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Dynamical large deviations for the boundary driven symmetric exclusion process with Robin boundary conditions

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Abstract. In this article, we consider a one-dimensional symmetric exclusion process in weak contact with reservoirs at the boundary. In the diffusive time-scaling the empirical measure evolves according to the heat equation with Robin boundary conditions. We prove the associated dynamical large deviations principle.

Contents

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1. Introduction

The investigation of the thermodynamic properties of stationary nonequilibrium states of interacting particle systems has been proven to be an important step in the understanding of nonequilibrium phenomena and a rich source of mathematical problems Derrida (2007); Bertini et al. (2015); Jack (2020).

In the context of lattices gases, the empirical measure is the only relevant thermodynamical quantity in the macroscopic description of the system, and the thermodynamical functionals, as the free energy, can be identified to the large deviations rate functional.

While in equilibrium the stationary state is given by the Gibbs distribution associated to the Hamiltonian, in nonequilibrium the construction of the stationary state requires solving a dynamical-variational problem which defines the so-called quasi-potential Freidlin and Wentzell (1998).

At the beginning of this century, Derrida et al. (2002) considered the one-dimensional symmetric exclusion process in strong contact with reservoirs. In this context, the empirical measure evolves in the diffusive time-scale according to the heat equation with Dirichlet boundary conditions (Kipnis and Landim, 1999). Expressing the stationary state as a product of matrices (Derrida et al., 1993), they obtained an explicit formula for the large deviations principle rate functional of the empirical measure under the stationary state, the so-called nonequilibrium free energy.

Later, Bertini et al. (2002) derived the Derrida-Lebowitz-Speer formula (in short DLS formula) for the nonequilibrium free energy extending to infinite dimensions the dynamical approach introduced in Freidlin and Wentzell (1998). More precisely, they first proved a dynamical large deviations principle for the empirical measure for symmetric exclusion processes in strong contact with reservoirs Bertini et al. (2003). Denote by $I_{[0,T]}(u)$ the large deviations rate function of the dynamical large deviations principle. Hence, $I_{[0,T]}(u)$ represents the cost of observing a trajectory $u(t)$ in the time-interval $[0,T]$. Let $\bar\rho$ be the stationary profile of the hydrodynamic equation, that is, the typical density profile under the stationary state. Define the quasi-potential as

$$V(\gamma) \;=\; \inf_{T>0} \inf_{u} I_{[0,T]}(u)\;,$$

where the second infimum is carried over all trajectories $u(t)$ connecting the stationary density profile $\bar\rho$ to a density profile γ in the time interval $[0,T]$: $u(0)=\bar\rho$, $u(T)=\gamma$. It is proven in

Bertini et al. (2002) that the quasi-potential V coincides with the nonequilibrium free energy, i.e., that it satisfies the DLS' equations.

In the sequel, Bodineau and Giacomin (2004); Farfan (2009) derived a large deviations principle for the empirical measure under the stationary state from the dynamical one, with rate functional given by the quasi-potential. This result was later extended to weakly symmetric exclusion processes in strong contact with reservoirs Enaud and Derrida (2004); Bertini et al. (2009b,a, 2011) and to reaction-diffusion models Landim and Tsunoda (2018); Farfán et al. (2019).

It has long been understood that these results extend to boundary-driven one-dimensional symmetric exclusion processes in weak contact with reservoirs Derrida (2016). But only recently, this result appeared in Derrida et al. (2021) through the matrix ansatz product method.

In this article, we accomplish the first step in the project of deriving the large deviations principle for the empirical measure under the stationary state, through the dynamical approach, for boundary-driven one-dimensional symmetric exclusion processes in weak contact with reservoirs. The law of large numbers has been obtained in Baldasso et al. (2017); Franco et al. (2021). We prove here the dynamical large deviations, while in the companion paper Bouley et al. (2021), it is shown that the quasi-potential satisfies the DLS' equations obtained in Derrida et al. (2021) for this model. In Franco et al. (2021), a large deviations principle is proved for symmetric exclusion processes in even weaker contact with reservoirs.

2. Notation and Results

The model. We consider one-dimensional, symmetric exclusion processes in a weak contact with boundary reservoirs. Fix $N \ge 1$, and let $\mathfrak{e}_N = 1/N$, $\mathfrak{r}_N = 1-(1/N)$, $\Lambda_N = \{\mathfrak{e}_N, \dots, (N-2)\,\mathfrak{e}_N, \mathfrak{r}_N\}$. The state-space is represented by $\Omega_N = \{0,1\}^{\Lambda_N}$ and the configurations by the Greek letters η, ξ so that η_x, $x \in \Lambda_N$, represents the number of particles at site x for the configuration η. Here and below all notation introduced in the text and not in displayed equations is indicated in blue.

Fix throughout this article, $0 < \alpha \le \beta < 1$, $A > 0$, $B > 0$. The generator of the Markov process, represented by $\mathcal{L}_N = \mathcal{L}_N^{\alpha,A,\beta,B}$, is given by

$$\mathcal{L}_N \;=\; L_N^{\rm lb} \,+\, L_N^{\rm bulk} \,+\, L_N^{\rm rb}\,.$$

In this formula, for every function $f : \Omega_N \to \mathbb{R}$,

$$(L_N^{\rm bulk} f)(\eta) \;=\; N^2 \sum_{x\in\Lambda_N^o} [\, f(\sigma^{x,x+\mathfrak{e}_N}\eta) \,-\, f(\eta)\,]\,, \tag{2.1}$$

where Λ_N^o represents the interior of Λ_N, $\Lambda_N^o := \Lambda_N \setminus \{\mathfrak{r}_N\} = \{\mathfrak{e}_N, \dots, (N-2)\,\mathfrak{e}_N\}$, and

$$\begin{aligned} (L_N^{\rm lb} f)(\eta) \;&=\; \frac{N}{A}\,\big[\,(1-\eta_{\mathfrak{e}_N})\,\alpha \,+\, (1-\alpha)\,\eta_{\mathfrak{e}_N}\,\big]\big[\, f(\sigma^{\mathfrak{e}_N}\eta) \,-\, f(\eta)\,\big]\,, \\ (L_N^{\rm rb} f)(\eta) \;&=\; \frac{N}{B}\,\big[\,(1-\eta_{\mathfrak{r}_N})\,\beta \,+\, (1-\beta)\,\eta_{\mathfrak{r}_N}\,\big]\big[\, f(\sigma^{\mathfrak{r}_N}\eta) \,-\, f(\eta)\,\big]\,. \end{aligned} \tag{2.2}$$

From now on, we omit the subindex N from $\mathfrak{e}_N$ and $\mathfrak{r}_N$. In the definitions above,

$$(\sigma^{x,x+\mathfrak{e}}\eta)_y \;=\; \begin{cases} \eta_y & \text{if } y \neq x, x+\mathfrak{e} \\ \eta_{x+\mathfrak{e}} & \text{if } y = x \\ \eta_x & \text{if } y = x+\mathfrak{e} \end{cases} \quad\text{and}\quad (\sigma^x\eta)_y \;=\; \begin{cases} \eta_y & \text{if } y \neq x \\ 1-\eta_x & \text{if } y = x\,. \end{cases} \tag{2.3}$$

For a metric space $\mathbb{X}$, denote by $D([0,T],\mathbb{X})$, $T>0$, the space of right-continuous functions $x\colon [0,T]\to\mathbb{X}$, with left-limits, endowed with the Skorohod topology and its associated Borel σ-algebra. The elements of $D([0,T],\Omega_N)$ are represent by $\boldsymbol{\eta}(\cdot)$.

For a probability measure μ on Ω_N, let $\mathbb{P}_\mu$ be the measure on $D([0,T],\Omega_N)$ induced by the continuous-time Markov process associated to the generator $\mathcal{L}_N$ starting from μ. When the measure μ is the Dirac measure concentrated on a configuration $\eta\in\Omega_N$, that is $\mu=\delta_\eta$, we represent $\mathbb{P}_{\delta_\eta}$ simply by $\mathbb{P}_\eta$. Expectation with respect to $\mathbb{P}_\mu$, $\mathbb{P}_\eta$ is denoted by $\mathbb{E}_\mu$, $\mathbb{E}_\eta$, respectively.

Hydrodynamic limit. Denote by $\mathcal{M}$ the set of non-negative measures on $[0,1]$ with total mass bounded by 1 endowed with the weak topology. Recall that this topology is metrisable and that, with this topology, $\mathcal{M}$ is a compact space. For a continuous function $F:[0,1]\to\mathbb{R}$ and a measure $\pi\in\mathcal{M}$, denote by $\langle\pi,F\rangle$ the integral of F with respect to π:

$$\langle\pi,F\rangle \;=\; \int F(x)\,\pi(dx)\;.$$

Given a configuration $\eta\in\Omega_N$, denote by $\pi=\pi(\eta)$ the measure in $\mathcal{M}$ obtained by assigning a mass N^{-1} to each particle:

$$\pi \;=\; \pi(\eta) \;=\; \tfrac{1}{N}\sum_{x\in\Lambda_N}\eta_x\,\delta_x\;.$$

The measure π is called the *empirical measure*. Denote by $\boldsymbol{\pi}: D([0,T],\Omega_N)\to D([0,T],\mathcal{M})$ the map which associates to a trajectory $\boldsymbol{\eta}(\cdot)$ its empirical measure:

$$\boldsymbol{\pi}(t) \;=\; \pi(\boldsymbol{\eta}(t)) \;=\; \tfrac{1}{N}\sum_{x\in\Lambda_N}\eta_x(t)\,\delta_x\;. \tag{2.4}$$

For a probability measure μ in Ω_N, let $\mathbb{Q}^N_\mu$ be the measure on $D([0,T],\mathcal{M})$ given by $\mathbb{Q}^N_\mu=\mathbb{P}^N_\mu\circ\boldsymbol{\pi}^{-1}$. The first result, due to Baldasso et al. (2017), establishes the hydrodynamic behavior of the empirical measure.

Theorem 2.1. *Fix $T>0$, a measurable density profile $\gamma\colon[0,1]\to[0,1]$, and a sequence $\{\nu^N\}_{N\ge1}$ of probability measures on Ω_N associated to γ in the sense that*

$$\lim_{N\to\infty}\nu^N\Big[\,\Big|\,\langle\pi\,,\,H\rangle-\int_0^1\gamma(x)\,H(x)\,dx\,\Big|>\delta\,\Big] \;=\; 0 \tag{2.5}$$

for all continuous functions $H\colon[0,1]\to\mathbb{R}$ and $\delta>0$. Then, the sequence of probability measures $\mathbb{Q}^N_{\nu^N}$ converges weakly to the probability measure $\mathbb{Q}$ concentrated on the trajectory $\pi(t,dx)=u(t,x)\,dx$, where u is the unique weak solution of the heat equation with Robin boundary conditions

$$\begin{cases}\partial_t u \;=\; \Delta u\\ (\nabla u)(t,0) \;=\; A^{-1}[\,u(t,0)-\alpha\,]\\ (\nabla u)(t,1) \;=\; B^{-1}[\,\beta-u(t,1)]\\ u(0,\cdot) \;=\; \gamma(\cdot)\;.\end{cases} \tag{2.6}$$

In this formula, ∇u stands for the partial derivative in space of u, $\partial_t u$ for its partial derivative in time and Δu for the Laplacian of u in the space variable. The definition of weak solutions of equation (2.6) and the proof of uniqueness of weak solutions is provided in Appendix B. It is also presented in Baldasso et al. (2017).

The energy. Denote by $\mathcal{M}_{\rm ac}$ the subset of $\mathcal{M}$ of all measures which are absolutely continuous with respect to the Lebesgue measure and whose density takes values in the interval $[0,1]$, that is, $\mathcal{M}_{\rm ac} = \{\pi \in \mathcal{M} : \pi(dx) = \gamma(x)\, dx \text{ and } 0 \le \gamma(x) \le 1\}$.

For $T>0$, let the *energy* $\mathcal{Q}_{[0,T]} : D([0,T],\mathcal{M}_{\rm ac}) \to [0,\infty]$ be given by

$$\begin{aligned} \mathcal{Q}_{[0,T]}(u) \;&:=\; \sup_G \mathcal{Q}^G_{[0,T]} \\ &:=\; \sup_G \Big\{ \int_0^T dt \int_0^1 u(t,x)\, (\nabla G)(t,x)\, dx \;-\; \frac{1}{2}\int_0^T dt \int_0^1 G(t,x)^2\, dx \Big\}\,, \end{aligned} \tag{2.7}$$

where the supremum is carried over all smooth functions $G: [0,T]\times(0,1)\to\mathbb{R}$ with compact support.

Remark 2.2. In this definition and below, for a functional $\Phi\colon D([0,T],\mathcal{M}_{\rm ac}) \to \mathbb{R}$, we often write $\Phi(u)$ instead $\Phi(\pi)$ when $\pi(t,dx) = u(t,x)\, dx$.

Clearly, the energy $\mathcal{Q}_{[0,T]}$ is convex and lower semicontinuous. Moreover, if $\mathcal{Q}_{[0,T]}(u)$ is finite, u has a generalized space derivative, denoted by ∇u, and

$$\mathcal{Q}_{[0,T]}(u) \;=\; \frac{1}{2}\int_0^T dt \int_0^1 (\nabla u_t)^2\, dx\,\cdot \tag{2.8}$$

In other words, u belongs to the Sobolev space $\mathcal{H}^1([0,T]\times[0,1])$, introduced below in Section 4. Denote by $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ the trajectories in $D([0,T],\mathcal{M}_{\rm ac})$ with finite energy.

The rate functional. For $T>0$ and positive integers m,n, denote by $C^{m,n}([0,T]\times[0,1])$ the space of functions $G\colon [0,T]\times[0,1]\to\mathbb{R}$ with m derivatives in time, n derivatives in space which are continuous up to the boundary. Denote by $C^{m,n}_0([0,T]\times[0,1])$ the functions G in $C^{m,n}([0,T]\times[0,1])$ such that $G(t,0)=G(t,1)=0$ for all $t\in[0,T]$, and by $C^{m,n}_c([0,T]\times(0,1))$ the functions in $C^{m,n}([0,T]\times[0,1])$ with compact support in $[0,T]\times(0,1)$.

For $0<\varrho<1$, $D>0$, $0<a<1$, $M\in\mathbb{R}$, let

$$\mathfrak{b}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{ [1-a]\,\varrho\,[e^M-1] \;+\; a\,[1-\varrho]\,[e^{-M}-1]\Big\}\,. \tag{2.9}$$

Fix a trajectory $\pi(t,dx) = u(t,x)\,dx \in D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$. Then, for almost all $t\in[0,T]$, $\int_0^1 (\nabla u_t)^2\, dx$ is finite, and therefore $u(t,\cdot)$ is Hölder-continuous. In particular, $u(t,0)$ and $u(t,1)$ are well defined for almost all t.

Denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the usual scalar product in $\mathcal{L}^2([0,1])$: $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$ for $f,g\in\mathcal{L}^2([0,1])$. Recall the convention established in Remark 2.2. For each H in $C^{1,2}([0,T]\times[0,1])$, let $J_{T,H}\colon D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac}) \longrightarrow \mathbb{R}$ be the functional given by

$$\begin{aligned} J_{T,H}(u) \;=\;& \big\langle u_T, H_T\big\rangle \;-\; \langle u_0, H_0\rangle \;-\; \int_0^T \big\langle u_t, \partial_t H_t\big\rangle\, dt \\ &-\; \int_0^T \big\langle u_t, \Delta H_t\big\rangle\, dt \;+\; \int_0^T u_t(1)\,\nabla H_t(1)\, dt \;-\; \int_0^T u_t(0)\,\nabla H_t(0)\, dt \\ &-\; \int_0^T \big\langle \sigma(u_t), \big(\nabla H_t\big)^2\big\rangle\, dt \\ &-\; \int_0^T \Big\{ \mathfrak{b}_{\alpha,A}\big(\, u_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(\, u_t(1)\,,\, H_t(1)\,\big)\Big\}\, dt\,. \end{aligned} \tag{2.10}$$

In this formula and below, $\sigma(a) = a(1-a)$ stands for the mobility of the exclusion process. Since trajectories in $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ have generalized space-derivatives, we may integrate by parts the second line and write the functional $J_{T,H}(\,\cdot\,)$ as

$$\begin{aligned} J_{T,H}(u) \;=\; & \big\langle u_T, H_T\big\rangle \;-\; \langle u_0, H_0\rangle \;-\; \int_0^T \big\langle u_t, \partial_t H_t\big\rangle \, dt \\ & + \int_0^T \big\langle \nabla u_t, \nabla H_t\big\rangle \, dt - \int_0^T \big\langle \sigma(u_t), \big(\nabla H_t\big)^2\big\rangle \, dt \\ & - \int_0^T \Big\{ \mathfrak{b}_{\alpha,A}\big(u_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(u_t(1)\,,\, H_t(1)\,\big) \Big\} \, dt\;. \end{aligned} \tag{2.11}$$

We extend the definition of $J_{T,H}(\,\cdot\,)$ to $D([0,T],\mathcal{M})$ by setting

$$J_{T,H}(\pi) \;=\; \infty \quad \text{if } \pi \notin D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})\;.$$

Remark 2.3. This definition differs from the one presented in Bertini et al. (2009b); Farfan et al. (2011); Franco et al. (2021) in the context of exclusion processes with Dirichlet boundary conditions. There, one defines $J_{T,H}(\,\cdot\,)$ in $D([0,T],\mathcal{M}_{\rm ac})$ by an equation similar to (2.10) with $u_t(0)$, $u_t(1)$ replaced by the densities α, β, respectively. Here, as the boundary values appear and are not fixed by the dynamics, in the definition of the functional $J_{T,H}$, one is forced to restrict the definition to trajectories with finite energy. Otherwise, the boundary values of a density profile are not defined.

Let $I_{[0,T]}(\,\cdot\,) \colon D([0,T],\mathcal{M}_{\rm ac}) \longrightarrow [0,+\infty]$ be the functional defined by

$$I_{[0,T]}(u) \;:=\; \sup_{H\in C^{1,2}([0,T]\times[0,1])} J_{T,H}(u)\;. \tag{2.12}$$

Fix a density profile γ in $\mathcal{M}_{\rm ac}$, and let $I_{[0,T]}(\,\cdot\,|\gamma) \colon D([0,T],\mathcal{M}) \to \mathbb{R}$ be given by

$$I_{[0,T]}(\pi|\gamma) \;=\; \begin{cases} I_{[0,T]}(\pi) & \text{if } \pi_0(dx) \;=\; \gamma(x)\, dx \;\text{ a.s.}\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{2.13}$$

Theorem 2.4. *Fix $T > 0$ and a measurable function $\gamma : [0,1] \to [0,1]$. The function $I_{[0,T]}(\cdot|\gamma) : D([0,T],\mathcal{M}) \to [0,\infty]$ is convex, lower semicontinuous and has compact level sets.*

This result is proved in Section 3, where we also show, in Lemma 3.1, that any path π with finite rate function, $I_{[0,T]}(\pi|\gamma) < \infty$, is weakly continuous in time. Moreover, Proposition 3.5 states that there exists a finite constant C_0 such that

$$\int_0^T dt \int_0^1 \frac{[\nabla u]^2}{\sigma(u)}\, dx \;\le\; C_0\, \{\, I_{[0,T]}(u) + 1 \,\}$$

for all u in $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$, where $I_{[0,T]}(u)$ is the rate functional introduced in (2.12).

In Section 4, we obtain an explicit formula for the action functional. Proposition 4.5 states that $I_{[0,T]}(\,\cdot\,)$ can be expressed as $I^{(1)}_{[0,T]}(\,\cdot\,) + I^{(2)}_{[0,T]}(\,\cdot\,)$. The first term provides the contribution to the rate function due to the evolution in the interior of the interval $[0,1]$, while the second one the contribution due to the evolution at the boundary.

In Section 5, we show that trajectories with finite rate function can be approximated by smooth ones. The precise statement requires some notation.

Definition 2.5. Given $\gamma \in \mathcal{M}_{\rm ac}$, let Π_γ be the collection of all paths $\pi(t,dx) = u(t,x)dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ such that

(a) There exists $\mathfrak{t} > 0$, such that u follows the hydrodynamic equation (2.6) in the time interval $[0,\mathfrak{t}]$. In particular, $u(0,\cdot) = \gamma(\cdot)$.

(b) For every $0 < \delta \le T$, there exists $\varepsilon > 0$ such that $\varepsilon \le u(t,x) \le 1-\varepsilon$ for all (t,x) in $[\delta,T]\times[0,1]$;

(c) u is smooth on $(0,T]\times[0,1]$.

Theorem 5.2 states that for all $\gamma : [0,1] \to [0,1]$, the set Π_γ is $I_{[0,T]}(\cdot|\gamma)$-dense. This means that any trajectory π in $D([0,T],\mathcal{M})$ with finite rate function can be approximated by a sequence of trajectories $\pi^n \in \Pi_\gamma$ in such a way that $I_{[0,T]}(\pi^n)$ converges to $I_{[0,T]}(\pi)$. This is one of the main technical difficulties in the proof of the lower bound.

We also provide in Section 4 an explicit formula for the rate function of trajectories in Π_γ. For $0<\varrho<1$, $D>0$, $0<a<1$, $M\in\mathbb{R}$, let

$$\mathfrak{p}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{[1-a]\,\varrho\, e^{M} \;-\; a\,[1-\varrho]\,e^{-M}\Big\}\,, \tag{2.14}$$

$$\mathfrak{c}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{[1-a]\,\varrho\,[1-e^{M}+Me^{M}] \;+\; a\,[1-\varrho]\,[1-e^{-M}-Me^{-M}]\Big\}\,.$$

Proposition 2.6. *Fix a density profile $\gamma:[0,1]\to[0,1]$ and a trajectory u in Π_γ. Then, for each $t>0$, the elliptic equation (for H)*

$$\begin{cases} \partial_t u \;=\; \Delta u \;-\; 2\,\nabla\{\sigma(u)\,\nabla H\}\,, \\ \nabla u_t(1) \;-\; 2\,\sigma(u_t(1))\,\nabla H_t(1) \;=\; \mathfrak{p}_{\beta,B}\big(\,u_t(1)\,,\,H_t(1)\,\big)\,, \\ \nabla u_t(0) \;-\; 2\,\sigma(u_t(0))\,\nabla H_t(0) \;=\; -\,\mathfrak{p}_{\alpha,A}\big(\,u_t(0)\,,\,H_t(0)\,\big)\,, \end{cases} \tag{2.15}$$

has a unique solution, denoted by H_t. The function H belongs to $C^{1,2}([0,T]\times[0,1])$, and the rate functional $I_{[0,T]}(u)$ takes the form

$$\begin{aligned} I_{[0,T]}(u) \;=\; &\int_0^T dt\int_0^1 \sigma(u_t)\,(\nabla H_t)^2\,dx \;+\; \int_0^T \mathfrak{c}_{\beta,B}\big(\,u_t(1)\,,\,H_t(1)\,\big)\,dt \\ &+\; \int_0^T \mathfrak{c}_{\alpha,A}\big(\,u_t(0)\,,\,H_t(0)\,\big)\,dt\,. \end{aligned} \tag{2.16}$$

Dynamical large deviations principle. The main result of this article reads as follows.

Theorem 2.7. *Fix $T>0$, $\gamma\in\mathcal{M}_{\rm ac}$ and let $\{\eta^N\}_{N\in\mathbb{N}}$ be a sequence of configurations. Assume that δ_{η^N} is associated to γ in the sense of (2.5). Then, the sequence of probability measures $\{\mathbb{Q}_{\eta^N}\}_{N\ge1}$ satisfies a large deviation principle with speed N and good rate function $I_T(\cdot|\gamma)$. Namely, for each closed set $\mathcal{C}\subset D([0,T],\mathcal{M})$ and each open set $\mathcal{O}\subset D([0,T],\mathcal{M})$*

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^N_{\eta^N}\big[\,\pi^N\in\mathcal{C}\,\big] \;\le\; -\inf_{\pi\in\mathcal{C}} I_{[0,T]}(\pi|\gamma)$$

$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}^N_{\eta^N}\big[\,\pi^N\in\mathcal{O}\,\big] \;\ge\; -\inf_{\pi\in\mathcal{O}} I_{[0,T]}(\pi|\gamma)\,.$$

Remark 2.8. In contrast to Kipnis et al. (1989); Bertini et al. (2009b), the large deviations principle is formulated here for the empirical measure and not for the empirical density. More precisely, the empirical measure $\pi(t)$, defined in (2.4), is a sum of Dirac measures, while in Kipnis et al. (1989); Bertini et al. (2009b) $\pi(t)$ is defined as a measure, absolutely continuous with respect to the Lebesgue measure and whose density takes values in the set $\{0,1\}$. Defining $\pi(t)$ as a singular measure requires to prove that $I_{[0,T]}(\pi) = +\infty$ if π_t is not absolutely continuous with respect to the Lebesgue measure. The proof presented in Subsection 6.3 of Farfan et al. (2011) applies to the dynamics considered here.

3. The rate functional $I_{[0,T]}(\,\cdot\,)$

In this section, we present some properties of the rate function $I_{[0,T]}(\,\cdot\,)$ and prove Theorem 2.4. Fix, once for all, a measurable density profile $\gamma : [0,1] \to [0,1]$.

Note: Throughout this article, given a function $u : [0,T] \times [0,1] \to \mathbb{R}$, we represent by u_t and $u(t)$ the function defined on $[0,1]$ and such that $u_t(x) = u(t,x)$.

We start with two elementary bounds. The first estimate asserts that the cost of a trajectory in a interval $[0,T]$ is bounded by the sum of its cost in the intervals $[0,S]$ and $[S,T]$. Let $\tau_r u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$, $r > 0$, be the function defined by $\tau_r u(t,x) = u(t+r,x)$. For all $\pi(t,dx) = u(t,x)\,dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ and $0 < S < T$,

$$I_{[0,T]}(u) \;\le\; I_{[0,S]}(u) \;+\; I_{[0,T-S]}(\tau_S u)\;. \tag{3.1}$$

The proof of this claim is elementary and left to the reader. It relies on the fact that $\sup_n\{a_n + b_n\} \le \sup_n a_n + \sup_n b_n$.

The second assertion states that the cost of a trajectory on a subinterval of $[0,T]$ is bounded by its total cost. For all $\pi(t,dx) = u(t,x)\,dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ and $0 < S < T$,

$$I_{[0,S]}(u) \;\le\; I_{[0,T]}(u)\;. \tag{3.2}$$

To prove this claim, assume that $I_{[0,S]}(u) < \infty$, and fix $\varepsilon > 0$. The same argument applies to the case $I_{[0,S]}(u) = \infty$. By definition of the rate function, there exists H in $C^{1,2}([0,S]\times[0,1])$ such that

$$I_{[0,S]}(u) \;\le\; J_{S,H}(u) \;+\; \varepsilon\;.$$

Extend smoothly the function H to $[0,S+\delta]\times[0,1]$ for some $\delta > 0$. Let $\sigma_n : [0,T] \to [0,1]$, $n \ge 1$, be a sequence of smooth, monotone functions such that $\sigma_n(t) = 1$ for $0 \le t \le S$ and $\sigma_n(t) = 0$ for $S + (1/n) \le t \le T$. Define the function $H_n : [0,T]\times[0,1] \to \mathbb{R}$ as $H_n(t,x) = H(t,x)\,\sigma_n(t)$ Then,

$$J_{S,H}(u) \;=\; \lim_{n\to\infty} J_{T,H_n}(u) \;\le\; I_{[0,T]}(u)\;,$$

as claimed.

A similar argument yields that the cost of a trajectory u in a time-interval $[R,R+S]$ is bounded by the total cost. More precisely,

$$I_{[0,S]}(\tau_R u) \;\le\; I_{[0,T]}(u)\;. \tag{3.3}$$

for all $S > 0$, $R > 0$ such that $R + S \le T$.

The proof of the next result is similar to the ones of Bertini et al. (2003, Lemma 3.5), Farfan et al. (2011, Lemma 4.1). We present it here in sake of completeness.

Lemma 3.1. *Fix $T > 0$ and $\gamma \in \mathcal{M}_{\rm ac}$. Let u be a path in $D([0,T],\mathcal{M}_{\rm ac})$ such that $I_{[0,T]}(u|\gamma) < \infty$. Then $u(0,x) = \gamma(x)$. Moreover, for each $M > 0$, g in $C^2([0,1])$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{u: I_T(u|\gamma)\le M}\ \sup_{|t-s|\le\delta} \big|\,\langle u_t, g\rangle - \langle u_s, g\rangle\,\big| \;\le\; \varepsilon\;.$$

In particular, u belongs to $C([0,T],\mathcal{M}_{\rm ac})$.

Proof: Fix $T > 0$, $\gamma \in \mathcal{M}_{\rm ac}$ and u in $D([0,T], \mathcal{M}_{\rm ac})$ such that $I_{[0,T]}(u|\gamma) < \infty$. We first show that $u(0,\cdot) = \gamma(\cdot)$.

As $I_{[0,T]}(u|\gamma) < \infty$, u has finite energy. For $\delta > 0$, consider the function $H_\delta(t,x) = h_\delta(t) g(x)$, where $h_\delta(t) = (1 - \delta^{-1} t)^+$ and g is a $C^2([0,1])$ function which vanishes at the boundary of the interval $[0,1]$. Here a^+ stands for the positive part of a. Of course, H_δ can be approximated by smooth functions. Since u is bounded and since $t \to u(t,\cdot)$ is right continuous for the weak topology,

$$\lim_{\delta \downarrow 0} J_{T,H_\delta}(u) \;=\; \langle u(0), g\rangle - \langle \gamma, g\rangle \;.$$

This proves that $u(0) = \gamma$ a.s. because $I_{[0,T]}(u|\gamma) < \infty$.

We turn to the second assertion of the lemma. Fix g in $C^2([0,1])$ and $0 \le s < t \le T$ such that $t - s < 1$. A convenient test function, depending only on time and similar to the one proposed after equation (4.3) in Farfan et al. (2011), yields that

$$\begin{aligned} \langle u(t)\,,\, g\rangle \;-\; \langle u(s)\,,\, g\rangle \;&\le\; \frac{1}{a}\, I_{[0,T]}(u|\gamma) \\ &+\; C_1(\|\Delta g\|_\infty, \|\nabla g\|_\infty)\, a\,(t-s) \;+\; C_2(A,B,\|g\|_\infty)\, a\,(t-s)\, e^{a\|g\|_\infty} \end{aligned}$$

for all $a > 0$. The exponential term comes from the $\mathfrak{b}_{\varrho,D}$ contribution to $J_{T,H}$ in the definition (2.10). Choose $a = -\,(1/2)\,(1 + \|g\|_\infty)^{-1} \log(t-s)$ to get that there exists a finite positive constant C_0, depending only on A, B, g, such that

$$\langle u(t)\,,\, g\rangle \;-\; \langle u(s)\,,\, g\rangle \;\le\; \frac{C_0}{\log(t-s)^{-1}}\,\big\{\, I_{[0,T]}(u|\gamma) \;+\; 1 \,\big\} \;.$$

This completes the proof of the lemma. □

The space $\mathcal{H}^1$. Let $\mathcal{H}^1$ be the Sobolev space of measurable functions $G : [0,1] \to \mathbb{R}$ with generalized derivatives ∇G in $\mathcal{L}^2([0,1])$. $\mathcal{H}^1$ endowed with the scalar product $\langle \cdot,\cdot\rangle_1$, defined by

$$\langle G\,,\, H\rangle_1 \;=\; \langle G\,,\, H\rangle \;+\; \langle \nabla G\,,\, \nabla H\rangle\;, \tag{3.4}$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{\mathcal{H}^1}$:

$$\|G\|^2_{\mathcal{H}^1} \;:=\; \int_0^1 |G(x)|^2\, dx \;+\; \int_0^1 |\nabla G(x)|^2\, dx\;.$$

Recall from (A.12) that any function γ in $\mathcal{H}^1$ has a continuous version. Hereafter, we always replace γ by its continuous version w.

Consider the function $\phi : \mathbb{R} \to [0,\infty)$ defined by

$$\phi(r) \;:=\; \begin{cases} \dfrac{1}{Z} \exp\{-\dfrac{1}{(1-r^2)}\} & \text{if } |r| < 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where the constant Z is chosen so that $\int_{\mathbb{R}} \phi(r) dr = 1$. For each $\delta > 0$, let

$$\phi^\delta(r) \;:=\; \frac{1}{\delta}\, \phi\Big(\frac{r}{\delta}\Big)\;, \tag{3.5}$$

whose support is contained in $[-\delta, \delta]$.

Denote by $f * g$ the space or time convolution of two functions f, g:

$$(f * g)(a) \;=\; \int_{\mathbb{R}} f(a-b)\, g(b)\, db\;.$$

Throughout this section, we adopt the following notation. Recall from Appendix A that we denote by $(P_t^{(R)} : t \geq 0)$ the semigroup associated to the Robin Laplacian. For a bounded measurable function $u : [0,T] \times [0,1] \to \mathbb{R}$, define the smooth approximation in space, time and space-time, respectively, by

$$u^\varepsilon(t,x) \;:=\; [\,P_\varepsilon^{(R)} u_t\,](x)\;, \quad u^\delta(t,x) \;:=\; [u(\cdot,x) * \phi^\delta](t) \;=\; \int_{-\delta}^{\delta} u(t+r,x)\,\phi^\delta(r)\,dr\;,$$
$$u^{\varepsilon,\delta}(t,x) \;:=\; [\,P_\varepsilon^{(R)} u_t^\delta\,](x) \;:=\; [\,P_\varepsilon^{(R)} u_t\,]^\delta(x)\;.$$

In the above formulas, we extend the definition of u to $[-1, T+1] \times [0,1]$ by setting $u_t = u_0$ for $-1 \leq t \leq 0$, $u_t = u_T$ for $T \leq t \leq T+1$.

Note that we use the same notation, u^ε and u^δ, for different objects. However, u^ε and u^δ always represent a smooth approximation of u in space and time, respectively. Moreover, the time-convolution commutes with the operator which explains the identity in the last displayed equation.

We summarize some properties of u^ε in the next result. Denote by $\nabla^n u$, $n \geq 1$, the n-th partial derivative in space of u and by $\mathcal{L}^2(0,T;\mathcal{H}^1)$ the space of square-integrable functions $F : [0,T] \to \mathcal{H}^1$, $\int_{[0,T]} \|F(t)\|_{\mathcal{H}^1}^2\, dt < \infty$.

Lemma 3.2. *Let $u : [0,T] \times [0,1] \to \mathbb{R}$ be a function in $\mathcal{L}^2(0,T;\mathcal{H}^1)$. Then, u^ε and ∇u^ε converge to u and ∇u in $\mathcal{L}^2([0,T] \times [0,1])$, respectively. Moreover, if u is bounded in $[0,T] \times [0,1]$ and the application $t \mapsto \langle u_t, g\rangle$ is continuous in the time interval $[0,T]$ for any function g in $C^\infty([0,1])$, then, for each $\varepsilon > 0$, $n \geq 1$, u^ε and $\nabla^n u^\varepsilon$ are uniformly continuous in $[0,T] \times [0,1]$.*

Proof: Recall the notation introduced in Appendix A. As u belongs to $\mathcal{L}^2(0,T;\mathcal{H}^1)$ and the norms $\mathcal{H}^1$, $\mathcal{H}_R$ are equivalent,

$$\int_0^T \|\, u_t \,\|_{\mathcal{H}_R}^2\, dt \;<\; \infty\;.$$

This relation can be rewritten in terms of the eigenfunctions $(f_k : k \geq 1)$ of the Robin Laplacian as

$$\int_0^T \sum_{k\geq 1} \lambda_k\, \langle u_t\,,\, f_k\rangle^2\, dt \;<\; \infty\;. \tag{3.6}$$

Since

$$u_t \;=\; \sum_{k\geq 1} \langle u_t\,,\, f_k\rangle\, f_k\;, \qquad u_t^\varepsilon \;=\; \sum_{k\geq 1} e^{-\lambda_k \varepsilon}\, \langle u_t\,,\, f_k\rangle\, f_k\;,$$

we have that

$$\int_0^T \|\, u_t^\varepsilon - u_t \,\|_2^2\, dt \;=\; \int_0^T \sum_{k\geq 1} \big[\, e^{-\lambda_k \varepsilon} \,-\, 1\,\big]^2\, \langle u_t\,,\, f_k\rangle^2\, dt$$

and, by (A.9),

$$\begin{aligned}\int_0^T \|\, \nabla u_t^\varepsilon - \nabla u_t \,\|_2^2\, dt \;&\leq\; C_0 \int_0^T \|u_t^\varepsilon - u_t\|_{\mathcal{H}_R}^2\, dt \\ &=\; C_0 \int_0^T \sum_{k\geq 1} \lambda_k\, \big[\, e^{-\lambda_k \varepsilon} \,-\, 1\,\big]^2\, \langle u_t\,,\, f_k\rangle^2\, dt\;.\end{aligned}$$

By (3.6), the left-hand side of the previous two displayed equations vanish as $\varepsilon \to 0$, which proves the first assertion of the lemma.

We turn to the second assertion. We may represent u^ε, $\nabla^n u^\varepsilon$ as

$$u^\varepsilon_t(x) \;=\; \sum_{k\ge 1} e^{-\lambda_k \varepsilon}\, \langle u_t\,,\, f_k\rangle\, f_k(x)\;, \quad (\nabla^n u^\varepsilon_t)(x) \;=\; \sum_{k\ge 1} e^{-\lambda_k \varepsilon}\, \langle u_t\,,\, f_k\rangle\, (\nabla^n f_k)(x)\;,$$

The second assertion follows from these identities and from the two hypotheses of the lemma. Indeed, the bound (A.5) on the eigenfunctions f_k permits to restrict the sum to a finite number of terms. □

For each $a > 0$, define the functions h_a and σ_a on $[0, 1]$ by

$$h_a(x) \;:=\; \frac{1}{2(1+2a)} \Big\{ (x+a)\log{(x+a)} + (1-x+a)\log{(1-x+a)} \Big\}\;,$$
$$\sigma_a(x) \;:=\; (x+a)(1-x+a)\;.$$

Note that $h''_a = (2\,\sigma_a)^{-1}$.

Until the end of this section, $0 < C_0 < \infty$ represents a constant independent of ε, δ and a and that may change from line to line.

Fix $T > 0$ and a path u in $D_{\mathcal{E}}([0, T], \mathcal{M}_{\rm ac})$. For a smooth function $G : [0, T] \times [0, 1] \to \mathbb{R}$ and a bounded function H in $\mathcal{L}^2(0, T; \mathcal{H}^1)$, define the functionals

$$L_G(u) \;=\; \langle u_T, G_T\rangle \;-\; \langle u_0, G_0\rangle \;-\; \int_0^T \langle u_t, \partial_t G_t\rangle \, dt\;,$$
$$B^1_H(u) \;=\; \int_0^T \langle \nabla u_t, \nabla H_t\rangle\, dt \;-\; \int_0^T \langle \sigma(u_t), (\nabla H_t)^2\rangle\, dt\;,$$
$$B^2_H(u) \;=\; \int_0^T \Big\{\, \mathfrak{b}_{\alpha,A}\big(\, u_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(\, u_t(1)\,,\, H_t(1)\,\big) \Big\}\, dt\;.$$

By (2.10), for paths u such that $u(0) = \gamma$,

$$\sup_{H\in C^{1,2}([0,T]\times[0,1])} \Big\{ L_H(u) + B^1_H(u) - B^2_H(u) \Big\} \;=\; I_{[0,T]}(u|\gamma)\;.$$

Lemma 3.3. *For $a > 0$, $\varepsilon > 0$, $\delta > 0$, let $H_{\varepsilon,\delta} = h'_a(u^{\varepsilon,\delta})$,*

$$R^{\varepsilon,\delta} \;=\; L_{H_{\varepsilon,\delta}}(u^{\varepsilon,\delta}) \;-\; L_{(H_{\varepsilon,\delta})^{\varepsilon,\delta}}(u)\;.$$

Then, for any fixed $a > 0$, $\varepsilon > 0$, $R^{\varepsilon,\delta}$ converges to 0 as $\delta \downarrow 0$.

Warning: Until the end of Proposition 3.5 proof's, we drop the dependence of $H = H_{\varepsilon,\delta} = h'_a(u^{\varepsilon,\delta})$ on ε, δ. Hence, H always stands for $H_{\varepsilon,\delta}$.

Proof of Lemma 3.3: Recall that C_0 represents a constant independent of ε, δ and a, that may change from line to line. As $P^{(R)}_\varepsilon$ is a self-adjoint operator in $\mathcal{L}^2([0, 1])$ and commutes with the time-derivative,

$$L_H(u^{\varepsilon,\delta}) \;=\; \langle u^\delta_T, H^\varepsilon_T\rangle - \langle u^\delta_0, H^\varepsilon_0\rangle - \int_0^T \langle u^\delta_t, \partial_t H^\varepsilon_t\rangle\, dt$$
$$=\; \langle u_T, H^{\varepsilon,\delta}_T\rangle - \langle u_0, H^{\varepsilon,\delta}_0\rangle - \int_0^T \langle u^\delta_t, \partial_t H^\varepsilon_t\rangle\, dt \;+\; R^{\varepsilon,\delta}_1\;,$$

where

$$R_1^{\varepsilon,\delta} := R^{\varepsilon,\delta,T} - R^{\varepsilon,\delta,0} \quad \text{and} \quad R^{\varepsilon,\delta,t} := \langle u_t^\delta - u_t, H_t^\varepsilon\rangle + \langle u_t, H_t^\varepsilon - H_t^{\varepsilon,\delta}\rangle$$

for $0 \le t \le T$.

A simple computation yields that

$$\int_0^T \langle u_t^\delta, \partial_t H_t^\varepsilon\rangle \, dt \;=\; \int_0^T \langle u_t, \partial_t H_t^{\varepsilon,\delta}\rangle \, dt \;+\; R_2^{\varepsilon,\delta}\;,$$

where $|R_2^{\varepsilon,\delta}| \le C_0\delta\|\partial_t H^\varepsilon\|_\infty$. To conclude the proof, it is enough to show that, for each fixed $a>0$, $\varepsilon>0$, $R_1^{\varepsilon,\delta}$ and $\delta\|\partial_t H^\varepsilon\|_\infty$ converge to zero as $\delta\downarrow 0$.

We first prove that $R_1^{\varepsilon,\delta}$ vanishes as $\delta\to 0$. We have to show that

$$\lim_{\delta\downarrow 0} R^{\varepsilon,\delta,t} \;=\; 0 \quad \text{for} \quad t=0 \;\text{ and }\; t=T\;. \tag{3.7}$$

We consider the case $t=T$, the argument being similar for $t=0$. As $P_t^{(R)}$ is symmetric,

$$R^{\varepsilon,\delta,T} \;=\; \langle u_T^{\varepsilon,\delta} - u_T^\varepsilon, H_T\rangle + \langle u_T^\varepsilon, H_T - H_T^\delta\rangle\;.$$

By Lemma 3.2, for each $x\in[0,1]$, $u^\varepsilon(\cdot,x)$ is continuous. Therefore, by definition of H, for any $(t,x)\in[0,T]\times[0,1]$,

$$\lim_{\delta\downarrow 0} u^{\varepsilon,\delta}(t,x) \;=\; u^\varepsilon(t,x)\;,$$

$$\lim_{\delta\downarrow 0} H^\delta(T,x) \;=\; \lim_{\delta\downarrow 0} h_a'(u^{\varepsilon,\delta})^\delta(T,x) \;=\; h_a'(u^\varepsilon)(T,x) \;=\; \lim_{\delta\downarrow 0} h_a'(u^{\varepsilon,\delta}) \;=\; \lim_{\delta\downarrow 0} H(T,x)$$

because h_a' is bounded and continuous on $[0,1]$. Note that the dependence on δ of the last term on the right-hand side is hidden, as H is actually $h_a'(u^{\varepsilon,\delta})$. Claim (3.7) follows from these results, from the boundedness of u and h_a', and the bounded convergence theorem.

It remains to show that $\delta\|\partial_t H^\varepsilon\|_\infty$ converges to 0 as $\delta\downarrow 0$. An elementary computation gives that, for any $t\in[0,T]$,

$$\partial_t H^\varepsilon(t) \;=\; P_\varepsilon^{(R)}\Big[\, h_a''(u^{\varepsilon,\delta}(t)) \int_{-\delta}^{\delta} u^\varepsilon(t-r)\,(\phi^\delta)'(r)\, dr\Big]\;.$$

Since ϕ^δ is an even function, a change of variables shows that

$$\int_{-\delta}^{\delta} u^\varepsilon(t-r)\,(\phi^\delta)'(r)\,dr \;=\; \int_0^\delta \{\, u^\varepsilon(t-r) \;-\; u^\varepsilon(t+r)\,\}\,(\phi^\delta)'(r)\,dr\;.$$

By Lemma 3.2, u^ε is uniformly continuous on $[-1,T+1]\times[0,1]$. On the other hand, $\delta\int_0^\delta(\phi^\delta)'(r)\,dr \;=\; -\,\phi(0)$. Therefore, the last expression multiplied by δ converges to 0 as $\delta\downarrow 0$ uniformly in $[0,T]\times[0,1]$. Since h_a'' is uniformly bounded, by the bounded convergence theorem, $\delta\|\partial_t H_\varepsilon\|_\infty$ converges to 0 as $\delta\downarrow 0$. □

Lemma 3.4. *There exists a positive constant $C_0<\infty$ such that*

$$\int_0^T dt\,\int_0^1 \frac{(\nabla u(t,x))^2}{\sigma_a(u(t,x))}\,dx \;\le\; C_0\, B^1_{h_a'(u)}(u)\;, \qquad |B^2_{h_a'(u)}(u)| \;\le\; C_0 \tag{3.8}$$

for all $u\in D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ and $0<a<1$. Moreover, for each $u\in D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ and $i=1,2$,

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} B^i_{H^{\varepsilon,\delta}}(u) \;=\; B^i_{h_a'(u)}(u)\;.$$

Proof: Let u be a path in $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$. We first show that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} B^1_{H^{\varepsilon,\delta}}(u) \;=\; B^1_{h'_a(u)}(u)\;. \tag{3.9}$$

By Lemma 3.2, ∇u^ε is uniformly continuous in $[0,T]\times[0,1]$. Therefore, for any $(t,x)\in[0,T]\times[0,1]$,

$$\lim_{\delta\downarrow 0}\nabla u^{\varepsilon,\delta}(t,x) \;=\; \nabla u^\varepsilon(t,x)\;.$$

Recall from the end of the Appendix B the definition of the semigroup $P^{(M)}_t$. By (B.11) $\nabla P^{(R)}_\varepsilon = P^{(M)}_\varepsilon\nabla$. Hence,

$$\lim_{\delta\downarrow 0}\nabla H^{\varepsilon,\delta}(t,x) \;=\; \lim_{\delta\downarrow 0} P^{(M)}_\varepsilon\Big[\, h''_a(u^{\varepsilon,\delta})\,\nabla u^{\varepsilon,\delta}\,\Big]^\delta(t,x) \;=\; P^{(M)}_\varepsilon\big[\, h''_a(u^\varepsilon_t)\,\nabla u^\varepsilon_t\,\big](x)\;.$$

Hence, as $(\nabla u)(t,x)\,dx\,dt$ is a finite measure on $[0,T]\times[0,1]$, by the bounded convergence theorem,

$$\lim_{\delta\downarrow 0} B^1_{H^{\varepsilon,\delta}}(u) \;=\; \int_0^T \Big\{ \big\langle \nabla u_t\,,\, G^\varepsilon_t\,\big\rangle \,-\, \big\langle\, \sigma(u_t)\,,\,[G^\varepsilon_t]^2\big\rangle \Big\}\,dt\;, \tag{3.10}$$

where $G^\varepsilon(t,x) = P^{(M)}_\varepsilon[\, h''_a(u^\varepsilon_t)\,\nabla u^\varepsilon_t\,](x)$.

On the one hand, since $P^{(M)}_\varepsilon$ is a contraction in $\mathcal{L}^2([0,1])$, h''_a is bounded, and since, by Lemma 3.2, ∇u^ε converges to ∇u in $\mathcal{L}^2([0,T]\times[0,1])$,

$$\lim_{\varepsilon\downarrow 0}\int_0^T dt\,\int_0^1 \big\{\, P^{(M)}_\varepsilon\big[\, h''_a(u^\varepsilon_t)\,\big(\nabla u^\varepsilon_t - \nabla u_t\,\big)\,\big]\big\}^2\,dx \;=\; 0\;.$$

Therefore, on the right-hand side of (3.10), in the formula for G^ε we may replace ∇u^ε_t by ∇u_t at a cost that vanishes as $\varepsilon\to 0$.

Since h''_a is Lipschitz continuous, by Lemma 3.2, as $\varepsilon\downarrow 0$, $h''_a(u^\varepsilon)$ converges in measure to $h''_a(u)$. In other words, for any $b>0$, the Lebesgue measure of the set $\{(t,x)\in[0,T]\times[0,1]; |h''_a(u^\varepsilon(t,x)) - h''_a(u(t,x))|\ge b\}$ converges to 0 as $\varepsilon\downarrow 0$. Therefore, as ∇u_t belongs to $\mathcal{L}^2([0,T]\times[0,1])$,

$$\lim_{\varepsilon\downarrow 0}\int_0^T dt\int_0^1 h''_a(u^\varepsilon_t)\,\big[\nabla u_t\big]^2\,dx \;=\; \int_0^T dt\int_0^2 h''_a(u_t)\,\big[\nabla u_t\big]^2\,dx\;.$$

In consequence, on the right-hand side of (3.10), in the formula for G^ε we may further replace $h''_a(u^\varepsilon)$ by to $h''_a(u)$.

To complete the proof of (3.9), it remains to recall that for any f in $\mathcal{L}^2([0,1])$, $P^{(M)}_\varepsilon f$ converges in $\mathcal{L}^2([0,1])$ to f as $\varepsilon\to 0$.

We turn to the proof that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} B^2_{H^{\varepsilon,\delta}}(u) \;=\; B^2_{h'_a(u)}(u)\;.$$

We examine the boundary condition at $x=0$, the other one being similar.

By Lemma 3.2, u^ε is uniformly continuous. Hence, as h'_a is continuous in the interval $[0,1]$, as $\delta\to 0$, $H^{\varepsilon,\delta}(t,0)$ converges to $P^{(R)}_\varepsilon[\, h'_a(u^\varepsilon_t)\,](0)$. Therefore,

$$\lim_{\delta\downarrow 0}\int_0^T \mathfrak{b}_{\alpha,A}\big(\, u_t(0)\,,\, H_{\varepsilon,\delta}(t,0)\,\big)\,dt \;=\; \int_0^T \mathfrak{b}_{\alpha,A}\big(\, u_t(0)\,,\, P^{(R)}_\varepsilon[\, h'_a(u^\varepsilon_t)\,](0)\,\big)\,dt\;. \tag{3.11}$$

To conclude the proof, we first replace on the right-hand side $P_\varepsilon^{(R)}[\,h'_a(u_t^\varepsilon)\,](0)$ by $h'_a(u_t^\varepsilon(0))$. Since h'_a is bounded, there exists a finite constant $C_1 = C_1(a, A, \alpha)$ such that

$$\Big|\int_0^T \mathfrak{b}_{\alpha,A}\big(\,u_t(0)\,,\,P_\varepsilon^{(R)}[\,h'_a(u_t^\varepsilon)\,](0)\,\big)\,dt \;-\; \int_0^T \mathfrak{b}_{\alpha,A}\big(\,u_t(0)\,,\,h'_a(u_t^\varepsilon(0))\,\big)\,dt\Big|$$
$$\le\; C_1 \int_0^T \big|\,P_\varepsilon^{(R)}[\,h'_a(u_t^\varepsilon)\,](0) \;-\; h'_a(u_t^\varepsilon(0))\,\big|\,dt\,.$$

It is easily seen that

By Lemma A.3 and (A.9),

$$\big|\,P_\varepsilon^{(R)}[\,h'_a(u_t^\varepsilon)\,](0) \;-\; h'_a(u_t^\varepsilon(0)\,)\,\big| \;\le\; C_0\,\varepsilon^{1/5}\big\|\,h'_a(u_t^\varepsilon\,)\,\big\|_{\mathcal{H}^1}$$

for some finite constant C_0. Hence, the term on the right-hand side in the penultimate displayed equation is bounded by

$$C_1\,\varepsilon^{1/5}\int_0^T \big\|\,h'_a(u_t^\varepsilon\,)\,\big\|_{\mathcal{H}^1}\,dt\,.$$

By Lemma 3.2, u^ε converges to u in $\mathcal{L}^2(0,T;\mathcal{H}^1)$. Since h'_a and h''_a are bounded, the previous integral is bounded uniformly in ε. In particular, the previous expression vanishes as $\varepsilon\to 0$.

It remains to estimate the right-hand side of (3.11) with $P_\varepsilon^{(R)}[\,h'_a(u_t^\varepsilon)\,](0)$ replaced by $h'_a(u_t^\varepsilon(0))$. By Lemma 3.2, u^ε converges to u in $\mathcal{L}^2(0,T,\mathcal{H}^1)$. Thus, by (A.7) and (A.9),

$$\lim_{\varepsilon\to 0}\int_0^T \big|\,u_t(0) \;-\; u_t^\varepsilon(0)\,\big|^2\,dt \;=\; 0\,.$$

Hence, as $|\,\mathfrak{b}_{\alpha,A}(a,y) - \mathfrak{b}_{\alpha,A}(a,x)\,| \le C_0\{\,|\,e^y - e^x\,| \,+\, |\,e^{-y} - e^{-x}\,|\}$ for some finite constant C_0, and h'_a belongs to $C^1([0,1])$, and since, by (A.16), u^ε is uniformly bounded,

$$\lim_{\varepsilon\to 0}\int_0^T \mathfrak{b}_{\alpha,A}\big(\,u_t(0)\,,\,h'_a(u_t^\varepsilon(0))\,\big)\,dt \;=\; \int_0^T \mathfrak{b}_{\alpha,A}\big(\,u_t(0)\,,\,h'_a(u_t(0))\,\big)\,dt\,,$$

which completes the proof of first assertion of the lemma.

We turn to the bounds (3.8). As $\sigma(x)\le\sigma_a(x)$, for each $\varepsilon>0$,

$$\int_0^T \langle\nabla u_t^\varepsilon\,,\,\nabla h'_a(u_t^\varepsilon)\rangle\,dt \;-\; \int_0^T \langle\sigma_a(u_t^\varepsilon)\,,\,(\nabla h'_a(u_t^\varepsilon))^2\rangle\,dt \;\le\; B^1_{h'_a(u^\varepsilon)}(u^\varepsilon)\,.$$

Compute the derivative $\nabla h'_a(u_t^\varepsilon)$ on the left-hand side to get that

$$\frac{1}{4}\int_0^T dt\int_0^1 \frac{(\nabla u^\varepsilon)^2}{\sigma_a(u^\varepsilon)}\,dx \;\le\; B^1_{h'_a(u^\varepsilon)}(u^\varepsilon)\,.$$

The arguments presented in the first part of the proof permit to let $\varepsilon\to 0$ on both sides of this inequality and yield the first estimate in (3.8).

To estimate $B^2_{h'_a(u)}(u)$, note that

$$\mathfrak{b}_{\varrho,D}\big(\,v\,,\,h'_a(v)\,\big) \;=\;$$
$$\frac{1}{D}\Big\{\,\varrho\,[1-v]\,\Big[\Big(\frac{v+a}{1+a-v}\Big)^{1/2+4a} - 1\Big] \,+\, [1-\varrho]\,v\,\Big[\Big(\frac{1+a-v}{v+a}\Big)^{1/2+4a} - 1\Big]\,\Big\}\,.$$

In particular, $\mathfrak{b}_{\varrho,D}$, as a function of v and a is bounded: for all $0<\varrho<1$, $D>0$, there exists a finite constant $C_0 = C_0(\varrho, D)$ such that

$$\sup_{0\le a<1}\ \sup_{v\in[0,1]} \big|\, \mathfrak{b}_{\varrho,D}\big(\, v\,,\, h_a'(v)\,\big)\,\big| \;\le\; C_0\;.$$

The second inequality in (3.8) follows from this estimate and the definition of $B^2_{h_a'(u)}(u)$. □

Proposition 3.5. *There exists a constant $C_0>0$ such that*

$$\int_0^T dt \int_0^1 \frac{|\nabla u(t,x)|^2}{\sigma(u(t,x))}\, dx \;\le\; C_0 \,\{\, I_{[0,T]}(u) \,+\, 1\,\}$$

for any path u in $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$.

Proof: We may assume, without loss of generality, that $I_{[0,T]}(u)$ is finite. By the variational formula (2.12) and with the notation of Lemma 3.3,

$$L_H(u^{\varepsilon,\delta}) + B^1_{H^{\varepsilon,\delta}}(u) - B^2_{H^{\varepsilon,\delta}}(u) - R^{\varepsilon,\delta} \;\le\; I_{[0,T]}(u)\;, \tag{3.12}$$

where, recall, H stands for the function $h_a'(u^{\varepsilon,\delta})$.

Since $u^{\varepsilon,\delta}$ is smooth, an integration by parts yields that

$$L_H(u^{\varepsilon,\delta}) \;=\; \int_0^1 h_a(u^{\varepsilon,\delta}_T)\, dx \;-\; \int_0^1 h_a(u^{\varepsilon,\delta}_0)\, dx\;.$$

There exists, therefore, a constant C_0, independent of ε, δ and a, such that

$$|L_H(u^{\varepsilon,\delta})| \;\le\; C_0\;.$$

In (3.12), let $\delta\downarrow 0$ and then $\varepsilon\downarrow 0$. It follows from the previous bound, and from Lemmata 3.3 and 3.4 that

$$B^1_{h_a'(u)}(u) \;-\; B^2_{h_a'(u)}(u) \;\le\; I_T(u) \;+\; C_0\;.$$

Thus, by (3.8),

$$\int_0^T dt \;\int_0^1 \frac{|\nabla u(t,x)|^2}{\sigma_a(u(t,x))}\, dx \;\le\; C_0\,\{I_T(u)+1\}\;.$$

It remains to let $a\downarrow 0$ and to apply Fatou's lemma. □

Note: Since the rate function is declared to be infinite on trajectories with infinite energy, this result is not meant to show that a trajectory has finite energy. Its interest lies on the fact that it provides a uniform bound of a strong version of the energy for trajectories with rate function bounded by a constant.

Corollary 3.6. *The density u of a path $\pi(t,dx) = u(t,x)\,dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ is the weak solution of the initial-boundary value problem (2.6) if, and only if $I_T(u|\gamma)=0$.*

Proof: Suppose that the density u of a path $\pi(t,dx) = u(t,x)\,dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ is the weak solution of the initial-boundary value problem (2.6). Then, by Lemma B.5, u has finite energy. On the other hand, by Definition B.3 and the equation following it, $u(0)=\gamma$ a.s. and for any G in $C^{1,2}([0,T]\times[0,1])$,

$$\begin{aligned} J_{T,G}(u) \;=\; & -\int_0^T \langle \sigma(u_t), (\nabla G_t)^2\rangle\, dt \\ & -\int_0^T \big\{\, \mathfrak{q}_{\beta,B}(u_t(1),G_t(1)) \,+\, \mathfrak{q}_{\alpha,A}(u_t(0),G_t(0))\,\big\}\, dt\;, \end{aligned}$$

where

$$\mathfrak{q}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{[1-a]\,\varrho\,\big[\,e^{M} \,-\, M \,-\, 1\,\big] \;+\; a\,[1-\varrho]\,\big[\,e^{-M} + M \,-\, 1\,\big]\Big\}\,. \tag{3.13}$$

Here we used the fact that $u-a$ can be written as $u(1-a)-(1-u)a$. As $\mathfrak{q}_{\varrho,D}(a,M)\ge 0$, $J_{T,G}(u)\le 0$. Hence, the supremum in the variational problem (2.12) is attained at $H=0$ and $I_{[0,T]}(u)=0$. Since $u(0)=\gamma$, $I_{[0,T]}(u|\gamma)=0$.

On the other hand, if $I_{[0,T]}(u|\gamma)=0$, then, for any G in $C^{1,2}([0,T]\times[0,1])$ and ε in $\mathbb{R}$, $J_{T,\varepsilon G}(u)\le 0$. Since $J_{T,0}(u)=0$, the derivative in ε of $J_{T,\varepsilon G}(u)$ at $\varepsilon=0$ is equal to 0. Therefore, by Definition B.3, the density u is a weak solution of the initial-boundary value problem (2.6). □

Let E_q, $q\ge 0$, be the level set of the rate function $I_{[0,T]}(\cdot|\gamma)$:

$$E_q \;:=\; \big\{\,\pi\in D([0,T],\mathcal{M})\,|\,I_{[0,T]}(\pi|\gamma)\;\le\; q\,\big\}\,.$$

Proof of Theorem 2.4: The rate function $I_{[0,T]}(\,\cdot\,|\,\gamma)$ is convex because the energy $\mathcal{Q}_{[0,T]}(\,\cdot\,)$ and the functionals $J_{T,H}(\,\cdot\,)$ are convex.

Let $\{\pi^n : n\ge 1\}$ be a sequence in $D([0,T],\mathcal{M})$ such that π^n converges to some element π in $D([0,T],\mathcal{M})$. We show that $I_{[0,T]}(\pi|\gamma)\le \liminf_{n\to\infty} I_{[0,T]}(\pi^n|\gamma)$. If $\liminf I_{[0,T]}(\pi^n|\gamma)$ is equal to ∞, the conclusion is clear. Therefore, we may assume that the set $\{\pi^n : n\ge 1\}$ is contained in E_q for some $q>0$. In particular, by definition of $I_{[0,T]}(\,\cdot\,|\gamma)$ and by Lemma 3.1, $\pi^n(t,dx)=u^n(t,x)\,dx$ for some $u^n\in C([0,T],\mathcal{M}_{\rm ac})$ with finite energy.

Since u^n belongs to $C([0,T],\mathcal{M}_{\rm ac})$ and $\pi^n(t,dx)=u^n(t,x)\,dx$ converges to $\pi(t,dx)$ in $D([0,T],\mathcal{M})$, $\pi(t,dx)=u(t,x)\,dx$ for some $u\in C([0,T],\mathcal{M}_{\rm ac})$. Moreover, by the lower semi-continuity of the energy $\mathcal{Q}_{[0,T]}$ and by Proposition 3.5,

$$\mathcal{Q}_{[0,T]}(u)\;\le\;\varliminf_{n\to\infty}\,\mathcal{Q}_{[0,T]}(u^n)\;\le\;C_0(q+1)\;<\;\infty$$

for some finite constant C_0.

Claim 1: The sequence $\{u^n : n\ge 1\}$ converges to u in $\mathcal{L}^2([0,T]\times[0,1])$.

Indeed, by the triangle inequality,

$$\begin{aligned}&\frac{1}{3}\int_0^T \|u_t-u^n_t\|_2^2\,dt\\ &\quad\le\;\int_0^T \|u_t-u^\varepsilon_t\|_2^2\,dt\;+\;\int_0^T \|u^\varepsilon_t-u^{n,\varepsilon}_t\|_2^2\,dt\;+\;\int_0^T \|u^{n,\varepsilon}_t-u^n_t\|_2^2\,dt\,,\end{aligned}$$

where $u^\varepsilon_t=P^{(R)}_\varepsilon u_t$, $u^{n,\varepsilon}_t=P^{(R)}_\varepsilon u^n_t$. By Lemma A.2 and (A.9), and since $\|u_t\|_\infty\le 1$, the first and the last terms are bounded by

$$C_0\,\varepsilon^{2/3}\int_0^T\big\{\,\|u_t\|^2_{\mathcal{H}^1}\;+\;\|u^n_t\|^2_{\mathcal{H}^1}\,\big\}\,dt\;\le\;C_0\,\varepsilon^{2/3}\,\{\,q+T+1\,\}\,.$$

On the other hand,

$$\int_0^T \|u^\varepsilon_t-u^{n,\varepsilon}_t\|_2^2\,dt\;\le\;\int_0^T\sum_{k\ge1}e^{-2\lambda_k\varepsilon}\,\langle u^n_t-u_t\,,\,f_k\rangle^2\,dt\,.$$

As π^n converges to π in $D([0,T],\mathcal{M})$, for all $g\in C([0,1])$, $\langle u^n_t-u_t\,,\,g\rangle\to 0$ for almost all $t\in[0,T]$. In particular, for every $\varepsilon>0$, the right-hand side of the previous displayed equation vanishes as $n\to\infty$, which proves Claim 1.

Claim 2: We have that

$$\lim_{n\to\infty} \int_0^T \big\{ \,|\, u_t(0) - u^n_t(0) \,|^2 \;+\; |\, u_t(1) - u^n_t(1) \,|^2 \,\big\} \; dt \;=\; 0 \;. \tag{3.14}$$

We consider the boundary $x = 0$, the argument for $x = 1$ being identical. The proof is similar to the one of Claim 1 and relies on Lemma A.3 instead of Lemma A.2. By the triangle inequality, the previous integral, for $x = 0$ only and divided by 3, is bounded by

$$\int_0^T |u_t(0) - u^\varepsilon_t(0)|^2 \; dt \;+\; \int_0^T |u^\varepsilon_t(0) - u^{n,\varepsilon}_t(0)|^2 \; dt \;+\; \int_0^T |u^{n,\varepsilon}_t(0) - u^n_t(0)|^2 \; dt \;.$$

As u_t, u^n_t are continuous for almost all t [because they have finite energy], we may repeat the argument of Claim 1, using Lemma A.3 instead of Lemma A.2, to show that the first and third integrals in the previous equation are bounded by $C_0\, \varepsilon^{2/5} \{\, q + T + 1\,\}$.

By (A.13), (A.5) and Schwarz inequality,

$$\begin{aligned} |u^\varepsilon_t(0) - u^{n,\varepsilon}_t(0)|^2 \;&\le\; \sum_{k\ge 1} e^{-\lambda_k \varepsilon} \,\langle u^n_t - u_t \,,\, f_k\rangle^2 \sum_{k\ge 1} e^{-\lambda_k \varepsilon} \\ &=\; C_0(\varepsilon) \sum_{k\ge 1} e^{-\lambda_k \varepsilon} \,\langle u^n_t - u_t \,,\, f_k\rangle^2 \;. \end{aligned}$$

At this point, we may repeat the arguments presented in Claim 1 to complete the proof of Claim 2.

By Claims 1, 2 and (2.11), for any function G in $C^{1,2}([0,T]\times[0,1])$,

$$\lim_{n\to\infty} J_G(\pi^n) \;=\; J_G(\pi) \;. \tag{3.15}$$

Therefore, $I_{[0,T]}(\pi|\gamma) \le \liminf_{n\to\infty} I_{[0,T]}(\pi^n|\gamma)$, proving that $I_{[0,T]}(\,\cdot\,|\gamma)$ is lower semicontinuous.

The same argument shows that E_q is closed in $D([0,T],\mathcal{M})$. By Lemma 3.7 below, E_q is relatively compact in $D([0,T],\mathcal{M})$. Thus, E_q is compact in $D([0,T],\mathcal{M})$, as claimed. □

The proof of the next result is similar to the one contained in the proof of Theorem 4.2 in Bertini et al. (2009b).

Lemma 3.7. *For each $q > 0$, the set E_q is relatively compact in $D([0,T],\mathcal{M})$.*

Proof: Fix $q > 0$ and let π^n be a sequence in E_q. By Lemma 3.1, $\pi^n \in C([0,T],\mathcal{M}_{\rm ac})$. Denote by u^n the density of π^n: $\pi^n(t,dx) = u^n(t,x)\, dx$. Since $0 \le u^n(t,x) \le 1$, there exists a subsequence, still denoted by $(u^n : n \ge 1)$, which converges weakly in $\mathcal{L}^2([0,T]\times[0,1])$ to some trajectory u. By the lower semicontinuity of $\mathcal{Q}_{[0,T]}$, $\mathcal{Q}_{[0,T]}(u) < \infty$.

The proofs of Claims 1 and 2 in Theorem 2.4 yield that u^n converges strongly, as $n\to\infty$, to u in $\mathcal{L}^2([0,T]\times[0,1])$ and that (3.14) holds. Therefore, by (3.15) and the fact that π^n belongs to E_q, $I_{[0,T]}(\pi|\gamma) \le \liminf_{n\to\infty} I_{[0,T]}(\pi^n|\gamma) \le q$. By Lemma 3.1, u^n, u are uniformly weakly continuous in time. In particular, strong convergence in $\mathcal{L}^2([0,T]\times[0,1])$ implies convergence in $C([0,T],\mathcal{M}_{\rm ac})$. □

4. Deconstructing the rate functional

The main result of this section, stated in Proposition 4.5 below, shows that the rate function $I_{[0,T]}(\,\cdot\,)$ can be decomposed as the sum of two rate functions. The first one measures the cost of the trajectory due to its evolution in the bulk, while the second one measures the costs due to the boundary evolution. This decomposition of the rate function is the main tool in the proof that any trajectory u with finite rate function can be approximated by a sequence of regular trajectories $(u^n : n \geq 1)$ in such a way that $I_{[0,T]}(u^n \,|\gamma) \to I_{[0,T]}(u\,|\gamma)$, the content of the next section.

Weighted Sobolev spaces. Let Ω_T be the cylinder $[0,T]\times[0,1]$. Fix a non-negative weight $\kappa : \Omega_T \to \mathbb{R}_+$, and denote by $\mathcal{L}^2(\kappa)$ the Hilbert space induced by the smooth functions in $C^\infty(\Omega_T)$ endowed with the scalar product defined by

$$\langle\!\langle G, H\rangle\!\rangle_\kappa \;=\; \int_0^T dt \int_0^1 \kappa_t \, G_t \, H_t \, dx \, .$$

Above and hereafter, induced means that we first declare two functions F, G in $C^\infty(\Omega_T)$ to be equivalent if $\langle\!\langle F-G, F-G\rangle\!\rangle_\kappa = 0$ and then we complete the quotient space with respect to the scalar product.

Denote by $C^\infty_K(\Omega_T)$ the space of smooth functions $H : \Omega_T \to \mathbb{R}$ with support contained in $(0,T)\times(0,1)$. Let $\mathcal{H}^1(\kappa)$, $\mathcal{H}^1_0(\kappa)$ be the Hilbert spaces induced by the sets $C^\infty(\Omega_T)$, $C^\infty_K(\Omega_T)$ endowed with the scalar products, $\langle\!\langle G, H\rangle\!\rangle_{1,2,\kappa}$, $\langle\!\langle G, H\rangle\!\rangle_{1,\kappa}$, respectively defined by

$$\begin{aligned} \langle\!\langle G, H\rangle\!\rangle_{1,2,\kappa} \;&=\; \langle\!\langle G, H\rangle\!\rangle_\kappa \;+\; \langle\!\langle \nabla G, \nabla H\rangle\!\rangle_\kappa \, , \\ \langle\!\langle G, H\rangle\!\rangle_{1,\kappa} \;&=\; \langle\!\langle \nabla G, \nabla H\rangle\!\rangle_\kappa \, . \end{aligned}$$

The Poincaré's inequality yields that the norms induced by the scalar products $\langle\!\langle G, H\rangle\!\rangle_{1,2,\kappa}$, $\langle\!\langle G, H\rangle\!\rangle_{1,\kappa}$ are equivalent in $\mathcal{H}^1_0(\kappa)$.

Denote by $\|\cdot\|_\kappa$, $\|\cdot\|_{1,\kappa}$ the norm associated to the scalar product $\langle\!\langle \cdot,\cdot\rangle\!\rangle_\kappa$, $\langle\!\langle \cdot,\cdot\rangle\!\rangle_{1,\kappa}$, respectively. Let $\mathcal{H}^{-1}(\kappa)$ be the dual of $\mathcal{H}^1_0(\kappa)$; it is a Hilbert space equipped with the norm $\|\cdot\|_{-1,\kappa}$ defined by

$$\|L\|^2_{-1,\kappa} \;=\; \sup_{G\in C^\infty_K(\Omega_T)} \big\{\, 2\, L(G) \;-\; \|G\|^2_{1,\kappa} \,\big\} \, . \tag{4.1}$$

By Riesz' representation theorem, an element L of $\mathcal{H}^{-1}(\kappa)$ can be written as $L(H) = \langle\!\langle \nabla G \, , \, \nabla H\rangle\!\rangle_\kappa$ for some G in $\mathcal{H}^1_0(\kappa)$.

When $\kappa \equiv 1$, we represent $\mathcal{L}^2(\kappa)$, $\mathcal{H}^1(\kappa)$, $\mathcal{H}^1_0(\kappa)$, $\mathcal{H}^{-1}(\kappa)$ as $\mathcal{L}^2(\Omega_T)$, $\mathcal{H}^1(\Omega_T)$, $\mathcal{H}^1_0(\Omega_T)$ $\mathcal{H}^{-1}(\Omega_T)$, respectively. Next result is Bertini et al. (2009b, Lemma 4.8). It states that $\mathcal{H}^{-1}(\kappa)$ is formally the space $\{\nabla P : P \in \mathcal{L}^2(\kappa^{-1})\}$. For an integrable function $H : [0,1] \to \mathbb{R}$, let $\langle H\rangle = \int_0^1 H(x)\, dx$.

Lemma 4.1. *A linear functional $L : \mathcal{H}^1_0(\kappa) \to \mathbb{R}$ belongs to $\mathcal{H}^{-1}(\kappa)$ if, and only if, there exists P in $\mathcal{L}^2(\kappa^{-1})$ such that $L(H) = \int_0^T dt \int_0^1 P_t \, \nabla H_t dx$ for every H in $C^\infty_K(\Omega_T)$. In this case,*

$$\|L\|^2_{-1,\kappa} \;=\; \int_0^T \big\{\, \langle P_t, P_t\rangle_{\kappa(t)^{-1}} \;-\; c_t \,\big\}\, dt \, ,$$

where $c_t = \{\, \langle P_t/\kappa_t\rangle^2 \,/\, \langle 1/\kappa_t\rangle \,\}\, \mathbf{1}\{\langle 1/\kappa_t\rangle < \infty\}$.

Representation theorems. Until the end of this section, $\pi(t,dx) = u(t,x)\, dx$ is a path in $D_{\mathcal{E}}([0,T], \mathcal{M}_{\mathrm{ac}})$. We assume that

$$\begin{gathered} u \text{ is continuous on } \Omega_T \text{ and smooth in time, there exists } \varepsilon > 0 \\ \text{such that } \varepsilon \leq u(t,x) \leq 1-\varepsilon \text{ for all } (t,x)\in\Omega_T, \text{ and } I_{[0,T]}(u) < \infty. \end{gathered} \tag{4.2}$$

These conditions are fulfilled in sets of the form $[\delta, T] \times [0,1]$, $\delta > 0$, by paths in $\mathfrak{R}_3$, a class of trajectories to be introduced in Section 5. As u is bounded away from 0 and 1, the spaces $\mathcal{L}^2(\sigma(u))$ and $\mathcal{L}^2(\Omega_T)$ coincide, as well as, the other Hilbert spaces introduced in the previous subsection with $\kappa = \sigma(u)$.

Denote by $\mathfrak{W}\colon C^{0,1}(\Omega_T) \to \mathbb{R}$ the functional given by

$$\mathfrak{W}(H) \;=\; \int_0^T dt \int_0^1 \sigma(u_t)\, |\nabla H_t|^2\, dx \;+\; \int_0^T \Psi(t, H_t(0), H_t(1))\, dt\;, \tag{4.3}$$

where

$$\Psi(t,M,N) \;=\; \mathfrak{b}_{\alpha,A}(u_t(0), M) \;+\; \mathfrak{b}_{\beta,B}(u_t(1), N)\;,$$

and $\mathfrak{b}_{\varrho,D}(a,M)$ has been introduced in (2.9). For each $0 \le t \le T$, $(M,N) \mapsto \Psi(t,M,N)$ is a smooth, convex function which takes negative values.

Fix a linear functional $L : C^{0,1}(\Omega_T) \to \mathbb{R}$. Denote by L_0 its restriction to $C^{0,1}_0(\Omega_T)$:

$$L_0(H) \;=\; L(H)\;, \quad H \,\in\, C^{0,1}_0(\Omega_T)\;, \tag{4.4}$$

where

$$C^{0,1}_0(\Omega_T) \;:=\; \big\{\, H \in C^{0,1}(\Omega_T) : H(t,0) = H(t,1) = 0\,,\, 0 \le t \le T \,\big\}\;.$$

Let $\Xi : \Omega_T \to \mathbb{R}$ the function given by

$$\Xi(t,x) \;=\; \frac{1}{\int_0^1 1/\sigma(u(t,y))\; dy} \int_0^x \frac{1}{\sigma(u(t,y))}\; dy\;. \tag{4.5}$$

Note that Ξ belongs to $C^{\infty,1}(\Omega_T)$, and that $\Xi(t,0) = 0$, $\Xi(t,1) = 1$ for all $0 \le t \le T$. Let $\ell^{(0)}$, $\ell^{(1)}\colon C([0,T]) \to \mathbb{R}$ be the linear functionals given by

$$\ell^{(0)}(h) \;=\; L\big(\, h(t)\,[1 - \Xi(t,x)]\,\big)\;, \quad \ell^{(1)}(h) \;=\; L\big(\, h(t)\, \Xi(t,x)\,\big)\;. \tag{4.6}$$

Note that the right-hand sides of the previous identities are well defined because Ξ belongs to $C^{0,1}(\Omega_T)$.

Note: The definition of ℓ^0, ℓ^1 explains why we defined L in $C^{0,1}(\Omega_T)$ and not in $C^{\infty,\infty}(\Omega_T)$. For $L\big(\, h(t)\,[1 - \Xi(t,x)]\,\big)$ to make sense, we need the map $(t,x) \mapsto h(t)\,[1 - \Xi(t,x)]$ to belong to the domain of definition of L.

Decompose a function $H\colon \Omega_T \to \mathbb{R}$ as $H = H^{(0)} + H^{(1)}$, where

$$H^{(1)}(t,x) \;=\; H(t,0) \;+\; [\, H(t,1) - H(t,0)\,]\; \Xi(t,x)\;. \tag{4.7}$$

Note that $H^{(0)}(t,0) = H^{(0)}(t,1) = 0$ for all $0 \le t \le T$. In particular, $H^{(0)}$ belongs to $C^{0,1}_0(\Omega_T)$ so that $L_0(H^{(0)})$ is well defined and $L_0(H^{(0)}) = L(H^{(0)})$.

By linearity and the previous paragraph, $L(H) = L_0(H^{(0)}) + L(H^{(1)})$. By definition of $H^{(1)}$, ℓ^0, ℓ^1, $L(H^{(1)}) = L(\, H(t,0)\,[\,1 - \Xi\,]\,) \,+\, L(\, H(t,1)\, \Xi\,) \;=\; \ell^0(H(\cdot,0)) \,+\, \ell^1(H(\cdot,1))$, Hence, for all H in $C^{0,1}(\Omega_T)$,

$$L(H) \;=\; L_0(H^{(0)}) \;+\; \ell^0(H(\cdot,0)) \;+\; \ell^1(H(\cdot,1))\;.$$

Lemma 4.2. *Let $L : C^{0,1}(\Omega_T) \to \mathbb{R}$ be a linear functional. Then,*

$$\sup_{H \in C^{0,1}(\Omega_T)} \big\{ L(H) \;-\; \mathfrak{W}(H)\,\big\} \;=\; S_1 \;+\; S_2\;, \tag{4.8}$$

where

$$S_1 \;=\; \sup_{G \in C^{0,1}_0(\Omega_T)} \Big\{\, L_0(G) \;-\; \int_0^T dt \int_0^1 \sigma(u_t)\, |\nabla G_t|^2\, dx \,\Big\}\;, \tag{4.9}$$

and

$$S_2 \;=\; \sup_{h,g\in C([0,T])} \Big\{\, \ell(g,h) \;-\; \int_0^T \zeta_t\, [\, h(t) - g(t)\,]^2\, dt \;-\; \int_0^T \Psi(t, g_t, h_t)\, dt \,\Big\} \;.$$

In this formula, $\zeta_t = 1/\langle 1/\sigma(u_t)\rangle$ *and* $\ell(g,h) = \ell^{(0)}(g) \,+\, \ell^{(1)}(h)$.

The first variational problem concerns the interior of Ω_T, while the second one the boundary of the cylinder Ω_T.

Proof of Lemma 4.2. Fix a linear functional $L : C^{0,1}(\Omega_T) \to \mathbb{R}$. Write $H = H^{(0)} + H^{(1)}$, as in (4.7). Since $H^{(0)}$ belongs to $C_0^{0,1}(\Omega_T)$, $L_0(H^{(0)})$ is well defined and $L_0(H^{(0)}) = L(H^{(0)})$.

By linearity, $L(H) = L_0(H^{(0)}) + L(H^{(1)})$. On the other hand, an elementary computation yields that $\nabla H^{(0)}$ and $\nabla H^{(1)}$ are orthogonal in $\mathcal{L}^2(\sigma(u))$:

$$\int_0^T dt \int_0^1 \sigma(u_t)\, \nabla H_t^{(0)}\, \nabla H_t^{(1)}\, dx \;=\; 0 \;.$$

Therefore, the supremum appearing in (4.8) can be written as

$$\sup_{H\in C^{0,1}(\Omega_T)} \Big\{ L_0(H^{(0)}) \;-\; \int_0^T dt \int_0^1 \sigma(u_t)\, |\nabla H_t^{(0)}|^2\, dx$$
$$+\; L(H^{(1)}) \;-\; \int_0^T \zeta_t\, [\, H(t,1) - H(t,0)\,]^2\, dt \;-\; \int_0^T \Psi(t, H_t(0), H_t(1))\, dt \Big\} \;.$$

The first line depends only on $H^{(0)}$, while the second one only on $H_t(0)$, $H_t(1)$. We may, therefore, split the supremum in two pieces. Recall the definition of the functionals ℓ to rewrite the previous supremum as

$$\sup_{G\in C_0^{0,1}(\Omega_T)} \Big\{\, L_0(G) \;-\; \int_0^T dt \int_0^1 \sigma(u_t)\, |\nabla G_t|^2\, dx \,\Big\}$$
$$+\; \sup_{h,g\in C([0,T])} \Big\{\, \ell(g,h) \;-\; \int_0^T \zeta_t\, [\, h(t) - g(t)\,]^2\, dt \;-\; \int_0^T \Psi(t, g_t, h_t)\, dt \,\Big\} \;,$$

as claimed. □

We apply Lemma 4.2 to the linear functionals appearing in the definition of the rate functional $I_{[0,T]}(\,\cdot\,)$. Denote $L^{(\partial_t)}$, $L^{(\nabla)} : C^{0,1}(\Omega_T) \to \mathbb{R}$ the linear functionals given by

$$L^{(\partial_t)}(G) \;=\; \int_0^T \langle\, \partial_t u_t, G_t \,\rangle\, dt \;, \quad L^{(\nabla)}(H) \;=\; \int_0^T \langle\, \nabla u_t, \nabla H_t \,\rangle\, dt \;, \tag{4.10}$$

and let $\mathfrak{L} = L^{(\partial_t)} + L^{(\nabla)}$. Denote by $\mathfrak{L}_0$, $\mathfrak{l}^0$, $\mathfrak{l}^1$ the linear functionals associated to $\mathfrak{L}$ by (4.4), (4.6), so that

$$\mathfrak{L}_0(G) \;=\; \int_0^T \langle\, \partial_t u_t, G_t \,\rangle\, dt \;+\; \int_0^T \langle\, \nabla u_t, \nabla G_t \,\rangle\, dt \;,$$
$$\mathfrak{l}^0(g) \;=\; \int_0^T a(t)\, g(t)\, dt \;, \quad \mathfrak{l}^1(g) \;=\; \int_0^T b(t)\; g(t)\, dt \;,$$

where

$$a(t) \;=\; \langle\, \partial_t u_t, [1 - \Xi_t]\,\rangle \;-\; \langle\, \nabla u_t, \nabla \Xi_t \,\rangle \;, \;\; b(t) \;=\; \langle\, \partial_t u_t \,,\, \Xi_t \,\rangle \;+\; \langle\, \nabla u_t, \nabla \Xi_t \,\rangle \;. \tag{4.11}$$

With this notation,

$$\mathfrak{L}(H) \;=\; \mathfrak{L}_0(H^{(0)}) \;+\; \mathfrak{l}^0(H(\cdot,0)) \;+\; \mathfrak{l}^1(H(\cdot,1)) \;. \tag{4.12}$$

Denote by $\Upsilon_t \colon \mathbb{R}^2 \to \mathbb{R}$, $0 \le t \le T$, the strictly convex map defined by

$$\Upsilon_t(x,y) \;=\; \zeta_t\,[\,x-y\,]^2 \;+\; \mathfrak{b}_{\alpha,A}(u_t(0),x) \;+\; \mathfrak{b}_{\beta,B}(u_t(1),y) \;,$$

and let $\Phi_t : \mathbb{R}^2 \to \mathbb{R}$, $t \ge 0$ be its Legendre transform:

$$\Phi_t(a,b) \;=\; \sup_{x,y\in\mathbb{R}} \big\{\, a\,x \;+\; b\,y \;-\; \Upsilon_t(x,y)\,\big\} \;. \tag{4.13}$$

Lemma 4.3. *Under the hypotheses* (4.2),

$$I_{[0,T]}(u) \;=\; I^{(1)}_{[0,T]}(u) \;+\; I^{(2)}_{[0,T]}(u) \;,$$

where

$$I^{(1)}_{[0,T]}(u) \;=\; \frac{1}{4}\,\|\,\mathfrak{L}_0\,\|^2_{-1,\sigma(u)} \;, \quad I^{(2)}_{[0,T]}(u) \;=\; \int_0^T \Phi_t(a_t,b_t)\; dt \;.$$

Proof: Recall the definition of the operators $\mathfrak{W}$ and $\mathfrak{L}$, introduced in (4.3) and (4.12), respectively. By definition of the rate functional $I_{[0,T]}$, given in (2.12),

$$I_{[0,T]}(u) \;=\; \sup_{H\in C^{0,1}(\Omega_T)} \big\{ \mathfrak{L}(H) \;-\; \mathfrak{W}(H)\,\big\} \;.$$

Hence, by Lemma 4.2, (4.1) and the definition of $\mathfrak{l}^0$, $\mathfrak{l}^1$, given above (4.11),

$$I_{[0,T]}(u) \;=\; I^{(1)}_{[0,T]}(u) \;+\; I^{(2)}_{[0,T]}(u) \;,$$

where

$$\begin{gathered} I^{(1)}_{[0,T]}(u) \;=\; \frac{1}{4}\,\|\,\mathfrak{L}_0\,\|^2_{-1,\sigma(u)} \;, \\ I^{(2)}_{[0,T]}(u) \;=\; \sup_{h,g\in C([0,T])} \Big\{\, \mathfrak{l}(g,h) \;-\; \int_0^T \Upsilon_t(g_t,h_t)\; dt\,\Big\} \end{gathered} \tag{4.14}$$

and $\mathfrak{l}(g,h) = \mathfrak{l}^0(g) + \mathfrak{l}^1(h)$. The second term can be written as

$$\begin{aligned} &\sup_{h,g\in C([0,T])} \int_0^T \big\{\, a(t)\,g(t) \;+\; b(t)\,h(t) \;-\; \Upsilon_t(g_t,h_t)\,\big\}\; dt \\ &\quad =\; \int_0^T \sup_{x,y\in\mathbb{R}} \big\{\, a(t)\,x \;+\; b(t)\,y \;-\; \Upsilon_t(x,y)\,\big\}\; dt \;=\; \int_0^T \Phi_t(a_t,b_t)\; dt \;. \end{aligned}$$

This completes the proof of the lemma. □

The function Φ_t is convex and continuous. Moreover, $\Phi_t(a,b) \ge 0$ [take $x=y=0$ in the supremum] and $\Phi_t(a,b) \le \Phi^0_t(a) + \Phi^1_t(b)$, where Φ^0_t, Φ^1_t are the Legendre transform of $\mathfrak{b}_{\alpha,A}(u_t(0),\cdot)$, $\mathfrak{b}_{\beta,B}(u_t(1),\,\cdot\,)$, respectively:

$$0 \;\le\; \Phi_t(a,b) \;\le\; \Phi^0_{u_t(0)}(a) \;+\; \Phi^1_{u_t(1)}(b) \;,$$

where

$$\Phi^0_u(a) \;=\; a\,\ln\Big\{ \frac{\sqrt{a^2 + 4\,\mathfrak{f}_{0,u}\,\mathfrak{g}_{0,u}} \;+\; a}{2\,\mathfrak{f}_{0,u}} \Big\} \;-\; \sqrt{a^2 + 4\,\mathfrak{f}_{0,u}\,\mathfrak{g}_{0,u}} \;+\; \mathfrak{f}_{0,u} \;+\; \mathfrak{g}_{0,u} \;,$$

$\mathfrak{f}_{0,u} = (1/A)\,[1-u]\,\alpha$, $\mathfrak{g}_{0,u} = (1/A)\,u\,[1-\alpha]$. The formula for Φ^1_u is similar. One just needs to replace A, α by B, β, respectively. In particular,

$$0 \;\le\; \Phi_t(a,b) \;\le\; C_0\,\{1 \,+\, |a|\,\ln^+|a| \,+\, |b|\,\ln^+|b|\,\}\;, \tag{4.15}$$

where $\ln^+ x = 0$ for $0 < x \le 1$ and $\ln^+ x = \ln x$ for $x \ge 1$.

Note: It might be disconcerting that $\Phi_t(0,0)$ is not equal to 0. This is a consequence of the fact that $\mathfrak{b}_{\beta,B}(a,\cdot)$ takes negative values. To remedy, one can add and subtract a linear term to $\mathfrak{b}_{\beta,B}(a,\cdot)$, transforming $\mathfrak{b}_{\beta,B}(a,\cdot)$ into $\mathfrak{q}_{\beta,B}(a,\cdot)$, given by (3.13). In constrast with $\mathfrak{b}_{\beta,B}(a,\cdot)$, $\mathfrak{q}_{\beta,B}(a,\cdot)$ is nonnegative and attains its minimum at 0.

After these modifications, Φ_t becomes

$$\Phi_t(a,b) \;=\; \widehat{\Phi}_t\Big(\,a \,+\, \frac{1}{A}\,[\alpha - u_t(0)]\,,\, b \,+\, \frac{1}{B}\,[\beta - u_t(1)]\,\Big)\;,$$

where

$$\widehat{\Phi}_t(a,b) \;=\; \sup_{x,y\in\mathbb{R}} \Big\{\, a\,x \,+\, b\,y \;-\; \zeta_t\,[\,x-y\,]^2 \,-\, \mathfrak{q}_{\alpha,A}(u_t(0),x) \,-\, \mathfrak{q}_{\beta,B}(u_t(1),y)\,\Big\}\;,$$

and $\widehat{\Phi}_t(a,b) \ge \widehat{\Phi}_t(0,0) = 0$.

Both functions Φ_t and $\widehat{\Phi}_t$ are convex and continuous. As Φ_t, $\widehat{\Phi}_t$ depend on the trajectory u, whenever we wish to stress this dependence, we represent $\Phi_t(a,b)$, $\widehat{\Phi}_t(a,b)$, by $\Phi^{(u)}_t(a,b)$, $\widehat{\Phi}^{(u)}_t(a,b)$, respectively.

Lemma 4.3 decomposes the rate function as the sum of two independent functionals. The first piece can still be simplified. This is the content of the next result. Under the hypotheses (4.2), $\|\, L^{(\nabla)}_0 \,\|^2_{-1,\sigma(u)} < \infty$. Since $\int_0^T \Phi_t(a_t,b_t)\,dt$ is finite as well, it follows from the previous lemma that

$$I_{[0,T]}(u) \;<\; \infty \;\text{ if, and only if, }\; \|\, L^{(\partial_t)}_0 \,\|^2_{-1,\sigma(u)} \;<\; \infty\;. \tag{4.16}$$

Suppose that $L^{(\partial_t)}_0$ belongs to $\mathcal{H}^{-1}(\sigma(u))$. By Lemma 4.1, there exists P in $\mathcal{L}^2(\sigma(u)^{-1})$ such that

$$L^{(\partial_t)}_0(H) \;=\; \int_0^T \langle\, P_t \,,\, \nabla H_t \,\rangle\, dt$$

for all H in $C^\infty_K(\Omega_T)$. This identity extends to $C^{0,1}_0(\Omega_T)$. Since H_t vanishes at the boundary $x=0$, $x=1$, the same identity holds if we replace P_t by $P_t - c_t$ for some function c in $\mathcal{L}^1([0,T])$. By choosing the right constant [that is $c_t = \langle P_t/\sigma(u_t)\rangle / \langle 1/\sigma(u_t)\rangle$], we may assume that $\langle P_t/\sigma(u_t)\rangle = 0$ for almost all $0 \le t \le T$. We denote by M the element of $\mathcal{L}^2(\sigma(u)^{-1})$ satisfying this condition and the previous displayed equation:

$$L^{(\partial_t)}_0(H) \;=\; \int_0^T \langle\, M_s \,,\, \nabla H_s \,\rangle\, ds\;, \quad \int_0^1 \frac{M_t}{\sigma(u_t)}\, dx \;=\; 0 \tag{4.17}$$

for all $H \in C^{0,1}_0(\Omega_T)$ and almost all $0 \le t \le T$. Moreover, as $\langle M_t/\sigma(u_t)\rangle = 0$ for almost all t, by Lemma 4.1,

$$\|\, L^{(\partial_t)}_0 \,\|^2_{-1,\sigma(u)} \;=\; \int_0^T dt \int_0^1 \frac{M_t^2}{\sigma(u_t)}\, dx\;.$$

Lemma 4.4. *Fix a trajectory satisfying the hypotheses* (4.2)*. Then,*

$$I^{(1)}_{[0,T]}(u) \;=\; \frac{1}{4}\int_0^T \big\{\,\|\, M_t \,+\, \nabla u_t \,\|^2_{\sigma(u_t)^{-1}} \,-\, R_t\,\big\}\, dt\;,$$

where

$$R_t \;=\; \Big\langle \frac{\nabla u_t}{\sigma(u_t)} \Big\rangle^2 \frac{1}{\langle \sigma(u_t)^{-1}\rangle} \;=\; \Big\{ \log \frac{u_t(1)}{1-u_t(1)} \;-\; \log \frac{u_t(0)}{1-u_t(0)} \Big\}^2 \frac{1}{\langle \sigma(u_t)^{-1}\rangle} \;.$$

Proof: As $I_{[0,T]}(u)$ is finite, by (4.16) and the paragraph preceding the statement of the lemma, $L_0^{(\partial_t)}$ belongs to $\mathcal{H}^{-1}(\sigma(u))$ and there exists M in $\mathcal{L}^2(\sigma(u)^{-1})$ satisfying (4.17). Therefore, for all H in $C_0^{0,1}(\Omega_T)$,

$$\mathfrak{L}_0(H) \;=\; L_0^{(\nabla)}(H) \;+\; L_0^{(\partial_t)}(H) \;=\; \int_0^T \langle\, M_t \;+\; \nabla u_t\,,\, \nabla H_t \,\rangle\, dt\;.$$

By Lemma 4.1,

$$\|\, \mathfrak{L}_0 \,\|^2_{-1,\sigma(u)} \;=\; \int_0^T \Big\{ \,\|\, M_t \;+\; \nabla u_t \,\|^2_{\sigma(u_t)^{-1}} \;-\; R_t \,\Big\}\, dt\;,$$

where R_t has been introduced in the statement of the lemma. This completes the proof of the lemma. □

We summarize the last two results in the next proposition.

Proposition 4.5. *Fix a path $\pi(t,dx) = u(t,x)\,dx$ in $D([0,T],\mathcal{M}_{\rm ac})$. Assume that u is continuous on Ω_T and smooth in time, that there exists $\varepsilon > 0$ such that $\varepsilon \le u(t,x) \le 1-\varepsilon$ for all $(t,x)\in\Omega_T$, and that $I_{[0,T]}(u) < \infty$. Then,*

$$I_{[0,T]}(u) \;=\; I^{(1)}_{[0,T]}(u) \;+\; I^{(2)}_{[0,T]}(u)\;,$$

where Lemma 4.4 provides a formula for first term and Lemma 4.3 for the second.

Remark 4.6. In the statement of Proposition 4.5, we imposed many regularity assumptions on u because this is the context in which this result is applied in the next section. The proof shows that they can be relaxed.

5. $I_{[0,T]}(\,\cdot\,)$-density

In this section, we prove that any trajectory $\pi\in D([0,T],\mathcal{M})$ with finite rate function can be approximated by a sequence of smooth trajectories $\{\pi^n : n\ge 1\}$ such that

$$\pi^n \longrightarrow \pi \quad\text{and}\quad I_{[0,T]}(\pi^n|\gamma) \longrightarrow I_{[0,T]}(\pi|\gamma)\;.$$

We follow an approach proposed in Quastel et al. (1999); Bertini et al. (2009b); Farfan et al. (2011). Here, and throughout this section, $\gamma\colon [0,1]\to[0,1]$ is a fixed density profile. We first introduce some terminology.

Definition 5.1. A subset A of $D([0,T],\mathcal{M})$ is said to be $I_{[0,T]}(\cdot|\gamma)$-dense if for any π in $D([0,T],\mathcal{M})$ such that $I_{[0,T]}(\pi|\gamma)<\infty$, there exists a sequence $\{\pi^n : n\ge 1\}$ in A such that π^n converges to π in $D([0,T],\mathcal{M})$ and $I_{[0,T]}(\pi^n|\gamma)$ converges to $I_{[0,T]}(\pi|\gamma)$.

Theorem 5.2. *For all $\gamma : [0,1]\to[0,1]$, the set Π_γ is $I_{[0,T]}(\cdot|\gamma)$-dense. If there exists $\varepsilon_0>0$ such that $\varepsilon_0\le\gamma\le 1-\varepsilon_0$, condition (b) in Definition 2.5 can be replaced by the existence of $\varepsilon>0$ such that $\varepsilon\le u(t,x)\le 1-\varepsilon$ for all $(t,x)\in[0,T]\times[0,1]$.*

The proof of Theorem 5.2 is divided into several steps. Throughout this section, denote by $u^{(\gamma)} : [0,T] \times [0,1] \to [0,1]$ the unique weak solution of the boundary-initial valued problem (2.6) with initial profile $u_0 = \gamma$.

Let $\mathfrak{R}_1$ be the set of all paths $\pi(t,dx) = u(t,x)\,dx$ in $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$, whose density u is a weak solution of the Cauchy problem (2.6) in some positive time interval. In other words, there exists $\delta > 0$ such that $u_t = u_t^{(\gamma)}$ for $0 \le t \le \delta$.

Lemma 5.3. *The set* $\mathfrak{R}_1$ *is* $I_{[0,T]}(\cdot|\gamma)$*-dense.*

Proof: Fix π in $D([0,T],\mathcal{M})$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. By definition of the rate function, π belongs to $D([0,T],\mathcal{M}_{\rm ac})$, $\pi(t,dx) = u(t,x)\,dx$, and $\mathcal{Q}_{[0,T]}(u) < \infty$.

For each $\delta > 0$, consider the path $\pi^\delta(t,dx) = u^\delta(t,x)\,dx$ defined by

$$u^\delta(t,x) \;=\; \begin{cases} u^{(\gamma)}(t,x) & \text{if } t \in [0,\delta]\;, \\ u^{(\gamma)}(2\delta - t,x) & \text{if } t \in [\delta,2\delta]\;, \\ u(t-2\delta,x) & \text{if } t \in [2\delta,T]\;. \end{cases}$$

Claim A: The trajectory π^δ belongs to $\mathfrak{R}_1$. Indeed, by definition, u^δ is the weak solution of the Cauchy problem (2.6) in the time-interval $[0,\delta]$. On the other hand, by definition of u^δ, $\mathcal{Q}_{[0,T]}(u^\delta) \le 2\,\mathcal{Q}_{[0,\delta]}(u^{(\gamma)}) + \mathcal{Q}_{[0,T]}(u)$. By Corollary 3.6, $\mathcal{Q}_{[0,\delta]}(u^{(\gamma)}) < \infty$. On the other hand, $\mathcal{Q}_{[0,T]}(u)$ is finite because $I_{[0,T]}(\pi|\gamma) < \infty$. Therefore, u^δ has finite energy, which completes the proof of Claim A.

It is clear that π^δ converges to π in $D([0,T],\mathcal{M})$ as $\delta \downarrow 0$. To conclude the proof of the lemma it is enough to show that $I_{[0,T]}(\pi^\delta|\gamma)$ converges to $I_{[0,T]}(\pi|\gamma)$ as $\delta \downarrow 0$.

Since the rate function is lower semicontinuous, $I_{[0,T]}(\pi|\gamma) \le \liminf_{\delta\to 0} I_{[0,T]}(\pi^\delta|\gamma)$. To prove that $\limsup_{\delta\to 0} I_{[0,T]}(\pi^\delta|\gamma) \le I_{[0,T]}(\pi|\gamma)$, decompose the rate function $I_{[0,T]}(\pi^\delta|\gamma)$ into the sum of the contributions on each time interval $[0,\delta]$, $[\delta,2\delta]$ and $[2\delta,T]$.

Recall the notation introduced at the beginning of Section 3. By (3.1), (3.2), since in the interval $[2\delta,T]$ π^δ is a time translation of the path π,

$$I_{[0,T]}(\pi^\delta|\gamma) \;\le\; I_{[0,\delta]}(\pi^\delta|\gamma) \;+\; I_{[0,\delta]}(\tau_\delta u^\delta\,|\,u_\delta^{(\gamma)}) \;+\; I_{[0,T]}(\pi|\gamma)\;.$$

Since the density u^δ is a weak solution of the equation (2.6) on the interval $[0,\delta]$, by Corollary 3.6, the first contribution is equal to 0. It remains to show that the second term on the right-hand side of the last display vanishes as $\delta \to 0$.

Let $v^\delta = \tau_\delta u^\delta$. As $v^\delta(t) = u^{(\gamma)}(\delta - t)$, the density v^δ solves the backward heat equation: $\partial_t v^\delta = -\Delta v^\delta$. Thus, by Definition B.1 and (2.11), for each H in $C^{1,2}([0,T]\times[0,1])$,

$$\begin{aligned} J_{\delta,H}(v^\delta) \;&=\; \int_0^\delta \big\{ 2\,\langle \nabla u_t^{(\gamma)}\,,\,\nabla H_t\,\rangle \;-\; \langle\,\sigma(u_t^{(\gamma)})\,,\,(\nabla H_t)^2\,\rangle \big\}\,dt \\ &\quad -\; \int_0^T \Big\{\, \widehat{\mathfrak{b}}_{\alpha,A}\big(\,u_t^{(\gamma)}(0)\,,\,H_t(0)\,\big) \;+\; \widehat{\mathfrak{b}}_{\beta,B}\big(\,u_t^{(\gamma)}(1)\,,\,H_t(1)\,\big) \Big\}\,dt\;, \end{aligned}$$

where

$$\widehat{\mathfrak{b}}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{ [1-a]\,\varrho\,[e^M - 1 + M] \;+\; a\,[1-\varrho]\,[e^{-M} - 1 - M] \Big\}\;.$$

By Schwarz inequality, the first integral on the right-hand side is bounded above by

$$\int_0^\delta dt\;\int_0^1 \frac{|\nabla u^{(\gamma)}(t,x)|^2}{\sigma(u^{(\gamma)}(t,x))}\,dx\;.$$

By (B.6), this expression vanishes as $\delta \to 0$. On the other hand, maximizing $\widehat{\mathfrak{b}}_{\varrho,D}(a, M)$ over M yields that the second integral is bounded above by

$$\int_0^\delta \Big\{ \frac{1}{B}\,[\,\beta - u_t^{(\gamma)}(1)\,]\,\log\frac{[1-u_t^{(\gamma)}(1)]\,\beta}{u_t^{(\gamma)}(1)\,[1-\beta]} \;+\; \frac{1}{A}\,[\,\alpha - u_t^{(\gamma)}(0)\,]\,\log\frac{[1-u_t^{(\gamma)}(0)]\,\alpha}{u_t^{(\gamma)}(0)\,[1-\alpha]} \Big\}\,dt$$
$$\le\; \int_0^\delta \Big|\, \frac{\beta - u_t^{(\gamma)}(1)}{B}\,\log\frac{[1-u_t^{(\gamma)}(1)]}{u_t^{(\gamma)}(1)} \;+\; \frac{\alpha - u_t^{(\gamma)}(0)}{A}\,\log\frac{[1-u_t^{(\gamma)}(0)]}{u_t^{(\gamma)}(0)}\,\Big|\,dt \;+\; C_0\delta$$

for some finite constant $C_0 = C_0(\alpha, \beta, A, B)$. By (B.6), this expression vanishes as $\delta \to 0$.

Putting together the previous estimates shows that there exists a function $c(\delta)$, independent of H, such that $\lim_{\delta\to 0} c(\delta) = 0$ and

$$J_{\delta,H}(\tau_\delta u^\delta) \;\le\; c(\delta)$$

for all $H \in C^{1,2}([0,T]\times[0,1])$. This shows that $\lim_{\delta\to 0} I_{[0,\delta]}(\tau_\delta u^\delta \,|\, u_\delta^{(\gamma)}) = 0$, and completes the proof of the lemma. □

Let $\mathfrak{R}_2$ be the set of all paths $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_1$ with the property that for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\varepsilon \le u(t,x) \le 1-\varepsilon$ for all $(t,x) \in [\delta, T]\times[0,1]$.

Lemma 5.4. *The set* $\mathfrak{R}_2$ *is* $I_T(\,\cdot\,|\gamma)$*-dense.*

Proof: Fix $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_1$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. For each $0 < \varepsilon < 1$, define the path $\pi^\varepsilon(t,dx) = u^\varepsilon(t,x)dx$ by $u^\varepsilon = (1-\varepsilon)u + \varepsilon u^{(\gamma)}$.

Claim A: For each $0 < \varepsilon < 1$, the trajectory π^ε belongs to $\mathfrak{R}_1$. Since π belongs to $\mathfrak{R}_1$, by definition, there exists $\delta > 0$ such that $\pi^\varepsilon_t = \pi_t$ for $0 \le t \le \delta$. Therefore, π^ε follows the hydrodynamic equation in the time-interval $[0,\delta]$. On the other hand, by the convexity of the energy, $\mathcal{Q}_{[0,T]}(u^\varepsilon) \le \varepsilon\,\mathcal{Q}_{[0,T]}(u^{(\gamma)}) + (1-\varepsilon)\,\mathcal{Q}_{[0,T]}(u)$. Hence, by lemma B.5, $\mathcal{Q}_{[0,T]}(u^\varepsilon) < \infty$. Therefore, π^ε belongs to $\mathfrak{R}_1$, as claimed.

Claim B: For each $0 < \varepsilon < 1$, the trajectory π^ε belongs to $\mathfrak{R}_2$. By Theorem B.4, for every $\delta > 0$ there exists $\kappa > 0$ such that $\kappa \le u_t^{(\gamma)} \le 1-\kappa$ for all $\delta \le t \le T$. This property is inherited by u^ε for a different $\kappa = \kappa(\varepsilon)$ because $0 \le u \le 1$, which proves Claim B.

It is clear that π^ε converges to π in $D([0,T],\mathcal{M})$ as $\varepsilon \downarrow 0$. Therefore, to conclude the proof it is enough to show that $I_{[0,T]}(\pi^\varepsilon|\gamma)$ converges to $I_{[0,T]}(\pi|\gamma)$ as $\varepsilon \downarrow 0$. Since the rate function is lower semicontinuous, $I_{[0,T]}(\pi|\gamma) \le \liminf_{\varepsilon\downarrow 0} I_{[0,T]}(\pi^\varepsilon|\gamma)$. On the other hand, as the rate function $I_{[0,T]}(\,\cdot\,|\,\gamma)$ is convex, by Corollary 3.6,

$$I_{[0,T]}(\pi^\varepsilon|\gamma) \;\le\; (1-\varepsilon)\,I_{[0,T]}(\pi|\gamma) \;+\; \varepsilon\,I_{[0,T]}(u^{(\gamma)}|\gamma) \;\le\; (1-\varepsilon)\,I_{[0,T]}(\pi|\gamma)\,.$$

This completes the proof of the lemma. □

Let $\mathfrak{R}_3$ be the set of all paths $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_2$ whose density u is continuous in $(0,T]\times[0,1]$ and smooth in time: for all $x \in [0,1]$, $u(x,\cdot\,)$ belongs to $C^\infty((0,T])$.

Lemma 5.5. *The set* $\mathfrak{R}_3$ *is* $I_{[0,T]}(\cdot|\gamma)$*-dense.*

Proof: Fix $\pi(t,dx) = u(t,x)dx$ in $\mathfrak{R}_2$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. Since π belongs to the set $\mathfrak{R}_1$, the density u solves the equation (2.6) in a time interval $[0,3\delta]$ for some $\delta > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth, nonnegative function such that

$$\operatorname{supp}\varphi \;\subset\; (0,1) \quad\text{and}\quad \int_0^1 \varphi(s)\,ds \;=\; 1\,.$$

Set $\varphi_\varepsilon(s) = \varepsilon^{-1}\varphi(s/\varepsilon)$.

Let $\chi : [0,T] \to [0,1]$ be a smooth, nondecreasing function such that

$$\begin{cases} \chi(t) \ = \ 0 & \text{if } \ t \in [0,\delta]\ , \\ 0 \ < \chi(t) \ < 1 & \text{if } \ t \in (\delta, 2\delta)\ , \\ \chi(t) \ = \ 1 & \text{if } \ t \in [2\delta, T]\ , \end{cases} \tag{5.1}$$

and set $\chi_n(t) = \chi(t)/n$ for $n \geq 1$. Hence, $\chi_n(t) = 1/n$ for $t \geq 2\delta$.

Let $\pi^n(t, dx) = u^n(t,x)\, dx$ where

$$u^n(t,x) \ = \ \int_0^1 u(t + \chi_n(t)\, s)\, \varphi(s)\, ds \ = \ \int_{\mathbb{R}} u(t+s)\, \varphi_{\chi_n(t)}(s)\, ds\ .$$

In the above formula, we extend the definition of u to $[0, T+1]$ by setting $u_t = u^{(u_T)}_{t-T}$ for $T \leq t \leq T+1$. This means that on the interval $[T, T+1]$, u_t follows the hydrodynamic equation (2.6) starting from the initial condition u_T. [If w represents the solution of equation (2.6) with $\gamma = u_T$, $u_{T+t} = w_t$ for $0 \leq t \leq 1$].

Claim A: The trajectory π^n belongs to $\mathfrak{R}_1$ for all $n > \delta^{-1}$.

Fix such $n \in \mathbb{N}$. By construction, the density u^n coincides with the solution $u^{(\gamma)}$ of the hydrodynamic in the time-interval $[0,\delta]$. To estimate the energy of u^n, we consider the time-intervals $[0,\delta]$, $[\delta, 2\delta]$ and $[2\delta, T]$ separately. On $[0,\delta]$, u^n coincides with $u^{(\gamma)}$. Therefore, by Lemma B.5, the energy of u^n in this interval is bounded (uniformly in n). In the interval $[\delta, 2\delta]$, u^n_t is a convex combination of u_{t+s} for $0 \leq s \leq 1/n \leq \delta$. Since u coincides with $u^{(\gamma)}$ in the interval $[\delta, 3\delta]$, and since the solution is smooth in this interval and bounded away from 0 and 1, the energy of u^n in this interval is bounded (uniformly in n). Finally, for $2\delta \leq t \leq T$,

$$u^n(t,x) \ = \ \int_0^{1/n} u(t+s)\, \varphi_{1/n}(s)\, ds\ .$$

By convexity of the energy,

$$Q_{[2\delta,T]}(\pi^n) \ \leq \ \int_0^{1/n} Q_{[2\delta,T]}(\tau_s \pi)\, \varphi_{1/n}(s)\, ds \ \leq \ Q_{[2\delta, T+1/n]}(\pi)\ ,$$

where the translation τ_s has been introduced in (3.1). This quantity is finite because u has finite energy and by Lemma B.5. This proves Claim A.

Claim B: The trajectory π^n belongs to $\mathfrak{R}_3$ for all $n > \delta^{-1}$.

As π belongs to $\mathfrak{R}_2$, by construction, so does π^n. By Definition B.3 and Theorem B.2, the function u is smooth in the set $(0, 3\delta) \times [0,1]$. Therefore, by definition, the function u^n is smooth in time on $(0,T] \times [0,1]$. As $n > \delta^{-1}$, and since $u = u^{(\gamma)}$ is continuous in $(0,3\delta) \times [0,1]$, by definition, u^n is continuous in $(0, 2\delta) \times [0,1]$. We turn to the set $[2\delta, T] \times [0,1]$. By convexity, for all $2\delta \leq t \leq T$,

$$\begin{aligned} \int_0^1 (\nabla u^n_t)^2\, dx \ &\leq \ \int_0^{1/n} ds\, \varphi_{1/n}(s) \int_0^1 [\, \nabla u_{t+s}\,]^2\, dx \\ &\leq \ C_n \int_t^{t+(1/n)} ds \int_0^1 [\, \nabla u_s\,]^2\, dx \leq \ C_n \int_0^{T+1} ds \int_0^1 [\, \nabla u_s\,]^2\, dx \end{aligned}$$

for some finite constant C_n. The last integral is finite for two reasons. By Lemma B.5, the integral restricted to $[T, T+1]$ is finite. The integral on $[0,T]$ is finite because π has finite energy as all elements of $\mathfrak{R}_2$. It follows from this bound and from its definition that u^n_t is continuous on $[2\delta, T] \times [0,1]$, which proves Claim B.

It is clear that π^n converges to π in $D([0,T], \mathcal{M})$. It remains to show that $I_{[0,T]}(\, u^n \,|\gamma) \to I_{[0,T]}(\, u \,|\gamma)$. As the rate-function $I_{[0,T]}(\,\cdot\,|\gamma)$ is lower semicontinuous, we turn to the bound $\limsup_{n\to\infty} I_{[0,T]}(\pi^n|\gamma) \le I_{[0,T]}(\pi|\gamma)$.

By (3.2), the cost of the trajectory π^n in the interval $[0,T]$ is bounded by the sum of its cost in the intervals $[0,\delta]$, $[\delta, 2\delta]$, $[2\delta, T]$. As $u^n = u$ in the time-interval $[0,\delta]$, and as u is the solution of the hydrodynamic equation in this interval,

$$I_{[0,\delta]}(\pi^n|\gamma) \;=\; 0\;. \tag{5.2}$$

Consider the contribution to $I_{[0,T]}(\pi^n|\gamma)$ of the piece of the trajectory corresponding to the time interval $[2\delta, T]$. Recall the definition of the functional τ_t, introduced just above (3.1). Since $\chi_n(t) = 1/n$ in this interval, by the concavity of $\sigma(\cdot)$, for any smooth function $H : [0, T-2\delta] \times [0,1] \to \mathbb{R}$,

$$\begin{aligned} J_{T-2\delta,H}(\,\tau_{2\delta} u^n\,) \;&\le\; \int \varphi_{1/n}(s)\, J_{T-2\delta,H}(\,\tau_{2\delta+s} u\,)\; ds \\ &\le\; \int \varphi_{1/n}(s)\, I_{[0,T-2\delta]}(\,\tau_{2\delta+s} u\,)\; ds\;. \end{aligned}$$

By (3.1), the right-hand side is bounded by

$$\int \varphi_{1/n}(s)\, \big\{\, I_{[0,T-2\delta-s]}(\,\tau_{2\delta+s} u\,) \;+\; I_{[0,s]}(\,\tau_T u\,)\,\big\}\; ds\;.$$

Since u solves the hydrodynamic equation on the interval $[T, T+1]$, by Corollary 3.6, $I_{[0,s]}(\,\tau_T u\,) = 0$ for $s \le 1$. Hence, by (3.3), the previous integral is bounded by

$$\int \varphi_{1/n}(s)\, I_{[0,T]}(\,u\,)\; ds \;\le\; I_{[0,T]}(\,u\,)\;.$$

Therefore, optimizing over H,

$$I_{[0,T-2\delta]}(\,\tau_{2\delta}\pi^n\,) \;\le\; I_{[0,T]}(\,u\,)\;. \tag{5.3}$$

We turn to the contribution to $I_{[0,T]}(\pi^n|\gamma)$ of the piece of the trajectory corresponding to the time interval $[\delta, 2\delta]$. Since u solves the hydrodynamic equation (2.6) on the time interval $[\delta, 3\delta]$, it is smooth in $(0,3\delta) \times [0,1]$. Hence, by definition of u^n,

$$\partial_t u^n(t,x) \;=\; \int_{\mathbb{R}} \partial_t u(t+s,x)\, \varphi_{\chi_n(t)}(s)\; ds \;+\; \int_{\mathbb{R}} u(t+s,x)\, \partial_t \varphi_{\chi_n(t)}(s)\; ds\;.$$

As u solves the hydrodynamic equation (2.6) on the time interval $[\delta, 3\delta]$, for any function G in $C^{1,2}([0,T]\times[0,1])$,

$$\begin{aligned} \langle u^n_{2\delta}, G_{2\delta}\rangle \;-\; \langle u^n_\delta, G_\delta\rangle \;-\; \int_\delta^{2\delta} \langle u^n_t, \partial_t G_t\rangle\; dt \;&=\; -\int_\delta^{2\delta} \langle \nabla u^n_t, \nabla G_t\rangle\; dt \\ +\; \int_\delta^{2\delta} \Big\{\, \frac{1}{B}\,[\beta - u^n_t(1)]\, G_t(1) \;+\; \frac{1}{A}\,[\alpha - u^n_t(0)]\, G_t(0)\,\Big\}\; dt \;&+\; \int_\delta^{2\delta} \langle r^n_t, G_t\rangle\; dt\;, \end{aligned}$$

where

$$r^n_t(x) \;=\; \int_{\mathbb{R}} u(t+s,x)\, \partial_t \varphi_{\chi_n(t)}(s)\; ds\;.$$

Therefore,

$$J_{\delta,G}(\tau_\delta u^n) \;\le\; \int_\delta^{2\delta} \langle r_t^n, G_t\rangle \, dt \;-\; \int_\delta^{2\delta} dt \int_0^1 \sigma(u_t^n)\, [\nabla G_t]^2 \, dx$$
$$-\; \int_\delta^{2\delta} \big\{\, \mathfrak{q}_{\beta,B}(u_t^n(1), G_t(1)) \;+\; \mathfrak{q}_{\alpha,A}(u_t^n(0), G_t(0)) \,\big\} \, dt \;,$$

where $\mathfrak{q}_{\varrho,D}(a,M)$ has been introduced in (3.13). Since u belongs to $\mathfrak{R}_2$, there exists $\varepsilon > 0$ such that $\varepsilon \le u(t,x) \le 1-\varepsilon$ for all $\delta \le t \le T$, $0 \le x \le 1$. By Theorem B.4, this bound extends to $T \le t \le T+1$, $0 \le x \le 1$. By definition, it is inherited by u^n. Therefore, there exists a positive constant $c_0 = c_0(\varepsilon)$ such that

$$J_{\delta,G}(\tau_\delta u^n) \;\le\; \int_\delta^{2\delta} \langle r_t^n, G_t\rangle \, dt \;-\; c_0 \int_\delta^{2\delta} dt \int_0^1 [\nabla G_t]^2 \, dx$$
$$-\; c_0 \int_\delta^{2\delta} \big\{\, G_t(1)^2 \;+\; G_t(0)^2 \,\big\} \, dt \;,$$

Adding and subtracting $G_t(0)$ to G_t in $\langle r_t^n, G_t\rangle$ yields, by Young's inequality, that this scalar product is bounded by $(1/2A_1)\langle (r_t^n)^2\rangle + A_1\langle [G_t - G_t(0)]^2\rangle + A_1 G_t(0)^2$ for all $A_1 > 0$. Hence, by choosing A_1 appropriately,

$$J_{\delta,G}(\tau_\delta u^n) \;\le\; C_0 \int_\delta^{2\delta} dt \int_0^1 (r_t^n)^2 \, dx \;,$$

so that

$$I_{[0,\delta]}(\tau_\delta u^n) \;\le\; C_0 \int_\delta^{2\delta} dt \int_0^1 (r_t^n)^2 \, dx \;, \tag{5.4}$$

It remains to show that $r^n(t,x)$ converges to 0, as $n \to \infty$, in $\mathcal{L}^2((\delta,2\delta)\times[0,1])$. Fix a point (t,x) in this set. Since $\int_{\mathbb{R}} \partial_t\, \varphi_{\chi_n(t)}(s)\, ds \;=\; \partial_t \int_{\mathbb{R}} \varphi_{\chi_n(t)}(s)\, ds \;=\; 0$, $r^n(t,x)$ can be written as

$$\int_{\mathbb{R}} [\, u(t+s,x) - u(t,x) \,]\, \partial_t\, \varphi_{\chi_n(t)}(s)\, ds \;.$$

Since u is Lipschitz continuous on $[\delta, 3\delta]\times[0,1]$, there exists a positive constant $C(\delta) > 0$, depending only on δ, such that

$$|\, u(t+s,x) - u(t,x) \,| \;\le\; C(\delta)\, s \;,$$

for any $(t,x) \in [\delta,2\delta]\times[0,1]$ and $s \in [0,\delta]$. Therefore $r^n(t,x)$ is bounded above by

$$C(\delta) \int_{\mathbb{R}} s \,\big|\, \partial_t\, \varphi_{\chi_n(t)}(s) \,\big|\, ds \;.$$

By the change of variables $s' = s/\chi_n(t)$,

$$\int_{\mathbb{R}} s \,\big|\, \partial_t\, \varphi_{\chi_n(t)}(s) \,\big|\, ds \;\le\; \frac{\|\, \chi' \,\|_\infty}{n} \int_0^1 \big\{\, s\, \varphi(s) + s^2\, |\varphi'(s)| \,\big\}\, ds \;.$$

Therefore, as $n \to \infty$, r^n converges to 0 uniformly in $(\delta,2\delta)\times[0,1]$, and, by (5.4),

$$\lim_{n\to\infty} I_{[0,\delta]}(\, \tau_\delta \pi^n \,) \;=\; 0 \;.$$

By (3.1), (5.2), (5.3) and the previous estimate, $\limsup_{n\to\infty} I_{[0,T]}(\, \pi^n \,|\, \gamma \,) \;\le\; I_{[0,T]}(\, \pi \,|\, \gamma \,)$, which completes the proof of the lemma. □

Let $\mathfrak{R}_4$ be the set of all paths $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_3$ whose density $u(t,\cdot)$ belongs to the space $C^\infty([0,1])$ for any $t \in (0,T]$. Note that $\mathfrak{R}_4 = \Pi_\gamma$, introduced in Definition 2.5.

Denote by $(P^{(D)}_t : t \ge 0)$, $(P^{(N)}_t : t \ge 0)$ the semigroup associated to the Laplacian on $[0,1]$ with Dirichlet, Neumann boundary conditions, respectively. The following property will be used many times below. For all $s \ge 0$ and function f in $C^1([0,1])$,

$$\nabla P^{(D)}_s f \;=\; P^{(N)}_s \nabla f \;. \tag{5.5}$$

To check this identity, fix f in $C^1([0,1])$, and let $u_s := P^{(D)}_s f$. Clearly u_s is the solution of the heat equation on $[0,1]$ with boundary conditions $u_s(0) = u_s(1) = 0$ and initial condition $u_0 = f$. Let $v_s := \nabla u_s$, Then, v_s solves the heat equation on $[0,1]$ with boundary conditions $\nabla v_s(0) = \nabla v_s(1) = 0$ and initial condition $v_0 = \nabla f$. Hence, v_s can be represented as $v_s = P^{(N)}_s \nabla f$, that is, $P^{(N)}_s \nabla f = v_s = \nabla u_s = \nabla P^{(D)}_s f$, as claimed.

Fix $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_3$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. Since π belongs to the set $\mathfrak{R}_1$, the density u solves the equation (2.6) in some time interval $[0,3\delta]$, $\delta > 0$. Recall the definition of the function $\chi_n(\cdot)$ introduced in (5.1). Let $\pi^n(t,dx) = u^n(t,x)\,dx$, where

$$u^n_t \;=\; w_t \;+\; P^{(D)}_{\chi_n(t)}[\, u_t - w_t\,]\;. \tag{5.6}$$

In this formula, $w_t(\cdot)$ is the smooth function given by $w_t(x) \,=\, u_t(0) \,+\, [\, u_t(1) \,-\, u_t(0)\,]\, x$.

Lemma 5.6. *Fix $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_3$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. Define u^n, $n \ge 1$, by (5.6). For each $n \ge 1$, $\pi^n(t,dx) = u^n(t,x)\,dx$ belongs to $\mathfrak{R}_4$ and the trajectory u^n has finite energy.*

Proof: *Claim A:* The trajectory π^n belongs to $\mathfrak{R}_1$.

By definition, $u^n_t = u_t = u^{(\gamma)}_t$ for $0 \le t \le \delta$. It remains to estimate its energy. As $u^n_t = u^{(\gamma)}_t$ for $0 \le t \le \delta$, by Lemma B.5, the contribution to the total energy of the evolution of u^n in the time interval $[0,\delta]$ is bounded. We turn to the contribution in the time interval $[\delta,T]$.

By definition and (5.5), $\nabla u^n_t \,=\, \nabla w_t \,+\, P^{(N)}_{\chi_n(t)} \nabla[\, u_t - w_t\,]$, so that $(\nabla u^n_t)^2 \le 2\,(\nabla w_t)^2 + 2\,\{\, P^{(N)}_{\chi_n(t)} \nabla[\, u_t - w_t\,]\,\}^2$. Therefore, as $\varepsilon(\delta) \le u^n_t \le 1 - \varepsilon(\delta)$ for $\delta \le t \le T$,

$$\begin{aligned}
\int_\delta^T dt \int_0^1 \frac{|\nabla u^n_t|^2}{\sigma(u^n_t)}\, dx \;&\le\; C_0(\varepsilon) \int_\delta^T dt \int_0^1 |\nabla u^n_t|^2\, dx \\
&\le\; C_0(\varepsilon) \int_\delta^T dt \int_0^1 (\nabla w_t)^2\, dx \;+\; C_0(\varepsilon) \int_\delta^T dt \int_0^1 \{\, P^{(N)}_{\chi_n(t)} \nabla[\, u_t - w_t\,]\,\}^2\, dx\;,
\end{aligned}$$

where the constant $C_0(\varepsilon)$ changed from line to line. The first term is bounded by the definition of w_t. As $P^{(N)}_s$ is a contraction in $\mathcal{L}^2([0,1])$, the second term is bounded by

$$C_0(\varepsilon) \int_\delta^T dt \int_0^1 (\nabla u_t)^2\, dx \;+\; C_0(\varepsilon) \int_\delta^T dt \int_0^1 (\nabla w_t)^2\, dx\;.$$

The first term is bounded because $\pi_t(dx) = u(t,x)\,dx$ belongs to $\mathfrak{R}_3$. We already estimated the second one. This completes the proof of Claim A.

Claim B: The trajectory π^n belongs to $\mathfrak{R}_2$. By Theorem B.4, and since π belongs to $\mathfrak{R}_2$, for every $\delta' > 0$, there exists $\varepsilon > 0$ such that $\varepsilon \le u_t \le 1 - \varepsilon$ for all $t \in [\delta',T]$. Denote by $\varepsilon(\delta)$ the constant ε when $\delta' = \delta$. As $u^n_t = u_t$ for $0 \le t \le \delta$, this property extends to u^n_t in the interval $[0,\delta]$: for every $0 < \delta' \le \delta$, there exists $\varepsilon > 0$ such that $\varepsilon \le u^n_t \le 1 - \varepsilon$ for all $t \in [\delta',\delta]$.

We turn to the interval $[\delta, T]$. Fix $\delta \le t \le T$. Let $v_s = v_s^{(t)} = w_t + P_s^{(D)}[u_t - w_t]$, $s \ge 0$. Note that $u_t^n = v^{(t)}_{\chi_n(t)}$. By definition, v is the unique solution of the heat equation with Dirichlet boundary conditions:

$$\begin{cases} \partial_s v \;=\; \Delta v \;, \\ v_s(0) \;=\; u_t(0) \;, \;\; v_s(1) \;=\; u_t(1) \\ v(0,\cdot) = u_t(\cdot) \;. \end{cases}$$

Here we used the fact that $w(t,0) = u(t,0)$, $w(t,1) = u(t,1)$ and that $\Delta w_t = 0$. By the maximum principle, for all $s \ge 0$, $\min_{0 \le x \le 1} u_t(x) \le \min_{0 \le x \le 1} v_s(x) \le \max_{0 \le x \le 1} v_s(x) \le \max_{0 \le x \le 1} u_t(x)$. Hence, the bound $\varepsilon(\delta) \le u_t \le 1 - \varepsilon(\delta)$, which holds for all $t \in [\delta, T]$ by definition of $\varepsilon(\delta)$, extends to $v^{(t)}_{\chi_n(t)} = u_t^n$. Therefore, π^n belongs to $\mathfrak{R}_2$, as claimed.

The condition $\Delta w_t = 0$ selects w_t among other possible choices. More precisely, in principle one could define w_t as $w_t(x) = u_t(0) + [\, u_t(1) - u_t(0)\,]\, f(x)$ for any smooth function $f(x)$ such that $f(0) = 0$, $f(1) = 1$. However, the proof that u^n belongs to $\mathfrak{R}_2$ is based on the maximum principle for the heat equation with Dirichlet boundary conditions. For v to be a solution we need $\Delta w_t = 0$ which imposes the choice $f(x) = x$.

It remains to examine the regularity in space and time of the trajectory u_t^n. Since u_t belongs to $\mathfrak{R}_3$ and as the time-derivative commutes with the operator $P_s^{(D)}$, by definition, the trajectory u_t^n also belongs to $\mathfrak{R}_3$. Furthermore, as w_t is smooth in space, by Theorem B.4 and its equivalent version for the heat equation with Dirichlet boundary conditions, $u_t^n \in C^\infty([0,1])$ for all $0 < t \le T$, and u_t^n belongs to $\mathfrak{R}_4$. This completes the proof of the lemma. □

Lemma 5.7. *The set* $\mathfrak{R}_4$ *is* $I_{[0,T]}(\cdot|\gamma)$*-dense.*

Proof: Fix $\pi(t,dx) = u(t,x)\,dx$ in $\mathfrak{R}_3$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. Keep in mind that u is continuous in $(0,T] \times [0,1]$. Define u^n, $n \ge 1$, by (5.6), and let $\pi^n(t,dx) = u^n(t,x)\,dx$. By Lemma 5.6, π^n belongs to $\mathfrak{R}_4$.

By definition, π^n converges to π in $D([0,T], \mathcal{M})$. Hence, by the lower semicontinuous of the rate function, it remains to show that $\limsup_{n\to\infty} I_{[0,T]}(\pi^n|\gamma) \le I_{[0,T]}(\pi|\gamma)$.

By (3.1), the cost of the trajectory π^n in the interval $[0,T]$ is bounded by the sum of its cost in the intervals $[0,\delta]$, $[\delta,T]$:

$$I_{[0,T]}(u^n) \;\le\; I_{[0,\delta]}(u^n) \;+\; I_{[0,T-\delta]}(\tau_\delta u^n) \;. \tag{5.7}$$

As $u^n = u^{(\gamma)}$ in the time-interval $[0,\delta]$,

$$I_{[0,\delta]}(u^n) \;=\; 0 \;. \tag{5.8}$$

We turn to the interval $[\delta, T]$. Recall the notation introduced in (3.1). The cost of the trajectory in this interval is given by $I_{[0,T-\delta]}(\tau_\delta\, u^n)$. Let $\widehat{\chi}_n(t) = \chi_n(t-\delta)$, $T_\delta = T - \delta$, $v = \tau_\delta\, u$, $v^n = \tau_\delta\, u^n$, $\widehat{w} = \tau_\delta\, w$, and observe that $v_t^n = \widehat{w}_t + P^{(D)}_{\widehat{\chi}_n(t)}[v_t - \widehat{w}_t]$, $0 \le t \le T_\delta$. Moreover,

$$\varepsilon(\delta) \;\le\; v_t^n \;\le\; 1 \,-\, \varepsilon(\delta) \tag{5.9}$$

for $0 \le t \le T_\delta$, where $\varepsilon(\delta)$ has been introduced at the beginning of the proof of Lemma 5.6. With this notation, $I_{[0,T-\delta]}(\tau_\delta u^n) = I_{[0,T_\delta]}(v^n)$.

By Lemma 4.3, $I_{[0,T_\delta]}(v^n) = I^{(1)}_{[0,T_\delta]}(v^n) + I^{(2)}_{[0,T_\delta]}(v^n)$. We estimate each term of this sum separately. The next observation will be useful in the argument.

Let $L^{(\partial_t)}$, $L_0^{(\partial_t)}$ be the functional introduced in (4.10), (4.4) with T, u_t replaced by T_δ, v_t, respectively. Keep in mind that these linear functionals depend on the trajectory $u(\cdot,\cdot)$, that is, on v. Since $I_{[0,T_\delta]}(v) = I_{[0,T-\delta]}(\tau_\delta u) \le I_{[0,T]}(u) < \infty$, by (4.16), (4.17), $L_0^{(\partial_t)}$ belongs to $\mathcal{H}^{-1}(\sigma(v))$ and there exists M in $\mathcal{L}^2(\sigma(v)^{-1})$ such that

$$L_0^{(\partial_t)}(H) \;=\; \int_0^{T_\delta} \langle\, M_s\,,\, \nabla H_s \,\rangle\, ds\;, \quad \int_0^1 \frac{M_s}{\sigma(v_s)}\, dx \;=\; 0 \tag{5.10}$$

for all H in $C_K^\infty(\Omega_{T_\delta})$, and almost all $0 \le s \le T_\delta$.

We turn to $I^{(1)}_{[0,T_\delta]}(v^n)$. By Lemma 4.3, $I^{(1)}_{[0,T_\delta]}(v^n) = (1/4)\|\mathfrak{L}_0\|^2_{-1,\sigma(v^n)}$. The linear functional $\mathfrak{L}_0$ introduced just below (4.10) is the sum of $L_0^{(\partial_t)}$ with $L_0^{(\nabla)}$. We first examine $L_0^{(\partial_t)}$.

The linear functional $L_0^{(\partial_t)}$. By definition, since $P_s^{(D)}$ is a symmetric operator in $\mathcal{L}^2([0,1])$, for every $H \in C_K^\infty(\Omega_T)$,

$$\begin{aligned}\int_0^{T_\delta} \langle\, \partial_t v_t^n\,,\, H_t \,\rangle\, dt \;&=\; \int_0^{T_\delta} \langle\, (\, I \,-\, P^{(D)}_{\widehat{\chi}_n(t)}\,)\, \partial_t \widehat{w}_t\,,\, H_t \,\rangle\, dt \\ &+\; \int_0^{T_\delta} \langle\, \partial_t v_t\,,\, P^{(D)}_{\widehat{\chi}_n(t)} H_t \,\rangle\, dt \;+\; \int_0^{\delta} \widehat{\chi}_n'(t)\, \langle\, \Delta P^{(D)}_{\widehat{\chi}_n(t)} [\, v_t - \widehat{w}_t\,]\,,\, H_t \,\rangle\, dt\;.\end{aligned} \tag{5.11}$$

The last integral runs from 0 to δ because $\widehat{\chi}_n'(t)$ vanishes for $t \ge \delta$.

As $(t,x) \mapsto (P^{(D)}_{\widehat{\chi}_n(t)} H_t)(x)$ is a smooth function that vanishes at $x=0$ and $x=1$, by (5.10), the second term on the right-hand side of (5.11) is equal to the time integral of $\langle\, M_t\,,\, \nabla\, P^{(D)}_{\widehat{\chi}_n(t)} H_t \,\rangle$. By (5.5), this scalar product is equal to

$$\langle\, M_t\,,\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla H_t \,\rangle \;=\; \langle\, P^{(N)}_{\widehat{\chi}_n(t)}\, M_t\,,\, \nabla H_t \,\rangle$$

because the operator $P^{(N)}_{\widehat{\chi}_n(t)}$ is symmetric in $\mathcal{L}^2([0,1])$.

On the other hand, as H_t vanishes at the boundary, an integration by parts yields that the third term on the right-hand side of (5.11) is equal to

$$-\int_0^{\delta} \widehat{\chi}_n'(t)\, \langle\, \nabla P^{(D)}_{\widehat{\chi}_n(t)} [\, v_t - \widehat{w}_t\,]\,,\, \nabla H_t \,\rangle\, dt \;=\; -\int_0^{\delta} \widehat{\chi}_n'(t)\, \langle\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla [\, v_t - \widehat{w}_t\,]\,,\, \nabla H_t \,\rangle\, dt\;,$$

where we apllied the identity (5.5) once more.

In conclusion,

$$\begin{aligned}\int_0^{T_\delta} \langle\, \partial_t v_t^n\,,\, H_t \,\rangle\, dt \;&=\; \int_0^{T_\delta} \langle\, (\, I \,-\, P^{(D)}_{\widehat{\chi}_n(t)}\,)\, \partial_t \widehat{w}_t\,,\, H_t \,\rangle\, dt \\ &+\; \int_0^{T_\delta} \langle\, P^{(N)}_{\widehat{\chi}_n(t)}\, M_t\,,\, \nabla H_t \,\rangle\, dt \;-\; \int_0^{\delta} \widehat{\chi}_n'(t)\, \langle\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla [\, v_t - \widehat{w}_t\,]\,,\, \nabla H_t \,\rangle\, dt\;.\end{aligned} \tag{5.12}$$

We estimate the first and the last term on the right-hand side of (5.12). By Young's inequality $xy \le (1/2A_1)x^2 + (A_1/2)y^2$, $A_1 > 0$, the first term on the right-hand side is bounded by

$$\frac{1}{2A_1} \int_0^{T_\delta} \langle\, [\, (\, I \,-\, P^{(D)}_{\widehat{\chi}_n(t)}\,)\, \partial_t \widehat{w}_t\,]^2 \,\rangle\, dt \;+\; \frac{A_1}{2} \int_0^{T_\delta} \langle\, H_t^2 \,\rangle\, dt$$

for all $A_1 > 0$. As H_t vanishes at the boundary of $[0,1]$, by Poincaré's inequality and (5.9), this sum is bounded by

$$\begin{aligned} &\frac{1}{2A_1}\int_0^{T_\delta}\langle\,[\,(\,I\,-\,P^{(D)}_{\widehat{\chi}_n(t)}\,)\,\partial_t\widehat{w}_t\,]^2\,\rangle\,dt\;+\;C_0\,A_1\int_0^{T_\delta}\langle\,(\,\nabla H_t\,)^2\,\rangle dt \\ &\quad\le\frac{1}{2A_1}\int_0^{T_\delta}\langle\,[\,(\,I\,-\,P^{(D)}_{\widehat{\chi}_n(t)}\,)\,\partial_t\widehat{w}_t\,]^2\,\rangle\,dt\;+\;C_0\,A_1\int_0^{T_\delta}dt\int_0^1\sigma(v^n_t)\,(\,\nabla H_t\,)^2\,dx \end{aligned} \tag{5.13}$$

for some finite constant $C_0 = C_0(u)$ which may change from line to line.

Since $\widehat{\chi}'_n(t) = (1/n)\,\chi'(t-\delta)$, by Young's inequality, the third term on the right-hand side of (5.12) is bounded by

$$\frac{C_0}{n}\int_0^{\delta}\langle\,\{\,P^{(N)}_{\widehat{\chi}_n(t)}\nabla[\,v_t-\widehat{w}_t\,]\,\}^2\,\rangle\,dt\;+\;\frac{1}{n}\int_0^{\delta}\langle\,(\,\nabla H_t\,)^2\,\rangle\,dt$$

for some finite constant C_0 which depends on $\chi(\cdot)$. As $P^{(N)}_s$, $s\ge 0$, is a contraction in $L^2([0,1])$ and since $\varepsilon(\delta)\le v^n_t\le 1-\varepsilon(\delta)$, this expression is less than or equal to

$$\frac{C_0}{n}\int_0^{\delta}dt\int_0^1[\,\nabla v_t-\nabla\widehat{w}_t\,]^2\,dx\;+\;\frac{C_1}{n}\int_0^{\delta}dt\int_0^1\sigma(v^n_t)[\,\nabla H_t\,]^2\,dx \tag{5.14}$$

for some finite constant $C_1 = C_1(u)$. We turn to the linear functional $L^{(\nabla)}_0$.

The linear functional $L^{(\nabla)}_0$. By definition of v^n_t,

$$\int_0^{T_\delta}\langle\,\nabla v^n_t\,,\,\nabla H_t\,\rangle\,dt\;=\;\int_0^{T_\delta}\langle\,(\,I\,-\,P^{(N)}_{\widehat{\chi}_n(t)}\,)\,\nabla\widehat{w}_t\,,\,\nabla H_t\,\rangle\,dt\;+\;\int_0^{T_\delta}\langle\,P^{(N)}_{\widehat{\chi}_n(t)}\,\nabla v_t\,,\,\nabla H_t\,\rangle\,dt\,.$$

For similar reasons to the ones presented above, the first term on the right-hand side is bounded by

$$\frac{1}{2A_2}\int_0^{T_\delta}\langle\,[\,(\,I\,-\,P^{(N)}_{\widehat{\chi}_n(t)}\,)\,\nabla\widehat{w}_t\,]^2\,\rangle\,dt\;+\;C_0\,A_2\int_0^{T_\delta}dt\int_0^1\sigma(v^n_t)\,(\,\nabla H_t\,)^2\,dx \tag{5.15}$$

for all $A_2 > 0$ and some finite constant $C_0 = C_0(u)$.

The linear functional $\mathfrak{L}_0$. We are now in a position to estimate $I^{(1)}_{[0,T_\delta]}(v^n) = (1/4)\|\mathfrak{L}_0\|^2_{-1,\sigma(v^n)}$. Let

$$r_1(n)\;=\;\int_0^{T_\delta}\langle\,[\,(\,I\,-\,P^{(D)}_{\widehat{\chi}_n(t)}\,)\,\partial_t\widehat{w}_t\,]^2\,\rangle\,dt\,,\quad r_2(n)\;=\;\frac{C_0}{n}\int_0^{\delta}dt\int_0^1[\,\nabla v_t-\nabla\widehat{w}_t\,]^2\,dx\,,$$
$$r_3(n)\;=\;\int_0^{T_\delta}\langle\,[\,(\,I\,-\,P^{(N)}_{\widehat{\chi}_n(t)}\,)\,\nabla\widehat{w}_t\,]^2\,\rangle\,dt\,.$$

As both semigroups $P^{(D)}$ and are $P^{(N)}$ continuous, $\lim_{n\to\infty} r_j(n) = 0$ for $j = 1, 3$. As u (and, thus, v) has finite energy, by definition of $\widehat{w}$, $\lim_{n\to\infty} r_2(n) = 0$. Set $A_j = \sqrt{r_j(n)} =: c_j(n)$,

$j = 1, 3$, and $c_2(n) := r_2(n)$. By (5.12), (5.13), (5.14), (5.15),

$$
\begin{aligned}
& 2 \int_0^{T_\delta} \langle \partial_t v_t^n \,,\, H_t \rangle \, dt \;+\; 2 \int_0^{T_\delta} \langle \nabla v_t^n \,,\, \nabla H_t \rangle \, dt \;-\; \int_0^{T_\delta} dt \int_0^1 \sigma(v_t^n) \, (\nabla H_t)^2 \, dx \\
& \quad \le\; 2 \int_0^{T_\delta} \langle P^{(N)}_{\widehat{\chi}_n(t)} M_t \,,\, \nabla H_t \rangle \, dt \;+\; 2 \int_0^{T_\delta} \langle P^{(N)}_{\widehat{\chi}_n(t)} \nabla v_t \,,\, \nabla H_t \rangle \, dt \\
& \quad -\; [1 - \varepsilon_n] \int_0^{T_\delta} dt \int_0^1 \sigma(v_t^n) \, (\nabla H_t)^2 \, dx \;+\; c_n \;,
\end{aligned}
$$

where $c_n = \sum_{1 \le j \le 3} c_j(n)$, $\varepsilon_n = C_0 \,[\, c_1(n) + c_2(n) + (1/n) \,]$ so that $\lim_{n\to\infty} \varepsilon_n = 0$. Note that c_n, ε_n do not depend on H. Hence, by definition of $I^{(1)}_{[0,T_\delta]}(\,\cdot\,)$ and (4.1),

$$
I^{(1)}_{[0,T_\delta]}(v^n) \;\le\; \frac{1}{4(1-\varepsilon_n)} \, \| \, L_0^n \, \|^2_{-1,\sigma(v^n)} \;+\; c_n \;,
$$

where $L_0^n(H) = \int_0^{T_\delta} \langle P^{(N)}_{\widehat{\chi}_n(t)} M_t \,+\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla v_t \,,\, \nabla H_t \rangle \, dt$.

By Lemma 4.1, the first term on the right-hand side of the previous displayed equation is equal to

$$
\frac{1}{4(1-\varepsilon_n)} \int_0^{T_\delta} \Big\{ \int_0^1 \frac{1}{\sigma(v_t^n)} \, [\, P^{(N)}_{\widehat{\chi}_n(t)} M_t \,+\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla v_t \,]^2 \, dx \,-\, R_t^n \Big\} \, dt \;,
$$

where $R_t^n = \langle \, [P^{(N)}_{\widehat{\chi}_n(t)} M_t \,+\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla v_t] / \sigma(v_t^n) \, \rangle^2 \,/\, \langle \, 1/\sigma(v_t^n) \rangle$.

Consider the limit, as $n \to \infty$, of the two previous displayed equations. Since $\varepsilon_n \to 0$ and $c_n \to 0$ we may ignore these constants. On the other hand, by (5.9), $\varepsilon(\delta) \,\le\, v_t^n \,\le\, 1 - \varepsilon(\delta)$. Therefore, as the semigroup $(P_t^{(N)} : t \ge 0)$ is continuous in $\mathcal{L}^2([0,1])$, we may replace in the previous equations $P^{(N)}_{\widehat{\chi}_n(t)} M_t$, $P^{(N)}_{\widehat{\chi}_n(t)} \nabla v_t$ by M_t, ∇v_t, respectively, at a cost which vanishes as $n \to \infty$. Finally, as $v_t^n \to v_t$ a.e., we conclude that

$$
\limsup_{n\to\infty} I^{(1)}_{[0,T_\delta]}(v^n) \;\le\; \frac{1}{4} \int_0^{T_\delta} \Big\{ \int_0^1 \frac{1}{\sigma(v_t)} \, [\, M_t \,+\, \nabla v_t \,]^2 \, dx \,-\, R'_t \Big\} \, dt \;,
$$

where $R'_t = \langle \, [\, M_t \,+\, \nabla v_t] / \sigma(v_t) \rangle^2 \,/\, \langle \, 1/\sigma(v_t) \rangle$. By (5.10), this expression is equal to R_t, where $R_t \,=\, \langle \, \nabla v_t / \sigma(v_t) \rangle^2 \,/\, \langle \, 1/\sigma(v_t) \rangle$. Hence, by Lemma 4.4 [with u_t replaced by v_t],

$$
\limsup_{n\to\infty} I^{(1)}_{[0,T_\delta]}(v^n) \;\le\; I^{(1)}_{[0,T_\delta]}(v) \;.
$$

We turn to $I^{(2)}_{[0,T_\delta]}(v^n)$. By Lemma 4.3,

$$
I^{(2)}_{[0,T_\delta]}(v^n) \;=\; \int_0^{T_\delta} \Phi^{v_n}(a_t^n, b_t^n) \, dt \;,
$$

where a_t^n, b_t^n are given by (4.11), (4.5) with u replaced by v^n. To stress the dependence of Φ on v_n, we denoted this functional by Φ^{v_n}. However, as $v^n(t,1) = v(t,1)$, $v^n(t,0) = v(t,0)$, $\Phi^{v_n} = \Phi^v$.

Let Ξ^n, Ξ be given by (4.5) with v^n, v in place of u, respectively. As $v^n \to v$ almost everywhere, and since $\varepsilon(\delta) \,\le\, v^n \,\le\, 1 - \varepsilon(\delta)$, the continuous function Ξ^n converges to Ξ pointwisely.

An elementary computation, similar to the one presented above when we examined the rate function $I^{(1)}_{[0,T_\delta]}$, yields that

$$a^n_t \;=\; \langle\, (\, I \,-\, P^{(D)}_{\widehat{\chi}_n(t)}\,)\, \partial_t \widehat{w}_t \,,\, 1 \,-\, \Xi^n_t \,\rangle \;-\; \langle\, (\, I \,-\, P^{(N)}_{\widehat{\chi}_n(t)}\,) \nabla \widehat{w}_t \,,\, \nabla \Xi^n_t \,\rangle \\ +\; \langle\, P^{(D)}_{\widehat{\chi}_n(t)}\, \partial_t v_t \,,\, 1 \,-\, \Xi^n_t \,\rangle \;-\; \langle\, P^{(N)}_{\widehat{\chi}_n(t)}\, \nabla v_t \,,\, \nabla \Xi^n_t \,\rangle \\ +\; \widehat{\chi}'_n(t) \,\langle\, v_t \,-\, \widehat{w}_t \,,\, \Delta P^{(D)}_{\widehat{\chi}_n(t)} [1 \,-\, \Xi^n_t] \,\rangle \;.$$

Note that in the last term the operator $\Delta P^{(D)}_{\widehat{\chi}_n(t)}$ is acting on $[1 - \Xi^n_t]$ instead of $v_t - \widehat{w}_t$, as in the first part of the proof. Here, we simply used the fact that the semigroup $P^{(D)}_r$ is symmetric.

As $v_t - \widehat{w}_t$ vanishes at the boundary, an integration by parts and (5.5) yield that the last term is equal to

$$-\, \widehat{\chi}'_n(t) \,\langle\, \nabla [v_t \,-\, \widehat{w}_t] \,,\, \nabla P^{(D)}_{\widehat{\chi}_n(t)} [1 \,-\, \Xi^n_t] \,\rangle \;=\; \widehat{\chi}'_n(t) \,\langle\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla [v_t \,-\, \widehat{w}_t] \,,\, \nabla \Xi^n_t \,\rangle \;,$$

where we used that the semigroup $P^{(N)}_s$ is symmetric in $\mathcal{L}^2([0,1])$.

Since $\varepsilon(\delta) \le v^n \le 1 - \varepsilon(\delta)$, there exists a finite constant C_0 such that $|\, \Xi^n_t \,| \le C_0$, $|\, \nabla \Xi^n_t \,| \le C_0$ for all $n \ge 1$, $0 \le t \le T_\delta$. Therefore, as $\widehat{\chi}'_n(t) = (1/n) \chi'(t - \delta)$ and since the operators $P^{(N)}_s$, $P^{(D)}_s$ are contractions in $\mathcal{L}^2([0,1])$, there exists a finite constant C_0 such that $|\, a^n_t \,|^2 \le C_0 \,\{\, 1 \,+\, \langle\, (\partial_t v_t)^2 \,\rangle \,+\, \langle\, (\nabla v_t)^2 \,\rangle \,\}$ for all $n \ge 1$, $0 \le t \le T_\delta$. Moreover,

$$\lim_{n\to\infty} \widehat{\chi}'_n(t) \,\langle\, P^{(N)}_{\widehat{\chi}_n(t)} \nabla [v_t \,-\, \widehat{w}_t] \,,\, \nabla \Xi^n_t \,\rangle \;=\; 0 \;,$$

and, as $\Xi^n_t \to \Xi_t$, $\nabla \Xi^n_t \to \nabla \Xi_t$ in $\mathcal{L}^2([0,1])$, for all $0 \le t \le T_\delta$,

$$\lim_{n\to\infty} a^n_t \;=\; a_t \;:=\; \langle\, \partial_t v_t \,,\, 1 \,-\, \Xi_t \,\rangle \;-\; \langle\, \nabla v_t \,,\, \nabla \Xi_t \,\rangle \;. \tag{5.16}$$

A similar bound and limit hold for the sequence b^n_t. Since Φ^v is continuous and the map $t \mapsto \langle\, (\partial_t v_t)^2 \,\rangle \,+\, \langle\, (\nabla v_t)^2 \,\rangle$ is integrable, by (4.15), (5.16) and the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^{T_\delta} \Phi^v(a^n_t, b^n_t)\; dt \;=\; \int_0^{T_\delta} \Phi^v(a_t, b_t)\; dt \;.$$

By Lemma 4.3, the right-hand side is equal to $I^{(2)}_{[0,T_\delta]}(v)$. Therefore,

$$\lim_{n\to\infty} I^{(2)}_{[0,T_\delta]}(v^n) \;=\; I^{(2)}_{[0,T_\delta]}(v) \;.$$

Since $v^n = \tau_\delta u^n$, $v = \tau_\delta u$, adding together the estimates on $I^{(1)}_{[0,T_\delta]}(v^n)$ and $I^{(2)}_{[0,T_\delta]}(v^n)$, yield that

$$\limsup_{n\to\infty} I_{[0,T_\delta]}(\tau_\delta u^n) \;\le\; I_{[0,T_\delta]}(\tau_\delta u) \;.$$

By (3.3), this expression is bounded by $I_{[0,T]}(u \,|\, \gamma)$, which completes the proof of the lemma in view of (5.7), (5.8). □

Proof of Theorem 5.2: The first assertion follows from Lemma 5.7 and the definition of the set $\mathfrak{R}_4$.

Assume that there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 \le \gamma \le 1 - \varepsilon_0$. Fix $\pi \in D([0,T], \mathcal{M})$ such that $I_{[0,T]}(\pi|\gamma) < \infty$. Let $\pi^n(t,dx) = u^n(t,x)\, dx$ be the sequence in Π_γ which $I_{[0,T]}(\,\cdot\,|\gamma)$-approximates π in the sense of Definition 5.1. Since π^n belongs to Π_γ, there exists $\delta > 0$ and

$\varepsilon > 0$ such that $u_t^n = u_t^{(\gamma)}$ for $0 \le t \le \delta$ and $\varepsilon \le u^n(t,x) \le 1-\varepsilon$ for all $(t,x) \in [\delta, T] \times [0,1]$. By (B.4), there exists $\varepsilon_1 > 0$ such that $\varepsilon_1 \le u^n(t,x) \le 1-\varepsilon_1$ for all $(t,x) \in [0,\delta] \times [0,1]$. This completes the proof of the theorem. □

Remark 5.8. The difference between the present context and Bertini et al. (2009b) is that here the rate function is convex. We used this property to restrict our attention to trajectories bounded away from 0 and 1 and smooth in time [that is to paths in $\mathfrak{R}_3$].

We conclude this section deriving the explicit formula for the rate functions of trajectories in Π_γ.

Proof of Proposition 2.6: As u belongs to Π_γ, u is smooth in $(0,T] \times [0,1]$, and for each $0 < t \le T$, there exists $\delta = \delta(t) > 0$ such that $\delta \le u(t,x) \le 1-\delta$. Therefore, equation (2.15) is strictly elliptic and can be solved explicitly. The solution H inherits the smoothness from u. In particular, it belongs to $C^{1,2}((0,T] \times [0,1])$.

As u belongs to Π_γ, u follows the hydrodynamic equation in a time-interval $[0,\mathfrak{t}]$ for some $\mathfrak{t} > 0$. Hence, for $0 < t \le \mathfrak{t}$, the solution of (2.15) vanishes: $H(t,x) = 0$ for all $(t,x) \in [0,\mathfrak{t}] \times [0,1]$. Hence, H actually belongs to $C^{1,2}([0,T] \times [0,1])$.

We turn to the formula for the rate function. For a function G in $C^{1,2}([0,T] \times [0,1])$, let

$$L_{T,G}(u) \;=\; \big\langle u_T, G_T \big\rangle \;-\; \langle u_0, G_0\rangle \;-\; \int_0^T \big\langle u_t, \partial_t G_t \big\rangle \, dt \;+\; \int_0^T \big\langle \nabla u_t, \nabla G_t \big\rangle \, dt \,.$$

Multiply equation (2.15) by G, integrate over space and time, integrate by parts in space, and recall the boundary conditions to get that

$$\begin{aligned} L_{T,G}(u) \;&=\; 2 \int_0^T \big\langle \sigma(u_t)\, \nabla H_t \,,\, \nabla G_t \big\rangle \, dt \\ &+\; \int_0^T \big\{ G_t(1)\, \mathfrak{p}_{\beta,B}\big(u_t(1) \,,\, H_t(1) \big) \;+\; G_t(0)\, \mathfrak{p}_{\alpha,A}\big(u_t(0) \,,\, H_t(0) \big) \big\} \, dt \,. \end{aligned}$$

Insert this expression in (2.11) and add and subtract some terms to get that

$$\begin{aligned} J_{T,G}(u) \;&=\; -\int_0^T \big\langle \sigma(u_t)\, [\, \nabla H_t \;-\; \nabla G_t \,]^2 \big\rangle \, dt \\ &-\; \frac{1}{A} \int_0^T [\, 1 - u_t(0) \,]\, \alpha \, \big[\, e^{G_t(0)} \;-\; e^{H_t(0)} \;-\; [G_t(0) - H_t(0)]\, e^{H_t(0)} \,\big] \, dt \\ &-\; \frac{1}{A} \int_0^T u_t(0)\, (1-\alpha) \, \big[\, e^{-G_t(0)} \;-\; e^{-H_t(0)} \;-\; [G_t(0) - H_t(0)]\, e^{-H_t(0)} \,\big] \, dt \\ &-\; \mathfrak{B} \;+\; \mathbb{I}_{[0,T]}(u) \,, \end{aligned}$$

where $\mathbb{I}_{[0,T]}(u)$ is the expression appearing on the right-hand side of (2.16) and $\mathfrak{B}$ is a term similar to the second and third lines of this formula with the left boundary conditions replaced by the right ones. Since the expressions inside the integrals are all positive, the supremum in G is attained at $G = H$, so that

$$I_{[0,T]}(u) \;=\; \sup_G J_{T,G}(u) \;=\; \mathbb{I}_{[0,T]}(u) \,,$$

which completes the proof of the lemma. □

6. Proof of Theorem 2.7

In this section, we prove the dynamical large deviations. The strategy is by now classical and we just indicate the main steps. The main point here is that the dynamics can be considered as a small perturbation of the exclusion process with Neumann boundary conditions (the process induced by the generator $L_N^{\rm bulk}$) because the boundary dynamics is speeded-up only by N.

The reversible stationary measures for the exclusion process with Neumann boundary conditions are the uniform measures with a fixed total number of particles. The grand canonical versions are the Bernoulli product measures with a fixed density. For this reason, we take one of these measures as reference measure.

There is an important difference between our model and the exclusion process with Dirichlet boundary conditions. Recall the definition the functional $J_{T,H}$ introduced in (2.10) and (2.11). For the sake of this discussion, denote by $J_{T,H}^{\rm DBC}$ the corresponding functional in the context of exclusion dynamics with Dirichlet boundary conditions Bertini et al. (2003, 2009b); Farfan et al. (2011); Franco et al. (2021). While $J_{T,H}^{\rm DBC}$ is defined on the set $D([0,T],\mathcal{M}_{\rm ac})$, in the present context, the functionals $J_{T,H}$ are defined only on the subset $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ of trajectories with finite energy because only for such trajectories are the boundary densities well defined. As a consequence, in the two-blocks estimate, the usual empirical density, $(2N\varepsilon+1)^{-1}\sum_{y\in\Lambda_N,|y-x|<\varepsilon}\eta_y$ which, as a function of x, has jumps needs to be replaced by a smooth approximation. See (6.1) below.

A super-exponential estimate. We follow the proofs presented in Bertini et al. (2009b, Section 3), Farfan et al. (2011, Section 6), Franco et al. (2021, Section 3). Denote by ν_N the Bernoulli product measure on Ω_N with density 1/2 and by $\mathbb{D}_N$ the Dirichlet form given by

$$\mathbb{D}_N(f) \;:=\; \langle -L_N^{\rm bulk} f\,,\, f\,\rangle_{\nu_N}\;, \quad f\colon\Omega_N\to\mathbb{R}_+\;.$$

Next result is Bertini et al. (2009b, Lemma 3.1) adapted to the present context. The proof is elementary and left to the reader. It relies on a Schwarz inequality.

Lemma 6.1. *There exists a finite constant C_0, which only depends on the parameters α, β, A, B, such that*

$$\langle\, \mathcal{L}_N f\,,\, f\,\rangle_{\nu_N} \;\le\; -\mathbb{D}_N(f) \;+\; C_0\, N\, E_{\nu_N}[\, f^2\,]$$

for all $f\colon\Omega_N\to\mathbb{R}_+$.

Given a cylinder function h, that is a function on $\{0,1\}^{\mathbb{Z}}$ depending on η_x, $x\in\mathbb{Z}$, only through finitely many x, denote by $\widetilde{h}(\alpha)$ the expectation of h with respect to ν_α, the Bernoulli product measure with density α:

$$\widetilde{h}(\alpha) \;=\; E_{\nu_\alpha}[h]\;.$$

Denote by $\{\tau_x : x\in\mathbb{Z}\}$ the group of translations in $\{0,1\}^{\mathbb{Z}}$ so that $(\tau_x\zeta)_z=\zeta_{x+z}$ for all x, z in $\mathbb{Z}$ and configuration ζ in $\{0,1\}^{\mathbb{Z}}$. Translations are extended to functions and measures in a natural way. They are also extended to configurations, functions and measures in Ω_N. In this case, for x, $y\in\{k/N : k\in\mathbb{Z}\}$ such that y, $x+y\in\Lambda_N$, $(\tau_x\eta)_y=\eta_{x+y}$.

Fix a strictly decreasing sequence $\{U_\varepsilon:\varepsilon>0\}$ converging to 1: $U_\varepsilon>U_{\varepsilon'}>1$ for $\varepsilon>\varepsilon'>0$, $\lim_{\varepsilon\to 0}U_\varepsilon=1$. Recall from (3.5) the definition of the approximation of the unity ϕ^δ. For $\varepsilon>0$, $\pi\in\mathcal{M}$, denote by $\Xi_\varepsilon(\pi)$ the measure in $\mathcal{M}_{\rm ac}$ defined by

$$\Xi_\varepsilon(\pi)\,(dx) \;=\; \frac{1}{U_\varepsilon}\int_0^1 \phi^\varepsilon(y-x)\,\pi(dy)\,dx\;. \quad \text{Let } \pi^{N,\varepsilon} \;=\; \Xi_\varepsilon(\pi^N)\;. \tag{6.1}$$

Clearly, $\pi^{N,\varepsilon}$ belongs to $\mathcal{M}_{\rm ac}$ for N sufficiently large because $U_\varepsilon > 1$. Denote its density by $u^{N,\varepsilon}$. We have just pointed out that $0 \le u^{N,\varepsilon}(x) \le 1$ for N large. The map $x \mapsto u^{N,\varepsilon}(x)$ is smooth, and, if x is at distance less than ε from the boundary of the interval $[0,1]$, $u^{N,\varepsilon}(x)$ does not represent the density of particles around x because the integral is carried over an interval which does not contain the support of $\phi(\,\cdot - x\,)$.

Let $H \in C([0,T]\times[0,1])$ and h a cylinder function. For $\varepsilon > 0$ and N large enough, define $V^{H,h}_{N,\varepsilon} : [0,T]\times\Omega_N \to \mathbb{R}$ by

$$V^{H,h}_{N,\varepsilon}(t,\eta) \;=\; \frac{1}{N}\sum_{x\in\Lambda_N} H(t,x/N)\big\{\tau_x h(\eta) \,-\, \widetilde{h}\big(u^{N,\varepsilon}(x)\big)\big\}\;.$$

The sum is carried over all $x \in \Lambda_N$ for which the support of $\tau_x h$ is contained in Λ_N. For a function $G \in C([0,T])$ and cylinder functions h, f whose supports are contained in $\mathbb{N}$, $-\mathbb{N}$, respectively, let $W^{\pm}_G : [0,T]\times\Lambda_N \to \mathbb{R}$ be defined by

$$W^{G,h,-}_{N,\varepsilon}(t,\eta) \;=\; G(t)\,\big\{\, h(\eta) - \widetilde{h}(u^{N,\varepsilon}(\varepsilon))\,\big\}\;,$$
$$W^{G,f,+}_{N,\varepsilon}(t,\eta) \;=\; G(t)\,\big\{\, (\tau_N f)(\eta) - \widetilde{f}(u^{N,\varepsilon}(1-\varepsilon))\,\big\}\;.$$

Theorem 6.2. *Fix H in $C([0,T]\times[0,1])$, G in $C([0,T])$, a cylinder function h whose support ia contained in $\mathbb{N}$, a sequence of configurations $\{\eta^N \in \Omega_N : N \ge 1\}$ and $\delta > 0$. Then*

$$\lim_{\varepsilon\to 0}\,\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\eta^N}\Big[\,\Big|\int_0^T V^{H,h}_{N,\varepsilon}(t,\eta_t)\,dt\,\Big| \,>\, \delta\Big] \;=\; -\infty\;,$$
$$\lim_{\varepsilon\to 0}\,\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\eta^N}\Big[\,\Big|\int_0^T W^{G,h,-}_{N,\varepsilon}(t,\eta_t)\,dt\Big| \,>\, \delta\,\Big] \;=\; -\infty\;.$$

A similar result holds if the cylinder functions h has support contained in $-\mathbb{N}$ and the minus sign in $W^{G,h,-}_{N,\varepsilon}(t,\eta_t)$ is replaced by a plus sign. The proof of this result follows from Lemma 6.1 and the computation presented in the proof of Landim (1992, Lemma 3.2).

An energy estimate. The next result is Lemma 3.3 and Corollary 3.4 in Bertini et al. (2009b). The proof is similar and the details are left to the reader.

Proposition 6.3. *Fix a sequence $\{G_j : j \ge 1\}$ of functions in $C^{0,1}([0,T]\times[0,1])$ with compact support in $[0,T]\times(0,1)$ and a sequence $\{\eta^N \in \Omega_N : N \ge 1\}$ of configurations. There exists a finite constant C_0, depending only on α, β, A, B, such that*

$$\limsup_{\varepsilon\to 0}\,\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^N_{\eta^N}\Big[\max_{1\le j\le k}\mathcal{Q}_{G_j}(u^{N,\varepsilon}) \,\ge\, \ell\Big] \;\le\; -\ell \,+\, C_0\,(T+1)\;.$$

for all k, $\ell \ge 1$.

Upper bound. The upper bound proof relies on the super-exponential estimate presented in Theorem 6.2 and on the energy estimate stated in Proposition 6.3. It is similar to the one presented in Farfan et al. (2011, Subsection 6.3). As a consequence of the argument, the rate function can be set as $+\infty$ for trajectories that are not absolutely continuous with respect to the Lebesgue measure or which do not have finite energy. In other words, in the proof of the upper bound one can set $I_{[0,T]}(\pi|\gamma) = +\infty$ for $\pi \notin D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$.

Lower bound. We follow the arguments presented in Bertini et al. (2009b, Subsection 3.4) and Farfan et al. (2011, Subsection 6.4). Fix an open set $\mathcal{G}$ of $D([0,T],\mathcal{M})$ and a density profile $\gamma:[0,1]\to[0,1]$. Recall the definition of the set Π_γ introduced in Definition 2.5. Fix a path $\pi(t,dx)=u(t,x)\,dx\in\Pi_\gamma\cap\mathcal{G}$.

Let $(\eta^N: N\ge 1)$ be a sequence of configurations associated to the density profile in the sense (2.5). Denote by $\mathbb{P}^H_{\eta^N}$ the probability measure on $D([0,T],\Omega_N)$ induced by the weakly asymmetric exclusion process with Robin boundary conditions defined in Section 7.

Given two probability measures P and Q we denote by $\text{Ent}\,(Q\,|\,P)$ the relative entropy of Q with respect to P. By Theorem 7.1, Proposition 2.6 and an elementary computation,

$$\lim_{N\to\infty}\frac{1}{N}\,\text{Ent}\,(\,\mathbb{P}^H_{\eta^N}\,|\,\mathbb{P}_{\eta^N}\,)\;=\;I_{[0,T]}(\pi|\gamma)\;.$$

Therefore, since $N^{-1}\log(d\mathbb{P}^H_{\eta^N}/d\mathbb{P}_{\eta^N})$ is absolutely bounded, by the proof of the lower bound presented at Kipnis and Landim (1999, page 277),

$$\liminf_{n\to\infty}\frac{1}{N}\,\log\mathbb{P}_{\eta^N}\big[\,\mathcal{G}\,\big]\;\ge\;-\inf_{\pi\in\mathcal{G}\cap\Pi_\gamma}I_{[0,T]}(u|\gamma)\;.$$

The lower bound follows from this result and the $I_{[0,T]}(\,\cdot\,|\gamma)$-density stated in Theorem 5.2.

7. Weakly asymmetric exclusion with Robin boundary conditions

Recall the notation introduced in Section 2. Fix $H\in C^{1,2}([0,T]\times[0,1])$. Consider the weakly asymmetric exclusion process induced by the external field H with Robin boundary conditions. The generator of this process, denoted by $\mathcal{L}^H_N$, is given by

$$\mathcal{L}^H_N\;=\;L^{H,\text{lb}}_N\;+\;L^{H,\text{bulk}}_N\;+\;L^{H,\text{rb}}_N\;,\tag{7.1}$$

where, for a function $f:\Omega_N\to\mathbb{R}$,

$$(L^{H,\text{bulk}}_N f)(\eta)\;=\;N^2\sum_{x\in\Lambda^0_N}e^{-\,(\eta_{x+\mathfrak{e}}-\eta_x)\,[\,H_t(x+\mathfrak{e})-H_t(x)\,]}\big\{\,f(\sigma^{x,x+\mathfrak{e}}\eta)\;-\;f(\eta)\,\big\}\,,$$

$$(L^{H,\text{lb}}_N f)(\eta)\;=\;\frac{N}{A}\Big\{\,e^{H_t(\mathfrak{e})}\,\alpha\,(1-\eta_{\mathfrak{e}})\;+\;e^{-H_t(\mathfrak{e})}\,(1-\alpha)\,\eta_{\mathfrak{e}}\,\Big\}\,\big\{\,f(\sigma^{\mathfrak{e}}\eta)\;-\;f(\eta)\,\big\}\,,$$

$$(L^{H,\text{rb}}_N f)(\eta)\;=\;\frac{N}{B}\Big\{\,e^{H_t(\mathfrak{r})}\,\beta\,(1-\eta_{\mathfrak{r}})\;+\;e^{-H_t(\mathfrak{r})}\,(1-\beta)\,\eta_{\mathfrak{r}}\,\Big\}\,\big\{\,f(\sigma^{\mathfrak{r}}\eta)\;-\;f(\eta)\,\big\}\,.$$

Denote by $\mathbb{P}^H_\mu$, μ a probability measure on Ω_N, the measure on $D([0,T],\Omega_N)$ induced by the Markov process with infinitesimal generator $\mathcal{L}^H_N$ and initial state μ. Let $\mathbb{Q}^H_\mu$ be the probability on $D([0,T],\mathcal{M})$ induced by the empirical measure $\boldsymbol{\pi}$ and the measure $\mathbb{P}^H_\mu$: $\mathbb{Q}^H_\mu=\mathbb{P}^H_\mu\circ\boldsymbol{\pi}^{-1}$.

Theorem 7.1. *Fix a measurable profile $\gamma:[0,1]\to[0,1]$. Let $\{\mu_N: N\ge 1\}$ be a sequence of probability measures on Ω_N associated to γ in the sense of (2.5). Then, the sequence of probability measures $\mathbb{Q}^H_{\mu_N}$ converges to the probability measure $\mathbb{Q}^H$ concentrated on the trajectory $\pi(t,dx)=u(t,x)\,dx$, whose density u is the unique weak solution of*

$$\begin{cases}\partial_t u\;=\;\Delta u\;-\;2\,\nabla\{\sigma(u)\,\nabla H\}\;,\\ \nabla u_t(1)\;-\;2\,\sigma(u_t(1))\,\nabla H_t(1)\;=\;\mathfrak{p}_{\beta,B}\big(\,u_t(1)\,,\,H_t(1)\,\big)\;,\\ \nabla u_t(0)\;-\;2\,\sigma(u_t(0))\,\nabla H_t(0)\;=\;-\,\mathfrak{p}_{\alpha,A}\big(\,u_t(0)\,,\,H_t(0)\,\big)\;,\\ u(0,\cdot)=\gamma(\cdot)\;.\end{cases}\tag{7.2}$$

The proof of this result is by now classical and divided in several steps. One first proves tightness of the sequence $(\mathbb{Q}^H_{\mu_N} : N \geq 1)$. Then, one shows that any limit point of the sequence $\mathbb{Q}^H_{\mu_N}$ is concentrated on trajectories $\pi(t, dx) = u(t, x)\, dx$ whose density belongs to $\mathcal{L}^2(0, T; \mathcal{H}^1)$, where the $\mathcal{H}^1$ is the Sobolev space introduced in Section 4. Finally, one shows that limit points of the sequence $\mathbb{Q}^H_{\mu_N}$ are concentrated on trajectories which satisfy the identity (B.7). It remains to invoke the uniqueness of weak solutions, stated in Theorem B.7, to complete the proof. The technical details are standard and the arguments rely on the bound presented in Lemma 6.1. We refer to Kipnis and Landim (1999); Baldasso et al. (2017); Franco et al. (2021).

Appendix A. The Robin Laplacian

We present in this section some results on the Robin Laplacian needed in the previous sections. We refer to Mikhaĭlov (1983); Strauss (2008) for details. Denote by Δ_R the Laplacian on $[0, 1]$ with Robin boundary conditions, sometimes called the Robin Laplacian Strauss (2008, Section 4.3). This is the symmetric linear operator defined on the Hilbert space $L^2([0, 1])$ whose domain $D(\Delta_R)$ is the set

$$D(\Delta_R) \;:=\; \big\{ f \in C^2([0,1]) : (\nabla f)(0) \;=\; A^{-1}\, f(0)\;,\;\; (\nabla f)(1) \;=\; -\, B^{-1} f(1) \;\big\}\;,$$

and such that

$$(\Delta_R f)(x) \;=\; f''(x)$$

for all $f \in D(\Delta_R)$. Here, A, $B > 0$ are fixed positive constants omitted from the notation,

Fix $\lambda \in \mathbb{R}$ and consider the eigenvalue problem

$$\begin{cases} -\,\Delta f = \lambda\, f\;, \\ (\nabla f)(0) \;=\; A^{-1}\, f(0)\;, \\ (\nabla f)(1) \;=\; -\, B^{-1} f(1)\;. \end{cases} \tag{A.1}$$

This problem has only the trivial solution $f = 0$ for $\lambda \leq 0$. For $\lambda > 0$, the equation $-\,\Delta f = \lambda\, f$ can be turned into a two-dimensional ODE which yields that the solutions of (A.1) are given by $f(x) = a\,[\cos(\sqrt{\lambda} x) + b\, \sin(\sqrt{\lambda} x)\,]$ for some a, $b \in \mathbb{R}$. The boundary conditions are satisfied if and only if

$$\tan\sqrt{\lambda} \;=\; (A + B)\, \frac{\sqrt{\lambda}}{\lambda AB - 1}\;, \tag{A.2}$$

in which case $b = (A\sqrt{\lambda})^{-1}$. This identity excludes $\lambda = 0$ from the set of eigenvalues of the Robin Laplacian.

An analysis of (A.2) shows that it has a countable set of solutions $\{\lambda_j : j \geq 1\}$, where $0 < \lambda_1$, $\lambda_j < \lambda_{j+1}$ and $\lambda_j \sim j^2$ in the sense that there exists $0 < c_0 < c_1 < \infty$ such that

$$c_0\, j^2 \;\leq\; \lambda_j \;\leq\; c_1\, j^2 \;\;\text{for all } j \geq 1\;. \tag{A.3}$$

Denote by $\{f_j : j \geq 1\}$ the associated orthonormal eigenvectors, which form a basis of $\mathcal{L}^2([0, 1])$. By the previous analysis,

$$f_j(x) \;=\; a_j\, \Big\{\, \cos(\sqrt{\lambda_j} x) \;+\; \frac{1}{A\sqrt{\lambda_j}}\, \sin(\sqrt{\lambda_j} x)\, \Big\}\;, \tag{A.4}$$

where a_j is chosen for f_j to have $\mathcal{L}^2$-norm equal to 1. It can be shown that $|a_j| \le C_0$ for all $j \ge 1$, where C_0 is a finite constant depending only on A and B. Therefore, by (A.3),

$$\| f_j \|_\infty \;\le\; C_0\,, \quad \| \nabla^n f_j \|_\infty \;\le\; C_0\,(\lambda_j)^{n/2} \;\le\; C_0\, j^n \tag{A.5}$$

for all $j \ge 1$, $n \ge 1$. A straightforward computation provides a formula for the Green function of the Robin Laplacian: Let $K_R : [0,1] \times [0,1] \to \mathbb{R}_+$ be given by

$$K_R(x,y) \;=\; \frac{1}{1+A+B} \begin{cases} (B+1-x)\,(A+y)\,, & 0 \le y \le x \le 1\,, \\ (B+1-y)\,(A+x)\,, & 0 \le x \le y \le 1\,. \end{cases} \tag{A.6}$$

Denote by K_R the integral operator defined by

$$(K_R f)(x) \;=\; \int_0^1 K_R(x,y)\, f(y)\, dy\,.$$

Then, $K_R = (-\Delta_R)^{-1}$.

Denote by $\mathcal{H}_R$ the Hilbert space obtained by completing the space $C^2([0,1])$ endowed with the scalar product $\langle \cdot\,, \cdot \rangle_{\mathcal{H}_R}$ defined by

$$\langle\, f\,,\, g\,\rangle_{\mathcal{H}_R} \;:=\; \frac{1}{A}\, f(0)\, g(0) \;+\; \int_0^1 (\nabla f)(x)\,(\nabla g)(x)\; dx \;+\; \frac{1}{B}\, f(1)\, g(1)\,. \tag{A.7}$$

Let $\|f\|_{\mathcal{H}_R}$ be the norm induced by the scalar product $\langle \cdot\,, \cdot \rangle_{\mathcal{H}_R}$. We have that

$$\|f\|^2_{\mathcal{H}_R} \;=\; \sum_{k\ge 1} \lambda_k\, \langle f\,,\, f_k \rangle^2\,, \tag{A.8}$$

for all $f \in \mathcal{H}_R$. Note that

$$\|f\|^2_{\mathcal{H}_R} \;=\; \langle\, f\,, (-\,\Delta_R f)\,\rangle$$

for all $f \in D(\Delta_R)$.

Recall from (3.4) the definition of the Sobolev space $\mathcal{H}^1$. The norms $\|\cdot\|_{\mathcal{H}_R}$ and $\|\cdot\|_{\mathcal{H}^1}$ are equivalent. There exist finite constants $0 < C_1 < C_2 < \infty$ such that

$$C_1\, \|f\|_{\mathcal{H}^1} \;\le\; \|f\|_{\mathcal{H}_R} \;\le\; C_2\, \|f\|_{\mathcal{H}^1} \tag{A.9}$$

for all $f \in C^2([0,1])$. In particular, the spaces $\mathcal{H}_R$ and $\mathcal{H}^1$ coincide.

In terms of the eigenfunctions f_k,

$$\|\, f\,\|^2_{\mathcal{H}_R} \;=\; \sum_{k\ge 1} \lambda_k\, |\,\langle\, f\,,\, f_k\,\rangle\,|^2\,. \tag{A.10}$$

Moreover, a straightforward computation yields that for all $f \in D(\Delta_R)$,

$$\|\, f\,\|^2_\infty \;\le\; 2\,(A \vee 1)\, \|f\|^2_{\mathcal{H}_R}\,. \tag{A.11}$$

Fix a function f in $\mathcal{H}^1$. It is well known that there exists a continuous function $f^{(c)} : [0,1] \to \mathbb{R}$ (actually Hölder continuous, $|f^{(c)}(y) - f^{(c)}(x)| \le \|f\|_2 |y-x|^{1/2}$) such that $f = f^{(c)}$ almost surely. Moreover, for all $h \in C^1([0,1])$,

$$\int_0^1 f\, \nabla h\; dx \;=\; f^{(c)}(1)\, h(1) \;-\; f^{(c)}(0)\, h(0) \;-\; \int_0^1 \nabla f\, h\; dx\,. \tag{A.12}$$

The next result provides an explicit formula for $f^{(c)}$ in terms of the eigenvectors f_k.

Lemma A.1. *There exists a finite constant C_0 such that*

$$\sum_{k\ge 1} \big|\, \langle f\,,\, f_k\rangle \,\big| \;\le\; C_0 \,\|\, f \,\|_{\mathcal{H}_R}$$

for all $f \in \mathcal{H}^1$. In particular, $\sum_{k\ge 1}\langle f\,,\, f_k\rangle\, f_k(\cdot)$ defines a continuous function, and, for almost all $x\in[0,1]$,

$$f(x) \;=\; \sum_{k\ge 1}\langle f\,,\, f_k\rangle\, f_k(x)\;. \tag{A.13}$$

Proof: By (A.9), f belongs to $\mathcal{H}_R$. By Schwarz inequality,

$$\Big(\sum_{k\ge 1} \big|\, \langle f\,,\, f_k\rangle \,\big|\Big)^2 \;\le\; \sum_{k\ge 1} \lambda_k \,\big|\, \langle f\,,\, f_k\rangle \,\big|^2 \sum_{k\ge 1} \frac{1}{\lambda_k}\;\cdot$$

The second sum is finite by (A.3) and the first one is finite by (A.10). This proves the first assertion.

Since each function f_k is continuous, and a summable sum of uniformly bounded continuous functions is continuous, $\sum_{k\ge 1}\langle f\,,\, f_k\rangle\, f_k(\cdot)$ defines a continuous function. As $(f_k : k\ge 1)$ forms an orthonormal basis of $\mathcal{L}^2([0,1])$, $f=\sum_{k\ge 1}\langle f\,,\, f_k\rangle\, f_k$ as an identity in $\mathcal{L}^2([0,1])$. In particular, these functions are equal almost everywhere. □

Denote by $(P^{(R)}_t : t\ge 0)$ the semigroup in $\mathcal{L}^2([0,1])$ generated by the Robin Laplacian: For any function $f\in \mathcal{L}^2([0,1])$, $t>0$,

$$P^{(R)}_t f \;=\; \sum_{k\ge 1} e^{-\lambda_k t}\, \langle f\,,\, f_k\rangle\, f_k\;. \tag{A.14}$$

In particular, for each $t\ge 0$, $P^{(R)}_t$ is a self-adjoint operator in $\mathcal{L}^2([0,1])$ and $P^{(R)}_t f\in C^\infty([0,1])$ for all $f\in \mathcal{L}^2([0,1])$. Moreover, as $P^{(R)}_t$ is symmetric, by (A.10), $P^{(R)}_t$ is a contraction in $\mathcal{H}_R$ and $\mathcal{L}^2([0,1])$:

$$\begin{aligned} \|\, P^{(R)}_t f \,\|^2_{\mathcal{H}_R} \;&=\; \sum_{k\ge 1} e^{-2\lambda_k t}\, \lambda_k \,|\, \langle f\,,\, f_k\rangle \,|^2 \;\le\; \|\, f\,\|^2_{\mathcal{H}_R}\;, \\ \|\, P^{(R)}_t f \,\|^2_{2} \;&=\; \sum_{k\ge 1} e^{-2\lambda_k t}\, |\, \langle f\,,\, f_k\rangle \,|^2 \;\le\; \|\, f\,\|^2_{2}\;. \end{aligned} \tag{A.15}$$

Let $f\in \mathcal{L}^2([0,1])$ be given by $f=\sum_{k\ge 1}\langle f\,,\, f_k\rangle\, f_k$. For each $t>0$, there exists a finite constant $C_0(t)$ such that

$$\|\, P^{(R)}_t f \,\|^2_{\infty} \;\le\; C_0(t)\, \|\, f\,\|^2_2\;, \quad \|\, P^{(R)}_t f \,\|^2_{\mathcal{H}_R} \;\le\; C_0(t)\, \|\, f\,\|^2_2\;. \tag{A.16}$$

Indeed, by (A.10) and since $P^{(R)}_t$ is symmetric and $P^{(R)}_t f_k = e^{-\lambda_k t} f_k$,

$$\|\, P^{(R)}_t f \,\|^2_{\mathcal{H}_R} \;=\; \sum_{k\ge 1} \lambda_k\, e^{-2\lambda_k t} \big|\, \langle f\,,\, f_k\rangle \,\big|^2 \;\le\; C_0(t) \sum_{k\ge 1} \big|\, \langle f\,,\, f_k\rangle \,\big|^2 \;=\; C_0(t)\, \|\, f\,\|^2_2$$

for some finite constant $C_0(t)$. On the other hand, by Schwarz inequality and (A.5),

$$\|\, P^{(R)}_t f \,\|^2_{\infty} \;=\; \Big\|\, \sum_{k\ge 1} e^{-\lambda_k t}\, \langle f\,,\, f_k\rangle\, f_k \,\Big\|^2_\infty \;\le\; \sum_{k\ge 1} e^{-2\lambda_k t} \sum_{k\ge 1} \langle f\,,\, f_k\rangle^2 \;=\; C_0(t)\, \|\, f\,\|^2_2$$

for some finite constant $C_0(t)$.

Lemma A.2. *There exists a finite constant C_0 such that*

$$\| P_t^{(R)} f - f \|_2 \le C_0 t^{1/3} \| f \|_{\mathcal{H}_R}$$

for all $t \ge 0$, $f \in \mathcal{H}_R$.

Proof: Since $(f_k : k \ge 1)$ is an orthonormal basis of $\mathcal{L}^2([0,1])$,

$$\| P_t^{(R)} f - f \|_2^2 = \sum_{k \ge 1} \big[e^{-\lambda_k t} - 1 \big]^2 \big| \langle f , f_k \rangle \big|^2 .$$

Fix $k_0 \ge 1$. Since the sequence λ_k increases and $| \exp\{-\lambda_k t\} - 1 | \le 1$, the right-hand side can be bounded by

$$\big[e^{-\lambda_{k_0} t} - 1 \big]^2 \sum_{k=1}^{k_0 - 1} \big| \langle f , f_k \rangle \big|^2 + \frac{1}{\lambda_{k_0}} \sum_{k \ge k_0} \lambda_k \big| \langle f , f_k \rangle \big|^2$$

for all $k_0 \ge 1$. The first sum is bounded by $\| f \|_2^2$. In view of (A.10), the second one is bounded by $\| f \|_{\mathcal{H}_R}^2$ so that

$$\| P_t^{(R)} f - f \|_2^2 \le \big[1 - e^{-\lambda_{k_0} t} \big]^2 \| f \|_2^2 + \frac{1}{\lambda_{k_0}} \| f \|_{\mathcal{H}_R}^2 .$$

As $1 - e^{-x} \le x$, $x > 0$, and since, by (A.9), $\| f \|_2 \le C_0 \| f \|_{\mathcal{H}_R}$ for some finite constant C_0,

$$\| P_t^{(R)} f - f \|_2^2 \le \Big\{ C_0 \, (\lambda_{k_0} t)^2 + \frac{1}{\lambda_{k_0}} \Big\} \| f \|_{\mathcal{H}_R}^2 .$$

To complete the proof, it remains to choose k_0 such that $\lambda_{k_0}^{-3} \sim t^2$. □

Lemma A.3. *There exists a finite constant C_0 such that*

$$\| P_t^{(R)} f - f \|_\infty \le C_0 t^{1/5} \| f \|_{\mathcal{H}_R}$$

for all $t \ge 0$, $f \in C([0,1]) \cap \mathcal{H}_R$.

Proof: Fix $x \in [0,1]$. Since f is continuous, by (A.13) and (A.5),

$$\big\{ P_t^{(R)} f(x) - f(x) \big\}^2 \le C_0^2 \Big(\sum_{k \ge 1} \big[1 - e^{-\lambda_k t} \big] \big| \langle f , f_k \rangle \big| \Big)^2$$

for some finite constant C_0. By Schwarz inequality and (A.10), the right-hand side is bounded by

$$C_0^2 \sum_{k \ge 1} \frac{1}{\lambda_k} \big[1 - e^{-\lambda_k t} \big]^2 \sum_{k \ge 1} \lambda_k \big| \langle f , f_k \rangle \big|^2 = C_0^2 \sum_{k \ge 1} \frac{1}{\lambda_k} \big[1 - e^{-\lambda_k t} \big]^2 \| f \|_{\mathcal{H}_R}^2 .$$

It remains to estimate the sum. Fix $k_0 \ge 1$. Since the sequence λ_k increases, as $1 - e^{-x} \le x$, $x > 0$, by (A.3), the sum is less than or equal to

$$C \big[1 - e^{-\lambda_{k_0} t} \big]^2 + \sum_{k \ge k_0} \frac{1}{\lambda_k} \le C \Big\{ (k_0^2 t)^2 + \frac{1}{k_0} \Big\}$$

for some finite constant C. It remains to choose k_0 such that $k_0^5 \sim t^{-2}$. □

Appendix B. Initial-value problems with Robin boundary conditions

We present in this section some result on the initial-boundary value problems (2.6), (7.2). Recall the definition of the Sobolev space $\mathcal{H}^1$ introduced in (3.4). Fix a function $\phi \in \mathcal{L}^2([0,1])$, and consider the initial-boundary problem

$$\begin{cases} \partial_t u \;=\; \Delta u \\ (\nabla u)(t,0) \;=\; A^{-1}\, u(t,0) \\ (\nabla u)(t,1) \;=\; -\,B^{-1}\, u(t,1) \\ u(0,\cdot) \;=\; \phi(\cdot)\,. \end{cases} \tag{B.1}$$

Definition B.1. A function u in $\mathcal{L}^2(0,T;\mathcal{H}^1)$ is said to be a generalized solution in the cylinder $[0,T]\times[0,1]$ of the equation (B.1) if

$$\begin{aligned} &\int_0^1 u_t\, H_t\, dx \;-\; \int_0^1 \phi\, H_0\, dx \;-\; \int_0^t ds \int_0^1 u_s\, \partial_s H_s\, dx \\ &\quad = \;-\int_0^t ds \int_0^1 \nabla u_s\, \nabla H_s\, dx \;-\; \int_0^t \Big\{ \frac{1}{B}\, u_s(1)\, H_s(1) \;+\; \frac{1}{A}\, u_s(0)\, H_s(0) \Big\}\, ds \end{aligned}$$

for every $0 < t \le T$, function H in $C^{1,2}([0,T]\times[0,1])$.

Next result is proved in Baldasso et al. (2017). We present it here in sake of completeness.

Theorem B.2. *For each $\phi \in \mathcal{L}^2([0,1])$, there exists one and only one generalized solution to (B.1). The solution is smooth in $(0,\infty)\times[0,1]$ and can be represented as $u(t,x) = (P_t^{(R)}\phi)(x)$, where $P_t^{(R)}$ is the semigroup associated to the Robin Laplacian. Moreover,*

$$\min\{\,0\,,\, \operatorname{ess\,inf} \phi\,\} \;\le\; u(t,x) \;\le\; \max\{\,0\,,\, \operatorname{ess\,sup} \phi\,\} \tag{B.2}$$

for all $(t,x) \in \mathbb{R}_+ \times [0,1]$. Finally, if $\phi(x) \le b$ for some $b > 0$, then, for each $t_0 > 0$ there exists $\varepsilon > 0$ such that $u(t,x) \le b - \varepsilon$ for all $(t,x) \in [t_0,\infty)\times[0,1]$. Analogously, if $\phi(x) \ge a$ for some $a < 0$, then, for each $t_0 > 0$ there exists $\varepsilon > 0$ such that $u(t,x) \ge a + \varepsilon$ for all $(t,x) \in [t_0,\infty)\times[0,1]$.

Proof: Existence and uniqueness of generalized solutions, as well as their representation in terms of the semigroup $P_t^{(R)}$ is the content of Theorems 1 and 3 in Mikhaĭlov (1983, Section VI.2).

We turn to (B.2). Assume first that ϕ belongs to $\mathcal{H}^1$. By (A.9), $\phi \in \mathcal{H}_R$, and, by Lemma A.3, $u(t)$ converges to ϕ in $\mathcal{L}^\infty([0,1])$ as $t \to 0$. Since the solution is smooth in $(0,\infty)\times[0,1]$, by the maximum principle stated in Theorems 2 and 3 of Protter and Weinberger (1984, Chapter 3),

$$\min\{\,0\,,\, \inf_{0\le y\le 1} u(t_0,y)\,\} \;\le\; u(t,x) \;\le\; \max\{\,0\,,\, \sup_{0\le y\le 1} u(t_0,y)\,\}$$

for all $(t,x) \in [t_0,\infty)\times[0,1]$. Letting $t_0 \to 0$, as $u(t_0)$ converges to ϕ in $\mathcal{L}^\infty([0,1])$, yields (B.2).

To extend this result to $\phi \in \mathcal{L}^2([0,1])$, we consider a sequence $\phi_n \in \mathcal{H}^1$ which converges to ϕ in $\mathcal{L}^2([0,1])$ and such that $\operatorname{ess\,inf}\phi \,\le\, \phi_n(x) \,\le\, \operatorname{ess\,sup}\phi$ for all $0 \le x \le 1$. Denote by u^n

the solution of (B.1) with initial condition ϕ_n. Fix $t > 0$. By the result for initial conditions in $\mathcal{H}^1$,

$$\min\{\,0\,,\,\operatorname{ess\,inf}\phi\,\}\;\le\;\min\{\,0\,,\,\inf_{0\le y\le 1}\phi_n(y)\,\}$$
$$\le\; u^n(t,x)\;\le\;\max\{\,0\,,\,\sup_{0\le y\le 1}\phi_n(y)\,\}\;\le\;\max\{\,0\,,\,\operatorname{ess\,sup}\phi\,\}\;.$$

for all $0 \le x \le 1$. By (A.16), $u^n(t)$ converges to $u(t)$ in $\mathcal{L}^\infty([0,1])$. This completes the proof of (B.2).

Assume that $\phi(x) \le b$ for some $b > 0$. By (B.2), $u(t,x) \le b$ for all $t \ge 0$, $0 \le x \le 1$. Fix $t_0 > 0$, and assume that $\max_{0\le x\le 1} u(t_0,x) = b$. As $b > 0$, the boundary conditions imply that the maximum cannot be attained at the boundary. On the other hand, if it is attained at the interior, by Theorem 2 of Protter and Weinberger (1984, Chapter 3) and by the smoothness of the solution, $u(t,x) = b$ for all $(t,x) \in (0,t_0] \times [0,1]$. This is not possible at the boundary. Therefore, $\max_{0\le x\le 1} u(t_0,x) < b$. By the maximum principle, this bound can be extended to all $(t,x) \in [t_0,\infty) \times [0,1]$. The same argument applies to the lower bound. □

Let $\bar\rho \in \mathcal{M}_{\rm ac}$ be the unique stationary solution of the equation (2.6). That, is $\bar\rho$ is the solution of the elliptic equation

$$\begin{cases} \Delta\rho \;=\; 0 \\ (\nabla\rho)(0) \;=\; A^{-1}[\,\rho(0)-\alpha\,] \\ (\nabla\rho)(1) \;=\; B^{-1}[\,\beta-\rho(1)]\;. \end{cases} \tag{B.3}$$

An elementary computation yields that $\bar\rho$ is given by

$$\bar\rho(x) \;=\; \frac{\alpha(1+B)+\beta A}{1+B+A} \;+\; \frac{(\beta-\alpha)\,x}{1+B+A}\;.$$

Note that $\bar\rho$ is the linear interpolation between $\bar\rho(-A) = \alpha$ and $\bar\rho(1+B) = \beta$.

Definition B.3. Fix $\gamma : [0,1] \to [0,1]$. A function u in $\mathcal{L}^2(0,T;\mathcal{H}^1)$ is said to be a generalized solution in the cylinder $[0,T]\times[0,1]$ of the equation (2.6) if $u(t,x)-\bar\rho$ is a generalized solution of the initial-boundary problem (B.1) with initial condition $\gamma - \bar\rho$.

Therefore, a function u in $\mathcal{L}^2(0,T;\mathcal{H}^1)$ is a generalized solution in the cylinder $[0,T]\times[0,1]$ of the equation (2.6) if

$$\begin{aligned}\int_0^1 u_t\,H_t\,dx \;-\; \int_0^1 \gamma\,H_0\,dx \;-\; \int_0^t ds\int_0^1 u_s\,\partial_s H_s\,dx = \;-\int_0^t ds\int_0^1 \nabla u_s\,\nabla H_s\,dx \\ -\;\int_0^t \Big\{\,\frac{1}{B}\,[\,u_s(1)-\beta\,]\,H_s(1) \;+\; \frac{1}{A}\,[\,u_s(0)-\alpha\,]\,H_s(0)\,\Big\}\,ds\end{aligned}$$

for every $0 < t \le T$ and function H in $C^{1,2}([0,T]\times[0,1])$.

Theorem B.4. *Fix a measurable density profile $\gamma : [0,1] \to [0,1]$. There exists a unique generalized solution of (2.6). The solution is smooth in $(0,T]\times[0,1]$ and satisfies the bounds*

$$\min\{\,\alpha\,,\,\operatorname{ess\,inf}\gamma\,\}\;\le\;u(t,x)\;\le\;\max\{\,\beta\,,\,\operatorname{ess\,sup}\gamma\,\} \tag{B.4}$$

for all $(t,x) \in [0,T]\times[0,1]$. Moreover, for all $0 < t_0 \le T$ there exists $\varepsilon > 0$ such that $\varepsilon \le u(t,x) \le 1-\varepsilon$ for all $(t,x) \in [t_0,T]\times[0,1]$.

Proof: The proof of this result is similar to the one of Theorem B.2. □

Fix $\gamma : [0,1] \to [0,1]$, and denote by $u^{(\gamma)}$ the unique weak solution of (2.6) with initial condition γ. Let $F_0\colon [0,1] \to \mathbb{R}$ be given by $F_0(r) = r \log r + (1-r)\log(1-r)$.

Lemma B.5. *There exists a finite constant C_0, which depends only on α, β, A, B such that*

$$\int_0^t ds \int_0^1 \frac{(\nabla u_s)^2}{\sigma(u_s)}\, dx + \frac{1}{A}\int_0^t \Big| [\, u_s(0) - \alpha\,] \log \frac{u_s(0)}{1-u_s(0)} \Big| \, ds \\ + \frac{1}{B}\int_0^t \Big| [\, u_s(1) - \beta\,] \log \frac{u_s(1)}{1-u_s(1)} \Big| \, ds \;\le\; C_0\, t \;+\; \int_0^1 F_0(\gamma)\, dx \;-\; \int_0^1 F_0(u_t)\, dx$$

for all $t>0$ and all $\gamma : [0,1] \to [0,1]$.

Proof: Fix $F \in C^2([0,1])$, an initial profile $\gamma : [0,1] \to [0,1]$, and denote by u the solution of (2.6). Since u is smooth on $(0,\infty)\times[0,1]$, integrating by parts and in view of the boundary conditions, for all $0<\delta<t<\infty$,

$$\int_0^1 F(u_t)\, dx \;-\; \int_0^1 F(u_\delta)\, dx \;=\; -\int_\delta^t ds \int_0^1 F''(u_s)\, (\nabla u_s)^2\, dx \\ -\int_\delta^t \frac{1}{A}\, [\, u_s(0)-\alpha\,]\, F'(u_s(0))\, ds \;-\; \int_\delta^t \frac{1}{B}\, [\, u_s(1)-\beta\,]\, F'(u_s(1))\, ds\,.$$

As u_δ converges to γ in $\mathcal{L}^2([0,1])$, letting $\delta\to 0$ yields that for all $t>0$,

$$\int_0^t ds \int_0^1 F''(u_s)\, (\nabla u_s)^2\, dx + \int_0^t \frac{1}{A}\, [\, u_s(0)-\alpha\,]\, F'(u_s(0))\, ds \\ + \int_0^t \frac{1}{B}\, [\, u_s(1)-\beta\,]\, F'(u_s(1))\, ds \;=\; \int_0^1 F(\gamma)\, dx \;-\; \int_0^1 F(u_t)\, dx\,.$$

Since for each $t>0$, there exists $\varepsilon>0$ such that $\varepsilon \le u(s,x) \le 1-\varepsilon$ for all $(s,x)\in[t,\infty)\times[0,1]$, the previous argument can be applied to the function F_0 introduced just before the statement of the lemma. It yields that

$$\int_0^t ds \int_0^1 \frac{(\nabla u_s)^2}{\sigma(u_s)}\, dx + \int_0^t \frac{1}{A}\, [\, u_s(0)-\alpha\,] \log \frac{u_s(0)}{1-u_s(0)}\, ds \\ + \int_0^t \frac{1}{B}\, [\, u_s(1)-\beta\,] \log \frac{u_s(1)}{1-u_s(1)}\, ds \;=\; \int_0^1 F_0(\gamma)\, dx \;-\; \int_0^1 F_0(u_t)\, dx \tag{B.5}$$

for all $t>0$. Clearly, for each $\varrho>0$, the function $f_\varrho : (0,1)\to\mathbb{R}$ defined by $f_\varrho(r) = [\, r-\varrho\,] \log[\, r/(1-r)\,]$ is bounded below by a finite constant, say $-c_1(\varrho)<0$. Hence, $|\, f_\varrho(r)\,| \le f_\varrho(r) + 2c_1$. Therefore, there exists a finite constant $C_0 = C_0(A,B,\alpha,\beta)$ such that

$$\int_0^t ds \int_0^1 \frac{(\nabla u_s)^2}{\sigma(u_s)}\, dx + \frac{1}{A}\int_0^t \Big| [\, u_s(0) - \alpha\,] \log \frac{u_s(0)}{1-u_s(0)} \Big| \, ds \\ + \frac{1}{B}\int_0^t \Big| [\, u_s(1) - \beta\,] \log \frac{u_s(1)}{1-u_s(1)} \Big| \, ds \;\le\; C_0\, t \;+\; \int_0^1 F_0(\gamma)\, dx \;-\; \int_0^1 F_0(u_t)\, dx$$

for all $t>0$, as claimed. □

As u_t converges to γ in $\mathcal{L}^2([0,1])$, letting $t \to 0$ in the previous lemma yields that

$$\lim_{t\to 0} \Big\{ \int_0^t ds \int_0^1 \frac{(\nabla u_s)^2}{\sigma(u_s)}\, dx + \frac{1}{A}\int_0^t \Big| [\, u_s(0) - \alpha\,] \log \frac{u_s(0)}{1-u_s(0)} \Big| ds \\ + \frac{1}{B}\int_0^t \Big| [\, u_s(1) - \beta\,] \log \frac{u_s(1)}{1-u_s(1)} \Big| ds \Big\} \;=\; 0\,. \tag{B.6}$$

Definition B.6. Fix a measurable density profile $\gamma : [0,1] \to [0,1]$ and a function H in $C^{0,1}([0,T]\times[0,1])$. A function u in $\mathcal{L}^2(0,T;\mathcal{H}^1)$ such that $0 \le u \le 1$ a.e. is said to be a generalized solution in the cylinder $[0,T]\times[0,1]$ of the equation (7.2) if

$$\begin{aligned} &\int_0^1 u_t\, G_t\, dx \;-\; \int_0^1 \gamma\, G_0\, dx \;-\; \int_0^t ds \int_0^1 u_s\, \partial_s G_s\, dx \\ &\quad = \int_0^t ds \int_0^1 \big\{ -\, \nabla u_s\, \nabla G_s \,+\, 2\,\sigma(u_s)\, \nabla H_s\, \nabla G_s \big\}\, dx \\ &\quad + \int_0^t \big\{ \mathfrak{p}_{\beta,B}(u_s(1), H_s(1))\, G_s(1) \,+\, \mathfrak{p}_{\alpha,A}(u_s(0), H_s(0))\, G_s(0) \big\}\, ds \end{aligned} \tag{B.7}$$

for every $0 < t \le T$ and function G in $C^{1,2}([0,T]\times[0,1])$.

Theorem B.7. *Fix $\gamma : [0,1] \to [0,1]$ and H in $C^{0,1}([0,T]\times[0,1])$. There exists a unique weak solution of (7.2).*

Proof: Existence follows from the hydrodynamic limit of the weakly asymmetric process with Robin boundary conditions introduced in Section 7. Uniqueness is based on the energy estimate. Fix two initial conditions $\gamma^{(1)}$, $\gamma^{(2)}$, and denote by $u^{(1)}$, $u^{(2)}$ two weak solutions of (7.2) with initial conditions $\gamma^{(1)}$, $\gamma^{(2)}$, respectively.

Before presenting a rigorous argument we provide an heuristic one. Approximate $w = u^{(2)} - u^{(1)}$ by a sequence of functions G in $C^{1,2}([0,T]\times[0,1])$. By (B.7), for all $0 < t \le T$,

$$\begin{aligned} &\frac12 \int_0^1 w_t^2\, dx \;-\; \frac12 \int_0^1 [\,\gamma^{(2)} - \gamma^{(1)}\,]^2\, dx \;+\; \int_0^t ds \int_0^1 (\nabla w_s)^2\, dx \\ &\quad = \; 2 \int_0^t ds \int_0^1 [\,\sigma(u_s^{(2)}) - \sigma(u_s^{(1)})\,]\, \nabla H_s\, \nabla w_s\, dx \\ &\quad - \int_0^t \Big\{ w_s(1)^2\, \overline{\mathfrak{p}}_{\beta,B}(H_s(1)) \,+\, w_s(0)^2\, \overline{\mathfrak{p}}_{\alpha,A}(H_s(0)) \Big\}\, ds\,, \end{aligned} \tag{B.8}$$

where $\overline{\mathfrak{p}}_{\varrho,D}(M) \,=\, D^{-1}\{\, \varrho\, e^{M} + (1-\varrho)e^{-M}\,\}$. As $\overline{\mathfrak{p}}_{\varrho,D}(M) \ge 0$, the last integral is negative. Therefore, by Young's inequality $2xy \le ax^2 + a^{-1}x^2$, $a > 0$, and since ∇H is uniformly bounded and σ Lipschitz continuous,

$$\begin{aligned} &\frac12 \int_0^1 w_t^2\, dx \;-\; \frac12 \int_0^1 [\,\gamma^{(2)} - \gamma^{(1)}\,]^2\, dx \;+\; \frac12 \int_0^t ds \int_0^1 (\nabla w_s)^2\, dx \\ &\quad \le \; C_0(H) \int_0^t ds \int_0^1 w_s^2\, dx\,. \end{aligned}$$

for some finite constant $C_0(H)$ which depends on H. It remains to apply Gronwal's inequality to conclude that

$$\int_0^1 w_t^2\, dx \;\le\; e^{C_0 t} \int_0^1 [\,\gamma^{(2)} - \gamma^{(1)}\,]^2\, dx\,,$$

which yields uniqueness.

We turn to a rigorous proof. Recall the notation introduced in the proof of Lemma 3.3. Fix a smooth function $F:\mathbb{R}\to\mathbb{R}$ and recall that $w=u^{(2)}-u^{(1)}$. As $w^{\varepsilon,\delta}$ is a smooth function, for $0<t\le T$,

$$\langle\, F(w^{\varepsilon,\delta}_t)\,\rangle \;-\; \langle\, F(w^{\varepsilon,\delta}_0)\,\rangle \;=\; \int_0^t ds \int_0^1 F'(w^{\varepsilon,\delta}_s)\, \partial_s w^{\varepsilon,\delta}_s \, dx \;.$$

Integrating by parts, the right-hand side becomes

$$\int_0^1 w^{\varepsilon,\delta}_t \, F'(w^{\varepsilon,\delta}_t)\; dx \;-\; \int_0^1 w^{\varepsilon,\delta}_0 \, F'(w^{\varepsilon,\delta}_0)\; dx \;-\; \int_0^t ds \int_0^1 w^{\varepsilon,\delta}_s\, \partial_s F'(w^{\varepsilon,\delta}_s)\, dx \;.$$

By Lemma 3.3, actually its proof since we changed the definition of w^ε, this expression is equal to

$$\int_0^1 w_t \, F'(w^{\varepsilon,\delta}_t)^{\varepsilon,\delta}\; dx \;-\; \int_0^1 w_0 \, F'(w^{\varepsilon,\delta}_0)^{\varepsilon,\delta}\; dx \;-\; \int_0^t ds \int_0^1 w_s\, \partial_s [F'(w^{\varepsilon,\delta}_s)^{\varepsilon,\delta}]\, dx \;+\; R_{\varepsilon,\delta}\;, \tag{B.9}$$

where for all $\varepsilon>0$, $\lim_{\delta\to 0} R_{\varepsilon,\delta}=0$.

Take $F(a)=(1/2)\,a^2$. Let $\phi^{(2)}$ be the convolution of ϕ with itself:

$$\phi^{(2)}(t) \;=\; \int_{\mathbb{R}} \phi(t-s)\, \phi(s)\; ds\;,$$

and set $\phi^{(2)}_\delta(t)=\delta^{-1}\,\phi^{(2)}(t/\delta)$. Since $P^{(R)}_t$ is a semigroup and since $P^{(R)}_t$ comutes with the time convolution, for any function $f\in\mathcal{L}^2([0,T]\times[0,1])$,

$$(f^{\varepsilon,\delta})^{\varepsilon,\delta}(t,x) \;=\; \int_{\mathbb{R}} [\, P^{(R)}_{2\varepsilon} f(t+s)\,](x)\, \phi^{(2)}_\delta(s)\; ds\;.$$

Therefore, the first three terms of (B.9) are equal to

$$\int_0^1 w_t\, w^{2\varepsilon,\delta}_t \; dx \;-\; \int_0^1 w_0\, w^{2\varepsilon,\delta}_0 \; dx \;-\; \int_0^t ds \int_0^1 w_s\, \partial_s w^{2\varepsilon,\delta}_s \, dx\;,$$

with the convention, starting from this equation and up to the end of the proof, that the superscript δ represent now convolution with $\phi^{(2)}_\delta$ instead of ϕ_δ.

By (B.7), this sum is equal to

$$\begin{aligned}&\int_0^t ds \int_0^1 \big\{ -\nabla w_s\, \nabla w^{2\varepsilon,\delta}_s \;+\; 2\,\{\, \sigma(u^{(2)}_s) - \sigma(u^{(1)}_s)\,\}\, \nabla H_s\, \nabla w^{2\varepsilon,\delta}_s \,\big\}\; dx \\ &\quad -\; \int_0^t \big\{\, \overline{\mathfrak{p}}_{\beta,B}(H_s(1))\, w_s(1)\, w^{2\varepsilon,\delta}_s(1) \;+\; \overline{\mathfrak{p}}_{\alpha,A}(H_s(0))\, w_s(0)\, w^{2\varepsilon,\delta}_s(0) \;\big\}\; ds\;,\end{aligned}$$

where $\overline{\mathfrak{p}}_{\varrho,D}(M)$ has been introduced in (B.8). By (A.16), (A.9), for each $\varepsilon>0$, $\nabla w^{2\varepsilon}$ belongs to $\mathcal{L}^2([0,T]\times[0,1])$. Therefore, as $\delta\to 0$, $\nabla w^{2\varepsilon,\delta}=(\nabla w^{2\varepsilon})^\delta\to\nabla w^{2\varepsilon}$ in $\mathcal{L}^2([0,T]\times[0,1])$.

On the other hand, by (A.7) and (A.9),

$$|\, w^{2\varepsilon,\delta}_t(1) \;-\; w^{2\varepsilon}_t(1)\,|^2 \;\le\; C_0\, \|\, w^{2\varepsilon,\delta}_t \;-\; w^{2\varepsilon}_t\,\|^2_{\mathcal{H}^1}$$

for some finite constant C_0 independent of ε and t. A similar inequality holds at $x=0$. Therefore, as $\nabla w^{2\varepsilon,\delta}\to\nabla w^{2\varepsilon}$ in $\mathcal{L}^2([0,T]\times[0,1])$ as $\delta\to 0$, $w^{2\varepsilon,\delta}_t(1)\to w^{2\varepsilon}_t(1)$ in $\mathcal{L}^2([0,T])$ as

$\delta \to 0$. In conclusion, letting $\delta \to 0$, the sum appearing in the penultimate displayed equation converges to

$$\int_0^t ds \int_0^1 \Big\{ -\nabla w_s \nabla w_s^{2\varepsilon} + 2\,\{\,\sigma(u_s^{(2)}) - \sigma(u_s^{(1)})\,\}\,\nabla H_s\,\nabla w_s^{2\varepsilon} \Big\}\,dx$$
$$-\int_0^t \Big\{ \overline{\mathfrak{p}}_{\beta,B}(H_s(1))\,w_s(1)\,w_s^{2\varepsilon}(1) + \overline{\mathfrak{p}}_{\alpha,A}(H_s(0))\,w_s(0)\,w_s^{2\varepsilon}(0) \Big\}\,ds\;.$$

By the first assertion of Lemma 3.2, as $\varepsilon \to 0$, $\nabla w^{2\varepsilon}$ converges to ∇w in $\mathcal{L}^2([0,T]\times[0,1])$. Therefore, as $\varepsilon \to 0$, the first line converges to

$$\int_0^t ds \int_0^1 \Big\{ -\nabla w_s \nabla w_s + 2\,[\,\sigma(u_s^{(2)}) - \sigma(u_s^{(1)})\,]\,\nabla H_s\,\nabla w_s \Big\}\,dx\;.$$

On the other hand, as $w \in \mathcal{L}^2(0,T;\mathcal{H}^1)$, by Lemma A.3, $w_t^\varepsilon(1) \to w_t(1)$ in $\mathcal{L}^2([0,T])$. Hence, by the dominated convergence theorem, as $\varepsilon \to 0$, the second line converges to

$$-\int_0^t \Big\{ \overline{\mathfrak{p}}_{\beta,B}(H_s(1))\,w_s(1)^2 + \overline{\mathfrak{p}}_{\alpha,A}(H_s(0))\,w_s(0)^2 \Big\}\,ds\;.$$

This proves that equation (B.8) is in force and completes the proof of the theorem. □

We conclude this section with a heat equation with mixed boundary equations. Fix a function $\phi \in \mathcal{L}^2([0,1])$, and consider the initial-boundary problem

$$\begin{cases} \partial_t u \,=\, \Delta u \\ (\Delta u)(t,0) \,=\, A^{-1}\,\nabla u(t,0) \\ (\Delta u)(t,1) \,=\, -\,B^{-1}\,\nabla u(t,1) \\ u(0,\cdot) \,=\, \phi(\cdot)\;. \end{cases} \tag{B.10}$$

One can define generalized solutions of this problem as in Definition B.1 and prove existence and uniqueness as stated in Theorem B.2. The solution can be represented as $u_t = P_t^{(M)}\phi$, where $(P_t^{(M)} : t \geq 0)$ represents the semigroup associated to the Laplacian with boundary conditions

$$(\Delta f)(0) \,=\, A^{-1}\,(\nabla f)(0)\;, \qquad (\Delta f)(1) \,=\, -\,B^{-1}(\nabla f)(1)\;.$$

Denote this operator by Δ_M. An elementary computation shows that the eigenvalues of Δ_M coincide with those of Δ_R.

We claim that for all $s \geq 0$ and function f in $C^1([0,1])$,

$$\nabla P_s^{(R)} f \,=\, P_s^{(M)} \nabla f\;. \tag{B.11}$$

To check this identity, fix f in $C^1([0,1])$, and let $u_s := P_s^{(R)} f$. Clearly u_s is the solution of (B.1) with initial condition $u_0 = f$. Let $v_s := \nabla u_s$, Then, v_s solves (B.10) initial condition $v_0 = \nabla f$. Hence, v_s can be represented as $v_s = P_s^{(M)}\nabla f$, that is, $P_s^{(M)}\nabla f = v_s = \nabla u_s = \nabla P_s^{(R)} f$, as claimed.

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