The Matsumoto-Yor property in free probability via subordination and Boolean cumulants

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Abstract. We study the Matsumoto-Yor property in free probability. We prove three characterizations of free-GIG and free Poisson distributions by freeness properties together with some assumptions about conditional moments. Our main tools are subordination and Boolean cumulants. In particular, we establish a new connection between the additive subordination function and Boolean cumulants.

1. Introduction

In Matsumoto and Yor (2001) the authors observed an interesting property of Gamma and Generalized Inverse Gaussian (GIG) laws that is now known in literature as the Matsumoto-Yor property: if X has the Generalized Inverse Gaussian law $GIG(-p, a, b)$, Y has the Gamma law $G(p, a)$ and X and Y are independent random variables, then

$$U = \frac{1}{X + Y} \text{ and } V = \frac{1}{X} - \frac{1}{X + Y}$$

are also independent and distributed according to $GIG(-p, b, a)$ and $G(p, b)$ laws respectively.

We recall that the Gamma law $G(p, a)$ with parameters $p, a > 0$ is a probability measure that has the density

$$\frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} \mathbb{1}_{(0,\infty)}(x)$$

and the Generalized Inverse Gaussian law $GIG(p, a, b)$ with parameters $a, b > 0$, $p \in \mathbb{R}$ is a probability measure that has density

$$\frac{(a/b)^{p/2}}{2K_p(2\sqrt{ab})} x^{p-1} e^{-ax - \frac{b}{x}} \mathbb{1}_{(0,\infty)}(x),$$

where K_p is s modified Bessel function of the third kind.

Later it was shown in Letac and Wesołowski (2000) that independence of X and Y and independence of U and V characterizes Gamma and GIG laws. In the same paper authors generalized

the Matsumoto-Yor property to the framework of real symmetric matrices. Further generalizations of different nature can be found for example in Massam and Wesołowski (2004), Bao and Noack (2021) and Kołodziejek (2017).

The analogue of the Matsumoto-Yor property in free probability was studied in Szpojankowski (2017). In this case the property states that if $\mathbf{X}, \mathbf{Y}$ are free non-commutative random variables and have free-GIG and Marchenko-Pastur distributions respectively (with suitably chosen parameters), then the random variables

$$\mathbf{U} = (\mathbf{X} + \mathbf{Y})^{-1} \text{ and } \mathbf{V} = \mathbf{X}^{-1} - (\mathbf{X} + \mathbf{Y})^{-1}$$

are also free and have free GIG and Marchenko-Pastur distributions. It was shown in Szpojankowski (2017) that freeness of $\mathbf{X}$ and $\mathbf{Y}$ and freeness of $\mathbf{U}$ and $\mathbf{V}$ characterizes free-GIG and Marchenko-Pastur laws.

In this paper we study regression versions of the above characterization, assuming only constant regressions

$$\begin{cases} \varphi\left(\mathbf{V}^k \mid \mathbf{U}\right) = m_k \mathbf{I}, \\ \varphi\left(\mathbf{V}^l \mid \mathbf{U}\right) = m_l \mathbf{I}, \end{cases} \tag{1.1}$$

for some non-zero $k, l \in \mathbb{Z}$, where $k \neq l$ and $m_k, m_l \in \mathbb{R}$ are some constants. The cases we consider are $(k, l) = (1, 2), (1, -1), (-1, -2)$. The case $(k, l) = (-1, 1)$ was also studied in Szpojankowski (2017) but the author used a different method based on the moment transform. In classical probability the same case $(k, l) = (-1, 1)$ was considered first in Wesołowski (2002) and the remaining cases were considered in Chou and Huang (2004)

Our main tools are subordination of free convolutions and Boolean cumulants. Subordination is a powerful technique first used in Biane (1998) and then enhanced considerably in Voiculescu (2000). Roughly speaking, for the free additive convolution one has that the conditional expectation of the resolvent $(z - \mathbf{X} - \mathbf{Y})^{-1}$ onto the algebra generated by $\mathbf{X}$ is the resolvent of $\mathbf{X}$ at a different point $\omega_1(z)$, where ω_1 is an analytic selfmap of the upper half-plane $\mathbb{C}^+$. Subordination proved to be very useful in studying properties of free convolutions (see e.g. Belinschi and Bercovici, 2007; Belinschi, 2006, 2008) and in random matrix theory (see Belinschi et al., 2017). It was also observed that subordination is useful in regression characterization problems (cf. Ejsmont et al., 2017). For an introduction to subordination results we recommend Chapter 2 of Mingo and Speicher (2017).

Powerful as it is, subordination itself does not allow to prove all the results which we are studying here. We take advantage of connections between free probability and Boolean cumulants established recently in Fevrier et al. (2020); Lehner and Szpojankowski (2021). We develop ideas from Lehner and Szpojankowski (2021), in particular we provide a new expansion of the reciprocal of the additive subordination function in terms of Boolean cumulants

$$\frac{1}{\omega_2(z)} = \sum_{n=0}^{\infty} \beta_{2n+1}\left((z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, \ldots, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}\right). \tag{1.2}$$

(See Lemma 4.1 for more details.)

One of the implicit results of this paper is a methodological remark, that Boolean cumulants prove to be a useful tool when dealing with conditional expectations of expressions involving free random variables. It confirms the observation already noted in Szpojankowski and Wesołowski (2020) in the context of regression versions of the Lukacs property.

The paper is organized as follows: In Sections 2 and 3 we introduce basic facts from free probability theory and recall the Matsumoto-Yor property in more details. In Section 4 we derive some formulas relating the subordination functions and Boolean cumulants and we relate regression conditions of the form (1.1) to some equations connecting the subordination functions and the Cauchy-Stieltjes transform of $\mathbf{X} + \mathbf{Y}$. In Section 5 we state and prove characterization theorems which are the main results of the paper.

2. Background and notation

In this section we introduce basic notions and facts from non-commutative probability theory that are needed to understand this paper. We assume we are given a C*-probability space $(\mathcal{A}, \varphi)$ i.e. $\mathcal{A}$ is a unital C*-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is positive, tracial and faithful linear functional (state) such that $\varphi(\mathbf{I}) = 1$ where $\mathbf{I}$ is the unit of $\mathcal{A}$.

Elements of $\mathcal{A}$ are called (non-commutative) random variables and in this paper are denoted as $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ etc. A random variable $\mathbf{X}$ is called centered if $\varphi(\mathbf{X}) = 0$.

2.1. *Freeness and cumulants.* Freeness is one of the basic concepts that serves as the analogue of independence from classic probability theory and was introduced by Voiculescu in Voiculescu (1986).

Definition 2.1. We say that unital subalgebras $\mathcal{A}_1, ..., \mathcal{A}_n$ of $\mathcal{A}$ are free if for every choice of centered random variables $\mathbf{X}_k \in \mathcal{A}_{i_k}$, $k = 1, 2, \ldots, n$, such that $i_1 \neq i_2 \neq \ldots \neq i_n$ we have

$$\varphi\left(\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n\right) = 0.$$

We say that random variables $\mathbf{X}, \mathbf{Y} \in \mathcal{A}$ are free if unital subalgebras generated by those elements are free.

The definition of freeness can be viewed as a rule for computing joint moments. For example if $\mathbf{X}, \mathbf{Y}$ are free, then $\varphi(\mathbf{XY}) = \varphi(\mathbf{X})\varphi(\mathbf{Y})$.

For positive integer n let us denote $[n] = \{1, 2, \ldots, n\}$.

Definition 2.2.
(1) A partition π of $[n]$ is a set $\pi = \{B_1, ..., B_k\}$ of non-empty and pairwise disjoint subsets of $[n]$ such that $[n] = \bigcup_{i=1}^{k} B_i$. Elements $B_1, \ldots, B_k$ are called blocks of π. The set of all partition of $[n]$ is denoted by $\mathcal{P}(n)$.
(2) A partition $\pi \in \mathcal{P}(n)$ is called an interval partition if every block B of π is of the form $[n] \cap I$ for some interval I. The set of all 2^{n-1} interval partitions of $[n]$ is denoted by $Int(n)$.
(3) A partition $\pi \in \mathcal{P}(n)$ is called a non-crossing partition if for every two blocks $B_1, B_2 \in \pi$ and every $i_1, i_2 \in B_1$ and $j_1, j_2 \in B_2$ such that $i_1 < j_1 < i_2 < j_2$ we have $B_1 = B_2$. The set of all non-crossing partitions of $[n]$ is denoted by $NC(n)$.

Remark 2.3. Both sets $Int(n)$ and $NC(n)$ have a lattice structure induced by the so-called reversed refinement order. We say that $\pi_1 \leq \pi_2$ if every block of the partition π_1 is contained in some block of π_2.

Definition 2.4. For $n \geq 1$ the Boolean cumulant functional $\beta_n : \mathcal{A}^n \to \mathbb{C}$ and the free cumulant functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ are defined recursively by

$$\forall \mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathcal{A} : \varphi\left(\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n\right) = \sum_{\pi \in Int(n)} \beta_\pi\left(\mathbf{X}_1, \ldots, \mathbf{X}_n\right),$$

$$\forall \mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathcal{A} : \varphi\left(\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n\right) = \sum_{\pi \in NC(n)} \kappa_\pi\left(\mathbf{X}_1, \ldots, \mathbf{X}_n\right),$$

where for $\pi = \{B_1, \ldots, B_k\}$

$$\beta_\pi\left(\mathbf{X}_1, \ldots, \mathbf{X}_n\right) = \prod_{j=1}^{k} \beta_{|B_j|}\left(\mathbf{X}_i : i \in B_j\right),$$

and

$$\kappa_\pi\left(\mathbf{X}_1, \ldots, \mathbf{X}_n\right) = \prod_{j=1}^{k} \kappa_{|B_j|}\left(\mathbf{X}_i : i \in B_j\right).$$

Remark 2.5. Boolean cumulants can also be defined directly via Möbius inversion formula as

$$\beta_n\left(\mathbf{X}_1,\ldots,\mathbf{X}_n\right) = \sum_{\pi \in Int(n)} (-1)^{|\pi|+1} \varphi_\pi\left(\mathbf{X}_1,\ldots,\mathbf{X}_n\right), \tag{2.1}$$

where $|\pi|$ is the number of blocks of π and φ_π is defined in a similar manner to β_π i.e.

$$\varphi_\pi\left(\mathbf{X}_1,\ldots,\mathbf{X}_n\right) = \prod_{j=1}^{k} \varphi\left(\prod_{i\in B_j} \mathbf{X}_i\right),$$

where $\prod_{i\in B_j}\mathbf{X}_i = \mathbf{X}_{k_1}\mathbf{X}_{k_2}\cdots\mathbf{X}_{k_m}$ if $B_j = \{k_1 < k_2 < \ldots < k_m\}$. In particular $\beta_1 = \varphi$ and $\beta_2(\mathbf{X},\mathbf{Y}) = \varphi(\mathbf{XY}) - \varphi(\mathbf{X})\varphi(\mathbf{Y})$.

We will need two formulas involving Boolean cumulants. They can be found in Fevrier et al. (2020) and Lehner and Szpojankowski (2021) and were used also in Szpojankowski and Wesołowski (2020).

Proposition 2.6. *Assume we are given two collections of random variables* $\{\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_{n+1}\}$ *and* $\{\mathbf{Y}_1,\mathbf{Y}_2,\ldots,\mathbf{Y}_n\}$ *that are free,* $n\geq 1$. *Then*

$$\varphi\left(\mathbf{X}_1\mathbf{Y}_1\cdots\mathbf{X}_n\mathbf{Y}_n\right) =$$
$$= \sum_{k=0}^{n-1} \sum_{0=j_0<j_1<\ldots<j_{k+1}=n} \varphi\left(\mathbf{Y}_{j_1}\mathbf{Y}_{j_2}\cdots\mathbf{Y}_{j_{k+1}}\right) \prod_{l=0}^{k} \beta_{2(j_{l+1}-j_l)-1}\left(\mathbf{X}_{j_l+1},\mathbf{Y}_{j_l+1},\ldots,\mathbf{Y}_{j_{l+1}-1},\mathbf{X}_{j_{l+1}}\right)$$
$$\tag{2.2}$$

and

$$\beta_{2n+1}\left(\mathbf{X}_1,\mathbf{Y}_1,\ldots,\mathbf{X}_n,\mathbf{Y}_n,\mathbf{X}_{n+1}\right) =$$
$$= \sum_{k=2}^{n+1} \sum_{1=j_1<\ldots<j_k=n} \beta_k\left(\mathbf{X}_{j_1},\ldots,\mathbf{X}_{j_k}\right) \prod_{l=1}^{k-1} \beta_{2(j_{l+1}-j_l)-1}\left(\mathbf{Y}_{j_l},\mathbf{X}_{j_l+1},\mathbf{Y}_{j_l+1},\ldots,\mathbf{X}_{j_{l+1}-1},\mathbf{Y}_{j_{l+1}-1}\right).$$
$$\tag{2.3}$$

Remark 2.7. Formula (2.3) will be used several times in this paper and it will be convenient for the reader if we write it down in the special case when $\{\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_{n+1}\} = \{\mathbf{Z}_1,\underbrace{\mathbf{X},\ldots,\mathbf{X}}_{n-1},\mathbf{Z}_2\}$ and

$\{\mathbf{Y}_1,\mathbf{Y}_2,\ldots,\mathbf{Y}_n\} = \{\underbrace{\mathbf{Y},\mathbf{Y},\ldots,\mathbf{Y}}_{n}\}$. In this case we have

$$\beta_{2n+1}\left(\mathbf{Z}_1,\mathbf{Y},\mathbf{X},\ldots,\mathbf{X},\mathbf{Y},\mathbf{Z}_2\right) =$$
$$= \sum_{k=2}^{n+1} \sum_{1=j_1<\ldots<j_k=n} \beta_k(\mathbf{Z}_1,\underbrace{\mathbf{X},\ldots,\mathbf{X}}_{k-2},\mathbf{Z}_2) \prod_{l=1}^{k-1} \beta_{2(j_{l+1}-j_l)-1}\left(\mathbf{Y},\mathbf{X},\mathbf{Y},\ldots,\mathbf{X},\mathbf{Y}\right).$$

After a change of indices this can be written in much simpler form

$$\beta_{2n+1}\left(\mathbf{Z}_1,\mathbf{Y},\mathbf{X},\ldots,\mathbf{X},\mathbf{Y},\mathbf{Z}_2\right)$$
$$= \sum_{k=1}^{n} \beta_{k+1}(\mathbf{Z}_1,\underbrace{\mathbf{X},\ldots,\mathbf{X}}_{k-1},\mathbf{Z}_2) \sum_{i_1+\ldots+i_k=n-k} \prod_{l=1}^{k} \beta_{2i_l+1}\left(\mathbf{Y},\mathbf{X},\mathbf{Y},\ldots,\mathbf{X},\mathbf{Y}\right). \tag{2.4}$$

We also recall two simple facts.

Proposition 2.8 (Fevrier et al., 2020). *Let* $n \geq 2$. *If either* $\mathbf{X}_1 = \mathbf{I}$ *or* $\mathbf{X}_n = \mathbf{I}$, *then* $\beta_n\left(\mathbf{X}_1,\ldots,\mathbf{X}_n\right) = 0$.

Proposition 2.9. *For* $n \geq 1$

$$\beta_n (\mathbf{X}_1 \cdot \mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_{n+1}) = \beta_{n+1} (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_{n+1}) + \beta_1 (\mathbf{X}_1) \beta_n (\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_{n+1}).$$

The last proposition is a special case of Proposition 2.12 from Fevrier et al. (2020).

2.2. *Conditional expectations.* Assume that $(\mathcal{A}, \varphi)$ is a W^* probability space, i.e., $\mathcal{A}$ is a finite von Neumann algebra and φ a faithful, normal, tracial state. If $\mathcal{B} \subset \mathcal{A}$ is von Neumann subalgebra, we denote by $\varphi (\cdot \mid \mathcal{B})$ the conditional expectation with respect to $\mathcal{B}$. That is $\varphi (\cdot \mid \mathcal{B}) : \mathcal{A} \to \mathcal{B}$ is a faithful, normal, projection such that $\varphi \circ [\varphi (\cdot \mid \mathcal{B})] = \varphi$. The map $\varphi (\cdot \mid \mathcal{B})$ is a $\mathcal{B}$-module map i.e.

$$\varphi (\mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2 \mid \mathcal{B}) = \mathbf{Y}_1 \varphi (\mathbf{X} \mid \mathcal{B}) \mathbf{Y}_2$$

for all $\mathbf{X} \in \mathcal{A}$ and $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{B}$. For the existence of the conditional expectation see e.g. (Takesaki, 2002, Proposition 2.36).

2.3. *Distribution of a random variable and analytic tools.*

Definition 2.10. The distribution of self-adjoint random variable $\mathbf{X} \in \mathcal{A}$ is a uniquely determined, compactly supported, probability measure $\mu_{\mathbf{X}}$ on the real line such that for all $n \geq 1$

$$\varphi (\mathbf{X}^n) = \int_{\mathbb{R}} x^n \mu_{\mathbf{X}}(dx).$$

We list now some analytic tools and their properties that we use in this paper.

(1) The Cauchy-Stieltjes transform of a compactly supported measure μ on the real line is the map

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z - x},$$

defined for $z \in \mathbb{C} \setminus \mathrm{supp}(\mu)$. It is known that the Cauchy-Stieltjes transform is an analytic map $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$. If $\mathbf{X}$ is a self-adjoint random variable we write $G_{\mathbf{X}}$ for the $G_{\mu_{\mathbf{X}}}$. Note that

$$G_{\mathbf{X}}(z) = \varphi \left((z\mathbf{I} - \mathbf{X})^{-1} \right) = \int_{\mathbb{R}} \frac{\mu_{\mathbf{X}}(dx)}{z - x}.$$

(2) The R-transform of $\mathbf{X}$ is the function

$$r_{\mathbf{X}}(z) = G_{\mathbf{X}}^{-1}(z) - \frac{1}{z},$$

where $G_{\mathbf{X}}^{-1}$ is the inverse function of $G_{\mathbf{X}}$, defined in some neighborhood of infinity. It is known that $r_{\mathbf{X}}$ is an analytic map and for sufficiently small z the following expansion holds

$$r_{\mathbf{X}}(z) = \sum_{k=0}^{\infty} \kappa_{k+1}(\mathbf{X}, ..., \mathbf{X}) z^k.$$

(3) The moment transform of $\mathbf{X}$ (which is not necessarily self-adjoint) is defined for all $z \in \mathbb{C}$ such that $\mathbf{I} - z\mathbf{X}$ is invertible as

$$M_{\mathbf{X}}(z) = \varphi \left(z\mathbf{X}(\mathbf{I} - z\mathbf{X})^{-1} \right).$$

$M_{\mathbf{X}}$ is an an analytic function in some neighborhood of 0 and one has

$$M_{\mathbf{X}}(z) = \sum_{k=1}^{\infty} \varphi(\mathbf{X}^k) z^k.$$

(4) The η-transform of $\mathbf{X}$ is defined by

$$\eta_{\mathbf{X}}(z) = \frac{M_{\mathbf{X}}(z)}{M_{\mathbf{X}}(z)+1}.$$

In some neighborhood of 0 one has

$$\eta_{\mathbf{X}}(z) = \sum_{k=1}^{\infty} \beta_k(\mathbf{X}, ..., \mathbf{X})z^k.$$

Each of these transformations uniquely determine moments of a self-adjoint random variable $\mathbf{X}$ and thus also uniquely determine its distribution.

2.4. *Subordination.* Let $\mathbf{X}$ and $\mathbf{Y}$ be free self-adjoint random variables. There is a fundamental relation between the R-transforms of $\mathbf{X}, \mathbf{Y}$ and $\mathbf{X} + \mathbf{Y}$, namely

$$r_{\mathbf{X}+\mathbf{Y}}(z) = r_{\mathbf{X}}(z) + r_{\mathbf{Y}}(z). \tag{2.5}$$

Consequently the distributions of $\mathbf{Y}$ and $\mathbf{X} + \mathbf{Y}$ determine the distribution of $\mathbf{X}$.

The relation between Cauchy-Stieltjes transforms of $\mathbf{X}, \mathbf{Y}$ and $\mathbf{X} + \mathbf{Y}$ is more complicated and was established by Biane in Biane (1998). It involves two functions ω_1, ω_2 that can be defined as unique analytic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ satisfying the following properties: $\mathrm{Im}(\omega_k(z)) \geq \mathrm{Im}(z)$, $\omega_k(iy)/iy \longrightarrow 1$ when $y \to +\infty$, $k = 1, 2$, and

$$G_{\mathbf{X}+\mathbf{Y}}(z) = G_{\mathbf{X}}(\omega_1(z)) = G_{\mathbf{Y}}(\omega_2(z)). \tag{2.6}$$

Because of the last property ω_1, ω_2 are called the subordination functions.

As a consequence of (2.5) and (2.6) the following equality holds for all $z \in \mathbb{C}^+$

$$z = \omega_1(z) + \omega_2(z) - \frac{1}{G_{\mathbf{X}+\mathbf{Y}}(z)}. \tag{2.7}$$

We also need the following theorems. The first one generalizes formula (2.6) in the framework of von Neumann algebras. The second gives an interesting series expansion of the subordination function $\omega_1(z)$ that involves Boolean cumulants.

Proposition 2.11 (Biane, 1998). *If $\mathbf{X}$ and $\mathbf{Y}$ are free self-adjoint random variables, then for all $z \in \mathbb{C}^+$*

$$\varphi\left((z\mathbf{I} - \mathbf{X} - \mathbf{Y})^{-1} \mid \mathbf{X}\right) = (\omega_1(z) - \mathbf{X})^{-1}. \tag{2.8}$$

Proposition 2.12 (Lehner and Szpojankowski, 2021). *If $\mathbf{X}$ and $\mathbf{Y}$ are free self-adjoint random variables, then*

$$\omega_1(z) = z - \sum_{n=0}^{\infty} \beta_{2n+1}\left(\mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, \ldots, (z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}\right) \tag{2.9}$$

in some neighborhood of infinity in $\mathbb{C}^+$.

3. Free Matsumoto-Yor property

In this section we recall some definitions needed to state the Matsumoto-Yor property in free probability.

3.1. *Free Poisson Distribution.* We say that the measure $\nu = \nu(\lambda, \gamma)$ with $\lambda \geq 0, \gamma > 0$ is free Poisson or Marchenko-Pastur distribution if

$$\nu = \max\{0, 1 - \lambda\}\delta_0 + \lambda\nu_1,$$

where ν_1 is a measure with density

$$\frac{1}{2\pi\gamma x}\sqrt{4\lambda\gamma^2 - (x - \gamma(1 + \lambda))^2} \; \mathbb{1}_{(\gamma(1-\sqrt{\lambda})^2, \gamma(1+\sqrt{\lambda})^2)}(x).$$

The R-transform of the free Poisson distribution $\nu(\lambda, \gamma)$ is equal to

$$r_{\nu(\lambda,\gamma)}(z) = \frac{\lambda\gamma}{1 - \gamma z}.$$

3.2. *Free-GIG distribution.* The free Generalized Inverse Gaussian distribution is a probability measure $\mu = \mu(\lambda, \alpha, \beta)$, with $\alpha, \beta > 0, \lambda \in \mathbb{R}$, which is compactly supported on the interval $[a, b]$ and has the density

$$\frac{d\mu}{dx} = \frac{1}{2\pi}\sqrt{(x - a)(b - x)}\left(\frac{\alpha}{x} + \frac{\beta}{\sqrt{ab}x^2}\right),$$

where the pair of numbers (a, b) is the unique solution of

$$\begin{cases} 1 - \lambda + \alpha\sqrt{ab} - \beta\dfrac{a + b}{ab} = 0, \\ 1 + \lambda + \dfrac{\beta}{\sqrt{ab}} - \alpha\dfrac{a + b}{2} = 0, \end{cases}$$

satisfying $0 < a < b$.

The Cauchy-Stieltjes transform of the free-GIG distribution $\mu = \mu(\lambda, \alpha, \beta)$ is equal to

$$G_\mu(z) = \frac{\alpha z^2 - (\lambda - 1)z - \beta - (\alpha z + \frac{\beta}{\sqrt{ab}})\sqrt{(z - a)(z - b)}}{2z^2}.$$

See Féral (2006) for more details.

It is easy to check that the Cauchy-Stieltjes transform $G = G(z)$ of the free-GIG distribution $\mu(\lambda, \alpha, \beta)$ satisfies the following quadratic equation

$$z^2 G^2 - (\alpha z^2 - (\lambda - 1)z - \beta)G + \alpha z + \delta = 0.$$

where δ depends on α, β, λ. The following lemma that can be extracted from the proof of (Szpojankowski (2017), Theorem 4.1.) shows the converse of this statement.

Lemma 3.1. *Suppose the function $G = G(z)$ satisfies the following equation*

$$z^2 G^2 - (\alpha z^2 - (\lambda - 1)z - \beta)G + \alpha z + \delta = 0$$

i.e.

$$G(z) = \frac{\alpha z^2 - (\lambda - 1)z - \beta \pm \sqrt{(\alpha z^2 - (\lambda - 1)z - \beta)^2 - 4z^2(\alpha z + \delta)}}{2z^2},$$

for some $\alpha, \beta, \delta > 0$ and $\lambda \in \mathbb{R}$. If G is the Cauchy-Stieltjes transform of a positive random variable $\mathbf{X}$, then δ is uniquely determined by α, β, λ and $\mathbf{X}$ has the free-GIG distribution $\mu(\lambda, \alpha, \beta)$.

3.3. *The free Matsumoto-Yor property.* The following independence property was observed by Matsumoto and Yor in Matsumoto and Yor (2001): If $X \sim GIG(-p, a, b)$ and $Y \sim G(p, a)$ are independent random variables, then

$$U = \frac{1}{X + Y} \text{ and } V = \frac{1}{X} - \frac{1}{X + Y}$$

are also independent and distributed $GIG(-p, b, a)$ and $G(p, b)$ respectively.

Later it was shown in Letac and Wesołowski (2000) that the Matsumoto-Yor property characterizes GIG and Gamma laws:

Theorem 3.2. *Let X and Y be positive, independent and non-degenerate random variables. If $U = \frac{1}{X+Y}$ and $V = \frac{1}{X} - \frac{1}{X+Y}$ are independent, then $X \sim GIG(-p, a, b)$ and $Y \sim G(p, a)$.*

The Matsumoto-Yor property in free probability was studied in Szpojankowski (2017) where the author proved the following theorems:

Theorem 3.3. *Let $\mathbf{X}$ and $\mathbf{Y}$ be self-adjoint random variables such that $\mathbf{X}$ has the free-GIG distribution $\mu(-\lambda, \alpha, \beta)$ and the distribution of $\mathbf{Y}$ is free-Poisson $\nu(\lambda, 1/\alpha)$. If $\mathbf{X}, \mathbf{Y}$ are free, then*

$$\mathbf{U} = (\mathbf{X} + \mathbf{Y})^{-1} \text{ and } \mathbf{V} = \mathbf{X}^{-1} - (\mathbf{X} + \mathbf{Y})^{-1} \tag{3.1}$$

are free. Moreover $\mathbf{U}$ and $\mathbf{V}$ have $\mu(-\lambda, \beta, \alpha)$ and $\nu(\lambda, 1/\beta)$ distributions respectively.

Theorem 3.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be free, positive, non-degenerate and self-adjoint random variables. If $\mathbf{U}, \mathbf{V}$ defined as in (3.1) are free, then $\mathbf{X}$ has the free-GIG distribution $\mu(-\lambda, \alpha, \beta)$ and the distribution of $\mathbf{Y}$ is free-Poisson $\nu(\lambda, 1/\alpha)$ for some parameters $\alpha, \beta > 0$ and $\lambda \in \mathbb{R}$.*

The following lemma will also be useful.

Lemma 3.5 (Szpojankowski, 2017, Remark 2.1). *Let $\mathbf{X}$ and $\mathbf{Y}$ be self-adjoint random variables such that $\mathbf{X}$ has the free-GIG distribution $\mu(-\lambda, \alpha, \beta)$ and the distribution of $\mathbf{Y}$ is free-Poisson $\nu(\lambda, 1/\alpha)$. Then the distribution of $\mathbf{X} + \mathbf{Y}$ is free-GIG $\mu(\lambda, \alpha, \beta)$.*

4. Analytic interpretation of regression conditions

In this section we prove a few auxiliary results that will be useful in the sequel.

4.1. *Subordination vs Boolean cumulants.*

Lemma 4.1. *Let $\mathbf{X}$ and $\mathbf{Y}$ be free self-adjoint and compactly supported random variables. Then for z in some neighborhood of infinity in $\mathbb{C}^+$*

$$\sum_{n=0}^{\infty} \beta_{2n+1}\left((z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}, \ldots, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}\right) = \frac{1}{\omega_2(z)}. \tag{4.1}$$

Remark 4.2. It's easy to check that for $|z| > ||\mathbf{X}||$ we have $||\,(z\mathbf{I} - \mathbf{X})^{-1}\,|| \leq (|z| - ||\mathbf{X}||)^{-1}$. The formula (2.1) implies that $|\beta_n(\mathbf{X}_1, \ldots, \mathbf{X}_n)| \leq 2^{n-1}||\mathbf{X}_1|| \cdots ||\mathbf{X}_n||$. Hence for $|z| > ||\mathbf{X}||$ we have

$$|\beta_{2n+1}\left((z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}, \ldots, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}\right)| \leq 2^{2n} \frac{||\mathbf{Y}||^n}{(|z| - ||\mathbf{X}||)^{n+1}}.$$

This implies that the series from Lemma 4.1 converges for $|z| > ||\mathbf{X}|| + 4||\mathbf{Y}||$ and represents a holomorphic function.

Proof: To simplify notation we will write $\mathcal{R}$ for the resolvent $(z\mathbf{I} - \mathbf{X})^{-1}$.

Let us denote the right hand side of (4.1) by $D(z)$, i.e.

$$D(z) = \sum_{n=0}^{\infty} \beta_{2n+1}\left(\mathcal{R}, \mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}, \mathcal{R}\right).$$

It is easy to check the result when $\mathbf{Y} = 0$. In this case $\omega_2(z) = \frac{1}{G_{\mathbf{X}}(z)}$ by (2.6) and the series consists of one non-zero element $\beta_1(\mathcal{R}) = \varphi\left((z\mathbf{I} - \mathbf{X})^{-1}\right) = G_{\mathbf{X}}(z)$. Thus for the rest of the proof we assume $\mathbf{Y} \neq 0$. This implies ω_1 is not the identity function on $\mathbb{C}^+$.

Formula (2.4) implies that for $n \geq 1$ the cumulant $\beta_{2n+1}(\mathcal{R}, \mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathbf{Y}, \mathcal{R})$ is equal to

$$\sum_{k=1}^{n} \beta_{k+1}(\mathcal{R}, \mathcal{R}, \ldots, \mathcal{R}) \sum_{i_1+\ldots+i_k=n-k} \prod_{l=1}^{k} \beta_{2i_l+1}(\mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}).$$

After changing the order of summation one can see that

$$\begin{aligned}
D(z) &= \beta_1(\mathcal{R}) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \beta_{k+1}(\mathcal{R}, \ldots, \mathcal{R}) \sum_{i_1+\ldots+i_k=n-k} \prod_{l=1}^{k} \beta_{2i_l+1}(\mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}) \\
&= \beta_1(\mathcal{R}) + \sum_{k=1}^{\infty} \beta_{k+1}(\mathcal{R}, \ldots, \mathcal{R}) \sum_{n=k}^{\infty} \sum_{i_1+\ldots+i_k=n-k} \prod_{l=1}^{k} \beta_{2i_l+1}(\mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}) \\
&= \beta_1(\mathcal{R}) + \sum_{k=1}^{\infty} \beta_{k+1}(\mathcal{R}, \ldots, \mathcal{R})C(z)^k,
\end{aligned}$$

where

$$C(z) = \sum_{n=0}^{\infty} \beta_{2n+1}(\mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}).$$

Thus, in view of (2.9) we can write $C(z) = z - \omega_1(z)$.

If $C(z) \neq 0$ one can write

$$D(z) = \frac{\eta_{\mathcal{R}}\left(C(z)\right)}{C(z)} = \frac{M_{\mathcal{R}}\left(C(z)\right)}{C(z)\left[M_{\mathcal{R}}\left(C(z)\right) + 1\right]}.$$

Easy algebraic manipulations and formula (2.6) show that

$$\begin{aligned}
M_{\mathcal{R}}\left(C(z)\right) = \varphi\left(C(z)\mathcal{R}(\mathbf{I} - C(z)\mathcal{R})^{-1}\right) &= C(z)\varphi\left((\mathcal{R}^{-1} - C(z)\mathbf{I})^{-1}\right) \\
&= C(z)\varphi\left((\omega_1(z)\mathbf{I} - \mathbf{X})^{-1}\right) = C(z)G_{\mathbf{X}+\mathbf{Y}}(z).
\end{aligned}$$

Thus

$$D(z) = \frac{G_{\mathbf{X}+\mathbf{Y}}(z)}{1 + (z - \omega_1(z))G_{\mathbf{X}+\mathbf{Y}}(z)} = \frac{G_{\mathbf{X}+\mathbf{Y}}(z)}{1 + \left(\omega_2(z) - G_{\mathbf{X}+\mathbf{Y}}(z)^{-1}\right)G_{\mathbf{X}+\mathbf{Y}}(z)} = \frac{1}{\omega_2(z)},$$

where we used formula (2.7). This proves the lemma for all sufficiently large $z \in \mathbb{C}^+$ such that $C(z) \neq 0$ but since $C(z)$ is a non-zero analytic function this last assumption can be dropped. □

Lemma 4.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be free self-adjoint and compactly supported random variables such that* $\mathbf{Y}$ *is invertible. Then for all sufficiently large* $z \in \mathbb{C}^+$

$$A(z) := \sum_{n=0}^{\infty} \beta_{2n+1}(\mathbf{Y}^{-1}, \underbrace{(z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, \ldots, (z\mathbf{I} - \mathbf{X})^{-1}}_{2n}, \mathbf{Y}) = \frac{1}{\omega_2(z)} + \frac{\varphi(\mathbf{Y}^{-1})}{\omega_2(z)G_{\mathbf{X}+\mathbf{Y}}(z)}. \quad (4.2)$$

$$B(z) := \sum_{n=1}^{\infty} \beta_{2n+1}(\mathbf{Y}^{-1}, (z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}, \dots, \mathbf{Y}, (z\mathbf{I} - \mathbf{X})^{-1}, \mathbf{Y}^{-1}) = \frac{\varphi(\mathbf{Y}^{-2}) - \varphi(\mathbf{Y}^{-1})A(z)}{\omega_2(z)}.$$

(4.3)

Proof: As before we will write $\mathcal{R}$ for $(z\mathbf{I} - \mathbf{X})^{-1}$ to simplify notation. Hence

$$A(z) = \sum_{n=0}^{\infty} \beta_{2n+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \dots, \mathcal{R}, \mathbf{Y}).$$

Applying formula (2.4) we can see that for $n \geq 1$ the Boolean cumulant $\beta_{2n+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \dots, \mathcal{R}, \mathbf{Y})$ is equal to

$$\sum_{k=1}^{n} \beta_{k+1}(\mathbf{Y}^{-1}, \mathbf{Y}, \dots, \mathbf{Y}) \sum_{i_1+\dots+i_k=n-k} \prod_{l=1}^{k} \beta_{2i_l+1}(\mathcal{R}, \mathbf{Y}, \mathcal{R}, \dots, \mathbf{Y}, \mathcal{R}).$$

The same argument as in the previous lemma shows that for sufficiently large $z \in \mathbb{C}^+$

$$A(z) = \beta_1(\mathbf{Y}^{-1}) + \sum_{k=1}^{\infty} \beta_{k+1}(\mathbf{Y}^{-1}, \mathbf{Y}, \dots, \mathbf{Y})D(z)^k,$$

(4.4)

where $D(z) = \sum_{n=0}^{\infty} \beta_{2n+1}(\mathcal{R}, \mathbf{Y}, \mathcal{R}, \dots, \mathbf{Y}, \mathcal{R}) = \frac{1}{\omega_2(z)}$ by Lemma 4.1. From Propositions 2.8, 2.9 and Remark 2.5 we can deduce that

$$\beta_{k+1}(\mathbf{Y}^{-1}, \mathbf{Y}, \dots, \mathbf{Y}) = \begin{cases} \varphi(\mathbf{Y}^{-1}), & k = 0 \\ 1 - \varphi(\mathbf{Y}^{-1})\beta_1(\mathbf{Y}), & k = 1 \\ -\varphi(\mathbf{Y}^{-1})\beta_k(\mathbf{Y}, \mathbf{Y}, \dots, \mathbf{Y}), & k \geq 2 \end{cases}.$$

Hence

$$A(z) = \varphi(\mathbf{Y}^{-1}) + (1 - \varphi(\mathbf{Y}^{-1})\beta_1(\mathbf{Y}))D(z) - \varphi(\mathbf{Y}^{-1})\sum_{k=2}^{\infty} \beta_k(\mathbf{Y}, \dots, \mathbf{Y})D(z)^k$$

$$= D(z) + \varphi(\mathbf{Y}^{-1})\left(1 - \eta_{\mathbf{Y}}(D(z))\right).$$

Now it's easy to check that $1 - \eta_{\mathbf{Y}}(z) = \frac{1}{\varphi((\mathbf{I}-z\mathbf{X})^{-1})} = \frac{z}{G_{\mathbf{X}}(\frac{1}{z})}$. Thus

$$A(z) = \frac{1}{\omega_2(z)} + \frac{\varphi(\mathbf{Y}^{-1})}{\omega_2(z)G_{\mathbf{Y}}(\omega_2(z))} = \frac{1}{\omega_2(z)} + \frac{\varphi(\mathbf{Y}^{-1})}{\omega_2(z)G_{\mathbf{X}+\mathbf{Y}}(z)}.$$

Now we can prove formula (4.3). Using formula (2.4) one more time one can see that

$$B(z) = \sum_{k=1}^{\infty} \beta_{k+1}(\mathbf{Y}^{-1}, \underbrace{\mathbf{Y}, \dots, \mathbf{Y}}_{k-1}, \mathbf{Y}^{-1})D(z)^k.$$

It follows from Propositions 2.8, 2.9 and the fact that Boolean cumulants are invariant under reflection (i.e. $\beta_n(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) = \beta_n(\mathbf{X}_n, \dots, \mathbf{X}_2, \mathbf{X}_1)$) that

$$\beta_{k+2}(\mathbf{Y}^{-1}, \mathbf{Y}, \dots, \mathbf{Y}, \mathbf{Y}^{-1}) = \begin{cases} \varphi(\mathbf{Y}^{-2}) - \varphi(\mathbf{Y}^{-1})^2, & k = 0 \\ -\varphi(\mathbf{Y}^{-1})\beta_{k+1}(\mathbf{Y}^{-1}, \mathbf{Y}, \dots, \mathbf{Y}, \mathbf{Y}), & k \geq 1 \end{cases}.$$

Consequently

$$B(z) = \frac{\varphi(\mathbf{Y}^{-2}) - \varphi(\mathbf{Y}^{-1})^2}{\omega_2(z)} - \frac{\varphi(\mathbf{Y}^{-1})}{\omega_2(z)}\left(\sum_{k=0}^{\infty} \beta_{k+1}(\mathbf{Y}^{-1}, \mathbf{Y}, \dots, \mathbf{Y})D(z)^k - \varphi(\mathbf{Y}^{-1})\right).$$

The series in the above expression is exactly $A(z)$ (formula (4.4).) This ends the proof of the lemma. $\square$

Remark 4.4. Consider (formal) power series

$$\eta_{\mathbf{Y}}^f(z) = \sum_{k \geq 0} \beta_{k+1}(f(\mathbf{Y}), \underbrace{\mathbf{Y}, \ldots, \mathbf{Y}}_{k}) \, z^k$$

and

$$\eta_{\mathbf{Y}}^{f,g}(z) = \sum_{k \geq 0} \beta_{k+2}(f(\mathbf{Y}), \underbrace{\mathbf{Y}, \ldots, \mathbf{Y}}_{k}, g(\mathbf{Y})) \, z^k$$

for $f, g : \mathcal{A} \to \mathcal{A}$, which seem to be important in relations between subordination and Boolean cumulants. In (Szpojankowski and Wesołowski (2020), Proposition 3.4) a general and rather complicated formula expressing $\eta_{\mathbf{Y}}^f$ and $\eta_{\mathbf{Y}}^{f,g}$ in terms of η_Y was proved for f and g being analytic functions in the unit disc. Consequently, in the special case when $0 \leq \mathbf{Y} < \mathbf{I}$ and $f(\mathbf{Y}) = g(\mathbf{Y}) = \psi(\mathbf{Y}) = \mathbf{Y}(1 - \mathbf{Y})^{-1}$ explicit expressions were derived there (see the proof of Szpojankowski and Wesołowski (2020), Proposition 3.7)

$$\eta_{\mathbf{Y}}^\psi(z) = \frac{\eta_{\mathbf{Y}}(z) - \eta_{\mathbf{Y}}(1)}{z - 1} \varphi((1 - \mathbf{Y})^{-1})$$

and

$$\eta_{\mathbf{Y}}^{\psi,\psi}(z) = \frac{\eta_{\mathbf{Y}}(z) - \eta_{\mathbf{Y}}(1) - (z - 1)\eta_{\mathbf{Y}}'(1)}{(z - 1)^2} \varphi^2((1 - \mathbf{Y})^{-1}).$$

It is interesting to note that in the proof of Lemma 4.3 we actually derived formulas for $\eta_{\mathbf{Y}}^h$ and $\eta_{\mathbf{Y}}^{h,h}$ for $h(\mathbf{Y}) = \mathbf{Y}^{-1}$ (which clearly is not analytic in the unit disc). Namely, the formula for $A(z)$ gives

$$\eta_{\mathbf{Y}}^h(z) = z + (1 - \eta_{\mathbf{Y}}(z))\varphi(\mathbf{Y}^{-1})$$

and the formula for $B(z)$ yields

$$\eta_{\mathbf{Y}}^{h,h}(z) = \varphi(\mathbf{Y}^{-2}) - z\varphi(\mathbf{Y}^{-1}) + \varphi^2(\mathbf{Y}^{-1})(\eta_{\mathbf{Y}}(z) - 1).$$

4.2. *Constant regressions and their implications.* From now on we assume that we are given a W^*-probability space $(\mathcal{A}, \varphi)$. We also assume $\mathbf{X}, \mathbf{Y} \in \mathcal{A}$ are free, self-adjoint and positive random variables and $\mathbf{U}, \mathbf{V}$ are defined as follows.

$$\mathbf{U} = (\mathbf{X} + \mathbf{Y})^{-1}, \quad \mathbf{V} = \mathbf{X}^{-1} - (\mathbf{X} + \mathbf{Y})^{-1}.$$

In this subsection we show that the condition of constant regression

$$\varphi(\mathbf{V}^k \mid \mathbf{U}) = m_k \mathbf{I}$$

in each of the considered cases, i.e. for $k \in \{-2, -1, 1, 2\}$, implies a certain equation that connects the Cauchy-Stieltjes transform $G_{\mathbf{X}+\mathbf{Y}}$ with the subordination functions ω_1 and ω_2. We will consider each case separately. The most challenging one is $k = -2$, where, contrary to the other cases, subordination does not suffice and we additionally have to employ the Boolean cumulants to get the result.

To simplify notation we will denote $\mathbf{T} = \mathbf{X} + \mathbf{Y} = \mathbf{U}^{-1}$. Note that $\varphi(\cdot \mid \mathbf{T}) = \varphi(\cdot \mid \mathbf{U})$ since we assumed $\mathbf{X}, \mathbf{Y}$ are positive.

We also introduce the families of rational functions $q_1(t, z)$ and $q_2(t, z)$ in the variable t and their partial fraction decompositions

$$q_1(t, z) = \frac{1}{t(z - t)} = \frac{1}{zt} + \frac{1}{z(z - t)}$$

and

$$q_2(t, z) = \frac{1}{t^2(z - t)} = \frac{1}{zt^2} + \frac{1}{z^2 t} + \frac{1}{z^2(z - t)}$$

where $z \in \mathbb{C}^+$.

Lemma 4.5. *Assume that*

$$\varphi\left(\mathbf{V}\mid\mathbf{U}\right)=c\mathbf{I}, \tag{4.5}$$

for some constant c. Then

$$\frac{1}{\omega_1(z)}\left(\varphi(\mathbf{U})+c+G_{\mathbf{X}+\mathbf{Y}}(z)\right)=\left(c+\frac{1}{z}\right)G_{\mathbf{X}+\mathbf{Y}}(z)+\frac{\varphi(\mathbf{U})}{z} \tag{4.6}$$

for all $z\in\mathbb{C}^+$.

Proof: We start by rewriting (4.5) as

$$\varphi\left(\mathbf{X}^{-1}\mid\mathbf{T}\right)=c\mathbf{I}+\mathbf{T}^{-1}. \tag{4.7}$$

When we multiply both sides from the right by $(z\mathbf{I}-\mathbf{T})^{-1}$ and apply φ we get

$$\varphi\left(\mathbf{X}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}\right)=c\varphi\left((z\mathbf{I}-\mathbf{T})^{-1}\right)+\varphi\left(\mathbf{T}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}\right). \tag{4.8}$$

Since $\mathbf{T}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}=q_1(\mathbf{T},z)=\frac{1}{z}\mathbf{T}^{-1}+\frac{1}{z}(z\mathbf{I}-\mathbf{T})^{-1}$ the right hand side of (4.8) becomes

$$\left(c+\frac{1}{z}\right)\varphi\left((z\mathbf{I}-\mathbf{T})^{-1}\right)+\frac{1}{z}\varphi\left(\mathbf{T}^{-1}\right)=\left(c+\frac{1}{z}\right)G_{\mathbf{X}+\mathbf{Y}}(z)+\frac{\varphi(\mathbf{U})}{z}.$$

Now we deal with the left hand side of (4.8). By conditioning on $\mathbf{X}$ we see that

$$\varphi\left(\mathbf{X}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}\right)=\varphi\left(\mathbf{X}^{-1}\varphi\left((z\mathbf{I}-\mathbf{X}-\mathbf{Y})^{-1}\mid\mathbf{X}\right)\right). \tag{4.9}$$

Formula (2.8) implies that this is equal to

$$\varphi\left(\mathbf{X}^{-1}\left(\omega_1(z)-\mathbf{X}\right)^{-1}\right)=\varphi\left(\frac{1}{\omega_1(z)}\mathbf{X}^{-1}+\frac{1}{\omega_1(z)}\left(\omega_1(z)-\mathbf{X}\right)^{-1}\right)$$

$$=\frac{1}{\omega_1(z)}\varphi(\mathbf{X}^{-1})+\frac{1}{\omega_1(z)}G_{\mathbf{X}}(\omega_1(z))$$

$$=\frac{1}{\omega_1(z)}\left(\varphi(\mathbf{U})+c+G_{\mathbf{X}+\mathbf{Y}}(z)\right).$$

by the subordination property (2.6) and by the equality $\varphi(\mathbf{X}^{-1})=\varphi(\mathbf{U})+c$ that follows from (4.7).
□

Lemma 4.6. *Assume that*

$$\varphi\left(\mathbf{V}^{-1}\mid\mathbf{U}\right)=d\mathbf{I}, \tag{4.10}$$

for some constant d. Then

$$\frac{1}{\omega_2(z)}\left(\varphi(\mathbf{Y}^{-1})+G_{\mathbf{X}+\mathbf{Y}}(z)\right)=\frac{d\varphi(\mathbf{U})}{z^2}+\frac{\varphi(\mathbf{Y}^{-1})}{z}+\left(\frac{d}{z^2}+\frac{1}{z}\right)G_{\mathbf{X}+\mathbf{Y}}(z) \tag{4.11}$$

for all $z\in\mathbb{C}^+$.

Proof: We start by noting that $\mathbf{TV}=(\mathbf{X}+\mathbf{Y})\left(\mathbf{X}^{-1}-(\mathbf{X}+\mathbf{Y})^{-1}\right)=\mathbf{YX}^{-1}$. This implies $\mathbf{V}^{-1}\mathbf{T}^{-1}=\mathbf{XY}^{-1}$ so if we multiply both sides of (4.10) from the right by $\mathbf{T}^{-1}$ we get

$$d\mathbf{T}^{-1}=\varphi\left(\mathbf{XY}^{-1}\mid\mathbf{T}\right)=\varphi\left((\mathbf{T}-\mathbf{Y})\mathbf{Y}^{-1}\mid\mathbf{T}\right)=\mathbf{T}\varphi(\mathbf{Y}^{-1}\mid\mathbf{T})-\mathbf{I}.$$

Hence

$$\varphi(\mathbf{Y}^{-1}\mid\mathbf{T})=\mathbf{T}^{-1}+d\mathbf{T}^{-2}. \tag{4.12}$$

After multiplying both sides by $(z\mathbf{I}-\mathbf{T})^{-1}$ and applying φ one can see that

$$\varphi\left(\mathbf{Y}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}\right)=\varphi\left(\mathbf{T}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}\right)+d\varphi\left(\mathbf{T}^{-2}(z\mathbf{I}-\mathbf{T})^{-1}\right). \tag{4.13}$$

The left hand side of the above expression is the same as the left hand side of (4.9) with $\mathbf{X}$ and $\mathbf{Y}$ swapped so by analogy we get

$$\varphi\left(\mathbf{Y}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \frac{1}{\omega_2(z)}\left(\varphi(\mathbf{Y}^{-1}) + G_{\mathbf{X}+\mathbf{Y}}(z)\right).$$

Now we calculate the right hand side of (4.13).

$$\varphi\left(\mathbf{T}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \varphi\left(q_1(\mathbf{T}, z)\right) = \frac{\varphi(\mathbf{T}^{-1})}{z} + \frac{1}{z}G_{\mathbf{X}+\mathbf{Y}}(z).$$

Similarly

$$\varphi\left(\mathbf{T}^{-2}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \varphi(q_2(\mathbf{T}, z)) = \frac{1}{z}\varphi(\mathbf{T}^{-2}) + \frac{1}{z^2}\varphi(\mathbf{T}^{-1}) + \frac{1}{z^2}G_{\mathbf{X}+\mathbf{Y}}(z).$$

Consequently the left hand side of (4.13) is equal to

$$\frac{\varphi(\mathbf{U})}{z} + \frac{1}{z}G_{\mathbf{X}+\mathbf{Y}}(z) + \frac{d}{z}\varphi(\mathbf{T}^{-2}) + \frac{d\varphi(\mathbf{U})}{z^2} + \frac{d}{z^2}G_{\mathbf{X}+\mathbf{Y}}(z).$$

This is exactly the right hand side of (4.11) as (4.12) implies that

$$\varphi(\mathbf{Y}^{-1}) = \varphi(\mathbf{U}) + d\varphi(\mathbf{T}^{-2}).$$

$\square$

Lemma 4.7. *Assume that*

$$\varphi\left(\mathbf{V}^2 \mid \mathbf{U}\right) = b\mathbf{I}, \tag{4.14}$$

for some constant b. Then

$$\frac{\varphi(\mathbf{X}^{-2})}{\omega_1(z)} + \frac{1}{\omega_1(z)}\left(\varphi(\mathbf{X}^{-1}) + G_{\mathbf{X}+\mathbf{Y}}(z)\right)\left(\frac{1}{\omega_1(z)} - \frac{2}{z}\right)$$
$$= \frac{\varphi(\mathbf{X}^{-2}) - b}{z} - \frac{\varphi(\mathbf{U})}{z^2} + \left(b - \frac{1}{z^2}\right)G_{\mathbf{X}+\mathbf{Y}}(z) \quad (4.15)$$

for all $z \in \mathbb{C}^+$.

Proof: We start by expanding $\mathbf{V}^2 = \left(\mathbf{X}^{-1} - \mathbf{T}^{-1}\right)^2 = \mathbf{X}^{-2} - \mathbf{X}^{-1}\mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{X}^{-1} + \mathbf{T}^{-2}$. The condition (4.14) implies now

$$\varphi\left(\mathbf{X}^{-2} \mid \mathbf{T}\right) - \mathbf{T}^{-1}\varphi\left(\mathbf{X}^{-1} \mid \mathbf{T}\right) - \varphi\left(\mathbf{X}^{-1} \mid \mathbf{T}\right)\mathbf{T}^{-1} + \mathbf{T}^{-2} = b\mathbf{I}$$

or equivalently

$$\varphi\left(\mathbf{X}^{-2} \mid \mathbf{T}\right) - 2\varphi\left(\mathbf{X}^{-1} \mid \mathbf{T}\right)\mathbf{T}^{-1} = b\mathbf{I} - \mathbf{T}^{-2}. \tag{4.16}$$

If we multiply both sides from the right by the resolvent $(z\mathbf{I} - \mathbf{T})^{-1}$ and apply φ we get

$$\varphi\left(\mathbf{X}^{-2}(z\mathbf{I} - \mathbf{T})^{-1}\right) - 2\varphi\left(\mathbf{X}^{-1}\mathbf{T}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right)$$
$$= b\varphi\left((z\mathbf{I} - \mathbf{T})^{-1}\right) - \varphi\left(\mathbf{T}^{-2}(z\mathbf{I} - \mathbf{T})^{-1}\right). \quad (4.17)$$

The right hand side of (4.17) equals

$$bG_{\mathbf{X}+\mathbf{Y}}(z) - \varphi\left(q_2(\mathbf{T}, z)\right) = bG_{\mathbf{X}+\mathbf{Y}}(z) - \frac{\varphi(\mathbf{U}^2)}{z} - \frac{\varphi(\mathbf{U})}{z^2} - \frac{1}{z^2}G_{\mathbf{X}+\mathbf{Y}}(z).$$

Now we evaluate the left hand side of (4.17). Note that

$$\varphi\left(\mathbf{X}^{-2}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \varphi\left(\mathbf{X}^{-2}\varphi\left(z\mathbf{I} - \mathbf{X} - \mathbf{Y}\right)^{-1} \mid \mathbf{X}\right) \tag{4.18}$$

Using (2.8) we see that the last expression is equal to

$$\varphi\left(\mathbf{X}^{-2}(w_1(z) - \mathbf{X})^{-1}\right) = \varphi\left(q_2(\mathbf{X}, \omega_1(z))\right)$$

$$= \frac{1}{\omega_1(z)}\varphi(\mathbf{X}^{-2}) + \frac{1}{\omega_1^2(z)}\varphi(\mathbf{X}^{-1}) + \frac{1}{\omega_1^2(z)}\varphi\left((\omega_1(z) - \mathbf{X})^{-1}\right)$$

$$= \frac{1}{\omega_1(z)}\varphi(\mathbf{X}^{-2}) + \frac{1}{\omega_1^2(z)}\left(\varphi(\mathbf{X}^{-1}) + G_{\mathbf{X}+\mathbf{Y}}(z)\right) \tag{4.19}$$

Similarly we have

$$\varphi\left(\mathbf{X}^{-1}\mathbf{T}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \varphi\left(\mathbf{X}^{-1} \cdot q_1(\mathbf{T}, z)\right)$$

$$= \frac{1}{z}\varphi\left(\mathbf{X}^{-1}\mathbf{T}^{-1}\right) + \frac{1}{z}\varphi\left(\mathbf{X}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right)$$

$$= \frac{1}{z}\varphi\left(\mathbf{X}^{-1}\mathbf{T}^{-1}\right) + \frac{1}{z\omega_1(z)}\left(\varphi(\mathbf{X}^{-1}) + G_{\mathbf{X}+\mathbf{Y}}(z)\right). \tag{4.20}$$

The result follows now by simple algebra and on noting that (4.16) yields

$$2\varphi\left(\mathbf{X}^{-1}\mathbf{T}^{-1}\right) = \varphi\left(\mathbf{X}^{-2}\right) + \varphi\left(\mathbf{U}^2\right) - b.$$

$\square$

Lemma 4.8. *Assume that*

$$\varphi\left(\mathbf{V}^{-2} \mid \mathbf{U}\right) = h\mathbf{I}, \tag{4.21}$$

for some constant h. Then

$$\varphi(\mathbf{X}^2)B(z) + A(z)^2\left(\omega_1^2(z)G_{\mathbf{X}+\mathbf{Y}}(z) - \omega_1(z) - \varphi(\mathbf{X})\right) =$$

$$= h\left(\frac{\varphi(\mathbf{U}^2)}{z} + \frac{\varphi(\mathbf{U})}{z^2} + \frac{1}{z^2}G_{\mathbf{X}+\mathbf{Y}}(z)\right) \tag{4.22}$$

for all $z \in \mathbb{C}^+$, where

$$A(z) = \frac{1}{\omega_2(z)} + \frac{\varphi(\mathbf{Y}^{-1})}{\omega_2(z)G_{\mathbf{X}+\mathbf{Y}}(z)},$$

$$B(z) = \frac{\varphi(\mathbf{Y}^{-2}) - \varphi(\mathbf{Y}^{-1})A(z)}{\omega_2(z)} = \frac{\varphi(\mathbf{Y}^{-2})}{\omega_2(z)} - \frac{\varphi(\mathbf{Y}^{-1})}{\omega_2^2(z)} - \frac{\varphi(\mathbf{Y}^{-1})^2}{\omega_2^2(z)G_{\mathbf{X}+\mathbf{Y}}(z)}.$$

Proof: As in the previous lemmas let us denote $\mathbf{T} = \mathbf{X} + \mathbf{Y} = \mathbf{U}^{-1}$. Since $\mathbf{TV} = \mathbf{YX}^{-1}$ and $\mathbf{VT} = \mathbf{X}^{-1}\mathbf{Y}$ we see that $\mathbf{T}^{-1}\mathbf{V}^{-2}\mathbf{T}^{-1} = \mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1}$. Hence, after multiplying (4.21) by $\mathbf{T}^{-1}$ from both sides, we get

$$\varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1} \mid \mathbf{U}\right) = h\mathbf{T}^{-2}. \tag{4.23}$$

Let us multiply (4.23) by $(z\mathbf{I} - \mathbf{T})^{-1}$ from the right and apply φ to obtain

$$\varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right) = h\varphi\left(\mathbf{T}^{-2}(z\mathbf{I} - \mathbf{T})^{-1}\right).$$

The right hand side equals

$$h\varphi(q_2(\mathbf{T}, z)) = h\left(\frac{1}{z}\varphi(\mathbf{T}^{-2}) + \frac{1}{z^2}\varphi(\mathbf{T}^{-1}) + \frac{1}{z^2}G_{\mathbf{X}+\mathbf{Y}}(z)\right),$$

which is exactly the right hand side of (4.22).

To calculate the left hand side we observe that

$$(z\mathbf{I} - \mathbf{T})^{-1} = (z\mathbf{I} - \mathbf{X} - \mathbf{Y})^{-1} = \left(\mathbf{I} - (z\mathbf{I} - \mathbf{X})^{-1}\mathbf{Y}\right)^{-1}(z\mathbf{I} - \mathbf{X})^{-1}.$$

When $|z| > ||\mathbf{X}|| + ||\mathbf{Y}||$ we can write

$$(z\mathbf{I} - \mathbf{T})^{-1} = \sum_{n=1}^{\infty} \left[(z\mathbf{I} - \mathbf{X})^{-1}\mathbf{Y}\right]^{n-1}(z\mathbf{I} - \mathbf{X})^{-1}.$$

Using the above expansion and traciality of φ we see that

$$\varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \sum_{n=1}^{\infty} \varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1}[\mathcal{R}\mathbf{Y}]^{n-1}\mathcal{R}\right)$$

$$= \sum_{n=1}^{\infty} \varphi\left(\mathbf{Y}^{-1}[\mathcal{R}\mathbf{Y}]^{n-1}\mathcal{R}\mathbf{Y}^{-1}\mathbf{X}^2\right), \quad (4.24)$$

where as before $\mathcal{R} = (z\mathbf{I} - \mathbf{X})^{-1}$.

Formula (2.2) for the collections $\{\underbrace{\mathbf{Y}^{-1}, \mathbf{Y}, \ldots, \mathbf{Y}, \mathbf{Y}^{-1}}_{n+1}\}$ and $\{\underbrace{\mathcal{R}, \mathcal{R}, \ldots, \mathcal{R}, \mathbf{X}^2}_{n+1}\}$ implies that

$$\varphi\left(\mathbf{Y}^{-1}[\mathcal{R}\mathbf{Y}]^{n-1}\mathcal{R}\mathbf{Y}^{-1}\mathbf{X}^2\right) = \varphi(\mathbf{X}^2)\beta_{2n+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1})+$$

$$\sum_{k=1}^{n} \varphi\left(\mathcal{R}^k\mathbf{X}^2\right) \sum_{i_1+\ldots+i_{k+1}=n-k} \beta_{2i_1+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}) \cdots \beta_{2i_{k+1}+1}(\mathbf{Y}, \mathcal{R}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1}),$$

where the middle terms, hidden behind three dots, have the form $\beta_{2i_l+1}(\mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y})$ for $l = 2, 3, \ldots, k$. Hence we get

$$\varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1}(z\mathbf{I} - \mathbf{T})^{-1}\right) = \varphi(\mathbf{X}^2)\sum_{n=1}^{\infty}\beta_{2n+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1})+$$

$$\varphi\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\sum_{i_1+\ldots+i_{k+1}=n-k}\beta_{2i_1+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}) \cdots \beta_{2i_{k+1}+1}(\mathbf{Y}, \mathcal{R}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1})\right)\mathcal{R}^k\mathbf{X}^2\right).$$

The inner expression is equal to

$$\mathcal{R}\mathbf{X}^2\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\left(\sum_{i_1+\ldots+i_{k+1}=n-k}\beta_{2i_1+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}) \cdots \beta_{2i_{k+1}+1}(\mathbf{Y}, \mathcal{R}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1})\right)\mathcal{R}^{k-1}$$

$$= A(z)\tilde{A}(z)\mathcal{R}\mathbf{X}^2\sum_{k=1}^{\infty}[C(z)\mathcal{R}]^{k-1} = A(z)\tilde{A}(z)\mathcal{R}\mathbf{X}^2\left(\mathbf{I} - C(z)\mathcal{R}\right)^{-1},$$

where we denoted

$$A(z) = \sum_{n=0}^{\infty}\beta_{2n+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}),$$

$$\tilde{A}(z) = \sum_{n=0}^{\infty}\beta_{2n+1}(\mathbf{Y}, \mathcal{R}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1})$$

and

$$C(z) = \sum_{n=0}^{\infty}\beta_{2n+1}(\mathbf{Y}, \mathcal{R}, \mathbf{Y}, \ldots, \mathcal{R}, \mathbf{Y}).$$

Let us additionally denote

$$B(z) = \sum_{n=1}^{\infty}\beta_{2n+1}(\mathbf{Y}^{-1}, \mathcal{R}, \mathbf{Y}, \ldots, \mathbf{Y}, \mathcal{R}, \mathbf{Y}^{-1}).$$

(Note that each of the above series is convergent for large z by the argument from Remark 4.2.)

So far we established that

$$\varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1}(z\mathbf{I}-\mathbf{T})^{-1}\right) = B(z)\varphi\left(\mathbf{X}^2\right) + A(z)\tilde{A}(z)\varphi\left(\mathcal{R}\mathbf{X}^2\left(\mathbf{I}-C(z)\mathcal{R}\right)^{-1}\right).$$

Since Boolean cumulants are invariant with respect to reflection we see that $\tilde{A}(z) = A(z)$. Moreover, Lemma 4.2 shows that $A(z)$ and $B(z)$ have the desired forms.

The remaining objective is to calculate $\varphi\left(\mathcal{R}\mathbf{X}^2\left(\mathbf{I}-C(z)\mathcal{R}\right)^{-1}\right)$. From formula (2.9) we know that $C(z) = z - \omega_1(z)$. Consequently

$$\mathcal{R}\mathbf{X}^2\left(\mathbf{I}-C(z)\mathcal{R}\right)^{-1} = \mathbf{X}^2\left(\mathcal{R}^{-1}-C(z)\mathbf{I}\right)^{-1} = \mathbf{X}^2\left(\omega_1(z)\mathbf{I}-\mathbf{X}\right)^{-1}$$
$$= -\omega_1(z)\mathbf{I} - \mathbf{X} + \omega_1^2(z)(\omega_1(z)\mathbf{I}-\mathbf{X})^{-1}.$$

Thus referring again to (2.6) we get

$$\varphi\left(\mathcal{R}\mathbf{X}^2\left(\mathbf{I}-C(z)\mathcal{R}\right)^{-1}\right) = \omega_1^2(z)G_{\mathbf{X}+\mathbf{Y}}(z) - \omega_1(z) - \varphi(\mathbf{X}).$$

This proves the result for all large enough z in $\mathbb{C}^+$. Since both sides of (4.22) are analytic functions (4.22) holds for all $z \in \mathbb{C}^+$.

$\square$

5. Characterization theorems

The aim of this section is to prove regression characterizations which are our main results. We will deal with each case $(k,l) = (1,-1), (1,2), (-1,-2)$ as given in (1.1) separately. The proof of each case will be broken into a series of lemmas and corollaries. Before we start with the first case, set once and for all, $\mathbf{U} = (\mathbf{X}+\mathbf{Y})^{-1}$ and $\mathbf{V} = \mathbf{X}^{-1} - (\mathbf{X}+\mathbf{Y})^{-1}$, where $\mathbf{X}$ and $\mathbf{Y}$ are free, positive, non-degenerate and self-adjoint random variables.

5.1. *The case* $(k,l) = (1,-1)$.

Theorem 5.1. *Assume that*
$$\varphi\left(\mathbf{V}\mid\mathbf{U}\right) = c\mathbf{I},$$
$$\varphi\left(\mathbf{V}^{-1}\mid\mathbf{U}\right) = d\mathbf{I},$$

for some constants c and d, then $cd > 1$ and $\mathbf{X}$ has free-GIG distribution $\mu\left(-\frac{cd}{cd-1}, \frac{\gamma}{cd-1}, \frac{d}{cd-1}\right)$ *and* $\mathbf{Y}$ *has free Poisson distribution* $\nu\left(\frac{cd}{cd-1}, \frac{cd-1}{\gamma}\right)$, *where γ is some positive constant.*

Proof: Under the assumptions of Theorem 5.1, Lemmas 4.5, 4.6 and equation (2.7) imply the following system of equations

$$\begin{cases} \dfrac{1}{\omega_1(z)}\left(\theta + c + G_{\mathbf{X}+\mathbf{Y}}(z)\right) = \left(c + \dfrac{1}{z}\right)G_{\mathbf{X}+\mathbf{Y}}(z) + \dfrac{\theta}{z} \\[2mm] \dfrac{1}{\omega_2(z)}\left(\gamma + G_{\mathbf{X}+\mathbf{Y}}(z)\right) = \dfrac{d\theta}{z^2} + \dfrac{\gamma}{z} + \left(\dfrac{d}{z^2} + \dfrac{1}{z}\right)G_{\mathbf{X}+\mathbf{Y}}(z) \quad , \\[2mm] z = \omega_1(z) + \omega_2(z) - \dfrac{1}{G_{\mathbf{X}+\mathbf{Y}}(z)} \end{cases} \qquad (5.1)$$

where $\theta = \varphi(\mathbf{U})$ and $\gamma = \varphi(\mathbf{Y}^{-1})$ are positive constants.

Moreover $cd = \varphi(\mathbf{V})\varphi(\mathbf{V}^{-1}) > 1$ by the Cauchy-Schwarz inequality. The remainder of the proof splits into two lemmas.

Lemma 5.2. *The R-transform of* $\mathbf{Y}$ *is equal to*

$$r_{\mathbf{Y}}(z) = \frac{cd}{\gamma - (cd - 1)z},$$

hence $\mathbf{Y}$ *has free Poisson distribution* $\nu\left(\frac{cd}{cd-1}, \frac{cd-1}{\gamma}\right)$.

Proof: From the first equation of (5.1) we see that

$$(\theta + G_{\mathbf{X}+\mathbf{Y}}(z))\left(\frac{1}{\omega_1(z)} - \frac{1}{z}\right) = c\left(G_{\mathbf{X}+\mathbf{Y}}(z) - \frac{1}{\omega_1(z)}\right).$$

Note that $\omega_1(z)$ is not the identity function because otherwise formula (2.6) would imply that $\mathbf{Y}$ has a degenerate distribution. This allows us to write

$$\theta + G_{\mathbf{X}+\mathbf{Y}}(z) = \frac{c\left(G_{\mathbf{X}+\mathbf{Y}}(z) - \frac{1}{\omega_1(z)}\right)}{\frac{1}{\omega_1(z)} - \frac{1}{z}}. \tag{5.2}$$

The second equation of (5.1) can be written in the following form

$$(\gamma + G_{\mathbf{X}+\mathbf{Y}}(z))\left(\frac{1}{\omega_2(z)} - \frac{1}{z}\right) = d\frac{\theta + G_{\mathbf{X}+\mathbf{Y}}(z)}{z^2}.$$

Using (5.2) and formula (2.7) we see that the right hand side of the above equation equals

$$\frac{cd}{z^2}\frac{\left(G - \frac{1}{\omega_1(z)}\right)}{\frac{1}{\omega_1(z)} - \frac{1}{z}} = \frac{cdG}{z}\frac{\omega_1(z) - \frac{1}{G}}{z - \omega_1(z)}$$

$$= \frac{cdG}{z}\frac{z - \omega_2(z)}{\omega_2(z) - \frac{1}{G}} = \frac{cdG}{1 - \frac{1}{\omega_2(z)G}}\left(\frac{1}{\omega_2(z)} - \frac{1}{z}\right),$$

where G stands for $G_{\mathbf{X}+\mathbf{Y}}(z)$ for simplicity of notation. Comparing both sides and noting that we are allowed to cancel out $\frac{1}{\omega_2(z)} - \frac{1}{z}$ we see that

$$\gamma + G = \frac{cdG}{1 - \frac{1}{\omega_2(z)G}}.$$

An easy calculation shows that

$$\omega_2(z) = \frac{\gamma + G_{\mathbf{X}+\mathbf{Y}}(z)}{G_{\mathbf{X}+\mathbf{Y}}(z)\left(\gamma - (cd - 1)G_{\mathbf{X}+\mathbf{Y}}(z)\right)}.$$

Recalling that $G_{\mathbf{X}+\mathbf{Y}}(z) = G_{\mathbf{Y}}(\omega_2(z))$, we can write the last equation as

$$\omega_2(z) = \frac{\gamma + G_{\mathbf{Y}}(\omega_2(z))}{G_{\mathbf{Y}}(\omega_2(z))\left(\gamma - (cd - 1)G_{\mathbf{Y}}(\omega_2(z))\right)}.$$

This proves that

$$G_{\mathbf{Y}}^{-1}(z) = \frac{\gamma + z}{z\left(\gamma - (cd - 1)z\right)}.$$

This allows us to determine the R-transform of $\mathbf{Y}$:

$$r_{\mathbf{Y}}(z) = \frac{\gamma + z}{z\left(\gamma - (cd - 1)z\right)} - \frac{1}{z} = \frac{cd}{\gamma - (cd - 1)z}.$$

$\square$

Lemma 5.3. *The distribution of* $\mathbf{X} + \mathbf{Y}$ *is free-GIG* $\mu\left(\frac{cd}{cd-1}, \frac{\gamma}{cd-1}, \frac{d}{cd-1}\right)$.

Proof: We already expressed $\omega_2(z)$ in terms of $G = G_{\mathbf{X}+\mathbf{Y}}(z)$ i.e. $\omega_2(z) = \frac{\gamma + G}{G(\gamma - (cd-1)G)}$. Plugging this formula into the second equation of (5.1) yields the following equation for G in terms of z:

$$(cd - 1)z^2 G^2 - \left(\gamma z^2 - z - d\right) G + \gamma z + d\theta = 0.$$

By positivity of $\mathbf{X}$ and $\mathbf{Y}$ we see that $\gamma, \theta, d > 0$. The result follows now from Lemma 3.1.

$\square$

Corollary 5.4. *The distribution of $\mathbf{X}$ is free-GIG $\mu\left(-\frac{cd}{cd-1}, \frac{\gamma}{cd-1}, \frac{d}{cd-1}\right)$.*

Proof: This follows from Lemma 3.5 and the fact that for free and compactly supported random variables $\mathbf{X}$ and $\mathbf{Y}$ the distribution of $\mathbf{X}$ is uniquely determined by distributions of $\mathbf{Y}$ and $\mathbf{X}+\mathbf{Y}$. $\square$

$\square$

5.2. The case $(k, l) = (1, 2)$.

Theorem 5.5. *Assume that*

$$\varphi\left(\mathbf{V} \mid \mathbf{U}\right) = c\mathbf{I}, \tag{5.3}$$
$$\varphi\left(\mathbf{V}^2 \mid \mathbf{U}\right) = b\mathbf{I}, \tag{5.4}$$

for some constants c and b, then $b > c^2$ and $\mathbf{X}$ has free-GIG distribution $\mu\left(-\frac{c^2}{b-c^2}, \frac{\rho}{b-c^2}, \frac{c}{b-c^2}\right)$ and $\mathbf{Y}$ has free Poisson distribution $\nu\left(\frac{c^2}{b-c^2}, \frac{b-c^2}{\rho}\right)$, where ρ is some positive constant.

Proof: Let us denote $\theta = \varphi(\mathbf{U})$ and $\alpha = \varphi(\mathbf{U}^2)$. Lemma 4.5 implies the following equality

$$\frac{1}{\omega_1(z)}\left(\theta + c + G_{\mathbf{X}+\mathbf{Y}}(z)\right) = \left(c + \frac{1}{z}\right) G_{\mathbf{X}+\mathbf{Y}}(z) + \frac{\theta}{z}. \tag{5.5}$$

The equation provided by Lemma 4.7 is

$$\frac{\varphi(\mathbf{X}^{-2})}{\omega_1(z)} + \frac{1}{\omega_1(z)}\left(\varphi(\mathbf{X}^{-1}) + G_{\mathbf{X}+\mathbf{Y}}(z)\right)\left(\frac{1}{\omega_1(z)} - \frac{2}{z}\right)$$
$$= \frac{\varphi(\mathbf{X}^{-2}) - b}{z} - \frac{\theta}{z^2} + \left(b - \frac{1}{z^2}\right) G_{\mathbf{X}+\mathbf{Y}}(z) \quad (5.6)$$

and contains two additional constants $\varphi(\mathbf{X}^{-1})$ and $\varphi(\mathbf{X}^{-2})$ that we want to express in terms of c, b, α and θ. First note that equality (4.7) i.e. $\varphi\left(\mathbf{X}^{-1} \mid \mathbf{T}\right) = c\mathbf{I} + \mathbf{U}$ implies $\varphi(\mathbf{X}^{-1}) = c + \theta$. Combining this with (4.16) yields

$$\varphi\left(\mathbf{X}^{-2} \mid \mathbf{T}\right) = 2\varphi\left(\mathbf{X}^{-1} \mid \mathbf{T}\right)\mathbf{T}^{-1} + b\mathbf{I} - \mathbf{T}^{-2}$$
$$= b\mathbf{I} + 2c\mathbf{U} + \mathbf{U}^2.$$

Hence $\varphi\left(\mathbf{X}^{-2}\right) = b + 2c\theta + \alpha$.

Replacing $\frac{1}{\omega_1(z)}\left(\varphi(\mathbf{X}^{-1}) + G_{\mathbf{X}+\mathbf{Y}}(z)\right)$ in (5.6) by the right hand side of (5.5) (and simple algebra) gives the following system of equations to work with:

$$\begin{cases} \dfrac{1}{\omega_1(z)}\left(\theta + c + G_{\mathbf{X}+\mathbf{Y}}(z)\right) = \left(c + \dfrac{1}{z}\right) G_{\mathbf{X}+\mathbf{Y}}(z) + \dfrac{\theta}{z} \\[2mm] \dfrac{1}{\omega_1(z)}\left(\delta + \dfrac{\theta}{z} + \left(c + \dfrac{1}{z}\right) G_{\mathbf{X}+\mathbf{Y}}(z)\right) = \dfrac{\alpha}{z} + b G_{\mathbf{X}+\mathbf{Y}}(z) + \left(\theta + G_{\mathbf{X}+\mathbf{Y}}(z)\right)\left(\dfrac{1}{z^2} + \dfrac{2c}{z}\right) \\[2mm] z = \omega_1(z) + \omega_2(z) - \dfrac{1}{G_{\mathbf{X}+\mathbf{Y}}(z)} \end{cases}, \tag{5.7}$$

where $\delta = b + 2c\theta + \alpha$.

Note, that this time both the first and the second equation in (5.7) involve only $\omega_1(z)$ and therefore we can easily calculate $G = G_{\mathbf{X}+\mathbf{Y}}(z)$. Namely let us divide the second equation by the first one to get

$$\frac{\delta + \frac{\theta}{z} + \left(c + \frac{1}{z}\right)G}{\theta + c + G} = \frac{\frac{\alpha}{z} + bG + (\theta + G)\left(\frac{1}{z^2} + \frac{2c}{z}\right)}{\left(c + \frac{1}{z}\right)G + \frac{\theta}{z}}.$$

(Note that the expression $\theta + c + G$ is non-zero as G takes values in $\mathbb{C}^-$ so this division is justified.)

Multiplying both sides by the denominators we arrive after some easy but tedious calculation at the following equation

$$(b - c^2)z^2G^2 - \left(\rho z^2 - (2c^2 - b)z - c\right)G + \rho z + \theta c = 0, \tag{5.8}$$

where $\rho = 2\theta c^2 + \alpha c - \theta b$. Note that $b = \varphi(\mathbf{V}^2) > c^2 = \varphi(\mathbf{V})^2$ by the Cauchy-Schwarz inequality.

Now we are ready to prove the following lemma.

Lemma 5.6. *The R-transform of $\mathbf{Y}$ is equal to*

$$r_{\mathbf{Y}}(z) = \frac{c^2}{\rho - (b - c^2)z},$$

in particular $\rho > 0$ and $\mathbf{Y}$ has free Poisson distribution $\nu\left(\frac{c^2}{b-c^2}, \frac{b-c^2}{\rho}\right)$.

Proof: From equations (2.7) and (5.5) we see that

$$R(z) := \omega_2(z) - \frac{1}{G_{\mathbf{X}+\mathbf{Y}}(z)} = z - \omega_1(z) = z - \frac{\theta + c + G}{\left(c + \frac{1}{z}\right)G + \frac{\theta}{z}},$$

where $G = G_{\mathbf{X}+\mathbf{Y}}(z)$. Hence

$$R(z) = \frac{cz(Gz - 1)}{\theta + G + czG}.$$

Now we write equation (5.8) as

$$(b - c^2)(z^2G^2 - Gz) - \rho z(Gz - 1) + c^2 zG + cG + \theta c = 0$$

or

$$c(\theta + G + czG) = \rho z(Gz - 1) - (b - c^2)zG(zG - 1).$$

After multiplying both sides by c and dividing by $\theta + G + czG$ we see that

$$c^2 = \rho R(z) - (b - c^2)R(z)G$$

or in other words

$$R(z) = \frac{c^2}{\rho - (b - c^2)G}.$$

Recalling the definition of $R(z)$ and the fact that $G = G_{\mathbf{Y}}(\omega_2(z))$ we get that

$$\omega_2(z) = \frac{1}{G_{\mathbf{Y}}(\omega_2(z))} + \frac{c^2}{\rho - (b - c^2)G_{\mathbf{Y}}(\omega_2(z))}.$$

This proves that $G_{\mathbf{Y}}^{-1}(z) = \frac{1}{z} + \frac{c^2}{\rho - (b-c^2)z}$ and that

$$r_{\mathbf{Y}}(z) = \frac{c^2}{\rho - (b - c^2)z}.$$

Since $r_{\mathbf{Y}}(0) = \varphi(\mathbf{Y}) > 0$ we see that $\rho > 0$ and $\mathbf{Y}$ has free Poisson distribution $\nu\left(\frac{c^2}{b-c^2}, \frac{b-c^2}{\rho}\right)$. □

Corollary 5.7. *The distribution of $\mathbf{X}$ is free-GIG $\mu\left(-\frac{c^2}{b-c^2}, \frac{\rho}{b-c^2}, \frac{c}{b-c^2}\right)$ where $\rho = 2\theta c^2 + \alpha c - \theta b$.*

Proof: The Cauchy-Stieltjes transform $G = G_{\mathbf{X+Y}}(z)$ satisfies the quadratic equation (5.8) i.e.

$$(b - c^2)z^2 G^2 - \left(\rho z^2 - (2c^2 - b)z - c\right)G + \rho z + \theta c = 0.$$

Since we know that $\rho > 0$ Lemma 3.1 implies that the distribution of $\mathbf{X + Y}$ is $\mu\left(\frac{c^2}{b-c^2}, \frac{\rho}{b-c^2}, \frac{c}{b-c^2}\right)$. The result follows from Lemma 3.5. $\square$

$\square$

5.3. *The case* $(k, l) = (-1, -2)$.

Theorem 5.8. *Assume that*

$$\varphi\left(\mathbf{V}^{-1} \mid \mathbf{U}\right) = d\mathbf{I}, \tag{5.9}$$

$$\varphi\left(\mathbf{V}^{-2} \mid \mathbf{U}\right) = h\mathbf{I}, \tag{5.10}$$

for some constants d and h, then $h > d^2$ and $\mathbf{X}$ has free-GIG distribution $\mu\left(-\frac{h}{h-d^2}, \frac{\gamma d^2}{h-d^2}, \frac{d^3}{h-d^2}\right)$ and $\mathbf{Y}$ has free Poisson distribution $\nu\left(\frac{h}{h-d^2}, \frac{h-d^2}{d^2\gamma}\right)$, where γ is some positive constant.

Proof: Let us denote $\gamma = \varphi(\mathbf{Y}^{-1})$, $\theta = \varphi(\mathbf{U})$ and $\alpha = \varphi(\mathbf{U}^2)$. Lemma 4.6 implies the following equality

$$\frac{1}{\omega_2(z)}\left(\gamma + G_{\mathbf{X+Y}}(z)\right) = \frac{d\theta}{z^2} + \frac{\gamma}{z} + \left(\frac{d}{z^2} + \frac{1}{z}\right)G_{\mathbf{X+Y}}(z)$$

that can be written also as

$$\left(\gamma + G_{\mathbf{X+Y}}(z)\right)\left(\frac{1}{\omega_2(z)} - \frac{1}{z}\right) = \frac{d}{z^2}\left(\theta + G_{\mathbf{X+Y}}(z)\right). \tag{5.11}$$

From Lemma 4.8 we get the second equation:

$$\varphi(\mathbf{X}^2)B(z) + A(z)^2\left(\omega_1^2(z)G_{\mathbf{X+Y}}(z) - \omega_1(z) - \varphi(\mathbf{X})\right) =$$

$$= h\left(\frac{\alpha}{z} + \frac{\theta}{z^2} + \frac{1}{z^2}G_{\mathbf{X+Y}}(z)\right), \quad (5.12)$$

where

$$A(z) = \frac{\gamma + G_{\mathbf{X+Y}}(z)}{\omega_2(z)G_{\mathbf{X+Y}}(z)} \quad \text{and} \quad B(z) = \frac{\varphi(\mathbf{Y}^{-2}) - \gamma A(z)}{\omega_2(z)}.$$

Our first goal is to simplify equation (5.12). First note that $\mathbf{V}^{-1}\mathbf{U} = \mathbf{X}\mathbf{Y}^{-1}$. Since $\mathbf{X}$ and $\mathbf{Y}^{-1}$ are free we get

$$\varphi(\mathbf{X})\varphi(\mathbf{Y}^{-1}) = \varphi(\mathbf{V}^{-1}\mathbf{U}) = \varphi(\varphi(\mathbf{V}^{-1} \mid \mathbf{U})\mathbf{U}) = d\theta.$$

Hence $\varphi(\mathbf{X}) = \frac{d\theta}{\gamma}$. Similarly $\mathbf{V}^{-1} = \mathbf{X}\mathbf{Y}^{-1}(\mathbf{X} + \mathbf{Y}) = \mathbf{X}\mathbf{Y}^{-1}\mathbf{X} + \mathbf{X}$. Taking the expectation we see that

$$\varphi(\mathbf{V}^{-1}) = \varphi(\mathbf{X}\mathbf{Y}^{-1}\mathbf{X}) + \varphi(\mathbf{X}) = \varphi(\mathbf{X}^2\mathbf{Y}^{-1}) + \varphi(\mathbf{X})$$

by traciality. From this we get $\varphi(\mathbf{X}^2)\gamma = d\left(1 - \frac{\theta}{\gamma}\right)$. Taking into account equation (4.23) i.e.

$$\varphi\left(\mathbf{Y}^{-1}\mathbf{X}^2\mathbf{Y}^{-1} \mid \mathbf{U}\right) = h\mathbf{U}^2,$$

we observe that $\varphi(\mathbf{X}^2)\varphi(\mathbf{Y}^{-2}) = h\alpha$. Hence

$$\varphi(\mathbf{X}^2)B(z) = \frac{\varphi(\mathbf{X}^2)\varphi(\mathbf{Y}^{-2}) - \varphi(\mathbf{X}^2)\gamma A(z)}{\omega_2(z)} = \frac{h\alpha}{\omega_2(z)} - d\left(1 - \frac{\theta}{\gamma}\right)\frac{A(z)}{\omega_2(z)}.$$

Thus the left hand side of (5.12) is equal to

$$\frac{h\alpha}{\omega_2(z)} - d\left(1 - \frac{\theta}{\gamma}\right)\frac{A(z)}{\omega_2(z)} + A(z)^2\left(\omega_1^2(z)G_{\mathbf{X+Y}}(z) - \omega_1(z)\right) - \frac{d\theta}{\gamma}A(z)^2.$$

An easy calculation shows that
$$d\left(1 - \frac{\theta}{\gamma}\right)\frac{A(z)}{\omega_2(z)} + \frac{d\theta}{\gamma}A(z)^2 = A(z)\frac{d(\theta + G_{\mathbf{X+Y}}(z))}{\omega_2(z)G_{\mathbf{X+Y}}(z)}$$

and that
$$\omega_1^2(z)G_{\mathbf{X+Y}}(z) - \omega_1(z) = \omega_1(z)G_{\mathbf{X+Y}}(z)\left(\omega_1(z) - \frac{1}{G_{\mathbf{X+Y}}(z)}\right) = G_{\mathbf{X+Y}}(z)\omega_1(z)\left(z - \omega_2(z)\right),$$

where in the last equality we used formula (2.7). Consequently, equation (5.12) takes on the following form
$$\frac{h\alpha}{\omega_2(z)} + G_{\mathbf{X+Y}}(z)\omega_1(z)\left(z - \omega_2(z)\right)A(z)^2 - A(z)\frac{d(\theta + G_{\mathbf{X+Y}}(z))}{\omega_2(z)G_{\mathbf{X+Y}}(z)} =$$
$$h\left(\frac{\alpha}{z} + \frac{\theta}{z^2} + \frac{1}{z^2}G_{\mathbf{X+Y}}(z)\right)$$

or equivalently
$$h\alpha\left(\frac{1}{\omega_2(z)} - \frac{1}{z}\right) + zG_{\mathbf{X+Y}}(z)\omega_1(z)\omega_2(z)\left(\frac{1}{\omega_2(z)} - \frac{1}{z}\right)A(z)^2 =$$
$$h\frac{\theta + G_{\mathbf{X+Y}}(z)}{z^2} + A(z)\frac{d(\theta + G_{\mathbf{X+Y}}(z))}{\omega_2(z)G_{\mathbf{X+Y}}(z)}.$$

We can now plug $\theta + G_{\mathbf{X+Y}}(z)$ calculated from (5.11) and cancel out the common term i.e. $\frac{1}{\omega_2(z)} - \frac{1}{z}$. The cancellation is allowed since both sides of the equation are analytic on $\mathbb{C}^+$ and $\omega_2(z)$ cannot be the identity function as it would contradict positivity of $\mathbf{X}$. This yields the following simpler equation
$$h\alpha + z\omega_1(z)\omega_2(z)G_{\mathbf{X+Y}}(z)A(z)^2 = \frac{h}{d}(\gamma + G_{\mathbf{X+Y}}(z)) + z^2 A(z)\frac{\gamma + G_{\mathbf{X+Y}}(z)}{\omega_2(z)G_{\mathbf{X+Y}}(z)}$$
$$= \frac{h}{d}(\gamma + G_{\mathbf{X+Y}}(z)) + z^2 A(z)^2. \qquad (5.13)$$

To proceed further we need to express α in terms of other constants. Note that
$$\mathbf{Y}^{-1} - \mathbf{U}\mathbf{V}^{-1}\mathbf{U} = \mathbf{U}\mathbf{U}^{-1}\mathbf{Y}^{-1} - \mathbf{U}\mathbf{X}\mathbf{Y}^{-1} = \mathbf{U}(\mathbf{U}^{-1} - \mathbf{X})\mathbf{Y}^{-1} = \mathbf{U}.$$

After taking the expectation and using the regression condition (5.9) we see that $\gamma - d\alpha = \theta$. In other words
$$\alpha = \frac{\gamma - \theta}{d}.$$

This fact and (5.11) imply that
$$\frac{h}{d}(\gamma + G_{\mathbf{X+Y}}(z)) - h\alpha = h\frac{\theta + G_{\mathbf{X+Y}}(z)}{d} = \frac{hz^2}{d^2}(\gamma + G_{\mathbf{X+Y}}(z))\left(\frac{1}{\omega_2(z)} - \frac{1}{z}\right).$$

The right hand side of the above expression can be written as
$$\frac{hzG_{\mathbf{X+Y}}(z)}{d^2}A(z)(z - \omega_2(z)).$$

This means that we can rewrite equation (5.13) as
$$z\omega_1(z)\omega_2(z)G_{\mathbf{X+Y}}(z)A(z)^2 = \frac{hzG_{\mathbf{X+Y}}(z)}{d^2}A(z)(z - \omega_2(z)) + z^2 A(z)^2$$

or, after canceling out the common term $zA(z) \neq 0$, as
$$\omega_1(z)\omega_2(z)G_{\mathbf{X+Y}}(z)A(z) = zA(z) + \frac{h}{d^2}G_{\mathbf{X+Y}}(z)(z - \omega_2(z)).$$

Equation (2.7) implies that $\omega_1(z)G_{\mathbf{X}+\mathbf{Y}}(z) = G_{\mathbf{X}+\mathbf{Y}}(z)(z - \omega_2(z)) + 1$. Plugging this into the above equation yields

$$G_{\mathbf{X}+\mathbf{Y}}(z)(z - \omega_2(z))\omega_2(z)A(z) = (z - \omega_2(z))A(z) + \frac{h}{d^2}G_{\mathbf{X}+\mathbf{Y}}(z)(z - \omega_2(z)).$$

Since $z - \omega_2(z)$ is a non-zero function and both sides are analytic on $\mathbb{C}^+$ we obtain

$$\gamma + G_{\mathbf{X}+\mathbf{Y}}(z) = \frac{\gamma + G_{\mathbf{X}+\mathbf{Y}}(z)}{\omega_2(z)G_{\mathbf{X}+\mathbf{Y}}(z)} + \frac{h}{d^2}G_{\mathbf{X}+\mathbf{Y}}(z). \qquad (5.14)$$

Lemma 5.9. *The R-transform of $\mathbf{Y}$ is equal to*

$$r_{\mathbf{Y}}(z) = \frac{h}{d^2\gamma - (h - d^2)z},$$

and hence $\mathbf{Y}$ has free Poisson distribution $\nu\left(\frac{h}{h-d^2}, \frac{h-d^2}{d^2\gamma}\right)$.

Proof: Equation (5.14) implies

$$\omega_2(z) = \frac{d^2(\gamma + G_{\mathbf{X}+\mathbf{Y}}(z))}{G_{\mathbf{X}+\mathbf{Y}}(z)\left(d^2\gamma - (h - d^2)G_{\mathbf{X}+\mathbf{Y}}(z)\right)}.$$

Since $G_{\mathbf{X}+\mathbf{Y}}(z) = G_{\mathbf{Y}}(\omega_2(z))$ we see that

$$G_{\mathbf{Y}}^{-1}(z) = \frac{d^2(\gamma + z)}{z\left(d^2\gamma - (h - d^2)z\right)}.$$

Thus

$$r_{\mathbf{Y}}(z) = G_{\mathbf{Y}}^{-1}(z) - \frac{1}{z} = \frac{h}{d^2\gamma - (h - d^2)z}.$$

Now it is enough to note that the Cauchy-Schwarz inequality implies $h > d^2$. $\qquad \square$

Lemma 5.10. *The distribution of $\mathbf{X}$ is free-GIG $\mu\left(-\frac{h}{h-d^2}, \frac{\gamma d^2}{h-d^2}, \frac{d^3}{h-d^2}\right)$.*

Proof: Equations (5.11) and (5.14) imply the following quadratic equation for $G = G_{\mathbf{X}+\mathbf{Y}}(z)$

$$(h - d^2)z^2G^2 - d^2(\gamma z^2 - z - d)G + d^2(\gamma z + d\theta) = 0.$$

Since all parameters are obviously positive and $h > d^2$ we see that Lemma 3.1 implies that the distribution of $\mathbf{X} + \mathbf{Y}$ is free-GIG $\mu\left(\frac{h}{h-d^2}, \frac{\gamma d^2}{h-d^2}, \frac{d^3}{h-d^2}\right)$.

The result follows now from Lemma 3.5. $\qquad \square$

$\qquad \square$

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