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Extreme values of critical and subcritical branching stable processes with positive jumps

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Abstract. We consider a branching stable process with positive jumps, i.e. a continuous-time branching process in which the particles evolve independently as stable Lévy processes with positive jumps. Assuming the branching mechanism is critical or subcritical, we compute the asymptotics of the maximum location ever reached by a particle of the process.

1. Statement of the main result

1.1. *Introduction.* We consider a one-dimensional branching stable Lévy process. It is a continuous-time particle system in which individuals move according to independent α-stable Lévy processes, and split at exponential times into a random number of children.

More precisely, the process starts at time $t = 0$ with a single particle located at the origin.

(1) When not branching, each particle moves independently as a strictly α-stable Lévy process L with positive jumps. We refer to Bertoin (1996, Chapter VIII) for an overview of such process. In particular, L may be parameterized by a scaling parameter α and a skewness parameter β (see Remark 1.2 below), and the existence of positive jumps implies the conditions :

$$\alpha \in (0,1) \cup (1,2) \text{ and } \beta \in (-1,1] \qquad \text{or} \qquad \alpha = 1 \text{ and } \beta = 0.$$

(2) Each particle lives for an exponentially distributed time of parameter 1, independently of the others. When it dies, it splits into a random number of children with distribution $\boldsymbol{p} = (p_n)_{n\geq 0}$. We assume that the distribution $\boldsymbol{p}$ is non trivial (i.e. $p_1 \neq 1$) and admits moments of order at least 3, i.e. $\mathbb{E}[\boldsymbol{p}^3] < +\infty$.

Such process may be constructed by first running a standard continuous-time Markov branching process Z (see for instance Athreya and Ney (1972, Chapter III)), and then running independent

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α-stable Lévy processes $(L^{(i)})$ along the edges. With this notation, for each $t > 0$, the number of particles alive at time t is thus given by $Z(t)$, and their locations by

$$\left\{L_t^{(1)}, \dots, L_t^{(Z(t))}\right\}. \tag{1.1}$$

It is classic that when $\mathbb{E}[\boldsymbol{p}] \leq 1$, the process will go extinct in finite time with probability one. As a consequence, one may define the overall maximum $\mathbf{M}_{\alpha,\beta}$ ever attained by one of the particle. The main result of the paper is the computation of the asymptotics of its tail distribution :

$$u(x) = \mathbb{P}\left(\mathbf{M}_{\alpha,\beta} \geq x\right).$$

Theorem 1.1. *Let $\kappa_{\alpha,\beta} > 0$ be the constant such that :*

$$\mathbb{P}\left(L_1 \geq x\right) \underset{x\to+\infty}{\sim} \kappa_{\alpha,\beta}\, x^{-\alpha}.$$

i) Assume that $\mathbb{E}[\boldsymbol{p}] < 1$. The asymptotics of $\mathbf{M}_{\alpha,\beta}$ is given by

$$\mathbb{P}(\mathbf{M}_{\alpha,\beta} \geq x) \underset{x\to+\infty}{\sim} \frac{\kappa_{\alpha,\beta}}{1 - \mathbb{E}[\boldsymbol{p}]} x^{-\alpha}.$$

ii) Assume that $\mathbb{E}[\boldsymbol{p}] = 1$. The asymptotics of $\mathbf{M}_{\alpha,\beta}$ is given by

$$\mathbb{P}(\mathbf{M}_{\alpha,\beta} \geq x) \underset{x\to+\infty}{\sim} \sqrt{\frac{2\kappa_{\alpha,\beta}}{\sigma^2}} x^{-\alpha/2}$$

where

$$\sigma^2 = \mathrm{Var}(\boldsymbol{p}) = \sum_{n=2}^{+\infty} n(n-1)p_n.$$

Remark 1.2. The constant $\kappa_{\alpha,\beta}$ may be computed explicitly, but depends on the normalization chosen for L. For instance, following Sato (2013, p.88), if the characteristic exponent of L is given by :

$$\ln\left(\mathbb{E}\left[e^{\mathbf{i}\lambda L_1}\right]\right) = \begin{cases} -c_{\alpha,\beta}|\lambda|^\alpha \left(1 - \mathbf{i}\beta \tan\left(\frac{\pi\alpha}{2}\right) \mathrm{sgn}(\lambda)\right) & \text{for } \alpha \neq 1 \\ -|\lambda| & \text{for } \alpha = 1 \end{cases} \tag{1.2}$$

with $c_{\alpha,\beta} = \cos\left(\frac{\pi\beta}{2} \min(\alpha,\ 2-\alpha)\right)$, then,

$$\kappa_{\alpha,\beta} = \begin{cases} \frac{1}{\pi}\Gamma(\alpha) \sin\left(\frac{\pi\alpha}{2}(1+\beta)\right) & \text{if } \alpha < 1 \\ \frac{1}{\pi} & \text{if } \alpha = 1 \\ \frac{1}{\pi}\Gamma(\alpha) \sin\left(\frac{\pi}{2}(\alpha + \alpha\beta - 2\beta)\right) & \text{if } \alpha > 1 \end{cases}$$

The occurrence of σ^2 is a classic feature of such asymptotics, and was already observed by Sawyer and Fleischman (1979) in the case of Branching Brownian motion, or by Lalley and Shao (2015) in the case of Branching random walks.

In the symmetric case (i.e. $\beta = 0$) and when $p_0 = p_2 = \frac{1}{2}$, Theorem 1.1 was first obtained by Lalley and Shao (2016), who prove that

$$\mathbb{P}\left(\mathbf{M}_{\alpha,0} \geq x\right) \underset{x\to+\infty}{\sim} \sqrt{\frac{2}{\alpha}} x^{-\alpha/2} \tag{1.3}$$

when choosing the normalization

$$\mathbb{E}\left[e^{\mathbf{i}\lambda L_1}\right] = \exp\left(-\int_{\mathbb{R}} (1 - e^{\mathbf{i}\lambda x}) \frac{dx}{|x|^{\alpha+1}}\right) = \exp\left(-\frac{\pi}{\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)} |\lambda|^\alpha\right).$$

In this case, from Remark 1.2 and the recurrence formula for the Gamma function, the constant $\kappa_{\alpha,0}$ equals

$$\kappa_{\alpha,0} = \frac{\pi}{\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)} \times \frac{1}{\pi}\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right) = \frac{1}{\alpha}$$

hence (1.3) agrees with Theorem 1.1 since $\sigma^2 = 1$ for this binary branching mechanism.

The starting point of Lalley and Shao (2016) was to show that u is the solution of a pseudo-differential equation involving the generator of the symmetric stable Lévy process. This in turn allows to obtain a Feynman-Kac representation of u, and the authors then deduce the asymptotics of Theorem 1.1 after a careful analysis of the jumps of the underlying stable Lévy process. We shall propose here another approach and rather work with an integral equation.

1.2. *An integral equation for u.* We start by writing down the integral equation satisfied by the function u. Let us denote by

$$S_t = \sup_{s\in[0,t]} L_s$$

the running supremum of the stable Lévy process L, and let $\mathbf{e}$ be a standard exponential random variable of parameter 1, independent from L.

Lemma 1.3. *The function u is a solution of the integral equation :*

$$u(x) = \mathbb{P}\left(S_{\mathbf{e}} \geq x\right) + \mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} u(x-L_{\mathbf{e}})\right] - \Phi_0(x) + \Phi_R(x) \tag{1.4}$$

where the main term Φ_0 is given by

$$\Phi_0(x) = (1-\mathbb{E}[\boldsymbol{p}])\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} u(x-L_{\mathbf{e}})\right] + \frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2 - \boldsymbol{p}\right]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} u^2(x-L_{\mathbf{e}})\right]$$

and the remainder Φ_R satisfies the bounds

$$0 \leq \Phi_R(x) \leq \mathbb{E}[\boldsymbol{p}^3]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} u^3(x-L_{\mathbf{e}})\right]. \tag{1.5}$$

Proof: We start by applying the Markov property at the first branching event :

$$\mathbb{P}(\mathbf{M}_{\alpha,\beta} < x) = p_0\mathbb{P}\left(S_{\mathbf{e}} < x\right) + \sum_{n=1}^{+\infty} p_n\,\mathbb{P}\left(S_{\mathbf{e}} < x,\ L_{\mathbf{e}} + \mathbf{M}_{\alpha,\beta}^{(1)} < x, \dots, L_{\mathbf{e}} + \mathbf{M}_{\alpha,\beta}^{(n)} < x\right)$$

where the random variables $(\mathbf{M}_{\alpha,\beta}^{(n)})_{n\in\mathbb{Z}_+}$ are independent copies of $\mathbf{M}_{\alpha,\beta}$, which are also independent of the pair $(L_{\mathbf{e}}, S_{\mathbf{e}})$. As a consequence, we obtain the integral equation :

$$1-u(x) = p_0\mathbb{P}\left(S_{\mathbf{e}} < x\right) + \sum_{n=1}^{+\infty} p_n\,\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}\left(1-u(x-L_{\mathbf{e}})\right)^n\right]. \tag{1.6}$$

Plugging the Taylor expansion with integral remainder

$$(1-u)^n = 1 - nu + \frac{n(n-1)}{2}u^2 - \frac{n(n-1)(n-2)}{6}u^3\int_0^1 (1-ut)^{n-3}(1-t)^2 dt$$

in (1.6), we deduce that

$$u(x) = \mathbb{P}\left(S_{\mathbf{e}} \geq x\right) + \mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} u(x-L_{\mathbf{e}})\right] - \Phi_0(x) + \Phi_R(x)$$

where the remainder Φ_R equals :

$$\Phi_R(x) = \sum_{n\geq 3} p_n \frac{n(n-1)(n-2)}{6}\int_0^1 \mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} u^3(x-L_{\mathbf{e}})(1-u(x-L_{\mathbf{e}})t)^{n-3}\right](1-t)^2 dt.$$

Since $0 \leq u(x) \leq 1$ for any $x \geq 0$, the upper bound for Φ_R is obtained by bounding the term to the power $n-3$ by 1.

□

Remark 1.4. Several terms in the equation (1.4) satisfied by u look like convolutions products. This will lead us to work with Laplace transforms and we thus set for a positive and bounded function $f : [0,+\infty) \to [0,+\infty)$:

$$\mathcal{L}[f](\lambda) = \int_0^{+\infty} e^{-\lambda x} f(x)dx.$$

We shall repeatedly use in the following the standard Karamata's Tauberian theorem (see for instance Korevaar (2004, Theorem 8.1)) which states that for $\gamma \geq 0$:

$$\mathcal{L}[f](\lambda) \underset{\lambda\to 0}{\sim} \frac{1}{\lambda^\gamma} h\left(\frac{1}{\lambda}\right) \quad \Longleftrightarrow \quad \int_0^x f(z)dz \underset{x\to+\infty}{\sim} \frac{1}{\Gamma(1+\gamma)} x^\gamma h(x), \tag{1.7}$$

where h is a slowly varying function.

Remark 1.5. It may be noted that Equation (1.4) involves the distributions of $L_{\mathbf{e}}$ and $S_{\mathbf{e}}$. The key observation is the following equivalence of asymptotics for strictly α-stable Lévy processes with positive jumps, see Bertoin (1996, p.221) :

$$\mathbb{P}\left(L_1 \geq x\right) \underset{x\to+\infty}{\sim} \mathbb{P}\left(S_1 \geq x\right) \underset{x\to+\infty}{\sim} \kappa_{\alpha,\beta}\, x^{-\alpha}. \tag{1.8}$$

We shall indeed prove, through Laplace transforms, that as $x \to +\infty$:

$$\mathbb{P}\left(L_1 \geq x\right) \leq (1-\mathbb{E}[\boldsymbol{p}])u(x) + \frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2 - \boldsymbol{p}\right] u^2(x) \leq \mathbb{P}\left(S_1 \geq x\right)$$

and the result will follow from the equivalence of both asymptotics.

The remainder of the paper is devoted to the proof of Theorem 1.1 : Section 2 is dedicated to the case $\alpha \in (0,1]$ and Section 3 to the case $\alpha \in (1,2)$. The general idea of the proof is the same in both cases, and is composed of three steps. We shall first write down some general inequalities involving Laplace transforms, then apply these inequalities to Equation (1.4) and finally pass to the limit and apply Karamata's Tauberian theorem. The main difference between both cases is the existence of the first moment of $L_{\mathbf{e}}$ when $\alpha \in (1,2)$. This will require us to make some extra computations, in order to remove the "first order" terms.

2. The case $0 < \alpha \leq 1$

2.1. *Preliminary lemma.* We start by writing some general bounds for the Laplace transform of the terms appearing in Equation (1.4). In the forthcoming proofs, we will frequently use the case $f = u$ and $f = u^2$. To simplify the notation, we set :

$$\eta_\alpha(\lambda) = \begin{cases} \Gamma(1-\alpha)\lambda^{\alpha-1} & \text{if } \alpha < 1 \\ -\ln(\lambda) & \text{if } \alpha = 1 \end{cases}$$

and $L_{\mathbf{e}}^+ = \max(0, L_{\mathbf{e}})$. Note that using (1.8), the Tauberian theorem and the scaling property to remove the independent exponential random variable $\mathbf{e}$, we have the asymptotics

$$\frac{1-\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]}{\lambda} \underset{\lambda\downarrow 0}{\sim} \frac{1-\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}{\lambda} \underset{\lambda\downarrow 0}{\sim} \kappa_{\alpha,\beta}\, \eta_\alpha(\lambda). \tag{2.1}$$

Lemma 2.1. *Assume that $\alpha \in (0,1]$ and let $f : [0,+\infty) \to [0,+\infty)$ be a positive and decreasing function. The following inequalities hold :*

i) Upper bound :

$$\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} f(x-L_{\mathbf{e}})\right] dx \leq \mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right] \mathcal{L}[f](\lambda)$$

ii) Lower bound :

$$\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx$$
$$\geq \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[f](\lambda)+f(0)\frac{\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]-\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}{\lambda}-\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z}f(z)dz\right].$$

iii) Assume that $\lim_{x\to+\infty} f(x)=0$*, then*

$$\lim_{\lambda\downarrow 0}\frac{1}{\eta_\alpha(\lambda)}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z}f(z)dz\right]=0.$$

Proof: The upper bound $i)$ is a direct consequence of the a.s. inequality $L_{\mathbf{e}}\leq S_{\mathbf{e}}$. Indeed, since f is positive and decreasing, the Laplace transform of the convolution product yields

$$\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx\leq\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}f(x-S_{\mathbf{e}})\right]dx=\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]\mathcal{L}[f](\lambda).$$

For the lower bound $ii)$, still using that f is decreasing and $L_{\mathbf{e}}\leq S_{\mathbf{e}}$ a.s.,

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx\\
&\quad=\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[(1_{\{S_{\mathbf{e}}<x\}}-1_{\{L_{\mathbf{e}}<x\}})f(x-L_{\mathbf{e}})\right]dx+\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx\\
&\quad\geq f(0)\int_0^{+\infty} e^{-\lambda x}\left(\mathbb{P}\left(S_{\mathbf{e}}<x\right)-\mathbb{P}\left(L_{\mathbf{e}}<x\right)\right)dx+\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx\\
&\quad=f(0)\frac{\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]-\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}{\lambda}+\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx.
\end{aligned}$$

We next decompose the remaining integral according as whether $L_{\mathbf{e}}<0$ or $L_{\mathbf{e}}\geq 0$:

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx\\
&\qquad=\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}}1_{\{L_{\mathbf{e}}\geq 0\}}\right]\mathcal{L}[f](\lambda)+\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}f(x-L_{\mathbf{e}})\right]dx.
\end{aligned}$$

Then, the Fubini-Tonelli theorem and a change of variable in the last integral yields :

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}f(x-L_{\mathbf{e}})\right]dx&=\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_{-L_{\mathbf{e}}}^{+\infty} e^{-\lambda z-\lambda L_{\mathbf{e}}}f(z)dz\right]\\
&\geq\mathbb{P}\left(L_{\mathbf{e}}<0\right)\mathcal{L}[f](\lambda)-\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z}f(z)dz\right]
\end{aligned}$$

and the result follows by gathering the two previous terms :

$$\int_0^{+\infty} e^{-\lambda x}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]dx\geq\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[f](\lambda)-\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z}f(z)dz\right].$$

It remains to compute the limit $iii)$. Notice first that if $\beta=1$ (and thus $\alpha<1$), then the process L is a subordinator, hence $L_{\mathbf{e}}\geq 0$ a.s and the expectation is null. We thus assume now that $\beta\in(-1,1)$.

Let $\varepsilon > 0$ and take A_ε large enough such that $f(x) \leq \varepsilon$ for $x \geq A_\varepsilon$. We decompose :

$$\begin{aligned}
&\frac{1}{\eta_\alpha(\lambda)}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z} f(z)dz\right] \\
&\qquad \leq \frac{1}{\eta_\alpha(\lambda)}\mathbb{E}\left[1_{\{-A_\varepsilon<L_{\mathbf{e}}<0\}}\int_0^{A_\varepsilon} e^{-\lambda z} f(z)dz\right] + \frac{\varepsilon}{\eta_\alpha(\lambda)}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<-A_\varepsilon\}}\int_{A_\varepsilon}^{-L_{\mathbf{e}}} e^{-\lambda z}dz\right] \\
&\qquad \leq \frac{A_\varepsilon}{\eta_\alpha(\lambda)} f(0) + \frac{\varepsilon}{\eta_\alpha(\lambda)}\int_0^{+\infty} e^{-\lambda z}\mathbb{P}(-L_{\mathbf{e}} > z)dz \xrightarrow[\lambda\downarrow 0]{} \varepsilon\kappa_{\alpha,-\beta}
\end{aligned}$$

where the limit of the integral follows from (2.1) since $-L$ is a stable Lévy process with parameter α and $-\beta$.

□

We now apply Lemma 2.1 to study Equation (1.4).

2.2. *Analysis of Equation (1.4).*

Lemma 2.2. *The Laplace transform of $\Phi_0 - \Phi_R$ satisfies the following bounds for $\lambda > 0$:*

$$\frac{\lambda}{1-\mathbb{E}[e^{-\lambda S_{\mathbf{e}}}]}\mathcal{L}[\Phi_0 - \Phi_R](\lambda) \leq 1$$

and

$$\frac{\lambda}{1-\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}\mathcal{L}[\Phi_0 - \Phi_R](\lambda) \geq 1 - \lambda\mathcal{L}[u](\lambda) - \frac{\lambda}{1-\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z}u(z)dz\right].$$

Proof: Taking the Laplace transform of (1.4) and using Point $i)$ of Lemma 2.1 with $f = u$, we deduce that

$$\mathcal{L}[u](\lambda) \leq \frac{1-\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]}{\lambda} + \mathcal{L}[u](\lambda)\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right] - \mathcal{L}[\Phi_0 - \Phi_R](\lambda)$$

which yields the upper bound

$$\frac{\lambda}{1-\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]}\mathcal{L}[\Phi_0 - \Phi_R](\lambda) \leq 1 - \lambda\mathcal{L}[u](\lambda) \leq 1.$$

To get the lower bound, we apply Point $ii)$ of Lemma 2.1 still with $f = u$. Since $u(0) = 1$, this yields

$$\mathcal{L}[u](\lambda) + \mathcal{L}[\Phi_0 - \Phi_R](\lambda) \geq \frac{1-\mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}{\lambda} + \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[u](\lambda) - \mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z}u(z)dz\right]$$

which gives the lower bound, after rearranging the terms.

□

It remains now to study the limit of both expressions when $\lambda \downarrow 0$.

2.3. *Proof of Theorem 1.1 when $\alpha \leq 1$.* Notice first that from a change of variable and the monotone convergence theorem,

$$\lambda\mathcal{L}[u](\lambda) = \int_0^{+\infty} e^{-z}u\left(\frac{z}{\lambda}\right)dz \xrightarrow[\lambda\downarrow 0]{} 0.$$

Then letting $\lambda \downarrow 0$ in Lemma 2.2 and using (2.1) and Point $iii)$ in Lemma 2.1 with $f = u$, we obtain

$$\mathcal{L}[\Phi_0](\lambda) - \mathcal{L}[\Phi_R](\lambda) \underset{\lambda\downarrow 0}{\sim} \kappa_{\alpha,\beta}\,\eta_\alpha(\lambda). \tag{2.2}$$

To simplify the notations, we set

$$\varphi_0(x) = (1 - \mathbb{E}[\boldsymbol{p}])u(x) + \frac{\mathbb{E}[\boldsymbol{p}^2 - \boldsymbol{p}]}{2} u^2(x) \tag{2.3}$$

so that

$$\mathcal{L}[\Phi_0](\lambda) = \int_0^{+\infty} e^{-\lambda x} \mathbb{E}\left[1_{\{S_{\mathbf{e}} < x\}} \varphi_0(x - L_{\mathbf{e}})\right] dx.$$

Note that the function φ_0 is positive and decreasing since $\mathbb{E}[\boldsymbol{p}^2 - \boldsymbol{p}] > 0$, as $\boldsymbol{p}$ is an integer-valued random variable. On the one hand, observe that since $\Phi_R \geq 0$, we deduce from Lemma 2.1 with $f = \varphi_0$ that

$$\mathcal{L}[\Phi_0](\lambda) - \mathcal{L}[\Phi_R](\lambda) \leq \mathcal{L}[\varphi_0](\lambda)$$

hence, from (2.2),

$$\kappa_{\alpha,\beta} \leq \liminf_{\lambda \downarrow 0} \frac{1}{\eta_\alpha(\lambda)} \mathcal{L}[\varphi_0](\lambda).$$

On the other hand, fix $\varepsilon > 0$ small enough. Since $\lim\limits_{x \to +\infty} u(x) = 0$, there exists $A_\varepsilon > 0$ such that $u(x) \leq \varepsilon$ for $x \geq A_\varepsilon$. This implies that there exists K, independent of ε, such that :

$$u^3(x) \leq \varepsilon K \varphi_0(x) \qquad \text{for} \qquad x \geq A_\varepsilon.$$

As a consequence, using (1.5) and Point $i)$ of Lemma 2.1 with $f = u^3$, we deduce

$$\begin{aligned}
\mathcal{L}[\Phi_R](\lambda) &\leq \mathbb{E}\left[\boldsymbol{p}^3\right] \left(\int_0^{A_\varepsilon} e^{-\lambda x} u^3(x) dx + \int_{A_\varepsilon}^{+\infty} e^{-\lambda x} u^3(x) dx \right) \\
&\leq \mathbb{E}\left[\boldsymbol{p}^3\right] \left(A_\varepsilon + \varepsilon K \mathcal{L}[\varphi_0](\lambda)\right).
\end{aligned}$$

Then, using Point $ii)$ of Lemma 2.1 with $f = \varphi_0$,

$$\begin{aligned}
\mathcal{L}[\Phi_0](\lambda) - \mathcal{L}[\Phi_R](\lambda) \geq \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right] \mathcal{L}[\varphi_0](\lambda) + \varphi_0(0) \frac{\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right] - \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]}{\lambda} \\
- \mathbb{E}\left[1_{\{L_{\mathbf{e}} < 0\}} \int_0^{-L_{\mathbf{e}}} e^{-\lambda z} \varphi_0(z) dz\right] - \mathbb{E}\left[\boldsymbol{p}^3\right] \left(A_\varepsilon + \varepsilon K \mathcal{L}[\varphi_0](\lambda)\right).
\end{aligned} \tag{2.4}$$

Dividing both sides by $\eta_\alpha(\lambda)$ and applying Point $iii)$ of Lemma 2.1, we deduce that

$$\kappa_{\alpha,\beta} \geq \left(1 - \varepsilon K \mathbb{E}\left[\boldsymbol{p}^3\right]\right) \limsup_{\lambda \downarrow 0} \frac{1}{\eta_\alpha(\lambda)} \mathcal{L}[\varphi_0](\lambda).$$

Finally, we have thus proven that

$$\mathcal{L}[\varphi_0](\lambda) \underset{\lambda \downarrow 0}{\sim} \kappa_{\alpha,\beta}\, \eta_\alpha(\lambda)$$

hence, by the Tauberian theorem,

$$\int_0^x \varphi_0(z) dz \underset{x \to +\infty}{\sim} \frac{\kappa_{\alpha,\beta}}{\Gamma(2-\alpha)} \eta_\alpha\left(\frac{1}{x}\right).$$

The result now follows by differentiation, since φ_0 is decreasing.

□

3. The case $1 < \alpha < 2$

In this case, the existence of the first moment of $S_{\mathbf{e}}$ prevents us from using directly the Tauberian theorem as in the previous case. Indeed, when $\alpha \in (1,2)$, letting $\lambda \downarrow 0$ in the first inequality of Lemma 2.2, one obtains :

$$\limsup_{\lambda\downarrow 0} \mathcal{L}[\Phi_0](\lambda) - \mathcal{L}[\Phi_R](\lambda) \leq \mathbb{E}[S_{\mathbf{e}}].$$

Going back to (2.4), this implies that

$$\left(1-\varepsilon K \mathbb{E}\left[\boldsymbol{p}^3\right]\right) \limsup_{\lambda\downarrow 0} \mathcal{L}[\varphi_0](\lambda) \leq (1+\varphi_0(0))\mathbb{E}[S_{\mathbf{e}}] + \mathbb{E}[\boldsymbol{p}^3]A_\varepsilon - \mathbb{E}[1_{\{L_{\mathbf{e}}<0\}}L_{\mathbf{e}}]\varphi_0(0) < +\infty \quad (3.1)$$

i.e. we can only deduce, by the monotone convergence theorem, that

i) if $\mathbb{E}[\boldsymbol{p}] < 1$,

$$\int_0^{+\infty} u(x)dx = \mathbb{E}[\mathbf{M}_{\alpha,\beta}] < +\infty, \quad (3.2)$$

ii) while if $\mathbb{E}[\boldsymbol{p}] = 1$,

$$\int_0^{+\infty} u^2(x)dx < +\infty.$$

We shall thus made a technical modification of the previous proof, and write down some new inequalities.

3.1. *Preliminary lemma.*

Lemma 3.1. *Assume that $\alpha \in (1,2)$ and let $f : [0,+\infty) \to [0,+\infty)$ be a positive and decreasing function. We write xf for the function $x \to xf(x)$. Then, the following inequalities hold :*

i) *Upper bound :*

$$\int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} f(x-L_{\mathbf{e}})\right] dx \leq \mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right] \mathcal{L}[xf](\lambda) + \mathbb{E}\left[S_{\mathbf{e}} e^{-\lambda S_{\mathbf{e}}}\right] \mathcal{L}[f](\lambda).$$

ii) *Lower bound :*

$$\int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}} f(x-L_{\mathbf{e}})\right] dx$$
$$\geq f(0)\int_0^{+\infty} e^{-\lambda x} x\left(\mathbb{P}\left(L_{\mathbf{e}} \geq x\right) - \mathbb{P}\left(S_{\mathbf{e}} \geq x\right)\right) dx + \int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}} f(x-L_{\mathbf{e}})\right] dx$$

and

$$\int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}} f(x-L_{\mathbf{e}})\right] dx$$
$$\geq \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right] \mathcal{L}[xf](\lambda) + \mathbb{E}\left[L_{\mathbf{e}} e^{-\lambda L_{\mathbf{e}}^+}\right] \mathcal{L}[f](\lambda) - \mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}} \int_0^{-L_{\mathbf{e}}} e^{-\lambda z} z f(z) dz\right].$$

iii) *Assume that $\lim_{x\to+\infty} f(x) = 0$, then*

$$\lim_{\lambda\downarrow 0} \lambda^{2-\alpha} \mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}} \int_0^{-L_{\mathbf{e}}} e^{-\lambda z} z f(z) dz\right] = 0.$$

Proof: The proof of Lemma 3.1 is similar to that of Lemma 2.1. Point *i*) follows from the decomposition $x = x - S_{\mathbf{e}} + S_{\mathbf{e}}$, using the fact that f is decreasing and computing the Laplace transforms of the convolution products. Point *ii*) follows similarly from the decomposition $x = x - L_{\mathbf{e}} + L_{\mathbf{e}}$, by separating the case $L_\mathbf{e} < 0$ and $L_\mathbf{e} \geq 0$. Finally, for Point *iii*), observe first that if $\beta = 1$, then the random variable $(-L_\mathbf{e})^+$ admits exponential moments, hence

$$\lambda^{2-\alpha}\mathbb{E}\left[1_{\{L_\mathbf{e}<0\}}\int_0^{-L_\mathbf{e}} e^{-\lambda z} z f(z)dz\right] \leq \lambda^{2-\alpha} f(0)\mathbb{E}\left[1_{\{L_\mathbf{e}<0\}}L_\mathbf{e}^2\right] \xrightarrow[\lambda\downarrow 0]{} 0.$$

Take now $\beta \in (-1,1)$ and let $\varepsilon > 0$. By assumption, there exists $A_\varepsilon > 0$ such that $f(x) \leq \varepsilon$ for $x \geq A_\varepsilon$. We then decompose :

$$\begin{aligned}
&\lambda^{2-\alpha}\mathbb{E}\left[1_{\{L_\mathbf{e}<0\}}\int_0^{-L_\mathbf{e}} e^{-\lambda z} z f(z)dz\right] \\
&\quad \leq \lambda^{2-\alpha}\mathbb{E}\left[1_{\{-A_\varepsilon<L_\mathbf{e}<0\}}\int_0^{A_\varepsilon} e^{-\lambda z} z f(z)dz\right] + \varepsilon\lambda^{2-\alpha}\mathbb{E}\left[1_{\{L_\mathbf{e}<-A_\varepsilon\}}\int_{A_\varepsilon}^{-L_\mathbf{e}} e^{-\lambda z} z dz\right] \\
&\quad \leq \lambda^{2-\alpha} A_\varepsilon^2 f(0) + \varepsilon\lambda^{2-\alpha}\int_0^{+\infty} e^{-\lambda z} z\mathbb{P}(-L_\mathbf{e} > z)dz \xrightarrow[\lambda\downarrow 0]{} \varepsilon\kappa_{\alpha,-\beta}
\end{aligned}$$

where the limit of the integral follows as before from (2.1) and the Tauberian theorem.

□

We now apply Lemma 3.1 to the equation (1.4) satisfied by u.

3.2. *Analysis of Equation (1.4) in the case $\alpha \in (1,2)$.*

Lemma 3.2. *The Laplace transform of $x(\Phi_0 - \Phi_R)$ satisfies the following bounds :*

$$\frac{\lambda^2}{1-\mathbb{E}\left[e^{-\lambda S_\mathbf{e}}\right] - \lambda\mathbb{E}\left[S_\mathbf{e} e^{-\lambda S_\mathbf{e}}\right]}\mathcal{L}[x(\Phi_0 - \Phi_R)](\lambda) \leq 1 + \frac{\lambda^2\,\mathbb{E}\left[S_\mathbf{e} e^{-\lambda S_\mathbf{e}}\right]}{1-\mathbb{E}\left[e^{-\lambda S_\mathbf{e}}\right] - \lambda\mathbb{E}\left[S_\mathbf{e} e^{-\lambda S_\mathbf{e}}\right]}\mathcal{L}[u](\lambda)$$

and

$$\begin{aligned}
&\frac{\lambda^2}{1-\mathbb{E}\left[e^{-\lambda L_\mathbf{e}^+}\right] - \lambda\mathbb{E}\left[L_\mathbf{e}^+ e^{-\lambda L_\mathbf{e}^+}\right]}\mathcal{L}[x(\Phi_0 - \Phi_R)](\lambda) \\
&\qquad\qquad \geq 1 - \lambda^2\mathcal{L}[xu](\lambda) - \frac{\lambda^2\,\Xi(\lambda)}{1-\mathbb{E}\left[e^{-\lambda L_\mathbf{e}^+}\right] - \lambda\mathbb{E}\left[L_\mathbf{e}^+ e^{-\lambda L_\mathbf{e}^+}\right]}
\end{aligned}$$

where

$$\Xi(\lambda) = \mathbb{E}\left[1_{\{L_\mathbf{e}<0\}}\int_0^{-L_\mathbf{e}} e^{-\lambda z} z u(z)dz\right] + \mathbb{E}\left[1_{\{L_\mathbf{e}<0\}}(-L_\mathbf{e})\right]\mathcal{L}[u](\lambda).$$

Proof: We first multiply (1.4) by x before taking the Laplace transform of both sides:

$$\mathcal{L}[xu](\lambda) = \int_0^{+\infty} e^{-\lambda x} x\mathbb{P}\left(S_\mathbf{e} \geq x\right)dx + \int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{S_\mathbf{e}<x\}}u(x - L_\mathbf{e})\right]dx - \mathcal{L}[x(\Phi_0 - \Phi_R)](\lambda).$$

Integrating by parts the first term on the right-hand side, we have

$$\int_0^{+\infty} e^{-\lambda x} x\mathbb{P}\left(S_\mathbf{e} \geq x\right)dx = \frac{1-\mathbb{E}\left[e^{-\lambda S_\mathbf{e}}\right] - \lambda\mathbb{E}\left[S_\mathbf{e} e^{-\lambda S_\mathbf{e}}\right]}{\lambda^2} \underset{\lambda\downarrow 0}{\sim} \kappa_{\alpha,\beta}\Gamma(2-\alpha)\lambda^{\alpha-2} \tag{3.3}$$

from (2.1) and the Tauberian theorem. To get the upper bound, we apply Point i) of Lemma 3.1 with $f = u$:

$$\begin{aligned}
&\mathcal{L}[x(\Phi_0 - \Phi_R)](\lambda) + \mathcal{L}[xu](\lambda) \\
&\qquad \leq \frac{1-\mathbb{E}\left[e^{-\lambda S_\mathbf{e}}\right] - \lambda\mathbb{E}\left[S_\mathbf{e} e^{-\lambda S_\mathbf{e}}\right]}{\lambda^2} + \mathbb{E}\left[e^{-\lambda S_\mathbf{e}}\right]\mathcal{L}[xu](\lambda) + \mathbb{E}\left[S_\mathbf{e}\, e^{-\lambda S_\mathbf{e}}\right]\mathcal{L}[u](\lambda).
\end{aligned}$$

Adding $\lambda\mathbb{E}[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}]\mathcal{L}[xu](\lambda)$ on the right-hand side and rearranging the terms yields the announced upper bound.

Similarly, to get the lower bound, we apply Point $ii)$ of Lemma 3.1 with $f = u$:

$$\begin{aligned}\mathcal{L}[x(\Phi_0 - \Phi_R)](\lambda) + \mathcal{L}[xu](\lambda) \\ \geq \frac{1 - \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right] - \lambda\mathbb{E}\left[L_{\mathbf{e}}^+ e^{-\lambda L_{\mathbf{e}}^+}\right]}{\lambda^2} + \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[xu](\lambda) \\ + \mathbb{E}\left[L_{\mathbf{e}}\, e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[u](\lambda) - \mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z} z u(z) dz\right]\end{aligned}$$

Adding $\lambda\mathbb{E}\left[L_{\mathbf{e}}^+ e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[xu](\lambda)$ in both sides and rearranging the terms, we obtain

$$\begin{aligned}\frac{\lambda^2}{1 - \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right] - \lambda\mathbb{E}\left[L_{\mathbf{e}}^+ e^{-\lambda L_{\mathbf{e}}^+}\right]}\mathcal{L}[x(\Phi_0 - \Phi_R)](\lambda) \\ \geq 1 - \lambda^2\mathcal{L}[xu](\lambda) - \frac{\lambda^2}{1 - \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right] - \lambda\mathbb{E}\left[L_{\mathbf{e}}^+ e^{-\lambda L_{\mathbf{e}}^+}\right]}R(\lambda)\end{aligned}$$

where the remainder $R(\lambda)$ is given by :

$$\begin{aligned}R(\lambda) = \mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z} z u(z) dz\right] - \mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}} L_{\mathbf{e}}\right]\mathcal{L}[u](\lambda) \\ + \mathbb{E}\left[L_{\mathbf{e}}^+ e^{-\lambda L_{\mathbf{e}}^+}\right]\left(\lambda\mathcal{L}[xu](\lambda) - \mathcal{L}[u](\lambda)\right).\end{aligned}$$

Note that the last term on the right-hand side is negative and may thus be removed since, integrating by parts,

$$\mathcal{L}[u](\lambda) - \lambda\mathcal{L}[xu](\lambda) = \mathbb{E}\left[\mathbf{M}_{\alpha,\beta}e^{-\lambda\mathbf{M}_{\alpha,\beta}}\right] \geq 0,$$

hence $R(\lambda) \leq \Xi(\lambda)$ which proves the lower bound.

□

3.3. *Proof of Theorem 1.1 when* $\alpha > 1$. We now want to let $\lambda \downarrow 0$ in Lemma 3.2. Notice first that, thanks to (2.1), the first terms on the left-hand side of both inequalities will converge towards the same quantity, which is given by (3.3). Also, by the monotone convergence theorem

$$\lambda^2\mathcal{L}[xu](\lambda) = \int_0^{+\infty} e^{-z} z u\left(\frac{z}{\lambda}\right) dz \xrightarrow[\lambda\downarrow 0]{} 0$$

and, applying Point $iii)$ of Lemma 3.1 with $f = u$,

$$\lim_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z} z u(z) dz\right] = 0.$$

As a consequence, it only remains to show that

$$\lim_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}[u](\lambda) = 0. \tag{3.4}$$

When $\mathbb{E}[\boldsymbol{p}] < 1$, this is a direct consequence of (3.2) since $\mathcal{L}[u](\lambda) \leq \mathbb{E}[\mathbf{M}_{\alpha,\beta}] < +\infty$. The situation is trickier when $\mathbb{E}[\boldsymbol{p}] = 1$, and we shall rely on the following Lemma, which gives an a priori bound on u.

Lemma 3.3. *Assume that $\alpha > 1$ and $\mathbb{E}[\boldsymbol{p}] = 1$. Then, there exists a constant $C > 0$ such that for x large enough,*

$$\mathbb{P}\left(\mathbf{M}_{\alpha,\beta} \geq x\right) \leq C x^{-\alpha/2}.$$

Proof: We mimic the arguments of Lalley and Shao (2016). Recall that since $\alpha > 1$, the positivity parameter $\rho = \mathbb{P}(L_1 \geq 0)$ of L belongs to the interval $[1 - \frac{1}{\alpha}, \frac{1}{\alpha}]$. Denote by $\underline{\mathbf{M}}^{(t)}_{\alpha,\beta}$ the maximum of the branching stable process on the interval $[0, t]$ and by $\overline{\mathbf{M}}^{(t)}_{\alpha,\beta}$ the maximum of the branching stable process on the interval $[t, +\infty]$. From the construction given in (1.1), we first deduce that

$$\begin{aligned}
\mathbb{P}\left(\mathbf{M}_{\alpha,\beta} \geq x\right) &\leq \mathbb{P}\left(\underline{\mathbf{M}}^{(t)}_{\alpha,\beta} \geq x\right) + \mathbb{P}\left(\overline{\mathbf{M}}^{(t)}_{\alpha,\beta} \geq x\right) \\
&\leq \mathbb{P}\left(\underline{\mathbf{M}}^{(t)}_{\alpha,\beta} \geq x\right) + \mathbb{P}\left(Z(t) \geq 1\right).
\end{aligned}$$

Let us now denote by T_x the first time at which a particle of the branching process reaches the level x. Then, by applying the Markov property at T_x, we deduce that conditionally to $\{T_x \leq t\} = \{\underline{\mathbf{M}}^{(t)}_{\alpha,\beta} \geq x\}$, the expected number of particles above x at time t is greater than ρ :

$$\mathbb{E}\left[\sum_{i=1}^{Z(t)} 1_{\{L_t^{(i)} \geq x\}} \middle| \underline{\mathbf{M}}^{(t)}_{\alpha,\beta} \geq x\right] \geq \rho \geq 1 - \frac{1}{\alpha}$$

which implies that

$$\mathbb{P}\left(\underline{\mathbf{M}}^{(t)}_{\alpha,\beta} \geq x\right) \leq \frac{\alpha}{\alpha - 1}\mathbb{E}\left[\sum_{i=1}^{Z(t)} 1_{\{L_t^{(i)} \geq x\}}\right] \leq \frac{\alpha}{\alpha-1}\mathbb{E}[Z(t)]\mathbb{P}\left(L_t \geq x\right) = \frac{\alpha}{\alpha-1}\mathbb{P}\left(t^{1/\alpha} L_1 \geq x\right)$$

since Z is independent from the positions $(L^{(i)})$, and $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$. Taking $t = x^{\alpha/2}$, we have thus proven that

$$\mathbb{P}\left(\mathbf{M}_{\alpha,\beta} \geq x\right) \leq \frac{\alpha}{\alpha-1}\mathbb{P}\left(L_1 \geq \sqrt{x}\right) + \mathbb{P}\left(Z(x^{\alpha/2}) \geq 1\right)$$

and the result follows by using (2.1) and the Kolmogorov's theorem, which states that

$$\mathbb{P}\left(Z(t) \geq 1\right) \underset{t \to +\infty}{\sim} \frac{2}{\sigma^2 t}$$

see for instance Asmussen and Hering (1983, Theorem 2.6).

□

We now come back to the limit (3.4). Applying Lemma 3.3 and the Tauberian theorem, we deduce that for λ small enough, there exists a constant $C > 0$ such that

$$\lambda^{2-\alpha}\mathcal{L}[u](\lambda) \leq C\lambda^{1-\frac{\alpha}{2}} \underset{\lambda \downarrow 0}{\longrightarrow} 0$$

since $\alpha \in (1, 2)$. As a consequence, by letting $\lambda \downarrow 0$ in Lemma 3.2, we obtain the asymptotics

$$\mathcal{L}[x\Phi_0](\lambda) - \mathcal{L}[x\Phi_R](\lambda) \underset{\lambda \downarrow 0}{\sim} \kappa_{\alpha,\beta}\Gamma(2-\alpha)\lambda^{\alpha-2}.$$

The remainder of the proof is now similar to the case $\alpha \in (0, 1]$. First, applying Point $i)$ of Lemma 3.1 with $f = \varphi_0$ and using that $\Phi_R \geq 0$, we deduce that

$$\mathcal{L}[x\Phi_0](\lambda) - \mathcal{L}[x\Phi_R](\lambda) \leq \mathcal{L}[x\varphi_0](\lambda) + \mathbb{E}[S_{\mathbf{e}}]\mathcal{L}[\varphi_0](\lambda)$$

which implies the lower bound

$$\liminf_{\lambda \downarrow 0} \lambda^{2-\alpha}\mathcal{L}[x\varphi_0](\lambda) \geq \kappa_{\alpha,\beta}\Gamma(2-\alpha).$$

Next, we deduce from Point *ii*) of Lemma 3.1 with $f=\varphi_0$ that

$$\begin{aligned}&\mathcal{L}[x\Phi_0](\lambda)\\&\qquad\geq \varphi_0(0)\int_0^{+\infty} e^{-\lambda x} x\left(\mathbb{P}\left(L_{\mathbf{e}}\geq x\right)-\mathbb{P}\left(S_{\mathbf{e}}\geq x\right)\right)dx+\int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}\varphi_0(x-L_{\mathbf{e}})\right]dx\end{aligned}$$

where

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda x} x\mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}\varphi_0(x-L_{\mathbf{e}})\right]dx\\&\qquad\geq \mathbb{E}\left[e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[x\varphi_0](\lambda)-\mathbb{E}\left[1_{\{L_{\mathbf{e}}<0\}}\int_0^{-L_{\mathbf{e}}} e^{-\lambda z} z\varphi_0(z)dz\right]+\mathbb{E}\left[L_{\mathbf{e}}e^{-\lambda L_{\mathbf{e}}^+}\right]\mathcal{L}[\varphi_0](\lambda).\end{aligned}$$

Take $\varepsilon>0$. As before, since $\lim\limits_{x\to+\infty} u(x)=0$, there exists $A_\varepsilon>0$ and $K>0$ such that

$$u(x)\leq\varepsilon \qquad \text{and} \qquad u^3(x)\leq \varepsilon K\varphi_0(x) \qquad \text{for} \quad x\geq A_\varepsilon.$$

As a consequence, using (1.5) and Point *i*) of Lemma 3.1 with $f=u^3$, we deduce

$$\begin{aligned}\mathcal{L}[x\Phi_R](\lambda)&\leq \mathbb{E}\left[\boldsymbol{p}^3\right]\left(\mathcal{L}[xu^3](\lambda)+\mathbb{E}\left[S_{\mathbf{e}}\right]\mathcal{L}[u^3](\lambda)\right)\\&\leq \mathbb{E}\left[\boldsymbol{p}^3\right]\left(A_\varepsilon^2+\mathbb{E}\left[S_{\mathbf{e}}\right]A_\varepsilon+\varepsilon K\left(\mathcal{L}[x\varphi_0](\lambda)+\mathbb{E}\left[S_{\mathbf{e}}\right]\mathcal{L}[\varphi_0](\lambda)\right)\right)\\&\leq C(1+\varepsilon\mathcal{L}[x\varphi_0](\lambda))\end{aligned}$$

for some constant C large enough since $\mathcal{L}[\varphi_0](\lambda)\leq\mathcal{L}[\varphi_0](0)<+\infty$ from (3.1). Then, using Point *iii*) of Lemma 3.1 with $f=\varphi_0$, we obtain

$$(1-\varepsilon C)\limsup_{\lambda\downarrow 0}\mathcal{L}[x\varphi_0](\lambda)\leq \kappa_{\alpha,\beta}\Gamma(2-\alpha)$$

which implies that

$$\mathcal{L}[x\varphi_0](\lambda)\underset{\lambda\downarrow 0}{\sim}\kappa_{\alpha,\beta}\Gamma(2-\alpha)\lambda^{\alpha-2}.$$

Finally, by the Tauberian theorem,

$$\int_0^x z\varphi_0(z)dz\underset{x\to+\infty}{\sim}\frac{\kappa_{\alpha,\beta}}{2-\alpha}x^{2-\alpha}$$

and the result now follows by differentiation, using the change of variable

$$\int_0^x z\varphi_0(z)dz=\int_0^{x^2/2}\varphi_0(\sqrt{2y})dy$$

and observing that the function $y\to\varphi_0(\sqrt{2y})$ is decreasing.

□

Remark 3.4. It might be noticed that the inequalities given in Lemma 2.2 and 3.2 are valid for any Lévy process, i.e. do not rely on any stability assumption. As a consequence, an analogue of Theorem 1.1 should remain valid for any Lévy process for which the tails of the distributions of $S_{\mathbf{e}}$ and $L_{\mathbf{e}}$ are asymptotically equivalent to the same regularly varying function. This is the case, under some mild assumptions, if the right tail of the Lévy measure is subexponential, see for instance Willekens (1987).

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