



## Asymptotics for Kendall’s renewal function

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**Abstract.** An elementary renewal theorem and a Blackwell theorem provided by Jasiulis-Gołdyn et al. (2020) in a setting of Kendall convolutions are proved under weaker hypothesis and extended to the Gamma class. Convergence rates of the limits concerned in these theorems are analyzed. Our theoretical results are illustrated by several examples involving novel probability distributions and extremes.

### 1. Introduction

Renewal theory and the corresponding expected number of renewal epochs have found numerous applications in finance, insurance, biostatistics, reliability theory, risk theory and even medicine. The renewal theorem is crucial in computing the expected number of renewals and is fundamental to developing risk-based asset management models. In actuarial sciences, insurance companies use asymptotic methods for computing the expected number of claims (renewals) and to estimate the failure risk. On the other hand, in medicine, the renewal function is the expected number of patients with illness cases in the interval  $[0, t]$ , where the first case was registered at moment  $t = 0$ . The widespread applications of renewal theory motivate the research presented here.

The paper deals with the development of renewal theory for extremal Markov chains with transition probability given by the Kendall convolution called Kendall random walks. Their structure is similar to Pareto processes and extremal processes.

Recently Jasiulis-Gołdyn, Misiewicz, Naskręć & Omeý (Jasiulis-Gołdyn et al., 2020) have formulated the corresponding renewal theory and proved an analogue of the Fredholm theorem for all regular generalized convolutions algebras. Using regularly varying functions they also proved a Blackwell theorem and renewal limit theorem for the renewal processes in the Kendall convolution

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case. Such a convolution that is based on the Williamson transform of a random variable is an example of a generalized convolution introduced by [Urbanik \(1963/64\)](#). We investigate here the rate of convergence for theorems mentioned above and the Gamma class of distributions that is famous in extreme value theory.

The origin of the Kendall convolution one can find in Stochastic Geometry since the corresponding generalized characteristic function is also an avoidance function for a random closed set (for details see [Kendall \(1974\)](#)).

The Kendall convolution  $\Delta_\alpha$ ,  $\alpha > 0$ , is defined for  $x \in [0, 1]$  and two probability measures  $\delta_x, \delta_1$  concentrated respectively at  $x$  and 1, in the following way:

$$\delta_x \Delta_\alpha \delta_1 = x^\alpha \delta_1 + (1 - x^\alpha) \pi_{2\alpha},$$

where  $\pi_{2\alpha}$  denotes the Pareto distribution with density  $2\alpha y^{-2\alpha-1} \mathbf{1}_{[1, \infty)}(y)$ . The corresponding generalized characteristic function for probability measure  $\lambda$  with cumulative distribution function  $F$  is the Williamson transform (for more details see [Williamson \(1956\)](#)) defined below:

$$\Phi_\lambda(t) := \int_0^\infty (1 - x^\alpha t^\alpha)_+ \lambda(dx),$$

where  $a_+ = a$  if  $a > 0$  and  $a_+ = 0$  if  $a \leq 0$ . The Williamson transform is easy to invert, hence it gives the opportunity to obtain interesting results, both asymptotic and direct mathematical formulas.

In [Arendarczyk et al. \(2022+\)](#), the authors pointed out connections between the concept of Slash distribution in statistics and Kendall convolution. They noticed that the cumulative distribution function of the Slash distribution is the generalized characteristic function of the Kendall convolution, which is the Williamson transform. It yields that the results of this paper on the Williamson transforms and Kendall convolution are very important to the development of Slash distributions and consequently stochastic modeling in finance.

In the classical probability theory, the renewal process is described by a sequence of occurrences of events  $T_1, T_2, \dots$ , called successive waiting times, where  $T_i$  are independent and identically distributed random variables with common distribution function  $F$ . The sums  $S_n = T_1 + \dots + T_n$ , with the convention  $S_0 = 0$ , then form a sequence called a renewal sequence if  $F(0) = 0$ . We count the appearance of phenomena (such as failures of machines, arrivals in a queue, lifetimes of systems) through the renewal function defined by the formula  $R(t) = \sum_{n=0}^\infty P(S_n \leq t)$ .

For more details on renewal theory, we refer the reader to [Asmussen \(2000\)](#) and e.g. [Mitov and Omey \(2014\)](#); [Rolski et al. \(1999\)](#). These authors studied  $R(t)$  when the usual convolutions involved in this function are interchanged by Kendall convolutions. In [Jasiulis-Gołdyn et al. \(2020\)](#) the renewal function was defined by  $R_\star(t) = \sum_{n=1}^\infty P(S_n \leq t)$ . Here we will investigate asymptotic behaviour of  $R(t) = 1 + R_\star(t)$  in the Kendall convolution sense, similarly to the classical renewal theory, which seems to be much more convenient for the new results. In this setting, among other results, assuming  $m(\alpha) = E(T_1^\alpha) < \infty$ , in [Jasiulis-Gołdyn et al. \(2020\)](#) the authors proved the elementary renewal theorem

$$\lim_{x \rightarrow \infty} \frac{R_\star(x)}{x^\alpha} = \frac{2}{m(\alpha)}, \quad (1.1)$$

and the Blackwell theorem

$$\lim_{x \rightarrow \infty} \left( \frac{R_\star(x+h)}{(x+h)^{\alpha-1}} - \frac{R_\star(x)}{x^{\alpha-1}} \right) = \frac{2h}{m(\alpha)}, \quad (1.2)$$

for all  $h \in \mathbb{R}$ . Interestingly, these results were obtained when  $\bar{F} = 1 - F$  is regularly varying.

In this paper we prove (1.1) without considering the assumption on regular variation mentioned above and prove (1.2) by introducing a general condition on  $\bar{F}$ . It is proved that such a general

condition is satisfied when  $\bar{F}$  is regularly varying or belongs to the Gamma class. We also analyze convergence rates of the limits (1.1) and (1.2) in the cases where  $\bar{F}$  is regularly varying or belongs to the Gamma class. Our new results on convergence rates contribute to other results obtained by a number of scholars in renewal theory, for instance Rogozin (1974), Kalashnikov (1978), Rogozin (1977) and Grübel (1983).

In the next section of the paper we give an overview of important definitions and properties that are going to be used. For instance, Kendall convolution, Williamson transform, renewal function, the class of regularly varying functions and the Gamma class of functions. In Section 3 we study rates of convergence in the renewal theorem. It turns out that we have to consider two important cases. The first case is the case where the tail of the random variable  $X$  is regularly varying. The second case is the case where the tail of  $X$  belongs to the Gamma class of distribution functions. Then, in Section 4 we study Blackwell type of results, i.e. we study the asymptotic behaviour of  $R(x+y) - R(x)$  and that of the derivative  $R'(x)$ . Again we study rates of convergence and we distinguish between the two cases as in the previous section. Section 5 presents these last results on rates of convergence. We close the paper with some concluding remarks given in the last section.

## 2. Notations and definitions

We write  $f(x) \sim g(x)$  by  $f(x)/g(x) \rightarrow 1$  as  $x \rightarrow \infty$ .

2.1. *Transforms.* In what follows  $X$  is a positive random variable (r.v.) with distribution function (d.f.)  $F(x) = P(X \leq x)$  and  $F(0-) = 0$ .  $F$  is assumed continuous. The tail of  $F$  is given by  $\bar{F}(x) = 1 - F(x)$ . Let  $\alpha$  denote a positive real number. We define truncated moments as follows

$$H_\alpha(x) = \int_0^x y^\alpha dF(y),$$

and

$$W_\alpha(x) = \int_0^x y^{\alpha-1} \bar{F}(y) dy.$$

Throughout we assume that the  $\alpha$ -th moment is finite, and we write  $m(\alpha) = H_\alpha(\infty) = \alpha W_\alpha(\infty)$ . The case of  $m(\alpha) = \infty$  has been treated in Jasiulis-Goldyn et al. (2020) and in Omey and Cadena (2022), Kevei (2021). Note that by dominated convergence we have  $\lim_{x \rightarrow \infty} x^{-\alpha} H_\alpha(x) = 0$ .

The Williamson or  $G$ -transform is given by

$$G_F(x) = \int_0^x \left(1 - \left(\frac{t}{x}\right)^\alpha\right) dF(t).$$

See Williamson (1956) and e.g. Jasiulis-Goldyn et al. (2020). The probabilistic interpretation of  $G_F(x)$  is as follows. Let  $Z$  denote a positive r.v. with  $P(Z \leq x) = x^\alpha, 0 \leq x \leq 1$ . In this case we see that  $G_F(x) = \int_0^x P(Z \geq t/x) dF(t) = P(Z \geq X/x)$  or  $G_F(x) = P(X/Z \leq x)$ . Hence  $G_F(x)$  is again a d.f. (with  $G_F(0-) = 0$ ).

For nondecreasing functions  $B(x)$  with  $B(0-) = 0$  we can also define its  $G$ -transform as  $G_B(x) = \int_0^x (1 - (\frac{t}{x})^\alpha) dB(t)$ .

2.2. *Relationships.* Using partial integration we have  $G_F(x) = \alpha x^{-\alpha} \int_0^x t^{\alpha-1} F(t) dt$ , and then also that

$$\bar{G}_F(x) = \alpha x^{-\alpha} \int_0^x t^{\alpha-1} \bar{F}(t) dt = \alpha x^{-\alpha} W_\alpha(x),$$

where  $\overline{G}_F(x) = 1 - G_F(x)$ . Also we have the following inversion formula, cf. [Jasiulis-Goldyn and Misiewicz \(2017\)](#) and [Jasiulis-Goldyn et al. \(2020\)](#):

$$F(x) = G_F(x) + \frac{x}{\alpha} G'_F(x).$$

For nondecreasing functions  $B(x)$  with  $B(0) = 0$ , we equally have

$$B(x) = G_B(x) + \frac{x}{\alpha} G'_B(x).$$

The functions introduced above are closely related to each other. Using partial integration we have

$$H_\alpha(x) = \alpha \int_0^x y^{\alpha-1} \overline{F}(y) dy - x^\alpha \overline{F}(x) = \alpha W_\alpha(x) - x^\alpha \overline{F}(x).$$

One can show cf [Omey and Cadena \(2022\)](#) that we can find back  $\overline{F}(x)$  from  $H_\alpha(x)$  and we have:

$$\overline{F}(x) = \alpha \int_x^\infty z^{-\alpha-1} H_\alpha(z) dz - x^{-\alpha} H_\alpha(x).$$

Also we have  $\alpha W_\alpha(x) = H_\alpha(x) + x^\alpha \overline{F}(x)$ . In terms of  $H_\alpha(x)$ , we have  $G_F(x) = F(x) - x^{-\alpha} H_\alpha(x)$  and

$$\overline{G}_F(x) = \overline{F}(x) + x^{-\alpha} H_\alpha(x).$$

Since  $m(\alpha) < \infty$ , we use the following notations and identities:

$$\begin{aligned} \overline{W}_\alpha(x) &= W_\alpha(\infty) - W_\alpha(x) = \int_x^\infty y^{\alpha-1} \overline{F}(y) dy, \\ \overline{H}_\alpha(x) &= H_\alpha(\infty) - H_\alpha(x) = \alpha \overline{W}_\alpha(x) + x^\alpha \overline{F}(x), \\ \alpha \overline{W}_\alpha(x) &= m(\alpha) - x^\alpha \overline{G}_F(x) \end{aligned}$$

We summarize some of the formulas that we obtained in the following propositions. Cf. [Omey and Cadena \(2022\)](#).

**Proposition 2.1.** *We have:*

- (i)  $H_\alpha(x) = \alpha \int_0^x t^{\alpha-1} \overline{F}(t) dt - x^\alpha \overline{F}(x)$ .
- (ii)  $\overline{F}(x) = \alpha \int_x^\infty z^{-\alpha-1} H_\alpha(z) dz - x^{-\alpha} H_\alpha(x)$ .
- (iii)  $\overline{F}(x) = \overline{G}_F(x) - x^{-\alpha} H_\alpha(x)$ .
- (iv)  $\overline{G}_F(x) = \alpha x^{-\alpha} W_\alpha(x)$ .

For the 'tails' we have the following relationships.

**Proposition 2.2.** *If  $m(\alpha) < \infty$ , we have:*

- (i)  $\overline{W}_\alpha(x) = \int_x^\infty y^{\alpha-1} \overline{F}(y) dy$ .
- (ii)  $\overline{H}_\alpha(x) = \alpha \int_x^\infty t^{\alpha-1} \overline{F}(t) dt + x^\alpha \overline{F}(x)$ .
- (iii)  $x^{-\alpha} m(\alpha) - \overline{G}_F(x) = \alpha x^{-\alpha} \overline{W}_\alpha(x)$ .

**2.3. Regular variation and the Gamma class.** For relevant background information about regular variation and the Gamma class, we refer to [Bingham et al. \(1989\)](#) and [Geluk and de Haan \(1987\)](#). Here we recall the main definitions.

**Definition 2.3.** A positive and measurable function  $f(x)$  is regularly varying with index  $\alpha \in \mathbb{R}$  if for all  $t > 0$  we have

$$\lim_{x \rightarrow \infty} \frac{f(tx)}{f(x)} = t^\alpha.$$

**Definition 2.4.** A positive and measurable function  $f(x)$  is in the class  $\Gamma(g)$  with auxiliary function  $g(x)$  if for all real numbers  $x$  we have

$$\lim_{x \rightarrow \infty} \frac{f(x + tg(x))}{f(x)} = e^{-t}.$$

The auxiliary function  $g(x)$  is self-neglecting (SN), i.e.  $\lim_{x \rightarrow \infty} g(x + tg(x))/g(x) = 1$ , and satisfies  $\lim_{x \rightarrow \infty} g(x)/x = 0$ .

2.4. *The Kendall convolution.* Starting from two d.f.'s  $F_1(x) = P(X_1 \leq x)$  and  $F_2(x) = P(X_2 \leq x)$  with  $F_1(0-) = F_2(0-) = 0$ , we can find their  $G$ -transforms  $G_1(x) = G_{F_1}(x)$  and  $G_2(x) = G_{F_2}(x)$ . Now we consider the product  $A(x) = G_1(x)G_2(x)$ . One can prove that there is a d.f.  $F_3(x) = P(X_3 \leq x)$  so that  $A(x)$  is its  $G$ -transform when  $F_3$  is the Kendall convolution of  $F_1$  and  $F_2$ , cf. Jasiulis-Góldyn and Misiewicz (2017) and Jasiulis-Góldyn et al. (2020). Details on the Kendall convolution can be found in Kendall (1974) and e.g. Kucharczak and Urbanik (1974) and Jasiulis-Góldyn and Misiewicz (2011). To stress the dependence of  $F_3(x)$  on  $F_1(x)$  and  $F_2(x)$ , we use the notation  $F_3(x) = F_1 \boxtimes F_2(x) = P(X_1 \boxtimes X_2 \leq x)$ . Alternatively we write  $X_3 \stackrel{d}{=} X_1 \boxtimes X_2$ .

If  $F_1(x) = F_2(x) = F(x)$ , we use the notation  $F \boxtimes F(x) = F^{\boxtimes 2}(x)$  and  $S_{\boxtimes 2} = X_1 \boxtimes X_2$ . In this way we can construct the  $n$ -fold convolution  $F^{\boxtimes n}(x) = P(S_{\boxtimes n} \leq x)$  which has  $G$ -transform given by  $G_F^n(x)$ .

2.5. *Kendall renewal function.* This subsection presents properties of the Kendall renewal function associated to the Kendall convolution, and provides relationships with the  $G$ -transform of  $F$ .

Using the Kendall convolution, the Kendall renewal function is given by

$$R(x) = \sum_{n=0}^{\infty} F^{\boxtimes n}(x). \quad (2.1)$$

where  $F^{\boxtimes 0}(x) = \delta_0(x)$ . Its  $G$ -transform is very simple:

$$G_R(x) = \sum_{n=0}^{\infty} G_{F^{\boxtimes n}}(x) = \sum_{n=0}^{\infty} G_F^n(x) = \frac{1}{\overline{G}_F(x)}.$$

The last equality comes from, for any  $k > 0$ ,  $(1 + G_F(x) + \dots + G_F^k(x))\overline{G}_F(x) = 1 - G_F^{k+1}(x)$  and the fact that  $G_F^{k+1}(x) \rightarrow 0$  as  $k \rightarrow \infty$  because  $G_F(x) < 1$ .

The inversion formula then gives

$$R(x) = \frac{2}{\overline{G}_F(x)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)}. \quad (2.2)$$

To see this, we make some calculations:

$$\begin{aligned} R(x) &= G_R(x) + \frac{x}{\alpha} G_R'(x) \\ &= \frac{1}{\overline{G}_F(x)} + \frac{1}{\overline{G}_F^2(x)} \frac{x}{\alpha} G_F'(x) \\ &= \frac{1}{\overline{G}_F(x)} + \frac{1}{\overline{G}_F^2(x)} (\overline{G}_F(x) - \overline{F}(x)) \\ &= \frac{2}{\overline{G}_F(x)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)}. \end{aligned}$$

Using (2.2) we have

$$\frac{R(x)}{x^\alpha} = \frac{2}{x^\alpha \overline{G}_F(x)} - \frac{x^\alpha \overline{F}(x)}{x^{2\alpha} \overline{G}_F^2(x)}.$$

Since  $x^\alpha \overline{G}_F(x) \rightarrow m(\alpha)$  and  $x^\alpha \overline{F}(x) \rightarrow 0$  as  $x \rightarrow \infty$ , we obtain that

$$\lim_{x \rightarrow \infty} \frac{R(x)}{x^\alpha} = \frac{2}{m(\alpha)}, \quad (2.3)$$

cf. Jasiulis-Góldyn et al. (2020).

We summarize our findings in the next proposition.

**Proposition 2.5.** *Let  $R(x)$  be the Kendall renewal function defined by (2.1). We have*

$$R(x) = \frac{2}{\overline{G}_F(x)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)},$$

and

$$\lim_{x \rightarrow \infty} \frac{R(x)}{x^\alpha} = \frac{2}{m(\alpha)}.$$

In the next sections we study the rate of convergence in (2.3).

## 2.6. Examples.

- (1) Let  $F_X(x) = \delta_1(x)$ . We have  $m(\alpha) = 1$  and  $G_F(x) = \int_0^x (1 - (\frac{t}{x})^\alpha) d\delta_1(t)$ . For  $x \geq 1$  we find that  $G_F(x) = 1 - x^{-\alpha}$  and  $\overline{G}_F(x) = x^{-\alpha}$ . Clearly for  $x \geq 1$  we have the following simple formula, by considering (2.2):

$$R(x) = \frac{2}{\overline{G}_F(x)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)} = 2x^\alpha.$$

- (2) Let  $F(x) = 1 - x^{-\beta}, x \geq 1$ . We take  $\beta > \alpha$  so that  $m(\alpha) < \infty$ . We have  $H_\alpha(x) = \int_0^x y^\alpha dF(y)$ . For  $x \geq 1$ , we find

$$H_\alpha(x) = \beta \int_1^x y^{\alpha-\beta-1} dy = \frac{\beta}{\beta-\alpha} (1 - x^{\alpha-\beta}).$$

Using  $G_F(x) = F(x) - x^{-\alpha} H_\alpha(x)$ , it follows that for  $x \geq 1$ :

$$G_F(x) = 1 + \frac{\alpha}{\beta-\alpha} x^{-\beta} - \frac{\beta}{\beta-\alpha} x^{-\alpha},$$

and

$$\overline{G}_F(x) = \frac{\beta}{\beta-\alpha} x^{-\alpha} - \frac{\alpha}{\beta-\alpha} x^{-\beta}.$$

Note that  $x^\alpha \overline{G}_F(x) \rightarrow m(\alpha) = \beta/(\beta-\alpha)$  as  $x \rightarrow \infty$ , and we can write  $\overline{G}_F(x) = m(\alpha)x^{-\alpha} + (1 - m(\alpha))x^{-\beta}$ . Also note that  $\overline{F}(x)/\overline{G}_F(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

Using (2.2) we have for  $x \geq 1$ ,

$$\begin{aligned} R(x) &= \frac{2}{\frac{m(\alpha)x^{-\alpha} + (1 - m(\alpha))x^{-\beta}}{x^{-\beta}}} \\ &= \frac{2x^\alpha}{\frac{m(\alpha) + (1 - m(\alpha))x^{\alpha-\beta}}{x^\alpha x^{\alpha-\beta}}} \\ &= \frac{2x^\alpha}{m(\alpha) + (1 - m(\alpha))x^{\alpha-\beta}}. \end{aligned}$$

It follows that  $x^{-\alpha}R(x) \rightarrow 2/m(\alpha)$  as  $x \rightarrow \infty$ .

As to the rate of convergence, we study the difference:

$$\begin{aligned} R(x) - \frac{2x^\alpha}{m(\alpha)} &= \frac{2x^\alpha}{m(\alpha) + (1 - m(\alpha))x^{\alpha-\beta}} - \frac{2x^\alpha}{m(\alpha)} \\ &= \frac{-2x^\alpha(1 - m(\alpha))x^{\alpha-\beta}}{m(\alpha)(m(\alpha) + (1 - m(\alpha))x^{\alpha-\beta})} \\ &= \frac{-2(1 - m(\alpha))}{m^2(\alpha)} - \frac{1}{m^2(\alpha)} \\ &= \frac{-3 + 2m(\alpha)}{m^2(\alpha)}. \end{aligned}$$

and then, as  $x \rightarrow \infty$ ,

$$\begin{aligned} x^{\beta-2\alpha} \left( R(x) - \frac{2x^\alpha}{m(\alpha)} \right) &\rightarrow \frac{-2(1 - m(\alpha))}{m^2(\alpha)} - \frac{1}{m^2(\alpha)} \\ &= \frac{-3 + 2m(\alpha)}{m^2(\alpha)}. \end{aligned}$$

- (3) Let  $F_X(x) = 1 - e^{-x}$ ,  $x \geq 0$  and  $\alpha = 1$ . We have  $m(1) = 1$  and straightforward calculations show that

$$G_F(x) = \int_0^x \left(1 - \frac{t}{x}\right) e^{-t} dt = 1 - \frac{F(x)}{x}.$$

Hence  $\overline{G}_F(x) = F(x)/x$ . For the renewal function, we find, by considering (2.2),

$$\begin{aligned} R(x) &= \frac{2}{\overline{G}_F(x)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)} = \frac{2x}{F(x)} - \frac{x^2\overline{F}(x)}{F^2(x)} \\ &= \frac{x}{F(x)} \left( 2 - \frac{x\overline{F}(x)}{F(x)} \right). \end{aligned}$$

It follows that  $R(x)/x \rightarrow 2$  as  $x \rightarrow \infty$ .

To find the rate of convergence, we proceed as follows. We have

$$\begin{aligned} R(x) - 2x &= 2x \frac{\overline{F}(x)}{F(x)} - \frac{x^2\overline{F}(x)}{F^2(x)} \\ &= \frac{x^2\overline{F}(x)}{F(x)} \left( \frac{2}{x} - \frac{1}{F(x)} \right) \sim -x^2e^{-x}. \end{aligned}$$

We find that  $R(x)/x - 2 \sim -x\bar{F}(x)$ . We also have, as  $x \rightarrow \infty$ ,

$$\begin{aligned} x \left( \frac{R(x) - 2x}{x^2\bar{F}(x)} + 1 \right) &= \frac{x}{F(x)} \left( \frac{2}{x} - \frac{1}{F(x)} \right) + 1 \\ &= \frac{2}{F(x)} - \frac{x\bar{F}(x)(1 + F(x))}{F^2(x)} \\ &\rightarrow 2. \end{aligned}$$

- (4) Let us consider distribution with the lack of memory property (for details see [Jasiulis-Góldyn et al. \(2020\)](#)) in the Kendall convolution algebra with  $F_X(x) = x^\alpha$ ,  $0 < x \leq 1$ . Then  $m(\alpha) = \frac{1}{2}$ ,

$$H_\alpha(x) = \int_0^x y^\alpha dF(y) = \frac{x^\alpha}{2} \mathbf{1}_{(0,1]}(x) + \frac{1}{2} \mathbf{1}_{[1,\infty)}(x)$$

and

$$G_F(x) = \frac{x^\alpha}{2} \mathbf{1}_{(0,1]}(x) + \left( 1 - \frac{1}{2x^\alpha} \right) \mathbf{1}_{[1,\infty)}(x).$$

Consequently

$$x^{-\alpha}R(x) = 4$$

and the rate of convergence is equal 0.

- (5) Let  $F_X(x) = (1 + x^{-\alpha})e^{-x^{-\alpha}}$ ,  $x \geq 0$ , where  $\alpha > 0$ . Notice that it is the limit distribution (see [Arendarczyk et al. \(2019\)](#)) in the Kendall convolution algebra corresponding to normal distribution in the classical case. Then the Williamson transform and truncated  $\alpha$ -moment for  $X$  are given by

$$G_F(x) = H(x)_\alpha = \exp\{-x^{-\alpha}\} \mathbf{1}_{(0,\infty)}(x).$$

Hence,  $m(\alpha) = 1$  and  $x^{-\alpha}R(x) \rightarrow 2$  as  $x \rightarrow \infty$ . Notice that we have

$$\begin{aligned} \bar{F}(x) &= 1 + (1 + x^{-\alpha})(\bar{G}_F(x) - 1) \\ &= (1 + x^{-\alpha})\bar{G}_F(x) - x^{-\alpha}. \end{aligned}$$

Notice that  $\bar{G}_F(x) = 1 - \exp\{-x^{-\alpha}\} \sim x^{-\alpha}$ . Using  $1 - \exp\{-z\} = z - z^2/2(1 + o(1))$  as  $z \rightarrow 0$ , we have

$$\bar{G}_F(x) = x^{-\alpha} - \frac{1}{2}x^{-2\alpha}(1 + o(1)),$$

and

$$1 - x^\alpha \bar{G}_F(x) = \frac{1}{2}x^{-\alpha}(1 + o(1))$$

Returning to  $R(x)$ , we have, by considering (2.2),

$$\begin{aligned} R(x) &= \frac{2}{\bar{G}_F(x)} - \frac{(1 + x^{-\alpha})\bar{G}(x) - x^{-\alpha}}{\bar{G}^2(x)} \\ &= \frac{1}{\bar{G}_F(x)} - \frac{1}{x^\alpha \bar{G}_F(x)} + \frac{1}{x^\alpha \bar{G}^2(x)} \end{aligned}$$

Hence

$$\begin{aligned} x^{-\alpha}R(x) - 2 &= \left( \frac{1}{x^\alpha \overline{G}_F(x)} - 1 \right) + \left( \frac{1}{x^{2\alpha} \overline{G}^2(x)} - 1 \right) - \frac{1}{x^{2\alpha} \overline{G}_F(x)} \\ &= \left( \frac{1}{x^\alpha \overline{G}_F(x)} - 1 \right) \left( \frac{1}{x^\alpha \overline{G}(x)} + 2 \right) - \frac{1}{x^{2\alpha} \overline{G}_F(x)} \\ &= \frac{1}{x^\alpha \overline{G}_F(x)} (1 - x^\alpha \overline{G}_F(x)) \left( 2 + \frac{1}{x^\alpha \overline{G}_F(x)} \right) - \frac{1}{x^{2\alpha} \overline{G}_F(x)} \end{aligned}$$

and

$$\begin{aligned} x^\alpha(x^{-\alpha}R(x) - 2) &= \frac{1}{x^\alpha \overline{G}_F(x)} x^\alpha (1 - x^\alpha \overline{G}_F(x)) \left( 2 + \frac{1}{x^\alpha \overline{G}_F(x)} \right) - \frac{1}{x^\alpha \overline{G}_F(x)} \\ &\rightarrow \frac{1}{2} \times 3 - 1 = \frac{1}{2} \end{aligned}$$

### 3. Rate of convergence in the renewal theorem

The goal of this section will be to prove the following results:

**Theorem 3.1.** *Let  $R(x)$  be the Kendall renewal function defined by (2.1). If  $\overline{F}(x) \in RV_{-\beta}$ ,  $\beta > \alpha$ , then*

$$\lim_{x \rightarrow \infty} \frac{1}{x^{2\alpha} \overline{F}(x)} \left( R(x) - \frac{2}{m(\alpha)} x^\alpha \right) = \frac{3\alpha - \beta}{(\beta - \alpha)m^2(\alpha)},$$

and

$$\lim_{x \rightarrow \infty} \frac{1}{x^{3\alpha} \overline{F}^2(x)} \left( R(x) - \frac{2x^\alpha}{m(\alpha)} - \frac{2\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} + \frac{x^{2\alpha} \overline{F}(x)}{m^2(\alpha)} \right) = \frac{2\alpha(2\alpha - \beta)}{m^3(\alpha)(\beta - \alpha)^2}.$$

and

**Theorem 3.2.** *Let  $R(x)$  be the Kendall renewal function defined by (2.1). If  $\overline{F}(x) \in \Gamma(g)$ , then*

$$\lim_{x \rightarrow \infty} \frac{1}{x^\alpha \overline{F}(x)} \left( \frac{2}{m(\alpha)} - x^{-\alpha} R(x) \right) = \frac{1}{m^2(\alpha)},$$

and

$$\frac{1}{x^\alpha \overline{F}(x)} \left( x^{-\alpha} R(x) - \frac{2}{m(\alpha)} \right) + \frac{1}{m^2(\alpha)} \sim \frac{g(x)}{x} \frac{2\alpha}{m^2(\alpha)}.$$

3.1. *A relationship of the Kendall renewal function.* To study the rate of convergence in the renewal theorem (2.3), we start from (2.2). We have  $x^{-\alpha}m(\alpha) - \overline{G}_F(x) = \alpha x^{-\alpha} \overline{W}_\alpha(x)$ , cf. Proposition 2.5. Now we write

$$\begin{aligned} &R(x) - \frac{2}{x^{-\alpha}m(\alpha)} + \frac{\overline{F}(x)}{(x^{-\alpha}m(\alpha))^2} \\ &= 2 \left( \frac{1}{\overline{G}_F(x)} - \frac{1}{x^{-\alpha}m(\alpha)} \right) - \overline{F}(x) \left( \frac{1}{\overline{G}_F^2(x)} - \frac{1}{(x^{-\alpha}m(\alpha))^2} \right) \\ &= 2I - \overline{F}(x) II. \end{aligned} \tag{3.1}$$

We consider the two terms in (3.1) separately. For the first term we have

$$I = \frac{\alpha \overline{W}_\alpha(x)}{m(\alpha)} \frac{1}{\overline{G}_F(x)}.$$

Using the same formula again, we also find that

$$I - \frac{\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} = \frac{\alpha \overline{W}_\alpha(x)}{m(\alpha)} \times I = \frac{\alpha^2 \overline{W}_\alpha^2(x)}{m^2(\alpha) \overline{G}_F(x)}.$$

Since  $x^\alpha \overline{G}_F(x) \rightarrow m(\alpha)$  as  $x \rightarrow \infty$ , it follows that

$$I \sim \frac{\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)},$$

and also that

$$I - \frac{\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} \sim \frac{\alpha^2 x^\alpha \overline{W}_\alpha^2(x)}{m^3(\alpha)}. \tag{3.2}$$

Now we consider the second part. We have

$$\begin{aligned} II &= I \times \left( \frac{1}{\overline{G}_F(x)} + \frac{1}{x^{-\alpha} m(\alpha)} \right) \\ &= \frac{\alpha \overline{W}_\alpha(x)}{m(\alpha)} \frac{x^{2\alpha}}{x^\alpha \overline{G}_F(x)} \left( \frac{1}{x^\alpha \overline{G}_F(x)} + \frac{1}{m(\alpha)} \right). \end{aligned}$$

Since  $x^\alpha \overline{G}_F(x) \rightarrow m(\alpha)$  as  $x \rightarrow \infty$ , it follows that

$$II \sim \frac{2\alpha}{m^3(\alpha)} x^{2\alpha} \overline{W}_\alpha(x). \tag{3.3}$$

Combining (3.1),(3.2) and (3.3), we find

$$\begin{aligned} R(x) &- \frac{2x^\alpha}{m(\alpha)} - \frac{2\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} + \frac{x^{2\alpha} \overline{F}(x)}{m^2(\alpha)} \\ &= 2 \left( I - \frac{\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} \right) - \overline{F}(x) \times II \\ &= (1 + o(1)) \frac{2\alpha^2}{m^3(\alpha)} x^\alpha \overline{W}_\alpha^2(x) - (1 + o(1)) \frac{2\alpha}{m^3(\alpha)} x^{2\alpha} \overline{W}_\alpha(x) \overline{F}(x). \end{aligned} \tag{3.4}$$

Now we analyse (3.4) further and we consider two important cases.

3.2. *The case of regular variation.* Let us prove Theorem 3.1.

First suppose that  $\overline{F}(x) \in RV_{-\beta}$ ,  $\beta > \alpha$ . In this case, applying Karamata’s theorem gives, see Karamata (1930) and e.g. de Haan (1970),

$$\overline{W}_\alpha(x) = \int_x^\infty y^{\alpha-1} \overline{F}(y) dy \sim \frac{1}{\beta - \alpha} x^\alpha \overline{F}(x).$$

Using (3.4), we have

$$\begin{aligned} &\frac{1}{x^{3\alpha} \overline{F}^2(x)} \left( R(x) - \frac{2x^\alpha}{m(\alpha)} - \frac{2\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} + \frac{x^{2\alpha} \overline{F}(x)}{m^2(\alpha)} \right) \\ &\rightarrow \frac{2\alpha^2}{(\beta - \alpha)^2 m^3(\alpha)} - \frac{2\alpha}{(\beta - \alpha) m^3(\alpha)} \\ &= \frac{2\alpha(2\alpha - \beta)}{m^3(\alpha)(\beta - \alpha)^2}. \end{aligned}$$

Note that the formula above, implies that, as  $x \rightarrow \infty$ ,

$$R(x) - \frac{2}{m(\alpha)} x^\alpha - \frac{2\alpha x^\alpha \overline{W}_\alpha(x)}{m^2(\alpha)} + \frac{1}{m^2(\alpha)} x^{2\alpha} \overline{F}(x) = o(1) x^{2\alpha} \overline{F}(x).$$

Using  $\bar{W}_\alpha(x) \sim \frac{1}{\beta-\alpha}x^\alpha\bar{F}(x)$ , we then find that, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \frac{1}{x^{2\alpha}\bar{F}(x)} \left( R(x) - \frac{2x^\alpha}{m(\alpha)} \right) &\rightarrow \frac{2\alpha}{(\beta-\alpha)m^2(\alpha)} - \frac{1}{m^2(\alpha)} \\ &= \frac{3\alpha - \beta}{(\beta-\alpha)m^2(\alpha)}. \end{aligned}$$

We see that  $x^{-\alpha}R(x) \rightarrow 2/m(\alpha)$  as  $x \rightarrow \infty$  with a rate of convergence determined by  $x^\alpha\bar{F}(x)$ .

3.3. *The case of the class  $\Gamma$ .* Let us prove Theorem 3.2.

In the case where  $\bar{F}(x) \in \Gamma(g)$ , we have  $\bar{F}(x+yg(x))/\bar{F}(x) \rightarrow e^{-y}$  as  $x \rightarrow \infty$ , where  $g(x) \in SN$  is an auxiliary function satisfying  $g(x)/x \rightarrow 0$  as  $x \rightarrow \infty$ . Clearly also  $x^{\alpha-1}\bar{F}(x) \in \Gamma(g)$ , cf. [Bingham et al. \(1989\)](#) (Chapter 3.10) or [Geluk and de Haan \(1987\)](#) (Chapter I) and we have the following property:

$$\bar{W}_\alpha(x) = \int_x^\infty y^{\alpha-1}\bar{F}(y)dy \sim \frac{g(x)}{x}x^\alpha\bar{F}(x). \quad (3.5)$$

Using (3.4) and (3.5) leads to

$$\begin{aligned} &R(x) - \frac{2x^\alpha}{m(\alpha)} - \frac{2\alpha x^\alpha \bar{W}_\alpha(x)}{m^2(\alpha)} + \frac{x^{2\alpha}\bar{F}(x)}{m^2(\alpha)} \\ &= (1+o(1))\frac{g^2(x)2\alpha^2x^{3\alpha}\bar{F}^2(x)}{x^2m^3(\alpha)} - (1+o(1))\frac{g(x)2\alpha}{xm^3(\alpha)}x^{3\alpha}\bar{F}^2(x) \\ &\sim -\frac{g(x)}{x}\frac{2\alpha}{m^3(\alpha)}x^{3\alpha}\bar{F}^2(x). \end{aligned}$$

The last line follows because the first term is dominated by the second term.

Since  $x^\alpha\bar{F}(x) \rightarrow 0$  as  $x \rightarrow \infty$ , it follows that

$$R(x) - \frac{2x^\alpha}{m(\alpha)} + \frac{x^{2\alpha}\bar{F}(x)}{m^2(\alpha)} = \frac{2\alpha x^\alpha \bar{W}_\alpha(x)}{m^2(\alpha)} + o(1)\frac{g(x)}{x}x^{2\alpha}\bar{F}(x).$$

Using  $\bar{W}_\alpha(x) \sim g(x)x^\alpha\bar{F}(x)/x$ , we find that, as  $x \rightarrow \infty$ ,

$$\frac{x}{g(x)x^{2\alpha}\bar{F}(x)} \left( R(x) - \frac{2x^\alpha}{m(\alpha)} + \frac{x^{2\alpha}\bar{F}(x)}{m^2(\alpha)} \right) \rightarrow \frac{2\alpha}{m^2(\alpha)}.$$

Among others we see that, as  $x \rightarrow \infty$ ,

$$\frac{1}{x^\alpha\bar{F}(x)} \left( x^{-\alpha}R(x) - \frac{2}{m(\alpha)} \right) + \frac{1}{m^2(\alpha)} \sim \frac{g(x)}{x}\frac{2\alpha}{m^2(\alpha)}.$$

and also that, as  $x \rightarrow \infty$ ,

$$\frac{1}{x^\alpha\bar{F}(x)} \left( \frac{2}{m(\alpha)} - x^{-\alpha}R(x) \right) \rightarrow \frac{1}{m^2(\alpha)},$$

and that also here  $x^{-\alpha}R(x) \rightarrow 2/m(\alpha)$  as  $x \rightarrow \infty$  at a rate determined by  $x^\alpha\bar{F}(x)$ .

#### 4. Renewal theorems of Blackwell type

The Blackwell renewal theorem studies the asymptotic behaviour of difference  $R(x+y) - R(x)$ . In our case, we take  $y > 0$ , and using (2.2), we find that

$$\begin{aligned} R(x+y) - R(x) &= 2 \left( \frac{1}{\overline{G}_F(x+y)} - \frac{1}{\overline{G}_F(x)} \right) - \left( \frac{\overline{F}(x+y)}{\overline{G}_F^2(x+y)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)} \right) \\ &= 2I - II. \end{aligned}$$

We consider the two terms separately.

On the first term we have

$$I = \frac{1}{\overline{G}_F(x+y)} - \frac{1}{\overline{G}_F(x)} = \frac{G_F(x+y) - G_F(x)}{\overline{G}_F(x)\overline{G}_F(x+y)}.$$

Using the mean value theorem, this gives

$$\frac{1}{\overline{G}_F(x+y)} - \frac{1}{\overline{G}_F(x)} = \frac{G'_F(z)}{\overline{G}_F(x)\overline{G}_F(x+y)}y,$$

where  $x \leq z \leq x+y$ . Using  $F(z) = G_F(z) + \frac{z}{\alpha}G'_F(z)$ , we obtain that

$$\begin{aligned} I &= \frac{\alpha}{z} \frac{\overline{G}_F(z) - \overline{F}(z)}{\overline{G}_F(x)\overline{G}_F(x+y)}y \\ &= \frac{\alpha}{z} \frac{z^{-\alpha}H_\alpha(z)}{\overline{G}_F(x)\overline{G}_F(x+y)}y. \end{aligned}$$

Since  $m(\alpha) < \infty$ , we have  $x^\alpha \overline{G}_F(x) \rightarrow m(\alpha)$  and  $(x+y)^\alpha \overline{G}_F(x+y) \rightarrow m(\alpha)$  as  $x \rightarrow \infty$ , and  $H_\alpha(\infty) = m(\alpha)$ . Since  $x \sim z \sim x+y$ , we obtain that

$$I = \frac{\alpha}{z} \frac{z^{-\alpha}H_\alpha(z)}{\overline{G}_F(x)\overline{G}_F(x+y)}y \sim \frac{\alpha x^{\alpha-1}}{m(\alpha)}y.$$

Now we consider the second term. We have

$$\begin{aligned} II &= \frac{\overline{F}(x+y)}{\overline{G}_F^2(x+y)} - \frac{\overline{F}(x)}{\overline{G}_F^2(x)} \\ &= \frac{\overline{F}(x+y) - \overline{F}(x)}{\overline{G}_F^2(x+y)} + \overline{F}(x) \left( \frac{1}{\overline{G}_F^2(x+y)} - \frac{1}{\overline{G}_F^2(x)} \right) \\ &= II_A + \overline{F}(x)II_B. \end{aligned}$$

First consider  $II_B$ . We have

$$\begin{aligned} II_B &= \left( \frac{1}{\overline{G}_F(x+y)} - \frac{1}{\overline{G}_F(x)} \right) \times \left( \frac{1}{\overline{G}_F(x+y)} + \frac{1}{\overline{G}_F(x)} \right) \\ &= I \times \left( \frac{1}{\overline{G}_F(x+y)} + \frac{1}{\overline{G}_F(x)} \right). \end{aligned}$$

Using the results of above, we find that

$$II_B \sim \frac{2x^\alpha}{m(\alpha)} \frac{\alpha x^{\alpha-1}}{m(\alpha)}y,$$

and hence that

$$\overline{F}(x)II_B \sim 2x^\alpha \overline{F}(x) \frac{\alpha x^{\alpha-1}}{m^2(\alpha)}y = o(1)x^{\alpha-1}.$$

Now consider  $II_A$ . We have

$$\begin{aligned} II_A &= \frac{\overline{F}(x+y) - \overline{F}(x)}{\overline{G}_F^2(x+y)} \\ &\sim \frac{x^{2\alpha}(\overline{F}(x+y) - \overline{F}(x))}{m^2(\alpha)} \\ &= x^{\alpha-1} \frac{1}{m^2(\alpha)} x^{\alpha+1} (\overline{F}(x+y) - \overline{F}(x)). \end{aligned}$$

If  $x^{\alpha+1}(\overline{F}(x+y) - \overline{F}(x)) \rightarrow 0$  as  $x \rightarrow \infty$ , we obtain that  $II_A = o(1)x^{\alpha-1}$ . We conclude.

**Theorem 4.1.** *Let  $R(x)$  be the Kendall renewal function defined by (2.1). Suppose that  $m(\alpha) < \infty$  and that  $x^{\alpha+1}(\overline{F}(x+y) - \overline{F}(x)) \rightarrow 0$  as  $x \rightarrow \infty$ . Then*

$$\lim_{x \rightarrow \infty} \frac{1}{x^{\alpha-1}} (R(x+y) - R(x)) = \frac{2\alpha}{m(\alpha)} y.$$

*In particular, if  $m(1+\alpha) < \infty$ , the result holds.*

Again we consider two important cases.

4.1. *Regular variation of the density.* Assume that  $F$  has a density  $f(x)$  and that  $xf(x)/\overline{F}(x) \rightarrow \beta > \alpha$  as  $x \rightarrow \infty$ . In this case, we have  $\overline{F}(x) \in RV_{-\beta}$ , and

$$\overline{F}(x+y) - \overline{F}(x) = f'(z)y \sim \beta \frac{\overline{F}(z)}{z} y \sim \beta \frac{\overline{F}(x)}{x} y.$$

It follows that  $x^{\alpha+1}(\overline{F}(x+y) - \overline{F}(x)) \sim \beta y x^\alpha \overline{F}(x) \rightarrow 0$  as  $x \rightarrow \infty$ , and Theorem 4.1 applies.

4.2. *The Gamma class.* For  $\overline{F} \in \Gamma(g)$  we have  $H_\sigma(\infty) = m(\sigma) < \infty$  for all  $\sigma > 0$  and Theorem 4.1 applies.

## 5. Rates of convergence in the Blackwell result

In this section, rates of convergence of the renewal theorem and Blackwell's result are given. We begin providing relationships of the derivative of  $R(x)$ .

To obtain these rate of convergence results, we assume that  $F(x)$  has a density  $f(x)$ .

5.1. *The derivative of  $R(x)$ .*

**Lemma 5.1.** *Assume that  $F(x)$  has a density  $f(x)$ . Let  $R(x)$  be the Kendall renewal function defined by (2.1). We have*

$$R'(x) = \frac{2\alpha x^{\alpha-1}}{m(\alpha)} (1 + o(1)) + \frac{1}{m^2(\alpha)} x^{2\alpha} f(x) (1 + o(1)).$$

*Proof:* Using (2.2), we find

$$R'(x) = 2 \frac{G'_F(x)}{G_F^2(x)} + \frac{f(x)}{G_F^2(x)} - 2 \frac{\overline{F}(x) G'_F(x)}{G_F^3(x)}.$$

Using  $G'_F(x) = \alpha x^{-1}(F(x) - G_F(x)) = \alpha x^{-\alpha-1}H_\alpha(x)$ , it follows that

$$\begin{aligned} R'(x) &= 2\frac{\alpha x^{-\alpha-1}H_\alpha(x)}{\overline{G}_F^2(x)} + \frac{f(x)}{\overline{G}_F^2(x)} - 2\frac{\alpha \overline{F}(x)x^{-\alpha-1}H_\alpha(x)}{\overline{G}_F^3(x)} \\ &= \frac{2\alpha x^{\alpha-1}H_\alpha(x)}{x^{2\alpha}\overline{G}_F^2(x)} + \frac{x^{2\alpha}f(x)}{x^{2\alpha}\overline{G}_F^2(x)} - 2\frac{\alpha x^\alpha \overline{F}(x)x^{\alpha-1}H_\alpha(x)}{x^{3\alpha}\overline{G}_F^3(x)} \\ &= \frac{2\alpha x^{\alpha-1}}{m(\alpha)}(1 + o(1)) + \frac{1}{m^2(\alpha)}(1 + o(1))x^{2\alpha}f(x). \end{aligned}$$

This proves the result. □

Specializing to the two cases, we have the following result.

**Proposition 5.2.** *Assume that  $F(x)$  has a density  $f(x)$ . Let  $R(x)$  be the Kendall renewal function defined by (2.1).*

- (i) *If  $xf(x)/\overline{F}(x) \rightarrow \beta > \alpha$  as  $x \rightarrow \infty$ , then  $R'(x) \sim 2\alpha x^{\alpha-1}/m(\alpha)$ .*
- (ii) *If  $f \in \Gamma(g)$  and  $\liminf_{x \rightarrow \infty} x^{\sigma-\alpha-1}g(x) > 0$ , then  $R'(x) \sim 2\alpha x^{\alpha-1}/m(\alpha)$ .*

*Proof:* (i) In the first case we have  $x^{2\alpha}f(x) = O(1)x^{2\alpha-1}\overline{F}(x) = o(1)x^{\alpha-1}$  and hence

$$R'(x) = \frac{2\alpha x^{\alpha-1}}{m(\alpha)}(1 + o(1)) + o(1)x^{\alpha-1} \sim \frac{2\alpha x^{\alpha-1}}{m(\alpha)}.$$

(ii) If  $f \in \Gamma(g)$  we have  $\overline{F}(x) \sim f(x)g(x)$  and  $x^{2\alpha}f(x) = O(1)x^{2\alpha-1}\overline{F}(x)/g(x)$ . Note that in this case all moments  $m(\sigma)$  are finite. It follows that

$$x^{1-\alpha}x^{2\alpha}f(x) = O(1)\frac{x^\sigma \overline{F}(x)}{x^{\sigma-\alpha-1}g(x)}.$$

If  $\liminf_{x \rightarrow \infty} x^{\sigma-\alpha-1}g(x) > 0$ , it follows that  $x^{1-\alpha}x^{2\alpha}f(x) \rightarrow 0$  as  $x \rightarrow \infty$ . We conclude that  $x^{1-\alpha}R'(x) \rightarrow 2\alpha/m(\alpha)$  as  $x \rightarrow \infty$ . This proves the result. □

*Remark 5.3.* Note that  $R'(x) \sim 2\alpha x^{\alpha-1}/m(\alpha)$  implies that  $R(x+y) - R(x) \sim 2\alpha x^{\alpha-1}y/m(\alpha)$ .

5.2. *Rate of convergence in the renewal theorem.* Now we look for the rate of convergence in the previous Proposition 5.2. Using the formula for  $R'(x)$ , earlier we have shown that

$$R'(x) = 2\frac{\alpha x^{-\alpha-1}H_\alpha(x)}{\overline{G}_F^2(x)} + \frac{f(x)}{\overline{G}_F^2(x)} - 2\frac{\alpha \overline{F}(x)x^{-\alpha-1}H_\alpha(x)}{\overline{G}_F^3(x)}.$$

It follows that

$$\begin{aligned} R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} &= 2\frac{\alpha x^{\alpha-1}H_\alpha(x)}{x^{2\alpha}\overline{G}_F^2(x)} - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} + \frac{f(x)}{\overline{G}_F^2(x)} - 2\frac{\alpha \overline{F}(x)x^{-\alpha-1}H_\alpha(x)}{\overline{G}_F^3(x)} \\ &= \frac{2\alpha x^{\alpha-1}}{m(\alpha)} \left( \frac{m(\alpha)H_\alpha(x)}{x^{2\alpha}\overline{G}_F^2(x)} - 1 \right) + \frac{f(x)}{\overline{G}_F^2(x)} - 2\frac{\alpha \overline{F}(x)x^{-\alpha-1}H_\alpha(x)}{\overline{G}_F^3(x)} \\ &= I + II - III. \end{aligned}$$

First we consider the first term. Recall that earlier we have also shown that

$$\frac{1}{\overline{G}_F^2(x)} - \frac{1}{(x^{-\alpha}m(\alpha))^2} \sim \frac{2\alpha}{m^3(\alpha)}x^{2\alpha}\overline{W}_\alpha(x),$$

or equivalently that

$$\frac{m^2(\alpha)}{x^{2\alpha}\overline{G}_F^2(x)} - 1 \sim \frac{2\alpha}{m(\alpha)}\overline{W}_\alpha(x).$$

We then have

$$\begin{aligned} I &= \frac{2\alpha x^{\alpha-1}}{m(\alpha)} \left( \frac{H_\alpha(x)}{m(\alpha)} \left( \frac{m^2(\alpha)}{x^{2\alpha}\overline{G}_F^2(x)} - 1 \right) + \frac{H_\alpha(x)}{m(\alpha)} - 1 \right) \\ &= \frac{2\alpha x^{\alpha-1}}{m(\alpha)} \left( (1 + o(1)) \frac{2\alpha}{m(\alpha)} \overline{W}_\alpha(x) - \frac{\overline{H}_\alpha(x)}{m(\alpha)} \right). \end{aligned}$$

It follows that

$$\frac{1}{\overline{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} I = (1 + o(1)) \frac{2\alpha}{m(\alpha)} - \frac{1}{m(\alpha)} \frac{\overline{H}_\alpha(x)}{\overline{W}_\alpha(x)}.$$

Looking at the last term, note that we have

$$\frac{\overline{H}_\alpha(x)}{\overline{W}_\alpha(x)} = \frac{\alpha \overline{W}_\alpha(x) + x^\alpha \overline{F}(x)}{\overline{W}_\alpha(x)} = \alpha + \frac{x^\alpha \overline{F}(x)}{\overline{W}_\alpha(x)}.$$

Now we distinguish two cases as before.

5.2.1. *Regularly varying case.* If  $\overline{F}(x) \in RV_{-\beta}$ ,  $\beta > \alpha$ , we have  $\overline{W}_\alpha(x) \sim x^\alpha \overline{F}(x) / (\beta - \alpha)$  and hence

$$\frac{\overline{H}_\alpha(x)}{\overline{W}_\alpha(x)} = \alpha + \frac{x^\alpha \overline{F}(x)}{\overline{W}_\alpha(x)} \sim \alpha + \beta - \alpha = \beta.$$

We conclude that

$$\frac{1}{\overline{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} I \rightarrow \frac{2\alpha}{m(\alpha)} - \frac{\beta}{m(\alpha)} = \frac{2\alpha - \beta}{m(\alpha)},$$

or equivalently that, as  $x \rightarrow \infty$ ,

$$\frac{1}{x^{2\alpha-1}\overline{F}(x)} I \rightarrow \frac{2\alpha(2\alpha - \beta)}{(\beta - \alpha)m^2(\alpha)}.$$

Now we consider the second term:

$$II = \frac{x^{2\alpha} f(x)}{x^{2\alpha}\overline{G}_F^2(x)}.$$

Using  $xf(x) \sim \beta \overline{F}(x)$ , we find

$$II \sim \frac{x^{2\alpha} f(x)}{m^2(\alpha)} \sim \frac{\beta}{m^2(\alpha)} x^{2\alpha-1} \overline{F}(x).$$

For the third term, we find

$$\begin{aligned} III &= 2 \frac{\alpha x^\alpha \overline{F}(x) x^{\alpha-1} H_\alpha(x)}{x^{3\alpha} \overline{G}_F^3(x)} \\ &\sim \frac{2\alpha}{m^2(\alpha)} x^{2\alpha-1} \overline{F}(x). \end{aligned}$$

Everything together, we conclude that

$$\begin{aligned} \frac{1}{x^{2\alpha-1}\overline{F}(x)} \left( R'(x) - \frac{2x^{\alpha-1}}{m(\alpha)} \right) &\rightarrow \frac{2\alpha(2\alpha-\beta)}{m^2(\alpha)(\beta-\alpha)} + \frac{\beta}{m^2(\alpha)} - \frac{2\alpha}{m^2(\alpha)} \\ &= \frac{2\alpha(2\alpha-\beta)}{m^2(\alpha)(\beta-\alpha)} + \frac{\beta-2\alpha}{m^2(\alpha)} \\ &= \frac{(2\alpha-\beta)}{m^2(\alpha)} \left( \frac{2\alpha}{\beta-\alpha} - 1 \right) \\ &= \frac{(2\alpha-\beta)(3\alpha-\beta)}{m^2(\alpha)(\beta-\alpha)}. \end{aligned}$$

As a conclusion we have the following result.

**Theorem 5.4.** *Assume that  $F(x)$  has a density  $f(x)$ . Let  $R(x)$  be the Kendall renewal function defined by (2.1). If  $xf(x)/\overline{F}(x) \rightarrow \beta > \alpha$  as  $x \rightarrow \infty$ , then*

$$\lim_{x \rightarrow \infty} \frac{1}{x^{2\alpha-1}\overline{F}(x)} \left( R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} \right) = C,$$

where  $C = \frac{(2\alpha-\beta)(3\alpha-\beta)}{m^2(\alpha)(\beta-\alpha)}$ .

5.2.2. *The Gamma class case  $f \in \Gamma(g)$ .* We reconsider the terms *I*, *II* and *III* from above.

For the first term. We have proved that

$$\frac{1}{\overline{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} I = (1 + o(1)) \frac{2\alpha}{m(\alpha)} - \frac{1}{m(\alpha)} \frac{\overline{H}_\alpha(x)}{\overline{W}_\alpha(x)}.$$

Also we have  $\overline{W}_\alpha(x) \sim g(x)x^{\alpha-1}\overline{F}(x)$  and  $\overline{F}(x) \sim f(x)g(x)$ .

Clearly we have

$$\frac{\overline{H}_\alpha(x)}{\overline{W}_\alpha(x)} = \alpha + \frac{x^\alpha \overline{F}(x)}{\overline{W}_\alpha(x)} = \alpha + (1 + o(1)) \frac{x}{g(x)} \sim \frac{x}{g(x)},$$

since  $g(x)/x \rightarrow 0$ . It follows that, as  $x \rightarrow \infty$ ,

$$\frac{g(x)}{x\overline{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} I \rightarrow -\frac{1}{m(\alpha)}.$$

For the second term. We have

$$II - \frac{x^{2\alpha} f(x)}{m^2(\alpha)} = x^{2\alpha} f(x) \left( \frac{1}{x^{2\alpha} \overline{G}_F^2(x)} - \frac{1}{m^2(\alpha)} \right).$$

Earlier we proved that

$$\frac{1}{x^{2\alpha} \overline{G}_F^2(x)} - \frac{1}{m^2(\alpha)} \sim \frac{2\alpha}{m^3(\alpha)} \overline{W}_\alpha(x).$$

Now we find that

$$\begin{aligned} II - \frac{x^{2\alpha} f(x)}{m^2(\alpha)} &\sim \frac{2\alpha x^{2\alpha} f(x)}{m^3(\alpha)} \overline{W}_\alpha(x) \\ &\sim \frac{2\alpha x^{2\alpha} \overline{F}(x)}{g(x)m^3(\alpha)} \overline{W}_\alpha(x), \end{aligned}$$

and hence, as  $x \rightarrow \infty$ ,

$$\frac{g(x)}{x\overline{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} \left( II - \frac{x^{2\alpha} f(x)}{m^2(\alpha)} \right) \sim \frac{x^\alpha \overline{F}(x)}{m^2(\alpha)} \rightarrow 0.$$

For the third term. We have

$$\begin{aligned} III &= 2 \frac{\alpha x^\alpha \bar{F}(x) x^{\alpha-1} H_\alpha(x)}{x^{3\alpha} \bar{G}_F^3(x)} \\ &\sim 2 \frac{\alpha x^\alpha \bar{F}(x) x^{\alpha-1} m(\alpha)}{m^3(\alpha)} \\ &\sim 2 \frac{\alpha x^\alpha \bar{W}_\alpha(x)}{g(x) m^2(\alpha)} \\ &\sim 2 \frac{x}{g(x)} \bar{W}_\alpha(x) \frac{\alpha x^{\alpha-1}}{m^2(\alpha)}, \end{aligned}$$

and hence, as  $x \rightarrow \infty$ ,

$$\frac{g(x)}{x \bar{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} III \rightarrow \frac{1}{m(\alpha)}.$$

Everything together we find, as  $x \rightarrow \infty$ ,

$$\frac{g(x)}{x \bar{W}_\alpha(x)} \frac{m(\alpha)}{2\alpha x^{\alpha-1}} \left( R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} - \frac{x^{2\alpha} f(x)}{m^2(\alpha)} \right) \rightarrow -\frac{2}{m(\alpha)}.$$

Equivalently, using  $\bar{W}_\alpha(x) \sim g(x) x^{\alpha-1} \bar{F}(x)$ , we have

$$\frac{1}{x^{2\alpha-1} \bar{F}(x)} \left( R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} - \frac{x^{2\alpha} f(x)}{m^2(\alpha)} \right) \rightarrow -\frac{4\alpha}{m^2(\alpha)}.$$

Note that we have, as  $x \rightarrow \infty$ ,

$$\frac{x^{2\alpha-1} \bar{F}(x)}{x^{2\alpha} f(x)} = \frac{\bar{F}(x)}{x f(x)} \sim \frac{g(x)}{x} \rightarrow 0.$$

It follows that, as  $x \rightarrow \infty$ ,

$$\frac{m^2(\alpha)}{x^{2\alpha} f(x)} \left( R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} \right) \rightarrow 1.$$

We have proved the following result.

**Theorem 5.5.** *Assume that  $F(x)$  has a density  $f(x)$ . Let  $R(x)$  be the Kendall renewal function defined by (2.1). Suppose that  $f \in \Gamma(g)$ . Then*

$$\lim_{x \rightarrow \infty} \frac{1}{x^{2\alpha-1} \bar{F}(x)} \left( R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} - \frac{x^{2\alpha} f(x)}{m^2(\alpha)} \right) = -\frac{4\alpha}{m^2(\alpha)},$$

and

$$\lim_{x \rightarrow \infty} \frac{m^2(\alpha)}{x^{2\alpha} f(x)} \left( R'(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} \right) = 1.$$

**5.3. Rate of convergence in Blackwell's result.** Earlier, we have proved a Blackwell type of result, i.e.  $R(x+y) - R(x) \sim 2\alpha x^{\alpha-1} y / m(\alpha)$ . We want to use Theorems 5.4 and 5.5 to find a rate of convergence result here.

5.3.1. *Regularly varying case.* Clearly for  $y > 0$  we have  $R(x+y) - R(x) = \int_x^{x+y} R'(t) dt$ . Using Theorem 5.4, we see that

$$\int_x^{x+y} \left( R'(t) - \frac{2\alpha t^{\alpha-1}}{m(\alpha)} \right) dt \sim C \int_x^{x+y} t^{2\alpha-1} \bar{F}(t) dt \sim C x^{2\alpha-1} \bar{F}(x) y.$$

It also follows that

$$\begin{aligned} & R(x+y) - R(x) - \frac{2\alpha}{m(\alpha)} x^{\alpha-1} y \\ &= \int_x^{x+y} \left( R'(t) - \frac{2\alpha t^{\alpha-1}}{m(\alpha)} \right) dt + \frac{2\alpha}{m(\alpha)} \int_x^{x+y} (t^{\alpha-1} - x^{\alpha-1}) dt. \end{aligned}$$

Hence we find that

$$\begin{aligned} & R(x+y) - R(x) - \frac{2\alpha}{m(\alpha)} x^{\alpha-1} y \\ &= (1+o(1)) C x^{2\alpha-1} \bar{F}(x) y + (1+o(1)) \frac{2\alpha(\alpha-1)}{m(\alpha)} x^{\alpha-2} y^2, \end{aligned}$$

or

$$\begin{aligned} & x^{1-\alpha} (R(x+y) - R(x)) - \frac{2\alpha}{m(\alpha)} y \\ &= (1+o(1)) C x^\alpha \bar{F}(x) y + (1+o(1)) \frac{2\alpha(\alpha-1)}{m(\alpha)} x^{-1} y^2. \end{aligned}$$

If  $x^{1+\alpha} \bar{F}(x) \rightarrow 0$ , we find that

$$x \left( x^{1-\alpha} (R(x+y) - R(x)) - \frac{2\alpha}{m(\alpha)} y \right) \rightarrow \frac{2\alpha(\alpha-1)}{m(\alpha)} y^2.$$

This is the case when  $m(1+\alpha) < \infty$ .

If  $x^{1+\alpha} \bar{F}(x) \rightarrow D$ , where  $0 < D \leq \infty$ , we find that

$$\begin{aligned} & \frac{1}{x^\alpha \bar{F}(x)} \left( x^{1-\alpha} (R(x+y) - R(x)) - \frac{2\alpha}{m(\alpha)} y \right) \\ & \rightarrow C y + \frac{2\alpha(\alpha-1)}{D m(\alpha)} y^2. \end{aligned}$$

5.3.2. *The Gamma class case.* Using Theorem 5.5, we have

$$\begin{aligned} \int_x^{x+y} \left( R'(t) - \frac{2\alpha t^{\alpha-1}}{m(\alpha)} \right) dt & \sim \frac{1}{m^2(\alpha)} \int_x^{x+y} t^{2\alpha} f(t) \\ & \sim \frac{x^{2\alpha}}{m^2(\alpha)} (F(x+y) - F(x)). \end{aligned}$$

It follows that

$$\begin{aligned} R(x+y) - R(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} y &= (1+o(1)) \frac{x^{2\alpha}}{m^2(\alpha)} (F(x+y) - F(x)) \\ & \quad + (1+o(1)) \frac{2\alpha(\alpha-1)}{m(\alpha)} x^{\alpha-2} y^2, \end{aligned}$$

and, as  $x \rightarrow \infty$ ,

$$\begin{aligned} & x^{2-\alpha} \left( R(x+y) - R(x) - \frac{2\alpha x^{\alpha-1}}{m(\alpha)} y \right) \\ &= (1+o(1)) \frac{x^{2+\alpha}}{m^2(\alpha)} (F(x+y) - F(x)) + (1+o(1)) \frac{2\alpha(\alpha-1)}{m(\alpha)} y^2 \\ &\rightarrow \frac{2\alpha(\alpha-1)}{m(\alpha)} y^2, \end{aligned}$$

since all moments are finite it follows that  $x^{2+\alpha}(F(x+y) - F(x)) \rightarrow 0$  as  $x \rightarrow \infty$ .

## 6. Concluding remarks

(1) We can also study weighted renewal functions of the form

$$WR(x) = \sum_{n=0}^{\infty} a_n F^{\boxtimes n}(x),$$

where  $(a_n)$  is a sequence of positive numbers. In this case the  $G$ -transform of  $WR(x)$  is given by

$$G_{WR}(x) = \sum_{n=0}^{\infty} a_n G_F^n(x) = A(G_F(x)),$$

where  $A(z) = \sum_{n=0}^{\infty} a_n z^n$  is the generating function of the sequence of weights.

- (2) Recall that  $F^{\boxtimes n}(x)$  is the d.f. of the Kendall sum  $S_{\boxtimes n}$  of independent and identically distributed random elements. If  $(a_n)$  is the probability density function of a discrete random variable  $N$ , then  $WR(x)$  is the d.f. of the random sum  $S_{\boxtimes N}$ .
- (3) The Williamson transform can be written as  $G_F(x) = \int_0^x P(Z \geq t/x) dF(t) = P(X/Z \leq x)$ , where  $Z$  denotes a positive r.v. with d.f.  $P(Z \leq x) = x^\alpha, 0 \leq x \leq 1$ . It could be interesting to study a transform where we replace this d.f. of  $Z$  by another d.f.

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