



# Markov limits of steady states of the KPZ equation on an interval

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**Abstract.** This paper builds upon the research of [Corwin and Knizel \(2021\)](#) who proved the existence of stationary measures for the KPZ equation on an interval and characterized them through a Laplace transform formula. [Bryc et al. \(2022\)](#) found a probabilistic description of the stationary measures in terms of a Doob transform of some Markov kernels; essentially at the same time, another description connecting the stationary measures to the exponential functionals of the Brownian motion appeared in [Barraquand and Le Doussal \(2022\)](#).

Our first main result clarifies and proves the equivalence of the two probabilistic description of these stationary measures. We then use the Markovian description to give rigorous proofs of some of the results claimed in [Barraquand and Le Doussal \(2022\)](#). We analyze how the stationary measures of the KPZ equation on  $[0, \tau]$  behave at large scale, as  $\tau$  goes to infinity. We also analyze the behaviour of the stationary measures of the KPZ equation on  $[0, \tau]$  without rescaling, when  $\tau$  goes to infinity. Finally, we analyze the measures on  $[0, \infty)$  at large scale, which according to [Barraquand and Le Doussal \(2022\)](#) should correspond to stationary measures of a hypothetical KPZ fixed point on  $[0, \infty)$ .

## 1. Notation and background

The Kardar-Parisi-Zhang (KPZ) equation was proposed in [Kardar et al. \(1986\)](#) as a model for the evolution of the profile of a growing interface driven by the space-time white noise  $\zeta$ . The interface profile is described by a height function  $H(t, s)$ , where  $t \geq 0$  is a time variable and  $s$  is a spatial variable, which in one spatial dimension formally satisfies

$$\partial_t H(t, s) = \frac{1}{2} \partial_s^2 H(t, s) + \frac{1}{2} (\partial_s H(t, s))^2 + \zeta(t, s). \quad (1.1)$$

We refer the reader to the reviews [Corwin \(2012\)](#); [Quastel \(2012\)](#) about the KPZ equation and its universality class.

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We are interested in (1.1) on a finite interval,  $s \in [0, 1]$ , with Neumann boundary conditions

$$\partial_s H(t, 0) = c/2, \quad \partial_s H(t, 1) = -a/2. \quad (1.2)$$

Since the KPZ equation is ill-posed, the standard notion of the solution is in the Cole-Hopf sense, i. e. that they are defined as  $H(t, s) = \log Z(t, s)$ , where  $Z(t, s)$  solves the stochastic heat equation

$$\partial_t Z(t, s) = \frac{1}{2} \partial_s^2 Z(t, s) + Z(t, s) \zeta(t, s), \quad t \geq 0, \quad s \in [0, 1] \quad (1.3)$$

with boundary conditions (1.2)

$$\partial_s Z(t, s)|_{s=0} = (c-1)Z(t, 0)/2, \quad \partial_s Z(t, s)|_{s=1} = (1-a)Z(t, 1)/2. \quad (1.4)$$

The solutions of (1.3) are not differentiable, so the meaning of (1.4) is not obvious. A proper definition was given in Corwin and Shen (2018, Definition 2.5), see also Gerencsér and Hairer (2019); Gonçalves et al. (2020).

A stationary measure for the open KPZ equation is the law of a random function  $(\tilde{H}_s)_{s \in [0, 1]}$  with  $\tilde{H}_0 = 0$ , defined by the property that if  $H(s, t)$  is a Cole-Hopf solution of equation (1.1) with boundary values (1.2) and with the initial condition  $H(0, s) = \tilde{H}_s$  for  $s \in [0, 1]$ , then the law of  $(H(t, s) - H(t, 0))_{s \in [0, 1]}$  does not depend on  $t$ . In a breakthrough paper, Corwin and Knizel (2021) proved existence of such stationary measures and determined their multivariate Laplace transform under the restriction  $a + c \geq 0$ . Papers Bryc et al. (2022) and Barraquand and Le Doussal (2022) inverted this Laplace transform. Two different representations of the stationary measure obtained in these papers are the starting point of this paper.

For ease of comparison we restate results in Bryc et al. (2022) and Barraquand and Le Doussal (2022) in common parametrization. To this end, as in Bryc et al. (2022) we replace two real parameters  $u, v$  from Barraquand and Le Doussal (2022); Corwin and Knizel (2021) by  $a = 2v, c = 2u$ . (Boundary parameters in Parekh (2019) are thus  $A = c/2, B = a/2$ .) Both papers find it convenient to replace interval  $[0, 1]$  with  $[0, \tau]$ ,  $\tau > 0$ . In our re-write, the length of the interval  $\tau$  corresponds to  $L$  in Barraquand and Le Doussal (2022), and it is four times longer than the length parameter in Bryc et al. (2022). Since time variable plays no role in this paper, and we are interested in Markov processes in the space variable, following Bryc et al. (2022) we will use variable  $t$  as the spatial index of the random fields on  $[0, \tau]$ . This index was denoted by  $x$  in Barraquand and Le Doussal (2022) and by  $X$  in Corwin and Knizel (2021).

In this notation, Barraquand and Le Doussal (2022) represent the stationary measure of the KPZ equation on an interval as

$$\left( \tilde{H}_t \right)_{t \in [0, \tau]} \stackrel{d}{=} (B_t + X_t)_{t \in [0, \tau]}, \quad (1.5)$$

where  $(B_t)$  is a Brownian motion of variance rate  $1/2$  and  $(X_t)_{t \in [0, \tau]}$  is an independent stochastic process with continuous trajectories such that the Radon-Nikodym derivative of its law  $\mathbb{P}_X$  on  $C[0, \tau]$  with respect to the law  $\mathbb{P}_B$  of Brownian motion  $(B_t)_{t \in [0, \tau]}$  with variance rate  $1/2$  is

$$\frac{d\mathbb{P}_X}{d\mathbb{P}_B} = \frac{1}{\mathfrak{K}_{a,c}^{(\tau)}} e^{-a\beta\tau} \left( \int_0^\tau e^{-2\beta t} dt \right)^{-a/2 - c/2}, \quad (1.6)$$

where we denoted by  $\beta = (\beta_t) \in C[0, \tau]$  the argument of the density function. Process  $(X_t)_{t \in [0, \tau]}$  depends on parameters  $a, c, \tau$  but for now we have suppressed this dependence in our notation.

Representation (1.5) is established in Barraquand and Le Doussal (2022) using the Laplace transform formula of Corwin and Knizel (2021). The argument is given for  $a, c > 0$  and  $\tau = 1$ , but it is conjectured that it represents the stationary measure for the KPZ equation on any interval  $\tau > 0$  with Neumann boundary conditions for any real  $a, c$ .

Representation (1.5) is also established in Bryc et al. (2022) but process  $X$  is described differently and only for  $a + c > 0$ . The argument there also relies on the Laplace transform formula of Corwin and Knizel (2021) with  $\tau = 1$ , with proof that covers also some negative values for the parameters,

as long as  $a + c > 0$  and  $\min\{a, c\} > -2$ . The stationary measure of the KPZ equation on an interval is represented there in the form which, after a change of the time scale and sign, compare [Bryc et al. \(2022, \(1.8\)\)](#), in present notation is

$$\left(\tilde{H}_t\right)_{t \in [0, \tau]} \stackrel{d}{=} (B_t + Y_t - Y_0)_{t \in [0, \tau]}, \tag{1.7}$$

where  $(B_t)$  is an independent Brownian motion of variance  $1/2$  and  $(Y_t)_{t \in [0, \tau]} = \left(Y_t^{(a,c)}\right)_{t \in [0, \tau]}$  is an  $\mathbb{R}$ -valued Markov process with transition probabilities

$$\mathbb{P}(Y_t^{(a,c)} = dy | Y_s^{(a,c)} = x) = \frac{H_t(y)}{H_s(x)} p_{t-s}(x, y) dy, \quad 0 \leq s < t \leq \tau \tag{1.8}$$

and with initial distribution

$$\mathbb{P}(Y_0^{(a,c)} = dx) = \frac{1}{C_{a,c}^{(\tau)}} e^{-cx} H_0(x) dx, \tag{1.9}$$

where

$$p_t(x, y) = \frac{2}{\pi} \int_0^\infty e^{-tu^2/4} K_{iu}(e^{-x}) K_{iu}(e^{-y}) \frac{du}{|\Gamma(iu)|^2}, \quad x, y \in \mathbb{R}, \quad t > 0, \tag{1.10}$$

is the Yakubovich heat kernel,

$$C_{a,c}^{(\tau)} := \int_{\mathbb{R}^2} e^{-ax-cy} p_\tau(x, y) dx dy \tag{1.11}$$

is the normalizing constant, and

$$H_t(x) = \int_{\mathbb{R}} e^{-ay} p_{\tau-t}(x, y) dy, \quad 0 \leq t < \tau, \tag{1.12}$$

(with  $H_\tau(x) := e^{-ax}$ ) is Doob’s  $h$ -transform. In (1.10),  $K_{iu}(e^{-x})$  is the modified Bessel K function (1.16) with positive argument  $e^{-x}$  and imaginary index  $iu$ .

Markov process  $(Y_t^{(a,c)})_{t \in [0, \tau]}$  is well defined for all  $a + c > 0$  and it is expected that (1.13) gives stationary solution (1.5) for all  $\tau > 0$  and all  $a + c > 0$ , so the assumptions  $\tau = 1$  and  $\min\{a, c\} > -2$  in [Bryc et al. \(2022, Proposition 1.4\)](#) should not be needed.

*Remark 1.1.* Markov process  $(Y_t^{(a,c)})_{t \in [0, \tau]}$  can be described as follows. Start with a Brownian motion of variance rate  $1/2$  and drift  $-at/2$  and kill it at rate  $\frac{1}{4}e^{-2x}$ . Next, condition this killed process to survive until time  $\tau$ , and we would obtain a Markov process with the same transition probabilities as  $(Y_t^{(a,c)})_{t \in [0, \tau]}$ . One can check that this description is correct by applying Girsanov’s theorem and Doob’s  $h$ -transform, see [Rogers and Williams \(1987\)](#); [Pinsky \(1985\)](#). We also note that the review paper [Corwin \(2022, page 7\)](#) explains the construction of process  $(Y_t^{(a,c)})_{t \in [0, \tau]}$  emphasizing nice symmetry between parameters  $a$  and  $c$ .

Formulas (1.5) and (1.7) indicate that the two representations are equivalent, and that process  $X$ , which is defined for all real  $a, c$ , can be represented as the process of Markov differences

$$(X_t)_{t \in [0, \tau]} \stackrel{d}{=} (Y_t^{(a,c)} - Y_0^{(a,c)})_{t \in [0, \tau]}, \tag{1.13}$$

for  $a + c > 0$ . This fact can be read out from [Barraquand and Le Doussal \(2022, formula \(16\)\)](#) when  $a, c > 0$ . We will give an argument that works for the parameters in the entire admissible range  $a + c > 0$  for process  $(Y_t^{(a,c)})$ .

**Theorem 1.2.** *For  $a + c > 0$ , process  $(Y_t^{(a,c)} - Y_0^{(a,c)})_{t \in [0, \tau]}$  has the same law as process  $(X_t)_{t \in [0, \tau]}$  defined by (1.6).*

The proof is in Section 3. In the proof, we relate the normalizing constants in (1.6) and (1.11) as follows:

$$\mathfrak{K}_{\mathbf{a},\mathbf{c}}^{(\tau)} = \frac{2^{1-\mathbf{a}/2-c/2} C_{\mathbf{a},\mathbf{c}}^{(\tau)}}{\Gamma(\frac{\mathbf{a}+\mathbf{c}}{2})}. \quad (1.14)$$

To point out the difference between the two representations (1.5) and (1.7), consider the case  $\mathbf{a} + \mathbf{c} = 0$ . By Cameron-Martin theorem, formula (1.6) implies that process  $(X_t)$  with  $\mathbf{a} + \mathbf{c} = 0$  has the same law as  $(B_t + ct/2)$ , so representation (1.5) is in agreement with Corwin and Knizel (2021, Theorem 1.2(3)). From Theorem 1.2, it follows that if  $\varepsilon = \mathbf{a}_\varepsilon + \mathbf{c}_\varepsilon \searrow 0$  while  $\mathbf{c}_\varepsilon \rightarrow \mathbf{c}$  then

$$(Y_t^{(\mathbf{a}_\varepsilon, \mathbf{c}_\varepsilon)} - Y_0^{(\mathbf{a}_\varepsilon, \mathbf{c}_\varepsilon)})_{t \in [0, \tau]} \Rightarrow (B_t + ct/2)_{t \in [0, \tau]} \text{ as } \varepsilon \rightarrow 0.$$

However, if  $\mathbf{c} \neq 0$  then Markov process  $(Y_t^{(\mathbf{a}_\varepsilon, \mathbf{c}_\varepsilon)})_{t \in [0, \tau]}$  cannot converge because

$$\varepsilon Y_0^{(\mathbf{a}_\varepsilon, \mathbf{c}_\varepsilon)} \Rightarrow \gamma_1 \text{ as } \varepsilon \searrow 0, \quad (1.15)$$

where  $\gamma_1$  is an exponential random variable of mean 1. So representation (1.7) does not extend “by continuity” to this case. (Verification of (1.15) appears in Section 4.7.1.)

The goal of this paper is to investigate which of the limits discovered in Barraquand and Le Doussal (2022) for process  $X$  extend to process  $Y$ . When this happens, we can describe the limits in Barraquand and Le Doussal (2022) as differences of explicit Markov processes. To identify a limiting Markov process, we only need to prove convergence of finite dimensional distributions, which follows from convergence of the initial laws and transition probabilities, which is a simpler task than deducing weak convergence from the convergence of the Feller semigroups as in Ethier and Kurtz (1986, Ch. 4, Thm. 2.5), and seems to be also easier than the approach based on Radon-Nikodym derivatives as in Barraquand and Le Doussal (2022); Hariya and Yor (2004).

*Notation.* Throughout the paper,  $(B_t)$  denotes Brownian motion of variance 1/2, i.e., process  $(B_t)_{t \geq 0}$  has the same law as process  $(W_{t/2})_{t \geq 0}$  or process  $(\frac{1}{\sqrt{2}}W_t)_{t \geq 0}$ , where  $(W_t)_{t \geq 0}$  is the standard Wiener process. By  $\gamma_p$  we denote a Gamma random variable with shape parameter  $p > 0$  and scale parameter one, i.e.

$$\mathbb{P}(\gamma_p \in dx) = \frac{1}{\Gamma(p)} x^{p-1} e^{-x} \mathbf{1}_{\{x>0\}} dx.$$

In particular,  $\gamma_1$  is the standard exponential random variable of mean 1. By  $\Rightarrow$  we denote weak convergence (of the corresponding measures on  $\mathbb{R}$  or on  $C[0, 1]$ ) and by  $\xrightarrow{\text{f.d.d.}}$  we denote the convergence of finite dimensional distributions. For argument  $x > 0$  and complex index  $z$  we denote by

$$K_z(x) = \int_0^\infty e^{-x \cosh w} \cosh(zw) dw \quad (1.16)$$

the modified Bessel K function.

## 2. Asymptotic results

We first describe the limits derived in Barraquand and Le Doussal (2022) for process  $X$  and then in Section 2.2 we discuss the corresponding limits for Markov process  $Y$ . In view of Theorem 1.2, limits for process  $Y$  give the corresponding limits for process  $X$ , but we shall see that the converse sometimes fails.

To indicate how Barraquand and Le Doussal scale the parameters for their limits, we need to write them explicitly, so in Section 2.1 we shall write  $(X_t^{(\mathbf{a}, \mathbf{c})})_{t \in [0, \tau]}$  for the stochastic process with density (1.6).

2.1. *Description of limits in Barraquand and Le Doussal (2022).* Barraquand and Le Doussal (2022) determine several interesting limits. Some of these limits are expected to correspond to the stationary measures of a hypothetical KPZ fixed points on an interval and on the half-line. This is the area of intense current research, and some new preprints appeared while this paper was undergoing a revision. At this time however, we note that the KPZ fixed point was defined rigorously in Matetski et al. (2021) only on  $\mathbb{R}$ .

The following describes the limits determined in Barraquand and Le Doussal (2022).

- (i) It is claimed in Barraquand and Le Doussal (2022) that the large scale limit  $\lim_{\tau \rightarrow \infty} \left( \frac{1}{\sqrt{\tau}} X_{t\tau}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \right)_{t \in [0,1]}$  should give the *stationary measures of the hypothetical KPZ fixed point on an interval*. The limit is described in Barraquand and Le Doussal (2022) by density (2.7) and also by the heat kernel and the Laplace transform Barraquand and Le Doussal (2022, supplement, formulas (51), (54)).

By universality, such limits should correspond to stationary measures of a hypothetical KPZ fixed point on an interval, and indeed a recent result in Bryc et al. (2022+) derives the same limit from an open Asymmetric Simple Exclusion Process.

At this time, the notion of the KPZ fixed point on an interval has not yet been defined.

- (ii) It is conjectured in Barraquand and Le Doussal (2022) that *the stationary solution of the half-line KPZ equation* is given by  $\lim_{\tau \rightarrow \infty} \left( X_t^{(a,c)} \right)_{t \in [0,\tau]}$ , a limit which was already studied in Hariya and Yor (2004, Theorem 1.3) for different reasons.

The form of the limit depends on the parameters  $a, c$  as follows:

- (a) If  $a \geq 0, c \geq 0$  then the limit  $(\tilde{Z}_t^{(c)})_{t \geq 0}$  does not depend on  $a$ , and is expressed by (2.11) with  $a = 0$ .
- (b) If  $a \leq 0, c \geq a$  then the limit  $(\tilde{Z}_t^{(a,c)})_{t \geq 0}$  is given by (2.11) (with  $a$  replaced by  $-a$ ) and depends on both  $a, c$ .
- (c) If  $a \geq c, c \leq 0$  then the limit is  $(B_t + tc/2)$  and does not depend on  $a$ .

According to a recent preprint Barraquand and Corwin (2022), these limits are indeed the stationary solutions for the KPZ equation on the half-line.

- (iii) It is claimed in Barraquand and Le Doussal (2022) that the large scale limit as  $T \rightarrow \infty$  of the stationary measures on the half-line from point (ii) should give the stationary measures of the *KPZ fixed point on the half-line*. These limits are:

- (a) If  $a > 0, c > 0$  then

$$\frac{1}{\sqrt{T}} \left( \tilde{Z}_{tT}^{(c/\sqrt{T})} \right)_{t \geq 0} \Rightarrow \left( B_t + \max \left\{ 0, -\frac{2}{c} \gamma_1 - 2 \min_{0 \leq s \leq t} B_s \right\} \right)_{t \geq 0} \text{ as } T \rightarrow \infty, \tag{2.1}$$

where  $\gamma_1$  is an independent exponential random variable.

- (b) If  $a < 0$  and  $c > a$ , then

$$\begin{aligned} \frac{1}{\sqrt{T}} \left( \tilde{Z}_{tT}^{(a/\sqrt{T}, c/\sqrt{T})} \right)_{t \geq 0} \\ \Rightarrow \left( B_t + at/2 + \max \left\{ 0, -\frac{2}{c-a} \gamma_1 - 2 \min_{0 \leq s \leq t} \{B_s + as/2\} \right\} \right)_{t \geq 0} \text{ as } T \rightarrow \infty, \end{aligned} \tag{2.2}$$

where  $\gamma_1$  is an independent exponential random variable.

- (c) For  $a \geq c, c \leq 0$ , process  $(B_t + tc/2)_{t \geq 0}$  is invariant under the scaling used in the limit so it should be the stationary measure of the hypothetical KPZ fixed point on the half-line.

At this time, the notion of the KPZ fixed point on the half-line has not yet been defined.

- (iv) For completeness, we also mention that according to [Barraquand and Le Doussal \(2022\)](#) , the small scale limit

$$\lim_{\tau \rightarrow 0} \left( \frac{1}{\sqrt{\tau}} X_{t\tau}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \right)_{t \in [0,1]}$$

is described in (2.15). We learned from the review that this limit should correspond to the stationary measure of the *Edwards-Wilkinson equation* that is the stochastic equation with additive noise

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + \zeta.$$

(Our techniques do not allow us to say anything about this limit.)

We shall represent some of these limits in a different form using the corresponding limits of the Markov process  $\left( Y_t^{(a,c)} \right)_{t \in [0,\tau]}$ .

## 2.2. Limits of Markov process $Y$ .

### 2.2.1. Markov process for the stationary measure of the (hypothetical) KPZ fixed point on an interval.

The following result is a version of point (i) in Section 2.1. It is related to [Bryc et al. \(2022+\)](#), who determined that up to an irrelevant scaling factor, the fluctuations of particle density for ASEP with varying parameters for fixed  $q$  are given by the same Markov process. We use the notation  $(\tilde{\eta}_t)$  and  $(\eta_t) = (\tilde{\eta}_t - \tilde{\eta}_0)$  to connect with that paper. (Strictly speaking, the process denoted by  $\eta$  in [Bryc et al. \(2022+\)](#) is  $\sqrt{2}\eta$  in our notation. But the actual process appearing in the limit [Bryc et al. \(2022+, Theorem 1.5\)](#) is  $\eta$ .)

**Theorem 2.1.** *For fixed  $a + c > 0$  we have*

$$\left( \frac{1}{\sqrt{\tau}} Y_{t\tau}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \right)_{t \in [0,1]} \xrightarrow{\text{f.d.d.}} \left( \tilde{\eta}_t^{(a,c)} \right)_{t \in [0,1]} \quad \text{as } \tau \rightarrow \infty,$$

where  $(\tilde{\eta}_t)_{t \in [0,1]}$  is a Markov process with the initial distribution

$$\mathbb{P}(\tilde{\eta}_0 = dx) = \frac{1}{\mathfrak{C}_{a,c}} e^{-cx} h_0(x) \mathbf{1}_{\{x>0\}} dx. \tag{2.3}$$

and transition probabilities

$$p_{s,t}(x, dy) = \frac{h_t(y)}{h_s(x)} g_{t-s}(x, y) dy, \quad 0 \leq s < t \leq 1, \tag{2.4}$$

where

$$g_t(x, y) = \frac{2}{\pi} \int_0^\infty e^{-tu^2/4} \sin(xu) \sin(yu) du = \frac{1}{\sqrt{\pi t}} \left( e^{-\frac{(x-y)^2}{t}} - e^{-\frac{(x+y)^2}{t}} \right), \quad x, y, t > 0, \tag{2.5}$$

the normalizing constant is

$$\mathfrak{C}_{a,c} = \int_{\mathbb{R}_+^2} e^{-cx-ay} g_1(x, y) dx dy \tag{2.6}$$

and Doob's  $h$ -transform is

$$h_t(x) = \int_{\mathbb{R}_+} g_{1-t}(x, y) e^{-ay} dy, \quad x > 0, \quad 0 \leq t < 1$$

with  $h_1(x) := e^{-ax} \mathbf{1}_{\{x>0\}}$ .

The proof of this result is in Section 4.2.

*Remark 2.2.* Function  $g_t(x, y)$  defined in (2.5) is the transition probability density function of a Brownian motion of variance rate 1/2 killed at the time when it first hits zero. Applying Girsanov’s theorem and Doob’s  $h$ -transform we conclude that  $p_{s,t}(x, dy)$  defined in (2.4) is the transition probability kernel of Brownian motion of variance rate 1/2 and drift  $-at/2$  and conditioned to stay positive for  $0 \leq t \leq 1$ .

We remark that according to Barraquand and Le Doussal (2022, formula (31)), for process  $(\eta_t)_{t \in [0,1]} := (\tilde{\eta}_t - \tilde{\eta}_0)_{t \in [0,1]}$ , we have

$$\frac{d\mathbb{P}_\eta}{d\mathbb{P}_B} = \frac{1}{(a + c)\mathfrak{C}_{a,c}} e^{-a\beta_1 + (a+c) \min_{0 \leq t \leq 1} \beta_t} \tag{2.7}$$

with the normalizing constant given by (2.6). This can be verified by combining Theorem 1.2 and Theorem 2.1.

2.2.2. *Markov processes for the stationary measure of KPZ on the half-line.* The following is a version of point ii(a) in Section 2.1. In view of Theorem 1.2, it also provides a different description of the limit process in Hariya and Yor (2004, Theorem 1.3.(i)), representing it as  $(Z_t^{(c)} - Z_0^{(c)})_{t \geq 0}$ .

**Theorem 2.3.** *If  $a, c > 0$  then*

$$\left( Y_t^{(a,c)} \right)_{t \in [0,\tau]} \xrightarrow{\text{f.d.d.}} (Z_t^{(c)})_{t \geq 0} \text{ as } \tau \rightarrow \infty,$$

where  $(Z_t^{(c)})$  is an  $\mathbb{R}$ -valued Markov process with the initial distribution

$$\mathbb{P}(Z_0^{(c)} = dx) = \frac{4e^{-cx} K_0(e^{-x})}{2^c \Gamma(c/2)^2} dx \tag{2.8}$$

and with transition probabilities

$$\mathbb{P}(Z_t^{(c)} = dy | Z_s^{(c)} = x) = \frac{K_0(e^{-y})}{K_0(e^{-x})} p_{t-s}(x, y) dy, \quad 0 \leq s < t, \tag{2.9}$$

where  $p_t(x, y)$  is given by (1.10).

The proof of this result is in Section 4.1. By comparing the Laplace transforms (see (4.4)) we notice that the law (2.8) is the law of  $-\log(2\sqrt{\gamma_{c/2}\tilde{\gamma}_{c/2}})$ , where  $\gamma_{c/2}$  and  $\tilde{\gamma}_{c/2}$  are independent Gamma random variables with shape parameter  $c/2$ .

From Theorem 1.2, and Hariya and Yor (2004) it follows that if  $a + c > 0$  with  $a < 0$ , then process  $(Y_t^{(a,c)} - Y_0^{(a,c)})_{t \in [0,\tau]}$  converges in distribution as  $\tau \rightarrow \infty$ . This suggests the following result related to point ii(b) in Section 2.1.

**Theorem 2.4.** *If  $a + c > 0$ , with  $a \leq 0$  then*

$$\left( Y_t^{(a,c)} \right)_{t \in [0,\tau]} \xrightarrow{\text{f.d.d.}} (Z_t^{(a,c)})_{t \geq 0} \text{ as } \tau \rightarrow \infty$$

where  $(Z_t^{(a,c)})_{t \geq 0}$  is an  $\mathbb{R}$ -valued Markov process with the initial distribution

$$\mathbb{P}(Z_0^{(a,c)} = dx) = \frac{4}{2^c \Gamma(\frac{c-a}{2}) \Gamma(\frac{a+c}{2})} e^{-cx} K_a(e^{-x}) dx \tag{2.10}$$

and with transition probabilities

$$\mathbb{P}(Z_t^{(a,c)} = dy | Z_s^{(a,c)} = x) = \frac{e^{-a^2t/4} K_a(e^{-y})}{e^{-a^2s/4} K_a(e^{-x})} p_{t-s}(x, y) dy, \quad 0 \leq s < t.$$

The proof is in Section 4.3. We note that Markov process  $(Z_t^{(a,c)})_{t \geq 0}$  is well defined when  $|a| < c$ , and it is clear that process  $(Z_t^{(c)})_{t \geq 0}$  from Theorem 2.3 has the same law as  $(Z_t^{(0,c)})_{t \geq 0}$ . From (4.4) it is easy to use the Laplace transform to identify the law (2.10) as the law of  $-\log\left(2\sqrt{\gamma_{\frac{a+c}{2}}\tilde{\gamma}_{\frac{c-a}{2}}}\right)$  for a pair of independent Gamma random variables. In light of Remark 1.1 the transition probabilities of the two processes identified in Theorems 2.3 and 2.4 can be described as the Brownian motion of variance rate 1/2 and drift  $-at/2$ , killed at rate  $\frac{1}{4}e^{-2x}$  and conditioned to survive for all  $t \geq 0$ . These asymptotic results should be compared with Theorem 1.2 in Hariya and Yor (2004).

Next, we confirm that Theorems 2.3 and 2.4 give Markov representations for the processes Hariya and Yor (2004, Theorem 1.3(i)+(iii)) when  $a+c > 0$ , i.e., when their parameter  $m > 0$ . (Barraquand and Le Doussal (2022, Section 5 of Supplement) shows how to relate the parametrizations.) Of course, this result follows from Hariya and Yor (2004, Theorem 1.3(i)+(iii)) and our previous results. However we include a somewhat more direct proof for completeness.

**Theorem 2.5.** *If  $a + c > 0$ ,  $|a| < c$ , then process  $(Z_t^{(a,c)} - Z_0^{(a,c)})_{t \geq 0}$  has the same distribution as process*

$$\left( B_t - at/2 + \log\left(1 + \gamma_{\frac{a+c}{2}} \int_0^t e^{-2B_s+as} ds\right) \right)_{t \geq 0}, \tag{2.11}$$

where  $\gamma_{\frac{a+c}{2}}$  is an independent gamma random variable of shape parameter  $\frac{a+c}{2}$  and  $(B_t)_{t \geq 0}$  is a Brownian motion of variance 1/2.

Note that both the initial distribution and the transition probability density function of the process  $(Z_t^{(a,c)})_{t \geq 0}$  (as described in Theorem 2.4) is invariant with respect to change  $a \mapsto -a$ , therefore, the same is true for the process given in (2.11). This latter result also follows from Hariya and Yor (2004). The proof of Theorem 2.4 is in Section 4.4.

We conclude with a remark about the limit in point ii(c) in Section 2.1. From Theorem 1.2, and Hariya and Yor (2004, Theorem 1.3.(ii)) it follows that if  $a + c > 0$ , with  $c \leq 0$  then

$$\left( Y_t^{(a,c)} - Y_0^{(a,c)} \right)_{t \in [0,\tau]} \xrightarrow{\text{f.d.d.}} (B_t + ct/2)_{t \geq 0} \text{ as } \tau \rightarrow \infty,$$

where  $(B_t)_{t \geq 0}$  is the Brownian motion of variance rate 1/2. However, when  $c < 0$  process  $(Y_t^{(a,c)})_{t \in [0,\tau]}$  does not converge as  $\tau \rightarrow \infty$  because under a different scaling  $Y_0^{(a,c)}$ , which depends on  $\tau$  through (1.9), is asymptotically normal:

$$\sqrt{\frac{2}{\tau}} \left( Y_0^{(a,c)} + \tau c/2 \right) \Rightarrow N(0, 1) \text{ as } \tau \rightarrow \infty. \tag{2.12}$$

The proof of (2.12) is in Section 4.7.2.

2.2.3. *Markov processes for the stationary measures of the (hypothetical) KPZ fixed point on the half-line.* The following is related to point iii(a) in Section 2.1.

**Theorem 2.6.** *Fix  $c > 0$  and denote by  $(Z_t^{(c)})_{t \geq 0}$  the Markov process from Theorem 2.3. Then*

$$\frac{1}{\sqrt{T}} \left( Z_{tT}^{(c/\sqrt{T})} \right)_{t \geq 0} \xrightarrow{\text{f.d.d.}} (\rho_t^{(c)})_{t \geq 0} \text{ as } \tau \rightarrow \infty,$$

where  $(\rho_t^{(c)})_{t \geq 0}$  is  $\frac{1}{\sqrt{2}}$  multiple of the 3-dimensional Bessel process with the initial distribution

$$\mathbb{P}(\rho_0^{(c)} = dx) = c^2 x e^{-cx} \mathbf{1}_{\{x>0\}} dx. \tag{2.13}$$

(Of course, (2.13) is the gamma law with shape parameter two and scale parameter  $c$ , which is the sum of two independent exponential variables,  $(\gamma_1 + \tilde{\gamma}_1)/c$ .) The proof is in Section 4.5.



*Remark 2.7.* According to Barraquand and Le Doussal (2022, (36)), process  $(\rho_t^{(c)} - \rho_0^{(c)})_{t \geq 0}$  has the same law as the right hand side of (2.1) and this reference also noted the connection to the Bessel 3 process.

The following is related to point iii(b) in Section 2.1.

**Theorem 2.8.** Let  $a + c > 0$ ,  $a < 0$ , and denote by  $(Z_t^{(a,c)})_{t \geq 0}$  the Markov process from Theorem 2.4. Then

$$\frac{1}{\sqrt{T}} \left( Z_{tT}^{(a/\sqrt{T}, c/\sqrt{T})} \right)_{t \geq 0} \xrightarrow{\text{f.d.d.}} (\rho_t^{(a,c)})_{t \geq 0} \text{ as } \tau \rightarrow \infty,$$

where  $(\rho_t^{(a,c)})_{t \geq 0}$  is a Markov process with transition probabilities

$$\frac{e^{-a^2t/4} \sinh(ay)}{e^{-a^2s/4} \sinh(ax)} g_{t-s}(x, y), \quad x > 0, y > 0, \tag{2.14}$$

where  $g_t(x, y)$  is given by (2.5), with the initial distribution

$$\mathbb{P}(\rho_0^{(a,c)} = dx) = \mu_{a,c}(dx),$$

where  $\mu_{a,c}(dx) = \frac{c^2 - a^2}{2a} (e^{(a-c)x} - e^{-(a+c)x}) \mathbf{1}_{\{x > 0\}} dx$  is the law of the linear combination  $\frac{1}{a+c} \gamma_1 + \frac{1}{c-a} \tilde{\gamma}_1$  for a pair of independent exponential random variables.

The proof is in Section 4.6. The transition probability density function (2.14) is the same as for the process  $B_t - \frac{a}{2}t$ , conditioned to stay positive for all  $t \geq 0$ , see Rogers and Pitman (1981).

*Remark 2.9.* According to Barraquand and Le Doussal (2022, (37)), when  $a < 0$  and  $a + c > 0$ , process  $(\rho_t^{(a,c)} - \rho_0^{(a,c)})_{t \geq 0}$  has the same law as the right hand side of (2.2).

2.2.4. *The Edwards-Wilkinson limit.* We conclude this section with a comment on the limit from point (iv) in Section 2.1. According to Barraquand and Le Doussal (2022, (29)), when  $a + c > 0$ , Theorem 1.2 implies that

$$\frac{1}{\sqrt{\tau}} \left( Y_{t\tau}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} - Y_0^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \right)_{t \in [0,1]} \xrightarrow{\text{f.d.d.}} \left( B_t + ct/2 - (a + c) \frac{t^2}{4} \right)_{t \in [0,1]} \text{ as } \tau \rightarrow 0. \tag{2.15}$$

We expect that process  $\frac{1}{\sqrt{\tau}} \left( Y_{t\tau}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \right)$  does not converge as  $\tau \rightarrow 0$ .

### 3. Proof of Theorem 1.2

For the proof we choose a probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$ , where  $\Omega$  is a space of continuous functions on  $[0, \tau]$  and  $\mathbb{P}$  is the law of Brownian motion  $(B_t) \cong (W_t)/\sqrt{2}$  with variance 1/2 which was denoted by  $\mathbb{P}_B$  in (1.6). Under the measure  $\mathbb{P}$  the coordinate process  $X_t(\omega) = \omega(t)$  is a Brownian motion with variance 1/2. We define the exponential functional as  $I_t := \int_0^t e^{-2X_s} ds$  and consider the two-dimensional process  $(X, I)$ . This process is a time-homogeneous diffusion and from Matsumoto and Yor (2005, Theorem 4.1) (see also Yor (1992, Proposition 2) or Donati-Martin et al. (2002, formula (2.6))) we know that its transitional probability density function is given by

$$\mu_t^{\mathbb{P}}(x_0, z_0; x_1, z_1) = \exp \left( - \frac{1}{2} \frac{e^{-2x_0} + e^{-2x_1}}{z_1 - z_0} \right) \theta \left( \frac{e^{-x_0 - x_1}}{z_1 - z_0}, \frac{t}{2} \right) \frac{1}{z_1 - z_0},$$

where  $t > 0$ ,  $x_i \in \mathbb{R}$  and  $z_1 > z_0 \geq 0$ . Here  $\theta(r, t)$  is the (unnormalized) Hartman-Watson density function, which is succinctly defined in Hartman and Watson (1974) by its Laplace transform

$$\int_0^\infty e^{-\lambda^2 t} \theta(\xi, t) dt = \mathbf{I}_\lambda(\xi), \quad \lambda > 0, \xi > 0, \tag{3.1}$$

where  $\mathbf{I}_\lambda(\xi)$  is the modified Bessel function. For literature on properties of the Hartman-Watson density we refer to [Matsumoto and Yor \(2005\)](#), [Wiśniewolski \(2020\)](#), and the references therein.

For  $t \in [0, \tau]$ ,  $x \in \mathbb{R}$  and  $z \geq 0$  we define

$$\phi_t(x, z) = \mathbb{E}^{\mathbb{P}} \left[ e^{-aX_t} I_t^{-a/2-c/2} \mid X_0 = x, I_0 = z \right].$$

For  $\tau > 0$  and  $a + c > 0$  we define

$$Z_\tau := \frac{1}{\mathfrak{R}_{a,c}^{(\tau)}} e^{-aX_\tau} I_\tau^{-a/2-c/2},$$

where  $\mathfrak{R}_{a,c}^{(\tau)} = \phi_\tau(0, 0)$  is a normalizing constant which ensures  $\mathbb{E}^{\mathbb{P}}[Z_\tau \mid X_0 = I_0 = 0] = 1$ . For  $t \in [0, \tau]$  we define a martingale  $Z_t := \mathbb{E}^{\mathbb{P}}[Z_\tau \mid \mathcal{F}_t]$  and compute

$$Z_t = \mathbb{E}^{\mathbb{P}}[Z_\tau \mid \mathcal{F}_t] = \frac{1}{\mathfrak{R}_{a,c}^{(\tau)}} \mathbb{E}^{\mathbb{P}} \left[ e^{-aX_\tau} I_\tau^{-a/2-c/2} \mid \mathcal{F}_t \right] = \frac{1}{\mathfrak{R}_{a,c}^{(\tau)}} \phi_{\tau-t}(X_t, I_t).$$

We define a change of measure  $d\mathbb{Q}_t/d\mathbb{P}_t = Z_t$  (where  $\mathbb{P}_t$  is the restriction of  $\mathbb{P}$  on  $\mathcal{F}_t$  and similarly for  $\mathbb{Q}_t$ ). Take an arbitrary bounded measurable function  $f(x, z)$  and compute for  $0 < s < t < \tau$

$$\mathbb{E}^{\mathbb{Q}}[f(X_t, I_t) \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}^{\mathbb{P}}[Z_t f(X_t, I_t) \mid \mathcal{F}_s] = \frac{\mathbb{E}^{\mathbb{P}}[\phi_{\tau-t}(X_t, I_t) f(X_t, I_t) \mid X_s, I_s]}{\phi_{\tau-s}(X_s, I_s)}.$$

Thus under the measure  $\mathbb{Q}$  the process  $(X, I)$  has transitional probability density function

$$\mu_{s,t}^{\mathbb{Q}}(x_0, z_0; x_1, z_1) = \frac{\phi_{\tau-t}(x_1, z_1)}{\phi_{\tau-s}(x_0, z_0)} \mu_{t-s}^{\mathbb{P}}(x_0, z_0; x_1, z_1).$$

The coordinate process  $X$  under measure  $\mathbb{Q}$  has the law denoted as  $\mathbb{P}_X$  in (1.6). Let  $0 = t_0 < t_1 < t_2 < \dots < t_n = \tau$ . The joint density function (under  $\mathbb{Q}$ ) of  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$  (given that  $X_0 = I_0 = x_0 = z_0 = 0$ ) is

$$\frac{e^{-a x_n}}{\mathfrak{R}_{a,c}^{(\tau)}} f_{\mathbf{t}}^{\mathbb{Q}}(\mathbf{x}) \tag{3.2}$$

with

$$f_{\mathbf{t}}^{\mathbb{Q}}(\mathbf{x}) = \int_{0 < z_1 < z_2 < \dots < z_n} z_n^{-a/2-c/2} \prod_{j=1}^n \mu_{t_j-t_{j-1}}^{\mathbb{P}}(x_{j-1}, z_{j-1}; x_j, z_j) dz_1 dz_2 \dots dz_n.$$

Here we denoted  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{t} = (t_1, t_2, \dots, t_n)$ . Changing the variable of integration  $z_j - z_{j-1} = w_j$  we have

$$\begin{aligned} f_{\mathbf{t}}^{\mathbb{Q}}(\mathbf{x}) &= \int_{(0,\infty)^n} (w_1 + w_2 + \dots + w_n)^{-a/2-c/2} \prod_{j=1}^n \mu_{t_j-t_{j-1}}^{\mathbb{P}}(x_{j-1}, 0; x_j, w_j) dw_1 dw_2 \dots dw_n \\ &= \int_{(0,\infty)^n} (w_1 + w_2 + \dots + w_n)^{-a/2-c/2} \\ &\quad \prod_{j=1}^n \exp\left(-\frac{1}{2} \frac{e^{-2x_{j-1}} + e^{-2x_j}}{w_j}\right) \theta\left(\frac{e^{-x_{j-1}-x_j}}{w_j}, (t_j - t_{j-1})/2\right) \frac{dw_j}{w_j}. \end{aligned}$$

Next we change the variable of integration  $\xi_j = (e^{-x_{j-1}-x_j})/w_j$  and we obtain

$$f_{\mathbf{t}}^{\mathbb{Q}}(\mathbf{x}) = \int_{(0,\infty)^n} \left[ \sum_{j=1}^n \xi_j^{-1} e^{-x_{j-1}-x_j} \right]^{-a/2-c/2} \prod_{j=1}^n \exp\left(-\xi_j \cosh(x_j - x_{j-1})\right) \theta\left(\xi_j, (t_j - t_{j-1})/2\right) \frac{d\xi_j}{\xi_j}.$$

Next we consider a Markov process  $Y$  with transition probabilities (1.8) and the initial distribution of  $Y_0$  is (1.9). Let  $0 = t_0 < t_1 < t_2 < \dots < t_n = \tau$ . The joint distribution of  $(Y_0, Y_{t_1}, \dots, Y_{t_n})$  is given by

$$\frac{e^{-cy_0}}{C_{a,c}^{(\tau)}} \prod_{j=1}^n p_{t_j-t_{j-1}}(y_{j-1}; y_j) e^{-ay_n} dy_0 dy_1 \dots dy_n. \tag{3.3}$$

Thus the joint density function of  $(Y_{t_1} - Y_0, \dots, Y_{t_n} - Y_0)$  is

$$\frac{e^{-ax_n}}{C_{a,c}^{(\tau)}} \tilde{f}_{\mathbf{t}}(\mathbf{x}) \tag{3.4}$$

with

$$\tilde{f}_{\mathbf{t}}(\mathbf{x}) = \int_{\mathbb{R}} e^{-(a+c)y_0} \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1} + y_0; x_j + y_0) dy_0. \tag{3.5}$$

Here we assume that  $x_0 = 0$ . The goal is to show that

$$\tilde{f}_{\mathbf{t}}(\mathbf{x}) = 2^{(a+c)/2-1} \Gamma((a+c)/2) f_{\mathbf{t}}(\mathbf{x}) \tag{3.6}$$

for all  $\mathbf{t} = (t_1, t_2, \dots, t_n)$  and  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ . Once (3.6) is established, the normalizing constants will satisfy (1.14) and hence the densities (3.2) and (3.4) will be identical.

It remains to prove (3.6). To do so, we express (3.5) in terms of the Hartman-Watson density function (3.1). From Matsumoto and Yor (2005, Remark 4.1), Sousa and Yakubovich (2018, formula (50)), or Bryc et al. (2022, formula (7.4)) we know that

$$\begin{aligned} p_t(x, y) &= \int_{\mathbb{R}} \exp\left(-\frac{1}{2}(e^{x-y-r} + e^{y-x-r} + e^{r-x-y})\right) \theta(e^{-r}, t/2) dr \\ &= \int_0^\infty \exp\left(-\frac{1}{2}(\xi(e^{x-y} + e^{y-x}) + \xi^{-1}e^{-x-y})\right) \theta(\xi, t/2) \frac{d\xi}{\xi} \\ &= \int_0^\infty \exp\left(-\frac{1}{2}(-\xi \cosh(x-y) - \frac{1}{2}\xi^{-1}e^{-x-y})\right) \theta(\xi, t/2) \frac{d\xi}{\xi}. \end{aligned}$$

Once we plug the latter into (3.5) we get

$$\begin{aligned} \tilde{f}_{\mathbf{t}}(\mathbf{x}) &= \int_{\mathbb{R}} e^{-(a+c)y_0} dy_0 \\ &\int_{(0,\infty)^n} \prod_{j=1}^n \exp\left(-\xi_j \cosh(x_j - x_{j-1}) - \frac{1}{2}\xi_j^{-1}e^{-x_{j-1}-x_j-2y_0}\right) \theta(\xi_j, (t_j - t_{j-1})/2) \frac{d\xi_j}{\xi_j}. \end{aligned}$$

The integral with respect to  $y_0$  is

$$\int_{\mathbb{R}} \exp\left(- (a+c)y_0 - \frac{1}{2}e^{-2y_0}D\right) dy_0 = D^{-(a+c)/2} 2^{(a+c)/2-1} \Gamma((a+c)/2),$$

where  $D := \sum_{j=1}^n \xi_j^{-1} e^{-x_{j-1}-x_j}$ . Thus we obtain

$$\begin{aligned} \tilde{f}_{\mathbf{t}}(\mathbf{x}) &= 2^{(a+c)/2-1} \Gamma((a+c)/2) \int_{(0,\infty)^n} \left[ \sum_{j=1}^n \xi_j^{-1} e^{-x_{j-1}-x_j} \right]^{-(a+c)/2} \\ &\prod_{j=1}^n \exp\left(-\xi_j \cosh(x_j - x_{j-1})\right) \theta(\xi_j, (t_j - t_{j-1})/2) \frac{d\xi_j}{\xi_j} = 2^{(a+c)/2-1} \Gamma((a+c)/2) f_{\mathbf{t}}(\mathbf{x}). \end{aligned}$$

This establishes (3.6) and completes the proof.

*Remark 3.1.* An alternative method of proving Theorem 1.2 is outlined on page 8 in a recent paper Corwin (2022).

### 4. Proofs of limit Theorems

We begin with some known facts that we will use in several proofs. We note that with  $\tilde{Y}_t := Y_{\tau-t}^{(a,c)}$ , Markov process  $(\tilde{Y}_t)_{t \in [0, \tau]}$  has the same law as process  $(Y_t^{(c,a)})_{t \in [0, \tau]}$ . Therefore in some proofs without loss of generality we may assume  $a > 0$ . We recall Bryc et al. (2022, Theorem 3.3) in our notation/parametrization. Specialized to the case  $a > 0$  it says that the normalizing constant (1.11) is given by expression

$$C_{a,c}^{(\tau)} = \frac{2^{a+c}}{8\pi} \int_0^\infty e^{-\tau u^2/4} \frac{|\Gamma(\frac{a+iu}{2}, \frac{c+iu}{2})|^2}{|\Gamma(iu)|^2} du + 2^{a+c} \frac{\Gamma(\frac{c+a}{2}, \frac{a-c}{2})}{2c\Gamma(-c)} \sum_{\{k \geq 0: c+2k < 0\}} e^{\tau(c+2k)^2/4} (c+2k) \frac{(\frac{c}{2}, \frac{a+c}{2})_k}{k!(1 + \frac{c-a}{2})_k}. \tag{4.1}$$

From (1.11) is clear that  $C_{c,a}^{(\tau)} = C_{a,c}^{(\tau)}$ . It is also clear that function  $(a, c, \tau) \mapsto C_{a,c}^{(\tau)}$  is continuous in the domain  $\tau > 0, a + c > 0$  in  $\mathbb{R}^3$ . In some proofs we will need to determine its behaviour near a point at the boundary.

We will need small index asymptotics for the modified Bessel K function.

**Lemma 4.1.** *Assume that  $\nu \neq 0$  is an arbitrary complex number. If  $x > 0$  then*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} K_{\nu/T}(e^{-xT}) = \sinh(\nu x)/\nu. \tag{4.2}$$

If  $x \leq 0$  then

$$\lim_{T \rightarrow +\infty} \frac{1}{T} K_{\nu/T}(e^{-xT}) = 0. \tag{4.3}$$

*Proof:* We know (Erdélyi et al. (1954, 6.8 (26))) that the Mellin transform of Bessel  $K$ -function is

$$\int_0^\infty K_\nu(x) x^{s-1} dx = 2^{s-2} \Gamma((s+\nu)/2) \Gamma((s-\nu)/2), \quad \text{Re}(s) > |\text{Re}(\nu)|. \tag{4.4}$$

Thus with  $x > 0$  we can write

$$K_\nu(x) = \frac{1}{8\pi i} \int_{c+i\mathbb{R}} \Gamma((s+\nu)/2) \Gamma((s-\nu)/2) (x/2)^{-s} ds,$$

where  $c$  is any positive number greater than  $|\text{Re}(\nu)|$ . We shift the contour of integration  $c + i\mathbb{R} \mapsto -1/2 + i\mathbb{R}$ . We pick up two residues at poles  $s = \pm\nu$ , coming from Gamma functions, compute the residues at these poles and obtain

$$K_\nu(x) = \frac{1}{2} \left( \Gamma(\nu) (x/2)^{-\nu} + \Gamma(-\nu) (x/2)^\nu \right) + \frac{1}{8\pi i} \int_{-1/2+i\mathbb{R}} \Gamma((s+\nu)/2) \Gamma((s-\nu)/2) (x/2)^{-s} ds. \tag{4.5}$$

In the strip  $-3/8 < \text{Re}(z) < -1/8$  we have a uniform bound  $|\Gamma(z)| < C e^{-\frac{\pi}{4}|\text{Im}(z)|}$  for some absolute constant  $C$  (this is true for any strip of finite width which does not contain the poles of the Gamma function and it follows from Stirling’s asymptotic approximation to the Gamma function), thus we can estimate the integral in the right-hand side of (4.5) as follows

$$\left| \int_{-1/2+i\mathbb{R}} \Gamma((s+\nu)/2) \Gamma((s-\nu)/2) (x/2)^{-s} ds \right| \leq C x^{1/2} \int_{\mathbb{R}} e^{-\frac{\pi}{4}|t|} dt = x^{1/2} \frac{8C}{\pi}$$

and thus we obtain the following result:

$$\left| K_\nu(x) - \frac{1}{2} \left( \Gamma(\nu)(x/2)^{-\nu} + \Gamma(-\nu)(x/2)^\nu \right) \right| < \tilde{C}x^{1/2},$$

which is valid for all  $x > 0$ ,  $\nu \in \mathbb{C} \setminus \{0\}$  with  $|\nu| < 1/4$  and some absolute constant  $\tilde{C}$ .

Since  $\Gamma(z) \sim 1/z$  as  $|z| \rightarrow 0$ , this implies that for  $x > 0$ , and complex  $\nu \neq 0$

$$\lim_{T \rightarrow +\infty} \frac{1}{T} K_{\nu/T}(e^{-xT}) = \lim_{T \rightarrow +\infty} \frac{1}{2\nu} \left( e^{\nu x} \frac{T}{\nu} \Gamma(\nu/T) 2^{\nu/T} + e^{-\nu x} \frac{T}{\nu} \Gamma(-\nu/T) 2^{-\nu/T} \right) = \sinh(\nu x)/\nu.$$

On the other hand, for  $x \leq 0$ , complex  $\nu \neq 0$  and large enough  $T$ , we have

$$|K_{\nu/T}(e^{-xT})| \leq K_{|\nu/T|}(e^{-xT}) \leq K_{1/2}(e^{-xT}) = \sqrt{\pi/2} \frac{\exp(-e^{-xT})}{e^{-xT/2}},$$

which gives (4.3). □

We will also need an explicit formula for the Doob’s  $h$ -transform (1.12), which relies on analytic extension from  $\mathbf{a} > 0$ .

**Lemma 4.2.** *For  $\mathbf{a} \in \mathbb{R}$ ,  $0 \leq t < \tau$  and  $x \in \mathbb{R}$  the expression (1.12) is finite and is given by*

$$H_t(x) = H_t^{(\tau)}(x) := \frac{2^{\mathbf{a}}}{2\pi} \int_0^\infty e^{-(\tau-t)u^2/4} K_{iu}(e^{-x}) \frac{|\Gamma((\mathbf{a} + iu)/2)|^2}{|\Gamma(iu)|^2} du + 2^{\mathbf{a}+1} \sum_{k \geq 0 : \mathbf{a}+2k < 0} e^{(\tau-t)(\mathbf{a}+2k)^2/4} K_{\mathbf{a}+2k}(e^{-x}) \frac{1}{\Gamma(-\mathbf{a} - 2k)}. \quad (4.6)$$

*Proof:* We fix  $\tau > 0$  and  $x \in \mathbb{R}$ . For complex argument  $\text{Re}(\mathbf{a}) > 0$  we define

$$G_\tau(\mathbf{a}) = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{\tau w^2/4} K_w(e^{-x}) \frac{\Gamma((\mathbf{a} + w)/2)\Gamma((\mathbf{a} - w)/2)}{\Gamma(w)\Gamma(-w)} dw. \quad (4.7)$$

The integral (4.7) converges absolutely: the function  $w \mapsto K_w(e^{-x})$  is bounded in any strip  $w_1 < \text{Re}(w) < w_2$ , the ratio of the gamma functions grows at most as an exponential function of  $|w|$ , thus the quadratic exponential factor  $e^{\tau w^2/4}$  guarantees convergence.

We use the same method of analytic continuation of (4.1) as was used in Bryc et al. (2022). The poles of the integrand occur at points  $\pm(\mathbf{a} + 2k)$  for  $k = 0, 1, 2, \dots$ . First we assume that  $0 < \text{Re}(\mathbf{a}) < 1$  and shift the contour of integration  $i\mathbb{R} \mapsto 1 + i\mathbb{R}$ , collecting the pole at  $w = \mathbf{a}$ :

$$G_\tau(\mathbf{a}) = \frac{1}{2\pi i} \int_{1+i\mathbb{R}} e^{\tau w^2/4} K_w(e^{-x}) \frac{\Gamma((\mathbf{a} + w)/2)\Gamma((\mathbf{a} - w)/2)}{\Gamma(w)\Gamma(-w)} dw + 2e^{\tau \mathbf{a}^2/4} K_{\mathbf{a}}(e^{-x}) \frac{1}{\Gamma(-\mathbf{a})}.$$

The above expression provides analytic continuation in the strip  $-1 < \text{Re}(\mathbf{a}) < 1$ . (Recall the convention  $1/\Gamma(0) = 0$ .) Now we assume that  $-1 < \text{Re}(\mathbf{a}) < 0$  and shift the contour of integration back to  $i\mathbb{R}$ . We collect the residue at  $w = -\mathbf{a}$  and obtain

$$G_\tau(\mathbf{a}) = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{\tau w^2/4} K_w(e^{-x}) \frac{\Gamma((\mathbf{a} + w)/2)\Gamma((\mathbf{a} - w)/2)}{\Gamma(w)\Gamma(-w)} dw + 4e^{\tau \mathbf{a}^2/4} K_{\mathbf{a}}(e^{-x}) \frac{1}{\Gamma(-\mathbf{a})}.$$

The above equation gives analytic continuation in the strip  $-2 < \text{Re}(\mathbf{a}) < 0$ . Continuing this process we obtain an expression for any  $\mathbf{a} < 0$

$$G_\tau(\mathbf{a}) = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{\tau w^2/4} K_w(e^{-x}) \frac{\Gamma((\mathbf{a} + w)/2)\Gamma((\mathbf{a} - w)/2)}{\Gamma(w)\Gamma(-w)} dw + \sum_{k \geq 0 : \mathbf{a}+2k < 0} 4e^{\tau(\mathbf{a}+2k)^2/4} K_{\mathbf{a}+2k}(e^{-x}) \frac{1}{\Gamma(-\mathbf{a} - 2k)}. \quad (4.8)$$

Note that the integral (1.12) is well defined and by (4.4) formula (4.6) holds for  $\text{Re}(\mathbf{a}) > 0$ . We see that  $H_t(x) = 2^{\mathbf{a}-1} G_{\tau-t}(\mathbf{a})$ . Thus formula (4.8) shows that for fixed  $0 \leq t < \tau$  and  $x \in \mathbb{R}$ , function

$\mathbf{a} \mapsto 2^{\mathbf{a}-1}G_{\tau-t}(\mathbf{a})$  is an analytic extension of (1.12) to complex plane. Since as a function of  $\mathbf{a}$ , expression (1.12) is a Laplace transform when  $\text{Re}(\mathbf{a}) > 0$ , its analytic extension (4.6) is the Laplace transform of the same non-negative function. (For a version of this fact in the language of analytic characteristic functions, see Lukacs and Szász (1952, Theorem 2).)  $\square$

4.1. *Proof of Theorem 2.3.* We prove convergence of finite dimensional densities by establishing pointwise convergence of the initial densities, and pointwise convergence of the transition densities.

4.1.1. *Convergence of initial densities.* Since  $\mathbf{a} > 0$ , from (4.6) we get

$$H_0(x) = \frac{2^{\mathbf{a}}}{2\pi} \int_0^\infty K_{iu}(e^{-x})e^{-\tau u^2/4} \frac{|\Gamma(\mathbf{a} + iu)/2|^2}{|\Gamma(iu)|^2} du.$$

So  $Y_0^{(\mathbf{a},c)}$  has density

$$\begin{aligned} f(x) &= \frac{2^{\mathbf{a}}}{2\pi C_{\mathbf{a},c}^{(\tau)}} e^{-cx} \int_0^\infty K_{iu}(e^{-x})e^{-\tau u^2/4} \frac{|\Gamma(\mathbf{a} + iu)/2|^2}{|\Gamma(iu)|^2} du \\ &= \frac{2^{\mathbf{a}}}{2\pi C_{\mathbf{a},c}^{(\tau)}} e^{-cx} \int_0^\infty K_{iv/\sqrt{\tau}}(e^{-x})e^{-v^2/4} \frac{|\Gamma(\mathbf{a} + iv/\sqrt{\tau})/2|^2}{\sqrt{\tau}|\Gamma(iv/\sqrt{\tau})|^2} dv. \end{aligned} \tag{4.9}$$

Clearly,  $|\Gamma(\mathbf{a} + iv/\sqrt{\tau})/2|^2 \rightarrow \Gamma(\mathbf{a}/2)^2$  and  $K_{iv/\sqrt{\tau}}(e^{-x}) \rightarrow K_0(e^{-x})$  as  $\tau \rightarrow \infty$ . It is well known that

$$\lim_{\tau \rightarrow \infty} \frac{\tau}{|\Gamma(iv/\sqrt{\tau})|^2} = v^2, \tag{4.10}$$

and it is clear that we can pass to the limit under the integral sign. So

$$\begin{aligned} \lim_{\tau \rightarrow \infty} f(x) &= \frac{2^{\mathbf{a}}\Gamma(\mathbf{a}/2)^2}{2\pi \lim_{\tau \rightarrow \infty} \tau^{3/2}C_{\mathbf{a},c}^{(\tau)}} e^{-cx} K_0(e^{-x}) \int_0^\infty v^2 e^{-v^2/4} dv \\ &= \frac{2^{\mathbf{a}}\Gamma(\mathbf{a}/2)^2}{\sqrt{\pi} \lim_{\tau \rightarrow \infty} \tau^{3/2}C_{\mathbf{a},c}^{(\tau)}} e^{-cx} K_0(e^{-x}). \end{aligned}$$

To end the proof, we note the following.

**Lemma 4.3.**

$$\lim_{\tau \rightarrow \infty} \tau^{3/2}C_{\mathbf{a},c}^{(\tau)} = \begin{cases} \frac{1}{4\sqrt{\pi}} 2^{\mathbf{a}+c}\Gamma(\mathbf{a}/2)^2\Gamma(c/2)^2 & c \geq 0, \\ \infty & c < 0. \end{cases}$$

*Proof:* We use (4.1). It is clear that the integral is non-negative, and that if  $c < 0$  then the discrete sum diverges to  $+\infty$  as  $\tau \rightarrow \infty$ , because the first term of the sum dominates. This gives the second limit in the statement.

If  $c \geq 0$ , then there is no discrete sum in (4.1), only the integral. We change the variable of integration and write

$$\tau^{3/2}C_{\mathbf{a},c}^{(\tau)} = \frac{2^{\mathbf{a}+c}}{8\pi} \int_0^\infty e^{-v^2/4} |\Gamma(\frac{\mathbf{a}+iv/\sqrt{\tau}}{2}, \frac{c+iv/\sqrt{\tau}}{2})|^2 \frac{\tau}{|\Gamma(iv/\sqrt{\tau})|^2} dv. \tag{4.11}$$

Passing to the limit under the integral sign, from (4.10) we get the answer.  $\square$

4.1.2. *Convergence of transition densities.* For  $a > 0$ , from (4.6), we get

$$\begin{aligned}
 H_t(x) &= \frac{2^a}{2\pi} \int_0^\infty K_{iu}(e^{-x}) e^{-(\tau-t)u^2/4} \frac{|\Gamma(\frac{a+iu}{2})|^2}{|\Gamma(iu)|^2} du \\
 &= \frac{2^a}{2\pi\sqrt{\tau}} \int_0^\infty K_{iv/\sqrt{\tau}}(e^{-x}) e^{-(1-t/\tau)v^2/4} \frac{|\Gamma(\frac{a+iv/\sqrt{\tau}}{2})|^2}{|\Gamma(iv/\sqrt{\tau})|^2} dv,
 \end{aligned}$$

where we used (4.4). As before, passing to the limit under the integral sign, for any fixed  $0 \leq s < t$  and  $x, y \in \mathbb{R}$  we get

$$\lim_{\tau \rightarrow \infty} \frac{H_t(y)}{H_s(x)} = \frac{K_0(e^{-y})}{K_0(e^{-x})}.$$

Therefore the transition probabilities converge (pointwise) to (2.9). □

4.2. *Proof of Theorem 2.1.* Passing to the process  $(Y_{\tau-t})_{t \in [0, \tau]}$  if necessary, without loss of generality we may assume that  $a > 0$ . Recall (3.3). For  $0 = t_0 < t_1 < \dots < t_n = 1$ , the joint density of vector  $\frac{1}{\sqrt{\tau}} \left( Y_{\tau t_j}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \right)_{j=0, \dots, n}$  is

$$\frac{\sqrt{\tau}}{C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}} e^{-cx_0} H_0(x_0/\sqrt{\tau}) e^{-ax_n} \prod_{k=1}^n \sqrt{\tau} p_{\tau(t_k - t_{k-1})}(\sqrt{\tau} x_{k-1}, \sqrt{\tau} x_k). \tag{4.12}$$

Since the joint density of  $(\tilde{\eta}_{t_j}^{(a,c)})_{j=0, \dots, n}$  is of similar product form

$$\frac{1}{\mathfrak{C}_{a,c}} e^{-cx_0} h_0(x_0) 1_{x_0 > 0} e^{-ax_n} \prod_{k=1}^n g_{t_{k-1} - t_k}(x_{k-1}, x_k) 1_{x_k > 0}, \tag{4.13}$$

we only need to prove convergence of the corresponding factors. With  $a > 0$ , the part of the first factor that depends on  $\tau$  is

$$\begin{aligned}
 f_\tau(x) &:= \frac{\sqrt{\tau}}{C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}} H_0(x/\sqrt{\tau}) = \frac{\sqrt{\tau} 2^{a/\sqrt{\tau}}}{2\pi C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}} \int_0^\infty K_{iu}(e^{-x\sqrt{\tau}}) e^{-\tau u^2/4} \frac{|\Gamma(\frac{a/\sqrt{\tau} + iu}{2})|}{|\Gamma(iu)|^2} du \\
 &= 2^{a/\sqrt{\tau}} \frac{\sqrt{\tau}}{2\pi C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}} \int_0^\infty \frac{K_{iv/\sqrt{\tau}}(e^{-x\sqrt{\tau}})}{\sqrt{\tau}} e^{-v^2/4} \frac{|\Gamma(\frac{a+iv}{2\sqrt{\tau}})|}{|\Gamma(iv/\sqrt{\tau})|^2} dv, \tag{4.14}
 \end{aligned}$$

compare (4.9). In the next lemmas, we verify that the integral and the multiplicative constant in the above expression converge.

Recall (2.6). We have

**Lemma 4.4.**

$$\lim_{\tau \rightarrow \infty} \frac{C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}}{\sqrt{\tau}} = \mathfrak{C}_{a,c}.$$

*Proof:* We invoke (4.1) with  $a/\sqrt{\tau}$  and  $c/\sqrt{\tau}$ . For large  $\tau$ , at most one atom may be present. With  $a > 0$ , the contribution of this atom to  $C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}$  is

$$2^{(a+c)/\sqrt{\tau}} \frac{\Gamma(\frac{c+a}{2\sqrt{\tau}}, \frac{a-c}{2\sqrt{\tau}})}{2\Gamma(-c/\sqrt{\tau})} e^{c^2/4} 1_{c < 0}.$$

Combining this with the integral part, we get

$$\begin{aligned} \frac{2\pi C_{a/\sqrt{\tau}, c/\sqrt{\tau}}^{(\tau)}}{\sqrt{\tau}} &= \frac{1}{4\sqrt{\tau}} \int_0^\infty e^{-\tau u^2/4} \frac{|\Gamma(\frac{a/\sqrt{\tau}+iu}{2}, \frac{c/\sqrt{\tau}+iu}{2})|^2}{|\Gamma(iu)|^2} du + \pi 2^{(a+c)/\sqrt{\tau}} \frac{\Gamma(\frac{c+a}{2\sqrt{\tau}}, \frac{a-c}{2\sqrt{\tau}})}{\sqrt{\tau}\Gamma(-c/\sqrt{\tau})} e^{c^2/4} 1_{c<0} \\ &= \frac{1}{4} \int_0^\infty e^{-v^2/4} \frac{|\Gamma(\frac{a+iv}{2\sqrt{\tau}}, \frac{c+iv}{2\sqrt{\tau}})|^2}{\tau|\Gamma(iv/\sqrt{\tau})|^2} dv + \pi 2^{(a+c)/\sqrt{\tau}} \frac{\Gamma(\frac{c+a}{2\sqrt{\tau}}, \frac{a-c}{2\sqrt{\tau}})}{\sqrt{\tau}\Gamma(-c/\sqrt{\tau})} e^{c^2/4} 1_{c<0} \\ &\rightarrow \int_0^\infty e^{-v^2/4} \frac{4v^2}{(a^2+v^2)(c^2+v^2)} dv - \frac{4\pi c e^{c^2/4}}{a^2-c^2} 1_{c<0}, \end{aligned}$$

as we can pass to the limit under the integral sign.

Here we use  $\Gamma(z) \sim 1/z$  as  $z \rightarrow 0$  which gives

$$\lim_{\tau \rightarrow \infty} \frac{|\Gamma(\frac{a+iv}{2\sqrt{\tau}}, \frac{c+iv}{2\sqrt{\tau}})|^2}{\tau|\Gamma(iv/\sqrt{\tau})|^2} = \frac{16v^2}{(a^2+v^2)(c^2+v^2)} \text{ and } \lim_{\tau \rightarrow \infty} \frac{\Gamma(\frac{c+a}{2\sqrt{\tau}}, \frac{a-c}{2\sqrt{\tau}})}{2\sqrt{\tau}\Gamma(-c/\sqrt{\tau})} = \frac{-2c}{a^2-c^2}.$$

To complete the proof we need the following. □

**Lemma 4.5.** *For  $a + c > 0$  and  $a > 0$ , we have*

$$\frac{1}{2\pi} \int_0^\infty e^{-v^2/4} \frac{4v^2}{(a^2+v^2)(c^2+v^2)} dv - \frac{2ce^{c^2/4}}{a^2-c^2} 1_{c<0} = \mathfrak{C}_{a,c}. \tag{4.15}$$

*Proof:* For non-zero  $a, c$  the integral in (4.15) is

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty e^{-v^2/4} \frac{4v^2}{(a^2+v^2)(c^2+v^2)} dv &= \frac{2}{\pi} \int_0^\infty e^{-v^2/4} \int_0^\infty e^{-|a|x} \sin(vx) dx \int_0^\infty e^{-|c|y} dy \sin(vy) dv \\ &= \frac{2}{\pi} \int_0^\infty \int_0^\infty e^{-|a|x-|c|y} \int_0^\infty e^{-v^2/4} \sin(vx) \sin(vy) v dv dx dy = \int_0^\infty \int_0^\infty e^{-|a|x-|c|y} g_1(x, y) dx dy \\ &= \mathfrak{C}_{|a|, |c|}, \end{aligned}$$

where we used (2.5) and Fubini’s theorem. By taking a limit (or modifying the above calculation) the formula extends to  $c = 0$ . This shows that formula (4.15) holds for  $a > 0, c \geq 0$ .

To extend the formula to  $c \in (-a, 0)$  we use the identity  $\operatorname{erfc}(x) + \operatorname{erfc}(-x) = 2$  and explicit form of (2.6), i.e.,

$$\mathfrak{C}_{a,c} = \begin{cases} \frac{ae^{a^2/4}\operatorname{erfc}(a/2) - ce^{c^2/4}\operatorname{erfc}(c/2)}{a^2 - c^2} & a \neq c, a + c > 0 \\ \frac{2+a^2/2}{4a} e^{a^2/4} \operatorname{erfc}(a/2) - \frac{1}{2\sqrt{\pi}} & a = c > 0. \end{cases}$$

A calculation verifies that

$$\mathfrak{C}_{a,c} = \mathfrak{C}_{a,-c} - \frac{2ce^{c^2/4}}{a^2 - c^2}.$$

□

Next we tackle convergence of the integral in (4.14). The factor  $e^{-v^2/4}$  allows us to pass to the limit under the integral sign. We use

$$\lim_{\tau \rightarrow \infty} \frac{|\Gamma((a+iv)/(2\sqrt{\tau}))|^2}{|\Gamma(iv/\sqrt{\tau})|^2} = \frac{4v^2}{a^2+v^2}$$

and the following. For  $v > 0$ ,

$$\lim_{\tau \rightarrow \infty} \frac{K_{iv/\sqrt{\tau}}(e^{-x\sqrt{\tau}})}{\sqrt{\tau}} = \begin{cases} \frac{\sin(vx)}{v} & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{4.16}$$



This follows from (4.3) when  $x \leq 0$ , and from (4.2) with  $\nu = iv$  when  $x > 0$ . For  $x > 0$ , passing to the limit under the integral and invoking (4.16), we get

$$\begin{aligned} e^{-cx} \lim_{\tau \rightarrow \infty} f_\tau(x) &= \frac{e^{-cx}}{2\pi \mathfrak{C}_{a,c}} \mathbf{1}_{\{x>0\}} \int_0^\infty e^{-v^2/4} \frac{4v}{v^2 + a^2} \sin(vx) dv \\ &= \frac{e^{-cx}}{\mathfrak{C}_{a,c}} \mathbf{1}_{\{x>0\}} \int_0^\infty \frac{2}{\pi} \int_0^\infty e^{-v^2/4} e^{-ay} \sin(vy) \sin(vx) dv dy = \frac{e^{-cx}}{\mathfrak{C}_{a,c}} \mathbf{1}_{\{x>0\}} \int_0^\infty e^{-ay} g_1(x, y) dy \\ &= \frac{e^{-cx} h_0(x)}{\mathfrak{C}_{a,c}}, \end{aligned}$$

which matches (2.3) and recovers the first factor in (4.13).

Next, we consider a single factor from the product expression in (4.12). For  $t > 0$  we have

$$\begin{aligned} \sqrt{\tau} p_{\tau t}(x\sqrt{\tau}, y\sqrt{\tau}) &= \sqrt{\tau} \frac{2}{\pi} \int_0^\infty e^{-t\tau u^2} K_{iu}(e^{x\sqrt{\tau}}) K_{iu}(e^{y\sqrt{\tau}}) \frac{du}{|\Gamma(iu)|^2} \\ &= \frac{2}{\pi} \int_0^\infty e^{-tv^2} \frac{1}{\sqrt{\tau}} K_{iv/\sqrt{\tau}}(e^{x\sqrt{\tau}}) \frac{1}{\sqrt{\tau}} K_{iv/\sqrt{\tau}}(e^{y\sqrt{\tau}}) \frac{\tau}{|\Gamma(iv/\sqrt{\tau})|^2} dv. \end{aligned}$$

So by the dominated convergence theorem, (4.10) and (4.16) give

$$\lim_{\tau \rightarrow \infty} \sqrt{\tau} p_{\tau t}(x\sqrt{\tau}, y\sqrt{\tau}) = \begin{cases} \frac{2}{\pi} \int_0^\infty e^{-tv^2/4} \sin(xv) \sin(yv) dv = g_t(x, y) & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Taking it all together, we see that density (4.12) converges to density (4.13) for all  $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ .

*Remark 4.6.* As expected, process  $\tilde{\eta}^{(a,c)}$  is a ‘‘fixed point’’ of the procedure in the following sense. Instead of taking  $t \in [0, 1]$ , consider a Markov process  $(\tilde{\eta}_t^{(a,c)})_{t \in [0, \tau]}$  with initial distribution (2.3) with the normalizing constant  $\int_{\mathbb{R}_+^2} e^{-cx-ay} g_\tau(x, y) dx dy$ , and transition probabilities (2.4) but with Doob’s  $h$ -transform given by

$$h_t(x) = \int_{\mathbb{R}_+} g_{\tau-t}(x, y) e^{-ay} dy, \quad 0 \leq t < \tau$$

(with  $h_\tau(x) := e^{-ax}$ ). Then the law of  $\frac{1}{\sqrt{\tau}} \{ \tilde{\eta}_{\tau t}^{(a/\sqrt{\tau}, c/\sqrt{\tau})} \}_{t \in [0, 1]}$  does not depend on  $\tau > 0$ . This is a consequence of scaling  $\sqrt{\tau} g_{\tau t}(x\sqrt{\tau}, y\sqrt{\tau}) = g_t(x, y)$  for the kernel (2.5).

**4.3. Proof of Theorem 2.4.** To make the dependence on  $\tau$  explicit, we write  $H_t^{(\tau)}(x)$  for expression (4.6). We fix  $0 = t_0 < t_1 < \dots < t_n$ . The joint distribution of the vector  $(Y_0, Y_{t_1}, \dots, Y_{t_n})$  has density

$$f_\tau(\mathbf{x}) = \frac{H_{t_n}^{(\tau)}(x_n)}{C_{a,c}^{(\tau)}} e^{-cx_0} \prod_{j=1}^n p_{t_j - t_{j-1}}(x_{j-1}, x_j). \tag{4.18}$$

We note that as  $\tau \rightarrow \infty$  the integral term in (4.6) converges to 0 and if  $a < 0$  then the dominant term in the finite sum is the one with  $k = 0$ . Thus we have the following result: if  $a < 0$  then

$$H_t^{(\tau)}(x) = 2^{a+1} e^{(\tau-t)a^2/4} \frac{K_a(e^{-x})}{\Gamma(-a)} (1 + o(1)), \quad \tau \rightarrow +\infty.$$

In the same way, from (4.1) we see that when  $a < 0$  then

$$C_{a,c}^{(\tau)} = 2^{a+c-1} \frac{\Gamma((c+a)/2) \Gamma((c-a)/2)}{\Gamma(-a)} e^{\tau a^2/4} (1 + o(1)), \quad \tau \rightarrow +\infty.$$

Thus we obtain

$$\lim_{\tau \rightarrow \infty} \frac{H_t^{(\tau)}(x)}{C_{a,c}^{(\tau)}} = \frac{4e^{-ta^2/4}K_a(e^{-x})}{2^c\Gamma((a+c)/2)\Gamma((c-a)/2)}. \tag{4.19}$$

If  $a = 0$ , changing the variable in the integral (4.6), compare (4.11), we get

$$H_t^{(\tau)}(x) = \frac{2}{\sqrt{\pi}}K_0(e^{-x}) \left( \frac{1}{\sqrt{\tau}} + o(1/\sqrt{\tau}) \right), \tau \rightarrow \infty$$

and

$$C_{0,c}^{(\tau)} = \frac{2^c}{2\sqrt{\pi}}\Gamma(c/2)^2 \left( \frac{1}{\sqrt{\tau}} + o(1/\sqrt{\tau}) \right), \tau \rightarrow \infty.$$

(Here we use  $4 \int_0^\infty \exp(-v^2/4)dv = 4\sqrt{\pi}$ .) Thus (4.19) holds also for  $a = 0$ .

This shows that the density (4.18) converges as  $\tau \rightarrow \infty$  to the joint density

$$\frac{4}{2^c\Gamma((a+c)/2)\Gamma((c-a)/2)} e^{-cx_0} \left( \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1}, x_j) \right) e^{-t_n a^2/4} K_a(e^{-x_n})$$

of vector  $(Z_0^{(a,c)}, Z_{t_1}^{(a,c)}, \dots, Z_{t_n}^{(a,c)})$ .

4.4. *Proof of Theorem 2.5.* Let  $z > 0$ . Following Donati-Martin et al. (2001) and Hariya and Yor (2004), consider the transformation

$$\mathbb{T}_z(X)(t) = X_t - \ln \left( 1 + z \int_0^t e^{2X_s} ds \right)$$

that maps continuous function  $\{X_s\}_{0 \leq s \leq t}$  into a continuous function.

For  $\nu \in \mathbb{R}$  and  $\alpha > 0$  we define the Generalized Inverse Gaussian distribution

$$\mathbb{P}(\text{GIG}(\nu, \alpha) \in dx) = \frac{\alpha^{-\nu} x^{\nu-1}}{2K_\nu(\alpha)} e^{-\frac{1}{2}(x+\alpha^2/x)} \mathbf{1}_{\{x>0\}} dx,$$

which in statistical literature (Jørgensen, 1982) comes with an additional scale parameter. From Proposition 4.1 in Hariya and Yor (2004), we get the following result: for any  $\alpha > 0$ ,  $\nu \in \mathbb{R}$ ,  $t \geq 0$  and every non-negative  $\mathcal{F}_t$ -measurable functional  $F$  one has

$$\mathbb{E}[F(\mathbb{T}_{\text{GIG}(\nu,\alpha)}(W^{(\nu)})(t), s \leq t)] = e^{-\nu^2 t/2} \mathbb{E} \left[ F(W_s, s \leq t) e^{-\frac{\alpha^2}{2} \int_0^t e^{2W_s} ds} \frac{K_\nu(\alpha e^{W_t})}{K_\nu(\alpha)} \right]. \tag{4.20}$$

Here  $W_t^{(\nu)} = W_t + \nu t$  is a standard Brownian motion with drift  $\nu t$ . We set  $\nu = -a$ ,  $\alpha = e^{-x}$ , scale time parameter  $t \mapsto t/2$ , use the fact that  $(W_t)_{t \geq 0}$  and  $(-W_t)_{t \geq 0}$  have the same distribution (when the Brownian motion is started from zero) and that the right hand side of (4.20) is invariant with respect to changing  $\nu$  to  $-\nu$ . Applying the above identity to functional  $F(-W_s, s \leq t)$ , we get

$$\begin{aligned} & \mathbb{E} \left[ F(-\mathbb{T}_{\frac{1}{2}\text{GIG}(-a,e^{-x})}(-X^{(a)})(t), s \leq t) \right] \\ &= \mathbb{E} \left[ F(X_s - x, s \leq t) e^{-\frac{1}{4} \int_0^t e^{-2X_s} ds} \frac{e^{-a^2 t/4} K_a(e^{-X_t})}{K_a(e^{-x})} \Bigg| X_0 = x \right], \tag{4.21} \end{aligned}$$

where  $(X_t^{(\nu)}) := (W_{t/2}^{(\nu)}) \stackrel{d}{=} (\frac{1}{\sqrt{2}}W_t + \nu t/2) \stackrel{d}{=} (B_t + \nu t/2)$  and  $(X_t) := (X_t^{(0)}) \stackrel{d}{=} (\frac{1}{\sqrt{2}}W_t) \stackrel{d}{=} (B_t)$ . Let  $\tilde{X}$  be the Markov process having transition probability density (1.10). As was discussed in Bryc et al. (2022, Section 3),  $\tilde{X}$  can be identified with the Brownian motion  $B$  (of variance 1/2) killed at a rate  $\frac{1}{4}e^{-2Bt}$ . More precisely, the semigroup of the process  $\tilde{X}$  is given by

$$\tilde{P}_t f(x) = \mathbb{E}_x[f(\tilde{X}_t)] = \mathbb{E} \left[ e^{-\frac{1}{4} \int_0^t e^{-2B_s} ds} f(B_t) \Big| B_0 = x \right].$$

Thus we can rewrite (4.21) in the form

$$\mathbb{E} \left[ F(-\mathbb{T}_{\frac{1}{2}} \text{GIG}(-\mathbf{a}, e^{-x}) (-X^{(\mathbf{a})})(t), s \leq t) \right] = e^{-\mathbf{a}^2 t/4} \mathbb{E} \left[ F(\tilde{X}_s - x, s \leq t) \frac{K_{\mathbf{a}}(e^{-\tilde{X}_t})}{K_{\mathbf{a}}(e^{-x})} \Big| \tilde{X}_0 = x \right]. \tag{4.22}$$

Now we start the process  $\tilde{X}$  from initial distribution

$$\mathbb{P}(\tilde{X}_0 = dx) = \frac{4e^{-cx} K_{\mathbf{a}}(e^{-x})}{2^c \Gamma((c - \mathbf{a})/2) \Gamma((c + \mathbf{a})/2)} dx$$

and we note that this gives us the process  $Z^{(\mathbf{a},c)}$  in Theorem 2.4. So (4.22) becomes

$$\int_{\mathbb{R}} \mathbb{E}[F(-\mathbb{T}_{\frac{1}{2}} \text{GIG}(-\mathbf{a}, e^{-x}) (-X^{(\mathbf{a})})(t), s \leq t)] \mathbb{P}(\tilde{X}_0 = dx) = \mathbb{E} \left[ F(Z_s^{(\mathbf{a},c)} - Z_0^{(\mathbf{a},c)}, s \leq t) \right]. \tag{4.23}$$

On the left-hand side of (4.23) we get:

$$\int_{\mathbb{R}} \frac{4e^{-cx} K_{\mathbf{a}}(e^{-x})}{2^c \Gamma((c - \mathbf{a})/2) \Gamma((c + \mathbf{a})/2)} \mathbb{E}[F(-\mathbb{T}_{\frac{1}{2}} \text{GIG}(-\mathbf{a}, e^{-x}) (-X^{(\mathbf{a})})(t), s \leq t)] dx.$$

We need to compute the distribution of the mixture

$$\int_{\mathbb{R}} \frac{4e^{-cx} K_{\mathbf{a}}(e^{-x})}{2^c \Gamma((c - \mathbf{a})/2) \Gamma((c + \mathbf{a})/2)} \mathbb{P}(\frac{1}{2} \text{GIG}(-\mathbf{a}, e^{-x}) \in dy) dx.$$

The density of this mixture of distributions can be written in the form

$$\int_{\mathbb{R}} \frac{4e^{-cx} K_{\mathbf{a}}(e^{-x})}{2^c \Gamma((c - \mathbf{a})/2) \Gamma((c + \mathbf{a})/2)} \frac{e^{-\mathbf{a}x} (2y)^{-\mathbf{a}-1}}{K_{\mathbf{a}}(e^{-x})} e^{-y - e^{-2x}/(4y)} dx = \frac{y^{(c-\mathbf{a})/2-1}}{\Gamma((c - \mathbf{a})/2)} e^{-y} \mathbf{1}_{y>0}$$

and this is the density of  $\gamma_{(c-\mathbf{a})/2}$  random variable. Thus we obtain

$$\mathbb{E} \left[ F(Z_s^{(\mathbf{a},c)} - Z_0^{(\mathbf{a},c)}, s \leq t) \right] = \mathbb{E}[F(-\mathbb{T}_{\gamma_{(c-\mathbf{a})/2}} (-X^{(\mathbf{a})})(t), s \leq t)],$$

which implies that the process  $(Z_t^{(\mathbf{a},c)} - Z_0^{(\mathbf{a},c)})$  has the same distribution as

$$-\mathbb{T}_{\gamma_{(c-\mathbf{a})/2}} (-X^{(\mathbf{a})})(t) = X_t^{(\mathbf{a})} + \ln \left( 1 + \gamma_{(c-\mathbf{a})/2} \int_0^t e^{-2X_s^{(\mathbf{a})}} ds \right),$$

where  $(X_t^{(\mathbf{a})}) \stackrel{d}{=} (\frac{1}{\sqrt{2}} W_t + \mathbf{a}t/2)$  and  $W$  is the standard Brownian motion. Using the fact that the distribution of  $(Z_t^{(\mathbf{a},c)})$  is invariant under change  $\mathbf{a} \mapsto -\mathbf{a}$ , we obtain the expression in (2.11) and thus conclude the proof of Theorem 2.5.

4.5. *Proof of Theorem 2.6.* Recall that the  $1/\sqrt{2}$  multiple of the 3-dimensional Bessel process  $BES^3$  Revuz and Yor (1999, Ch VI §3) has transition probabilities

$$\mathbb{P}(\rho_t^{(\mathbf{c})} = dy | \rho_s^{(\mathbf{c})} = x) = \frac{y}{x} g_{t-s}(x, y) dy, \quad 0 \leq s < t, \quad x, y > 0, \tag{4.24}$$

with kernel (2.5).

*Proof of Theorem 2.6:* It is known Olver et al. (2010, (10.25.3) and (10.30.3)) that

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \text{ as } z \rightarrow \infty, \text{ and } K_0(z) \sim -\log z \text{ as } z \searrow 0.$$

We get

$$\lim_{T \rightarrow \infty} K_0(e^{-x\sqrt{T}})/\sqrt{T} = x \mathbf{1}_{\{x>0\}}.$$

Using this in (2.8) we see that the densities of  $Z_0^{(c/\sqrt{T})}$  converge pointwise. We get  $\frac{1}{\sqrt{T}}Z_0^{(c/\sqrt{T})} \Rightarrow c^2xe^{-cx}\mathbf{1}_{\{x>0\}}dx$ , which is the density of  $\frac{1}{c}\gamma_2$ . From (2.9) and (4.17), we see that the transition densities

$$\frac{K_0(e^{-y\sqrt{T}})}{K_0(e^{-x\sqrt{T}})}\sqrt{T}p_{T(t-s)}(x\sqrt{T},y\sqrt{T})$$

for the process  $\frac{1}{\sqrt{T}}\left(Z_{tT}^{(c/\sqrt{T})}\right)_{t\geq 0}$  converge to (4.24) when  $x, y > 0$ .

A minor technical issue is an undefined expression 0/0 in case  $x < 0$ , which can be handled by considering joint multivariate density as in proof of Theorem 2.1. We omit the details.  $\square$

4.6. *Proof of Theorem 2.8.* Recall that the initial law (2.10) is the law of  $-\log\left(2\sqrt{\gamma_{\frac{a+c}{2}}\tilde{\gamma}_{\frac{c-a}{2}}}\right)$ . Since  $-\varepsilon\log\gamma_\varepsilon \Rightarrow \gamma_1$  as  $\varepsilon \rightarrow 0$ , the initial law  $\mu_{a,c}(dx)$  is the law of  $\frac{1}{a+c}\gamma_1 + \frac{1}{c-a}\tilde{\gamma}_1$ . In particular, when  $a = 0$  here we get (2.13).

For transition probabilities, we would only need to find the asymptotics of  $e^{-a^2t}K_{a/\sqrt{T}}\left(e^{-x\sqrt{T}}\right)$ . For  $a \neq 0$ , the limit is

$$\lim_{T\rightarrow+\infty}\frac{1}{\sqrt{T}}K_{a/\sqrt{T}}\left(e^{-\sqrt{T}x}\right) = \begin{cases} \frac{1}{a}\sinh(ax) & x > 0, \\ 0 & x \leq 0. \end{cases}$$

For  $x \leq 0$ , the limit follows from (4.3). For  $x > 0$  we use (4.2) with  $\nu = a$ . To avoid undefined expression 0/0 when  $x \leq 0$ , we need to consider joint multivariate density as in the proof of Theorem 2.1. We omit the details.

4.7. *Proofs of the observations about convergence of univariate laws.* In this section we collect proofs of observations on the limits of the initial laws which served as justification that some of the limits in Barraquand and Le Doussal (2022) do not extend to the Markov process level. A convenient tool for this task are identities for the Laplace transform, which we state in slightly more general form than what we need.

Recall (1.11). Since the joint law of  $(Y_0, Y_\tau)$  is

$$\frac{1}{C_{a,c}^{(\tau)}}e^{-cx-ay}p_\tau(x,y)dxdy,$$

we have

$$\mathbb{E}\left[\exp\left(s_0Y_\tau^{(a,c)} + s_1Y_\tau^{(a,c)}\right)\right] = \frac{C_{a-s_1,c-s_0}^{(\tau)}}{C_{a,c}^{(\tau)}}.$$

This gives

$$\mathbb{E}\left[\exp\left(-sY_0^{(a,c)}\right)\right] = \frac{C_{a,s+c}^{(\tau)}}{C_{a,c}^{(\tau)}}. \tag{4.25}$$

4.7.1. *Proof of (1.15) (convergence to the exponential law).* By symmetry, if  $a \neq 0$  without loss of generality we may assume that  $a < 0$  so  $\min\{a_\varepsilon, c_\varepsilon\} < 0$  for all  $\varepsilon > 0$  small enough.

We now use (4.25) to verify that  $\varepsilon Y_0^{(a_\varepsilon, c_\varepsilon)} \Rightarrow \gamma_1$ . Of course  $c_\varepsilon \rightarrow -a$  so without loss of generality, we assume  $a > 0$ . (The proof requires minor modifications if  $a < 0$ .) From (4.1), as  $\varepsilon \rightarrow 0$  we get

$$C_{a_\varepsilon, c_\varepsilon + \varepsilon s}^{(\tau)} \sim \frac{1}{8\pi} \int_0^\infty e^{-\tau u^2/4} \frac{|\Gamma(\frac{a+iu}{2}, \frac{-a+iu}{2})|^2}{|\Gamma(iu)|^2} du + \frac{\Gamma(\frac{(1+s)\varepsilon}{2}, a)}{2c\Gamma(a)} \sum_{\{k \geq 0: -a+2k < 0\}} e^{\tau(c+2k)^2/4} (c+2k) \frac{(c, \frac{\varepsilon}{2})_k}{k!(1-a)_k}.$$

Since the integral converges and  $\Gamma(x) \sim \frac{1}{x} - \gamma$  as  $x \rightarrow 0$ , the leading term in the asymptotics comes from the coefficient in front of the sum and the first term in the sum. We see that

$$\mathbb{E} \left[ \exp \left( -s\varepsilon Y_0^{(a_\varepsilon, c_\varepsilon)} \right) \right] = \frac{C_{a_\varepsilon, c_\varepsilon + \varepsilon s}^{(\tau)}}{C_{a_\varepsilon, c_\varepsilon}^{(\tau)}} \sim \frac{\Gamma(\frac{c_\varepsilon + a_\varepsilon + \varepsilon s}{2}, \frac{a_\varepsilon - c_\varepsilon - \varepsilon s}{2})}{\Gamma(-c_\varepsilon - \varepsilon s)} e^{\tau(c_\varepsilon + \varepsilon s)^2/4} \times \left[ \frac{\Gamma(\frac{c_\varepsilon + a_\varepsilon}{2}, \frac{a_\varepsilon - c_\varepsilon}{2})}{\Gamma(-c_\varepsilon)} e^{\tau c_\varepsilon^2/4} \right]^{-1} \sim \frac{\Gamma((1+s)\varepsilon/2)}{\Gamma(\varepsilon/2)} \rightarrow \frac{1}{1+s},$$

which is the Laplace transform of the exponential  $\gamma_1$  law. □

4.7.2. *Proof of asymptotic normality (2.12).* We note that as indicated in (1.9), the law of  $Y_0^{(a,c)}$  depends on  $\tau$  though the normalizing constant and though  $H_0$ .

*Proof of (2.12):* We use (4.25) to determine the Laplace transform.

$$L_\tau(s) := \mathbb{E} \left[ e^{-s(Y_0^{(a,c)} + \tau c/2)/\sqrt{\tau}} \right] = e^{-sc\sqrt{\tau}/2} \frac{C_{a, c+s/\sqrt{\tau}}^{(\tau)}}{C_{a, c}^{(\tau)}}.$$

It is enough to determine how the numerator depends on  $s$  for  $s < 0$ , discarding all multiplicative constants that will cancel out with the denominator. From (4.1) we know that  $C_{a, c+s/\sqrt{\tau}}^{(\tau)} = I_\tau(s) + D_\tau(s)$  is the sum of the integral

$$I_\tau(s) = \frac{1}{8\pi} \int_0^\infty e^{-\tau u^2/4} \frac{|\Gamma(\frac{a+iu}{2}, \frac{c+s/\sqrt{\tau}+iu}{2})|^2}{|\Gamma(iu)|^2} du \leq \frac{1}{8\pi} \int_0^\infty e^{-u^2} \frac{|\Gamma(\frac{a+iu}{2}, \frac{c+s/\sqrt{\tau}+iu}{2})|^2}{|\Gamma(iu)|^2} du \rightarrow \frac{1}{8\pi} \int_0^\infty e^{-u^2} \frac{|\Gamma(\frac{a+iu}{2}, \frac{c+iu}{2})|^2}{|\Gamma(iu)|^2} du,$$

which is bounded in  $\tau$ , and the finite sum

$$D_\tau(s) = 2^{a+c+s/\sqrt{\tau}} \frac{\Gamma(\frac{a+c+s/\sqrt{\tau}}{2}, \frac{a-c-s/\sqrt{\tau}}{2})}{2(c+s/\sqrt{\tau})\Gamma(-c-s/\sqrt{\tau})} \times \sum_{\{k \geq 0: c+s/\sqrt{\tau}+2k < 0\}} e^{\tau(c+s/\sqrt{\tau}+2k)^2/4} (c+s/\sqrt{\tau}+2k) \frac{(c+s/\sqrt{\tau}, \frac{a+c+s/\sqrt{\tau}}{2})_k}{k!(1+\frac{c+s/\sqrt{\tau}-a}{2})_k} = 2^{a+c+s/\sqrt{\tau}-1} \frac{\Gamma(\frac{a+c+s/\sqrt{\tau}}{2}, \frac{a-c-s/\sqrt{\tau}}{2})}{\Gamma(-c-s/\sqrt{\tau})} \times \left( e^{\tau(c+s/\sqrt{\tau})^2/4} + \frac{(c+s/\sqrt{\tau}+2)(a+c+s/\sqrt{\tau})}{2+c+s/\sqrt{\tau}-a} e^{\tau(c+s/\sqrt{\tau}+2)^2/4} + \dots \right).$$

With  $s < 0$ , we see that  $e^{-sc\sqrt{\tau}}I_{\tau} \rightarrow 0$  and that the leading term in the asymptotics comes from the first term in the sum  $D_{\tau}$ . We get

$$\lim_{\tau \rightarrow \infty} L_{\tau}(s) = \lim_{\tau \rightarrow \infty} e^{-sc\sqrt{\tau}/2} \frac{D_{\tau}(s)}{D_{\tau}(0)} = \lim_{\tau \rightarrow \infty} \frac{e^{\tau(c+s/\sqrt{\tau})^2/4 - sc\sqrt{\tau}/2}}{e^{\tau c^2/4}} = e^{s^2/4}.$$

□

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## References

- Barraquand, G. and Corwin, I. Stationary measures for the log-gamma polymer and KPZ equation in half-space. *ArXiv Mathematics e-prints* (2022). [arXiv: 2203.11037](#).
- Barraquand, G. and Le Doussal, P. Steady state of the KPZ equation on an interval and Liouville quantum mechanics. *Europhysics Letters*, **137** (6), 61003 (2022). ArXiv preprint with Supplementary material: [arXiv: 2105.15178](#). DOI: [10.1209/0295-5075/ac25a9](#).
- Bryc, W., Kuznetsov, A., Wang, Y., and Wołowski, J. Markov processes related to the stationary measure for the open KPZ equation. *Probab. Theory Related Fields*, pp. 1–37 (2022). DOI: [10.1007/s00440-022-01110-7](#).
- Bryc, W., Wang, Y., and Wołowski, J. From the asymmetric simple exclusion processes to the stationary measures of the KPZ fixed point on an interval (2022+). [arXiv: 2202.11869](#). To appear in *Ann. Inst. Henri Poincaré Probab. Stat.*
- Corwin, I. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, **1** (1), 1130001, 76 (2012). [MR2930377](#).
- Corwin, I. Some recent progress on the stationary measure for the open KPZ equation. *ArXiv Mathematics e-prints* (2022). [arXiv: 2202.01836](#).
- Corwin, I. and Knizel, A. Stationary measure for the open KPZ equation. *ArXiv Mathematics e-prints* (2021). [arXiv: 2103.12253](#).
- Corwin, I. and Shen, H. Open ASEP in the weakly asymmetric regime. *Comm. Pure Appl. Math.*, **71** (10), 2065–2128 (2018). [MR3861074](#).
- Donati-Martin, C., Matsumoto, H., and Yor, M. Some absolute continuity relationships for certain anticipative transformations of geometric Brownian motions. *Publ. Res. Inst. Math. Sci.*, **37** (3), 295–326 (2001). [MR1855425](#).
- Donati-Martin, C., Matsumoto, H., and Yor, M. The law of geometric Brownian motion and its integral, revisited; application to conditional moments. In *Mathematical finance—Bachelier Congress, 2000 (Paris)*, Springer Finance, pp. 221–243. Springer, Berlin (2002). [MR1960566](#).
- Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. *Tables of integral transforms. Vol. I*. McGraw-Hill Book Co., Inc., New York-Toronto-London (1954). [MR0061695](#).
- Ethier, S. N. and Kurtz, T. G. *Markov processes. Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York (1986). ISBN 0-471-08186-8. [MR838085](#).
- Gerencsér, M. and Hairer, M. Singular SPDEs in domains with boundaries. *Probab. Theory Related Fields*, **173** (3-4), 697–758 (2019). [MR3936145](#).
- Gonçalves, P., Perkowski, N., and Simon, M. Derivation of the stochastic Burgers equation with Dirichlet boundary conditions from the WASEP. *Ann. H. Lebesgue*, **3**, 87–167 (2020). [MR4060852](#).

- Hariya, Y. and Yor, M. Limiting distributions associated with moments of exponential Brownian functionals. *Studia Sci. Math. Hungar.*, **41** (2), 193–242 (2004). [MR2082657](#).
- Hartman, P. and Watson, G. S. “Normal” distribution functions on spheres and the modified Bessel functions. *Ann. Probability*, **2**, 593–607 (1974). [MR370687](#).
- Jørgensen, B. *Statistical properties of the generalized inverse Gaussian distribution*, volume 9 of *Lecture Notes in Statistics*. Springer-Verlag, New York-Berlin (1982). ISBN 0-387-90665-7. [MR648107](#).
- Kardar, M., Parisi, G., and Zhang, Y.-C. Dynamic Scaling of Growing Interfaces. *Phys. Rev. Lett.*, **56**, 889–892 (1986). DOI: [10.1103/PhysRevLett.56.889](#).
- Lukacs, E. and Szász, O. On analytic characteristic functions. *Pacific J. Math.*, **2**, 615–625 (1952). [MR51461](#).
- Matetski, K., Quastel, J., and Remenik, D. The KPZ fixed point. *Acta Math.*, **227** (1), 115–203 (2021). [MR4346267](#).
- Matsumoto, H. and Yor, M. Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.*, **2**, 312–347 (2005). [MR2203675](#).
- Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W., editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge (2010). ISBN 978-0-521-14063-8. [MR2723248](#).
- Parekh, S. The KPZ limit of ASEP with boundary. *Comm. Math. Phys.*, **365** (2), 569–649 (2019). [MR3907953](#).
- Pinsky, R. G. On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.*, **13** (2), 363–378 (1985). [MR781410](#).
- Quastel, J. Introduction to KPZ. In *Current developments in mathematics, 2011*, pp. 125–194. Int. Press, Somerville, MA (2012). [MR3098078](#).
- Revuz, D. and Yor, M. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition (1999). ISBN 3-540-64325-7. [MR1725357](#).
- Rogers, L. C. G. and Pitman, J. W. Markov functions. *Ann. Probab.*, **9** (4), 573–582 (1981). [MR624684](#).
- Rogers, L. C. G. and Williams, D. *Diffusions, Markov processes, and martingales. Vol. 2. Itô calculus*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York (1987). ISBN 0-471-91482-7. [MR921238](#).
- Sousa, R. and Yakubovich, S. The spectral expansion approach to index transforms and connections with the theory of diffusion processes. *Commun. Pure Appl. Anal.*, **17** (6), 2351–2378 (2018). [MR3814379](#).
- Wiśniewolski, M. K-Hartman-Watson distributions: a study on distributional dependencies between functionals of geometric Brownian motion, GIG and Hartman-Watson distributions. *J. Math. Anal. Appl.*, **482** (2), 123579, 14 (2020). [MR4018567](#).
- Yor, M. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.*, **24** (3), 509–531 (1992). [MR1174378](#).