ALEA, Lat. Am. J. Probab. Math. Stat. 19, 1471–1495 (2022)
DOI: 10.30757/ALEA.v19-59

# Slow, ordinary and rapid points for Gaussian Wavelets Series and application to Fractional Brownian Motions

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**Abstract.** We study the Hölderian regularity of Gaussian wavelets series and show that they display, almost surely, three types of points: slow, ordinary and rapid. In particular, this fact holds for the Fractional Brownian Motion. Finally, we remark that the existence of slow points is specific to these functions.

## 1. Introduction

Let $B$ denote the standard Brownian motion on $\mathbb{R}$. The Khinchin law of the iterated logarithm Khintchine (1924) allows to control the behaviour of $B$ at a given point, in the sense that for every $t \in \mathbb{R}$, it holds

$$\limsup_{r\to 0} \frac{|B(t+r)-B(t)|}{\sqrt{|r|\log\log|r|^{-1}}} = \sqrt{2} \tag{1.1}$$

on an event of probability one. As a direct application of Fubini's theorem, one obtains that almost surely, the set of points $t \in \mathbb{R}$ such that (1.1) holds, called ordinary points, has full Lebesgue measure. This contrasts with the uniform Hölder condition obtained by Paul Lévy in 1937 which states that the uniform modulus of continuity of $B$ is of larger order: almost surely, one has

$$\limsup_{r\to 0} \sup_{t\in[0,1]} \frac{|B(t+r)-B(t)|}{\sqrt{|r|\log|r|^{-1}}} = \sqrt{2}.$$

In particular, there exist exceptional points, called fast points, where the law of the iterated logarithm fails. In 1974, Oray and Taylor studied how often this exceptional behaviour holds and proved

*Received by the editors March 10th, 2022; accepted October 25th, 2022.*

2010 *Mathematics Subject Classification.* 42C40, 26A16, 60G15, 60G22, 60G17.

*Key words and phrases.* Random Wavelets Series, Fractional Brownian motion, modulus of continuity, slow/ordinary/rapid points.

Second Author is supported by the FNR OPEN grant APOGEe at University of Luxembourg.

especially that the Hausdorff dimension of the set

$$\Big\{t \in [0,1] : \limsup_{r\to 0} \frac{|B(t+r)-B(t)|}{\sqrt{|r|\log|r|^{-1}}} \geq \lambda\sqrt{2}\Big\}$$

is given almost surely by $1-\lambda^2$ for every $\lambda \in [0,1]$, see Orey and Taylor (1974).

In the meanwhile, Kahane proposed in Kahane (1985) an easy way to study the regularity and irregularity properties of the Brownian motion. Its method relies on the expansion of $B$ on $[0,1]$ in the so-called Faber-Schauder system. If $\Lambda$ is the triangular function

$$\Lambda \,:\, x \mapsto \begin{cases} x & \text{if } \frac{1}{2} \leq x < 1 \\ 1-x & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

and $\xi$, $\xi_{j,k}$ ($j \in \mathbb{N}_0$ and $k \in \{0,\dots,2^j-1\}$) are independent $\mathcal{N}(0,1)$ random variables then, we have

$$B(t) = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} \xi_{j,k} 2^{-j/2} \Lambda(2^j t - k) + \xi t \tag{1.2}$$

where the convergence is almost surely uniform for $t \in [0,1]$. Working with this expression, Kahane recovered the law of the iterated logarithm and the estimation of the modulus of continuity of the Brownian motion. Furthermore, Kahane obtained the existence of a third category of points, presenting a slower oscillation. These points, called slow points, satisfy the condition

$$\limsup_{r\to 0} \frac{|B(t+r)-B(t)|}{\sqrt{|r|}} < +\infty.$$

The law of iterated logarithm and the study of the set of fast points has naturally been studied and extended since then for more general classes of Gaussian processes such as Gaussian processes with stationary increments, see e.g. Marcus (1968); Orey (1972); Bingham (1986); Marcus and Rosen (1992); Monrad and Rootzén (1995); Khoshnevisan and Shi (2000). In particular, given a fractional Brownian motion $B_h$ of index $h \in (0,1)$, one has almost surely

$$\limsup_{r\to 0} \frac{|B_h(t+r)-B_h(t)|}{|r|^h\sqrt{\log\log|r|^{-1}}} = \sqrt{2}$$

and

$$\limsup_{r\to 0} \sup_{t\in[0,1]} \frac{|B_h(t+r)-B_h(t)|}{|r|^h\sqrt{\log|r|^{-1}}} = \sqrt{2}.$$

However, unlike the Brownian motion, for more general Gaussian processes the existence of points that present slower oscillations was not highlighted. The aim of this paper is to investigate this question. In particular, we prove the fractional Brownian motion admits slow points.

In 1999, Meyer, Sellan and Taqqu introduced a famous decomposition of the fractional Brownian motion using a Lemarié-Meyer or sufficiently smooth Daubechies wavelet $\psi$, which decorrelates the high frequencies Meyer et al. (1999). More precisely, any fractional Brownian motion $B_h$ of Hurst index $h \in (0,1)$ can be written as

$$B_h(t) = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} 2^{-hj} \xi_{j,k} \psi_{h+1/2}(2^j t - k) + R(t) \tag{1.3}$$

where $R$ is a smooth process, $(\xi_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{Z}}$ is a sequence of independent $\mathcal{N}(0,1)$ random variables, and $\psi_\alpha$ is defined by

$$\hat{\psi}_\alpha(\xi) = \frac{1}{|\xi|^\alpha}\hat{\psi}(\xi).$$

Note that such a function leads to a biorthogonal wavelet basis, see Section 2.

Motivated by the study of fractional Brownian motions using this particular decomposition, we develop in this paper a systematic study of the different pointwise behaviours of random wavelets series of the form

$$f_h = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \xi_{j,k} 2^{-hj}\psi(2^j\cdot -k) \tag{1.4}$$

where $(\xi_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{Z}}$ denotes a sequence of independent $\mathcal{N}(0,1)$ random variables, $h\in(0,1)$ is fixed and $\psi$ is any compactly supported or smooth wavelet, see Section 2 for a precise definition. Note that, even if the expression (1.4) is very similar to (1.2), dealing with it requires much more technical arguments. Indeed, due to the symmetry of the function $\Lambda$, most of the terms vanish in the expansion of the increments $B(t)-B(s)$ if $s$ and $t$ are close enough. This fact can not be used anymore while working with an arbitrary wavelet and compensations of different terms may occur.

Concerning the regularity of the function $f_h$, one can show, see Proposition 2.3 below, that, for all $s,t\in\mathbb{R}$,

$$\mathbb{E}[(f_h(t)-f_h(s))^2] \leq C|s-t|^{2h},$$

for some deterministic constant $C>0$. Applying Kolmogorov continuity Theorem for Gaussian processes, one can deduce that almost surely, for every $t\in\mathbb{R}$, one has

$$\limsup_{r\to 0}\frac{|f_h(t+r)-f_h(t)|}{|r|^{h-\varepsilon}} < +\infty$$

for every $\varepsilon>0$. The aim of this paper is to characterize more precisely the pointwise behaviour of such a wavelet series, in the spirit of what is known for the Brownian motion. Recently, generalized Hölder spaces have been introduced to address this kind of questions Kreit and Nicolay (2018); Loosveldt and Nicolay (2021) as well as the regularity of solutions of partial differential equations Loosveldt and Nicolay (2020). This article is the continuation of the work done in Ayache (2019); Ayache et al. (2019); Kahane (1985) for the Brownian motion.

As a consequence of our results, we get that almost surely, the Hölder exponent of the random wavelet series $f_h$ is $h$ while it does not belong to the uniform Hölder space of order $h$. Nevertheless, if $t$ is a slow point, $f_h$ belongs to the **pointwise** Hölder space of order $h$ at $t$. One can therefore wonder if this feature is common or if it is specific to the functions under study in this paper. To address this question, in Section 6 we recall two commonly used notions of genericity: the prevalence and the Baire category point of view. We obtain that, in both senses, the existence of slow points is a specific property of Gaussian random wavelets series.

The paper is structured as follows: in Section 2, before stating our main result, we recall the most important tools used in the paper: discrete wavelet transform and modulus of continuity as well as some fundamental inequalities. Section 3 is devoted to exploring the regularity of the Gaussian random wavelet series (1.4) by identifying three precise pointwise estimates. In Section 4, we focus on the irregularity that we deduce from the asymptotic behaviour of the wavelet coefficients. Then, using the obtained results of regularity and irregularity, we prove our main result concerning the existence of slow, ordinary and rapid points in Section 5. Finally, Section 6, focuses on the results of generecity for slow points. In this paper, $C$ stands for a deterministic constant not necessary the same in different lines.

## 2. Preliminaries and statement of the main result

In this section, we present the notions needed for the statement of the main theorem and the first result concerning the convergence of the wavelets series defined in (1.4).

Let us first briefly recall some definitions and notations about wavelets and biortogonal wavelets (for more precisions, see e.g. Daubechies (1992); Meyer (1992); Mallat (2009); Cohen et al. (1992)). Under some general assumptions, there exist two functions $\phi$ and $\psi$, called wavelets, which generate two orthonormal bases of $L^2(\mathbb{R})$, namely

$$\{\phi(\cdot - k)\}_{k\in\mathbb{Z}} \cup \{\psi(2^j \cdot -k) : j \in \mathbb{N}, k \in \mathbb{Z}\}$$

and

$$\{\psi(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}.$$

Any function $f \in L^2(\mathbb{R})$ can be decomposed as follows,

$$f = \sum_{k\in\mathbb{Z}} C_k \phi(\cdot - k) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot -k) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot -k)$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbb{R}^n} f(x)\phi(x-k)\, dx.$$

Let us remark that we do not choose the $L^2(\mathbb{R})$ normalization for the wavelets, but rather an $L^\infty(\mathbb{R})$ normalization, which is better fitted to the study of the Hölderian regularity. Amoung the families of wavelet basis that exist, we are mostly interested in two classes:

- The Lemarié-Meyer wavelets for which $\phi$ and $\psi$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$, which means that, for all $m \in \mathbb{N}_0$ and $L > 0$ we have
$$\sup_{x\in\mathbb{R}} \left\{ (3+|x|)^L \left| D^m \psi(x) \right| \right\} < +\infty. \tag{2.1}$$
- Daubechies wavelets for which $\phi$ and $\psi$ are compactly supported functions.

In both cases, the first moment of the wavelet $\psi$ vanishes.

The setting in which we work is more general that just orthogonal wavelet basis, so that it allows to cover the important example supply by the fractional Brownian motion. Biorthogonal wavelet bases are a couple of two Riesz wavelet bases of $L^2(\mathbb{R})$ generated respectively by $\psi$ and $\widetilde{\psi}$ and such that

$$2^{j/2} 2^{j'/2} \int_{\mathbb{R}} \psi(2^j x - k) \widetilde{\psi}(2^{j'} x - k') dx = \delta_{j,j'} \delta_{k,k'}.$$

In that case, any function $f \in L^2(\mathbb{R})$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot -k)$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x) \widetilde{\psi}(2^j x - k) dx.$$

Wavelet bases and biorthogonal wavelet bases give a powerful tool to study the regularity and irregularity of functions or signals belonging to numerous functional spaces, see e.g. Jaffard and Meyer (1996); Meyer (1992); Bastin et al. (2016); Jaffard (2004a); Aubry and Bastin (2010); Jaffard (2004b); Clausel and Nicolay (2011); Esser et al. (2017); Kreit and Nicolay (2018); Loosveldt and Nicolay (2021). In this paper, they will be used to get irregularity properties in Section 4.

Let us now present two lemmata that allow to prove the uniform convergence on any compact set of the series defined in (1.4), where $\psi$ comes from any wavelet basis or biorthogonal wavelet basis. The first one is very classical and the second one gives informations about the asymptotic behaviour of a sequence of i.i.d Gaussian random variables.

**Lemma 2.1.** *There exists a constant $C_1 > 0$ such that, for all $x \in \mathbb{R}$*

$$\sum_{k\in\mathbb{Z}} \frac{1}{(3+|x-k|)^4} \leq C_1.$$

**Lemma 2.2.** *(Ayache and Bertrand, 2010; Ayache and Taqqu, 2003) Let $(\xi_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{Z}}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables. There are an event $\Omega^*$ of probability 1 and a positive random variable $C_2$ of finite moment of every order such that, for all $\omega \in \Omega^*$ and $(j,k) \in \mathbb{Z}^2$, the inequality*

$$|\xi_{j,k}(\omega)| \leq C_2(\omega)\sqrt{\log(3+j+|k|)} \tag{2.2}$$

*holds.*

Let $f_h$ denote the process defined in (1.4). For all $j \in \mathbb{N}$, we set

$$f_{h,j} = \sum_{k\in\mathbb{Z}} \xi_{j,k} 2^{-hj}\psi(2^j\cdot -k).$$

Remark that, if inequality (2.2) holds then, thanks to Lemma 2.1 and the fast decay of the wavelet and its derivative (or using the compactness of the support of $\psi$), the sum in the right-hand side converges uniformly on any compact set, as well as the sum

$$\sum_{k\in\mathbb{Z}} \xi_{j,k} 2^{(1-h)j} D\psi(2^j\cdot -k). \tag{2.3}$$

Therefore, for all $j$, $f_{h,j}$ is continuously differentiable with derivative $Df_{h,j}$ given by (2.3). In particular, the process $f_h$ is well defined and bounded on the event $\Omega^*$ of probability 1. In the following, we will hence work on this event without mentioning it explicitly.

Using similar arguments, the following Proposition gives us a first information concerning the regularity of the function $f_h$.

**Proposition 2.3.** *Let $f_h$ be the random wavelet series defined in (1.4). If $\psi$ is continuously differentiable, there exists a constant $C > 0$ such that, for all $s, t \in \mathbb{R}$*

$$\mathbb{E}[(f_h(t) - f_h(s))^2] \leq C|s-t|^{2h}.$$

*Proof*: From the independence of the centred random variables in $(\xi_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{Z}}$, we have

$$\mathbb{E}[(f_h(t) - f_h(s))^2] = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} 2^{-2hj}(\psi(2^jt-k) - \psi(2^js-k))^2.$$

Let us fix $t$ and assume that $s$ and $\nu \in \mathbb{Z}$ are such that

$$2^{-\nu} < |t-s| \leq 2^{-\nu+1}.$$

For all $j \leq \nu$, we set

$$F_{j,t} : x \mapsto \sum_{k\in\mathbb{Z}} 2^{-2hj}(\psi(2^jt-k) - \psi(2^jx-k))^2.$$

Let us then remark that $F_{j,t}$ is continuously differentiable on any compact set. If $\psi$ is compactly supported, this is obvious. Otherwise, it comes from the fast decay of $\psi$ and Lemma 2.1. In any case, by the mean value theorem, there exists $x_1$ between $s$ and $t$ such that

$$\begin{aligned}|F_{j,t}(s) - F_{j,t}(t)| &= \left|(s-t)\sum_{k\in\mathbb{Z}} 2^{(1-2h)j} 2(\psi(2^jt-k) - \psi(2^jx_1-k))D\psi(2^jx_1-k)\right| \\ &\leq C\left|(s-t)\sum_{k\in\mathbb{Z}} 2^{(1-2h)j}(\psi(2^jt-k) - \psi(2^jx_1-k))\right|.\end{aligned}$$

Applying again the mean value theorem to the function

$$g_j \,:\, x \mapsto \sum_{k\in\mathbb{Z}} 2^{(1-2h)j}\psi(2^j x - k),$$

we get

$$|F_{j,t}(s) - F_{j,t}(t)| \le C|s-t|^2 2^{(2-2h)j}$$

because

$$\sum_{k\in\mathbb{Z}} D\psi(2^j t - k)$$

can be uniformly bounded, by the fast decay and Lemma 2.1 or using the compact support of $\psi$. With the same argument, we also get, for all $j > \nu$,

$$\begin{aligned}\Big|\sum_{k\in\mathbb{Z}} 2^{-2hj}(\psi(2^j t-k) - \psi(2^j s - k))^2\Big| &\le 2^{-2hj}\sum_{k\in\mathbb{Z}} 2((\psi(2^j t - k))^2 + (\psi(2^j s-k))^2)\\ &\le C2^{-2hj}.\end{aligned}$$

Putting all of these together, we have

$$\begin{aligned}\mathbb{E}[(f_h(t) - f_h(s))^2] &= \sum_{j\le\nu}\sum_{k\in\mathbb{Z}} 2^{-2hj}(\psi(2^j t-k) - \psi(2^j s-k))^2\\ &\quad + \sum_{j>\nu}\sum_{k\in\mathbb{Z}} 2^{-2hj}(\psi(2^j t-k) - \psi(2^j s-k))^2\\ &\le C(|s-t|^2\sum_{j\le\nu} 2^{(2-2h)j} + \sum_{j>\nu} 2^{-2hj})\\ &\le C(|s-t|^2 2^{(2-2h)\nu} + 2^{-2h\nu})\\ &\le C|s-t|^{2h}.\end{aligned}$$

□

From the last proposition and Kolmogorov continuity Theorem for Gaussian processes, we know that the sample paths of $f_h$ are almost surely locally Hölder-continuous of order $h-\varepsilon$ for every $\varepsilon > 0$. Our aim in this paper is to give more precise information concerning the regularity of $f_h$. In order to state it, we recall finally that a modulus of continuity is an increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\omega(0) = 0$ and for which there is $C > 0$ such that $\omega(2x) \le C\omega(x)$ for all $x \in \mathbb{R}^+$.

Wavelet characterizations of regularity require the following additional regularity property for moduli of continuity, see Jaffard and Meyer (1996): A modulus of continuity $\omega$ is regular if there is $N \ge 0$ such that

$$\begin{cases} \displaystyle\sum_{j=J}^{\infty} 2^{Nj}\omega(2^{-j}) \le C2^{NJ}\omega(2^{-J}) \\ \displaystyle\sum_{j=-\infty}^{J} 2^{(N+1)j}\omega(2^{-j}) \le C2^{(N+1)J}\omega(2^{-J}) \end{cases} \tag{2.4}$$

for all $J \ge 0$. Our main result will use three different regular moduli of continuity:

- the modulus of continuity $\omega_r$ of the rapid points is defined by

$$\omega_r^{(h)}(x) = |x|^h\sqrt{\log|x|^{-1}}$$

- the modulus of continuity $\omega_o$ of the ordinary points is defined by

$$\omega_o^{(h)}(x) = |x|^h\sqrt{\log\log|x|^{-1}}$$

- the modulus of continuity $\omega_s$ of the slow points is defined by
$$\omega_s^{(h)}(x) = |x|^h.$$

**Theorem 2.4.** *Almost surely, the random wavelets series defined in* (1.4) *satisfies the following property for every non-empty interval $I$ of $\mathbb{R}$:*

- *For almost every $t \in I$,*
$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_o^{(h)}(|s-t|)} < +\infty \tag{2.5}$$
*and if $\omega$ is a modulus of continuity such that $\omega = o(\omega_o^{(h)})$, then*
$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega(|s-t|)} = +\infty, \tag{2.6}$$
*Such points are called* ordinary points*.*
- *There exists $t \in I$ such that*
$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_r^{(h)}(|s-t|)} < +\infty \tag{2.7}$$
*and if $\omega$ is a modulus of continuity such that $\omega = o(\omega_r^{(h)})$, then*
$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega(|s-t|)} = +\infty, \tag{2.8}$$
*Such points are called* rapid points*.*
- *There exists $t \in I$ such that*
$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_s^{(h)}(|s-t|)} < +\infty. \tag{2.9}$$
*and if $\omega$ is a modulus of continuity such that $\omega = o(\omega_s^{(h)})$, then*
$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega(|s-t|)} = +\infty. \tag{2.10}$$
*Such points are called* slow points*.*

*Remark* 2.5. Theorem 2.4 is stated in full generality but let us already emphasize that it can be improved while considering compactly supported wavelets. Indeed, in this case, one can show the strict positiveness of the limits in (2.5), (2.7) and (2.9), see Remark 5.2 below.

Theorem 2.4 generalises the famous result of Kahane about the different pointwise behaviours of the Brownian motion, see Kahane (1985, Theorem 3 in Chapter 16). Moreover, it can be applied to the random wavelet series in (1.3), hence it proves the existence of the three types of points for the fractional Brownian motion. As already stated in the introduction of this paper, the existence of rapid and ordinary points for the fractional Brownian motion is already known. Thus, our contribution concerning the knowledge of the pointwise behaviour of this process is the existence of slow points. In particular, the reader should note that wavelet theory is, up to now and our knowledge, the only tool to emphasize the existence of such points for the fractional Brownian motion.

*Remark* 2.6. A good tool to determine the regularity of a locally bounded function $f$ at a point $t$ is to compute its Hölder exponent $h_f(t)$. If $\alpha > 0$, we say that $f$ belongs to the *pointwise Hölder space* $C^\alpha(t)$ if there exists a polynomial $P_t$ of degree strictly less than $\alpha$ and a constant $C > 0$ such that, for all $j \in \mathbb{N}$,
$$\sup_{x\in B(x_0,2^{-j})} |f(x) - P_t(x)| \leq C 2^{-\alpha j}.$$

It is straightforward to show that, as soon as $\alpha < \beta$, $C^{\beta}(t) \subseteq C^{\alpha}(t)$. Therefore, we define the *pointwise Hölder exponent* of $f$ at point $t$ by

$$h_f(t) = \sup\{\alpha > 0 \,:\, f \in C^{\alpha}(t)\}.$$

Theorem 2.4 tells us in particular that, if $t$ is an ordinary or rapid point, then, for all $\varepsilon > 0$, $f_h \in C^{h-\varepsilon}(t)$ and $f_h \notin C^h(t)$ which gives $h_{f_h}(t) = h$. On the contrary, if $t$ is a slow point, $f_h \in C^h(t)$ while, for all $\varepsilon > 0$, $f_h \notin C^{h+\varepsilon}(t)$ and so, again, $h_{f_h}(t) = h$. In fact, the finiteness of the limits (2.5), (2.7) and (2.9) means that the function $f$ belongs to a generalized pointwise Hölder space Kreit and Nicolay (2018); Loosveldt and Nicolay (2021) associated to the corresponding modulus of continuity.

The next sections are dedicated to the proof of this result. Any open interval in $\mathbb{R}$ can be written as a countable union of dyadic intervals. Then, to prove Theorem 2.4, it is sufficient to show that, for all dyadic interval of the form $\lambda_{j,k} = [k2^{-j}, (k+1)2^{-j})$ with $j \in \mathbb{N}, k \in \mathbb{Z}$, there exist an event $\Omega_{j,k}$ of probability 1 such that, for all $\omega \in \Omega_{j,k}$, almost every $t \in \lambda_{j,k}$ is ordinary and there exist $t_r \in \lambda_{j,k}$ which is rapid and $t_s \in \lambda_{j,k}$ which is slow. For the sake of simpleness in notation, we will only do the proofs in full details for $\lambda_{0,0} = [0,1)$. In fact, after dilatations and translations, our proofs hold true for any arbitrary dyadic interval. Note also that the proofs will be done in the case where the wavelet $\psi$ is in the Schwartz class; it can easily be adapted and simplified if $\psi$ is compactly supported.

## 3. Regularity properties

In this section, we establish inequalities (2.5), (2.7) and (2.9). Concerning rapid and ordinary points, the conduct of the proof is similar. First we use Lemma 2.2 to bound the coefficients in (1.4). Then we use the fast decay of the wavelet to measure the contribution of the coefficients associated to dyadic intervals that are far away from the point of interest $t$ in the difference $|f_h(s) - f_h(t)|$. These are rather classic techniques of wavelet theory. The more challenging part concerns the proof of the existence of slow points. In this purpose, we take advantage of a procedure initiated by Kahane in Kahane (1985) to identify points for which we can obtain more precise information concerning the coefficients of the "closest" intervals while we still use Lemma 2.2 and the fast decay for the "furthest" one. This part uses more specific arguments and highlight the benefit we have to work with a wavelet expansion with independent coefficients.

3.1. *Rapid points.* To prove the existence of rapid points, we apply Lemma 2.2 and get an uniform modulus of continuity for the function $f_h$. We deal with the coefficients associated to furthest intervals thanks to the following lemma.

**Lemma 3.1.** *There exists a constant $C_1 > 0$ such that, for all $j \in \mathbb{N}_0$ and $x \in (0,1)$,*

$$\sum_{|k|>2^{j+1}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j x - k|)^5} \leq C_1$$

*Proof*: Let us fix $j \in \mathbb{N}_0$ and $x \in (0,1)$. As $|k| > 2^{j+1}$, obviously, $\frac{k}{2^j} \notin (-2,2)$ so let $n \in \mathbb{Z} \setminus \{0, \pm 1, -2\}$ be such that $n2^j \leq k < (n+1)2^j$. Now, as $x \in (0,1)$, we have

$$|2^j x - k| \geq 2^j |x - n| - 1 \geq 2^j(|n| - 1) - 1 \geq \frac{|k|}{2} - 1.$$

Thus, for all such $j, k$ and $x$, we have

$$\frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j x - k|)} \leq 2\frac{\sqrt{\log(3+2|k|)}}{|k|}$$

and we conclude using the boundedness of the function $x \mapsto \frac{\sqrt{\log(3+2x)}}{x}$ on $[1, +\infty[$ and Lemma 2.1. □

**Proposition 3.2.** *Almost surely, there exists a constant $C_1 > 0$ such that, for all $t, s \in (0,1)$ we have*

$$|f_h(s) - f_h(t)| \leq C_1 |t-s|^h \sqrt{\log|t-s|^{-1}}.$$

*In particular, one has almost surely*

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{\omega_r^{(h)}(|s-t|)} < +\infty$$

*for every $t \in (0,1)$.*

*Proof*: Let us assume that $t, s$ and $\nu \in \mathbb{N}$ are such that

$$2^{-\nu} < |t-s| \leq 2^{-\nu+1}.$$

For all $j \leq \nu$, by the mean value theorem, there exists $x$ between $s$ and $t$ such that

$$|f_{h,j}(t) - f_{h,j}(s)| \leq |t-s||Df_{h,j}(x)| \tag{3.1}$$

and, it follows that

$$\begin{aligned}|Df_{h,j}(x)| \leq CC_2 2^{(1-h)j} \Big( &\sum_{|k| \leq 2^{j+1}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j x - k|)^4} \\ &+ \sum_{|k| > 2^{j+1}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j x - k|)^5} \Big),\end{aligned}$$

using (2.2) and the fast decay property (2.1) with $m = 1$ and $L = 4$, for the first sum, and $L = 5$, for the second one.

We bound the second sum by Lemma 3.1 while, for the first sum, we have

$$\sum_{|k| \leq 2^{j+1}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|2^j x - k|)^4} \leq C \sum_{|k| \leq 2^{j+1}} \frac{\sqrt{j}}{(3+|2^j x - k|)^4} \leq C\sqrt{j}$$

by Lemma 2.1. Thus, we obtain

$$\begin{aligned}|\sum_{j \leq \nu} (f_{h,j}(t) - f_{h,j}(s))| &\leq CC_2 |t-s| 2^{(1-h)\nu} \sqrt{\nu} \\ &\leq CC_2(\omega) |t-s|^h \sqrt{\log|t-s|^{-1}}.\end{aligned}$$

Now, if $j > \nu$, by splitting the sums in the same way, we also have

$$|f_{h,j}(t)| \leq CC_2 2^{-hj} \sqrt{j} \text{ and } |f_{h,j}(s)| \leq CC_2 2^{-hj} \sqrt{j}$$

which obviously leads to

$$|\sum_{j > \nu} f_{h,j}(t)| \leq CC_2 |t-s|^h \sqrt{\log|t-s|^{-1}}$$

and

$$|\sum_{j > \nu} f_{h,j}(s)| \leq CC_2 |t-s|^h \sqrt{\log|t-s|^{-1}}.$$

The conclusion follows immediately. □

3.2. *Ordinary points.* To establish the existence of ordinary points, we need to introduce some notations. If $j \in \mathbb{N}_0$, and $t \in (0,1)$, we denote by $k_j(t)$ the unique positive integer in $\{0, \dots, 2^j - 1\}$ such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j})$. We also define, for all $n \in \mathbb{N}$ the set

$$\kappa_j^t(n) = \{k \in \mathbb{Z} : |k - k_j(t)| \leq n\}$$

The main proof of this section consists in applying Lemma 2.2 to a sequence of independent $\mathcal{N}(0,1)$ random variables indexed from $k_j(t)$. Then, Lemmata 3.3 and 3.4 help us to deal with the coefficients associated with the furthest intervals.

**Lemma 3.3.** *There exists a constant $C_1 > 0$ such that, for all $j \in \mathbb{N}_0$, $t \in (0,1)$ and $x \in (0,1)$ such that $|t - x| \leq 2^{-j+1}$ we have*

$$\sum_{k \notin \kappa_j^t(j)} \frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(3 + |2^j x - k|)^5} \leq C_1$$

*Proof*: Let us fix $j \in \mathbb{N}_0$, $t \in (0,1)$ and $x \in (0,1)$ such that $|t - x| \leq 2^{-j+1}$. If $k \notin \kappa_j^t(j)$, we immediately have $|2^j x - k_j(t)| \leq 3$ and thus $|2^j x - k| \geq |k_j(t) - k| - 3$. Therefore,

$$\frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(3 + |2^j x - k|)} \leq \frac{\sqrt{\log(3 + 2|k - k_j(t)|)}}{|k_j(t) - k|}$$

and the conclusion follows just as in Lemma 3.1. □

**Lemma 3.4.** *There exists a constant $C_1 > 0$ such that, for all $j \in \mathbb{N}_0$, $t \in (0,1)$ and $s \in (0,1)$ such that $|2^j s - k_j(t)| \leq j$ we have*

$$\sum_{k \notin \kappa_j^t(2j)} \frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(3 + |2^j s - k|)^5} \leq C_1$$

*Proof*: Let us fix $j \in \mathbb{N}_0$ $t \in (0,1)$ and $s \in (0,1)$ such that $|2^j s - k_j(t)| \leq j$. If $k \notin \kappa_j^t(2j)$, of course, $|2^j s - k_j(t)| < \frac{|k - k_j(t)|}{2}$ and thus $|2^j s - k| \geq \frac{|k - k_j(t)|}{2}$. It follows that

$$\frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{3 + |2^j s - k|} \leq 2\frac{\sqrt{\log(3 + 2|k - k_j(t)|)}}{|k - k_j(t)|}$$

and, again, we conclude just as in Lemma 3.1. □

**Proposition 3.5.** *Almost surely, for almost every $t \in (0,1)$,*

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{\omega_o^{(h)}(|t - s|)} < +\infty.$$

*Proof*: Let us fix $t \in (0,1)$, and $j \in \mathbb{N}_0$ and let $0 \leq k_j(t) < 2^j$ be such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j}[$. A simple modification of Lemma 2.2 insure us the existence of a positive random variable $C_t$ of finite moment of every order such that, almost surely

$$|\xi_{j,k}| \leq C_t \sqrt{\log(3 + j + |k - k_j(t)|)}. \tag{3.2}$$

As previously, if $s \in (0,1)$ and $\nu \in \mathbb{N}$ is such that

$$2^{-\nu} < |t - s| \leq 2^{-\nu+1},$$

we first start by considering, for all $j \leq \nu$ ,$|Df_{h,j}(x)|$ for a $x$ between $s$ and $t$. In this case, we have

$$|Df_{h,j}(x)| \leq CC_t 2^{(1-h)j}\Big( \sum_{k\in\kappa_j^t(j)} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^jx-k|)^4} + \sum_{k\notin\kappa_j^t(j)} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^jx-k|)^5}\Big),$$

using (3.2) and the fast decay property (2.1) with $m = 1$ and $L = 4$, for the first sum, and $L = 5$, for the second one.

We bound the second sum by Lemma 3.3, while, for the first sum, we have

$$\sum_{k\in\kappa_j^t(j)} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^jx-k|)^4} \leq C \sum_{k\in\kappa_j^t(j)} \frac{\sqrt{\log(j)}}{(3+|2^jx-k|)^4} \leq C\sqrt{\log(j)}$$

by Lemma 2.1. These inequalities lead to

$$|\sum_{j\leq\nu}(f_{h,j}(t)-f_{h,j}(s))| \leq CC_t|t-s|^h\sqrt{\log\log|t-s|^{-1}}. \tag{3.3}$$

To bound $|f_{h,j}(t)|$, for all $j > \nu$, we use the same techniques and get

$$|\sum_{j>\nu} f_{h,j}(t)| \leq CC_t|t-s|^h\sqrt{\log\log|t-s|^{-1}}.$$

The bound for $|f_{h,j}(s)|$ is a little bit more tricky. As $|2^js - k_j(t)| \leq 2^{j-\nu+2}$, we first consider the case when $2^{j-\nu+2} \leq j$. Then, using again (3.2) and the fast decay property (2.1) in the same way, we get

$$|f_{h,j}(s)| \leq CC_t 2^{-hj}\Big( \sum_{k\in\kappa_j^t(2j)} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^js-k|)^4} + \sum_{k\notin\kappa_j^t(2j)} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^js-k|)^5}\Big).$$

The second sum is this time bounded using Lemma 3.4 while we again use Lemma 2.1 to get

$$\sum_{k\in\kappa_j^t(2j)} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^js-k|)^4} \leq C\sqrt{\log(j)}.$$

Now, if $j < 2^{j-\nu+2}$, we use one last time (3.2) and the fast decay property (2.1) in the same way to get

$$|f_{h,j}(s)| \leq CC_t 2^{-hj}\Big( \sum_{k\in\kappa_j^s(2^{j-\nu+2})} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^js-k|)^4} + \sum_{k\notin\kappa_j^s(2^{j-\nu+2})} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^js-k|)^5}\Big).$$

As $|k_j(s) - k_j(t)| \leq 2^{j-\nu+1} + 2$, for all $k \in \kappa_j^s(2^{j-\nu+2})$ we have

$$\log(3+j+|k-k_j(t)|) \leq \log(3+j+|k_j(s)-k_j(t)|+|k-k_j(s)|) \leq \log(4.2^{j-\nu+2})$$

while, for all $k \notin \kappa_j^s(2^{j-\nu+2})$,

$$\begin{aligned}\frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^j s-k|)} &\leq \frac{\log(3+j+|k_j(s)-k_j(t)|+|k-k_j(s)|)}{|k-k_j(s)|}\\ &\leq \frac{\log(4|k-k_j(s)|)}{|k-k_j(s)|}\\ &\leq C\end{aligned}$$

which gives us, thanks to Lemma 2.1,

$$|f_{h,j}(s)| \leq CC_t 2^{-hj}\sqrt{j-\nu}. \tag{3.4}$$

In total, using (3.3) and (3.4), we get

$$\begin{aligned}|\sum_{j>\nu} f_{h,j}(s)| &\leq \sum_{j>\nu} CC_t 2^{-hj}\sqrt{j-\nu}\sqrt{\log(j)}\\ &\leq CC_t|t-s|^h\sqrt{\log\log|t-s|^{-1}}.\end{aligned}$$

The conclusion comes from Fubini's Theorem. □

3.3. *Slow points.*

3.3.1. *An iterative procedure for slow points.* The following procedure is inspired by the one initially described by Kahane in Kahane (1985) to identify slow points for the Brownian motion. We generalize it here by introducing an extra parameter $m \in \mathbb{N}$ in order to use it for any[1] $h > 0$. Some clarifications are also made.

If $\mu \in \mathbb{N}$, $\xi \sim \mathcal{N}(0,1)$ and $l \in \mathbb{N}_0$, we set

$$p_l(\mu) = \mathbb{P}(2^l\mu < |\xi| \leq 2^{l+1}\mu).$$

For all $j, l \in \mathbb{N}_0$ and $0 \leq k < 2^j$, we define

$$S^\mu_{j,l} = \{k \in \{0,\ldots,2^j-1\} : 2^l\mu < |\xi_{j,k}| \leq 2^{l+1}\mu\}$$

and

$$\Lambda_{j,l}(k) = \{k' \in \{0,\ldots,2^j-1\} : |k-k'| \leq 2^{ml}\}.$$

Note that $\#\Lambda_{j,l}(k) \leq 2^{ml+1}+1$. For all $j \in \mathbb{N}_0$, we define a closed set from the dyadic intervals $[k2^{-j},(k+1)2^{-j}]$ for which $k$ belongs to the set

$$I^\mu_j = \{k \in \{0,\ldots,2^j-1\} : \forall l \in \mathbb{N}_0,\ \Lambda_{j,l}(k) \cap S^\mu_{j,l} = \emptyset\},$$

namely, we consider

$$F^\mu_j = \bigcup_{k\in I^\mu_j}[k2^{-j},(k+1)2^{-j}].$$

We want to show that, almost surely, there exists $\mu \in \mathbb{N}$ such that

$$S^\mu_{\text{low}} = \bigcap_{j\in\mathbb{N}_0} F^\mu_j \neq \emptyset$$

which is equivalent to the fact that, for all $J \in \mathbb{N}_0$,

$$S^\mu_{\text{low},J} = \bigcap_{j\leq J} F^\mu_j$$

[1]Kahane only considered the case $h = \frac{1}{2}$ and so its construction is made with $m = 3$ which is sufficient to fulfil the condition $h > \frac{1}{m}$ we will need afterwards.

is non-empty, as $(S^\mu_{\text{low},J})_{J\in\mathbb{N}_0}$ is a decreasing sequence of compact sets. In other words, if we denote by $N^\mu_J$ the number of subintervals of $S^\mu_{\text{low},J}$, we want to show that

$$\mathbb{P}(\bigcup_{\mu\in\mathbb{N}}\bigcap_{J\in\mathbb{N}_0}\{N^\mu_J \geq 1\}) = 1.$$

For this purpose, we will consider sufficiently large $\mu$ such that the inequality

$$\sum_{l=0}^{+\infty}(2^{ml+1}+1)(p_l(\mu)+l\sqrt{p_l(\mu)(1-p_l(\mu))}) < \frac{1}{4} \tag{3.5}$$

holds. Moreover, if $J_1$ is fixed, let us remark that the construction of $S^\mu_{\text{low},J_1}$ needs to consider at most $\sum_{j=0}^{J_1} 2^j$ i.i.d. $\mathcal{N}(0,1)$ random variables and, by increasing $\mu$ if necessary, one can choose to remove the intervals $[0,2^{-J_1}]$ and $[1-2^{-J_1},1]$ from $S^\mu_{\text{low},J_1}$ at this step if necessary while making $\mathbb{P}(N^\mu_{J_1} \geq (\frac{3}{2})^{J_1})$ as close to 1 as we want. This trick helps us to start our construction with a arbitrary close to 1 "initial value" of probability and to make sure that, at the end, the resulting points will differ from 0 and 1.

**Lemma 3.6.** *For all $\mu\in\mathbb{N}$ sufficiently large such that condition (3.5) holds, the sequence $(N^\mu_J)_{J\in\mathbb{N}_0}$ of random variables satisfies the formula*

$$\mathbb{P}(N^\mu_{J+1} \geq (\frac{3}{2})^{J+1}) \geq (1-(\frac{2}{3})^J)\mathbb{P}(N^\mu_J \geq (\frac{3}{2})^J), \quad \forall J\in\mathbb{N}_0. \tag{3.6}$$

*Proof*: For all $J$, we define[2]

$$\mathcal{I}^\mu_J = \{k\in\{0,\ldots,2^J-1\} : [k2^{-J},(k+1)2^{-J}] \subseteq S^\mu_{\text{low},J}\},$$

and remark that $\mathcal{I}^\mu_{J+1}$ is obtained by removing from $2\mathcal{I}^\mu_J \cup 2\mathcal{I}^\mu_J+1$ the elements $k\in\{0,\ldots,2^{J+1}-1\}$ such that $\Lambda_{J+1,l}(k)\cap S^\mu_{J+1,l} \neq \emptyset$ for a $l\in\mathbb{N}_0$. But now, for all such a $l$, if $\mathcal{B}(n,p)$ stands for a binomial random variable with $n$ trials with success probability $p$, and $N^\mu_J = N$,

$$\#(S^\mu_{J+1,l}\cap(2\mathcal{I}^\mu_J\cup 2\mathcal{I}^\mu_J+1)) \sim \mathcal{B}(2N, p_l(\mu)).$$

Therefore, we have, by Chebyshev's inequality, that

$$\#(S^\mu_{J+1,l}\cap(2\mathcal{I}^\mu_J\cup 2\mathcal{I}^\mu_J+1)) \leq 2N(p_l(\mu)+(l+1)\sqrt{p_l(\mu)(1-p_l(\mu))})$$

with probability greater than $1-(l+1)^{-2}N^{-1}$. Thus the number of removed $k$ is bounded by

$$2N\sum_{l=0}^{+\infty}(2^{ml+1}+1)(p_l(\mu)+l\sqrt{p_l(\mu)(1-p_l(\mu))})$$

with probability greater than $1-N^{-1}$. Now, with condition (3.5), we get, for all $J$,

$$\mathbb{P}(N^\mu_{J+1} \geq \frac{3}{2}N_J | N^\mu_J = N) \geq 1-N^{-1}$$

[2] We obviously have $N^\mu_J = \#\mathcal{I}^\mu_J$.

which gives us

$$\begin{aligned}\mathbb{P}(N^\mu_{J+1} \geq (\frac{3}{2})^{J+1}) &\geq \mathbb{P}((N^\mu_{J+1} \geq \frac{3}{2}N^\mu_J) \cap (N^\mu_J \geq (\frac{3}{2})^J)) \\ &= \sum_{N \geq (\frac{3}{2})^J} \mathbb{P}(N^\mu_{J+1} \geq \frac{3}{2}N_J | N^\mu_J = N)\mathbb{P}(N^\mu_J = N) \\ &\geq \sum_{N \geq (\frac{3}{2})^J} (1 - N^{-1})\mathbb{P}(N^\mu_J = N) \\ &\geq (1 - (\frac{2}{3})^J) \sum_{N \geq (\frac{3}{2})^J} \mathbb{P}(N^\mu_J = N) \\ &= (1 - (\frac{2}{3})^J)\mathbb{P}(N^\mu_J \geq (\frac{3}{2})^J).\end{aligned}$$

□

**Proposition 3.7.** *Almost surely, there exists $\mu \in \mathbb{N}$ such that $(0,1) \cap S^\mu_{low} \neq \emptyset$.*

*Proof*: From formula (3.6), if $\mu$ is large enough we deduce by induction that for all $J_1, J$ with $J_1 \leq J$,

$$\begin{aligned}\mathbb{P}(N^\mu_J \geq 1) &\geq \mathbb{P}(N^\mu_J \geq (\frac{3}{2})^J) \\ &\geq \mathbb{P}(N^\mu_{J_1} \geq (\frac{3}{2})^{J_1})\Big(\prod_{j=J_1}^{J-1} (1 - (\frac{2}{3})^j)\Big).\end{aligned}$$

Let us remark that, as $\sum_{j=1}^\infty (\frac{2}{3})^j < \infty$, the infinite product $\prod_{j=1}^\infty (1 - (\frac{2}{3})^j)$ converges to a non-zero limit and

$$\lim_{J_1 \to +\infty} \prod_{j=J_1}^{+\infty} (1 - (\frac{2}{3})^j) = 1.$$

Now, for all $0 < \varepsilon < \frac{1}{2}$, one can choose $J_1$ such that

$$\prod_{j=J_1}^{+\infty} (1 - (\frac{2}{3})^j) > 1 - \varepsilon$$

and, by increasing $\mu$ if necessary, we can choose to remove the intervals $[0, 2^{-J_1}]$ and $[1 - 2^{-J_1}, 1]$ from $S^\mu_{\mathrm{low},J_1}$, if necessary and assume

$$\mathbb{P}(N^\mu_{J_1} \geq (\frac{3}{2})^{J_1}) > 1 - \varepsilon.$$

Therefore, as the sequence of events $(\{N^\mu_J \geq 1\})_{J \in \mathbb{N}_0}$ is decreasing, we get

$$\mathbb{P}\big(\bigcap_{J \in \mathbb{N}_0} (N^\mu_J \geq 1)\big) \geq \mathbb{P}(N^\mu_{J_1} \geq (\frac{3}{2})^{J_1})\Big(\prod_{j=J_1}^{\infty} (1 - (\frac{2}{3})^j)\Big) > (1 - \varepsilon)^2.$$

In total, we have shown that, for all $0 < \varepsilon < \frac{1}{2}$,

$$\mathbb{P}\big(\bigcup_{\mu \in \mathbb{N}} \bigcap_{J \in \mathbb{N}_0} (N^\mu_J \geq 1)\big) > (1 - \varepsilon)^2$$

and the conclusion follows immediately. □

3.3.2. *Existence of slow points.* Now, using this iterative procedure, we establish the existence of points satisfying inequality (2.9). In this section, it will be convenient to write

$$f_{h,j} = \sum_{n\in\mathbb{Z}} f_{h,j}^{[n]} \tag{3.7}$$

where, for all $n \in \mathbb{Z}$,

$$f_{h,j}^{[n]} = \sum_{k=n2^j}^{(n+1)2^j-1} \xi_{j,k} 2^{-hj}\psi(2^j\cdot -k)$$

is the partial sum involving the $k$ for which the corresponding dyadic interval $[k2^{-j},(k+1)2^{-j})$ is included in $[n,n+1)$.

**Proposition 3.8.** *Almost surely, there exists $t\in(0,1)$ such that*

$$\limsup_{s\to t} \frac{|f_h(s)-f_h(t)|}{\omega_s^{(h)}(|t-s|)} < +\infty.$$

*Proof*: Let us fix $m\in\mathbb{N}_0$ such that $h\geq 1/m$. The iterative procedure with $m$ gives that almost surely, $(0,1)\cap S_{\text{low}}^{\mu}\neq\emptyset$. Now let $t\in(0,1)\cap S_{\text{low}}^{\mu}$. There exists $r>0$ such that $[t-r,t+r]\subset(0,1)$ and so, let us take $s\in[t-r,t+r]$ and $\nu\in\mathbb{N}$ be such that

$$2^{-\nu} < |t-s| \leq 2^{-\nu+1}.$$

We are going to estimate from above $|f_h(t)-f_h(s)|$.

For all $j\leq\nu$, once again, by the mean value theorem, there exists $x$ between $s$ and $t$ such that

$$|f_{h,j}(t)-f_{h,j}(s)| \leq |s-t||Df_{h,j}(x)|.$$

To bound $|Df_{h,j}(x)|$, we will use the decomposition (3.7). For $n=0$, if $0\leq l\leq\lceil j/m\rceil$, $0\leq k<2^j$ and $|k_j(t)-k|\leq 2^{ml}$, then necessarily $|\xi_{j,k}|\leq 2^l\mu$. Thus, if we set

$$\Lambda_j^0(t) = \{0\leq k<2^j \,:\, |k_j(t)-k|\leq 1\}$$

and, for all $1\leq l\leq\lceil j/m\rceil$

$$\Lambda_j^l(t) = \{0\leq k<2^j \,:\, 2^{m(l-1)} < |k_j(t)-k| \leq 2^{ml}\},$$

we have, using the fast decay property (2.1) with $m=1$ and $L=5$,

$$|Df_{h,j}^{[0]}(x)| \leq 2^{(1-h)j} \sum_{l=0}^{\lceil j/m\rceil} \sum_{k\in\Lambda^l(t)} |\xi_{j,k}||D\psi(2^jx-k)| \tag{3.8}$$

$$\leq C2^{(1-h)j} \sum_{l=0}^{\lceil j/m\rceil} \sum_{k\in\Lambda_j^l(t)} 2^l\mu \frac{1}{(3+|2^jx-k|)^5}. \tag{3.9}$$

As, for $l\geq 1$ and $k\in\Lambda_j^l(t)$, $|2^jx-k|\geq|k_j(t)-k|-3>2^{m(l-1)}-3$, we obtain

$$|Df_{h,j}^{[0]}(x)| \leq C\mu 2^{(1-h)j} \sum_{k=0}^{2^j-1} \frac{1}{(3+|2^jx-k|)^4}.$$

Now, there exists[3] $C_t > 0$, such that, for all $n \neq 0$ and $n2^j \leq k < (n+1)2^j - 1$, we have $|2^j x - k| \geq C_t |n| 2^j$, which gives, using similarly the fast decay property (2.1),

$$\begin{aligned}|Df_{h,j}^{[n]}(x)| &\leq CC_2 2^{(1-h)j} \sum_{k=n2^j}^{(n+1)2^j-1} \frac{1}{(3+|2^j x - k|)^4} \frac{\sqrt{\log(3+j+|k|)}}{C_t|n|2^j} \\ &\leq CC_2 2^{(1-h)j} \sum_{k=n2^j}^{(n+1)2^j-1} \frac{1}{(3+|2^j x - k|)^4}.\end{aligned}$$

Let us define the random variable $C_\mu = \max(C_2, \mu)$. By Lemma 2.1, we obtain, for all $j \leq \nu$,

$$|f_{h,j}(t) - f_{h,j}(s)| \leq C_\mu 2^{(1-h)j} |s-t|$$

and thus

$$\begin{aligned}|\sum_{j \leq \nu} (f_{h,j}(t) - f_{h,j}(s))| &\leq CC_\mu 2^{(1-h)\nu} |s-t| \\ &\leq CC_\mu^* |s-t|^h. \end{aligned} \tag{3.10}$$

Now, we consider the terms for $j > \nu$ and we will bound separately $|f_{h,j}(t)|$ and $|f_{h,j}(s)|$. For $|f_{h,j}(t)|$, we just have to repeat the same procedure, using the set $\Lambda_j^l(t)$ to estimate $|f_{h,j}^{[0]}(t)|$ and Lemma 2.2 for $|f_{h,j}^{[n]}(t)|$ with $n \neq 0$. We then conclude that

$$\begin{aligned}|\sum_{j>\nu} f_{h,j}(t)| &\leq C \sum_{j>\nu} C_\mu 2^{-hj} \\ &\leq CC_\mu 2^{-h\nu} \\ &\leq CC_\mu |t-s|^h\end{aligned}$$

For $|f_{h,j}(s)|$, the strategy remains the same: nothing changes to bound $|f_{h,j}^{[n]}(s)|$ with $n \neq 0$ while, for $n = 0$, if $l$ is the greatest integer such that $|s-t| \geq 2^{ml} 2^{-j}$, the construction insures that, for all $1 \leq l' \leq \lceil j/m \rceil$ and $k \in \Lambda_j^{l'}(s)$,

$$|\xi_{j,k}| \leq 2^l 2^{l'} \mu$$

and we get, from the fast decay property (2.1) wil $L = 4$,

$$|f_{h,j}^{[0]}(s)| \leq C 2^l \mu 2^{-hj} \sum_{k=0}^{2^j-1} \frac{1}{(3+|2^j s - k|)^4}.$$

But as $|t-s| \leq 2^{-\nu+1}$, we have $l \leq \frac{1}{m}(j+1-\nu)$ and thus

$$2^l \mu 2^{-hj} \leq 2^{\frac{1}{m}} \mu 2^{(\frac{1}{m}-h)(j-\nu)} 2^{-h\nu}.$$

It gives, as we took $\frac{1}{m} < h$, combined with Lemma 2.1,

$$|\sum_{j>\nu} f_{h,j}(s)| \leq CC_\mu |t-s|^h. \tag{3.11}$$

Finally, combining (3.10) with (3.11) allows to obtain

$$|f_h(t) - f_h(s)| \leq CC_\mu |t-s|^h$$

as desired. □

[3] We can make this constant only dependant of $t$. Note that, as $t$ is random, $C_t$ is a random variable as well.

## 4. Asymptotic behaviour of wavelet coefficients

In this section we study the asymptotic behaviour of the wavelet coefficients of the Gaussian wavelet series. It will allow to get irregularity properties for the random wavelets series $f_h$, i.e. to prove that the three corrections obtained in the previous section characterize exactly three possible pointwise behaviours. Let us start by recalling the following result from Ayache (2019, Lemma A.27).

**Lemma 4.1.** *(Ayache, 2019) Almost surely, for every $t \in \mathbb{R}$, one has*

$$\limsup_{j \to +\infty} |\xi_{j,k_j(t)}| \geq 2^{-3/2}\sqrt{\pi}.$$

Before stating the next lemma, let us recall that if $\xi \sim \mathcal{N}(0,1)$, then one has

$$\lim_{x \to +\infty} \frac{\mathbb{P}\big(|\xi| > x\big)}{(2\pi^{-1})^{1/2} x^{-1} e^{-x^2/2}} = 1\,. \tag{4.1}$$

The following result follows the lines of Ayache et al. (2019).

**Lemma 4.2.** *Almost surely, for almost every $t \in \mathbb{R}$, one has*

$$\limsup_{j \to +\infty} \frac{|\xi_{j,k_j(t)}|}{\sqrt{\log j}} > 0\,. \tag{4.2}$$

*Proof*: By Fubini's theorem, it is enough to prove that for every $t \in \mathbb{R}$, (4.2) holds almost surely. Let us fix $t \in \mathbb{R}$. For every $m \in \mathbb{N}$, we consider the event

$$A_m(t) = \Big\{ \max_{2^m \leq j < 2^{m+1}} |\xi_{2^j,k_j(t)}| \geq \sqrt{m \log 2} \Big\}.$$

Using the independence of the random variables $\xi_{j,k}$, $(j,k) \in \mathbb{N} \times \mathbb{Z}$, we have

$$\begin{aligned}
\mathbb{P}\Big(A_m(t)\Big) &= 1 - \prod_{2^m \leq j < 2^{m+1}} \mathbb{P}\Big(|\xi_{2^j,k_j(t)}| < \sqrt{m \log 2}\Big) \\
&= 1 - \Big(1 - \mathbb{P}\big(|\xi| > \sqrt{m \log 2}\,\big)\Big)^{2^m}
\end{aligned}$$

where $\xi \sim \mathcal{N}(0,1)$. Using (4.1) and the fact that $\log(1-x) \leq -x$ if $x \in (0,1)$, we obtain for $m$ large enough

$$\mathbb{P}\Big(A_m(t)\Big) \geq 1 - \Big(1 - C\frac{2^{-\frac{m}{2}}}{\sqrt{m \log 2}}\Big)^{2^m} \geq 1 - \exp\Big(-C\frac{2^{\frac{m}{2}}}{\sqrt{m \log 2}}\Big)$$

where $C = \frac{1}{2}\sqrt{\frac{2}{\pi}}$. Hence, it follows that

$$\sum_{m=0}^{+\infty} \mathbb{P}\Big(A_m(t)\Big) = +\infty$$

and since the events $A_m(t)$, $m \in \mathbb{N}$, are independents, the Borel-Cantelli lemma implies that

$$\mathbb{P}\Big(\bigcap_{M \in \mathbb{N}} \bigcup_{m \geq M} A_m(t)\Big) = 1\,.$$

It gives that almost surely, for infinitely many $m \in \mathbb{N}$, there is $j \in \{2^m, \ldots, 2^{m+1}-1\}$ such that

$$|\xi_{2^j,k_j(t)}| \geq \sqrt{m \log 2} \geq \sqrt{\log j - \log 2}.$$

The conclusions follows. $\square$

The last lemma we need is proved in Ayache et al. (2019). Note that in this paper, the authors work in a more general context. Indeed, it is only required that the sequence of standard Gaussian random variables satisfy the following condition: there is $N \in \mathbb{N}$ such that for every $(j_1, k_1), (j_2, k_2) \in \mathbb{N} \times \mathbb{Z}$ satisfying

$$\Big(\frac{k_1 - N}{2^{j_1}}, \frac{k_1 + N}{2^{j_1}}\Big) \cap \Big(\frac{k_2 - N}{2^{j_2}}, \frac{k_2 + N}{2^{j_2}}\Big) = \emptyset\,,$$

the random variables $\xi_{j_1,k_1}$ and $\xi_{j_2,k_2}$ are independent.

**Lemma 4.3.** *(Ayache et al., 2019) Almost surely, for every non-empty open interval $I$ of $\mathbb{R}$, there is $t \in I$ such that*

$$\limsup_{j\to+\infty} \left\{ \frac{|\xi_{j,k_j(t)}|}{\sqrt{j}} \right\} > 0\,.$$

## 5. Proof of the main result

Putting together Propositions 3.2, 3.5, 3.8 and Lemmata 4.1, 4.2 and 4.3, we can summarize the results obtained in the previous sections as follows.

**Corollary 5.1.** *Almost surely,*

- *there exists $t \in (0,1)$ such that*

$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_r^{(h)}(|t-s|)} < +\infty \quad \text{and} \quad \limsup_{j\to+\infty} \frac{|c_{j,\lfloor 2^j t\rfloor}|}{\omega_r^{(h)}(2^{-j})} > 0,$$

- *for almost every $t \in (0,1)$,*

$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_o^{(h)}(|t-s|)} < +\infty \quad \text{and} \quad \limsup_{j\to+\infty} \frac{|c_{j,\lfloor 2^j t\rfloor}|}{\omega_o^{(h)}(2^{-j})} > 0,$$

- *there exists $t \in (0,1)$ such that*

$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_s^{(h)}(|t-s|)} < +\infty \quad \text{and} \quad \limsup_{j\to+\infty} \frac{|c_{j,\lfloor 2^j t\rfloor}|}{\omega_s^{(h)}(2^{-j})} > 0.$$

We are now ready to prove our main result.

*Proof of Theorem 2.4:* Using Corollary 5.1, it suffices to prove the second part of the result. Let us consider $t \in (0,1)$ for which there is $v \in \{r, o, s\}$ such that

$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega_v^{(h)}(|t-s|)} < +\infty \quad \text{and} \quad \limsup_{j\to+\infty} \frac{|c_{j,\lfloor 2^j t\rfloor}|}{\omega_v^{(h)}(2^{-j})} > 0. \tag{5.1}$$

Let us fix a modulus of continuity $\omega$ such that $\omega = o(\omega_v^{(h)})$. Assume by contradiction that

$$\limsup_{s\to t} \frac{|f_h(s) - f_h(t)|}{\omega(|s-t|)} < +\infty. \tag{5.2}$$

Given an arbitrary fixed $j \in \mathbb{N}$, we set $k = \lfloor 2^j t \rfloor$. The first vanishing moment of the wavelet allows to write

$$\begin{aligned} |c_{j,k}| &= \Big| 2^j \int_{\mathbb{R}} \psi(2^j x - k)\big(f_h(x) - f_h(t)\big)\,dx \Big| \\ &\le 2^j \int_{B(t,2^{-j})} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx \\ &\quad + 2^j \sum_{l=0}^{j-1} \int_{B_l} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx \\ &\quad + 2^j \int_{\mathbb{R}\setminus B(t,1)} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx \end{aligned} \tag{5.3}$$

where for every $l \in \{0, \dots, j-1\}$, $B_l$ denotes the set

$$(t - 2^{-l}, t + 2^{-l}) \ \setminus \ (t - 2^{-l-1}, t + 2^{-l-1}).$$

The first term can be controlled by using the modulus of continuity $\omega$ in the following way

$$\begin{aligned} 2^j \int_{B(t,2^{-j})} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx &\le \sup_{x \in B(t,2^{-j})} \big|f(x) - f(t)\big| \|\psi\|_{L^1} \\ &\le C\omega(2^{-j}) \end{aligned} \tag{5.4}$$

for some positive constant $C$ by assumption (5.2). In order to deal with the second term, we use in addition the fast decay property of the wavelet (2.1) to get the existence of a natural number $N$ and a constant $C$ such that

$$2^j \sum_{l=0}^{j-1} \int_{B_l} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx \le C \sum_{l=0}^{j-1} \omega(2^{-l}) \int_{B_l} \frac{2^j}{(3 + |2^j x - k|)^{2N}}\,dx.$$

Notice then that if $x \in B_l$, then

$$3 + |2^j x - k| \ge 3 + 2^j |x - t| - |2^j t - k| \ge 2^{j-l-1}.$$

It implies that

$$\begin{aligned} 2^j \sum_{l=0}^{j-1} \int_{B_l} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx &\le C \sum_{l=0}^{j-1} \omega(2^{-l}) 2^{-N(j-l-1)} \int_{B_l} \frac{2^j}{(3 + |2^j x - k|)^{N}}\,dx \\ &\le C \sum_{l=0}^{j-1} \omega(2^{-l}) 2^{-N(j-l-1)} \int_{\mathbb{R}} \frac{1}{(3 + |y|)^{N}}\,dy \\ &= C' 2^{-Nj} \sum_{l=0}^{j-1} \omega(2^{-l}) 2^{Nl} \\ &\le C'' \omega(2^{-j}) \end{aligned} \tag{5.5}$$

for some constants $C', C''$ and using (2.4). It remains to bound the last term. Again, the fast decay of the wavelet together with the boundedness of the random wavelets series $f_h$ lead to

$$\begin{aligned} 2^j \int_{\mathbb{R}\setminus B(t,1)} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big|\,dx &\le 2\|f_h\|_\infty \int_{\mathbb{R}\setminus B(t,1)} \frac{2^j}{(3 + |2^j x - k|)^{2N}}\,dx \\ &\le 2\|f_h\|_\infty 2^{-Nj} \int_{\mathbb{R}} \frac{2^j}{(3 + |y|)^{N}}\,dy \\ &\le C 2^{-Nj} \end{aligned} \tag{5.6}$$

for some positive constant $C$, since $|2^j x - k| \geq 2^j$ if $|t-x| \geq 1$. Putting (5.3), (5.4), (5.5) and (5.6) together, we finally obtain

$$|c_{j,k}| \leq C\big(\omega(2^{-j}) + 2^{-Nj}\big)$$

for some positive constant $C$, so that

$$\limsup_{j\to+\infty} \frac{|c_{j,k}|}{\omega(2^{-j})} < +\infty$$

if $N$ is chosen large enough. This contradicts the second part of (5.1) since $\omega = o(\omega_v^{(h)})$. □

*Remark* 5.2. As mentioned earlier, let us notice that if the wavelet $\psi$ is compactly supported, and if $t$ is a rapid, ordinary or slow point, then $\omega_v$ gives the exact pointwise behaviour of $f$ at $t$, meaning that

$$0 < \limsup_{s\to t} \frac{|f(s)-f(t)|}{\omega_v^{(h)}(|t-s|)} < +\infty$$

where $v = r$, $v = o$ or $v = s$ respectively. Indeed, in this case, one has directly

$$\begin{aligned} |c_{j,k}| &\leq 2^j \int_{B(t,R2^{-j})} |\psi(2^j x - k)|\big|f_h(x) - f_h(t)\big| dx \\ &\leq C \sup_{x:|x-t|<R2^{-j}} |f_h(x) - f_h(t)| \end{aligned}$$

where $R$ can be computed via the support of the wavelet and $C$ is a positive constant.

The last remark applies in particular to the fractional Brownian motion thanks to the representation (1.3). Indeed, as $R$ is a smooth process, it does not modify the pointwise regularity and irregularity properties.

**Corollary 5.3.** *Almost surely, the fractional Brownian motion satisfies the following property for every non-empty interval $I$ of $\mathbb{R}$:*

- *almost every $t \in I$ is ordinary:*
$$0 < \limsup_{s\to t} \frac{|f_h(s)-f_h(t)|}{|t-s|^h \sqrt{\log\log|t-s|^{-1}}} < +\infty,$$
- *there exists $t \in I$ which is fast:*
$$0 < \limsup_{s\to t} \frac{|f_h(s)-f_h(t)|}{|t-s|^h \sqrt{\log|t-s|^{-1}}} < +\infty,$$
- *there exists $t \in I$ which is slow:*
$$0 < \limsup_{s\to t} \frac{|f_h(s)-f_h(t)|}{|t-s|^h} < +\infty.$$

## 6. Genericity of the non-existence of slow points

The aim of this section is to prove that the results obtained in the previous section are specific to these Gaussian random wavelet series. On this purpose, let us define the Fréchet space

$$C^{\nearrow h} := \bigcap_{\alpha<h} C^\alpha([0,1]),$$

where for every $\alpha \in (0,h)$, $C^\alpha([0,1])$ denote the Hölder space of order $\alpha$. This space and its topology can be equivalently defined using sequence of wavelet coefficients as follows: If we consider $\alpha > 0$, $\alpha \notin \mathbb{N}$, then for any $f \in C^\alpha([0,1])$, one has

$$\sup_{j\in\mathbb{N}} \sup_{k\in\{0,\dots,2^j-1\}} 2^{\alpha j}|c_{j,k}| < +\infty \tag{6.1}$$

where $(c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ denotes the sequence of wavelet coefficients of $f$ on $[0,1]$. It allows to identify (algebraically and topologically) the Hölder space $C^\alpha([0,1])$ with the space of complex sequences $(c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ satisfying (6.1), see Meyer (1992). When $\alpha \in \mathbb{N}$, we will also denote by $C^\alpha([0,1])$ the space of functions satisfying the condition (6.1).

We will prove that in $C^{\nearrow h}$, "most of" the functions do not present a slow pointwise behaviour as the one exhibited for Gaussian wavelet series, i.e. does not belong to the space $C^h(t)$ formed by the functions $f$ such that

$$\limsup_{s\to t} \frac{|f(s)-f(t)|}{|s-t|^h} < +\infty.$$

We will use in this purpose two notions of genericity: the prevalence and the Baire category points of view.

The notion of prevalence supplies an extension of the notion of "almost everywhere" (for the Lebesgue measure) in infinite dimensional spaces. In a metric infinite dimensional vector space, no measure is both $\sigma$-finite and translation invariant. However, one can consider a natural extension of the notion of "almost everywhere" which is translation invariant, see Christensen (1974); Hunt et al. (1992).

**Definition 6.1.** Let $E$ be a complete metric vector space. A Borel set $A \subset E$ is Haar-null if there exists a compactly supported probability measure $\mu$ such that

$$\forall x \in E, \quad \mu(x+A) = 0. \tag{6.2}$$

If this property holds, the measure $\mu$ is said to be transverse to $A$. A subset of E is called Haar-null if it is contained in a Haar-null Borel set. The complement of a Haar-null set is called a prevalent set.

In order to prove that a set is Haar-null in a functional space $E$, one can often use for transverse measure the law of a stochastic process, see Clausel and Nicolay (2010); Esser and Jaffard (2018) for some applications of this method. If $\mathcal{P}$ is a property that can be satisfied by points of $E$, one can prove that $\mathcal{P}$ holds only on a Haar-null set by exhibiting a stochastic process $X$ whose sample paths lies in a compact subset of $E$ and such that for all $f \in E$ almost surely the property $\mathcal{P}$ does not hold for $X+f$.

In our setting, the stochastic process that will be used is a random wavelet series. The following result allows to get that the sample paths of this series are almost surely in a compact set of $C^{\nearrow h}$. Let us first describe this subset.

**Lemma 6.2.** *Let $h>0$ and let $(\alpha_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of $(0,h)$ with tends to $h$. The subset*

$$K = \left\{ f \in C^{\nearrow h} : \max_{k\in\{0,\dots,2^j-1\}} |c_{j,k}| \le 2^{-\alpha_j j} \ \forall j \in \mathbb{N} \right\}$$

*is compact in $C^{\nearrow h}$, where $(c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ denotes the sequence of wavelet coefficients of $f$.*

*Proof*: Clearly, $K$ is closed in $C^{\nearrow h}$. Since this last space is a Fréchet space, it suffices now to prove that $K$ is totally bounded. Let us fix $\varepsilon > 0$ and $\alpha < h$. Then there exists $J \in \mathbb{N}$ such that $2^{-\alpha_j j} < \varepsilon 2^{-\alpha j}$ for every $j \ge J$, which implies that

$$\sup_{j\ge J,k\in\{0,\dots,2^j-1\}} 2^{\alpha j}|c_{j,k}| < \varepsilon$$

for the sequence of wavelet coefficients $(c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ of any $f \in K$. Moreover, since the product

$$P = \prod_{j=0}^{J-1}\prod_{k=0}^{2^j-1} [-2^{-\alpha_{j,k} j}, 2^{\alpha_{j,k} j}]$$

where we have set $\alpha_{j,k} = \alpha_j$ for every $k \in \{0, \dots, 2^j - 1\}$, is compact, one can find a finite number of sequences $c^1, \dots, c^L$ with support included in $\{(j,k) : j \leq J-1, k \in \{0, \dots, 2^j - 1\}\}$ such that

$$P \subseteq \bigcup_{l=1}^{L} \prod_{j=0}^{J-1} \prod_{k=0}^{2^j-1} (c^l_{j,k} - \delta, c^l_{j,k} + \delta),$$

where $\delta < 2^{-\alpha J}\varepsilon$. For every $l$, we define $f^l = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} c^l_{j,k} \psi(2^j \cdot -k)$. In order to conclude, let us show that

$$K \subseteq \bigcup_{l=1}^{L} \Big\{ f \in C^{\nearrow h} : \|f - f^l\|_\alpha < \varepsilon \Big\}.$$

If $f \in K$, its truncated sequence $(c_{j,k})_{j \leq J-1, k \in \{0,\dots,2^j-1\}}$ of wavelet coefficients belongs to $P$. Hence, for a $l \in \{1, \dots, L\}$, one has

$$2^{\alpha j} |c_{j,k} - c^l_{j,k}| \leq \delta 2^{\alpha j} < \varepsilon$$

if $j \leq J-1$, and

$$2^{\alpha j} |c_{j,k} - c^l_{j,k}| = 2^{\alpha j} |c_{j,k}| < \varepsilon$$

if $j \geq J$. □

Before stating our result, we need to recall the following wavelet "almost characterization" of the pointwise Hölder regularity. It relies on alternative quantities, namely the wavelet leaders. In order to define them, we need to introduce the notation $c_\lambda$ to denote the wavelet coefficient $c_{j,k}$, where $\lambda$ is the dyadic interval

$$\lambda = \lambda(j,k) = \Big[\frac{k}{2^j}, \frac{k+1}{2^j}\Big)$$

Then, if $3\lambda$ denote the interval with the same center as $\lambda$ but three times larger, the wavelet leader $d_\lambda$ is defined by

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|. \tag{6.3}$$

Note that this supremum is finite as soon as $f$ is locally bounded. In Jaffard (2004b), the author proved that if $f \in C^h(t)$ for some $h > 0$, then there exists a constant $C > 0$ such that

$$d_{\lambda(j,k_j(t))} \leq C 2^{-\alpha j}. \tag{6.4}$$

This inequality will allow us to construct wavelet series which do not belong to $C^h(t)$ for every $t \in [0,1]$.

**Proposition 6.3.** *Let $h > 0$. The set of functions $f$ such that $f \notin C^h(t)$ for every $t \in [0,1]$ is prevalent in $C^{\nearrow h}$.*

*Proof*: Let us consider a sequence $(j_n)_{n \in \mathbb{N}}$ satisfying $j_{n+1} > j_n + \lfloor \log_2 j_n^2 \rfloor + 1$ and let us set $\alpha_n = h - \frac{1}{\sqrt{j_n}}$ for every $n \in \mathbb{N}$. Let us define the random wavelets series

$$f = \sum_{n \in \mathbb{N}} \sum_{j=j_n}^{j_{n+1}-1} \sum_{k=0}^{2^j-1} 2^{-\alpha_n j} \varepsilon_{j,k} \psi_{j,k}$$

where $(\varepsilon_{j,k})_{j \in \mathbb{N}, k \in \{0,\dots,2^j-1\}}$ is a sequence of independent $\mathcal{U}([-1,1])$ random variables. Clearly, for every $\alpha < h$, one has

$$2^{-\alpha_n j} \leq 2^{-\alpha j}$$

if $j \in \{j_n, \dots, j_{n+1} - 1\}$ for $n$ large enough. Using the characterization given in (6.1), it follows that $f \in C^\alpha([0,1])$. Moreover, from Lemma 6.2, we know that the process $f$ takes its values in a compact subset of $C^{\nearrow h}$.

Let us now show that almost surely $f \notin C^h(t)$ for every $t \in [0,1]$. Let us fix $M \in \mathbb{N}$. For every $n \in \mathbb{N}$, if one consider the subintervals of scale $j' = j_n + \lfloor \log_2 j_n^2 \rfloor + 1$ in the supremum appearing in the definition (6.3) of the wavelet leaders, one has

$$\begin{aligned}\mathbb{P}\Big(\inf_{k\in\{0,\dots,2^{j_n}-1\}} d_{j_n,k} \leq M2^{-hj_n}\Big) &\leq \sum_{k=0}^{2^{j_n}-1} \mathbb{P}\Big(\sup_{\lambda'\subseteq\lambda_{j_n,k}} |2^{-\alpha_n j'}\varepsilon_{\lambda'}| \leq M2^{-hj_n}\Big)\\ &\leq 2^{j_n}\Big(2^{\alpha_n j'} M 2^{-hj_n}\Big)^{2^{j'-j_n}}\\ &\leq 2^{j_n + j_n^2 h} j_n^{2hj_n^2} 2^{-j_n^{5/2}} M^{j_n^2}.\end{aligned}$$

The Borel-Cantelli Lemma implies that almost surely, one has

$$d_{j_n,k} > M2^{-hj_n}$$

for every $n$ large enough and every $k \in \{0,\dots,2^{j_n}-1\}$. Since $M \in \mathbb{N}$ is arbitrary, (6.4) gives the announced result. In order to conclude, it suffices to prove that the previous result is still valid if we replace the wavelet series $f$ by

$$\tilde{f} = f + g$$

for a function $g \in C^{\nearrow h}$. In this case, the wavelet coefficients $2^{-\alpha_n j}$ of $f$ are replaced by

$$2^{-\alpha_n j}\varepsilon_{j,k} + c_{j,k} = 2^{-\alpha_n j}(\varepsilon_{j,k} + 2^{\alpha_n j} c_{j,k}).$$

It implies that the random variables defining the wavelets series are still independent but no more centred since they are shifted by a deterministic quantity. Clearly, the probabilities computed before can only become smaller, hence the Borel-Cantelli lemma still holds. □

To end the paper, we show that the same result holds true if one replaces the notion of prevalence by the genericity in the sense supplied by the Baire category theorem. Let us recall that a subset $A$ of a Baire space $X$ is of first category (or meagre) if it is included in a countable union of closed sets of $X$ with empty interior. The complement of a set of first category is Baire-residual; it contains a countable union of dense open sets of $X$.

**Proposition 6.4.** *Let $h > 0$. The set of functions $f$ such that $f \notin C^h(t)$ for every $t \in [0,1]$ is Baire-residual in $C^{\nearrow h}$.*

*Proof*: Let us consider the non-decreasing sequence $(\alpha_j)_{j\in\mathbb{N}}$ of $(0,h)$ with converges to $h$ defined by $\alpha_j = h - \frac{1}{\sqrt{j}}$. For every $J \in \mathbb{N}$, the set $\mathcal{C}_J$ is formed by the functions $f \in C^{\nearrow h}$ whose sequence of wavelet coefficients $(c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ satisfies

$$2^{\alpha_j j}|c_{j,k}| \in \mathbb{N}\setminus\{0\} \quad \forall j \geq J\,, \forall k \in \{0,\dots,2^j-1\}.$$

Finally, we define the open sets $U_J$ by

$$U_J = \bigcup_{j\geq J}\Big\{g \in C^{\nearrow h} : \exists f \in \mathcal{C}_j \text{ such that } \|f-g\|_{\alpha_j} < \frac{1}{2}\Big\}.$$

Notice that if $g \in U_J$ with wavelet coefficients $(e_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$, then there is $f \in \mathcal{C}_{j_0}$ for a $j_0 \geq J$, with wavelet coefficients $(c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ such that

$$|e_{j_0,k}| \geq |c_{j_0,k}| - |e_{j_0,k} - c_{j_0,k}| \geq 2^{-\alpha_{j_0} j_0} - \frac{1}{2}2^{-\alpha_{j_0} j_0} = \frac{1}{2}2^{-\alpha_{j_0} j_0}$$

for every $k \in \{0,\dots,2^{j_0}-1\}$. Now, assume that $g$ belongs to the set $\mathcal{R}$ defined by

$$\mathcal{R} = \bigcap_{J\in\mathbb{N}} U_J.$$

If there exists $t \in [0,1]$ such that $g \in C^h(t)$, (6.4) gives the existence of a constant $C > 0$ such that

$$d_{j,k_j(t)} \leq C 2^{-hj}.$$

Since $g \in \mathcal{R}$, one gets

$$\frac{1}{2} 2^{-\alpha_j j} = \frac{1}{2} 2^{-hj+\sqrt{j}} \leq C 2^{-hj}$$

for infinitely many $j$, which is impossible. Consequently, $g \notin C^h(t)$ for every $t \in [0,1]$.

To conclude, it suffices to prove that the open sets $U_J$ are dense in $C^{\nearrow h}$. Let us fix $J \in \mathbb{N}$, $g \in C^{\nearrow h}$, $\alpha < h$ and $\varepsilon > 0$. Let $J_0 \geq J$ be large enough to ensure that both $\alpha < \alpha_{J_0}$ and $2^{(\alpha - \alpha_{J_0})J_0} < \varepsilon$ are satisfied. We construct the function $f$ via its sequence of wavelet coefficients by setting

$$c_{j,k} = \begin{cases} e_{j,k} & \text{if } j < J_0 \\ 2^{-\alpha_j j} [2^{\alpha_j j} e_{j,k}] & \text{if } j \geq J_0 \text{ and } 2^{\alpha_j j} |e_{j,k}| \geq 2, \\ 2^{-\alpha_j j} & \text{if } j \geq J_0 \text{ and } 2^{\alpha_j j} |e_{j,k}| < 2, \end{cases}$$

so that $f \in \mathcal{C}_{J_0} \subseteq U_J$ and $|2^{\alpha_j j} c_{j,k} - 2^{\alpha_j j} e_{j,k}| \leq 1$ for every $j \geq J_0$. Hence one has

$$2^{\alpha j} |e_{j,k} - c_{j,k}| \leq 2^{(\alpha - \alpha_j) j} \leq 2^{(\alpha - \alpha_{J_0}) J_0} < \varepsilon$$

for every $j \geq J_0$, which allows to conclude. □

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