall the definition of the complete orthogonal normal basis $\{e_k; k \in \mathbb{Z}\}$ introduced before (3.1).

Because $\rho_t(\theta)$ is a weak solution to the Cauchy problem (3.4), for any weak solution $\widetilde{\rho}_t(\theta) = \widetilde{\rho}(t, \theta)$ to the Cauchy problem (3.4), with the initial condition $\widetilde{\rho}_0$, we have

$$d(\widetilde{\rho}_t, \rho_t) \le d(\widetilde{\rho}_0, \rho_0) + \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \left| \frac{1}{2} \int_0^t ds \, \langle [\widetilde{\rho}_s - \rho_s], \Delta e_k \rangle + \int_0^t ds \, \langle [F(\widetilde{\rho}_s) - F(\rho_s)], e_k \rangle \right|.$$

The first term of the right-hand side is bounded by $\beta/2$ if $\widetilde{\rho}_0 \in \mathcal{B}[\beta/2; \rho_0]$. However, the sum is less than or equal to

$$t \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \left\{ (2\pi k)^2 + 2\|F\|_\infty \right\} =: C_1 \, t,$$

because $\rho_s, \widetilde{\rho}_s$ are bounded by 1, F is bounded, and $\|e_k\|_2 = 1$. Hence, if we set $T_1 = \beta/2C_1$, we have

$$\widetilde{\rho}_t(\theta)d\theta \in \mathcal{B}[\beta; \rho_t], \tag{A.1}$$

for any $\widetilde{\rho}_0(\theta)d\theta \in \mathcal{B}[\beta/2; \rho_0]$ and any $0 \le t \le T_1$,

Let P_t be the semigroup on $L^2(\mathbb{T})$ generated by $(1/2)\Delta$. Then, by Duhamel's formula, we have

$$\begin{aligned} \|\widetilde{\rho}_t - \rho_t\|_2 &\le \|P_t(\widetilde{\rho}_0 - \rho_0)\|_2 + \int_0^t \|P_{t-s}\left[F(\widetilde{\rho}_s) - F(\rho_s)\right]\|_2 \, ds \\ &\le \|P_t(\widetilde{\rho}_0 - \rho_0)\|_2 + t\|F'\|_\infty. \end{aligned} \tag{A.2}$$

Let $T_2 = \min\{T, T_1, \beta/(6e^{C_0 T}\|F'\|_\infty)\}$.

Because $P_t(\widetilde{\rho}_0 - \rho_0)$ is a solution to the heat equation, there exists some $0 < \beta_0 < \beta/2$, depending only on β and T, such that for any $\widetilde{\rho}_0(\theta)d\theta \in \mathcal{B}[\beta_0; \rho_0]$

$$\|P_{T_2}(\widetilde{\rho}_0 - \rho_0)\|_2 \le \beta/(6e^{C_0 T}). \tag{A.3}$$

See the paragraph after (3.10) of Farfán et al. (2019) for details. Let C_0 be the constant appearing in Lemma A.1. Then, by (A.2), (A.3), and Lemma A.1, we have

$$\|\widetilde{\rho}_t - \rho_t\|_2 \le e^{C_0(t-T_2)}\|\widetilde{\rho}_{T_2} - \rho_{T_2}\|_2 \le \beta/3, \tag{A.4}$$

for any $\widetilde{\rho}_0(\theta)d\theta \in \mathcal{B}[\beta_0; \rho_0]$ and $T_2 \le t \le T$.

Therefore, it follows from $T_2 \le T_1$, (3.2), (A.1), and (A.4) that $\widetilde{\rho}_t(\theta)d\theta \in \mathcal{B}[\beta; \rho_t]$ for any $0 \le t \le T$ provided $\widetilde{\rho}_0(\theta)d\theta \in \mathcal{B}[\beta_0; \rho_0]$, which completing the proof of Proposition A.2. □

Because of Proposition A.2, we can obtain the following corollary.

Corollary A.3. *Under the notations of the proof of Lemma 4.3, the application $\rho(\theta)d\theta \in \mathcal{B}_i \cap \mathcal{M}_{+,1} \mapsto \tau^{(i)}(\rho)$ is lower semicontinuous with respect to the weak topology. Namely, for any fixed $\rho(\theta)d\theta \in \mathcal{B}_i \cap \mathcal{M}_{+,1}$ and for any sequence $\{\rho_n(\theta)d\theta\}_n$ in $\mathcal{B}_i \cap \mathcal{M}_{+,1}$, which converges to $\rho(\theta)d\theta$ in the weak topology, we have*

$$\tau^{(i)}(\rho) \le \liminf_{n\to\infty} \tau^{(i)}(\rho_n).$$

Proof: This corollary is a direct consequence of Proposition A.2. To see this, take any $\rho(\theta)d\theta \in \mathcal{B}_i \cap \mathcal{M}_{+,1}$ and any sequence $\{\rho_n(\theta)d\theta\}_n$ in $\mathcal{B}_i \cap \mathcal{M}_{+,1}$ converging to $\rho(\theta)d\theta$ in the weak topology. We can assume the loss of generality that $\tau^{(i)}(\rho) > 0$.

Let $\rho(t)$ and $\rho_n(t)$ be the unique weak solutions to the Cauchy problem (3.4) with the initial conditions ρ and ρ_n, respectively. Since $\mathcal{B}[\alpha_i; \varrho_i]$ is closed, it follows from the definition of $\tau^{(i)}(\rho)$ that $\rho(\tau^{(i)}(\rho)) \in \mathcal{B}[\alpha_i; \varrho_i]$ and $\rho(t) \notin \mathcal{B}[\alpha_i; \varrho_i]$ for any $0 \le t < \tau^{(i)}(\rho)$.

Fix the small $\varepsilon > 0$ and let $t_\varepsilon = \tau^{(i)}(\rho) - \varepsilon > 0$. Since $\mathcal{B}[\alpha_i; \varrho_i]$ is closed and $\rho(t_\varepsilon) \notin \mathcal{B}[\alpha_i; \varrho_i]$, we have

$$\alpha_\varepsilon := \inf\{d(\rho(t_\varepsilon), \widetilde{\rho}) : \widetilde{\rho} \in \mathcal{B}[\alpha_i; \varrho_i]\} > 0.$$

By applying Proposition A.2 for $\beta = \alpha_\varepsilon/2$ and $T = t_\varepsilon$, there exists $\beta_\varepsilon < \alpha_\varepsilon/2$ such that for any measurable function $\widetilde{\rho}_0 : \mathbb{T} \to [0,1]$ with $\widetilde{\rho}_0(\theta)d\theta \in \mathcal{B}[\beta_\varepsilon; \rho_0]$, we have $\widetilde{\rho}(t,\theta)d\theta \in \mathcal{B}[\alpha_\varepsilon/2; \rho_t]$ for all $0 \le t \le t_\varepsilon$, where $\widetilde{\rho}(t,\theta)$ is a unique weak solution to the Cauchy problem (3.4) with the initial condition $\widetilde{\rho}_0$. Then we have $\rho_n(t) \notin \mathcal{B}[\alpha_i; \varrho_i]$ for any $0 \le t \le t_\varepsilon$ if $\rho_n(\theta)d\theta \in \mathcal{B}[\beta_\varepsilon; \rho_0]$. Therefore, for any large enough n, we have

$$t_\varepsilon \le \tau^{(i)}(\rho_n).$$

Taking $n \to \infty$ and $\varepsilon \to 0$ completes the proof of Corollary A.3. □

Recall the definitions of $\rho_i, \bar{\rho}_i$, and ϱ_i from Section 2. Note that $x \in (0,1)$ is a limit point of the dynamical system

$$\frac{d}{dt}x_t = -F(x_t),$$

if, and only if, $x = \rho_i, i = 1, \ldots, \ell$. The following result shows that the constant function $\bar{\rho}_i$ is a local attractor of the dynamical system defined by (3.4) with respect to the weak topology.

Lemma A.4 (Farfán et al. (2019, Lemma 11))**.** *Let $\varepsilon > 0$ and $i = 1, \ldots, \ell$. There exists $\gamma_i > 0$ such that for any density condition $\rho : \mathbb{T} \to [0,1]$ such that $\rho(\theta)d\theta \in \mathcal{B}(\gamma_i; \varrho_i)$, ρ_t converges in the supremum norm to $\bar{\rho}_i$, as $t \to \infty$, where $\rho_t(\theta) = \rho(t,\theta)$ is a unique weak solution to the Cauchy problem (3.4) with the initial condition ρ. Moreover, $\pi_t(d\theta) = \rho(t,\theta)d\theta$ belongs to $\mathcal{B}(\varepsilon; \varrho_i)$ for all $t \ge 0$.*

Lemma A.4 immediately implies the following result.

Corollary A.5. *For each $i = 1, \dots, \ell$, we have*

$$\inf \{d(\bar{\varrho}_i, \bar{\varrho}) : \bar{\varrho} \in \mathcal{M}_{\text{sol}}, \bar{\varrho} \neq \bar{\varrho}_i\} > 0.$$

Appendix B. Dynamical rate function

In this appendix, we collect miscellaneous lemmata regarding the rate function of the dynamical large deviation principle

Lemma B.1 (Landim and Tsunoda (2018, Proposition 4.1))**.** *Fix $T > 0$ and a measurable function $\rho : \mathbb{T} \to [0,1]$. Let π be a trajectory in $D([0,T], \mathcal{M}_+)$ such that $I_T(\pi|\rho)$ is finite. Then π belongs to $C([0,T], \mathcal{M}_{+,1})$ and $\pi(0, d\theta) = \rho(\theta)d\theta$.*

Lemma B.2 (Landim and Tsunoda (2018, Corollary 4.6))**.** *Fix $T > 0$. The density ρ of a trajectory $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0,T], \mathcal{M}_{+,1})$ is the weak solution to the Cauchy problem (3.4) with initial condition ρ_0 if, and only if, $I_T(\pi|\rho_0) = 0$.*

Recall the definition of $\mathbf{D}_{T,\beta}$ defined before (4.9).

Lemma B.3 (Farfán et al. (2019, Lemma 14))**.** *For each $\beta > 0$ there exists $T = T(\beta) > 0$ such that*

$$\inf_{\pi \in \mathbf{D}_{T,\beta}} I_T(\pi) > 0.$$

In particular, for each $\beta > 0$ and each $A > 0$ there exists $T = T(\beta, A) > 0$ such that

$$\inf_{\pi \in \mathbf{D}_{T,\beta}} I_T(\pi) \geq A.$$

The first assertion of Lemma B.3 is proven in Farfán et al. (2019, Lemma 14), whereas the second assertion follows from the first one and noting that for each $\beta > 0, T > 0$ and $k \in \mathbb{N}$

$$\inf_{\pi \in \mathbf{D}_{kT,\beta}} I_{kT}(\pi) \geq k \inf_{\pi \in \mathbf{D}_{T,\beta}} I_T(\pi).$$

Recall the definitions of $\mathcal{M}_{\text{sol}}(\alpha)$ and $\widetilde{H}_N(\alpha)$, which are defined after (4.3).

Lemma B.4 (Farfán et al. (2019, Lemma 21))**.** *For each $\alpha > 0$, there exist $T_0, C_0, N_0 > 0$, depending on α, such that, for all $N \geq N_0$ and all $k \geq 1$,*

$$\sup_{\eta \in X_N} \mathbb{P}_\eta \left[\widetilde{H}_N(\alpha) \geq kT_0\right] \leq e^{-kC_0 N}.$$

Lemma B.5 (Farfán et al. (2019, Lemma 30))**.** *For each $i = 1, \dots, \ell$ and each $\alpha > 0$, we have*

$$\inf \{V_i(\varrho) : \varrho \notin \mathcal{B}(\alpha; \bar{\varrho}_i)\} > 0.$$

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