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Undirected Polymers in Random Environment: path properties in the mean field limit.

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Abstract. We consider the problem of undirected polymers (tied at the endpoints) in random environment, also known as the unoriented first passage percolation on the hypercube, in the limit of large dimensions. By means of the multiscale refinement of the second moment method we obtain a fairly precise geometrical description of optimal paths, i.e. of polymers with minimal energy. We give the distribution of the backsteps as a function of the Hamming distance from the origin. The picture which emerges can be loosely summarized as follows. The energy of the polymer is, to first approximation, uniformly spread along the strand. The polymer's bonds carry however a lower energy than in the directed setting, and are reached through the following geometrical evolution. Close to the origin, the polymer proceeds in oriented fashion – it is thus as stretched as possible. The tension of the strand decreases however gradually, with the polymer allowing for more and more backsteps as it enters the core of the hypercube. Backsteps, although increasing the length of the strand, allow the polymer to connect reservoirs of energetically favorable edges which are otherwise unattainable in a fully directed regime. These reservoirs lie at mesoscopic distance apart, but in virtue of the high dimensional nature of the ambient space, the polymer manages to connect them through approximate geodesics with respect to the Hamming metric: this is the key strategy which leads to an optimal energy/entropy balance. Around halfway, the mirror picture sets in: the polymer tension gradually builds up again, until full orientedness close to the endpoint. The approach yields, as a corollary, a constructive proof of the result by Martinsson [*Ann. Appl. Prob.* **26** (2016), *Ann. Prob.* **46** (2018)] concerning the leading order of the ground state (or first passage percolation).

In memory of Dima Ioffe.

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Contents

1. Introduction

We denote by $G_n = (V_n, E_n)$ the n-dimensional hypercube. $V_n = \{0,1\}^n$ is thus the set of vertices, and E_n the set of edges connecting nearest neighbours. We write $\mathbf{0} = (0,0,\dots,0)$ and $\mathbf{1} = (1,1,\dots,1)$ for diametrically opposite vertices. For $l \in \mathbb{Z}_+$ we let

$$\widetilde{\Pi}_{n,l} \equiv \text{the set of polymers, i.e. paths from } \mathbf{0} \text{ to } \mathbf{1} \text{ of length } l\,,$$

as well as

$$\widetilde{\Pi}_n \equiv \bigcup_{l=1}^{\infty} \widetilde{\Pi}_{n,l}.$$

For $\pi \in \tilde{\Pi}_n$ a polymer going through two vertices $\boldsymbol{v}, \boldsymbol{w}$ of the hypercube, we denote by $l_\pi(\boldsymbol{v}, \boldsymbol{w})$ the length of the connecting substrand, also shortening $l_\pi \equiv l_\pi(\mathbf{0}, \mathbf{1})$.

Every edge of the n-hypercube is parallel to some unit vector $e_j \in \mathbb{R}^n$, where e_j connects

$$(0,\dots,0) \text{ and } (0,\dots,0,\underbrace{1}_{j^{th}-\text{coordinate}},0,\dots,0)\,.$$

We write $e_{-j} \equiv -e_j$. The quantity $\pi_j \in \{1,n\} \cup \{-1,-n\}$ then specifies the direction of a π-path at step j. A *forward step* occurs if $\pi_j \in \{1,n\}$; if $\pi_j \in \{-1,-n\}$ we refer to this as a *backstep*.

Remark that the endpoint of the (sub)path $\pi_1\pi_2\dots\pi_i$ coincides with the vertex given by $\sum_{j\le i} e_{\pi_j}$. The edge traversed in the j-th step by the π-path will be denoted $[\pi]_j$.

To each edge we attach independent, standard (mean one) exponential random variables ξ, the random environment, and assign to a polymer $\pi \in \widetilde{\Pi}_{n,l}$ its *weight/energy* according to

$$X_\pi \equiv \sum_{j=1}^{l} \xi_{[\pi]_j}.$$

The question we wish to address concerns the ground state of undirected polymers in random environment[1], to wit:

$$m_n \equiv \min_{\pi \in \widetilde{\Pi}_n} X_\pi, \tag{1.1}$$

in the mean field limit $n \uparrow \infty$, and the statistical/geometrical properties of optimal paths.

A first remark is in place: since polymers with loops cannot achieve the ground state (their energy can always be reduced by removing the loops), we will henceforth focus on the set of *loopless* paths of length $l \in \mathbb{Z}_+$, denoted $\Pi_{n,l}$, and shortening, in full analogy,

$$\Pi_n \equiv \bigcup_{l=1}^{\infty} \Pi_{n,l},$$

for the set of all loopless paths.

Looplessness will be very useful: it guarantees, in particular, that the energy of a polymer of length, say, l, is indeed given by the sum of l independent standard exponentials. On the other hand, loopless paths are not necessarily directed, see Figure 1.1 for a graphical rendition.

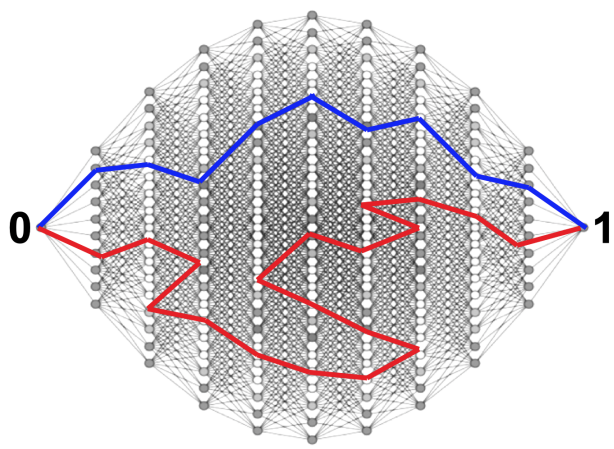

FIGURE 1.1. The 10-dim hypercube with two polymers. The blue polymer is *directed*: its length coincides with the dimension ($l = n = 10$), and it is thus as stretched as possible. The red polymer is *undirected*: it performs backsteps, which account for a lower "tension", and for the long excursions ($l = 20$).

It is clear that a major issue here will be that of *path counting*. For the hypercube, the following beautiful formula is available. We denote by $M_{n,l,d}$ the number of polymers of length l between two

[1]This problem also appears in the literature under the name of unoriented first passage percolation, FPP for short. In mathematical biology it bears relevance to the issue of fitness landscapes. in which case it is dubbed accessibility percolation, see Berestycki et al. (2016, 2017); Hegarty and Martinsson (2014); Martinsson (2016, 2018, 2015); Krug (2021); Hwang et al. (2018); Schmiegelt and Krug (2019) and references therein. We adopt here the polymer terminology since it is arguably more suitable to convey the type of results we obtain.

points at Hamming distance d, i.e. points thus disagree in exactly d coordinates. It then holds :

$$M_{n,l,d} = \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{d} \binom{d}{j} \binom{n-d}{i-j} (-1)^j (n-2i)^l \mathbb{1}_{j \leq i}. \tag{1.2}$$

(This formula concerns all paths of given length: loops, in particular, are also allowed). A proof of this formula, which relies on the classical approach via adjancency matrices, can be found in the monograph by Stanley (2013). We will refer to (1.2) as *Stanley's formula*.

No less remarkable is the following *Stanley's identity*, relating $M_{n,l,d}$ to hyperbolic functions. For $x \in \mathbb{R}$, it holds:

$$\sum_{l=0}^{\infty} M_{n,l,d} \frac{x^l}{l!} = \sinh(x)^d \cosh(x)^{n-d} . \tag{1.3}$$

Assuming the validity of (1.2), the proof of (1.3) only requires the binomial theorem and elementary Taylor expansions: it will be given in the Appendix for completeness. The formula (1.2) and (1.3) appeared in relation to the first passage percolation on the hypercube in Berestycki et al. (2017), Durrett (1988) and Li (2018). Lightening notations further by setting $M_{n,l} \equiv M_{n,l,n}$ for the number of polymers of length l between two opposite vertices on the hypercube, it thus follows from (1.3) that

$$\sum_{l=0}^{\infty} M_{n,l} \frac{x^l}{l!} = \sinh(x)^n . \tag{1.4}$$

This relation will allow for precise asymptotical analysis. Before seeing a first, key application, we shall recall yet another technical input concerning *tail estimates* for the distribution of the sum of independent standard exponentials as appearing in the problem at hand: denoting by $\{\xi_i\}_{i \in \mathbb{Z}_+}$ a family of such random variables and with $X_l \equiv \sum_{i \leq l} \xi_i$, it then holds:

$$\mathbb{P}\left(X_l \leq x\right) = \left(1 + K(x,l)\right) \frac{e^{-x} x^l}{l!}, \tag{1.5}$$

for $x > 0$, and with $0 \leq K(x,l) \leq e^x x/(l+1)$. (The proof is truly elementary, but see e.g. Kistler et al. (2020a, Lemma 5) for details).

Some notational convention: for $a_n, b_n \geq 0$ we write $a_n \lesssim b_n$ if $a_n \leq C b_n$ for some numerical constant $C > 0$ and $a_n \propto b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

Armed with Stanley's formula and the tail estimates, we are now ready to make the aforementioned key observation concerning the ground state of undirected polymers: denoting by $N_{n,l,x} \equiv \#\{\pi \in \Pi_{n,l}, X_\pi \leq x\}$ the number of polymers of length l and energies at most x, by union bounds and Markov inequality we have

$$\mathbb{P}\left(m_n \leq x\right) = \mathbb{P}\left(\cup_{l=0}^{\infty} \{N_{n,l,x} \geq 1\}\right) \leq \sum_{l=0}^{\infty} \mathbb{E}(N_{n,l,x}). \tag{1.6}$$

Remark that we are considering polymers with no loops, in which case the energies are indeed sums of l independent random variables. Furthermore, it clearly holds that $\#\Pi_{n,l} \leq M_{n,l}$, since allowing loops can only increase the cardinality[2]. All in all, we have

$$\mathbb{E}(N_{n,l,x}) \leq M_{n,l} \mathbb{P}\left(X_l \leq x\right) \lesssim M_{n,l} \frac{x^l}{l!} , \tag{1.7}$$

the second inequality by the tail estimates.

[2]Here and henceforth we use Stanley's formula although we will be mostly considering loopless polymers: in hindsight, the error/overshooting will turn out to be negligible. This is course due to the high dimensionality of the problem at hand.

Performing now the sum over all polymer-lengths in (1.6) and then using (1.3), we thus obtain

$$\mathbb{P}\left(m_n \leq x\right) \lesssim \sinh(x)^n \,. \tag{1.8}$$

The sinh-function is increasing, therefore, denoting by

$$\mathsf{E} \equiv \operatorname{arcsinh}(1) = \log(1+\sqrt{2}), \tag{1.9}$$

we deduce from (1.8), and the Borel-Cantelli lemma, a *lower bound* to the ground state, to wit:

$$\mathbb{P}\left(\lim_{n\to\infty} m_n \geq \mathsf{E}\right) = 1. \tag{1.10}$$

As it turns out, this bound is tight.

Martinsson's Theorem (Martinsson, 2016, 2018). *For undirected polymers on the hypercube, it holds*

$$\lim_{n\to\infty} m_n = \mathsf{E}, \tag{1.11}$$

in probability.

In other words, a "mean field trivialization" occurs in the limit of large dimensions, and the model of unoriented polymers in random environment thus falls in the so-called *REM class* Gayrard and Kistler (2015). Given the simple derivation of the lower bound, which eventually relies on the Markov inequality only, one is perhaps tempted to tackle the missing upper bound via the Second Moment Method [3]. This is however not the route taken by Martinsson who, in fact, has found *two* rather distinct proofs.

The historically first proof has appeared in Martinsson (2016). In that paper, Martinsson builds upon ideas of Durrett (1988) and work by Fill and Pemantle (1993), and settles the issue of the upper bound through a delicate comparison with the so-called Branching Translation Process, BTP for short. The BTP is a hierarchical model amenable to an explicit analysis and which, crucially, stochastically dominates the model of unoriented polymers. Note that Martinsson proved something stronger than (1.11) in Martinsson (2016): He proved (1.11) in L^p and he also proved that the subleading order correction is $O(\frac{1}{n})$ for exponential random variables.

In the second proof of the above theorem, Martinsson proceeds through some ingenious use of the FKG inequality, and (related) subadditivity/monotonicity properties of paths with optimal energies, see Martinsson (2018) for details.

Both proofs naturally come with their own strengths and weaknesses: the first one not only provides a solution of the problem at hand, but also insights into the structure of the BTP which are interesting in their own right, whereas the second proof settles the FPP on Cartesian power graphs, and thus applies in vast generality.

It seems however fair to say that, by their own nature, both approaches shed little light on the physical phenomena which eventually lead to the mean field trivialization. It is the purpose of this article to fill this gap by providing the distribution of the backsteps, and then obtain a precise geometrical description of optimal paths. As a corollary, we then obtain a third proof of the upper bound for the ground state, and hence of Martinsson's Theorem.

To this end, we will implement the *multiscale refinement of the second moment method* Gayrard and Kistler (2015), a tool which forces us to identify the mechanisms allowing polymers to reach minimal energies. (As will become clear in our treatment, the choice of an exponentially distributed random environment presents no loss of generality). Unfortunately, the formulation of our main result, Theorem 2 below, requires not a little infrastructure: this will be provided in the next Section 2. In order the justify (and de-mystify) some otherwise odd looking formulas, concepts, *etc.* we will proceed gradually, increasing the amount of details concerning the geometry of optimal paths through simple observations and elementary computations. The upshot of these findings will be

[3] Li used also a second moment in Li (2018) for accessibility percolation on the hypercube for particular paths.

recorded in the form of Insights. A cautionary note is here due. The computations underlying Insight 2.1-2.6 below are rigorous yet *per se* not necessarily conclusive: indeed, they all rely on the *existence* of paths with the established geometric properties, but this will be, in fact, the content of Theorem 2 itself.

Our new approach leads to a proof of Martinsson's theorem which is much longer than those already available. It does however yield a detailed geometrical description of optimal polymers, and this in turn opens a gateway towards the unsettled issue of fluctuations and weak limits.

2. Drawing the picture

As we have seen, a reasonable candidate for the ground state eventually follows from an application of the Markov inequality. Albeit crucial, the ground state encodes however only some limited information. Another fundamental quantity is of course the *length* of an optimal polymer: as it turns out, a simple computation, allows to make an educated guess.

2.1. *A candidate optimal length.* Due to the high dimensionality of the problem, in order to identify the optimal length it seems natural to analyze the asymptotics of $\mathbb{E}(N_{n,l,x})$, the expected number of polymers with energies at most $x \in \mathbb{R}_+$, and prescribed length $l \in \mathbb{Z}_+$. To this end, we recall Stanley's identity (1.4) which states that

$$\sum_{l=0}^{\infty} M_{n,l} \frac{x^l}{l!} = \sinh(x)^n. \tag{2.1}$$

Restricting to $x > 0$ implies that

$$M_{n,l} \frac{x^l}{l!} \leq \sinh(x)^n \,, \tag{2.2}$$

and therefore, by optimizing, we obtain,

$$M_{n,l} \leq \inf_{x>0} \left[\sinh(x)^n \frac{l!}{x^l} \right] . \tag{2.3}$$

Consistently with our terminology, we refer to (2.2) and (2.3) as *Stanley's M-bounds.*

Recall that $N_{n,l,E}$ is the number of paths of length l between two opposite vertices, and energy at most $\mathsf{E} = \log(1+\sqrt{2})$ as given in (1.9). By the tail estimates, and the above Stanley's M-bound, we thus have

$$\mathbb{E}(N_{n,l,\mathsf{E}}) \lesssim M_{n,l} \frac{\mathsf{E}^l}{l!} \leq \mathsf{E}^l \inf_{x>0} \frac{\sinh(x)^n}{x^l} = \mathsf{E}^l \frac{\sinh(x^*)^n}{x^{*l}}, \tag{2.4}$$

where $x^* = x^*(l)$ is the minimizer of the r.h.s. above; taking the derivative of the target function, we see that this is the (unique) solution of

$$\frac{x}{\tanh(x)} = \frac{l}{n}. \tag{2.5}$$

At this point one is perhaps tempted to revert the line of reasoning: with the natural candidate for the optimal energy in mind, we choose $x^* \equiv \mathsf{E}$, in which case it follows from (2.5) that $l = \sqrt{2}\mathsf{E}n$, as an elementary computation shows. Changing the order of extremization is of course not quite justified[4], but the upshot turns out to be correct:

[4]One can prove that for all $l \in \mathbb{Z}_+$, and x^* satisfying (2.5), it holds that

$$\sinh(x^*)^n \frac{\mathsf{E}^l}{x^{*l}} \leq 1,$$

with the bound being saturated at $x^* = E$. As a matter of fact, we will prove an even stronger statement, namely that the length of optimal polymers indeed strongly concentrates on $\mathsf{L}n$, asymptotically in n. As we will see, this concentration follows from a key property of the power expansion (2.1), when evaluated at $x = \mathsf{E}$: in this case, the

Insight 2.1. On the n-dim hypercube, the (candidate) length of optimal polymers is
$$\sqrt{2}\mathsf{E}n.$$

Henceforth, we will shorten
$$\mathsf{L} \equiv \sqrt{2}\mathsf{E}\,, \tag{2.6}$$
and always assume, without loss of generality, that $\mathsf{L}n \in \mathbb{Z}_+$. The concentration of length was first discovered by Martinsson in Martinsson (2016) and later also investigated in the case of accesssibility percolation on cartesian power graphs in Schmiegelt and Krug (2019). We emphasize that the first insight identified by Martinsson concern the "global length of optimal paths". We, on the other hand, will increase the resolution by establishing also the "evolution of the length" at a macroscopic and a microscopic scale to obtain a geometrical picture of the optimal paths.

2.2. *Uniform distribution of the energy.* Having found natural candidates for the minimal energy and optimal length, a further question naturally arises:

how is an E*-energy distributed along the polymer?*

To formalize, let us consider $\alpha \in [0,1]$, and shorten $\underline{\alpha} \equiv 1-\alpha$; furthermore let $\lambda \in [0,1]$ and similarly shorten $\underline{\lambda} = 1-\lambda$. We denote by
$$N^{\lambda,\alpha}_{n,\mathsf{L}n} := \#\left\{\pi \in \Pi_{n,\mathsf{L}n} : \sum_{i=1}^{\alpha\mathsf{L}n} \xi_{[\pi]_i} \le \lambda\mathsf{E}, \sum_{i=\underline{\alpha}\mathsf{L}n+1}^{\mathsf{L}n} \xi_{[\pi]_i} \le \underline{\lambda}\mathsf{E}\right\}. \tag{2.7}$$
the number of polymers with the property that an λ-fraction of the energy E is carried by an α-fraction of the length (and similarly for the remaining part of the strand).

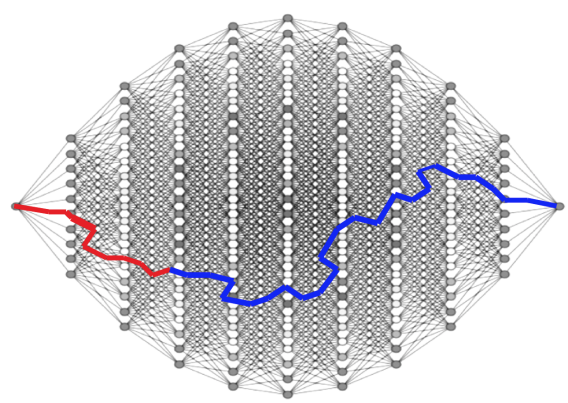

FIGURE 2.2. A polymer with (λ,α)-distribution of the energy E: the red strand has lenght $\alpha\mathsf{L}n$ and carries an energy $\lambda\mathsf{E}$, whereas the blue strand has length $\underline{\alpha}Ln$ and carries the remaining energy $\underline{\lambda}\mathsf{E}$.

$(\mathsf{L}n)^{th}$ Taylor-term carries virtually the whole "mass" (whence the saturation). Such a result also provides intriguing clues about the issue of fluctuations, but since it is not instrumental for the rest of the discussion, we postpone the precise formulation, see Proposition 3 below.

Since polymers are loopless, and by independence, we have

$$\begin{aligned}\mathbb{E}\left(N^{\lambda,\alpha}_{n,\mathsf{L}n}\right) &\le M_{n,\mathsf{L}n}\mathbb{P}\left(\sum_{i=1}^{\alpha\mathsf{L}n}\xi_{[\pi]_i}\le\lambda\mathsf{E},\ \sum_{i=\underline{\alpha}\mathsf{L}n+1}^{\mathsf{L}n}\xi_{[\pi]_i}\le\underline{\lambda}\mathsf{E}\right)\\ &= M_{n,\mathsf{L}n}\mathbb{P}\left(\sum_{i=1}^{\alpha\mathsf{L}n}\xi_{[\pi]_i}\le\lambda\mathsf{E}\right)\times\mathbb{P}\left(\sum_{i=\underline{\alpha}\mathsf{L}n+1}^{Ln}\xi_{[\pi]_i}\le\underline{\lambda}\mathsf{E}\right)\\ &\lesssim M_{n,\mathsf{L}n}\frac{(\lambda\mathsf{E})^{\alpha\mathsf{L}n}}{(\alpha\mathsf{L}n)!}\times\frac{(\underline{\lambda}\mathsf{E})^{\underline{\alpha}\mathsf{L}n}}{(\underline{\alpha}\mathsf{L}n)!}\,,\end{aligned}\tag{2.8}$$

the last inequality by the usual tail estimates. By *Stanley's M-bound* (2.2), this time with $x=\mathsf{E}$, we have

$$M_{n,\mathsf{L}n}\le\sinh(\mathsf{E})^n\frac{(\mathsf{L}n)!}{\mathsf{E}^{\mathsf{L}n}}=\frac{(\mathsf{L}n)!}{\mathsf{E}^{\mathsf{L}n}},\tag{2.9}$$

the last step since $\sinh(\mathsf{E})=1$. Using this in (2.8) we thus get

$$\begin{aligned}\mathbb{E}\left(N^{\lambda,\alpha}_{n,\mathsf{L}n}\right)&\lesssim\frac{(\mathsf{L}n)!}{\mathsf{E}^{\mathsf{L}n}}\frac{(\lambda\mathsf{E})^{\alpha\mathsf{L}n}}{(\alpha\mathsf{L}n)!}\frac{(\underline{\lambda}\mathsf{E})^{\underline{\alpha}\mathsf{L}n}}{(\underline{\alpha}\mathsf{L}n)!}\\ &=\binom{\mathsf{L}n}{\alpha\mathsf{L}n}(\lambda)^{\alpha\mathsf{L}n}(\underline{\lambda})^{\underline{\alpha}\mathsf{L}n},\end{aligned}\tag{2.10}$$

where in the last step we have used that $\mathsf{E}^{\alpha}\mathsf{E}^{\underline{\alpha}}=\mathsf{E}$, and simplified. By elementary Stirling approximation (to first order) of the binomial factor in (2.10), and again recalling that $\underline{\alpha}=1-\alpha$, and similarly for $\underline{\lambda}$, we thus arrive at the inequality

$$\mathbb{E}\left(N^{\lambda,\alpha}_{n,\mathsf{L}n}\right)\lesssim\left\{\left(\frac{\lambda}{\alpha}\right)^{\alpha}\left(\frac{1-\lambda}{1-\alpha}\right)^{1-\alpha}\right\}^{\mathsf{L}n}.\tag{2.11}$$

Note that $x\mapsto x^y(1-x)^{1-y}$ is strictly concave with a unique critical point at $x=y$. Therefore, $\mathbb{E}N^{\lambda,\alpha}_{n,\mathsf{L}n}$ vanishes exponentially fast as soon as $\lambda\neq\alpha$. Borel-Cantelli then implies the following, loosely formulated summary of the current section:

Insight 2.2. The energy E is spread *uniformly* along the polymer.

This insight is of course in complete agreement with the phenomenon of mean field trivialization, see Gayrard and Kistler (2015) for more on this issue.

2.3. *Length vs. distance: the macroscopic picture.* We address here the loosely formulated question:

at which Hamming distance from the origin
do we find a strand of prescribed length?

It is clear that the answer will yield profound insights into the geometry of optimal polymers. To formalize, consider as before $\alpha\in[0,1]$. (We stick to the convention $\underline{\alpha}=1-\alpha$). For $d\in[0,1]$, let $d_n=\lfloor dn\rfloor$ and denote by

$$H_{d_n}:=\{\boldsymbol{v}\in V_n:\ d(\mathbf{0},\boldsymbol{v})=d_n\}\,,\tag{2.12}$$

the *hyperplane* consisting of all vertices at Hamming distance d_n from the origin. (Remark that $\sharp H_{d_n}=\binom{n}{d_n}$: indeed, in order to specify a point on the hyperplane we simply need to switch d_n coordinates of $\mathbf{0}=(0,0,\dots,0)$ into 1).

For $\boldsymbol{w}\in H_{d_n}$ we denote by $\Pi^d_{\alpha\mathsf{L}n}[\mathbf{0}\to\boldsymbol{w}]$ the set of paths connecting $\mathbf{0}$ to $\boldsymbol{w}$ in αLn steps. In full analogy, $\Pi^d_{\underline{\alpha}\mathsf{L}n}[\boldsymbol{w}\to\mathbf{1}]$ stands for the set of path connecting $\boldsymbol{w}$ to $\mathbf{1}$ in $\underline{\alpha}\mathsf{L}n$ steps. Lastly, we denote

by $\Pi^{d,\alpha}_{\mathsf{L}n}[\mathbf{0} \to \mathbf{1}]$ the set of paths of length $\mathsf{L}n$ from $\mathbf{0}$ to $\mathbf{1}$, which are in H_{d_n} after $\alpha\mathsf{L}n$ steps. (Note that these paths can cross the hyperplane multiple times, see Figure 2.3 for a graphical rendition).

The goal is now to compute the expected number of these polymers after distributing the energy, in line with the Insight from the previous section, *uniformly* along the path. To this end, introduce the cardinalities

$$N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0} \to \boldsymbol{w}] = \#\left\{\pi \in \Pi^d_{\alpha\mathsf{L}n}[0 \to \boldsymbol{w}] : \sum_{i=1}^{\alpha\mathsf{L}n} \xi_{[\pi]_i} \le \alpha\mathsf{E}\right\},$$

$$N^{d,\underline{\alpha}}_{n,\underline{\mathsf{L}}n}[\boldsymbol{w} \to \mathbf{1}] = \#\left\{\pi \in \Pi^d_{\underline{\alpha}\mathsf{L}n}[\boldsymbol{w} \to 1], \sum_{i=1}^{\underline{\alpha}\mathsf{L}n} \xi_{[\pi]_i} \le \underline{\alpha}\mathsf{E}\right\},$$

and

$$N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0} \to \mathbf{1}] = \#\left\{\pi \in \Pi^{d,\alpha}_{\mathsf{L}n}[\mathbf{0} \to \mathbf{1}], \sum_{i=1}^{\mathsf{L}n} \xi_{[\pi]_i} \le \mathsf{E}\right\}.$$

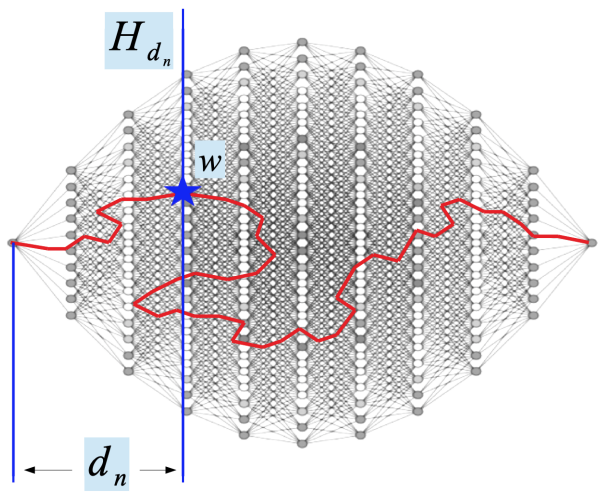

FIGURE 2.3. Path-decomposition with an hyperplane H_{d_n} at Hamming distance d_n from $\mathbf{0}$. The strand up to the first crossing of the hyperplane has an α-fraction of length, and carries an α-fraction of energy. The rest of the strand has length $\underline{\alpha}\mathsf{L}n$, and carries the remaining $\underline{\alpha}$-fraction of energy.

Since polymers are loopless, and by independence, it holds

$$\begin{aligned}
\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0} \to \mathbf{1}]\right) &= \sum_{\boldsymbol{w} \in H_{d_n}} \mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0} \to \boldsymbol{w}]\right) \mathbb{E}\left(N^{d,\underline{\alpha}}_{n,\underline{\mathsf{L}}n}[\boldsymbol{w} \to \mathbf{1}]\right) \\
&= \binom{n}{d_n} \mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0} \to \boldsymbol{w}]\right) \mathbb{E}\left(N^{d,\underline{\alpha}}_{n,\underline{\mathsf{L}}n}[\boldsymbol{w} \to \mathbf{1}]\right) \qquad (2.13) \\
&\lesssim \binom{n}{d_n} M_{n,\alpha\mathsf{L}n,d_n} \frac{(\alpha\mathsf{E})^{\alpha\mathsf{L}n}}{(\alpha\mathsf{L}n)!} M_{n,\underline{\alpha}\mathsf{L}n,n-d_n} \frac{(\underline{\alpha}\mathsf{E})^{\underline{\alpha}\mathsf{L}n}}{(\underline{\alpha}\mathsf{L}n)!},
\end{aligned}$$

the last inequality by the usual tail estimates.

In full analogy with (2.3), which is a consequence of Stanley's identity (1.4), the following *Stanley's M-bound* is a consequence of Stanley's identity (1.3): for $x > 0$, it holds

$$M_{n,l,d} \le \sinh(x)^d \cosh(x)^{n-d} \frac{l!}{x^l}. \qquad (2.14)$$

Using this for the r.h.s. of (2.13) we see that for arbitrary $y_1, y_2 > 0$, it holds:

$$\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \binom{n}{d_n} \frac{\sinh(y_1)^{d_n}\cosh(y_1)^{n-d_n}}{\left(\frac{y_1}{\alpha\mathsf{E}}\right)^{\alpha Ln}} \frac{\sinh(y_2)^{n-d_n}\cosh(y_2)^{d_n}}{\left(\frac{y_2}{\underline{\alpha}\mathsf{E}}\right)^{\underline{\alpha}\mathsf{L}n}}. \tag{2.15}$$

Taking $y_1 = \alpha\mathsf{E}$ and $y_2 = \underline{\alpha}\mathsf{E}$, and by elementary Stirling approximation (to first order),

$$\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \left(\frac{\cosh(\alpha E)\sinh(\underline{\alpha}E)}{1-\frac{d_n}{n}}\right)^{n-d_n} \left(\frac{\sinh(\alpha E)\cosh(\underline{\alpha}E)}{\frac{d_n}{n}}\right)^{d_n}. \tag{2.16}$$

We will now slightly modify the form of the r.h.s. above. In order to do so, we recall that

$$\begin{aligned} 1 = \sinh(\mathsf{E}) &= \sinh\left(\alpha\mathsf{E} + \underline{\alpha}\mathsf{E}\right) \\ &= \cosh(\alpha E)\sinh(\underline{\alpha}\mathsf{E}) + \sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})\,, \end{aligned} \tag{2.17}$$

the last step by the addition formula for hyperbolic functions, hence

$$\cosh(\alpha\mathsf{E})\sinh(\underline{\alpha}\mathsf{E}) = 1 - \sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})\,. \tag{2.18}$$

This allows to reformulate (2.16) as

$$\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \left\{\left(\frac{1-\sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})}{1-\frac{d_n}{n}}\right)^{1-\frac{d_n}{n}} \left(\frac{\sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})}{\frac{d_n}{n}}\right)^{\frac{d_n}{n}}\right\}^n. \tag{2.19}$$

One plainly checks that the function

$$[0,1] \ni \alpha \mapsto \sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E}) \tag{2.20}$$

is bijective, whereas $x \mapsto (1-x)^{1-y}x^y$ is strictly concave with a unique critical point at $x = y$. It thus steadily follows that the r.h.s. of (2.19) is exponentially small if $\frac{d_n}{n} \neq \sinh(\alpha E)\cosh(\underline{\alpha}E)$. We may thus summarize these findings as follows:

Insight 2.3. After an α-fraction of the total length, an optimal polymer finds itself at a typical (normalized) Hamming distance

$$d = \sinh(\alpha\mathsf{E})\cosh((1-\alpha)\mathsf{E}) \tag{2.21}$$

from the origin.

The above Insight is both intriguing and delicate. Indeed, a polymer of length greater than the dimension can (must) cross multiple times certain hyperplanes, yet the map $\alpha \mapsto d(\alpha)$ as in (2.21) is increasing: for consistency, we must therefore deduce that excursions can only happen on mesoscopic (if not microscopic) scales. In other words, and loosely:

Insight 2.4. Backsteps must be relatively rare, and spread out.

Not surprisingly, this additional Insight will play a key role, and guide us through the next steps, but before proceeding any further, a comparison with the directed case is perhaps in place. To better visualize, we re-parametrize in terms of the (normalised) *length* of the polymer: with $\alpha E \hookrightarrow l$, and recalling that $\mathsf{L} = \sqrt{2}\mathsf{E}$, we see that the "Hamming depth" $d_{\mathrm{un}}(l)$ reached by the unoriented polymer at length l is then given by

$$l \in [0,\mathsf{L}] \mapsto d_{\mathrm{un}}(l) \equiv \sinh\left(\frac{l}{\sqrt{2}}\right)\cosh\left(\frac{\mathsf{L}-l}{\sqrt{2}}\right). \tag{2.22}$$

In case of oriented polymers, the Hamming depth as a function of the length is simply

$$l \in [0,1] \mapsto d_{\mathrm{or}}(l) \equiv l\,. \tag{2.23}$$

The two functions are plotted in Figure 2.4, whereas a rendition of the emerging picture at the level of the strands is given in Figure 2.5.

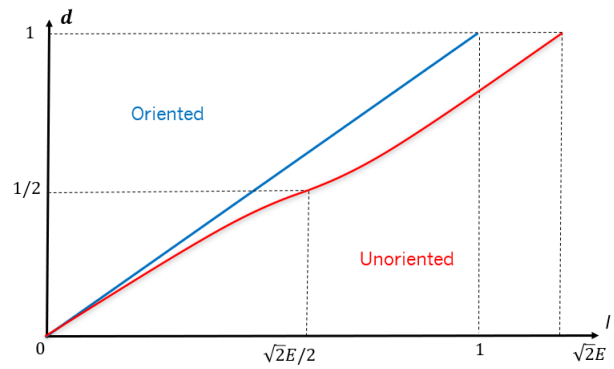

FIGURE 2.4. Hamming-depth as a function of the length: directed (blue) vs. undirected (red) polymers. For small lengths, the depths are comparable: close to the origin, the undirected polymer is thus as directed as possible. The slope of the red curve decreases however gradually as the polymer approaches the core of the hypercube: the further the polymer goes, the "loser" it becomes. Due to the inherent symmetry of the hypercube, a mirror picture sets in, of course, at half-length.

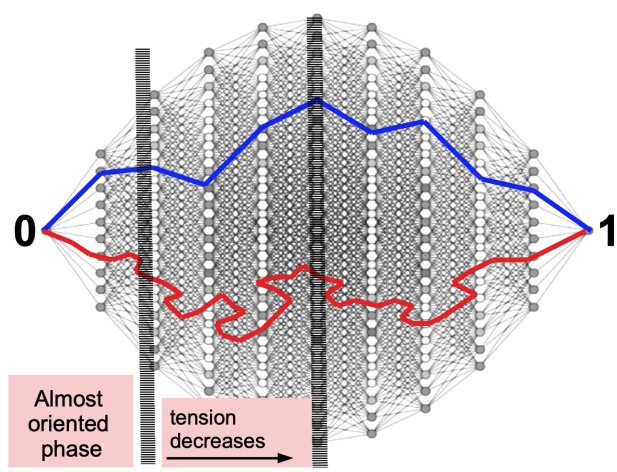

FIGURE 2.5. Directed (blue) vs. undirected (red) polymers. The red strand starts off as stretched as possible, but allows for more and more backsteps as it approaches the core of the hypercube. The phenomena are amplified for better visualisation only: in line with Insight 2.4, backsteps live on meso/microscopic scale only. In particular, long excursions as in Figure 1.1 are, in fact, ruled out.

2.4. *Length vs. distance: the mesoscopic picture.* As mentioned in the introduction, our approach will eventually rest on a multiscale analysis: in this section, inspired by the previous Insights, we introduce the necessary *coarse graining* Gayrard and Kistler (2015). To see how this goes, we denote by $K \in \mathbb{Z}_+$ the numbers of "scales", and shorten henceforth $\hat{n}_K \equiv n/K$ (assuming w.l.o.g. that

$\hat{n}_K \in \mathbb{Z}_+$). We then split the hypercube into K "slabs", i.e. hyperplanes equidistributed w.r.t. the Hamming distance: for $i = 1 \dots K$ we let

$$H_i \equiv \{v \in V_n, d(0,v) = i\hat{n}_K\} \ . \tag{2.24}$$

We will refer to these hyperplanes as *H-planes*. Accordingly, we split a polymer of length $\mathsf{L}n$ into K substrands of length $\alpha_i Ln$, for $i = 1 \dots K$, with the normalization $\sum_{i \le K} \alpha_i = 1$. We shorten $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_K) \in [0,1]^K$ for such a vector, $\overline{\boldsymbol{\alpha}}_i \equiv \sum_{j=1}^i \alpha_j$ for the (fraction of) length of the strand when the polymer crosses the ith H-plane, and $\underline{\boldsymbol{\alpha}}_i \equiv 1 - \sum_{j=1}^i \alpha_j$ for the length of the remaining strand. A graphical rendition is given in Figure 2.6.

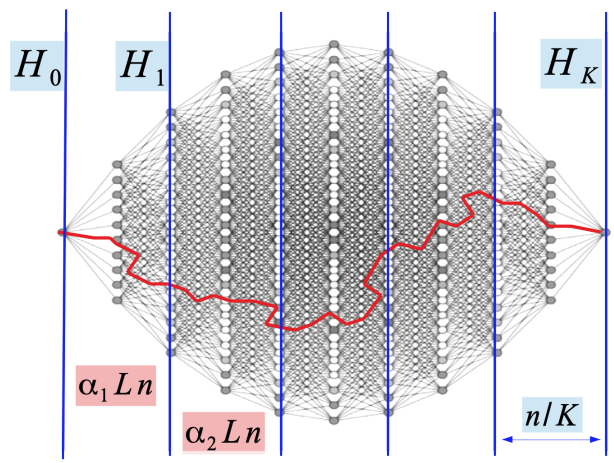

FIGURE 2.6. K-levels coarse graining: the Hamming distance between any two (successive) hyperplanes is $\hat{n}_K = n/K$. Remark that by (2.26)-(2.27), the length of the substrand from hyperplane to hyperplane is a function of E and K only.

By the above Insight 2.3, length of substrands and Hamming-depth must satisfy the fundamental relation

$$\sinh(\overline{\boldsymbol{\alpha}}_i \mathsf{E}) \cosh(\underline{\boldsymbol{\alpha}}_i \mathsf{E}) = \frac{i}{K} \ , \quad i = 1 \dots K. \tag{2.25}$$

The function $x \in [0,1] \mapsto \sinh(x\mathsf{E}) \cosh((1-x)\mathsf{E})$ is invertible, and one can even construct explicitly the solutions of the above equation: recalling that $\operatorname{arcsinh}(x) = \log(x + \sqrt{1+x^2})$ one plainly checks that these are given by

$$\overline{\boldsymbol{\alpha}}_i = \frac{1}{2}\left(1 + \frac{1}{\mathsf{E}} \operatorname{arcsinh}\left(2\frac{i}{K} - 1\right)\right) \ . \tag{2.26}$$

This also uniquely identifies the length of the substrands, to wit:

$$\alpha_i = \overline{\boldsymbol{\alpha}}_i - \overline{\boldsymbol{\alpha}}_{i-1} \ , \tag{2.27}$$

for $i = 1 \dots K$, see Figure 2.7 for a plot.

In particular, it follows from (2.26) and (2.27) that

$$\alpha_i = \alpha_{K+1-i}, \tag{2.28}$$

which is in full agreement with the inherent symmetry of the problem at hand, and $\sum_{j \le K} \alpha_j = 1$. Furthermore, since arcsinh is 1-Lipschitz we also immediately see that

$$\alpha_i \le \frac{1}{K\mathsf{E}}. \tag{2.29}$$

In order to emphasize that the $\alpha's$ are no longer arbitrary, we will write henceforth $\mathsf{a} = \mathsf{a}(\mathsf{E}, K)$ for the solutions of the equations (2.26), (2.27).

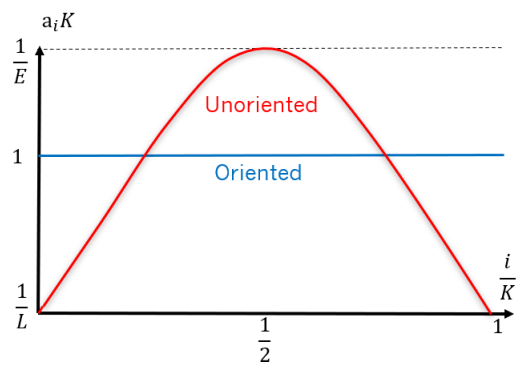

FIGURE 2.7. Substrand-length as function of the depth, $i \in \{1, \dots, K\} \mapsto \mathsf{a}_i$. This plot simply restates the key property of optimal polymers: substrands between equidistant hyperplanes become longer as the polymer enters the core of the hypercube.

A straightforward large-K Taylor expansion (with $i/K = const.$) yields that

$$\mathsf{a}_{i+1} - \mathsf{a}_i = \frac{2}{K^2 \mathsf{E}} \left(1 - \frac{2i}{K}\right) + O\left(\frac{1}{K^3}\right) , \tag{2.30}$$

which is manifestly different from the case of directed polymers, where the differential would necessarily vanish. Thus a fundamental question immediately arises:

how do substrands of undirected polymers connect the coarse graining-hyperplanes?

To shed light on this issue we consider $\boldsymbol{d} = (d_1, d_2, \dots, d_K) \in [0,1]^K$ and introduce

$$\begin{aligned} \Pi_i^{\boldsymbol{d}}[\boldsymbol{v} \to \boldsymbol{w}] \equiv\ & \text{all loopless paths connecting} \\ & \text{two vertices } \boldsymbol{v} \in H_{i-1},\ \boldsymbol{w} \in H_i \\ & \text{which are at Hamming distance } d(\boldsymbol{v}, \boldsymbol{w}) = d_i n\,, \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \Pi_{\{1\dots K\}}^{\boldsymbol{d}}[\boldsymbol{0} \to \boldsymbol{1}] \equiv\ & \text{all loopless paths connecting } \boldsymbol{0} \text{ to } \boldsymbol{1}\,, \\ & \text{and that cover a } d_i n\text{-Hamming distance} \\ & \text{while connecting the H-hyperplanes, } i = 1 \dots K. \end{aligned} \tag{2.32}$$

A graphical rendition is given in Figure 2.8.

For $\pi \in \Pi_{\{1\dots K\}}^{\boldsymbol{d}}[\boldsymbol{0} \to \boldsymbol{1}]$, and two vertices $\boldsymbol{v} \in H_{i-1}, \boldsymbol{w} \in H_i$ (for some $i = 1 \dots K$), we furthermore shorten

$$X_\pi(\boldsymbol{v}, \boldsymbol{w}) \equiv \text{energy of the substrand which connects } \boldsymbol{v},\ \boldsymbol{w}\ . \tag{2.33}$$

and denote by

$$N_i^{\boldsymbol{d}}[\boldsymbol{v} \to \boldsymbol{w}] = \#\left\{\pi \in \Pi_i^{\boldsymbol{d}}[\boldsymbol{v} \to \boldsymbol{w}], X_\pi(\boldsymbol{v}, \boldsymbol{w}) \le \mathsf{a}_i \mathsf{E}\right\}\ , \tag{2.34}$$

the number of substrands with energies at most $\mathsf{a}_i E$ connecting such vertices. Finally, let

$$N_{\{1\dots K\}}^{\boldsymbol{d}}[\boldsymbol{0} \to \boldsymbol{1}] = \#\left\{\pi \in \Pi_{\{1\dots K\}}^{\boldsymbol{d}}[\boldsymbol{0} \to \boldsymbol{1}],\ X_\pi(\boldsymbol{0}, \boldsymbol{1}) \le \mathsf{E}\right\} \tag{2.35}$$

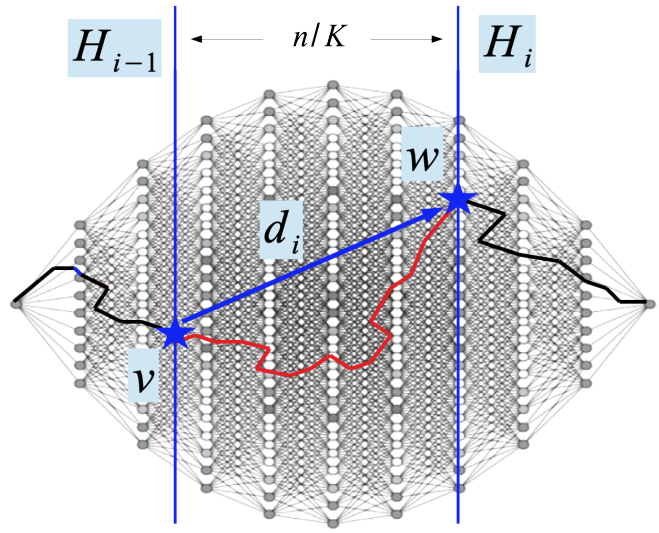

FIGURE 2.8. A polymer between two hyperplanes: the two vertices $\boldsymbol{v}$ and $\boldsymbol{w}$ are at a Hamming distance $d(\boldsymbol{v},\boldsymbol{w}) = d_i$. Remark that, in particular, $d(H_{i-1}, H_i) = 1/K \leq d_i \leq l_\pi(\boldsymbol{v},\boldsymbol{w})$.

stand for the number of paths with prescribed evolutions[5]. The goal is to compute the expectation of this random set, as this will provide fundamental insights into the possible choices of $\boldsymbol{d}$, which are the only degrees of freedom left. As we will see shortly, there is only one reasonable choice. Before that we need however to introduce some key concepts.

Definition 2.5. Let $\boldsymbol{v} \in H_{i-1}$ and $\boldsymbol{w} \in H_i$.

- The *effective forward steps* are given by
$$\mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) \equiv \frac{1}{n}\#\left\{0's \text{ in } \boldsymbol{v} \text{ which switch into } 1's \text{ in } \boldsymbol{w}\right\} .$$
- The *effective backsteps* are given by
$$\mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w}) \equiv \frac{1}{n}\#\left\{1's \text{ in } \boldsymbol{v} \text{ which switch into } 0's \text{ in } \boldsymbol{w}\right\} .$$
- The *detours* are given by
$$\gamma_\pi(\boldsymbol{v},\boldsymbol{w}) \equiv \frac{1}{n}\left\{l_\pi(\boldsymbol{v},\boldsymbol{w}) - d(\boldsymbol{v},\boldsymbol{w})\right\} .$$

Some comments concerning the above terminology are perhaps in place: we note that the *effective forward steps* encode the fraction of steps forward which are not undone by backsteps in the reverse direction; similarly, the *effective backsteps* encode the (fraction of) backsteps which are not undone by steps forward in the reverse direction (or vice versa). Finally, the *detours* capture the amount of forward steps in a path π which are cancelled by backsteps in the reverse direction (or vice versa): the smaller γ_π, the higher the "tension" of the substrand. For this reason, we call a substrand *stretched* if the detours vanish. A stretched path is, in fact, a *geodesic*.

[5]We shall perhaps emphasize that the above prescription of the evolution involves the Hamming-depths and energies, but *not* the length of the connecting substrands. This is because in (2.34) we are spreading the energies uniformly along the length of the polymer, very much in line with Insight 2.2: energies and optimal lengths are two sides of the same coin.

The above quantities are all intertwined. Indeed, it holds:

$$d_i = \mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) + \mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w}) \quad \text{and} \quad \frac{1}{K} = \mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) - \mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w})\,. \tag{2.36}$$

In particular, it follows from the above relations that

$$\mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) = \frac{d_i}{2} + \frac{1}{2K} \quad \text{and} \quad \mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w}) = \frac{d_i}{2} - \frac{1}{2K}\,. \tag{2.37}$$

In other words, effective forward- and backsteps along a substrand depend on the number of scales, and the remaining degrees of freedom $\boldsymbol{d}$ (which we are going to identify shortly), but *not* on the endpoints. An equally simple line of reasoning shows that detours, *as soon as the polymer-length is specified*, do not depend on the specific form of the π-path, neither: in fact, $\gamma_{i,\pi}n + d_i n = \mathsf{a}_i L n$.

As mentioned, the goal is to compute the expected number of paths connecting $\mathbf{0}$ to $\mathbf{1}$. Since polymers are loopless, and by independence, it holds:

$$\mathbb{E}\left(N^{\boldsymbol{d}}_{\{1\dots K\}}[\mathbf{0}\to\mathbf{1}]\right) = \sum_{(\star)}\prod_{i=1}^{K}\mathbb{E}N_i^{\boldsymbol{d}}\left[\boldsymbol{v}^{(i-1)}\to\boldsymbol{v}^{(i)}\right], \tag{2.38}$$

where the $(\star)$-sum runs over all possible vertices $\boldsymbol{v}^{(i)}\in H_i, i=1\dots K$. But by (2.37), none of the expectations on the r.h.s. depend on the specific $\boldsymbol{v}$-choice. The cardinality of $(\star)$ is easily computed: shortening

$$[0,\infty)\ni x\mapsto\varphi(x)\equiv x^x, \tag{2.39}$$

one plainly checks that

$$\begin{aligned}\#(\star) &= \prod_{i=1}^{K}\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{\left(1-\frac{i-1}{K}\right)n}{\mathsf{ef}_i n}\\ &\lesssim \prod_{i=1}^{K}\left(\frac{\varphi\left(\frac{i-1}{K}\right)\varphi\left(1-\frac{i-1}{K}\right)}{\varphi(\mathsf{eb}_i)\varphi\left(\frac{i-1}{K}-\mathsf{eb}_i\right)\varphi\left(\mathsf{ef}_i\right)\varphi\left(1-\frac{i-1}{K}-\mathsf{ef}_i\right)}\right)^n,\end{aligned} \tag{2.40}$$

the last step by elementary Stirling-approximation to first order.

By the tail estimates, and Stanley's M-bound (2.14) with $x=\mathsf{a}_i\mathsf{E}$, it holds

$$\mathbb{E}N_i^{\boldsymbol{d}}\left[\boldsymbol{v}^{(i-1)}\to\boldsymbol{v}^{(i)}\right]\lesssim\sinh(\mathsf{a}_i\mathsf{E})^{d_i n}\cosh(\mathsf{a}_i\mathsf{E})^{(1-d_i)n}\,, \tag{2.41}$$

for $i=1\dots K$.

Plugging (2.40) and (2.41) into (2.38), and rearranging, we thus get the *upperbound*

$$\mathbb{E}\left(N^{\boldsymbol{d}}_{\{1\dots K\}}[\mathbf{0}\to\mathbf{1}]\right)\lesssim\mathcal{F}_{\mathsf{a},K}(\boldsymbol{d})^n, \tag{2.42}$$

where we have shortened

$$\mathcal{F}_{\mathsf{a},K}(\boldsymbol{d})\equiv\prod_{i=1}^{K}\frac{\sinh(\mathsf{a}_iE)^{d_i}\cosh(\mathsf{a}_iE)^{(1-d_i)}\varphi\left(\frac{i-1}{K}\right)\varphi\left(1-\frac{i-1}{K}\right)}{\varphi(\mathsf{eb}_i)\varphi\left(\frac{i-1}{K}-\mathsf{eb}_i\right)\varphi\left(\mathsf{ef}_i\right)\varphi\left(1-\frac{i-1}{K}-\mathsf{ef}_i\right)}\,. \tag{2.43}$$

Since $\mathbf{a}=\mathbf{a}(\mathsf{E},K)$ are solutions of (2.26)-(2.27), the $\boldsymbol{d}'s$ appearing in the $\mathcal{F}$-function are the only degrees of freedom left (By (2.37), we recall that ef_i and eb_i are function of d_i). The next result shows that even for these, there is in fact one reasonable choice only.

Theorem 1. *(Optimal Hamming distance)* Let $\mathbf{d}=(\mathsf{d}_1,\dots,\mathsf{d}_K)$, with

$$\mathsf{d}_i\equiv\sinh(\mathsf{a}_i\mathsf{E})\cosh((1-\mathsf{a}_i)\mathsf{E}). \tag{2.44}$$

It then holds:

$$\mathcal{F}_{\mathbf{a},K}(\mathbf{d})=1, \tag{2.45}$$

and

$$\mathcal{F}_{\mathbf{a},K}(\boldsymbol{d}) < 1, \quad \text{for} \quad \boldsymbol{d} \neq \mathbf{d}. \tag{2.46}$$

By (2.42) and (2.46), the expected number of polymers connecting a sequence of prescribed vertices on the H-planes is thus exponentially small, *unless* the Hamming distance of the considered vertices satisfies (2.44): of course, the latter will henceforth be the value of our choice.

Theorem 1 is absolutely crucial for our approach. The proof, which requires a fair amount of work, is postponed. For the remaining part of this section we dwell rather informally on some of its far-reaching implications.

We anticipate that we will eventually consider a large (yet finite) number of scales for the coarse graining, in which case an elementary large-K Taylor expansion (together with the fact that $\mathsf{L} = \sqrt{2}\mathsf{E}$) shows that to first approximation, Hamming distance between two vertices on the H-planes and substrand-legth do, in fact, coincide:

$$\mathsf{d}_i = \sinh(\mathsf{a}_i\mathsf{E})\cosh((1-\mathsf{a}_i)\mathsf{E}) = \mathsf{a}_i\mathsf{L} + O\left(\frac{1}{K^2}\right). \tag{2.47}$$

A minute's thought suggests that the above may be reformulated as follows:

Insight 2.6. Optimal polymers connect the coarse graining H-planes through essentially stretched paths.

This is a somewhat surprising feature, which at first sight may even appear non-sensical. The devil is however in the details: by (2.26), and large-K Taylor expansions (again with $i/K = const$), one can check that

$$\mathsf{a}_i = \frac{1}{K\mathsf{E}\sqrt{1+(\frac{2i}{K}-1)^2}} + O\left(\frac{1}{K^2}\right), \tag{2.48}$$

which combined with (2.47), and recalling $\mathsf{L} = \sqrt{2}\mathsf{E}$, leads to

$$\mathsf{d}_i = \frac{\sqrt{2}}{K\sqrt{1+\left(\frac{2i}{K}-1\right)^2}} + O\left(\frac{1}{K^2}\right). \tag{2.49}$$

From this we may evince that:

- for small i (say $i = sK$, and $s \ll 1/2$) it holds that

$$\mathsf{a}_i = \frac{1}{K\mathsf{E}\sqrt{2}} + O\left(\frac{1}{K^2}\right) = \frac{1}{K\mathsf{L}} + O\left(\frac{1}{K^2}\right), \tag{2.50}$$

  as well as

$$\mathsf{d}_i = \frac{1}{K} + O\left(\frac{1}{K^2}\right) = d(H_{i-1}, H_i) + O\left(\frac{1}{K^2}\right), \tag{2.51}$$

  the latter confirming that close to the origin, unoriented polymers proceed in almost directed fashion;
- for large i (say $i = sK$, and $s \uparrow 1/2$) it holds that $\mathsf{d}_i \approx \sqrt{2}/K \gg 1/K$, which is much larger than the Hamming distance between two successive H-planes. Substrands of optimal polymers close to the core of the hypercube therefore reach, through approximate geodesics, vertices which are otherwise unattainable in a fully directed regime. Although the length of the substrand is increased, this strategy allows undirected polymers to gain access to a reservoir of energetically favorable edges. A graphical rendition of this feature, which encodes the key strategy of optimal polymers, is given in Figure 2.9.

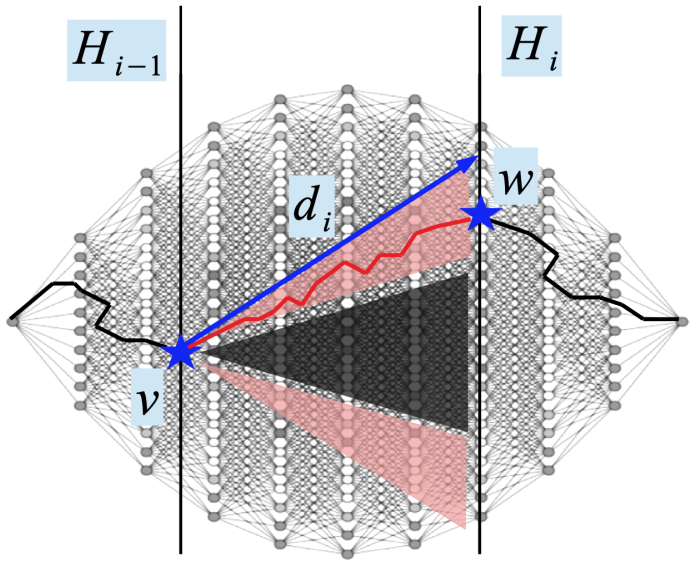

FIGURE 2.9. The black-shaded cone corresponds to the region where a fully directed polymer would lie. In virtue of Theorem 1, the optimal, undirected polymers evolve however in the red-shaded cones, thereby reaching vertices which are at larger Hamming distance. (Note also that black and red vertical boundaries of these cones are disjunct). For large hyperplane-density, the substrands (in red) of optimal polymers are, in first approximation, geodesics.

The feature according to which undirected polymers proceed through approximate geodesics [6] is absolutely fundamental. On the one hand it neatly explains the deeper mechanisms eventually responsible for the onset of the mean field trivialization. On a more technical level, this property will lead to a dramatic simplification of some otherwise daunting combinatorial estimates, eventually enabling us to implement the second moment method. In fact, in a (fully) stretched regime, a backstep cannot be cancelled by a forward step (and vice versa). This entails, in particular, a natural representation of paths connecting say $\boldsymbol{v} \in H_{i-1}$ to $\boldsymbol{w} \in H_i$ in terms of permutations of the $\boldsymbol{v}$-coordinates which must be changed in order to obtain $\boldsymbol{w}$, see in particular Lemma 6.10 below for a clear manifestation of this feature.

2.5. *Main result.* We now specify a subset of polymers with path properties capturing all Insights gathered so far: our main result, which is at last formulated in this section, simply states that such a subset is, in fact, non-empty. Towards this goal, some additional observations/notation is needed.

For arbitrary $\boldsymbol{d} = (d_1, \dots, d_K) \in [0,1)^K$ (the Hamming-depths) and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_K) \in [0,\infty)^K$ (the detours), consider the subset

$$\begin{aligned} \mathcal{P}_{n,K}\{\boldsymbol{d},\boldsymbol{\gamma}\} \equiv\ & \text{all paths connecting } \mathbf{0} \text{ to } \mathbf{1}\ , \\ & \text{and that cover a normalized } d_i\text{-Hamming distance,} \\ & \text{with } \gamma_i \text{ detours,} \\ & \text{while connecting the H-hyperplanes, } i = 1 \dots K\ . \end{aligned} \tag{2.52}$$

[6] It is important to emphasize that Martinsson already uses the word "geodesics", but his meaning radically differs from ours. Ours is a truly geometric property: we prove that polymers evolve as geodesics w.r.t. Hamming distance, whereas Martinsson refers to paths of minimal energy as geodesics.

We now make a specific choice of the free parameters, $\boldsymbol{d}$ and $\boldsymbol{\gamma}$, which is naturally justified by the picture canvassed in the above sections. As a matter of fact, we will force polymers to reflect an "extreme" version of the picture. Precisely:

- instead of considering polymers which are *essentially* directed close to the endpoints (recall in particular Figure 2.4) we will consider polymers which are *fully* directed in these regimes. We will achieve this by fixing a small $m = 205 \ll K$ (as already mentioned, we will choose K large enough). With $\mathbf{d} = (\mathsf{d}_1, \dots, \mathsf{d}_K)$ the optimal Hamming distance as in (2.44) from Theorem 1 we then set

$$\mathbf{d}_{opt} = \left(\underbrace{1/K, \dots, 1/K}_{m-\text{times}}, \mathsf{d}_{m+1}, \mathsf{d}_{m+2}, \dots, \mathsf{d}_{K-m}, \underbrace{1/K, \dots, 1/K}_{m-\text{times}} \right) , \tag{2.53}$$

- instead of considering polymers which are *essentially* stretched between the coarse graining H-planes (recall in particular Insight 2.6), we will consider polymers which proceed through *exact geodesics*; this will be achieved by setting

$$\boldsymbol{\gamma}_{opt} \equiv (0, \dots, 0) . \tag{2.54}$$

Denoting by L_{opt} the normalized length of paths in $\mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \boldsymbol{\gamma}_{opt}\right\}$, it holds that

$$\mathsf{L}_{opt} = \|\mathbf{d}_{opt}\|_1 . \tag{2.55}$$

We then focus on the ensuing subset $\mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \boldsymbol{\gamma}_{opt}\right\} \subset \widetilde{\Pi}_{n,\mathsf{L}_{opt}n}$. A graphical rendition of these polymers, which are only marginally shorter than $\mathsf{L} = \sqrt{2}\mathsf{E}$ (see (2.59) below for more on this), is given in Figure 2.10.

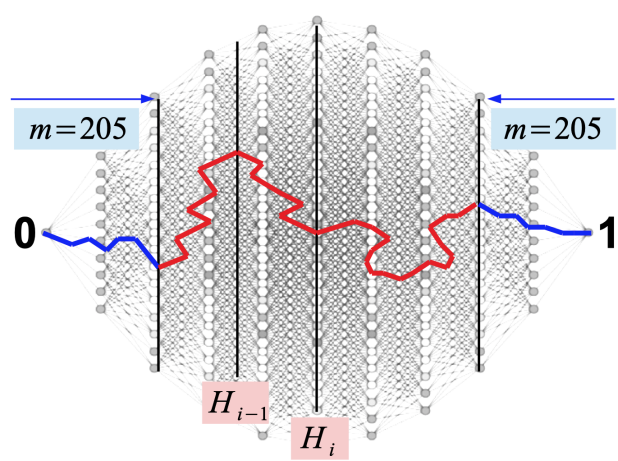

FIGURE 2.10. A polymer in $\mathcal{P}_{n,K}$: the blue substrand is fully directed. The red substrands connect the H-planes of the coarse graining through stretched paths, i.e. geodesics.

Since Hamming-depths and detours are specified, we lighten henceforth notation by

$$\mathcal{P}_{n,K} \equiv \mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \boldsymbol{\gamma}_{opt}\right\} . \tag{2.56}$$

Let now $\epsilon > 0$, and consider the subset of polymers

$$\mathcal{E}^{\epsilon}_{n,K} \equiv \pi \in \mathcal{P}_{n,K} \text{ with energies } X_\pi \leq \mathsf{E} + \epsilon , \tag{2.57}$$

namely those paths which *i)* are fully directed close to the endpoints, *ii)* connect the coarse graining H-planes in the core of the hypercube through geodesics, *iii)* and which reach an ϵ-neighborhood of the ground state energy. Our main result states that such polymers do, in fact, exist:

Theorem 2. *(The geometry of optimal polymers).* For $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{Z}_+$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\# \mathcal{E}^{\epsilon}_{n,K} \geq 1\right) = 1. \tag{2.58}$$

The proof of Theorem 2, which eventually boils down to an application of the Paley-Zygmund inequality, is both technically demanding and long, and will be given in the next sections. Before seeing how this goes, some comments are in order.

First, we remark that the length of the substrands connecting the H-planes (which is related to the $\mathsf{a}'s$) does not appear explicitly in the statement of Theorem 2, and neither do the sub-energies. This is again due to the fact that, in line with Insight 2.2, uniformly spread lengths/energies will be hiding behind the optimal Hamming-depths.

Second, we point out that Theorem 2, when combined with the simple *lower bound* discussed in the Introduction, yields a constructive proof of Martinsson's Theorem.

Lastly, and with the unsettled issue of fluctuations in mind, we shall dwell on a conceptually intricate aspect of the theorem, namely the nature of the parameter K encoding the density of hyperplanes for the coarse graining. One perhaps expects that larger constants lead to more accurate pictures, but this is only to some extent correct. In fact, too large hyperplane-density would even lead to inconsistencies: higher and higher densities "unbend" the strands, ultimately to the point of complete directedness, but this, in turn, would starkly contradict the crucial feature of optimal polymers, namely that their length is *larger* than the dimension. A delicate balance must therefore be met. As we will see in the course of the second moment implementation, see (6.52), (6.82), (6.87) and (6.104) below, for the present purpose of analyzing the ground state to leading order, it indeed suffices to take a large but *finite* $K = \max\left(2 \times 10^7, m\epsilon^{-2}\right)$. How fast (in the dimension n) the hyperplane-density can be allowed to grow is an interesting, and important issue, which unfortunately eludes us.

We conclude this section with the aforementioned result concerning the concentration of the length of optimal polymers, as it provides a neat round-off of the picture. We emphasize that this result has first been proved by Martinsson via BTP-comparison Martinsson (2016), whereas our short proof will rely on Laplace method/saddle point analysis.

To formulate, remark that Theorem 2 involves paths of length L_{opt}; by a more detailed study of Taylor's remainder term in (2.47), (2.50) and (2.51), and recalling that $\mathsf{L} = \sqrt{2}\mathsf{E}$, it can be plainly checked that

$$0 \leq \mathsf{L} - \mathsf{L}_{opt} \leq \frac{m}{K}\,. \tag{2.59}$$

In other words, for large hyperplane density, the difference between L_{opt} and L is vanishing. Our second main result states that the length L is, in fact, optimal:

Theorem 3. *(Concentration of the polymer's length).* For $\epsilon > 0$ and $a > \frac{\mathsf{E}}{2} + \sqrt{2}\mathsf{E} + \frac{1}{\sqrt{2}}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\# \left\{\pi \in \Pi_n :\ X_\pi \leq \mathsf{E} + \epsilon^2, \frac{1}{n}|l_\pi(\mathbf{0},\mathbf{1}) - \mathsf{L}n| \geq a\epsilon\right\} \geq 1\right) = 0\,. \tag{2.60}$$

Remark 2.7. The proof of the above Theorem, which is given below, suggests (albeit feebly) that the (ϵ^2, ϵ)-scaling in (2.60) is, in fact, optimal, and this in turn suggests that a central limit theorem applies for the optimal length. Martinsson proves in Martinsson (2016) that the optimal polymer has length $Ln + o(n)$ almost surely. In passing, we emphasize that we could take an n-dependent ϵ in the proof of theorem 3 in order to obtain that the optimal polymer has length $Ln + O(n^{1-\alpha})$ for $\alpha < 1/2$.

Proof of Theorem 3: Recall that claim (2.60) reads

$$\lim_{n\to\infty} \mathbb{P}\left(\# \left\{ \pi \in \Pi_n : \; X_\pi \le E + \epsilon^2, \frac{|l_\pi - \mathsf{L}n|}{n} \ge a\epsilon \right\} > 0 \right) = 0\,, \tag{2.61}$$

for $a > 0$ large enough. The proof, which is (vaguely) inspired by the *saddle point method* Flajolet and Sedgewick (2009), exploits the strong concentration of the expansion of the sinh-function on specific Taylor-terms. To see how this goes, in virtue of the by now "classical" route (union bounds and Markov's inequality / independence / tail estimates) it holds

$$\mathbb{P}\left(\# \left\{ \pi \in \Pi_n : \; X_\pi \le E + \epsilon^2, \frac{|l_\pi - \mathsf{L}n|}{n} \ge a\epsilon \right\} \right) \lesssim \sum_{\frac{|l-\mathsf{L}n|}{n} \ge a\epsilon} M_{n,l} \frac{(E+\epsilon^2)^l}{l!}\,. \tag{2.62}$$

Splitting the above sum

$$\sum_{\frac{|l_\pi-\mathsf{L}n|}{n} \ge a\epsilon} M_{n,l} \frac{\left(E+\epsilon^2\right)^l}{l!} = \sum_{l=0}^{(\mathsf{L}-a\epsilon)n} M_{n,l} \frac{\left(E+\epsilon^2\right)^l}{l!} + \sum_{l=(\mathsf{L}+a\epsilon)n}^{\infty} M_{n,l} \frac{\left(E+\epsilon^2\right)^l}{l!}\,, \tag{2.63}$$

we claim that both contributions vanish in the large-n limit.

Concerning the first sum, by Stanley's M-bound (2.14), and for *any* $x > 0$, we have that

$$\sum_{l=0}^{(\mathsf{L}-a\epsilon)n} M_{n,l} \frac{\left(E+\epsilon^2\right)^l}{l!} \le \sinh(x)^n \sum_{l=0}^{(\mathsf{L}-a\epsilon)n} \left(\frac{E+\epsilon^2}{x} \right)^l. \tag{2.64}$$

We choose $x \equiv E + \epsilon^2 - \epsilon$, in which case the largest term in the above sum is given by $l = \mathsf{L} - a\epsilon$, and therefore

$$\begin{aligned} (2.64) &\lesssim \sinh\left(E+\epsilon^2-\epsilon\right)^n \left(\frac{E+\epsilon^2}{E+\epsilon^2-\epsilon} \right)^{(\mathsf{L}-a\epsilon)n} \times n \\ &= n \exp\left(n \log\sinh(E+\epsilon^2-\epsilon) - (\mathsf{L}-a\epsilon)\log\left(1 - \frac{\epsilon}{E+\epsilon^2}\right)\right)\,. \end{aligned} \tag{2.65}$$

To get a handle on the above exponent we proceed by Taylor expansions around E:

$$\begin{aligned} \sinh(E+\epsilon^2-\epsilon) &= \sinh(E) + (\epsilon^2-\epsilon)\cosh(E) + (\epsilon^2-\epsilon)^2 \frac{\sinh(E)}{2} + o(\epsilon^2) \\ &= 1 + (\epsilon^2-\epsilon)\sqrt{2} + \frac{\epsilon^2}{2} + o(\epsilon^2) \qquad (\epsilon \downarrow 0). \end{aligned} \tag{2.66}$$

Further using that $\log(1-x) = 1 - x - \frac{x^2}{2} + o(x^2)$ for $x \downarrow 0$, we thus get

$$\begin{aligned} &\log\sinh(E+\epsilon^2-\epsilon) - (\mathsf{L}-a\epsilon)\log\left(1 - \frac{\epsilon}{E+\epsilon^2}\right) \\ &\quad = (\epsilon^2-\epsilon)\sqrt{2} + \frac{\epsilon^2}{2} - (\mathsf{L}-a\epsilon)\left(-\frac{\epsilon}{E} - \frac{\epsilon^2}{2E^2}\right) + o(\epsilon^2) \\ &\quad = \epsilon^2\left(\frac{1}{2} + \sqrt{2} + \frac{1}{\sqrt{2}E} - \frac{a}{E}\right) + o(\epsilon^2)\,, \end{aligned} \tag{2.67}$$

for $\epsilon \downarrow 0$. But the r.h.s. (2.67) is clearly negative as soon as $a > \frac{E}{2} + \sqrt{2}E + \frac{1}{\sqrt{2}}$, implying that the first sum in (2.63) yields no contribution in the large-n limit, as claimed.

We proceed in full analogy for the second sum, but this time around via Stanley's M-bound with $x \equiv E + \epsilon^2 + \epsilon$: an elementary estimate of the ensuing geometric series yields

$$\begin{aligned}\sum_{l=(\mathsf{L}+a\epsilon)n}^{\infty} M_{n,l}\frac{\left(E+\epsilon^2\right)^l}{l!} &\lesssim \sinh\left(E+\epsilon^2+\epsilon\right)^n\left(\frac{E+\epsilon^2}{E+\epsilon^2+\epsilon}\right)^{(\mathsf{L}+a\epsilon)n}\frac{E+\epsilon^2+\epsilon}{\epsilon}\\ &\lesssim \exp n\left(\log\sinh(E+\epsilon^2+\epsilon)-(\mathsf{L}+a\epsilon)\log\left(1+\frac{\epsilon}{E+\epsilon^2}\right)\right),\end{aligned} \tag{2.68}$$

recalling in the last step the definition of $l_{\epsilon,n} = \mathsf{L} + a\epsilon$. Once again Taylor-expanding the exponent (around E) we get

$$\log\sinh(\mathsf{E}+\epsilon^2+\epsilon)-(\mathsf{L}+a\epsilon)\log\left(1+\frac{\epsilon}{E+\epsilon^2}\right) = \epsilon^2\left(\frac{1}{2}+\sqrt{2}+\frac{1}{\sqrt{2}E}-\frac{a}{E}\right)+o(\epsilon^2), \tag{2.69}$$

for $\epsilon \downarrow 0$: as this is also negative for $a > \frac{E}{2} + \sqrt{2}E + \frac{1}{\sqrt{2}}$, the second claim is also settled, and the proof of the Theorem 3 follows. □

The rest of the paper is organised as follows. In the next Section 3 we will provide a proof of Theorem 1. In Section 4, and for technical reasons which will become clear in the course of the treatment, some additional restrictions on the candidate optimal polymers will be specified: this will lead to the identification of a *subset* $\mathcal{E}^{\epsilon}_{n,K,K'} \subset \mathcal{P}_{n,K}$ on which we will henceforth focus our attention, proving, in particular its non-emptiness. The *subset* $\mathcal{E}^{\epsilon}_{n,K,K'}$ contains two additional restrictions: the first one, which is explained in Section 4.1, concerns the *geometry* of paths (encoded by their combinatorial properties): we will introduce a *finer* sequence of hyperplanes which will allow us to force the paths to be *stretched* while crossing the original hyperplanes. The second restriction, explained in Section 4.2, concerns the way *energies* are distributed along the paths: The key step is to allow the first and last edges of the polymers to carry an unusually large fraction of the energy. Specifying these additional requirements will have an impact on the first moment as controlled in Theorem 1, and these modifications will be dealt with in Section 5. Section 6 forms the main body of the paper: there we will set up the second moment approach, postponing, however, the highly technical issues concerning the required path-counting to the Appendix. When implementing the second moment of $\mathcal{E}^{\epsilon}_{n,K,K'}$, we will

i) distinguish between the paths which share a few edges (small overlap) and the paths which share many edges (large overlap).
ii) In the case of small overlaps, i.e. when two paths share few edges, we will discriminate between two additional cases, i.e. dealing on separate footing the case where the shared edges are close to the origin/endpoint of the hypercube, or in the core of the hypercube (where the entropic cost to meet turns out to be exceedingly high and thus non-viable) .

3. The optimal Hamming distance: proof of Theorem 1

Recall that $\varphi(x) = x^x$ for $x \geq 0$, with the convention $0^0 = 1$. We shorten

$$g_{j,K}(x) \equiv \frac{\sinh(\mathsf{a}_j\mathsf{E})^x\cosh(\mathsf{a}_j\mathsf{E})^{(1-x)}\varphi\left(\frac{j-1}{K}\right)\varphi\left(1-\frac{j-1}{K}\right)}{\varphi\left(\frac{x}{2}-\frac{1}{2K}\right)\varphi\left(\frac{j-1}{K}-\left(\frac{x}{2}-\frac{1}{2K}\right)\right)\varphi\left(\frac{x}{2}+\frac{1}{2K}\right)\varphi\left(1-\frac{j-1}{K}-\left(\frac{x}{2}+\frac{1}{2K}\right)\right)}, \tag{3.1}$$

in which case, in virtue of (2.37), we may represent the $\mathcal{F}$-function as

$$\mathcal{F}_{\mathbf{a},K}(\boldsymbol{d}) = \prod_{j=1}^{K} g_{j,K}(d_j)\,. \tag{3.2}$$

Since the terms in the product on the r.h.s. are non-interacting, we clearly have

$$\max_{\boldsymbol{d}}\{\mathcal{F}_{\mathsf{a},K}(\boldsymbol{d})\} = \prod_{j=1}^{K} \max_{x\geq 0}\{g_{j,K}(x)\}. \tag{3.3}$$

We now claim that

$$\prod_{j=1}^{K} \max_{x\geq 0}\{g_{j,K}(x)\} = 1, \tag{3.4}$$

and

$$\arg\max_{x\geq 0}\ \{g_{j,K}(x)\} = \mathsf{d}_j, \tag{3.5}$$

with d_j as in (2.44).

We will prove (3.5) first. We begin with the cases $j = 1, K$ and claim that

$$\arg\max_{x\geq 0} g_{1,K}(x) = \arg\max_{x\geq 0} g_{K,K}(x) = \frac{1}{K}\,, \tag{3.6}$$

and

$$\frac{1}{K} = \mathsf{d}_1 = \mathsf{d}_K. \tag{3.7}$$

In fact, $g_{1,K}(x)$ involves the terms

$$\begin{gathered}\varphi\left(\frac{x}{2}-\frac{1}{2K}\right),\\ \varphi\left(\frac{j-1}{K}-\left(\frac{x}{2}-\frac{1}{2K}\right)\right)\Big|_{j=1} = \varphi\left(\frac{1}{2K}-\frac{x}{2}\right),\end{gathered} \tag{3.8}$$

but for both to be properly defined it must hold

$$\frac{x}{2}-\frac{1}{2K}\geq 0, \quad \text{and} \quad \frac{1}{2K}-\frac{x}{2}\geq 0, \tag{3.9}$$

implying $x = \frac{1}{K}$. A similar reasoning applies to $g_{K,K}$, and (3.6) is settled. Claim (3.7) follows from (2.25) for the $j = 1$ case, whereas the $j = K$ case follows by symmetry, see in particular (2.28).

Concerning the other indices, we fix $j \in \{2, \dots, K-1\}$ and shorten, for $x \geq 0$,

$$g_{j,K}(x) \equiv \frac{N_{j,K}(x)}{D_{j,K}(x)}, \tag{3.10}$$

where

$$N_{j,K}(x) \equiv \sinh(\mathsf{a}_j\mathsf{E})^x \cosh(\mathsf{a}_j\mathsf{E})^{(1-x)} \varphi\left(\frac{j-1}{K}\right)\varphi\left(1-\frac{j-1}{K}\right), \tag{3.11}$$

and

$$D_{j,K}(x) \equiv \varphi\left(\frac{x}{2}-\frac{1}{2K}\right)\varphi\left(\frac{j}{K}-\frac{x}{2}-\frac{1}{2K}\right)\varphi\left(\frac{x}{2}+\frac{1}{2K}\right)\varphi\left(1-\frac{j}{K}-\frac{x}{2}+\frac{1}{2K}\right). \tag{3.12}$$

Taking the x-derivative, we see that

$$g_{j,K}(x)' > 0 \iff N_{j,K}(x)' D_{j,K}(x) > N_{j,K}(x) D_{j,K}(x)'. \tag{3.13}$$

An elementary computation then yields

$$N_{j,K}(x)' = N_{j,K}(x)\log(\tanh(\mathsf{a}_j\mathsf{E})), \tag{3.14}$$

and

$$D_{j,K}(x)' = \frac{1}{2} D_{j,K}(x) \log\left(\frac{(\frac{x}{2}-\frac{1}{2K})(\frac{x}{2}+\frac{1}{2K})}{(\frac{j}{K}-\frac{x}{2}-\frac{1}{2K})(1-\frac{j}{K}-\frac{x}{2}+\frac{1}{2K})}\right). \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we therefore get

$$g_{i,K}(x)' > 0 \iff \tanh(\mathsf{a}_j\mathsf{E})^2 > \frac{(\frac{x}{2}-\frac{1}{2K})(\frac{x}{2}+\frac{1}{2K})}{(\frac{j}{K}-\frac{x}{2}-\frac{1}{2K})(1-\frac{j}{K}-\frac{x}{2}+\frac{1}{2K})}. \tag{3.16}$$

Consider now

$$\tanh(\mathsf{a}_j\mathsf{E})^2 = \frac{(\frac{x}{2}-\frac{1}{2K})(\frac{x}{2}+\frac{1}{2K})}{\left(\frac{j}{K}-\frac{x}{2}-\frac{1}{2K}\right)(1-\frac{j}{K}-\frac{x}{2}+\frac{1}{2K})}. \tag{3.17}$$

This is a quadratic equation (in x), whose unique positive solution is given by

$$\hat{x} \equiv -\sinh(\mathsf{a}_j\mathsf{E})^2 + \sqrt{\sinh(\mathsf{a}_j\mathsf{E})^4 + 4\sinh(\mathsf{a}_j\mathsf{E})^2\left(\frac{2j-1}{2K}-\frac{j(j-1)}{K^2}\right)+\frac{1}{K^2}}. \tag{3.18}$$

A straightforward analysis shows that the quotient on the r.h.s. of (3.16) is, in fact, increasing in x: in other words, the x-derivative $g'_{i,K}$ is positive for $x < \hat{x}$ and negative for $x > \hat{x}$, implying that $\hat{x}$ is indeed the extremal point. To finish the proof of (3.5) it thus remains to show that $\hat{x} = \mathsf{d}_j$, i.e. that $\hat{x} = \sinh(\mathsf{a}_j\mathsf{E})\cosh((1-\mathsf{a}_j)\mathsf{E})$. In order to do so, we will avoid the use of the explicit formulation (3.18), but rely rather on the expression (3.17) and the following

Lemma 3.1. *Let $d \in \mathbb{R}$ satisfy*

$$\frac{d}{2}-\frac{1}{2K} = \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})\,. \tag{3.19}$$

Then the above, and the following relations are all equivalent:

$$1-\frac{j}{K}-\frac{d}{2}+\frac{1}{2K} = \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E}), \tag{3.20}$$

$$\frac{d}{2}+\frac{1}{2K} = \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}), \tag{3.21}$$

$$\frac{j}{K}-\frac{d}{2}-\frac{1}{2K} = \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}). \tag{3.22}$$

It follows in particular, that for such d it holds $d = \hat{x}$, and $d = \mathsf{d}_j$.

Proof of Lemma 3.1: We first prove the equivalence of

$$(3.19) \iff (3.20) \iff (3.21) \iff (3.22), \tag{3.23}$$

Indeed, by (2.25) and the fact that

$$\sinh(\overline{\mathsf{a}_j}\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}) + \cosh(\bar{\mathsf{a}}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E}) = 1, \tag{3.24}$$

it holds:

$$\sinh(\underline{\mathsf{a}}_j\mathsf{E})\cosh(\bar{\mathsf{a}}_j\mathsf{E}) = 1-\frac{j}{K} \tag{3.25}$$

for all $j = 1\dots K$. Relation (3.19) therefore implies that

$$\begin{aligned} 1-\frac{j}{K}-\frac{d}{2}+\frac{1}{2K} &= 1-\frac{j}{K}-\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E}) \\ &= (\cosh(\bar{\mathsf{a}}_j\mathsf{E})-\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E}))\sinh(\underline{\mathsf{a}}_j\mathsf{E}) \\ &= \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})\,, \end{aligned} \tag{3.26}$$

the second equality with (3.25) and the last by the addition formula $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$. Thus,

$$(3.19) \iff (3.20). \tag{3.27}$$

A similar computation gives that

$$(3.21) \iff (3.22). \tag{3.28}$$

It remains to prove that

$$(3.19) \iff (3.21). \tag{3.29}$$

To see this we note that (3.19) yields

$$\frac{d}{2} + \frac{1}{2K} = \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E}) + \frac{1}{K}\,, \tag{3.30}$$

but combining the fundamental r.h.s (2.25) and (3.25) gives that

$$\frac{1}{K} = \sinh(\underline{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) - \sinh(\underline{\mathsf{a}}_j\mathsf{E})\cosh(\bar{\mathsf{a}}_j\mathsf{E})\,. \tag{3.31}$$

Thus, by (3.31), we see that

$$\begin{aligned}(3.30) &= \sinh(\underline{\mathsf{a}}_j\mathsf{E})\left(\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_jE) - \cosh(\bar{\mathsf{a}}_j\mathsf{E})\right) + \sinh(\underline{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \\ &= -\sinh(\underline{\mathsf{a}}_j\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) + \sinh(\underline{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}),\end{aligned} \tag{3.32}$$

the last equality again by the addition formula $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$. Hence

$$\begin{aligned}(3.32) &= \left(-\sinh(\underline{\mathsf{a}}_j\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E}) + \sinh(\underline{\mathsf{a}}_j\mathsf{E} + \mathsf{a}_j\mathsf{E})\right)\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \\ &= \cosh(\underline{\mathsf{a}}_j\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\,,\end{aligned} \tag{3.33}$$

and (3.23) is established.

Let now d satisfy any of the equivalent (3.19)-(3.22). It holds:

$$\begin{aligned}&\frac{(\frac{d}{2} - \frac{1}{2K})(\frac{d}{2} + \frac{1}{2K})}{(\frac{j}{K} - \frac{d}{2} - \frac{1}{2K})(1 - \frac{j}{K} - \frac{d}{2} + \frac{1}{2K})} = \\ &\quad = \frac{\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E}) \times \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E})}{\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}) \times \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})} \\ &\quad = \tanh(\mathsf{a}_j\mathsf{E})^2\,,\end{aligned} \tag{3.34}$$

hence, by uniqueness of the (positive) solution of (3.17), we deduce that $d = \hat{x}$.

Finally, it holds:

$$\begin{aligned}d &= \frac{d}{2} + \frac{1}{2K} + \frac{d}{2} - \frac{1}{2K} \\ &= \sinh(\mathsf{a}_i\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}) + \sinh(\mathsf{a}_i\mathsf{E})\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})\,,\end{aligned} \tag{3.35}$$

the last equality by (3.19) and (3.21), hence

$$\begin{aligned}d &= \sinh(\mathsf{a}_i\mathsf{E}) \times \left(\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}) + \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})\right) \\ &= \sinh(\mathsf{a}_j\mathsf{E}) \times \cosh((1-\mathsf{a}_j)\mathsf{E}),\end{aligned} \tag{3.36}$$

by the addition formula for hyperbolic functions (and using that $\bar{\mathsf{a}}_{j-1} + \underline{\mathsf{a}}_j = 1 - \mathsf{a}_j$, by definition), settling the claim that $d = \mathsf{d}_j$.

□

The remaining Claim (3.4) is taken care of by the following Lemma, which tracks the evolution of the g-product while changing the hyperplane-index.

Lemma 3.2 (Evolution Lemma)**.** *For any $i = 1 \dots K$, it holds:*

$$\prod_{j=1}^{i} g_{j,K}(\mathsf{d_j}) = \left[\frac{\sinh(\bar{\mathsf{a}}_i\mathsf{E})}{\frac{i}{K}}\right]^{\frac{i}{K}} \left[\frac{\cosh(\bar{\mathsf{a}}_i\mathsf{E})}{1-\frac{i}{K}}\right]^{1-\frac{i}{K}}. \tag{3.37}$$

Furthermore,

$$\prod_{j=1}^{K} g_{j,K}(\mathsf{d_j}) = 1\,. \tag{3.38}$$

Proof: We will proceed by induction over i. The cases $K = 1, 2$ are trivial, so let $K \geq 3$. Recalling that $\mathsf{d}_1 = \frac{1}{K}$, we therefore have that

$$g_{1,K}(\mathsf{d}_1) = \left[\frac{\sinh(\mathsf{a}_1\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}} \left[\frac{\cosh(\mathsf{a}_1\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}, \tag{3.39}$$

which settles the base case $i = 1$. We thus assume that (3.37) holds for an $i \in \{1, K-2\}$, and show that this implies the validity of the $(i+1)$-case, namely that

$$\left[\frac{\sinh(\bar{\mathsf{a}}_i\mathsf{E})}{\frac{i}{K}}\right]^{\frac{i}{K}} \left[\frac{\cosh(\bar{\mathsf{a}}_i\mathsf{E})}{1-\frac{i}{K}}\right]^{1-\frac{i}{K}} g_{i+1,K}(\mathsf{d}_{i+1}) = \left[\frac{\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{\frac{i+1}{K}}\right]^{\frac{i+1}{K}} \left[\frac{\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{1-\frac{i+1}{K}}\right]^{1-\frac{i+1}{K}}. \tag{3.40}$$

Remark that by (3.17),

$$\begin{aligned}\sinh(\mathsf{a}_{i+1}\mathsf{E})^{\mathsf{d}_{i+1}}\cosh(\mathsf{a}_{i+1}\mathsf{E})^{1-\mathsf{d}_{i+1}} &= \tanh(\mathsf{a}_{i+1}\mathsf{E})^{\mathsf{d}_{i+1}}\cosh(\mathsf{a}_{i+1}\mathsf{E}) \\ &= \left[\frac{(\frac{d_{i+1}}{2}-\frac{1}{2K})(\frac{d_{i+1}}{2}+\frac{1}{2K})}{\left(\frac{i+1}{K}-\frac{d_{i+1}}{2}-\frac{1}{2K}\right)(1-\frac{i+1}{K}-\frac{d_{i+1}}{2}+\frac{1}{2K})}\right]^{\frac{d_{i+1}}{2}} \cosh(\mathsf{a}_{i+1}\mathsf{E})\,.\end{aligned} \tag{3.41}$$

By definition of $g_{i+1,K}$, the above, and simple rearrangements, we thus have

$$g_{i+1,K}(\mathsf{d}_{i+1}) = \frac{(\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K})^{\frac{1}{2K}}\cosh(\mathsf{a}_{i+1}E)\left(\frac{i}{K}\right)^{\frac{i}{K}}(1-\frac{i}{K})^{1-\frac{i}{K}}}{(\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K})^{\frac{i+1}{K}-\frac{1}{2K}}(\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K})^{\frac{1}{2K}}(1-\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K})^{1-\frac{i+1}{K}+\frac{1}{2K}}}. \tag{3.42}$$

Thus (3.40) is equivalent to prove that

$$\begin{aligned}&\left[\frac{\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K}}{\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K}}\right]^{\frac{1}{2K}} \left[\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K}\right]^{\frac{i+1}{K}-\frac{1}{2K}} \left[1-\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K}\right]^{1-\frac{i+1}{K}+\frac{1}{2K}} \\ &\qquad = \frac{\sinh(\bar{\mathsf{a}}_i\mathsf{E})^{\frac{i}{K}}\cosh(\bar{\mathsf{a}}_i\mathsf{E})^{1-\frac{i}{K}}\cosh(\mathsf{a}_{i+1}\mathsf{E})}{\left[\frac{\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{\frac{i+1}{K}}\right]^{\frac{i+1}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{1-\frac{i+1}{K}}\right]^{1-\frac{i+1}{K}}}\,.\end{aligned} \tag{3.43}$$

We now rewrite the term on the l.h.s. (3.43) as

$$\begin{aligned}&\left[\frac{(\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K})(\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K})}{(\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K})(1-\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K})}\right]^{\frac{1}{2K}} \\ &\qquad\times\left[\frac{\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}-\frac{1}{2K}}{1-\frac{i+1}{K}-\frac{\mathsf{d}_{i+1}}{2}+\frac{1}{2K}}\right]^{\frac{i}{K}} \\ &\qquad\qquad\times\left[1-\frac{i+1}{K}-\frac{d_{i+1}}{2}+\frac{1}{2K}\right],\end{aligned} \tag{3.44}$$

and the term on the r.h.s. of (3.43) as

$$\begin{aligned}
&\left[\left(\frac{\frac{i+1}{K}\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{(1-\frac{i+1}{K})\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}\right)^2\right]^{\frac{1}{2K}} \\
&\qquad\times\left[\frac{\frac{i+1}{K}\tanh(\bar{\mathsf{a}}_i\mathsf{E})\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{(1-\frac{i+1}{K})\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}\right]^{\frac{i}{K}} \\
&\qquad\qquad\times\frac{\cosh(\bar{\mathsf{a}}_iE)\cosh(\mathsf{a}_{i+1}E)(1-\frac{i+1}{K})}{\cosh(\bar{\mathsf{a}}_{i+1}E)} \qquad (3.45)\\
&=\left[\left(\frac{\cosh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}{\sinh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}\right)^2\right]^{\frac{1}{2K}} \\
&\qquad\times\left[\frac{\tanh(\bar{\mathsf{a}}_i\mathsf{E})\cosh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}{\sinh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}\right]^{\frac{i}{K}} \\
&\qquad\qquad\times\cosh(\bar{\mathsf{a}}_i\mathsf{E})\cosh(\mathsf{a}_{i+1}\mathsf{E})\sinh(\underline{\mathsf{a}}_{i+1}\mathsf{E})\,,
\end{aligned}$$

the last step by (2.25) and (3.25). But by (3.19), (3.20), (3.21) and (3.22), the terms raised to the same powers in (3.44) and the r.h.s. of (3.45) coincide, settling the induction step.

We now move to (3.38). It holds:

$$\begin{aligned}
\prod_{j=1}^{K} g_{j,K}(\mathsf{d}_j) &= \prod_{j=1}^{K-1} g_{j,K}(\mathsf{d}_j)\; g_{K,K}(\mathsf{d}_K) \\
&= \left[\frac{\sinh(\bar{\mathsf{a}}_{K-1}\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_{K-1}\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}}\sinh(\mathsf{a}_K\mathsf{E})^{\frac{1}{K}}\cosh(\mathsf{a}_K\mathsf{E})^{1-\frac{1}{K}} \qquad (3.46)\\
&= \left[\frac{\sinh(\bar{\mathsf{a}}_{K-1}\mathsf{E})\cosh(\mathsf{a}_K\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_{K-1}\mathsf{E})\sinh(\mathsf{a}_K\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}},
\end{aligned}$$

the second equality by the induction step, and the third by simple rearrangements. By the $\mathsf{a}'s$ symmetry (2.28), and the normalization $\sum_{i=1}^{K}\mathsf{a}_i=1$, it thus holds

$$(3.46)=\left[\frac{\sinh(\bar{\mathsf{a}}_{K-1}\mathsf{E})\cosh(\underline{\mathsf{a}}_{K-1}\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}\left[\frac{\cosh((1-\mathsf{a}_1)\mathsf{E})\sinh(\mathsf{a}_1\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}}=1, \qquad (3.47)$$

the last equality by the fundamental (2.25). □

4. Taming optimal polymers

In order to prove our main result Theorem 2, we will show non-emptiness of a *subset* of $\mathcal{P}_{n,K}$, whose paths satisfy additional properties. As a matter of fact, we will introduce two additional restrictions: the first one, which is explained in Section 4.1, concerns the *geometry* of paths, i.e. their combinatorial properties. The second restriction, explained in Section 4.2, concerns the way *energies* are distributed along the paths. Both restrictions will be of course inspired by/in line with the above Insights. We emphasize that the reason for restricting the candidate polymers further is here chiefly technical: the additional requirements we are about to introduce will in fact lead to a considerable simplification of some otherwise daunting combinatorial estimates.

4.1. *A sprinckle of microstructure.* We introduce yet another coarse graining: for $i = 0 \dots K-1$, we split the region between two consecutive hyperplanes H_{i-1} and H_i further, into K' additional slabs:

$$H'_{i,j} \equiv \left\{ v \in V_n, d(0,v) = \left(i + \frac{j}{K'} \right) \hat{n}_K \right\}, \qquad j = 0 \dots K', \tag{4.1}$$

(remark that $H'_{i,0} = H_i$ and $H'_{i,K'} = H_{i+1}$), and focus henceforth on the subset

$$\begin{aligned} \mathcal{P}_{n,K,K'} \equiv\ & \text{all polymers } \pi \in \mathcal{P}_{n,K} \text{ which cover} \\ & \text{a (normalized) Hamming distance } (\mathsf{ef_i} + \mathsf{eb_i})/K' \\ & \text{while connecting the hyperplanes } H'_{i,j} \text{ and } H'_{i,j+1}, \\ & \text{for } j = 0 \dots K' \text{ and } i = 1 \dots K-1. \end{aligned} \tag{4.2}$$

The subset $\mathcal{P}_{n,K,K'}$ is of course motivated by Insight 2.4: adding an additional level of coarse graining and spreading the backsteps as evenly as possible among the K'-slabs, allows to rule out polymers where backsteps tend to accumulate, cfr. Figure 4.11 and 4.12.

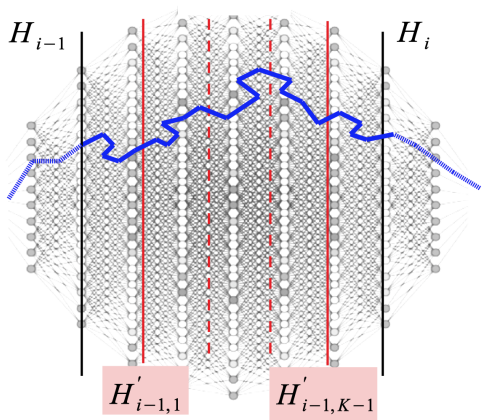

FIGURE 4.11. The backsteps (five in total) are spread as evenly as possible: one for each sublayer H'.

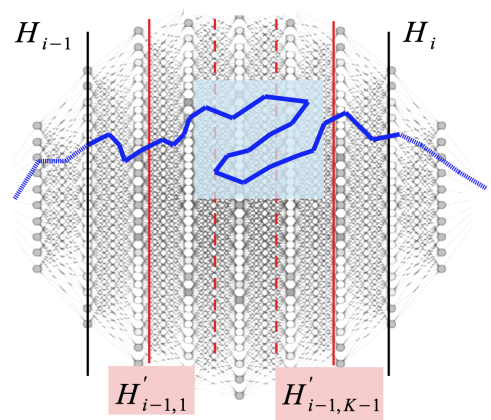

FIGURE 4.12. The five backsteps are lumped together: this polymer wouldn't belong to $\mathcal{P}_{n,K,K'}$.

Finally, we render the H-hyperplanes (of the coarser layer) *repulsive*, i.e. we force paths to cross them only once. As we will see shortly, see Lemma 4.1 below, this can be achieved by considering the following (sub)subset of polymers:

$$\begin{aligned} \mathcal{P}^{\mathsf{rep}}_{n,K,K'} \equiv\ & \text{all polymers } \pi \in \mathcal{P}_{n,K,K'} \text{ which connect the hyperplanes } H'_{i,0} \text{ and } H'_{i,1} \\ & \text{by first making } (\mathsf{ef}_i\ \hat{n}_{K'}) \text{ steps forward and only then } (\mathsf{eb}_i\ \hat{n}_{K'}) \text{ backsteps,} \\ & \text{and which connect the hyperplanes } H'_{i,K'-1} \text{ and } H'_{i,K'} \\ & \text{by first making } (\mathsf{eb}_i\ \hat{n}_{K'}) \text{ backsteps, and only then } (\mathsf{ef}_i\ \hat{n}_{K'}) \text{ steps forward,} \\ & \text{for } i = 1 \dots K. \end{aligned} \tag{4.3}$$

Note that $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ is still a deterministic set. A graphical rendition is given in Figure 4.13.

Remark that, by construction,

$$\mathcal{P}^{\mathsf{rep}}_{n,K,K'} \subset \mathcal{P}_{n,K,K'} \subset \mathcal{P}_{n,K} \left\{ \mathbf{d}_{opt}, \boldsymbol{\gamma}_{opt} \right\}. \tag{4.4}$$

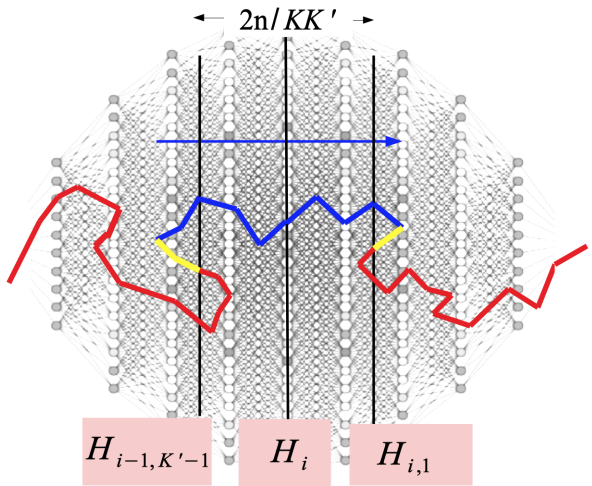

FIGURE 4.13. A path in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$: red edges correspond to the free evolution of the path, yellow edges are backsteps, and blue edges are forward steps.

Our main Theorem 2 will therefore follow as soon as we prove that one can find polymers in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ which reach the ground state energy. Before seeing how this goes, here is the aforementioned result stating that H-hyperplanes are indeed repulsive:

Lemma 4.1. *For $K \geq 1$ the following holds true: a polymer $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ crosses the hyperplanes $H_1, \dots, H_K$ only once.*

Proof: The statement is trivial in the directed phase, so let $i \in \{m \dots K-m\}$.

There is of course a certain *directivity* in the polymers' evolution: this is captured by the fact that $\mathsf{ef}_i > \mathsf{eb}_i$ for all $i = 1 \dots K$ (see in particular the second relation in (2.36)), and graphically represented by evolutions "from the left to the right".

Sticking to this graphical convention, we begin with the case "to the right of the H_i-hyperplane": after crossing this hyperplane, a path $\pi \in \mathcal{P}_{n,K,K'}$ is bound to first make $(\mathsf{ef}_i\ \hat{n}_{K'})$ steps to the right (forward) and only then to make $(\mathsf{eb}_i\ \hat{n}_{K'})$ steps to the left (backwards). At this point, and by construction, the polymer will find itself on $H'_{i,1}$. Continuing its evolution, the polymer will eventually reach from there the next hyperplane $H'_{i,2}$, again through $(\mathsf{ef}_i\ \hat{n}_{K'})$ steps to the right, and $(\mathsf{eb}_i\ \hat{n}_{K'})$ steps to the left. Since in this phase no restriction is imposed on the *order* of back- and forwardsteps, it could thus happen that the polymer first performs all available steps to the left, in one fell swoop: this would increase the proximity of the polymer to H_i, with the hyperplane potentially even crossed for a second time. However, we claim that even in such worst case scenario, the polymer will find itself well to the right of H_i. In other words we claim that

$$\mathsf{ef}_i n_{K'} - 2\mathsf{eb}_i n_{K'} > 0, \tag{4.5}$$

or, which is the same, that

$$\mathsf{ef}_i - 2\mathsf{eb}_i > 0. \tag{4.6}$$

Indeed, it follows from (2.37) that

$$\begin{aligned}\mathsf{ef}_i - 2\mathsf{eb}_i &= \frac{\mathsf{d}_i}{2} + \frac{1}{2K} - 2\left(\frac{\mathsf{d}_i}{2} - \frac{1}{2K}\right)\\ &= \frac{1}{K} - \left(\frac{\mathsf{d}_i}{2} - \frac{1}{2K}\right)\\ &= \frac{1}{K} - \mathsf{eb}_i,\end{aligned} \tag{4.7}$$

the last step again by (2.37). Our new claim thus states that for large enough K,

$$\frac{1}{K} - \mathsf{eb}_i > 0. \tag{4.8}$$

To see this, we recall that by (3.19), the number of effective backsteps between hyperplanes in the stretched phase satisfies

$$\mathsf{eb}_i = \sinh(\bar{\mathsf{a}}_{i-1}\mathsf{E})\sinh(\mathsf{a}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E}). \tag{4.9}$$

Real analysis shows that

$$\arg\max_{y\in[0,1]} \sinh(y\mathsf{E})\sinh((1-y)\mathsf{E}) = \frac{1}{2}. \tag{4.10}$$

Furthermore, by (2.29),

$$\mathsf{a}_i\mathsf{E} \le \frac{1}{K}, \tag{4.11}$$

which, together with an elementary large-K Taylor expansion, implies that

$$\sinh(\mathsf{a}_i\mathsf{E}) = \mathsf{a}_i\mathsf{E} + \frac{(\mathsf{a}_i\mathsf{E})^3}{6} \le \frac{1}{K} + \frac{1}{6K^3} \le \frac{2}{K}, \tag{4.12}$$

for $K \ge 1$. Using (4.12) in (4.9) we get

$$\begin{aligned}\mathsf{eb}_i &\le \sinh(\bar{\mathsf{a}}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E}) \times \frac{2}{K}\\ &\le \sinh\left(\frac{\mathsf{E}}{2}\right)^2 \times \frac{2}{K}.\end{aligned} \tag{4.13}$$

the second inequality by (4.10). The first term on the r.h.s. above can be easily estimated:

$$\begin{aligned}\sinh\left(\frac{\mathsf{E}}{2}\right)^2 &= \frac{1}{4}\left(e^{\mathsf{E}/2} - e^{-\mathsf{E}/2}\right)^2 = \frac{1}{4}\left(e^{\mathsf{E}} - 2 + e^{-\mathsf{E}}\right)\\ &= \frac{1}{2}\left(\cosh(\mathsf{E}) - 1\right) = \frac{1}{2}\left(\sqrt{1+\sinh^2(\mathsf{E})} - 1\right)\\ &= \frac{1}{2}\left(\sqrt{2} - 1\right),\end{aligned} \tag{4.14}$$

the step before last by the Pythagorean's identity for hyperbolic functions, and the last since $\sinh(\mathsf{E}) = 1$ by definition. In particular, we see that

$$\sinh\left(\frac{\mathsf{E}}{2}\right)^2 \le \frac{1}{4}. \tag{4.15}$$

Using this in (4.13) we thus get $\mathsf{eb}_i \le \frac{1}{2K}$, hence

$$\frac{1}{K} - \mathsf{eb}_i \ge \frac{1}{2K} > 0, \tag{4.16}$$

settling claim (4.8), and therefore (4.6).

Summarizing the upshot of these considerations, we thus see that *after* crossing an H-plane for the first time, the polymer will forever remain "to its right". But by symmetry, a similar line of reasoning

holds also for the case "to the left", i.e. for paths making $(\mathsf{eb}_i \hat{n}_{K'})$ steps to the left, and then $(\mathsf{ef}_i \hat{n}_{K'})$ steps to the right *before* reaching such hyperplane. Lemma 4.1 is therefore established. □

Remark 4.2. Polymers in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ are, in fact, loopless: this follows from Lemma 4.1, and the property that paths make no detours between H-planes.

4.2. *Partitioning the energy.* We will eventually implement the multiscale refinement of the second moment method Gayrard and Kistler (2015), a procedure which involves a number of steps. The first, and key, step is to break the self-similarity of the underlying random field: this can be achieved here by allowing the first and last edges of the polymers to carry an unusually large fraction of the energy, and handling these on different footing. This procedure has already been succesfully implemented for the problem of (directed) first passage percolation in Kistler et al. (2020a), see also Remark 6.2 below for more on this issue.

We need some additional notation: since a path $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ consists of a set of edges which uniquely characterises the vertices visited by the polymer, by a a slight abuse of notation we will denote by $\pi \cap H_i$ the vertices that lie both in H_i and between two edges of the π-path.

For a polymer $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$, we begin by writing its energy as

$$X_\pi = \mathcal{F}_\pi + \left(X_m(\pi) + \left[\sum_{j=m+1}^{K-m} X_{j-1,j}(\pi) \right] + X_{K-m+1}(\pi) \right) + \mathcal{L}_\pi \,, \tag{4.17}$$

with the following notational conventions:

- $\mathcal{F}_\pi \equiv X_{[\pi]_1}$ is the energy of the first edge of the path;
- $X_m(\pi) \equiv \sum_{j=2}^{m\hat{n}_K} \xi_{[\pi]_j}$ is the energy of the substrand connecting the second visited vertex to the mth-hyperplane, i.e. $\mathbf{0}$ to the mth-hyperplane, *but with the first edge excluded*;
- For $i = m+1 \dots K-m$,
$$X_{i-1,i}(\pi) \equiv X_\pi(\pi \cap H_{i-1}, \pi \cap H_i) \tag{4.18}$$
is the energy of the substrand connecting consecutive H-hyperplanes;
- $X_{K-m+1}(\pi)$ is the energy of the substrand connecting the $(K-m)^{th}$-hyperplane to $\mathbf{1}$, *but with the last edge excluded*;
- $\mathcal{L}_\pi$ is the energy of the last edge of the path.

For $\epsilon > 0$, recalling $\{\mathsf{a}_i\}_{i=1}^K$ solutions of (2.26) and the convention $\bar{\mathsf{a}}_m = \sum_{i \le m} \mathsf{a}_i$, we set

$$\tilde{\mathsf{a}}_{m,\epsilon} \equiv \bar{\mathsf{a}}_m \left(\mathsf{E} + \frac{\epsilon}{5} \right) + \frac{\epsilon}{5} \,, \tag{4.19}$$

and

$$\tilde{\mathsf{a}}_{K-m+1,\epsilon} \equiv \tilde{\mathsf{a}}_{m,\epsilon} \,, \tag{4.20}$$

and for $i = m+1 \dots K-m$,

$$\mathsf{a}_{i,\epsilon} \equiv \mathsf{a}_i \left(\mathsf{E} + \frac{\epsilon}{5} \right) \,. \tag{4.21}$$

We then introduce the following subsets of polymers:

$$\mathcal{E}^{1,\epsilon}_{n,K,K'} \equiv \pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'} \text{ such that } \mathcal{F}_\pi,\ \mathcal{L}_\pi \le \epsilon/5. \tag{4.22}$$

$$\begin{aligned} \mathcal{E}^{2,\epsilon}_{n,K,K'} \equiv \pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'} &\text{ such that} \\ & X_m(\pi),\ X_{K-m+1}(\pi) \le \tilde{\mathsf{a}}_{m,\epsilon}, \\ & X_{i-1,i}(\pi) \le \mathsf{a}_{i,\epsilon} \text{ for } i = m+1 \dots K-m \,. \end{aligned} \tag{4.23}$$

Recalling that $\bar{\mathsf{a}}_m + \sum_{i=m+1}^{K-m} \mathsf{a}_i + \underline{\mathsf{a}}_{K-m} = 1$, we emphasize that the newly constructed subset consists of polymers with sub-energies

$$\overline{X}_m^{K-m+1}(\pi) \equiv X_m(\pi) + \left[\sum_{j=m+1}^{K-m} X_{j-1,j}(\pi)\right] + X_{K-m+1}(\pi) \leq \mathsf{E} + \frac{3}{5}\epsilon\,, \tag{4.24}$$

and with first resp. last edges carrying unusually large an energy (potentially up to $\epsilon/5$). At last, we consider the sub-subset

$$\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \equiv \mathcal{E}^{1,\epsilon}_{n,K,K'} \cap \mathcal{E}^{2,\epsilon}_{n,K,K'}\,. \tag{4.25}$$

Thus, by definition, the polymers in $\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}$ have energies less than $\mathsf{E} + \epsilon$. A graphical rendition of this set is given in Figure 4.14.

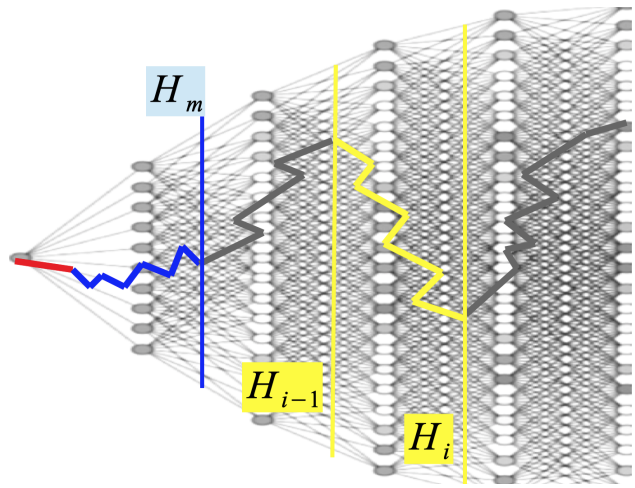

FIGURE 4.14. Distributing the energy in the first half of the hypercube. The first edge (red) has energy less than $\epsilon/5$. The blue strand is in the directed phase, and corresponds to $X_m(\pi) \leq \bar{\mathsf{a}}_m$. The yellow strand is in the stretched phase, it connects two consecutive H-hyperplanes with sub-energy less than $\mathsf{a}_{i,\epsilon}$. For the second half of the hypercube, an analogous (mirror) picture holds.

4.3. *Connecting first and last region.* By definition, and recalling the inclusions (4.4), it clearly holds that

$$\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \subset \mathcal{E}^{\epsilon}_{n,K}\,. \tag{4.26}$$

In particular, non-emptiness of $\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}$ will immediately yield our main Theorem 2, and this is indeed the route we take. Precisely, we will show that one can connect the first and last edges through polymers satisfying the energy requirements in the directed/stretched phases. To see how this goes, we begin with the observation that

$$\begin{aligned}\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1\right) &\geq \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1,\ \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor\right) \\ &= \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor\right) \mathbb{P}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor\right).\end{aligned} \tag{4.27}$$

By independence, it clearly holds that

$$\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathbb{P}\left(\mathcal{F}_\pi \leq \frac{\epsilon}{5}\right)\mathbb{P}\left(\mathcal{L}_\pi \leq \frac{\epsilon}{5}\right)\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} = C(\epsilon)^2\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}, \tag{4.28}$$

where

$$C(\epsilon) \equiv 1 - \exp(-\epsilon/5)\,. \tag{4.29}$$

We now claim that

$$\lim_{n\to\infty} \mathbb{P}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor \right) = 1. \tag{4.30}$$

Indeed, by Chebycheff's inequality, and for $\delta > 0$,

$$\mathbb{P}\left(\left| \frac{\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{\mathbb{E}(\#\mathcal{E}^{1,\epsilon}_{n,K,K'})} - 1 \right| \geq \delta \right) \leq \frac{1}{\delta^2} \left(\frac{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2 \right)}{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \right)^2} - 1 \right)\,. \tag{4.31}$$

Let now $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ and denote by

$$f_\pi(n,k) \equiv \text{the number of paths in } \mathcal{P}^{\mathsf{rep}}_{n,K,K'} \text{ sharing } k \text{ weigthed edges with } \pi. \tag{4.32}$$

Since for paths in $\mathcal{E}^{1,\epsilon}_{n,K,K'}$ only the first and the last edges are weighted,

$$\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2 \right) \leq \mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \right)^2 + \#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} \left(C(\epsilon)^3 f_\pi(n,1) + C(\epsilon)^2 f_\pi(n,2) \right)\,, \tag{4.33}$$

the first term on the r.h.s. corresponding to the case of $k = 0$ shared edges. Using that $C(\epsilon) \leq 1$ and that $f_\pi(n,2) \leq f_\pi(n,1)$, the above becomes

$$\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2 \right) \leq \mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \right)^2 + 2\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} f_\pi(n,1)\,. \tag{4.34}$$

Therefore, for the r.h.s. of (4.31) we have

$$\frac{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2 \right)}{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \right)^2} - 1 \leq \frac{2\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} f_\pi(n,1)}{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \right)^2} = \frac{2}{C(\epsilon)^4} \frac{f_\pi(n,1)}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}}\,. \tag{4.35}$$

Let now $f^l_\pi(n,1)$ be the number of paths which share one edge with π on the *left* of the hypercube. Clearly, $f^l_\pi(n,1) = 2f_\pi(n,1)$, hence

$$\frac{f_\pi(n,1)}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} = 2\frac{f^l_\pi(n,1)}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} \leq 2\frac{(m\hat{n}_K - 1)!}{(m\hat{n}_K)!} = \left(\frac{2K}{m}\right)\frac{1}{n}, \tag{4.36}$$

where for the key inequality we have used that there are $(m\hat{n}_K)!$ possibilities to reach a given (admissible) vertex on the H_m-plane, but specifying the first edge reduces such possibilities to $(m\hat{n}_K - 1)!$. Using (4.36) in (4.35) and then (4.31) we thus obtain

$$\mathbb{P}\left(\left| \frac{\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{\mathbb{E}(\#\mathcal{E}^{1,\epsilon}_{n,K,K'})} - 1 \right| \geq \delta \right) \lesssim \frac{1}{n} \longrightarrow 0\,, \tag{4.37}$$

as $n \uparrow \infty$, which settles claim (4.30). Using the latter in (4.27) then yields

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \right) \geq \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor \right) - o_n(1). \tag{4.38}$$

Now, for any $\mathsf{J} \leq \#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$, it holds that

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \mathsf{J} \right) \geq \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J} \right)\,, \tag{4.39}$$

since the more paths survive the "thinning procedure" via the energy condition on first and last edge, the higher the chance to find at least a connecting polymer which satisfies the imposed energy requirements. See Figure 4.15 for a graphical rendition.

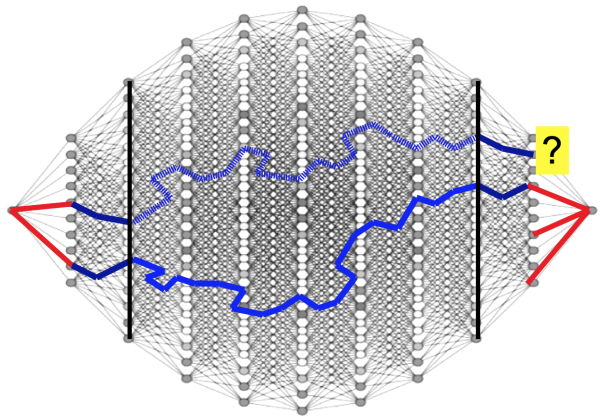

FIGURE 4.15. The first and last edges carrying an energy less than $\epsilon/5$ (hence surviving the thinning procedure) are drawn in red. The continuous blue strand manages to connect these edges while satisfying the energy constraints, whereas the dashed strand does not.

Using (4.39) with

$$\mathsf{J} \equiv \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor , \tag{4.40}$$

and by the Paley-Zygmund inequality, we thus get

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right) \geq \frac{\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right)^2}{\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}{}^2 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right)}. \tag{4.41}$$

Consider now *any* deterministic set $\mathcal{J} \subset \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ with cardinality $\#\mathcal{J} = \mathsf{J}$, and the subset

$$\mathcal{E}^{\epsilon}_{n,K,K'} \equiv \mathcal{E}^{2,\epsilon}_{n,K,K'} \cap \mathcal{J} , \tag{4.42}$$

which is obtained from $\mathcal{E}^{2,\epsilon}_{n,K,K'}$ via thinning procedure. We shorten $\#\mathcal{E}^{\epsilon}_{n,K,K'} \equiv \mathcal{N}^{\epsilon}_{n,K,K'}$. By independence of the sigma algebras issued from first and last edges, and the sigma algebra involving all other edges, we clearly have that

$$\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right) = \mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \tag{4.43}$$

and

$$\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}{}^2 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right) = \mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}{}^2\right). \tag{4.44}$$

Using (4.43) and (4.44) in (4.41), and by (4.38), we see that

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1\right) \geq \frac{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}{}^2\right)} - o_n(1). \tag{4.45}$$

Therefore, our main result Theorem 2, will be an immediate consequence of

Theorem 2′. For $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{Z}_+$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}{}^2\right)} = 1, \tag{4.46}$$

for any $K' > 2\log(2)\mathsf{L}K^2$.

5. Π vs. $\mathcal{P}$, and a lower bound to the first moment

In Sections 4.1-4.2 we have altered the path-properties derived in Section 2, and this of course has relevant consequences. The following result precisely quantifies the changes to the first moment as given in Theorem 1 (which has been instrumental to all our considerations so far) once these modifications have been taken into account.

Theorem 1′. For $\epsilon > 0$, shorten

$$\epsilon_E \equiv \frac{\epsilon}{5\mathsf{E}}, \qquad \epsilon_{m,\mathsf{E}} \equiv \frac{\epsilon}{5\mathsf{E}} + \frac{\epsilon}{5\bar{\mathsf{a}}_m\mathsf{E}}\,. \tag{5.1}$$

Let furthermore

$$S_{n,K,m} \equiv \exp -n\left(\frac{1}{\sqrt{2}K} + \frac{\sqrt{2}m(m-1)}{K^2}\right), \qquad R_{n,K} \equiv \exp\left(-\frac{n}{K^2}\right), \tag{5.2}$$

and set

$$C_{n,K,m} \equiv R_{n,K} \times S_{n,K,m}\,. \tag{5.3}$$

Then for any $K' > 2\log(2)\mathsf{L}K^2$,

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \geq C_{n,K,m}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\frac{Q_n}{P_n}\,, \tag{5.4}$$

where Q_n and P_n are finite degree polynomials.

Remark 5.1. It will become clear in the course of the proof that the S-term in Theorem 1′ encodes the entropic cost for *stretching* the paths in Π in order to construct $\mathcal{P}_{n,K}$, whereas the R-term relates to the entropic cost for rendering the H-planes *repulsive*, i.e. in order to construct $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ out of $\mathcal{P}_{n,K}$.

Proof of Theorem 1′: We begin by computing the cardinality of $\mathcal{P}_{n,K}$. To do so, we recall that paths in this set are *directed* in the m first (and last) H-planes: since there are $(m\hat{n}_K)!$ ways to reach a vertex on the mth-hyperplane starting from $\mathbf{0}$, and

$$\binom{n}{m\hat{n}_K} \tag{5.5}$$

vertices on such hyperplane, we have, altogether,

$$(m\hat{n}_K)!\binom{n}{m\hat{n}_K} \tag{5.6}$$

subpaths connecting $\mathbf{0}$ to H_m. Furthermore, there are are

$$(m\hat{n}_K)! \tag{5.7}$$

subpaths connecting a given vertex in H_{K-m} to $\mathbf{1}$.

As for the *stretched* phase, we will heavily rely on the fact already mentioned in Figure 2.10, namely that a natural representation of paths in terms of permutations is available. First we remark that for any two vertices $\boldsymbol{v}, \boldsymbol{w}$ of the hypercube,

$$\#\text{ stretched paths between } \boldsymbol{v} \text{ and } \boldsymbol{w} = (nd(\boldsymbol{v},\boldsymbol{w}))!\,, \tag{5.8}$$

and therefore, by definition of $\mathcal{P}_{n,K}$,

$$\#\mathcal{P}_{n,K} = \underbrace{(m\hat{n}_K)!\binom{n}{m\hat{n}_K}}_{\text{directed}} \underbrace{\left(\sum_{(\star_m)} \prod_{i=m+1}^{K-m} (n\mathsf{d}_i)!\right)}_{\text{stretched}} \underbrace{(m\hat{n}_K)!}_{\text{directed}}, \tag{5.9}$$

where the $(\star_m)$-sum runs over all possible vertices $\boldsymbol{v} \in H_i$. By definition, the subpaths in $\mathcal{P}_{n,K}$ going through a given vertex of the H_{i-1}-plane can reach the same number of vertices on the H_i-plane as the subpaths in $\Pi^{\boldsymbol{d}}_{\{1,\dots,K\}}[\mathbf{0} \to \mathbf{1}]$: the $(\star_m)$-sum thus runs over the same vertices as the $(\star)$-sum in (2.40), hence

$$\#(\star_m) = \prod_{i=m+1}^{K-m} \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{\left(1-\frac{i-1}{K}\right)n}{\mathsf{ef}_i n}. \tag{5.10}$$

Combining (5.9) and (5.10) thus yields

$$\#\mathcal{P}_{n,K} = (m\hat{n}_K)!^2\binom{n}{m\hat{n}_K}\prod_{i=m+1}^{K-m}\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{\left(1-\frac{i-1}{K}\right)n}{\mathsf{ef}_i n}(n\mathsf{d}_i)!\,. \tag{5.11}$$

We now quantify the difference in cardinality between $\mathcal{P}_{n,K}$ and $\mathcal{P}_{n,K,K'}$, and then, in a second step, between $\mathcal{P}_{n,K,K'}$ and $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$. To do so, the following observation is helpful: in the stretched phase, since by (2.36) it holds that $\mathsf{eb}_i + \mathsf{ef}_i = \mathsf{d}_i$, we may re-write the r.h.s. of (5.8) as

$$(n\mathsf{d}_i)! = (n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{n\mathsf{d}_i}{n\mathsf{eb}_i}. \tag{5.12}$$

This elementary algebraic identity can be given an *interpretation* which proves useful for the purpose of computing the cardinality of $\mathcal{P}_{n,K,K'}$. To see this, let us assume that each step of the polymer is a ball which is both coloured *and* labeled: backsteps are red whereas forward steps are blue; the labels correspond to which coordinate switches its value during the considered step: there are thus $(n\mathsf{eb}_i)$ labels for the red balls, and $(n\mathsf{ef}_i)$ labels for the blue balls. The first factorial on the r.h.s. of (5.12) then stands for the number of possible ways of listing the red balls while discriminating according to the labels, and similarly for the second factorial corresponding to the blue balls. Finally, the binomial factor on the r.h.s. of (5.12) accounts for the number of ways to place the red and blue balls, but *without* discriminating among labels.

Now, the subset $\mathcal{P}_{n,K,K'}$ is constructed out of $\mathcal{P}_{n,K}$ by adding an additional layer of coarse graining, and modifying the order of appearance of balls while discriminating according to their colors, but disregarding the labels. Adapting the interpretation of (5.12) discussed in the previous paragraph, it is clear that there are now

$$(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'} \tag{5.13}$$

subpaths in $\mathcal{P}_{n,K,K'}$ connecting two vertices in H_{i-1} and H_i at Hamming distance $n\mathsf{d}_i$.

The subset $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ differs from $\mathcal{P}_{n,K,K'}$ in that the order of backsteps and forward steps between H_{i-1} and $H'_{i-1,1}$, and between $H'_{i-1,K'-1}$ and H_i, is totally specified. This evidently reduces the cardinality: instead of (5.13), there are only

$$(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'-2} \tag{5.14}$$

subpaths between any two given vertices connecting the H_{i-1} and H_i hyperplanes.

To compare quantitatively the cardinality of all these sets we write

$$\frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} = \frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}_{n,K,K'}} \times \frac{\#\mathcal{P}_{n,K,K'}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}}. \tag{5.15}$$

By (5.13), it holds that

$$\begin{aligned}\frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}_{n,K,K'}} &= \prod_{i=m+1}^{K-m} \frac{(n\mathsf{d}_i)!}{(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'}} \\ &= \prod_{i=m+1}^{K-m} \frac{(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{n\mathsf{d}_i}{n\mathsf{eb}_i}}{(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'}} \\ &\lesssim \prod_{i=m+1}^{K-m} \frac{\sqrt{2\pi n\mathsf{d}_i}}{\sqrt{2\pi n\mathsf{eb}_i}\sqrt{2\pi n\mathsf{ef}_i}} \left(\frac{\sqrt{2\pi\mathsf{eb}_i\hat{n}_{K'}}\sqrt{2\pi\mathsf{ef}_i\hat{n}_{K'}}}{\sqrt{2\pi\mathsf{d}_i\hat{n}_{K'}}}\right)^{K'},\end{aligned} \tag{5.16}$$

the last step by elementary Stirling approximation (this time including the lower order, polynomial terms). The r.h.s. of (5.16) is, up to irrelevant numerical constant, *at most*

$$(5.16) \lesssim \prod_{i=m+1}^{K-m} n^{\frac{K'-1}{2}} = n^{\frac{(K'-1)(K-2m)}{2}}. \tag{5.17}$$

Furthermore, one has

$$\begin{aligned}\frac{\#\mathcal{P}_{n,K,K'}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} &= \prod_{i=m+1}^{K-m} \binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^2 \\ &\lesssim \left(\prod_{i=m+1}^{K-m} \left(1-\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{\mathsf{d}_i-\mathsf{ef}_i} \left(\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{\mathsf{ef}_i}\right)^{-2\hat{n}_{K'}} \prod_{i=m+1}^{K-m} \frac{K'\mathsf{d}_i}{2\pi n\mathsf{eb}_i\mathsf{ef}_i},\end{aligned} \tag{5.18}$$

the last inequality again by Stirling approximation. Since the term in the curly bracket is raised to a negative power, we will use the following lower bound

$$\prod_{i=m+1}^{K-m} \left(\left(1-\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{1-\frac{\mathsf{ef}_i}{\mathsf{d}_i}} \left(\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{\frac{\mathsf{ef}_i}{\mathsf{d}_i}}\right)^{\mathsf{d}_i} \geq \prod_{i=m+1}^{K-m} \left(\frac{1}{2}\right)^{\mathsf{d}_i} \geq \left(\frac{1}{2}\right)^{\mathsf{L}}, \tag{5.19}$$

where the second inequality holds true since the function $x \mapsto (1-x)^{1-x}x^x$ is convex, and attains its minimal value $1/2$ in $x = 1/2$, as can be plainly checked. Plugging the bound (5.19) in (5.18) then yields

$$\frac{\#\mathcal{P}_{n,K,K'}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} \lesssim n^{-(K-2m)} \exp\left(\frac{\mathsf{L}n}{K'}2\log 2\right). \tag{5.20}$$

Remark that for any $K' > 2K^2L\log 2$, it holds that

$$\exp\left(\frac{\mathsf{L}n}{K'}2\log 2\right) \leq \exp\left(\frac{n}{K^2}\right). \tag{5.21}$$

Combining (5.15), (5.17), (5.20) and (5.21) therefore implies that the entropic cost for rendering the hyperplanes repulsive is

$$\frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} \lesssim n^{(\frac{K'-1}{2}-1)(K-2m)} \times \exp\left(\frac{n}{K^2}\right). \tag{5.22}$$

Using (5.11) in (5.22) then yields

$$\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} \gtrsim (m\hat{n}_K)!^2 \binom{n}{m\hat{n}_K} \prod_{i=m+1}^{K-m} (n\mathsf{d}_i)! \binom{\frac{i-1}{K}n}{\mathsf{e}b_i n} \binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \times \frac{R_{n,K}}{P_n}, \tag{5.23}$$

where we have shortened

$$P_n \equiv n^{(\frac{K'-1}{2}-1)(K-2m)}, \qquad R_{n,K} \equiv \exp\left(-\frac{n}{K^2}\right). \tag{5.24}$$

Recall that by Remark 4.2, polymers in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ are loopless: this property, the ensuing independence of the sub-energies, and (5.23) thus yield

$$\begin{aligned}\mathbb{E}(\#\mathcal{E}^{2,\epsilon}_{n,K,K'}) \gtrsim (m\hat{n}_K)!\mathbb{P}\left(X_m(\pi) \le \tilde{\mathsf{a}}_{m,\epsilon}\right) \binom{n}{m\hat{n}_K} \\ \times \prod_{i=m+1}^{K-m} (n\mathsf{d}_i)!\mathbb{P}\left(X_{i-1,i} \le \mathsf{a}_{i,\epsilon}\right) \binom{\frac{i-1}{K}n}{\mathsf{e}b_i n} \binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \\ \times (m\hat{n}_K)!\mathbb{P}\left(X_{K-m+1} \le \tilde{\mathsf{a}}_{m,\epsilon}\right) \frac{R_{n,K}}{P_n}.\end{aligned} \tag{5.25}$$

Further, recalling that by the thinning procedure, it holds

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) = \frac{C(\epsilon)^2}{2} E(\#\mathcal{E}^{2,\epsilon}_{n,K,K'}), \tag{5.26}$$

and by the usual tail estimates, we thus see that

$$\mathbb{E}(\mathcal{N}^{\epsilon}_{n,K,K'}) \gtrsim \prod_{i=m+1}^{K-m} (\mathsf{a}_{i,\epsilon})^{nd_i} \binom{\frac{i-1}{K}n}{\mathsf{e}b_i n} \binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \binom{n}{m\hat{n}_K} \tilde{\mathsf{a}}^{2m\hat{n}_K-2}_{m,\epsilon} (m\hat{n}_K)^2 \frac{R_{n,K}}{P_n}. \tag{5.27}$$

But since $(m\hat{n}_K)^2\tilde{\mathsf{a}}^{-2}_{m,\epsilon} > 1$ for n large enough, we have, altogether, that

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=m+1}^{K-m} (\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i} \binom{\frac{i-1}{K}n}{\mathsf{e}b_i n} \binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \times \\ \times \binom{n}{m\hat{n}_K} (\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K} (1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i} (1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K} \frac{R_{n,K}}{P_n}.\end{aligned} \tag{5.28}$$

The first term on the r.h.s. of (5.28) is reminiscent of the expression appearing in Theorem 1, but contrary to the latter, we are facing here a product which runs over the indices $i = m+1 \dots K-m$ only. The natural idea is thus to modify and then extend this partial product to a full product in order to exploit the control already established in Theorem 1. To do so we first note that, since on the positive axis it holds that $x \ge \tanh(x)$,

$$(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i} \ge \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}, \quad \text{and} \quad (\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K} \ge \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K}. \tag{5.29}$$

Using this in (5.28) yields

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=m+1}^{K-m} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i} \binom{\frac{i-1}{K}n}{\mathsf{e}b_i n} \binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \\ \times \binom{n}{m\hat{n}_K} \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K} (1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i} (1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K} \frac{R_{n,K}}{P_n}.\end{aligned} \tag{5.30}$$

The new (partial) product is closer yet not quite the same as that appearing in Theorem 1, so we artificially introduce some cosh-terms which however leave the r.h.s. above as a whole unaltered.

Precisely, we rewrite (5.30) as

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=m+1}^{K-m} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\left(\frac{\cosh(\mathsf{a}_i\mathsf{E})}{\cosh(\mathsf{a}_i\mathsf{E})}\right)^n \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}\times$$
$$\times\binom{n}{m\hat{n}_K}\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K}\left(\frac{\cosh(\bar{\mathsf{a}}_m\mathsf{E})}{\cosh(\bar{\mathsf{a}}_m\mathsf{E})}\right)^{2n}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\frac{R_{n,K}}{P_n}; \tag{5.31}$$

We can now move to the aforementioned procedure of extending the product to *all* indices $i = 1\dots K$. This naturally requires a good control of the missing terms, i.e. for $i \le m$ (a case which is referred to below as First), and for $i \ge K-m+1$ (Second case).

First case. We begin noting that by the Evolution Lemma 3.2,

$$\begin{aligned}\left[\prod_{i=1}^{m} g_{i,K}(\mathsf{d}_i)\right]^n &= \left(\frac{\sinh(\bar{\mathsf{a}}_m\mathsf{E})}{\frac{m}{K}}\right)^{m\hat{n}_K}\left(\frac{\cosh(\bar{\mathsf{a}}_m\mathsf{E})}{1-\frac{m}{K}}\right)^{n-m\hat{n}_K}\\ &= \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\frac{n^n}{(m\hat{n}_K)^{m\hat{n}_K}(n-m\hat{n}_K)^{n-m\hat{n}_K}}\,,\end{aligned} \tag{5.32}$$

the second equality by elementary rearrangement. But by "reverse" Stirling-approximation,

$$\frac{n^n}{(m\hat{n}_K)^{m\hat{n}_K}(n-m\hat{n}_K)^{n-m\hat{n}_K}} \propto \sqrt{n}\binom{n}{m}\hat{n}_K, \tag{5.33}$$

and therefore

$$\left[\prod_{i=1}^{m} g_{i,K}(\mathsf{d}_i)\right]^n \propto \sqrt{n}\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\binom{n}{m\hat{n}_K}. \tag{5.34}$$

Furthermore, by definition of the g-functions, and taking into account the lower orders in the Stirling approximation of the binomial factors, one also plainly checks that

$$\left[\prod_{i=1}^{m} g_{i,K}(\mathsf{d}_i)\right]^n \propto \sqrt{n}n^{m-1}\prod_{i=1}^{m}\tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}. \tag{5.35}$$

Equating (5.34) and (5.35) therefore yields the asymptotic identity

$$\begin{aligned}&\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\binom{n}{m\hat{n}_K}\\ &\qquad\propto n^{m-1}\prod_{i=1}^{m}\tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}.\end{aligned} \tag{5.36}$$

Remark, in particular, that what lies behind the l.h.s. above (these are terms contributing to (5.31)) are thus the first m-terms (up to irrelevant, for our purposes below) polynomial factors, of the product analysed in Theorem 1.

Second case. Again by the Evolution Lemma 3.2 it holds that

$$1 = \prod_{i=1}^{K-m} g_{i,K}(\mathsf{d}_i)\times\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i)\,, \tag{5.37}$$

and therefore

$$\begin{aligned}\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i) &= \left[\prod_{i=1}^{K-m} g_{i,K}(\mathsf{d}_i)\right]^{-1} \\ &= \left[\left(\frac{\sinh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{\frac{K-m}{K}}\right)^{\frac{K-m}{K}}\left(\frac{\cosh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{1-\frac{K-m}{K}}\right)^{1-\frac{K-m}{K}}\right]^{-1},\end{aligned} \tag{5.38}$$

the second equality in virtue of (3.37). In order to get a handle on the r.h.s. above we use the fundamental relation (2.25) which states that

$$\sinh\left(\bar{\mathsf{a}}_{K-m}\mathsf{E}\right)\cosh\left(\underline{\mathsf{a}}_{K-m}\mathsf{E}\right) = \frac{K-m}{K}, \tag{5.39}$$

implying, in particular, that

$$\left[\left(\frac{\sinh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{\frac{K-m}{K}}\right)^{\frac{K-m}{K}}\right]^{-1} = \cosh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{\frac{K-m}{K}}. \tag{5.40}$$

Furthermore, the following "mirror" version of (5.39) holds in virtue of the addition formula for hyperbolic functions (see (3.24) for the detailed derivation):

$$\cosh\left(\bar{\mathsf{a}}_{K-m}\mathsf{E}\right)\sinh\left(\underline{\mathsf{a}}_{K-m}\mathsf{E}\right) = 1-\frac{K-m}{K}, \tag{5.41}$$

hence

$$\left[\left(\frac{\cosh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{1-\frac{K-m}{K}}\right)^{1-\frac{K-m}{K}}\right]^{-1} = \sinh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{1-\frac{K-m}{K}}. \tag{5.42}$$

Using (5.40) and (5.42) in (5.38) we thus have

$$\begin{aligned}\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i) &= \cosh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{\frac{K-m}{K}}\sinh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{1-\frac{K-m}{K}} \\ &= \cosh(\bar{\mathsf{a}}_{m}\mathsf{E})^{\frac{K-m}{K}}\sinh(\bar{\mathsf{a}}_{m}\mathsf{E})^{1-\frac{K-m}{K}},\end{aligned} \tag{5.43}$$

the second identity since $\sum_{i=1}^{K}\mathsf{a}_i = 1$ and by symmetry of the $\mathsf{a}'s$. Raising (5.43) to the nth- power, and by simple rearrangement, we thus see that

$$\left[\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i)\right]^{n} = \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^{n}. \tag{5.44}$$

Again by the definition of the g-functions, and taking into account the lower orders in the Stirling approximation of the binomial factors, one plainly checks that

$$\left[\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i)\right]^{n} \propto n^{m-1}\prod_{i=K-(m-1))}^{K}\tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^{n}\binom{\frac{i-1}{K}n}{\mathsf{eb_i}n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}, \tag{5.45}$$

and therefore, equating (5.44) and (5.45), we also obtain the following asymptotic equivalence

$$\begin{aligned}&\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^{n} \\ &\qquad\propto n^{m-1}\prod_{i=K-(m-1)}^{K}\tanh(\mathsf{a}_i\mathsf{E})^{nd_i}\cosh(\mathsf{a}_i\mathsf{E})^{n}\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}.\end{aligned} \tag{5.46}$$

In full analogy to (5.36), we therefore see that behind the l.h.s. above (these are also terms contributing to (5.31)) hide in fact the last m-terms of the product analysed in Theorem 1.

Thanks to both (5.36) and (5.46), we may now replace the corresponding terms on the r.h.s. of (5.31): this indeed allows to extend the product to all indices $i = 1, \dots, K$, and seamlessly leads to the lower bound

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=1}^{K} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n \binom{\frac{i-1}{K}n}{\mathsf{eb_i}n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n} \\ \times \frac{Q_n R_{n,K}}{P_n \cosh(\bar{\mathsf{a}}_m\mathsf{E})^{2n}} \prod_{i=m+1}^{K-m} \frac{1}{\cosh(\mathsf{a}_i\mathsf{E})}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K},\end{aligned} \tag{5.47}$$

where $Q_n \equiv n^{2(m-1)}$ is yet another polynomial term.

The full product in the first line of the r.h.s. of (5.47) is easily taken care of. In fact, by elementary rearrangement, it holds that

$$\begin{aligned}\prod_{i=1}^{K} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n \binom{\frac{i-1}{K}n}{\mathsf{eb_i}n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n} \\ = \prod_{i=1}^{K} \sinh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^{n(1-\mathsf{d}_i)} \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n},\end{aligned} \tag{5.48}$$

and by Stirling approximation to second order, the r.h.s. of (5.48) equals

$$\prod_{i=1}^{K}\left(\frac{\sinh(\mathsf{a}_i\mathsf{E})^{\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^{1-\mathsf{d}_i}\varphi\left(\frac{i-1}{K}\right)\varphi\left(1-\frac{i-1}{K}\right)}{\varphi(\mathsf{eb}_i)\varphi\left(\frac{i-1}{K}-\mathsf{eb}_i\right)\varphi\left(\mathsf{ef}_i\right)\varphi\left(1-\frac{i-1}{K}-\mathsf{ef}_i\right)}\right)^n \times S_{n,K}, \tag{5.49}$$

where $S_{n,K}$ corresponds to the lower order (polynomial) terms in the approximation. But by Theorem 1, the first term of (5.49), i.e. the full product, equals unity, whereas an elementary inspection of the polynomial terms further shows that

$$\begin{aligned}S_{n,K} &\gtrsim \frac{1}{\sqrt{\frac{2\pi n}{K}(1-\frac{1}{K})}} \prod_{i=2}^{K-1}\left(\frac{(2\pi n)^2(\frac{i-1}{K})(1-\frac{i-1}{K})}{(2\pi n)^4\mathsf{eb}_i(\frac{i}{K}-\mathsf{ef}_i)\mathsf{ef}_i(1-\frac{i}{K}-\mathsf{eb}_i)}\right)^{\frac{1}{2}} \\ &\gtrsim \frac{1}{n^{K-2+\frac{1}{2}}}.\end{aligned} \tag{5.50}$$

Using all this in (5.47) yields

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \geq \frac{Q_n R_{n,K}}{P_n \cosh(\bar{\mathsf{a}}_m\mathsf{E})^{2n}} \prod_{i=m+1}^{K-m} \frac{1}{\cosh(\mathsf{a}_i\mathsf{E})^n}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{\bar{\mathsf{a}}_m,\mathsf{E}})^{2m\hat{n}_K}, \tag{5.51}$$

where $P_n \equiv P_n n^{K-2+\frac{1}{2}}$ is yet another polynomial term.

It thus remains to control the cosh-terms in (5.51). To see how this goes we observe that by Taylor expanding the cosh-function to second order,

$$\begin{aligned}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^{-1} &= \exp\left[-\log\cosh(\bar{\mathsf{a}}_m\mathsf{E})\right] \\ &\geq \exp -\log\left(1+\frac{(\bar{\mathsf{a}}_m\mathsf{E})^2\cosh(\bar{\mathsf{a}}_m\mathsf{E})}{2}\right) \\ &\geq \exp\left(-\frac{(\bar{\mathsf{a}}_m\mathsf{E})^2}{\sqrt{2}}\right),\end{aligned} \tag{5.52}$$

the second inequality since $\log(1+x) \le x$, and using that $\cosh(\bar{\mathsf{a}}_m \mathsf{E}) \le \cosh(\mathsf{E}) = \sqrt{2}$. Moreover, by (2.29) it holds that $\mathsf{a}_i \mathsf{E} \le \frac{1}{K}$: summing over $i = 1 \dots m$ thus leads to $\bar{\mathsf{a}}_m \mathsf{E} \le \frac{m}{K}$, which combined with (5.52) yields

$$\cosh(\bar{\mathsf{a}}_m \mathsf{E})^{-1} \ge \exp\left(-\frac{m^2}{\sqrt{2}K^2}\right). \tag{5.53}$$

A similar reasoning evidently yields

$$\cosh(\mathsf{a}_i E)^{-1} \ge \exp\left(-\frac{1}{\sqrt{2}K^2}\right), \tag{5.54}$$

for any $i = 1 \dots K$. By (5.53) and (5.54) we thus have that

$$\frac{1}{\cosh(\bar{\mathsf{a}}_m \mathsf{E})^{2n}} \times \prod_{i=m+1}^{K-m} \frac{1}{\cosh(\mathsf{a}_i \mathsf{E})^n} \ge \exp\left(-\frac{n}{\sqrt{2}K} - \frac{2nm(m-1)}{\sqrt{2}K^2}\right), \tag{5.55}$$

which we recognize as the $S_{n,K,m}$-term announced in (5.2): *the entropic cost for stretching the paths.* Using (5.55) in (5.51) finally yields

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \ge (1+\epsilon_E)^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i} (1+\epsilon_{\bar{\mathsf{a}}_m,\mathsf{E}})^{2m\hat{n}_K} \frac{S_{n,K,m} R_{n,K} Q_n}{P_n}, \tag{5.56}$$

and Theorem 1′ is thus settled. □

6. The second moment, and proof of Theorem 2′

The goal of this section is to provide a proof of Therem 2′. We begin with a technical input, concerning tail estimates for the probability of two correlated sums of exponentials.

Lemma 6.1 (Overlap probability)**.** *Consider independent standard exponentials $\{\xi_i\}$, and let $X_l \equiv \sum_{i=1}^{l} \xi_i$. Denote by X'_l the sum of l such ξ-exponentials, and assume that X'_l shares exactly k edges with X_l. Then for $x > 0$, it holds:*

$$\mathbb{P}\left(X_l \le x, X'_l \le x\right) \propto \frac{x^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l. \tag{6.1}$$

where

$$\gamma \in [0,1] \mapsto g(\gamma) \equiv \frac{(4(1-\gamma))^{1-\gamma}}{(2-\gamma)^{2-\gamma}}. \tag{6.2}$$

In particular, $\|g\|_\infty \le 1$.

Proof: Without loss of generality we may write

$$X'_l = \sum_{i=1}^{k} \xi_i + \sum_{i=k+1}^{l} \xi'_i, \tag{6.3}$$

for independent ξ''s, which are also independent of the ξ-family. Remark that the first sum, the common trunk, is a $\Gamma(k,1)$-distributed r.v., whereas the second sum is $\Gamma(l-k,1)$-distributed. By conditioning on the common trunk, and by independence, it thus holds:

$$\begin{aligned}
\mathbb{P}\left(X_l \le x, X'_l \le x\right) &= \int_0^{+\infty} \mathbb{P}\left(t + X_{l-k} \le x\right)^2 \mathbb{P}(X_k \in dt) \\
&= \int_0^{+\infty} \mathbb{P}\left(X_{l-k} \le x - t\right)^2 \frac{t^{k-1}e^{-t}}{(k-1)!} dt \\
&\propto \frac{1}{(l-k)!^2 (k-1)!} \int_0^x (x-t)^{2(l-k)} t^{k-1} dt,
\end{aligned} \tag{6.4}$$

the last step by the standard tail-estimates. Integration by parts then yields

$$\int_0^x (x-t)^{2(l-k)} t^{k-1} dt = \frac{(k-1)!(2(l-k))!}{(2l-k)!} x^{2l-k}\,, \tag{6.5}$$

and therefore

$$\begin{aligned}\mathbb{P}\left(X_l \le x, X_l' \le x\right) &\propto \frac{(2(l-k))!}{(2l-k)!(l-k)!^2} x^{2l-k} \\ &\propto \frac{x^{2l-k}}{(l-k)!l!} \frac{l!(2(l-k))!}{(2l-k)!(l-k)!}. \\ &\propto \frac{x^{2l-k}}{(l-k)!l!} \frac{(1-\frac{k}{l})^{l-k}}{2^k(1-\frac{k}{2l})^{2l-k}}\,,\end{aligned} \tag{6.6}$$

the last inequality by Stirling approximation.

Remark that with $\gamma \equiv k/l \in [0,1]$, the second factor in the last term above can be written as

$$\frac{(1-\frac{k}{l})^{l-k}}{2^k(1-\frac{k}{2l})^{2l-k}} = \left(\frac{(4(1-\gamma))^{(1-\gamma)}}{(2-\gamma)^{(2-\gamma)}}\right)^l \equiv g(\gamma)^l\,, \tag{6.7}$$

and using this in (6.6) yields

$$\mathbb{P}\left(X_l \le x, X_l' \le x\right) \propto \frac{x^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l\,, \tag{6.8}$$

concluding the proof of the estimate for the overlap probability. □

We now address the second moment of $\mathcal{N}^{\epsilon}_{n,K,K'}$, as required for a proof of Theorem 2′. For this, some notation is needed: recall from (4.42) that $\mathcal{J}$ is a deterministic subset of polymers with cardinality $\#\mathcal{J} = \mathsf{J} = \left\lfloor \mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}/2 \right\rfloor$. Given a path $\pi \in \mathcal{J}$, we shorten:

$$\begin{aligned}\mathcal{J}_\pi(n,k) \equiv\ &\text{all paths } \pi' \in \mathcal{J} \\ &\text{which share } k \text{ edges with } \pi, \\ &\text{without considering the first and the last edge,}\end{aligned} \tag{6.9}$$

and for its cardinality

$$f_\pi(n,k) \equiv \#\mathcal{J}_\pi(n,k)\,. \tag{6.10}$$

Analogously we shorten

$$\begin{aligned}\mathcal{J}^{(d)}_\pi(n,k) \equiv\ &\text{all paths } \pi' \in \mathcal{J} \text{ which share } k \text{ edges with} \\ &\pi \text{ only in the directed phase, i.e between} \\ &\mathbf{0} \text{ and } H_m \text{ or } H_{K-m} \text{ and } \mathbf{1}, \\ &\text{but without considering first and last edge,}\end{aligned} \tag{6.11}$$

and let

$$f^{(d)}_\pi(n,k) \equiv \#\mathcal{J}^{(d)}_\pi(n,k)\,, \tag{6.12}$$

denote its cardinality.

And finally,

$$\begin{aligned}\mathcal{J}^{(s)}_\pi(n,k) \equiv\ &\text{number of paths } \pi' \in \mathcal{J} \text{ which share } k \text{ edges with} \\ &\pi \text{ with at least one common edge in the stretched} \\ &\text{phase, i.e between } H_m \text{ and } H_{K-m}, \\ &\text{but without considering first and last edge,}\end{aligned} \tag{6.13}$$

analogously shortening for its cardinality

$$f_\pi^{(s)}(n,k) \equiv \#\mathcal{J}_\pi^{(s)}(n,k)\,. \tag{6.14}$$

Remark that

$$f_\pi(n,k) = f_\pi^{(d)}(n,k) + f_\pi^{(s)}(n,k)\,. \tag{6.15}$$

We will also need the "worst case scenarios"

$$\begin{aligned} f(n,k) &\equiv \sup_{\pi\in\mathcal{J}} f_\pi(n,k)\,,\\ f^{(d)}(n,k) &\equiv \sup_{\pi\in\mathcal{J}} f_\pi^{(d)}(n,k)\,,\\ f^{(s)}(n,k) &\equiv \sup_{\pi\in\mathcal{J}} f_\pi^{(s)}(n,k)\,, \end{aligned} \tag{6.16}$$

in which case it holds, in particular, that

$$f(n,k) \le f^{(d)}(n,k) + f^{(s)}(n,k). \tag{6.17}$$

For $i = m+1\dots K-m$, and two polymers $\pi,\pi'\in\mathcal{J}$, we shorten

$$\mathbb{P}_i(\pi) \equiv \mathbb{P}\left(X_{i-1,i}(\pi) \le \mathsf{a}_{i,\epsilon}\right), \tag{6.18}$$

and

$$\mathbb{P}_i(\pi,\pi') \equiv \mathbb{P}\left(X_{i-1,i}(\pi) \le \mathsf{a}_{i,\epsilon},\ X_{i-1,i}(\pi') \le \mathsf{a}_{i,\epsilon}\right)\,. \tag{6.19}$$

Furthermore, we shorten

$$\begin{aligned} \mathbb{P}_m(\pi) &\equiv \mathbb{P}\left(X_m(\pi) \le \tilde{\mathsf{a}}_{m,\epsilon}\right),\\ \mathbb{P}_{K-m+1}(\pi) &\equiv \mathbb{P}\left(X_{K-m+1}(\pi) \le \tilde{\mathsf{a}}_{K-m+1,\epsilon}\right), \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} \mathbb{P}_m(\pi,\pi') &\equiv \mathbb{P}\left(X_m(\pi), X_m(\pi') \le \tilde{\mathsf{a}}_{m,\epsilon}\right)\,,\\ \mathbb{P}_{K-m+1}(\pi,\pi') &\equiv \mathbb{P}\left(X_{K-m+1}(\pi), X_{K-m+1}(\pi') \le \tilde{\mathsf{a}}_{K-m+1,\epsilon}\right)\,, \end{aligned} \tag{6.21}$$

as well as

$$\mathbb{P}(\pi) \equiv \mathbb{P}\Big(X_m(\pi) \le \tilde{\mathsf{a}}_{m,\epsilon},\ X_{i-1,i}(\pi) \le \mathsf{a}_{i,\epsilon}\ i = m+1\dots K-m,\ X_{K-m+1}(\pi) \le \tilde{\mathsf{a}}_{K-m+1,\epsilon}\Big), \tag{6.22}$$

and

$$\begin{aligned} \mathbb{P}(\pi,\pi') \equiv &\mathbb{P}(X_m(\pi),\ X_m(\pi') \le \tilde{\mathsf{a}}_{m,\epsilon},\ X_{i-1,i}(\pi), X_{i-1,i}(\pi') \le \mathsf{a}_{i,\epsilon} \text{ for}\\ &i = m+1\dots K-m,\ X_{K-m+1}(\pi), X_{K-m+1}(\pi') \le .\tilde{\mathsf{a}}_{K-m+1,\epsilon})\,. \end{aligned} \tag{6.23}$$

Remark that for loopless paths the substrand-energies are independent, hence, and with the above notation,

$$\mathbb{P}(\pi) = \prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi), \qquad \mathbb{P}(\pi,\pi') = \prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi,\pi')\,. \tag{6.24}$$

In particular, it holds that

$$\mathbb{E}\left(\mathcal{N}_{n,K,K'}^{\epsilon}\right) = \mathsf{J}\mathbb{P}(\pi) = \mathsf{J}\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi)\,. \tag{6.25}$$

Concerning the second moment, we write

$$\begin{aligned} \mathbb{E}\left({\mathcal{N}_{n,K,K'}^{\epsilon}}^2\right) &= \sum_{\pi,\pi'\in\mathcal{J}} \mathbb{P}(\pi,\pi')\\ &= \sum_{\pi\in\mathcal{J}} \sum_{k=0}^{\mathsf{L}_{opt}n-2} \sum_{\pi'\in\mathcal{J}_\pi(n,k)} \mathbb{P}(\pi,\pi')\,, \end{aligned} \tag{6.26}$$

by arranging the sum according to the possible overlap-regimes.

The case $k = 0$ is both crucial and easily taken care of by the following observations: first remark that the distribution of the energies of a pair of polymers depends solely on the number of common edges; furthermore the number of pairs of polymers with zero common edges is at most J^2. Therefore, for any $(\hat{\pi}, \tilde{\pi}) \in (\mathcal{J}, \mathcal{J}_{\hat{\pi}}(n, 0))$ it holds:

$$\sum_{\pi \in \mathcal{J}} \sum_{\pi' \in \mathcal{J}_\pi(n,0)} \mathbb{P}(\pi, \pi') \leq \mathsf{J}^2 \mathbb{P}(\hat{\pi}, \tilde{\pi}) = \mathsf{J}^2 \mathbb{P}(\hat{\pi})^2 \,, \tag{6.27}$$

the last equality holding true since in case of non-overlapping paths, the $\hat{\pi}, \tilde{\pi}$-energies are independent and identically distributed. Using (6.25) in (6.27) therefore yields

$$\sum_{\pi \in \mathcal{J}} \sum_{\pi' \in \mathcal{J}_\pi(n,0)} \mathbb{P}(\pi, \pi') \leq \mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2 \,. \tag{6.28}$$

This settles the $k = 0$ regime.

Remark 6.2. Recovering the first moment squared as in (6.28) is absolutely crucial for the whole approach, and the main reason for treating first and last edge on different footing. Without such different treatment, one would get the first moment squared *up to a constant only*, and this would nullify the proof of Theorem 2. This feature is common to virtually all models in the REM-class, see Gayrard and Kistler (2015) for more on this delicate issue.

As for the remaining overlap-regimes, we will distinguish between

- $1 \leq k \leq 200\hat{n}_K$: this corresponds to the case of weak correlations (the overlap between the two polymers is small);
- $k > 200\hat{n}_K$: this corresponds to the case of strong correlations (the two polymers strongly overlap).

We now rearrange the second moment according to the above dichotomy. Henceforth, given $\pi \in \mathcal{J}$, and with $k \in \mathbb{Z}_+$, we denote by $\pi_k^{(d)} \in \mathcal{J}_\pi^{(d)}(n, k)$ a polymer which shares k edges with π, and in full analogy for $\pi_k^{(s)} \in \mathcal{J}_\pi^{(s)}(n, k)$ and $\pi_k \in \mathcal{J}_\pi(n, k)$. With this notation, again using that specifying the number of common edges fixes the distribution of the pair of paths, and by (6.28), we thus have

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}{}^2\right) \leq \mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2 &+ \\ &+ \mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k) \mathbb{P}\left(\pi, \pi_k^{(d)}\right) \\ &+ \mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k) \mathbb{P}\left(\pi, \pi_k^{(s)}\right) \\ &+ \mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt} n - 2} f(n,k) \mathbb{P}\left(\pi, \pi_k\right) \,.\end{aligned} \tag{6.29}$$

On the other hand, by Jensen inequality it holds

$$\mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}{}^2\right) \geq \mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2 . \tag{6.30}$$

In order to establish Theorem 2′ it therefore suffices to show that the last three sums on the r.h.s. of (6.29) are of lower order when compared with the first moment squared. This is indeed our key claim: since its proof is long and technical, we formulate it in the form of three Propositions.

Proposition 6.3. *For any $K > m\epsilon^{-2}$, it holds*

$$\mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(d)}\right) = o\left(\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2\right), \tag{6.31}$$

for $n \to \infty$.

Proposition 6.4. *For any $K > \max(2 \times 10^7, m\epsilon^{-2})$ and $K' > 2\log(2)\mathsf{L}K^2$, it holds*

$$\mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} f(n,k)\mathbb{P}\left(\pi, \pi_k\right) = o\left(\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2\right), \tag{6.32}$$

for $n \to \infty$.

Proposition 6.5. *For any $K > 2 \times 10^7$ and $K' > 2\log(2)\mathsf{L}K^2$, it holds*

$$\mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(s)}\right) = o\left(\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2\right), \tag{6.33}$$

for $n \to \infty$.

The following three sections are devoted to the proofs of the above statements. We anticipate that each proposition/treatment will require a good control of the asymptotics of the $f^{(d)}, f$- and $f^{(s)}$-terms: these will be formulated in the form of Lemmata whose proofs, relying on extremely technical combinatorial estimates, are however postponed to the Appendix.

The reason for tackling the f-regime before the $f^{(s)}$-one is that the treatment of the latter will require some technical inputs which are obtained in the analysis of the the former.

6.1. *Proof of Proposition 6.3.* The goal is to prove that

$$\lim_{n\to\infty} \frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(d)}\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} = 0. \tag{6.34}$$

The combinatorial input here is the following

Lemma 6.6. *For all $k \le 200\hat{n}_K$, one has*

$$f^{(d)}(n,k) \le \frac{\mathsf{J}(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor)!(n-1-\lceil \frac{k}{2} \rceil)!}{(m\hat{n}_K)!n!} l(k), \tag{6.35}$$

where

$$l(k) \equiv \begin{cases} 32(k+1)^3 & k \le n^{1/4} \\ 16n^{13}(k+1) & \textit{otherwise.} \end{cases} \tag{6.36}$$

The proof of this Lemma is postponed to the Appendix. Coming back to the task of proving (6.34), by (6.24) and (6.25) we write

$$\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(d)}\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} = \frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\prod_{i=m}^{K-m+1}\mathbb{P}_i\left(\pi, \pi_k^{(d)}\right)}{\mathsf{J}^2\prod_{i=m}^{K-m+1}\mathbb{P}_i(\pi)^2}, \tag{6.37}$$

In the considered regime, polymers share no edges but in the directed phase: the probabilities indexed by $i \in \{m+1, \dots, K-m\}$ therefore factor out in virtue of the ensuing independence, and the r.h.s. of (6.37) then takes the neater form

$$\sum_{k=1}^{200\hat{n}_K} \frac{f^{(d)}(n,k)\mathbb{P}_m\left(\pi, \pi_k^{(d)}\right)\mathbb{P}_{K-m+1}\left(\pi, \pi_k^{(d)}\right)}{\mathsf{J}\mathbb{P}_m(\pi)^2\,\mathbb{P}_{K-m+1}(\pi)^2}. \tag{6.38}$$

Now, for $\pi \in \mathcal{J}$ and $\pi_k^{(d)} \in \mathcal{J}_\pi^{(d)}(n,k)$, let us denote by k_l the number of common edges between $\mathbf{0}$ and H_m, and by k_r the number of common edges between H_{K-m} and $\mathbf{1}$ (in which case it evidently holds that $k = k_l + k_r$). By the estimates for the overlap probabilities from Lemma 6.1 (using the rough bound $\|g\|_\infty \leq 1$), it steadily follows that

$$\mathbb{P}_m\left(\pi, \pi_k^{(d)}\right)\mathbb{P}_{K-m+1}\left(\pi, \pi_k^{(d)}\right) \lesssim \frac{\tilde{\mathsf{a}}_{m,\epsilon}^{4(m\hat{n}_K-1)-k}}{(m\hat{n}_K-1-k_l)!(m\hat{n}_K-1-k_r)!(m\hat{n}_K-1)!^2}. \tag{6.39}$$

We now proceed by worst case scenario and *maximize* the r.h.s. over all possible (k_l, k_r)-choices. This can be seamlessly identified thanks to the well-known log-convexity of factorials, which we recall is the property that for any $a \geq b \geq j \geq 0$ it holds

$$(a+j)!(b-j)! \geq a!b!\,. \tag{6.40}$$

Using (6.40) with

$$a \equiv m\hat{n}_K - 1 - \lfloor \frac{k}{2} \rfloor, \qquad \text{and} \qquad b \equiv m\hat{n}_K - 1 - \lceil \frac{k}{2} \rceil, \tag{6.41}$$

we see that the worst case on the r.h.s. of (6.39) is attained in $k_r \in \{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\}$, which is equivalent to $k_l \in \{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\}$ because $k = k_l + k_r$, hence

$$\mathbb{P}_m\left(\pi, \pi_k^{(d)}\right)\mathbb{P}_{K-m+1}\left(\pi, \pi_k^{(d)}\right) \leq \frac{\tilde{\mathsf{a}}_{m,\epsilon}^{4(m\hat{n}_K-1)-k}}{(m\hat{n}_K-1-\lfloor \frac{k}{2} \rfloor)!(m\hat{n}_K-1-\lceil \frac{k}{2} \rceil)!(m\hat{n}_K-1)!^2}. \tag{6.42}$$

Using the latter in (6.38), and by the usual tail estimates, we obtain

$$\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(d)}\right)}{\mathbb{E}\left(\mathcal{N}_{n,K,K'}^{\epsilon}\right)^2} \lesssim \sum_{k=1}^{200\hat{n}_K} \frac{f^{(d)}(n,k)(m\hat{n}_K-1)!^2}{\mathsf{J}(m\hat{n}_K-1-\lfloor \frac{k}{2} \rfloor)!(m\hat{n}_K-1-\lceil \frac{k}{2} \rceil)!(\tilde{\mathsf{a}}_{m,\epsilon})^k}. \tag{6.43}$$

To get a handle on the factorials in the r.h.s. above we employ the bound

$$\frac{(m\hat{n}_K-1)!^2}{(m\hat{n}_K-1-\lfloor \frac{k}{2} \rfloor)!(m\hat{n}_K-1-\lceil \frac{k}{2} \rceil)!} \leq \frac{(m\hat{n}_K)!^2}{(m\hat{n}_K-\lfloor \frac{k}{2} \rfloor)!(m\hat{n}_K-\lceil \frac{k}{2} \rceil)!}\,, \tag{6.44}$$

which can be plainly checked by writing out, and simplifying. Using (6.44), and the combinatorial estimates of Lemma 6.6 for the $f^{(d)}$-term, yields

$$\begin{aligned}(6.43) &\lesssim \sum_{k=1}^{200\hat{n}_K} \frac{(m\hat{n}_K-\lfloor \frac{k}{2} \rfloor)!(n-1-\lceil \frac{k}{2} \rceil)!l(k)(m\hat{n}_K)!^2}{(m\hat{n}_K)!n!(m\hat{n}_K-\lfloor \frac{k}{2} \rfloor)!(m\hat{n}_K-\lceil \frac{k}{2} \rceil)!(\tilde{\mathsf{a}}_{m,\epsilon})^k} \\ &= \sum_{k=1}^{200\hat{n}_K} \frac{(n-1-\lceil \frac{k}{2} \rceil)!l(k)(m\hat{n}_K)!}{n!(m\hat{n}_K-\lceil \frac{k}{2} \rceil)!(\tilde{\mathsf{a}}_{m,\epsilon})^k}\,,\end{aligned} \tag{6.45}$$

the second step in virtue of elementary, term by term, simplifications.

Using $(a-1)! = a!/a$ for the first factorial-term in the numerator on the r.h.s. above yields

$$\begin{aligned}(6.45) &= \sum_{k=1}^{200\hat{n}_K} \frac{(n-\lceil \frac{k}{2} \rceil)!l(k)(m\hat{n}_K)!}{(n-\lceil \frac{k}{2} \rceil)n!(m\hat{n}_K-\lceil \frac{k}{2} \rceil)!(\tilde{\mathsf{a}}_{m,\epsilon})^k} \\ &\lesssim \sum_{k=1}^{200\hat{n}_K} \frac{l(k)}{(n-\lceil \frac{k}{2} \rceil)} \cdot \frac{(1-\frac{1}{n}\lceil \frac{k}{2} \rceil)^{n-\lceil \frac{k}{2} \rceil}(\frac{m}{K})^{m\hat{n}_K}}{(\frac{m}{K}-\frac{1}{n}\lceil \frac{k}{2} \rceil)^{(m\hat{n}_K-\lceil \frac{k}{2} \rceil)}} \cdot \frac{1}{(\tilde{\mathsf{a}}_{m,\epsilon})^k}\,,\end{aligned} \tag{6.46}$$

the last inequality by Stirling's approximation.

We now focus on the middle term on the r.h.s. above. Omitting the rounding operation, and shortening

$$Q(x) \equiv (1-x)\log(1-x) - \frac{m}{K}(1-x\frac{K}{m})\log\left(1-x\frac{K}{m}\right), \tag{6.47}$$

we may rewrite this middle term as

$$\frac{(1-\frac{k}{2n})^{n-\frac{k}{2}}(\frac{m}{K})^{m\hat{n}_K}}{(\frac{m}{K}-\frac{k}{2n})^{(m\hat{n}_K-\frac{k}{2})}} = \left(\sqrt{\frac{m}{K}}\right)^k \exp nQ\left(\frac{k}{n}\right). \tag{6.48}$$

It is plainly checked that, for $k/n \in [0,1]$, the Q-function is in fact *negative* (for $K > m$), hence

$$(6.48) \leq \left(\sqrt{\frac{m}{K}}\right)^k. \tag{6.49}$$

By definition,

$$(\tilde{\mathsf{a}}_{m,\epsilon})^k = (\bar{\mathsf{a}}_m(\mathsf{E}+\epsilon)+\epsilon)^k \geq \epsilon^k, \tag{6.50}$$

the inequality by elementary minorization: this, as well as the bound (6.49), imply that (6.46) is *at most*

$$\sum_{k=1}^{200\hat{n}_K} \frac{l(k)}{(n-\frac{k}{2})}\frac{1}{\left(\sqrt{\frac{K}{m}}\epsilon\right)^k} = \left(\sum_{k=1}^{n^{\frac{1}{4}}} + \sum_{k=n^{\frac{1}{4}}+1}^{200\hat{n}_K}\right)\frac{l(k)}{(n-\frac{k}{2})}\frac{1}{\left(\sqrt{\frac{K}{m}}\epsilon\right)^k}. \tag{6.51}$$

If we now take K large enough such that $\sqrt{\frac{K}{m}}\epsilon > 1$, to wit:

$$K > m\epsilon^{-2}, \tag{6.52}$$

and recalling the definition of $l(k)$ as in (6.36), we obtain

$$(6.51) \lesssim \frac{1}{(n-n^{\frac{1}{4}})}\sum_{k=1}^{n^{\frac{1}{4}}}\frac{(k+1)^3}{(\sqrt{\frac{K}{m}}\epsilon)^k} + \sum_{k=n^{\frac{1}{4}}+1}^{200\hat{n}_K} n^{13}\frac{(n+1)}{(\sqrt{\frac{K}{m}}\epsilon)^k}. \tag{6.53}$$

The first sum on the r.h.s is, in the large-n limit, obviously convergent: its contribution therefore vanishes in virtue of the $(n-n^{1/4})$-normalization. The second sum converges exponentially fast to 0. All in all, the r.h.s. of (6.53) tends to 0 as $n \to \infty$: this settles the proof of claim (6.34), and therefore of Proposition 6.3.

□

6.2. *Proof of Proposition 6.4.* We will need here two technical inputs. The first one is similar in nature to Lemma 6.1, and provides tail-estimates for the energies of overlapping polymers. As the proof is short and elementary, it will be given right away.

Lemma 6.7. *Consider independent standard exponentials $\{\xi_i\}$, and let $X_l \equiv \sum_{i=1}^{l}\xi_i$. Denote by X_l' the sum of l such ξ-exponentials, and assume that X_l' shares exactly k edges with X_l. Then, for $a, b > 0$, it holds:*

$$\mathbb{P}\left(X_l \leq a+b,\ X_l' \leq a+b\right) \lesssim \mathbb{P}\left(X_l \leq a,\ X_l' \leq a\right)\left(1+\frac{b}{a}\right)^{2l-k}. \tag{6.54}$$

Proof: Recalling that

$$g(\gamma) \equiv \frac{(4(1-\gamma))^{1-\gamma}}{(2-\gamma)^{2-\gamma}}, \tag{6.55}$$

by Lemma 6.1, it holds

$$\mathbb{P}\left(X_l \le a+b,\ X_l' \le a+b\right) \lesssim \frac{(a+b)^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l . \tag{6.56}$$

Using that $(a+b)^{2l-k} = a^{2l-k}\left(1+\frac{b}{a}\right)^{2l-k}$, we rephrase the r.h.s. of (6.56), to wit

$$\mathbb{P}\left(X_l \le a+b,\ X_l' \le a+b\right) \lesssim \frac{a^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l \left(1+\frac{b}{a}\right)^{2l-k} . \tag{6.57}$$

Again by Lemma 6.1, for the first two terms on the r.h.s. above we have that

$$\frac{a^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l \lesssim \mathbb{P}\left(X_l \le a,\ X_l' \le a\right) , \tag{6.58}$$

and plugging this in (6.57) yields the claim of the Lemma. □

The second technical input concerns the asymptotic of the f-terms. Here and below, we will denote by P_n, Q_n finite degree polynomials, not necessarily the same at different occurences, and which depend on the hypercube dimension only.

Lemma 6.8. *For all $k \le \mathsf{L}_{opt} n$, it holds*

$$\begin{aligned} f(n,k) \le \tanh\left(\mathsf{E}\left(1-\frac{k}{\mathsf{L}_{opt} n}\right)\right)^{\max\left(n-k, \frac{\mathsf{L}_{opt} n-k}{4}\right)} \\ \cosh\left(\mathsf{E}\left(1-\frac{k}{\mathsf{L}_{opt} n}\right)\right)^n \left(\frac{\mathsf{L}_{opt} n}{e\mathsf{E}}\right)^{\mathsf{L}_{opt} n-k} n^{Kn^\alpha} P_n, \end{aligned} \tag{6.59}$$

where P_n is polynomial with finite degree and $\alpha \equiv \frac{5}{6}$.

The proof of this Lemma is also postponed to the Appendix: here we will use it for the

Proof of Proposition 6.4: By (6.24), it holds that

$$\frac{\mathsf{J}\sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt} n-2} f(n,k)\mathbb{P}\left(\pi,\pi_k\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} = \frac{\mathsf{J}\sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt} n-2} f(n,k)\prod_{i=m}^{K-m+1}\mathbb{P}_i\left(\pi,\pi_k\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} . \tag{6.60}$$

We claim that the r.h.s. of (6.60) converges to 0 as $n \to \infty$. To see this, some notation is needed: given two paths $\pi, \pi' \in \mathcal{J}$ which share k edges, we denote by

- k_l the number of common edges between $\mathbf{0}$ and H_m,
- k_m the number of common edges between H_m and H_{K-m},
- k_r the number of shared edges between H_{K-m} and $\mathbf{1}$.

It clearly holds that $k = k_l + k_m + k_r$. Using Lemma 6.7, we obtain

$$\begin{aligned} \prod_{i=m}^{K-m+1} \mathbb{P}_i\left(\pi,\pi_k\right) \lesssim \mathbb{P}\left(X_m(\pi), X_m(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \times \\ \times \prod_{i=m+1}^{K-m} \mathbb{P}\left(X_{i-1,i}(\pi), X_{i-1,i}(\pi_k) \le \mathsf{a}_i \mathsf{E}\right) \times \\ \times \mathbb{P}\left(X_{K-m+1}(\pi), X_{K-m+1}(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \times \\ \times (1+\epsilon_{\mathsf{E}})^{2\sum_{i=m+1}^{K-m} \mathsf{n} d_i - k_m} (1+\epsilon_{m,\mathsf{E}})^{4m\hat{n}_K - 2 - k_l - k_r} . \end{aligned} \tag{6.61}$$

By definition of $\epsilon_{m,\mathsf{E}}$ and ϵ_{E}, see (5.1), the following lower bound plainly holds

$$1+\epsilon_{m,\mathsf{E}} \ge 1+\epsilon_{\mathsf{E}} . \tag{6.62}$$

Using the independence of sub-energies we rewrite

$$\begin{aligned}
&\mathbb{P}\left(X_m(\pi), X_m(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \times \\
&\qquad \times \prod_{i=m+1}^{K-m} \mathbb{P}\left(X_{i-1,i}(\pi), X_{i-1,i}(\pi_k) \le \mathsf{a}_i \mathsf{E}\right) \times \\
&\qquad \times \mathbb{P}\left(X_{K-m+1}(\pi), X_{K-m+1}(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \\
&= \mathbb{P}\Bigg(X_m(\pi), X_m(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}, \\
&\qquad\qquad X_{i-1,i}(\pi), X_{i-1,i}(\pi_k) \le \mathsf{a}_i \mathsf{E},\ i = m+1 \dots K-m, \\
&\qquad X_{K-m+1}(\pi), X_{K-m+1}(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E} \Bigg).
\end{aligned} \tag{6.63}$$

Since $\sum_{i=1}^{K} \mathsf{a}_i = 1$, and by monotonicity of the probabilities, the r.h.s. of (6.63) is *at most*

$$\mathbb{P}\left(\overline{X}_m^{K-m+1}(\pi), \overline{X}_m^{K-m+1}(\pi_k) \le \mathsf{E}\right). \tag{6.64}$$

Using (6.62) and (6.64) in (6.61) thus yields

$$\begin{aligned}
\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi, \pi_k) \le \mathbb{P}\left(\overline{X}_m^{K-m+1}(\pi), \overline{X}_m^{K-m+1}(\pi_k) \le \mathsf{E}\right) \times \\
\times \frac{(1+\epsilon_E)^{2\sum_{i=m+1}^{K-m} \mathsf{n}d_i} (1+\epsilon_{\bar{\mathsf{a}}_m,E})^{4m\hat{n}_K - 2}}{(1+\epsilon_{\mathsf{E}})^k},
\end{aligned} \tag{6.65}$$

which no longer depends on k_l, k_r, k_m, but only on their total sum. Using Lemma 6.1 in (6.65) we thus obtain

$$\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi, \pi_k) \lesssim \frac{\mathsf{E}^{2\mathsf{L}_{opt}n-2-k} g(\frac{k}{\mathsf{L}_{opt}n-2})^{\mathsf{L}_{opt}n-2}}{(\mathsf{L}_{opt}n-2)!(\mathsf{L}_{opt}n-2-k)!} \frac{(1+\epsilon_{\mathsf{E}})^{2\sum_{i=m+1}^{K-m} \mathsf{n}d_i} (1+\epsilon_{m,\mathsf{E}})^{4m\hat{n}_K-2}}{(1+\epsilon_{\mathsf{E}})^k}. \tag{6.66}$$

We now come back to (6.60): using the lower bound to the first moment of $\mathcal{N}^{\epsilon}_{n,K,K'}$ established in Theorem 1′ for the denominator, and (6.66) for the numerator, we see that

$$(6.60) \le \frac{P_n^2}{Q_n^2} \mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} \frac{f(n,k)\mathsf{E}^{2\mathsf{L}_{opt}n-2-k} g(\frac{k}{\mathsf{L}_{opt}n-2})^{\mathsf{L}_{opt}n-2}}{(1+\epsilon_E)^k (\mathsf{L}_{opt}n-2)!(\mathsf{L}_{opt}n-2-k)! C^2_{n,K,m}}. \tag{6.67}$$

(Recall the convention that P_n stands for some finite degree polynomial, not necessarily the same at different occurences). It is immediate to check that the following inequality holds

$$g\left(\frac{k}{\mathsf{L}_{opt}n-2}\right)^{\mathsf{L}_{opt}n-2} < g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} P_n\,. \tag{6.68}$$

Furthermore,

$$(\mathsf{L}_{opt}n-2)! = \frac{(\mathsf{L}_{opt}n)!}{(\mathsf{L}_{opt}n)(\mathsf{L}_{opt}n-1)} = \frac{(\mathsf{L}_{opt}n)!}{P_n}, \tag{6.69}$$

where P_n is a polynomial of finite (quadratic) degree, and analogously

$$(\mathsf{L}_{opt}n-2-k)! = \frac{(\mathsf{L}_{opt}n-k)!}{P_n}. \tag{6.70}$$

Using (6.68), (6.69), and (6.70), we thus see that

$$(6.67) \leq \frac{P_n}{Q_n} \mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} \frac{f(n,k)\mathsf{E}^{2\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k (\mathsf{L}_{opt}n)!(\mathsf{L}_{opt}n-k)! C^2_{n,K,m}}, \tag{6.71}$$

for some (modified, but still finite degree) polynomials P_n, Q_n.

The inclusion $\mathcal{J} \subset \Pi_{n,\mathsf{L}_{opt}n}$ holds by construction, hence

$$\mathsf{J} \leq M_{n,\mathsf{L}_{opt}n} \leq \sinh(\mathsf{E})^n \frac{(\mathsf{L}_{opt}n)!}{\mathsf{E}^{\mathsf{L}_{opt}n}} = \frac{(\mathsf{L}_{opt}n)!}{\mathsf{E}^{\mathsf{L}_{opt}n}}, \tag{6.72}$$

the second inequality by Stanley's M-bound (2.14) with $x := \mathsf{E}$, and the last step since E satisfies $\sinh(\mathsf{E}) = 1$. Plugging (6.72) into (6.71), we obtain

$$\begin{aligned}(6.71) &\leq \frac{P_n}{C^2_{n,K,m}Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} \frac{f(n,k)\mathsf{E}^{\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k (\mathsf{L}_{opt}n-k)!} \\ &\leq \frac{P_n}{C^2_{n,K,m}Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n} \frac{f(n,k)(e\mathsf{E})^{\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k (\mathsf{L}_{opt}n-k)^{\mathsf{L}_{opt}n-k}},\end{aligned} \tag{6.73}$$

the last inequality by Stirling's approximation, and extending the sum up to $\mathsf{L}_{opt}n$ (the terms are positive anyhow). The estimates of Lemma 6.8 applied to (6.73) yield

$$\begin{aligned}(6.73) \leq \frac{n^{Kn^\alpha}P_n}{C^2_{n,K,m}Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n} \Bigg[&\tanh\left(\mathsf{E}\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)} \times \\ &\times \frac{\cosh\left(\mathsf{E}\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)\right)^n g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{(1+\epsilon_{\mathsf{E}})^k \left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} \Bigg].\end{aligned} \tag{6.74}$$

Recalling the definition (6.2) of the g-function, one plainly checks that

$$\frac{g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} = \left[\frac{4^{1-\frac{k}{\mathsf{L}_{opt}n}}}{\left(2-\frac{k}{\mathsf{L}_{opt}n}\right)^{2-\frac{k}{\mathsf{L}_{opt}n}}}\right]^{\mathsf{L}_{opt}n}. \tag{6.75}$$

We lighten notation by setting, for $x \in [0,1]$,

$$\widehat{\Theta}(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}} \tanh\left(\mathsf{E}\left(1-x\right)\right)^{\max\left(\frac{1}{\mathsf{L}_{opt}}-x,\frac{1-x}{4}\right)} \cosh\left(\mathsf{E}\left(1-x\right)\right)^{\frac{1}{\mathsf{L}_{opt}}}. \tag{6.76}$$

With this notation, the r.h.s. of (6.74) then reads

$$\begin{aligned}&\frac{n^{Kn^\alpha}P_n}{C^2_{n,K,m}Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n} \frac{1}{(1+\epsilon_E)^k} \widehat{\Theta}\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} \\ &= \frac{n^{Kn^\alpha}P_n}{C^2_{n,K,m}Q_n} \left(\sum_{k=200\hat{n}_K+1}^{\frac{\mathsf{L}_{opt}n}{5}} + \sum_{k=\frac{\mathsf{L}_{opt}n}{5}+1}^{\mathsf{L}_{opt}n} \right) \frac{1}{(1+\epsilon_E)^k} \widehat{\Theta}\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} \\ &=: (A) + (B),\end{aligned} \tag{6.77}$$

say. In order to prove that these two terms vanish as $n \uparrow \infty$, we need the following

Lemma 6.9. *It holds:*

$$\sup_{x\le 1} \widehat{\Theta}(x) \le 1\,. \tag{6.78}$$

Furthermore, for $x \le \frac{1}{5}$,

$$\widehat{\Theta}(x) \le \exp\left(-\frac{x}{100}\right)\,. \tag{6.79}$$

The proof of Lemma 6.9 is given in the Appendix. We first use it to conclude the proof of Proposition 6.4: using the bound (6.79) for the (A)-term yields

$$\begin{aligned}(A) &\le \frac{n^{Kn^\alpha}P_n}{C^2_{n,K,m}Q_n}\sum_{k=200\hat{n}_K+1}^{\frac{\mathsf{L}_{opt}n}{5}} \frac{\exp-\frac{k}{100}}{(1+\epsilon_E)^k}\\ &\le \frac{\exp-\frac{200n}{100K}}{C^2_{n,K,m}}\frac{n^{Kn^\alpha}P_n}{Q_n}\sum_{k=200\hat{n}_K+1}^{\frac{\mathsf{L}_{opt}n}{5}}\frac{1}{(1+\epsilon_E)^k},\end{aligned} \tag{6.80}$$

since $x\mapsto \exp(-x)$ is decreasing. Furthermore using that the above sum is convergent we thus see that

$$(A) \lesssim \frac{\exp -2\frac{n}{K}}{C^2_{n,K,m}}\frac{n^{Kn^\alpha}P_n}{Q_n}\,. \tag{6.81}$$

Finally plugging the definition (5.3) of $C_{n,K,m}$ into (6.81), yields

$$(A) \le \exp n\left[\frac{\sqrt{2}-2}{K}+\frac{2\sqrt{2}m(m-1)+2}{K^2}\right]\times\frac{n^{Kn^\alpha}P_n}{Q_n}\,. \tag{6.82}$$

But for $K>10^7$, the exponent on the r.h.s. above is <0, hence the (A)-term vanishes as $n\uparrow\infty$, settling the first claim.

As for the (B)-term , using (6.78) yields

$$\begin{aligned}(B) &\le \frac{n^{Kn^\alpha}P_n}{C^2_{n,K,m}Q_n}\times\sum_{k=\frac{\mathsf{L}_{opt}n}{5}+1}^{\mathsf{L}_{opt}n}\frac{1}{(1+\epsilon_{\mathsf{E}})^k}\\ &\le \frac{n^{Kn^\alpha}P_n}{C^2_{n,K,m}Q_n}\times\frac{\mathsf{L}_{opt}n}{(1+\epsilon_{\mathsf{E}})^{\frac{\mathsf{L}_{opt}n}{5}}},\end{aligned} \tag{6.83}$$

the last inequality majorizing with the largest term of the sum. Again plugging the definition (5.3) of $C_{n,K,m}$ in (6.83), and absorbing the n-factor in the P-polynomial, yields

$$(B) \le \exp n\left[\frac{\sqrt{2}}{K}+\frac{2\sqrt{2}m(m-1)+2}{K^2}\right]\times\frac{1}{(1+\epsilon_{\mathsf{E}})^{\frac{\mathsf{L}_{opt}n}{5}}}\times\frac{n^{Kn^\alpha}P_n}{Q_n}. \tag{6.84}$$

By (2.59), it holds that $\mathsf{L}_{opt}>\mathsf{L}-\frac{m}{K}$, clearly implying that for any $K>10^5$,

$$1.25 \ge \mathsf{L}_{opt} \ge 1.24\,. \tag{6.85}$$

Using this in (6.84) yields

$$\begin{aligned}(B) &\le \exp n\left[\frac{\sqrt{2}}{K}+\frac{2\sqrt{2}m(m-1)+2}{K^2}\right]\times\frac{1}{(1+\epsilon_{\mathsf{E}})^{\frac{1.24n}{5}}}\times\frac{n^{Kn^\alpha}P_n}{Q_n}\\ &= \exp n\left[\frac{\sqrt{2}}{K}+\frac{2\sqrt{2}m(m-1)+2}{K^2}-\frac{1.24n}{5}\log(1+\epsilon_{\mathsf{E}})\right]\times\frac{n^{Kn^\alpha}P_n}{Q_n}.\end{aligned} \tag{6.86}$$

Using the lower bound $\log(1+x) \geq x - \frac{x^2}{2}$ in (6.86) finally yields

$$(B) \leq \exp n \left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} - \left(\epsilon_{\mathsf{E}} - \frac{\epsilon_{\mathsf{E}}^2}{2}\right) \frac{1.24}{5} \right] \times \frac{n^{Kn^\alpha} P_n}{Q_n} . \tag{6.87}$$

But for $K > \max(10^7, \epsilon^{-2})$, the exponent is definitely strictly negative, hence the (B)-terms also vanishes as $n \uparrow \infty$, concluding the proof of the second claim. □

6.3. *Proof of Proposition 6.5.* We first state the technical input concerning the asymptotic of the $f^{(s)}$-terms. (As usual, P_n, Q_n stand for finite degree polynomials, not necessarily the same at different occurences).

Lemma 6.10. *For any $k \leq 200\hat{n}_K$, it holds*

$$\begin{aligned} f^{(s)}(n,k) \leq & \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \tanh\left(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n})\right)^{n-k} \times \\ & \times \cosh\left(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n})\right)^{n} \left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{\mathsf{L}_{opt}n-k} n^{Kn^\alpha} P_n . \end{aligned} \tag{6.88}$$

The proof of this Lemma is also postponed to the Appendix.

Proof of Proposition 6.5: . By (6.24), it holds that

$$\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(s)}\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} = \frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\prod_{i=m}^{K-m+1}\mathbb{P}_i\left(\pi, \pi_k^{(s)}\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} . \tag{6.89}$$

We claim that the r.h.s. of (6.89) converges to 0 as $n \to \infty$. To see this, we follow *exactly* the same steps which from (6.60) lead to (6.73), this time of course with $f^{(s)}$ instead of f. Omitting the details, the upshot is that the r.h.s. of (6.89) is *at most*

$$\frac{P_n}{C^2_{n,K,m}Q_n} \sum_{k=1}^{200\hat{n}_K} \frac{f^{(s)}(n,k)(e\mathsf{E})^{\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k(\mathsf{L}_{opt}n-k)^{\mathsf{L}_{opt}n-k}} . \tag{6.90}$$

The estimates from Lemma 6.10 applied to (6.90) then yield

$$(6.90) \leq \frac{(\frac{3}{4})^{(m-200)\hat{n}_K} n^{Kn^\alpha} P_n}{C^2_{n,K,m}Q_n} \sum_{k=1}^{200\hat{n}_K} \frac{\tanh(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n}))^{n-k}\cosh(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n}))^n g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{(1+\epsilon_{\mathsf{E}})^k\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} . \tag{6.91}$$

As in (6.75), it holds that

$$\frac{g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} = \left[\frac{4^{1-\frac{k}{\mathsf{L}_{opt}n}}}{\left(2-\frac{k}{\mathsf{L}_{opt}n}\right)^{2-\frac{k}{\mathsf{L}_{opt}n}}}\right]^{\mathsf{L}_{opt}n} . \tag{6.92}$$

We lighten notation by setting, for $x \in [0, 1/\mathsf{L}_{opt}]$,

$$\Theta(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}} \tanh\left(\mathsf{E}(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}}-x} \cosh\left(\mathsf{E}(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}}} . \tag{6.93}$$

Using this, together with (6.92), the r.h.s. of (6.91) then takes the neater form

$$\left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m}Q_n} \sum_{k=1}^{200\hat{n}_K} \frac{1}{(1+\epsilon_{\mathsf{E}})^k} \Theta\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} . \tag{6.94}$$

We recall that

$$K > 2 \times 10^7. \tag{6.95}$$

Thus, in the regime $k \le 200\hat{n}_K$, and since $\mathsf{L}_{opt} \ge 1$, we have

$$\frac{k}{\mathsf{L}_{opt} n} \le \frac{200n}{\mathsf{L}_{opt} K n} \le \frac{200}{K} \le 10^{-5}. \tag{6.96}$$

We now claim that for all $x \le 10^{-5}$,

$$\Theta(x) = \widehat{\Theta}(x)\,. \tag{6.97}$$

In fact, for any $x \le 10^{-5}$,

$$\max\left(\frac{1}{\mathsf{L}_{opt}} - x, \frac{1-x}{4}\right) = \frac{1}{\mathsf{L}_{opt}} - x\,, \tag{6.98}$$

as a simple numerical inspection shows: this proves (6.97).

Combining Lemma 6.9 and (6.97), thus yields

$$\sup_{x \le 10^{-5}} \Theta(x) \le 1. \tag{6.99}$$

Using (6.99) in (6.94) then gives that

$$\begin{aligned}(6.94) &\le \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n} \sum_{k=1}^{200\hat{n}_K} \frac{1}{(1+\epsilon_{\mathsf{E}})^k} \\ &\lesssim \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n}\,,\end{aligned} \tag{6.100}$$

since the sum is evidently convergent. Furthermore recalling the definition (5.3) of $C_{n,K,m}$, we thus see that

$$\begin{aligned}(6.100) &\lesssim \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \times \exp n \left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2}\right] \times \frac{n^{Kn^\alpha} P_n}{Q_n} \\ &= \exp n \left[\frac{1}{K}\left((m-200)\log\left(\frac{3}{4}\right) + \sqrt{2}\right) + \frac{2\sqrt{2}m(m-1)+2}{K^2}\right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}\,.\end{aligned} \tag{6.101}$$

Since $m = 205$,

$$(m-200)\log\left(\frac{3}{4}\right) + \sqrt{2} < -\frac{1}{100}\,, \tag{6.102}$$

(this bound is, as a matter of fact, the reason for choosing m as we do), plugging (6.102) in (6.100), yields

$$(6.100) \lesssim \exp n \left[-\frac{1}{100K} + \frac{2\sqrt{2}m(m-1)+2}{K^2}\right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}. \tag{6.103}$$

But again in virtue of (6.95), and with $m = 205$,

$$-\frac{1}{100K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} < 0\,, \tag{6.104}$$

as can be immediately checked: the r.h.s. of (6.103) is therefore vanishing as $n \uparrow \infty$, and the proof of Proposition 6.5 is concluded. □

7. Appendix

7.1. *Stanley's formula* (1.3). We give for completeness the short proof of Stanley's formula (1.3), which states that

$$\sinh(x)^d \cosh(x)^{n-d} = \sum_{l=0}^{\infty} M_{n,l,d} \frac{x^l}{l!} \,. \tag{7.1}$$

Indeed, by the Binomial Theorem, it holds

$$\begin{aligned}\sinh(x)^d \cosh(x)^{n-d} &= \frac{1}{2^n}\left(e^x - e^{-x}\right)^d \left(e^x + e^{-x}\right)^{n-d} \\ &= \frac{1}{2^n}\left(\sum_{j=0}^{d} \binom{d}{j} (-1)^j e^{(d-2j)x}\right)\left(\sum_{i=0}^{n-d} \binom{n-d}{i} e^{(n-d-2i)x}\right) \\ &= \frac{1}{2^n} \sum_{j=0}^{d} \sum_{i=0}^{n-d} \binom{n-d}{i}\binom{d}{j} (-1)^j \exp\left(n - 2(i+j)x\right) \,.\end{aligned} \tag{7.2}$$

Taylor expanding the exponential function, we get that the r.h.s. above equals

$$\begin{aligned}&\sum_{l=0}^{\infty} \frac{1}{2^n} \sum_{i=0}^{n-d} \sum_{j=0}^{d} \binom{d}{j}\binom{n-d}{i} (-1)^j (n - 2(i+j))^l \frac{x^l}{l!} \\ &\qquad = \sum_{l=0}^{\infty} \left(\frac{1}{2^n} \sum_{i'=j}^{n-d+j} \sum_{j=0}^{d} \binom{d}{j}\binom{n-d}{i'-j} (-1)^j (n - 2i')^l \mathbb{1}_{j \le i'} \right) \frac{x^l}{l!} \,,\end{aligned} \tag{7.3}$$

the last step by the substitution $i' \hookrightarrow i + j$. By definition of the $M's$, Stanley's formula thus follows .

□

7.2. *Proof of Lemma 6.9.* We first address claim (6.79): since $\mathsf{L}_{opt} \le \sqrt{2}\mathsf{E} \le 1.25$, one plainly checks that for all $x \le \frac{1}{5}$ it holds

$$\max\left(\frac{1}{\mathsf{L}_{opt}} - x, \frac{1-x}{4}\right) = \frac{1}{\mathsf{L}_{opt}} - x \,, \tag{7.4}$$

therefore

$$\begin{aligned}\widehat{\Theta}(x) &= \frac{4^{1-x}}{(2-x)^{2-x}} \tanh\left(\mathsf{E}\,(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}} - x} \cosh\left(\mathsf{E}\,(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}}} \\ &= \frac{4^{1-x}}{(2-x)^{2-x}} \sinh\left(\mathsf{E}\,(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}} - x} \cosh\left(\mathsf{E}\,(1-x)\right)^{x} \,.\end{aligned} \tag{7.5}$$

The following inequalities can be easily checked using the convexity of $x \mapsto \sinh\left(\mathsf{E}(1-x)\right)$, and of $x \mapsto \cosh\left(\mathsf{E}(1-x)\right)$, and constructing the corresponding chords between $x = 0$ and $x = 1$: it holds

$$\sinh\left(\mathsf{E}(1-x)\right) \le (1-x) \ , \text{ and } \cosh\left(\mathsf{E}(1-x)\right) \le \sqrt{2} + (1-\sqrt{2})x \,. \tag{7.6}$$

Combining (7.5) and (7.6), we obtain

$$\begin{aligned}\widehat{\Theta}(x) &\le \frac{4^{1-x}}{(2-x)^{2-x}} (1-x)^{\frac{1}{\mathsf{L}_{opt}} - x} \left(\sqrt{2} + (1-\sqrt{2})x\right)^x \\ &= \frac{2^{2(1-x)-(2-x)} (1-x)^{\frac{1}{\mathsf{L}_{opt}} - x} \left(\sqrt{2} + (1-\sqrt{2})x\right)^x}{\left(1 - \frac{x}{2}\right)^{(2-x)}} ,\end{aligned} \tag{7.7}$$

the last step by rearrangement. Moreover, it holds that

$$1 - x \leq \left(1 - \frac{x}{2}\right)^2 . \tag{7.8}$$

Simplifying the exponent of the first term in the numerator on the r.h.s. of (7.7), and using (7.8) for the middle term, yields

$$\begin{aligned} \widehat{\Theta}(x) &\leq \frac{2^{-x}\left(1 - \frac{x}{2}\right)^{2\left(\frac{1}{\mathsf{L}_{opt}} - x\right)}\left(\sqrt{2} + (1-\sqrt{2})x\right)^x}{\left(1 - \frac{x}{2}\right)^{(2-x)}} \\ &= \left(\frac{1 + \frac{(1-\sqrt{2})}{\sqrt{2}}x}{\sqrt{2}(1 - \frac{x}{2})}\right)^x \times \frac{1}{\left(1 - \frac{x}{2}\right)^{2\left(1 - \frac{1}{\mathsf{L}_{opt}}\right)}}, \end{aligned} \tag{7.9}$$

the last step again by simple rearrangements.

Elementary inspection of the first derivative shows that, on the interval $[0, 1/5]$, the function

$$x \mapsto \frac{1 + \frac{(1-\sqrt{2})}{\sqrt{2}}x}{(1 - \frac{x}{2})} \tag{7.10}$$

is, in fact, increasing: bounding the function with its largest value attained in $x = 1/5$, and plugging in (7.9), yields

$$\begin{aligned} \widehat{\Theta}(x) &\leq \left(\frac{1 + \frac{(1-\sqrt{2})}{\sqrt{2}}\frac{1}{5}}{\sqrt{2}\frac{9}{10}}\right)^x \times \frac{1}{\left(1 - \frac{x}{2}\right)^{2\left(1 - \frac{1}{\mathsf{L}_{opt}}\right)}} \\ &\leq \left(\frac{3}{4}\right)^x \times \frac{1}{\left(1 - \frac{x}{2}\right)^{2\left(1 - \frac{1}{\mathsf{L}_{opt}}\right)}}, \end{aligned} \tag{7.11}$$

the second inequality by elementary numerical estimates. Exponentiating the second term on the r.h.s. above then leads to

$$\begin{aligned} \widehat{\Theta}(x) &\leq \left(\frac{3}{4}\right)^x \exp\left[-2(1 - \frac{1}{\mathsf{L}_{opt}}) \log\left(1 - \frac{x}{2}\right)\right] \\ &\leq \left(\frac{3}{4}\right)^x \exp\left[x\left(1 - \frac{1}{\mathsf{L}_{opt}}\right) 2\log(2)\right], \end{aligned} \tag{7.12}$$

where in the second step we have used that

$$-\log(1 - \frac{x}{2}) \leq x\log(2), \tag{7.13}$$

which is an immediate consequence of the convexity of $x \mapsto -\log(1 - \frac{x}{2})$. Recalling (2.59), and the ensuing elementary estimate $\mathsf{L}_{opt} < \sqrt{2}\mathsf{E} < 1.25$, we thus see that

$$\widehat{\Theta}(x) \leq \exp x\left[\log\left(\frac{3}{4}\right) + \left(1 - \frac{1}{1.25}\right) 2\log(2)\right] \leq \exp\left[-\frac{x}{100}\right], \tag{7.14}$$

the second inequality by straightforward numerical evaluation: claim (6.79) is thus settled.

We now move to claim (6.78). We recall that

$$\begin{aligned}\widehat{\Theta}(x) &= \frac{4^{1-x}}{(2-x)^{2-x}}\tanh\left(\mathsf{E}\left(1-x\right)\right)^{\max\left(\frac{1}{\mathsf{L}_{opt}}-x,\frac{1-x}{4}\right)}\cosh\left(\mathsf{E}\left(1-x\right)\right)^{\frac{1}{\mathsf{L}_{opt}}}\\ &= \frac{4^{1-x}}{(2-x)^{2-x}}\sinh\left(\mathsf{E}\left(1-x\right)\right)^{\frac{1}{\mathsf{L}_{opt}}-x}\cosh\left(\mathsf{E}\left(1-x\right)\right)^{x}1_{\{\max\left(\frac{1}{\mathsf{L}_{opt}}-x,\frac{1-x}{4}\right)=\frac{1}{\mathsf{L}_{opt}}-x\}}\\ &+ \frac{4^{1-x}}{(2-x)^{2-x}}\sinh\left(\mathsf{E}\left(1-x\right)\right)^{\frac{1-x}{4}}\cosh\left(\mathsf{E}\left(1-x\right)\right)^{\frac{1}{\mathsf{L}_{opt}}-\frac{1-x}{4}}1_{\{\max\left(\frac{1}{\mathsf{L}_{opt}}-x,\frac{1-x}{4}\right)=\frac{1-x}{4}\}}.\end{aligned} \tag{7.15}$$

By (2.59), it holds that $\mathsf{L}_{opt} > \mathsf{L} - \frac{m}{K}$ and this implies that for any $K > 10^5$,

$$\frac{1}{1.24} \geq \frac{1}{\mathsf{L}_{opt}} \geq \frac{1}{1.25}\,. \tag{7.16}$$

Let now

$$g_1(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}}\sinh\left(\mathsf{E}(1-x)\right)^{\frac{1}{1.25}-x}\cosh\left(\mathsf{E}(1-x)\right)^{x}\,, \tag{7.17}$$

$$g_2(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}}\sinh\left(\mathsf{E}(1-x)\right)^{\frac{1-x}{4}}\cosh\left(\mathsf{E}(1-x)\right)^{\frac{1}{1.24}-\frac{1-x}{4}}\,. \tag{7.18}$$

In virtue of (7.16), g_1 is larger than the first term in (7.15), whereas g_2 ls larger than the second one. In particular, setting $g_3 \equiv \min(g_1, g_2)$, we see that in order to establish (6.78) it suffices to prove that

$$\sup_{x\in[0,1]} g_3(x) \leq 1\,, \tag{7.19}$$

which is our new claim. A plot of these two functions is given in Figure 7.16.

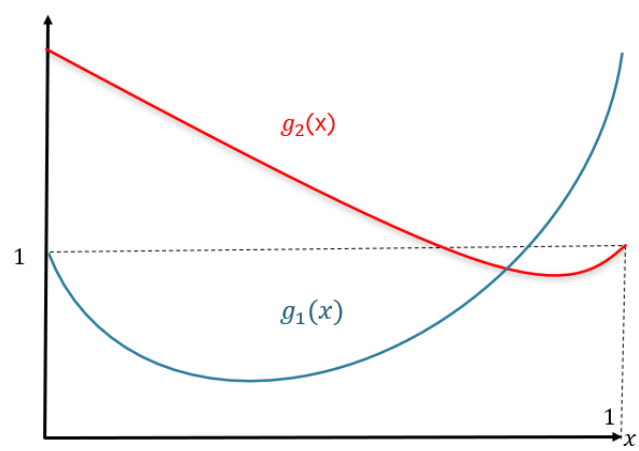

FIGURE 7.16. The functions g_1 and g_2. One clearly sees that the minimum of these functions is always below 1.

To see this, we first note that (6.79) already shows that

$$\sup_{x\in[0,1/5]} g_1(x) \leq 1\,. \tag{7.20}$$

We now claim that

$$g_1 \text{ is convex on } [0.12, 0.73],\ g_1(0.12) \leq 1 \text{ and } g_1(0.73) \leq 1\,, \tag{7.21}$$

and that

$$g_2 \text{ is convex on } [0.71, 1],\ g_2(0.71) \leq 1 \text{ and } g_2(1) = 1. \tag{7.22}$$

Assuming the validity of these two claims for the time being, it follows that

$$g_1(x) \leq 1\ \forall x \leq 0.73\,, \tag{7.23}$$

and

$$g_2(x) \leq 1\ \forall x \geq 0.71\,. \tag{7.24}$$

Combining (7.23) and (7.24) thus yields

$$\sup_{x\in[0,1]} g_3(x) \leq 1, \tag{7.25}$$

and claim (6.78) is verified.

To conclude the proof of Lemma 6.9 it thus remains to prove (7.21) and (7.22). We begin with the convexity of g_1 on the interval $[0.12, 0.73]$. Since $g_1 > 0$,

$$\frac{d^2 \log(g_1)}{dx^2} = \frac{g_1'' g_1 - g_1'^2}{g_1^2} \geq 0 \Longrightarrow g_1''(x) \geq 0, \tag{7.26}$$

hence convexity of $\log(g_1)$ implies convexity of g_1: we will check the former by showing positivity of its second derivative. It holds:

$$\begin{aligned}\frac{d^2 \log(g_1(x))}{dx^2} =& \frac{d^2}{dx^2}\Big[(1-x)\log(4) + (-2+x)\log(2-x)\Big] + \\ &+ \frac{d^2}{dx^2}\Big[x\log(\cosh(\mathsf{E}(1-x)))\Big] + \\ &+ \frac{d^2}{dx^2}\Big[\Big(-x + \frac{1}{1.25}\Big)\log\sinh(\mathsf{E}(1-x))\Big]\,.\end{aligned} \tag{7.27}$$

By elementary computations, we see that:

$$\frac{d^2}{dx^2}\Big[(1-x)\log(4) + (-2+x)\log(2-x)\Big] = \frac{-1}{2-x}, \tag{7.28}$$

$$\begin{aligned}\frac{d^2}{dx^2}\Big[x\log(\cosh(\mathsf{E}(1-x)))\Big] &= \frac{d}{dx}\Big[\log(\cosh(\mathsf{E}(1-x))) - x\mathsf{E}\tanh(\mathsf{E}(1-x))\Big] \\ &= -2E\tanh(\mathsf{E}(1-x)) + \frac{x\mathsf{E}^2}{\cosh(\mathsf{E}(1-x))^2}\,,\end{aligned} \tag{7.29}$$

and finally

$$\begin{aligned}&\frac{d^2}{dx^2}\Big[\Big(-x + \frac{1}{1.25}\Big)\log\sinh(\mathsf{E}(1-x))\Big] \\ &\quad= \frac{d}{dx}\Big[-\log(\sinh(\mathsf{E}(1-x))) + \mathsf{E}\Big(x - \frac{1}{1.25}\Big)\coth(\mathsf{E}(1-x))\Big] \\ &\quad= 2\mathsf{E}\coth(\mathsf{E}(1-x)) + \frac{\mathsf{E}^2(x - \frac{1}{1.25})}{\sinh(\mathsf{E}(1-x))^2}\,.\end{aligned} \tag{7.30}$$

Since $1/5 \geq 0.12$, say, by the previous considerations we see that $g_1(0.12) \leq 1$. We may thus restrict to to $x \in [0.12, 0.73]$: we first note that the first function on the r.h.s. of (7.28) is decreasing. In particular, it holds that

$$\frac{-1}{2-x} \geq \frac{-1}{2-0.73} \geq -0.8\,. \tag{7.31}$$

Plugging (7.28)-(7.30) in (7.27), and then using (7.31) and the fact that $\frac{x\mathsf{E}^2}{\cosh(\mathsf{E}(1-x))^2} \geq 0$, thus yields

$$(7.27) \geq -0.8 - 2\mathsf{E}\tanh\left(\mathsf{E}(1-x)\right) + 2\mathsf{E}\coth\left(\mathsf{E}(1-x)\right) + \frac{\mathsf{E}^2(x-\frac{1}{1.25})}{\sinh\left(\mathsf{E}(1-x)\right)^2}\,. \tag{7.32}$$

We now make two observations:

- First of all we note that the r.h.s. of (7.32) consists of three increasing functions.
- Furthermore, by Taylor expansions to fifth order, and some elementary yet tedious numerical estimates (which will be here omitted) one plainly checks that in $x = 0.12$ the r.h.s. of (7.32) is, in fact, *positive*, whereas $g_1(0.73) \leq 1$.

Combining the above items we see, in particular, that g_1 is indeed convex on $[0.12, 0.73]$, and the proof of claim (7.21) is therefore concluded.

We now move to the analysis of g_2. Simple computations show that

$$\begin{aligned}\frac{d^2}{dx^2}\Big[\log(g_2(x))\Big] &= \frac{-1}{2-x} - \frac{\mathsf{E}}{2}\tanh\left(\mathsf{E}(1-x)\right) + \frac{\mathsf{E}^2(\frac{1}{1.24}+\frac{x-1}{4})}{\cosh\left(\mathsf{E}(1-x)\right)^2} + \frac{\mathsf{E}}{2}\coth\left(\mathsf{E}(1-x)\right)\\ &\qquad + \frac{\mathsf{E}^2(x-1)}{4\sinh\left(\mathsf{E}(1-x)\right)^2}\\ &\geq -1 - \frac{\mathsf{E}}{2}\tanh\left(\mathsf{E}(1-x)\right) + \frac{\mathsf{E}^2(\frac{1}{1.24}+\frac{x-1}{4})}{\cosh\left(\mathsf{E}(1-x)\right)^2} + \frac{\mathsf{E}}{2}\coth\left(\mathsf{E}(1-x)\right)\\ &\qquad + \frac{\mathsf{E}^2(x-1)}{4\sinh\left(\mathsf{E}(1-x)\right)^2}\,,\end{aligned} \tag{7.33}$$

the last inequality using that $\frac{-1}{2-x} \geq -1$. We now proceed in full analogy to (7.32):

- First we note that the r.h.s. of (7.33) consists of four increasing functions.
- Furthermore, and again by some tedious yet elementary numerical estimates via Taylor expansions to fifth order (also omitted), one plainly checks that in $x = 0.71$, say, the r.h.s. of (7.33) is, in fact, *positive*, and $g_2(0.71) \leq 1$.

Since the above items clearly imply, in particular, that g_2 is convex on $[0.71, 1]$, the second claim (7.22) is also settled, and the proof of Lemma 6.9 is thus concluded. □

To control the asymptotics of the $f^{(d)}, f^{(s)}$ and f-terms requires some delicate path-counting.

7.3. *Counting directed paths, and proof of Lemma 6.6.* Key to the whole treatment are estimates for the number of pairs of *directed* paths with prescribed overlaps which are formulated in Lemma 7.1 below. We shall emphasize that the estimates (7.35) and (7.36) have been established by Fill and Pemantle Fill and Pemantle (1993, Lemma 2.2, 2.4), whereas (7.37) can be found in Kistler et al. (2020b, Lemma 6).

Lemma 7.1 (Path counting directed, Fill and Pemantle)**.** *Let π' be any reference path on the n-dim hypercube connecting $\mathbf{0}$ and $\mathbf{1}$, say $\pi' = 12\ldots n$. For $k \geq 1$, denote by $F(n,k)$ the number of directed paths π that share* precisely *k edges with π', and by $F^*(n,k)$ the number of paths that share k edges with π', without considering the first and the last edge. Finally, shorten $\mathfrak{n}_\mathfrak{e} \equiv n - 5e(n+3)^{2/3}$. It holds:*

- *For all $k \geq \mathfrak{n}_\mathfrak{e}$, we have*

$$F(n,k) \leq (n-k)!\binom{n}{\mathfrak{n}_\mathfrak{e}}\,, \tag{7.34}$$

- *suppose $k \le \mathfrak{n}_{\mathfrak{e}}$ for $n \ge 25$. Then, it holds*

$$F(n,k) \le (n-k)!n^6\,. \tag{7.35}$$

- *For $k \le n^{1/4}$, the stronger bounds hold*

$$F(n,k) \le (n-k)!(k+1)(1+o_n(1)), \tag{7.36}$$

and

$$F^*(n,k) \le (n-k-1)!(k+1)(1+o_n(1)), \tag{7.37}$$

as $n \uparrow \infty$, uniformly in k.

Proof: As mentioned, we only need to prove (7.34): to this end, consider a directed path π which shares precisely k edges with the reference path $\pi' = 12\ldots n$. We set $r_i = l$ if the lth traversed edge by π is the ith edge shared by π and π'. (We set by convention $r_0 \equiv 0$, and $r_{k+1} \equiv n+1$). Furthemore let $\mathbf{r} \equiv \mathbf{r}(\pi) = (r_0, \ldots, r_{k+1})$. For any sequence $\mathbf{r}_0 = (r_0, \ldots, r_{k+1})$ with $0 = r_0 < r_1 < \cdots < r_k < r_{k+1} = n+1$, let $C(\mathbf{r}_0)$ denote the number of paths π with $\mathbf{r}(\pi) = \mathbf{r}_0$. Since the values $\pi_{r_i+1}, \ldots, \pi_{r_{i+1}-1}$ must be a permutation of $\{r_i+1, \ldots, r_{i+1}-1\}$, it clearly holds that $C(\mathbf{r}) \le G(\mathbf{r})$, where

$$G(\mathbf{r}) \equiv \prod_{i=0}^{k} (r_{i+1} - r_i - 1)!\,. \tag{7.38}$$

Iterating the log-convexity (6.40) of factorials in its simplest form: $a!b! \le (a+b)!$, yields

$$G(\mathbf{r}) \le \left(\sum_{i=0}^{k} r_{i+1} - r_i - 1\right)! = (n+1-(k+1))! = (n-k)!\,, \tag{7.39}$$

which implies, in particular, that there are at most $(n-k)!$ paths sharing k edges with a reference-path π' for *given* $\mathbf{r}$-sequence. But since there are $\binom{n}{k}$ ways to choose such $\mathbf{r}$-sequences we obtain

$$F(n,k) \le (n-k)!\binom{n}{k}\,. \tag{7.40}$$

Since the factorial term on the r.h.s. above is decreasing in k for $k \ge \lceil \frac{n}{2} \rceil$, we deduce that for $k \ge \mathfrak{n}_{\mathfrak{e}} \gg \frac{n}{2}$,

$$(n-k)!\binom{n}{k} \le (n-k)!\binom{n}{\mathfrak{n}_{\mathfrak{e}}}, \tag{7.41}$$

settling the proof of (7.34). □

Armed with the above estimates on the number of directed paths with prescribed overlaps, we can move to the

Proof of Lemma 6.6: For $\pi \in \mathcal{J}$ and $\pi_k^{(d)} \in \mathcal{J}_\pi^{(d)}(n,k)$, let us denote by k_l the number of common edges between $\mathbf{0}$ and H_m, and by k_r the number of common edges between H_{K-m} and $\mathbf{1}$ (in which case it evidently holds that $k = k_l + k_r$). Furthermore, let

$$\begin{aligned} f_\pi^{(d)}(n,k,k_l) \equiv\ & \text{all paths } \pi' \in \mathcal{J} \text{ which share } k \text{ edges with} \\ & \pi \text{ only in the directed phase, i.e between} \\ & \mathbf{0} \text{ and } H_m \text{ or } H_{K-m} \text{ and } 1, \\ & \text{with } k_l \text{ edges in common between } \mathbf{0} \text{ and } H_m, \\ & \text{but without considering first and last edge.} \end{aligned} \tag{7.42}$$

We have

$$\begin{aligned} f_\pi^{(d)}(n,k) &= \sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l) + \sum_{k_l < k_r} f_\pi^{(d)}(n,k,k_l) \\ &\leq \sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l) + \sum_{k_l \leq k_r} f_\pi^{(d)}(n,k,k_l). \end{aligned} \tag{7.43}$$

We claim that

$$\sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l) = \sum_{k_l \leq k_r} f_\pi^{(d)}(n,k,k_l). \tag{7.44}$$

This claim is perhaps surprising at first sight, as k_l and k_r cannot be simply swapped. The idea is over to work through bijections relating the (pair) of paths appearing in the first sum to those in the second one.

Indeed, each vertex on the right side of the hypercube stands in one to one correspondence with a vertex on the left side: the (trivial) bijection here amounts to changing the $1's$ into $0's$ (and the $0's$ into $1's$).

Furthermore, by (2.28), backsteps and forward steps are symmetric around the center of the hypercube, meaning that for $i \in \{m+1, K-m\}$,

$$\mathsf{eb}_i = \mathsf{eb}_{K-i+1} \qquad \text{and} \qquad \mathsf{ef}_i = \mathsf{eb}_{K-i+1}\,. \tag{7.45}$$

This, together with the fact that polymers are stretched, implies that the number of subpaths reaching two given vertices between H_i and H_{i+1}, and the number of those between $H_{K-(i+1)}$ and H_{K-i} do in fact coincide.

Finally, we note that the "cone" of vertices in H_{i+1} which are attainable from a vertex in H_i in the first half of the hypercube is in one-to-one correspondence with the vertices in $H_{K-(i+1)}$ which lead to a given vertex in H_{K-i} (this can immediately seen by changing the 1-coordinates of a vertex into 0, or the other way around). Thus, for each cone on the left side of the hypercube, we find a cone on the right side which evolves in the opposite direction, settling claim (7.44).

Using (7.44) in (7.43) yields

$$f_\pi^{(d)}(n,k) \leq 2 \sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l). \tag{7.46}$$

We now make the following key observation: counting the number of directed subpaths which share k_l edges with π (disregarding the first edge) between $\mathbf{0}$ and any admissible point of H_m is equivalent to counting the number of directed subpaths π' that share k_l edges with the directed subpath of π, but on a hypercube of dimension $m\hat{n}_K$ (again disregarding the first edge). By symmetry, the same of course holds true for the number of subpaths between H_{K-m} and $\mathbf{1}$ (this time disregarding the last edge). The new goal is thus to solve the path-counting problem on these hypercubes of smaller dimensions. In order to do so, we focus on the rightmost edge shared by both polymers, and denote by

$$d_l \equiv d\left(\pi_{r_{k_l}}, H_m\right), \tag{7.47}$$

its Hamming distance to the H_m-plane. We now distinguish between two cases: the first case concerns the situation where $d_l = 0$, whereas the second case concerns $d_l > 0$.

If $d_l = 0$, the rightmost common edge leads directly into the H_m-plane. Any subpath sharing k_l edges with π can thus reach one vertex only on the target plane: counting the number of subpaths connecting $\mathbf{0}$ and this prescribed vertex, *while disregarding the first edge*, is therefore equivalent to estimating the number of directed paths which share $k_l - 1$ edges on a hypercube of dimension $m\hat{n}_K - 1$, also disregarding the first edge. We will solve the latter problem with the help of F, in which case a small detail must be taken into account. In fact, contrary to our current situation, the first edge does matter in the definition of F. We thus have to distinguish between the case whether the first edge is shared, respectively: not shared, by both paths. In both cases we need to specify

$k_l - 1$ common edges disregarding first and "last" edge: in the first case the number of commond edges is, in fact, $(k_l - 1) + 1 = k_l$, and this leads to at most $F(m\hat{n}_K - 1, k_l)$ ways to choose them. In the second case the problem of the "hidden" (first) shared edge is not present, and we simply have *at most* $F(m\hat{n}_K - 1, k_l - 1)$ possibilities to choose the common edges. All in all, for the number of directed paths sharing k_l common edges (first one excluded), and $d_l = 0$, we have the rough bound

$$F(m\hat{n}_K - 1, k_l) + F(m\hat{n}_K - 1, k_l - 1) \leq 2F(m\hat{n}_K - 1, k_l - 1), \tag{7.48}$$

using for the inequality that $j \mapsto F(n, j)$ is decreasing.

We now move to the case $d_l > 0$ and first note that by definition of $f^{(d)}(n, k)$, neither first nor the last edges can be a common edge. The number of subpaths, which are sharing k_l edges between $\mathbf{0}$ and H_m with π without considering the first and the last edge is thus at most

$$(\#\text{ admissible vertices in } H_m) \times F^*(m\hat{n}_K, k_l). \tag{7.49}$$

We claim that

$$\#\text{ admissible vertices in } H_m = \binom{n - (m\hat{n}_K - d_l)}{d_l}. \tag{7.50}$$

Indeed, of the n possible 1-coordinates, $(m\hat{n}_K - d_l)$ many are already specified by the rightmost common edge; furthermore, in order to reach any of the admissible points on H_m we may switch, out of $n - (m\hat{n}_K - d_l)$ 0-coordinates, d_l many into $1's$: (7.50) thus follows by simple counting.

Next we claim that $j \mapsto \binom{n+j}{j}$ is increasing. To see this, we write

$$\binom{n+j}{j} = \frac{(n+j)\dots(j+1)}{n!}, \tag{7.51}$$

and observe that the term in the numerator on the r.h.s. above is increasing. It follows in particular, that the r.h.s. of (7.50) is maximized for $d_l = m\hat{n}_K - k_l - 1$ (recall that we are not considering the first edge), and therefore

$$(7.49) \leq \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1} \times F^*(m\hat{n}_K, k_l). \tag{7.52}$$

Combining (7.48) and (7.52), we thus see that the overall number of subpaths sharing k_l edges on the "left side" of the hypercube (i.e. between $\mathbf{0}$ and H_m, but without considering the first edge) with a reference path π is less than

$$2F(m\hat{n}_K - 1, k_l - 1) + F^*(m\hat{n}_K, k_l) \times \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1}. \tag{7.53}$$

We next move to the "right side" of the hypercube: in full analogy to the considerations leading to (7.48), one sees that the number of subpaths sharing k_r edges between a point on H_{K-m} and $\mathbf{1}$ with a given reference path π (disregarding, in this case, the last edge), is less than

$$F(m\hat{n}_K, k_r) + F(m\hat{n}_K, k_r + 1) \leq 2F(m\hat{n}_K, k_r). \tag{7.54}$$

The bounds (7.53) and (7.54) address "left" and "right" side of the hypercube on separate footing: for these bounds to be of any use in estimating the $f^{(d)}(n, k, k_l)$-terms appearing in (7.46), left and right side must be connected. We will do so by slightly "overshooting", insofar we do not take into account the fact that the number of subpaths connecting H_m and H_{K-m} is *reduced*, as soon as shared edges on the right region are specified. Recalling that $\mathsf{J} = \#\mathcal{J}$ takes the form

$$\mathsf{J} = \underbrace{(m\hat{n}_K)!\binom{n}{m\hat{n}_K}}_{\text{directed}} \times \underbrace{\mathsf{J}_s}_{\text{stretched}} \times \underbrace{(m\hat{n}_K)!}_{\text{directed}}, \tag{7.55}$$

with J_s denoting the number of subpaths between a given vertex on H_m and the H_{K-m}-plane, it follows from (7.53), (7.54) and the aforementioned overshooting, that

$$f_\pi^{(d)}(n,k,k_l) \le \left(2F(m\hat{n}_K-1,k_l-1)+F^*(m\hat{n}_K,k_l)\times\binom{n-k_l-1}{m\hat{n}_K-k_l-1}\right)\times \\ \mathsf{J}_s\times 2F(m\hat{n}_K,k_r)\,. \tag{7.56}$$

The above is our fundamental estimate. Remark in particular, that it holds uniformly over $k=k_l+k_r$. To proceed further we will now distinguish two cases: either $k\le n^{\frac14}$ or $k>n^{\frac14}$.

First case: $k\le n^{\frac14}$. We begin with an estimate for the terms in the large brackets of the r.h.s. of (7.56). In the considered k-regime, we may use the bounds provided by Lemma 7.1: display (7.36) yields the bound

$$F(m\hat{n}_K-1,k_l-1)\le (k_l+1)(m\hat{n}_K-k_l)!\,[1+o_n(1)]\le 2(k_l+1)(m\hat{n}_K-k_l)!\,, \tag{7.57}$$

for n large enough, whereas display (7.37) of the same Lemma yields, for the F^*-term on the r.h.s. of (7.56) the bound

$$F^*(m\hat{n}_K,k_l)\le (k_l+1)(m\hat{n}_K-k_l-1)!\,[1+o_n(1)]\le 2(k_l+1)(m\hat{n}_K-k_l-1)!\,, \tag{7.58}$$

which holds again for large enough n. Combining (7.57) and (7.58) we thus get that the terms in the large brackets of the r.h.s. of (7.56) are *at most*

$$4(k_l+1)(m\hat{n}_K-k_l)!+2(k_l+1)(m\hat{n}_K-k_l-1)!\binom{n-k_l-1}{m\hat{n}_K-k_l-1} \\ \le 4(k_l+1)(m\hat{n}_K-k_l-1)!\times\binom{n-k_l-1}{m\hat{n}_K-k_l-1}, \tag{7.59}$$

the second inequality since $n-k_l-1\ge m\hat{n}_K-k_l-1\ge 5\hat{n}_K-1$ (see $m=205$ and $k\le 200\hat{n}_K$) implies that the second term on the l.h.s. above is (exponentially) larger than the first one.

We may again use the bounds provided by Lemma 7.1, display (7.36), akin to (7.57), and we obtain

$$2F(m\hat{n}_K,k_r)\le 4(k_r+1)(m\hat{n}_K-k_r)!. \tag{7.60}$$

Plugging the estimates (7.59) and (7.60) into (7.56), we obtain

$$\begin{aligned} f_\pi^{(d)}(n,k,k_l) &\le 16(k_l+1)(m\hat{n}_K-k_l-1)!\binom{n-k_l-1}{m\hat{n}_K-k_l-1}\mathsf{J}_s(k_r+1)(m\hat{n}_K-k_r)! \\ &= 16(k_l+1)(m\hat{n}_K-k_l-1)!\binom{n-k_l-1}{m\hat{n}_K-k_l-1}\frac{\mathsf{J}(k_r+1)(m\hat{n}_K-k_r)!}{(m\hat{n}_K)!^2\binom{n}{m\hat{n}_K}}, \end{aligned} \tag{7.61}$$

the last equality expressing J_s as a function of J via the relation (7.55). Writing out the binomials, and after some elementary simplifications, (7.61) becomes

$$f_\pi^{(d)}(n,k,k_l)\le 16(k_l+1)(k_r+1)(n-k_l-1)!(m\hat{n}_K-k_r)!\frac{\mathsf{J}}{(m\hat{n}_K)!n!}. \tag{7.62}$$

In order to estimate the r.h.s. of (7.62), we recall that $k=k_r+k_l$, hence

$$(k_l+1)\,(k_r+1)\le (k+1)^2\,. \tag{7.63}$$

Furthermore, we claim that

$$(n-k_l-1)!(m\hat{n}_K-k_r)!\le\left(n-\lceil\frac{k}{2}\rceil-1\right)!\left(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor\right)!\,. \tag{7.64}$$

To see this, we will make use of the log-convexity (6.40) with $a \equiv \lceil n-k_l-1 \rceil$ and $b \equiv \lfloor m\hat{n}_K - k_r \rfloor$, in which case it clearly holds that $a > b$ for any choice of $k_l = k - k_r$ with $k \leq 200\hat{n}_K$. By log-convexity we thus obtain

$$\begin{aligned}(n-k_l-1)!(m\hat{n}_K-k_r)! &\leq (n-k_l-1+1)!(m\hat{n}_K-k_r-1)! \\ &= \left(n-k_l'-1\right)!\left(m\hat{n}_K-k_r'\right)!\,,\end{aligned} \tag{7.65}$$

where $k_l' \equiv k_l - 1$ and $k_r' \equiv k_r + 1$. Iterating the log-convexity as in (7.65) and taking into account that $k_l \geq k_r$ gives that the r.h.s. of (7.65) is maximized in $k_l = \lceil \frac{k}{2} \rceil$, settling the claim (7.64).

Plugging (7.63) and (7.64) into (7.62) then yields

$$f_\pi^{(d)}(n,k,k_l) \leq 16(k+1)^2 \left(n - \lceil \frac{k}{2} \rceil - 1\right)! \left(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!}. \tag{7.66}$$

All in all, using (7.46) and (7.66), we have

$$\begin{aligned}f_\pi^{(d)}(n,k) &\leq 2 \sum_{k_l \geq k_r} 16(k+1)^2 \left(n - \lceil \frac{k}{2} \rceil - 1\right)! \left(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!} \\ &\leq 32(k+1)^3 \left(n - \lceil \frac{k}{2} \rceil - 1\right)! \left(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!},\end{aligned} \tag{7.67}$$

the last inequality since $k_l + k_r = k$, implying that the sum consists at most of $k+1$ terms.

Second case: $k > n^{\frac{1}{4}}$. Note that we additionally require that $k \leq 200\hat{n}_K$. On the other hand, $200\hat{n}_K \leq \mathfrak{n}_\mathfrak{e}$, by definition. This implies, in particular, that $k \leq \mathfrak{n}_\mathfrak{e}$: we are thus in the (7.35)-regime. Recalling the definition of F^*, the upperbound clearly holds

$$\begin{aligned}F^*(m\hat{n}_K,k_l) &\leq F(m\hat{n}_K,k_l) + F(m\hat{n}_K,k_l+1) + F(m\hat{n}_K,k_l+2) \\ &\leq n^6(n-k_l)!\left(1 + \frac{1}{(n-k_l)} + \frac{1}{(n-k_l)(n-k_l-1)}\right) \\ &\leq n^7(n-k_l-1)!2, \qquad \text{(n large enough)}\end{aligned} \tag{7.68}$$

the second inequality by (7.35). Following *exactly* the same steps which lead from (7.56) to (7.67), again using the Lemma 7.1 but this time with the estimate (7.35) and replacing (7.58) by (7.68), one immediately obtains

$$f_\pi^{(d)}(n,k) \leq 16n^{13}(k+1)\left(n - 1 - \lceil \frac{k}{2} \rceil\right)! \left(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!}, \tag{7.69}$$

for all $\pi \in \mathcal{J}$, concluding the proof of Lemma 6.6. □

7.4. *Counting undirected paths, and proofs of Lemmata 6.8 and 6.10.* Thanks to the repulsive nature of the H-planes, if two paths share two edges between a different pair of H-planes, the common edge with the smaller Hamming distance to $\mathbf{0}$ is evidently crossed first. Given that paths eventually proceed according to the inherent directivity of the problem ("from left to right"), one may ask a similar question for the way two (or more) common edges between two successive H-planes (in the stretched phase) are crossed. To address this question, we will distinguish between two concepts: *i)* directionality, i.e. whether the path performs, while crossing the considered edge, a forward- or a backstep, and *ii)* order in which the considered edges are crossed[7].

[7]In hindsight, we only need two distinctions here: either the two paths cross the edges in the same, or in reverse order. We will avoid explicit definitions for this intuitive concept, but provide an example: assuming that the common edges are labeled a,b,c,d, etc., the order in which a path crosses them is simply the order of the labels: assume the path π crosses the edges in the order a-b-c-d; the path π' can cross the same edges either in exactly the same order a-b-c-d, or in reverse order d-c-b-a.

Lemma 7.2. *Let $\pi, \pi' \in \mathcal{J}$ share edges between the H_{i-1}- and the H_i-plane, for some $i \in \{m+1, \dots, K-m\}$, and assume that the π-path crosses the common edges in a certain directionality and order. Then the π'-path has to cross the edges either*

- *in the same directionality* and *order,*
  or
- *in opposite directionality* and *reverse order.*

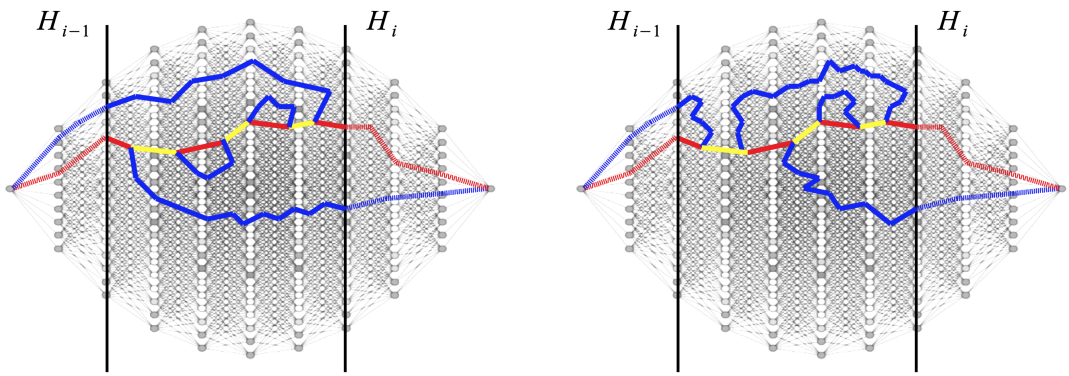

FIGURE 7.17. The yellow edges are shared by both polymers. The picture on the left satisfies the directionality: the red polymer crosses the yellow edges in graphical order "from left to right", while the blue polymer crosses the yellow edges in reversed order and opposite directionality. The picture on the right does not: the blue polymer first crosses the first common edge, but then reverts both order and directionality.

Proof of Lemma 7.2: Consider a path π, and the associated directionality/order in which it crosses the prescribed, common edges. A second path π' which does not follow such directionality and order (nor its complete reversal) will move away from one of the shared edges which are bound to be crossed in a future step. The second path will thus have to make up for this "departure", eventually, but this can only happen if it performs, during its evolution, a *detour*, i.e. if it goes through an edge (parallel to one of the unit vectors) in *both* directions. Since detours are not possible in the stretched phase at hand, the claim follows repeating the line of reasoning.

□

Proof of Lemma 6.8: Consider $\pi \in \mathcal{J}$, and $\boldsymbol{k} = (k_1, k_2, \dots, k_K) \in {\mathbb{Z}_+}^K$, such that $k_1 + k_2 + \dots + k_K = k$. By a slight abuse of notation we denote by $f_\pi(n, \boldsymbol{k})$ the number of paths which share k_i edges with π between the hyperplanes H_{i-1} and H_i, $i \in \{1, \dots, K\}$. It then holds

$$f_\pi(n, k) = \sum_{\boldsymbol{k}} f_\pi(n, \boldsymbol{k}). \tag{7.70}$$

If $k_i > 0$, let v_i^{fi} be the *first* vertex which π hits when crossing the first common edge between H_{i-1} and H_i, and v_i^{la} the *last* vertex from which π departs after crossing the last common edge (also between H_{i-1} and H_i). Furthermore, denote by

$$l_i^{\mathrm{fi}}(\pi) \equiv d\left(H_{i-1} \cap \pi, v_i^{\mathrm{fi}}\right), \qquad l_i^{\mathrm{la}}(\pi) \equiv d\left(v_i^{\mathrm{la}}, H_i \cap \pi\right), \tag{7.71}$$

the Hamming distance from (resp. to) the first (resp. last) vertex to the previous (resp. next) H-plane. If $k_i = 0$, we simply set $l_i^{\mathrm{fi}}(\pi) \equiv d(H_{i-1} \cap \pi, H_i \cap \pi)$ and $l_i^{\mathrm{la}} \equiv 0$.

Finally, consider the whole list (vector) of Hamming distances

$$\boldsymbol{l}(\pi) \equiv \left(l_1^{\mathrm{fi}}(\pi), l_1^{\mathrm{la}}(\pi), l_2^{\mathrm{fi}}(\pi), l_2^{\mathrm{la}}(\pi), \dots, l_K^{\mathrm{fi}}(\pi), l_K^{\mathrm{la}}(\pi)\right) \in {\mathbb{Z}_+}^{2K}. \tag{7.72}$$

Let $f_\pi(n, \boldsymbol{k}, \boldsymbol{l})$ the number of paths sharing k_i edges with π between the hyperplanes H_{i-1} and H_i, $i = 1 = \dots K$, and with prescribed $\boldsymbol{l}$-vector. It then holds

$$f_\pi(n, k) = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} f_\pi(n, \boldsymbol{k}, \boldsymbol{l}) \,. \tag{7.73}$$

By Lemma 7.2, a path $\hat\pi \in \mathcal{J}_\pi(n, k)$ has two ways only to travel through the common edges between successive H-planes: either in identical, or opposite directionality/order. In order to keep track of this, we consider the $\boldsymbol{\sigma} \equiv (\sigma_1, \dots, \sigma_K) \in \{-1, 1\}^K$ with coordinates given by

$$\sigma_i \equiv +1, \qquad \text{if } k_i = 0, \tag{7.74}$$

and

$$\sigma_i \equiv \begin{cases} +1, & \text{if } \hat\pi \text{ crosses first } v_i^{\mathrm{fi}}, \\ -1, & \text{if } \hat\pi \text{ crosses first } v_i^{\mathrm{la}}. \end{cases} \qquad \text{and } k_i > 0. \tag{7.75}$$

We need some additional notation: if $k_i > 0$ and in case of identical directionality/order, i.e. $\sigma_i = +1$, we set

$$\begin{aligned} \hat l_i^{\mathrm{fi}}(\hat\pi) &\equiv \text{length of the substrand connecting the vertices } H_{i-1} \cap \hat\pi \text{ and } v_i^{\mathrm{fi}}\,, \\ \hat l_i^{\mathrm{la}}(\hat\pi) &\equiv \text{length of the substrand connecting } v_i^{\mathrm{la}} \text{ and } H_i \cap \hat\pi, \\ \hat v_i^{\mathrm{fi}} &\equiv v_i^{\mathrm{fi}}, \\ \hat v_i^{\mathrm{la}} &\equiv v_i^{\mathrm{la}}\,. \end{aligned} \tag{7.76}$$

If $k_i > 0$ and in case of reverse directionality/order, i.e. $\sigma_i = -1$, we set

$$\begin{aligned} \hat l_i^{\mathrm{fi}}(\hat\pi) &\equiv \text{length of the substrand connecting the vertices } H_{i-1} \cap \hat\pi \text{ and } v_i^{\mathrm{la}}\,, \\ \hat l_i^{\mathrm{la}}(\hat\pi) &\equiv \text{length of the substrand connecting } H_i \cap \hat\pi \text{ and } v_i^{\mathrm{fi}}, \\ \hat v_i^{\mathrm{fi}} &\equiv v_i^{\mathrm{la}}, \\ \hat v_i^{\mathrm{la}} &\equiv v_i^{\mathrm{fi}}\,. \end{aligned} \tag{7.77}$$

If $k_i = 0$, we simply set

$$\begin{aligned} \hat l_i^{\mathrm{fi}}(\hat\pi) &\equiv l_\pi(H_{i-1} \cap \pi, H_i \cap \pi), \\ \hat l_i^{\mathrm{la}}(\hat\pi) &\equiv 0. \end{aligned} \tag{7.78}$$

Furthermore, let

$$\begin{aligned} \hat v_0^{\mathrm{la}} &\equiv \mathbf{0}, \\ \hat v_{K+1}^{\mathrm{fi}} &\equiv \mathbf{1}. \end{aligned} \tag{7.79}$$

In full analogy with $\boldsymbol{l}$, we denote by $\hat{\boldsymbol{l}}$ the list (vector) of $\hat l$-lengths.

Let us now go back to (7.73): with $f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}})$ standing for the number of $\hat\pi$-paths which share k_i edges with π between the hyperplanes H_{i-1} and H_i with prescribed lengths $\boldsymbol{l}$ (for π), $\hat{\boldsymbol{l}}$ (for $\hat\pi$) and with $\boldsymbol{\sigma}$ directionality/order, it holds

$$f_\pi(n, k) = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\hat{\boldsymbol{l}}} f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}). \tag{7.80}$$

We will now derive a formula for the f_π-summands on the r.h.s. above in terms of the number of paths satisfying the prescriptions *locally*: this requires discriminating between the cases where first and last common edge both lie within the same slab (i.e. between successive H-planes), or in two different slabs. Let $h(i) \equiv \min\{a, a \geq i, k_a > 0\}$ and $h(i) = K + 1$ if $\{a, a \geq i, k_a > 0\}$ is empty or $i = K + 1$. Finally, $h(0) \equiv 0$.

- Same slab.

  - For $k_{h(i)} \geq 1$, we denote by $\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)$ the number of stretched subpaths sharing $k_{h(i)}$ edges with π between $v^{\mathrm{fi}}_{h(i)}$ and $v^{\mathrm{la}}_{h(i)}$, knowing that first and last edge are in common.
- Different slabs.
  - We denote by $\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)$ the number of paths connecting $\hat{v}^{\mathrm{la}}_{h(i)}$ to $\hat{v}^{\mathrm{fi}}_{h(i+1)}$.

See figure 7.18 for a graphical rendition.

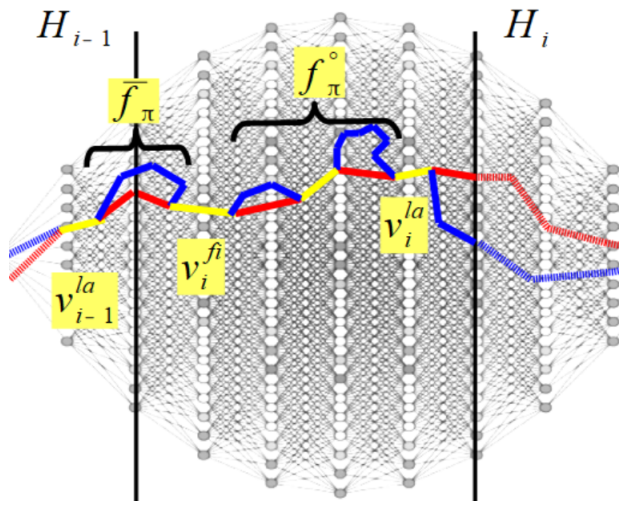

FIGURE 7.18. The blue and the red paths are admissible paths, which cross the different common edges in yellow.

With these definitions, denoting by $b \equiv \#\{i : \ k_i > 0\}$, it clearly holds that

$$f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) = \prod_{i=1}^{b} \mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \prod_{i=0}^{b} \overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)\,. \tag{7.81}$$

The new goal is to get a handle on the $\mathring{f}_\pi$ and $\overline{f}_\pi$-terms. As for the former, we claim that for n big enough, for i, $k_{h(i)} > 0$ and with $\alpha \equiv \frac{5}{6}$,

$$\begin{aligned} \mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq \tanh &\left(\mathsf{E} \frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt} n} \right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \\ &\times \cosh \left(\mathsf{E} \frac{\left(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - k_{h(i)})}{\mathsf{L}_{opt} n} \right)^{n} \\ &\times \left(\frac{\mathsf{L}_{opt} n}{e \mathsf{E}} \right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} n^{n^\alpha} n^{\frac{1}{2}}\,. \end{aligned} \tag{7.82}$$

In order to see this, we first observe that substrands are stretched between successive H-planes: the number of subpaths which share $k_{h(i)} \geq 2$ edges with π between $v^{\mathrm{fi}}_{h(i)}$ and $v^{\mathrm{la}}_{h(i)}$ therefore equals the number of directed subpaths that share $k_{h(i)} - 2$ edges with the subpath of π between $v^{\mathrm{fi}}_{h(i)}$ and $v^{\mathrm{la}}_{h(i)}$ on a hypercube of dimension $d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - 2$. Hence

$$\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq F\left(d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - 2, k_{h(i)} - 2 \right). \tag{7.83}$$

Next we note that for n large enough,

$$n^6 \leq \binom{n}{\mathfrak{n}_{\mathfrak{e}}}, \tag{7.84}$$

and therefore, by Lemma 7.1, the following rough bound holds *for all* $k \leq n$:

$$F(n,k) \leq (n-k)! \binom{n}{\mathfrak{n}_{\mathfrak{e}}}. \tag{7.85}$$

Using this in (7.83) yields

$$F\left(d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - 2, k_{h(i)} - 2\right) \leq \left(d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - k_{h(i)}\right)! \binom{n}{\mathfrak{n}_{\mathfrak{e}}}. \tag{7.86}$$

Furthermore,

$$\binom{n}{\mathfrak{n}_{\mathfrak{e}}} \leq \frac{n!}{\mathfrak{n}_{\mathfrak{e}}!} = \frac{n!}{(n - 5e(n+3)^{2/3})!} \leq n^{5e(n+3)^{2/3}} \leq n^{n^{\alpha}}, \tag{7.87}$$

for n big enough, where $\alpha \equiv \frac{5}{6}$. Using this in (7.86), and plugging the ensuing estimates in (7.83) we obtain

$$\mathring{f}_{\pi}(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq (d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})! n^{n^{\alpha}}. \tag{7.88}$$

By elementary Stirling approximation,

$$\begin{aligned}(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})! &\lesssim \left(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}\right)^{1/2} \left[\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{e}\right]^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \\ &\lesssim n^{1/2} \left[\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{e}\right]^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}},\end{aligned} \tag{7.89}$$

the last inequality using that the dimension of an hypercube embedded between two hyperplanes is bounded above by their distance, i.e $d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) \leq \frac{n}{K} < n$.

Plugging (7.89) in (7.88) yields

$$\mathring{f}_{\pi}(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{n^{\alpha}+1/2} \left[\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{e}\right]^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}. \tag{7.90}$$

The above bound strongly depends on local specifications, which turn out to be rather untractable especially when it comes to the full product (7.81). We will circumvent this problem by means of a series of tricks: in a first step we recognize the term involving the $d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)})$ in (7.90) as a constituent part of a Stanley's bound, which we thus introduce artificially. In a second step, we will perform a rather elementary asymptotic analysis of the product (7.81) which is enabled by some monotonicity properties of the hyperbolic functions. To see how the first step comes about, we note that $\sinh(x) \geq x$ and $\cosh(x) \geq 1$ for all $x > 0$, hence the following holds

$$1 \leq \frac{\sinh(y)^d}{y^d} \cosh(y)^{n-d} = \tanh(y)^d \cosh(y)^n \frac{1}{y^d}, \tag{7.91}$$

for any $y > 0$ and $d \leq n$. We use this inequality with

$$y := \mathsf{E} \frac{(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})}{\mathsf{L}_{opt} n}, \quad d := d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}, \tag{7.92}$$

in which case we see that

$$1 \leq \tanh\left(\mathsf{E}\frac{(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})}{\mathsf{L}_{opt}n}\right) \times \cosh\left(\mathsf{E}\frac{(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})}{\mathsf{L}_{opt}n}\right)^n \times \left[\frac{\mathsf{L}_{opt}n}{\mathsf{E}(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})}\right]^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}. \tag{7.93}$$

Artificially upperbounding with the help of this estimate the r.h.s. of (7.90), and factoring out the $(d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)})^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}$-terms then yields

$$\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{n^\alpha + 1/2} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \times \cosh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}. \tag{7.94}$$

Claim (7.82) is therefore settled for $k_{h(i)} \geq 2$ and easily holds for $k_{h(i)} = 1$.

We now move to estimating the $\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)$-terms. Note that $\boldsymbol{l}$ fixes the vertices $v^{\mathrm{fi}}_i, v^{\mathrm{la}}_i$, and in particular the Hamming distance between two successive commons edges, which are not between the same H-planes, $\boldsymbol{\sigma}$ fixes $\hat{v}^{\mathrm{fi}}_i, \hat{v}^{\mathrm{la}}_i$, while $\hat{\boldsymbol{l}}$ gives the length of the subpaths $\hat{\pi}$ between these common edges.
For all $i \in \{0 \dots K\}$, we set

$$\hat{l}_i \equiv l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}). \tag{7.95}$$

We claim that

$$\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right)} \cosh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\hat{l}_i} n^{\frac{1}{2}}. \tag{7.96}$$

Indeed, it clearly holds that

$$\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq M_{n, \hat{l}_i, d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right)}. \tag{7.97}$$

To get a handle on the r.h.s. above we make use of the following estimate, the derivation of which follows the by now classical route[8], and is thus omitted:

$$M_{n,l,nd} \lesssim n^{\frac{1}{2}} \tanh\left(\frac{\mathsf{E}l}{\mathsf{L}_{opt}n}\right)^{nd} \cosh\left(\frac{\mathsf{E}l}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^l. \tag{7.98}$$

Using (7.98) with $l := \hat{l}_i$, $nd := d(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)})$ in (7.97) steadily yields the claim (7.96).

Having obtained explicit estimates for the $\mathring{f}_\pi$ and $\overline{f}_\pi$-terms, we need bounds to their products as appearing in (7.81). This will be done exploiting the aforementioned monotonicity properties of hyperbolic functions: for any $y_i, d_i \geq 0$, and $k \in \mathbb{Z}_+$ it holds

$$\prod_{i=1}^k \tanh(y_i)^{d_i} \leq \prod_{i=1}^k \tanh\left(\sum_{i=1}^k y_i\right)^{d_i} = \tanh\left(\sum_{i=1}^k y_i\right)^{\sum_{i=1}^k d_i}, \tag{7.99}$$

[8]Stanley's bound (2.14) with $x := \frac{l\mathsf{E}}{\mathsf{L}_{opt}n}$ / Stirling approximation / some elementary rearrangements.

since tanh is increasing, and

$$\prod_{i=1}^{k} \cosh(y_i) \leq \cosh\left(\sum_{i=1}^{k} y_i\right), \tag{7.100}$$

which can be steadily checked iterating $\cosh(a+c) = \cosh(a)\cosh(c) + \sinh(a)\sinh(c) \geq \cosh(a)\cosh(c)$, for $a, c > 0$.

These bounds allow to remove most of the local dependencies appearing in the products (7.81): shortening

$$\mathcal{D}_b \equiv \sum_{i=1}^{b} \left[d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - k_{h(i)} \right], \tag{7.101}$$

and combining (7.99), (7.100) and (7.82) we get

$$\prod_{i=1}^{b} \mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{Kn^\alpha + \frac{K}{2}} \tanh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b} \cosh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{e\mathsf{E}}\right)^{\mathcal{D}_b}. \tag{7.102}$$

On the other hand, shortening

$$\widehat{\mathcal{D}}_b \equiv \sum_{i=0}^{b} d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right), \quad \widehat{L}_b \equiv \sum_{i=0}^{b} \hat{l}_i, \tag{7.103}$$

and combining (7.99), (7.100) with (7.96) we obtain

$$\prod_{i=0}^{b} \overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{\frac{K+1}{2}} \tanh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\widehat{\mathcal{D}}_b} \cosh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E}e}\right)^{\widehat{L}_b}. \tag{7.104}$$

Plugging (7.102) and (7.104) in (7.81) thus leads to

$$\begin{aligned} f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) \lesssim n^{\frac{2K+1}{2} + Kn^\alpha} &\tanh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b} \cosh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E}e}\right)^{\mathcal{D}_b} \times \\ &\times \tanh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\widehat{\mathcal{D}}_b} \cosh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E}e}\right)^{\widehat{L}_b}. \end{aligned} \tag{7.105}$$

The above estimate still involves the product of two tanh-, and two cosh-terms: using once more the monotonicity tricks (7.99) and (7.100) we get

$$\begin{aligned} &f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) \lesssim \\ &\quad n^{\frac{2K+1}{2} + Kn^\alpha} \tanh\left(\mathsf{E}\frac{\mathcal{D}_b + \widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b + \widehat{\mathcal{D}}_b} \cosh\left(\mathsf{E}\frac{\mathcal{D}_b + \widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E}e}\right)^{\mathcal{D}_b + \widehat{L}_b}. \end{aligned} \tag{7.106}$$

But paths in $\mathcal{J}$ have the same, prescribed length, and it holds that

$$\mathcal{D}_b + \widehat{L}_b = \mathsf{L}_{opt} n - k. \tag{7.107}$$

Using this, (7.106) simplifies to

$$\begin{aligned} &f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) \lesssim \\ &\quad n^{\frac{2K+1}{2} + Kn^\alpha} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b + \widehat{\mathcal{D}}_b} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt} n - k}. \end{aligned} \tag{7.108}$$

Remark, in particular, that the r.h.s. above depends on the local prescriptions *only through the* tanh-*exponent*. It will come hardly as a surprise that this feature leads to a dramatic simplification

of the computations. As a matter of fact, even the exponent depends only very mildly on the local prescriptions: indeed, we claim that

Lemma 7.3.

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq \max\left(n-k, \frac{\mathsf{L}_{opt}n-k}{4}\right). \tag{7.109}$$

Proving this claim will unfortunately require a fair amount of work, so we assume its validity for the time being.

By monotonicity,

$$\tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right) \leq \tanh\left(\mathsf{E}\right) = \frac{1}{\sqrt{2}} < 1, \tag{7.110}$$

hence Lemma 7.3 applied to (7.108) yields the upperbound

$$f_\pi(n,\boldsymbol{k},\boldsymbol{l},\boldsymbol{\sigma},\hat{\boldsymbol{l}}) \lesssim n^{\frac{2K+1}{2}+Kn^\alpha}\tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)}\cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^n\left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}, \tag{7.111}$$

no longer depends on $\boldsymbol{l},\boldsymbol{\sigma},\hat{\boldsymbol{l}},\boldsymbol{k}$; plugging this in (7.81), and the ensuing estimate in (7.80) therefore leads to

$$f_\pi(n,k) \lesssim n^{\frac{2K+1}{2}+Kn^\alpha}\sum_{\boldsymbol{l}}\sum_{\boldsymbol{\sigma}}\sum_{\hat{\boldsymbol{l}}}\sum_{\boldsymbol{k}}\mathfrak{T}(n,k), \tag{7.112}$$

where

$$\mathfrak{T}(n,k) \equiv \tanh\left(\frac{\mathsf{E}(\mathsf{L}_{opt}n-k)}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)}\cosh\left(\frac{\mathsf{E}(\mathsf{L}_{opt}n-k)}{\mathsf{L}_{opt}n}\right)^n\left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}. \tag{7.113}$$

Since $\mathfrak{T}(n,k)$ depends on the number of common edges, but not on the local prescriptions, we thus only need estimates on the cardinalities of the sums appearing in (7.112). As for the first sum, since $v_\cdot^{\mathrm{fi}}$ can only move along the path π between two successive hyperplanes, the number of ways to place such $v_\cdot^{\mathrm{fi}}$'s is *at most* n (the same of course holds true for $v_\cdot^{\mathrm{la}}$), hence

$$\sum_{\boldsymbol{l}} \leq n^{2K}, \tag{7.114}$$

and by analogous reasoning

$$\sum_{\boldsymbol{l}'} \leq n^{2K}. \tag{7.115}$$

Moreover, it clearly holds that

$$\sum_{\boldsymbol{\sigma}} \leq 2^K. \tag{7.116}$$

Finally,

$$\sum_{\boldsymbol{k}} = \sum_{\substack{k_i,\\ k_1+k_2+\cdots+k_K=k}} = \binom{k+K-1}{K-1} \lesssim \frac{(k+K-1)^{k+K-1}}{(K-1)^{K-1}k^k}, \tag{7.117}$$

by Stirling approximation. Since $(K-1)^{K-1} \geq 1$, and $\log(1+x) \leq x$, we see that

$$\begin{aligned}(7.117) &\leq k^{K-1}\left(1+\frac{K-1}{k}\right)^{k+K-1} = k^{K-1}\exp\left[(k+K+1)\log\left(1+\frac{K-1}{k}\right)\right]\\ &\leq k^{K-1}\exp\left[(k+K-1)\frac{K-1}{k}\right] \leq k^{K-1}\exp K\,(K-1).\end{aligned} \tag{7.118}$$

Combining (7.112), (7.114), (7.115), (7.116) and (7.118), we obtain

$$f_\pi(n,k) \le P_n n^{Kn^\alpha} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}, \tag{7.119}$$

where P_n is a finite degree polynomial, which is indeed the claim of Lemma 6.8. □

Proof of Lemma 7.3: . Recall that the claim reads

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \ge \max\left(n-k, \frac{\mathsf{L}_{opt}n-k}{4}\right). \tag{7.120}$$

The validity of the first inequality, to wit

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \ge n-k, \tag{7.121}$$

relies on a self-evident fact, namely that the total distance of shared edges in the directed case is a lower bound for the undirected case. More precisely, since common edges contribute to the number of steps performed while connecting $\mathbf{0}$ to $\mathbf{1}$, as soon as a backstep acts on a shared edge, the total distance between shared edges is bound to increase: the path has eventually to make up for the "lost ground". Another way to put it: the contribution $\mathcal{D}_b + \widehat{\mathcal{D}}_b$ is smallest when *all* shared edges are steps forward, in which case the total distance between these edges must be *at least* the minimal number of steps required to connect $\mathbf{0}$ to $\mathbf{1}$. Since this minimal number is clearly the dimension minus the number of shared (prescribed) edges, i.e. $n-k$, (7.121) is settled.

The second inequality

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \ge \frac{\mathsf{L}_{opt}n-k}{4}, \tag{7.122}$$

requires more work and depends on some key properties of paths in $\mathcal{J}$. We begin with a couple of observations:

i) First we note that

$$d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) = d\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right),$$

since inverting directionality clearly has no impact on the distance.

ii) Furthemore, in a (fully) stretched phase distance and length do, in fact, coincide:

$$d\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) = l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right).$$

iii) Finally, and by definition,

$$\sum_{i=1}^{b} k_{h(i)} = k.$$

Plugging items i-iii) above in the $\mathcal{D}_b$-definition (7.101) yields

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b = \sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k + \sum_{i=0}^{b} d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right). \tag{7.123}$$

We now claim that for $i \in \{0,1,\dots b\}$, it holds:

$$d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right) \ge \frac{1}{4} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}). \tag{7.124}$$

This is, in fact, our key technical claim, but since its proof requires some involved analysis, we assume its validity for the time being, and first show how it implies (7.122): plugging (7.124) in (7.123) we obtain

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq \sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k + \frac{1}{4}\sum_{i=0}^{b} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)})\,. \tag{7.125}$$

But by construction,

$$l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) \geq k_{h(i)}, \tag{7.126}$$

hence

$$\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k \geq \sum_{i=1}^{b} k_{h(i)} - k \geq 0, \tag{7.127}$$

the last inequality by item iii) above. This positivity implies, in particular, that

$$\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k \geq \frac{1}{4}\left(\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k\right)\,, \tag{7.128}$$

and using this lower bound in (7.125) then yields

$$\begin{aligned}\mathcal{D}_b + \widehat{\mathcal{D}}_b &\geq \frac{1}{4}\left(\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k\right) + \frac{1}{4}\sum_{i=0}^{b} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) \\ &= \frac{1}{4}\left(\underbrace{\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) + \sum_{i=0}^{b} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)})}_{=\mathsf{L}_{opt}n} - k\right)\,,\end{aligned} \tag{7.129}$$

which settles our key claim (7.122).

It thus remains to prove (7.124). Recall that we are considering the situation where shared edges are separated by (at least) one H-plane[9]. Since by definition an H-plane is also an H'-plane, prescribing the number of separating H'-planes allows to discriminate among different scenarios. Indeed, introducing, for $i = 0 \dots b$,

$$c_{\hat{\pi}}(i) \equiv \quad \text{number of } H'\text{-planes which lie between } \hat{v}^{\mathrm{la}}_{h(i)} \text{ and } \hat{v}^{\mathrm{fi}}_{h(i+1)}\,, \tag{7.130}$$

a minute's thought suggests that there are three scenarios which are "structurally" manifestly different:

- $c_{\hat{\pi}}(i) > 2$: the common edges are separated by at least one H-plane, and multiple H'-planes. We will refer to this as the H'HH'-case.
- $c_{\hat{\pi}}(i) = 2$: in this case the common edges are separated by one H-plane, and one H'-plane (which is however not an H-plane). We will refer to this as the HH'-case.
- $c_{\hat{\pi}}(i) = 1$: the separating hyperplane must be an H-plane: we will refer to this as the H-case.

We will establish the validity of (7.124) in all three possible scenarios. We anticipate that (7.124) becomes more delicate the less hyperplanes are separating the common edges: this is due to the fact that the larger the number of separating hyperplanes the further apart (in terms of Hamming distance d) the common edges must lie, a feature which renders (7.124) all the more likely. In line with this observation, the $c_{\hat{\pi}}(i) = 1$ will turn out to be the most delicate. We emphasize that the

[9] as otherwise the claim would be trivial anyhow: if the shared edges lie within two successive H-planes, the polymer is in a stretched phase in which case distance (d) and length (l) coincide, with the inequality (7.124) thus trivially holding.

index i is given and fixed. To lighten notation we will thus omit it in the expressions, whenever no confusion can possibly arise.

A number of insights are common to the treatment of all three scenarios. Given the nature of the inequality we are aiming to prove, it will not come as a surprise that we will need a good control - in the form of *lower bounds* - on the distance of two common edges, as well as a good control - this time around in the form of *upper bounds* - on the length of the substrands connecting the shared edges.

A reasonably tight, but what is more: valid for any of the three $c_{\hat{\pi}}$-scenarios, lower bound for the distance is provided by technical input **(T1)** below. Let H'_{fi} be the first hyperplane on the right of $\hat{v}^{\text{la}}_{h(i)}$ and H'_{la} be the last hyperplane on the left of $\hat{v}^{\text{fi}}_{h(i+1)}$, and shorten $d^{\text{fi}}_{\hat{\pi}} \equiv d(\hat{v}^{\text{la}}_{h(i)}, H'_{\text{fi}})$, and $d^{\text{la}}_{\hat{\pi}} \equiv d(H'_{\text{la}}, \hat{v}^{\text{fi}}_{h(i+1)})$. A graphical depiction of this is given in Figure 7.19. The following estimate

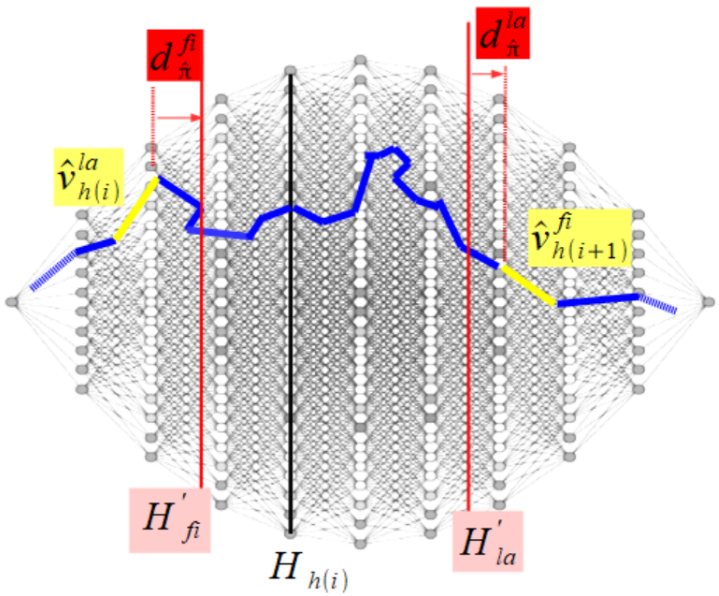

FIGURE 7.19. $\hat{v}^{\text{la}}_{h(i)}$ and $\hat{v}^{\text{fi}}_{h(i+1)}$ separated by three hyperplanes.

holds by definition/construction[10]

$$d\left(\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}\right) \geq d^{\text{fi}}_{\hat{\pi}} + \frac{c_{\hat{\pi}}(i) - 1}{KK'} n + d^{\text{la}}_{\hat{\pi}} \qquad \textbf{(T1)}\,.$$

(We note in passing that equality holds if and only if $\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}$ are connected by a directed substrand; since a stretched substrand may have to perform backsteps while connecting these two vertices, **(T1)** is in general only a lower bound).

As mentioned, the second technical input, **(T2)** below, concerns *upperbounds* on the length of a substrand connecting H'-planes. To see how these come about, let us denote by $\boldsymbol{v} \in H'_{i,j}, \boldsymbol{w} \in H'_{i,j+1}$ *the* vertices by which the $\hat{\pi}$-substrand connects the finer mesh. It is important to observe that in virtue of (4.2), there is no absolutely no ambiguity in the way we identify these vertices: in fact,

$$\begin{gathered}\text{these vertices are unequivocally identified through the } \textit{length} \text{ of} \\ \text{the substrand connecting the successive } H' \text{ -planes.}\end{gathered} \tag{7.131}$$

[10]it can also immediately evinced from Figure 7.19.

We now claim that

$$l_{\hat{\pi}}(\boldsymbol{v},\boldsymbol{w}) \leq \frac{1.46}{KK'}n \qquad \textbf{(T2)}\,.$$

The proof of **(T2)** is rather immediate: first recall that in virtue of (4.2),

$$l_{\hat{\pi}}(\boldsymbol{v},\boldsymbol{w}) = (\mathsf{ef}_i + \mathsf{eb}_i)\,\frac{n}{K'} = \left(\frac{1}{K} + 2\mathsf{eb}_i\right)\frac{n}{K'}, \tag{7.132}$$

the second equality by (2.36). But by (3.19) (and again (2.36)), the number of effective backsteps between H-planes in the stretched phase satisfies

$$\mathsf{eb}_i = \sinh(\overline{\mathsf{a}}_{i-1}\mathsf{E})\sinh(\mathsf{a}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E}), \tag{7.133}$$

and by (4.12),

$$\sinh(\mathsf{a}_i\mathsf{E}) \leq \frac{1}{K} + \frac{1}{6K^3}\,, \tag{7.134}$$

which combined with (7.133) yields

$$\begin{aligned}\mathsf{eb}_i &\leq \sinh(\overline{\mathsf{a}}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E})\left(\frac{1}{K} + \frac{1}{6K^3}\right)\\ &\leq \sinh\left(\frac{\mathsf{E}}{2}\right)^2\left(\frac{1}{K} + \frac{1}{6K^3}\right)\,,\end{aligned} \tag{7.135}$$

the second inequality by (4.10). Since

$$\sinh\left(\frac{\mathsf{E}}{2}\right)^2 \overset{(4.14)}{=} \frac{\sqrt{2}-1}{2} \leq 0.22, \tag{7.136}$$

and using that $K > 10^7$, one plainly checks that

$$\mathsf{eb}_i \leq 0.23 \times \frac{1}{K}. \tag{7.137}$$

Plugging (7.137) in (7.132) settles **(T2)**.

If it is true that there is no ambiguity in the way *vertices* on the H'-plane are identified (recall remark (7.131) above), it is nonetheless true there there is a certain amount of uncertainty in the way the polymer *connects* these planes. This is due to the fact that (contrary to the H-planes) the H'-planes are not repulsive, hence a polymer might cross them multiple times. Such excursions increase of course the length of the substrand, and introduce some "fuzziness" into the picture. Notwithstanding, we claim that

during one such excursion a polymer can overshoot,
in terms of Hamming distance, an H'-plane by at most
$\frac{0.23}{KK'}n$ units.

(T3)

Figure 7.20 provides an elementary proof of this fact.

The above insight, captured by **(T3)**, suggests to introduce the following set

$$\mathfrak{F}_{i,j} \equiv \left\{\boldsymbol{v} \in V_n,\, d(v, H'_{i,j}) \leq \frac{0.23}{KK'}n\right\}\,. \tag{7.138}$$

We emphasize that whenever a common edge lies in this set, it can be crossed by a substrand which *either* connects $H'_{i,j-1}$ with $H'_{i,j}$ *or* $H'_{i,j}$ with $H'_{i,j+1}$: for this reason, we refer to $\mathfrak{F}_{i,j}$ (which is nothing but "twice" the blue-shaded region in Figure 7.20) as the fuzzy zone.

We now record two useful consequences of **(T2)** and **(T3)** on the lengths of substrand which will play a role in the proof of (7.124). For reasons which will become clear, we will only need to

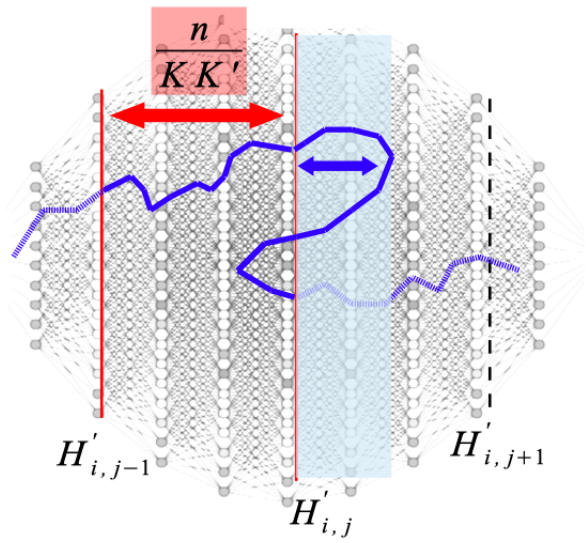

FIGURE 7.20. The proof of **(T3)** relies on two observations: *i)* By **(T2)**, the length of the path connecting first and second H'-planes (the continuous blue strand) is at most $\frac{1.46}{KK'}n$. *ii)* By construction, the Hamming distance of these planes is $\frac{n}{KK'}$. Taking into account that the polymer must return to the second H'-plane, we see that the blue arrow is at most half the difference of these quantities, indeed $\frac{0.23}{KK'}n$, as claimed by **(T3)**. (Remark that this case corresponds to a worst-case scenario: the polymer performs first all available forward steps, and only then all availble backsteps).

consider the case where the first common edge lies in the fuzzy zone of the $H'-$plane which is on the left of H'_{fi}, and/or the other common edge lies on the right of H'_{la}. There are two cases: either shared edges lie outside the fuzzy zone, OuF for short, or inside, InF.

(InF) Remark that $\hat{v}^{\mathrm{la}}_{h(i)}$ being in a fuzzy zone is equivalent to $d^{\mathrm{fi}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$. Analogously, $\hat{v}^{\mathrm{fi}}_{h(i+1)}$ is in a fuzzy zone if and only if $d^{\mathrm{la}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$. Furthermore, a path crossing $\hat{v}^{\mathrm{la}}_{h(i)}$ (or $\hat{v}^{\mathrm{fi}}_{h(i+1)}$) can cross multiple H'-planes besides that to which this vertex belongs: by **(T3)**, this phenomenon can contribute to the length of the substrand at most $\frac{0.46}{KK'}n$ units.

(OuF) If neither $\hat{v}^{\mathrm{la}}_{h(i)}$ nor $v^{\mathrm{fi}}_{h(i+1)}$ are in a fuzzy zone, by **(T2)**, the connecting substrands satisfy

$$l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) \leq \frac{\left(c_{\hat{\pi}(i)}+1\right)1.46}{KK'}n\,.$$

We can finally move to the proof of (7.124): this will be done via case-by-case analysis of the three possible $c_{\hat{\pi}}$-scenarios.

<u>The H'HH'-case.</u>

This case is graphically summarized in Figure 7.21: combining (OuF) and (InF), we immediately evince from this picture that

$$l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) \leq \frac{(c_{\hat{\pi}}(i)+1)\,1.46}{KK'}n + \frac{0.46}{KK'}n\left(1_{d^{\mathrm{fi}}_{\hat{\pi}} \geq \frac{0.77n}{KK'}} + 1_{d^{\mathrm{la}}_{\hat{\pi}} \geq \frac{0.77n}{KK'}}\right)\,. \tag{7.139}$$

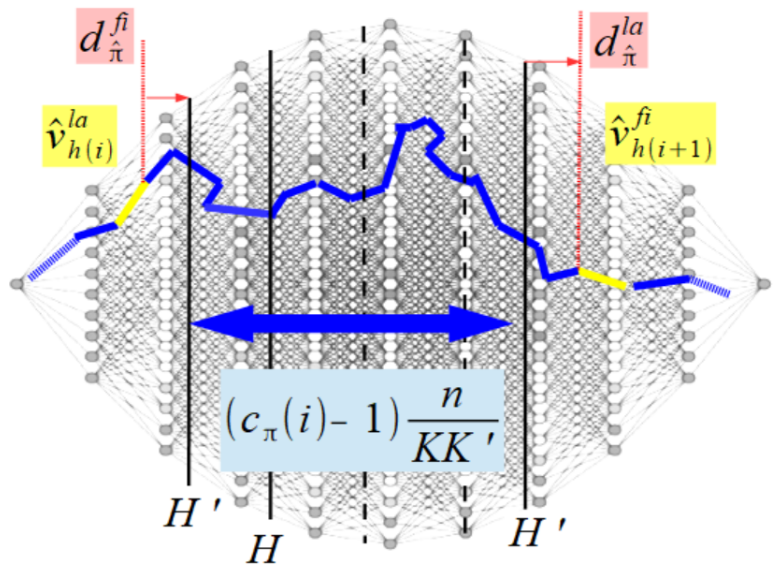

FIGURE 7.21. $c(i) \geq 3$: at least three hyperplanes, i.e. at least two H' and one H, separating the common edges.

The H'HH'-scenario at hand is characterized by $c_{\hat\pi}(i) > 2$, in which case the following inequality is immediate:

$$\frac{(c_{\hat\pi}(i)+1)\,1.46}{KK'} \leq \frac{4(c_{\hat\pi}(i)-1)}{KK'}. \tag{7.140}$$

Using this in (7.139) we obtain

$$l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) \leq \frac{4(c_{\hat\pi}(i)-1)}{KK'} + \frac{0.46}{KK'}n\left(1_{d^{\mathrm{fi}}_{\hat\pi}\geq\frac{0.77n}{KK'}} + 1_{d^{\mathrm{la}}_{\hat\pi}\geq\frac{0.77n}{KK'}}\right). \tag{7.141}$$

Concerning the last two terms on the r.h.s. above, we first observe that obviously

$$d \geq \frac{0.77}{KK'}n \Longrightarrow 4d \geq \frac{0.46}{KK'}n\,, \tag{7.142}$$

hence

$$\frac{0.46}{KK'}n1_{d^{\mathrm{fi}}_{\hat\pi}\geq\frac{0.77}{KK'}n} \leq 4d^{\mathrm{fi}}_{\hat\pi}(\hat\pi), \qquad \frac{0.46}{KK'}n1_{d^{\mathrm{la}}_{\hat\pi}\geq\frac{0.77}{KK'}n} \leq 4d^{\mathrm{la}}_{\hat\pi}(\hat\pi). \tag{7.143}$$

Plugging this in (7.141) yields

$$\begin{aligned} l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) &\leq \frac{4(c_{\hat\pi(i)}-1)}{KK'}n + 4d^{\mathrm{fi}}_{\hat\pi}(\hat\pi) + 4d^{\mathrm{la}}_{\hat\pi}(\hat\pi) \\ &\leq 4d\left(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}\right), \end{aligned} \tag{7.144}$$

the last step by **(T1)**. Claim (7.124) is therefore settled for the H'HH'-case.

<u>The HH'-case.</u>

In this case, see Figure 7.22 for a graphical rendition, a subpath connecting $\hat v^{\mathrm{la}}_{h(i)}$ and $\hat v^{\mathrm{fi}}_{h(i+1)}$, crosses $c_{\hat\pi(i)} = 2$ many H'-planes, one of which is also an H-plane. Without loss of generality, we assume that H'_{fi} is the H-plane. We will here distinguish two subcases: $d^{\mathrm{fi}}_{\hat\pi} \geq \frac{0.77}{KK'}n$, and its complement. It holds:

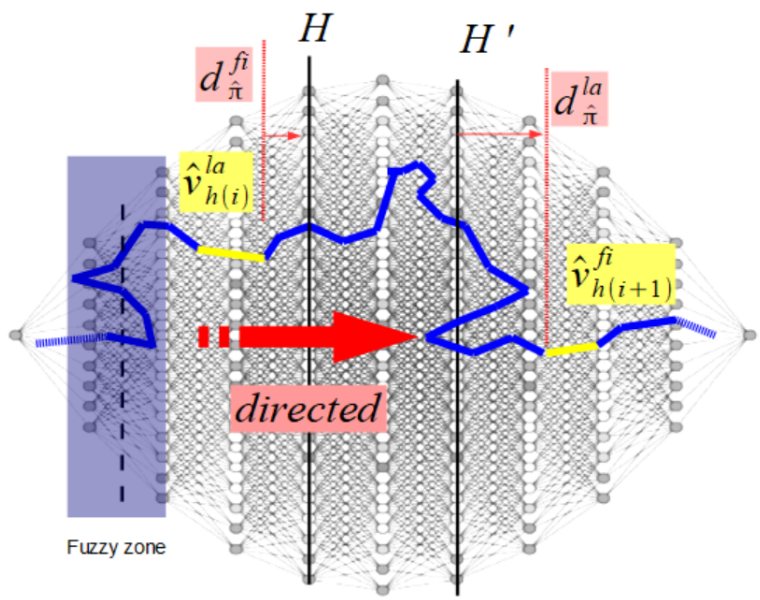

FIGURE 7.22. The common edges are separated by $c_{\hat\pi}(i) = 2$.

- If $d^{\mathrm{fi}}_{\hat\pi} \geq \frac{0.77}{KK'}n$, i.e. the vertex $\hat v^{\mathrm{la}}_{h(i)}$ is in the fuzzy zone, it follows from (OuF) and (InF) (cfr. also with Figure 7.22) that

$$\begin{aligned} l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) &\leq \frac{3\times 1.46}{KK'}n + \frac{0.46}{KK'}n\left(1_{d^{\mathrm{fi}}_{\hat\pi}\geq \frac{0.77n}{KK'}} + 1_{d^{\mathrm{la}}_{\hat\pi}\geq \frac{0.77n}{KK'}}\right), \\ &= \frac{4.38}{KK'}n + \frac{0.46}{KK'}n1 + \frac{0.46}{KK'}n1_{d^{\mathrm{la}}_{\hat\pi}\geq \frac{0.77n}{KK'}} \\ &\overset{(7.142)}{\leq} \frac{4.84}{KK'}n + 4d^{\mathrm{la}}_{\hat\pi} \\ &\leq \frac{4}{KK'}n + 4\frac{0.77}{KK'}n + 4d^{\mathrm{la}}_{\hat\pi} \\ &\overset{(\mathbf{T1})}{\leq} 4d(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}). \end{aligned} \tag{7.145}$$

- If $d^{\mathrm{fi}}_{\hat\pi} < \frac{0.77}{KK'}n$, the vertex $\hat v^{\mathrm{la}}_{h(i)}$ is no longer in the fuzzy zone. However, and crucially, the "complement" of the fuzzy zone is necessarily the repulsive phase, cfr. Figure 7.22. This in particular implies that the substrand will connect $\hat v^{\mathrm{la}}_{h(i)}$ with the H-plane in a directed fashion, and therefore

$$\begin{aligned} l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) &= l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, H_{h(i)}\cap\hat\pi) + l_{\hat\pi}(H_{h(i)}\cap\hat\pi, \hat v^{\mathrm{fi}}_{h(i+1)}) \\ &= d^{\mathrm{fi}}_{\hat\pi} + l_{\hat\pi}(H_{h(i)}\cap\hat\pi, \hat v^{\mathrm{fi}}_{h(i+1)}). \end{aligned} \tag{7.146}$$

As before, we estimate the last term on the r.h.s. above by OuF and InF. Here is the upshot:

$$\begin{aligned} l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) &\le d^{\mathrm{fi}}_{\hat\pi} + \frac{2\times 1.46}{KK'}n + \frac{0.46}{KK'}n 1_{d^{\mathrm{la}}_{\hat\pi}\ge \frac{0.77}{KK'}n} \\ &\overset{(7.142)}{\le} d^{\mathrm{fi}}_{\hat\pi} + \frac{2.92}{KK'}n + 4d^{\mathrm{la}}_{\hat\pi} \\ &\le 4\frac{0.77}{KK'}n + \frac{4}{KK'}n + 4d^{\mathrm{la}}_{\pi} \\ &\overset{(\mathbf{T_1})}{\le} 4d(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}). \end{aligned} \tag{7.147}$$

The claim (7.124) is thus settled for the HH'-case.

<u>The H-case.</u>

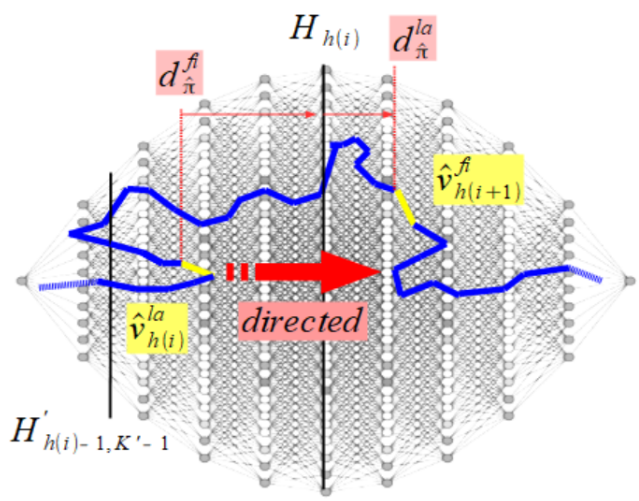

FIGURE 7.23. The common edges are separated by $c_{\hat\pi}(i) = 1$.

In this case, see Figure 7.23, a subpath connecting $\hat v^{\mathrm{la}}_{h(i)}$ and $\hat v^{\mathrm{fi}}_{h(i+1)}$, crosses $c_{\hat\pi(i)} = 1$ many H'-planes which is also an H-plane. Four subcases are possible:

- $d^{\mathrm{fi}}_{\hat\pi} < \frac{0.77}{KK'}n$ and $d^{\mathrm{la}}_{\hat\pi} < \frac{0.77}{KK'}n$, i.e. both vertices $\hat v^{\mathrm{la}}_{h(i)}$ and $\hat v^{\mathrm{fi}}_{h(i+1)}$ are in the (same) *repulsive phase*: the substrand thus connects them in directed fashion, in which case length and distance coincide, and

$$l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) = d(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) \le 4d(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}). \tag{7.148}$$

- $d^{\mathrm{fi}}_{\hat\pi} < \frac{0.77}{KK'}n$ and $d^{\mathrm{la}}_{\hat\pi} \ge \frac{0.77}{KK'}n$. In this case:

  - the vertex $\hat v^{\mathrm{la}}_{h(i)}$ is in the repulsive phase (cfr. with the second subcase in the HH'-regime above): in this first part of the journey, the substrand thus connects it with the H-plane in directed fashion, where again, and crucially, length and distance coincide.
  - as for the "rest of the journey", i.e. in order to deal with the length of the strand connecting H-plane and target vertex $\hat v^{\mathrm{fi}}_{h(i+1)}$, we proceed exactly as in (7.146).

Splitting the substrand in first/second part of the journey, and then by these observations, we get

$$\begin{aligned} l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) &= l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, H_{h(i)} \cap \hat\pi) + l_{\hat\pi}(H_{h(i)} \cap \hat\pi, \hat v^{\mathrm{fi}}_{h(i+1)}) \\ &\le d^{\mathrm{fi}}_{\hat\pi} + \frac{1.46}{KK'}n + \frac{0.46}{KK'}n \\ &\le 4d^{\mathrm{fi}}_{\hat\pi} + 4\frac{0.77}{KK'}n \\ &\overset{(\mathbf{T1})}{\le} 4d(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}). \end{aligned} \tag{7.149}$$

- $d^{\mathrm{fi}}_{\hat\pi} \ge \frac{0.77}{KK'}n$ and $d^{\mathrm{la}}_{\hat\pi} < \frac{0.77}{KK'}n$: this case is, by symmetry, equivalent to the previous.
- $d^{\mathrm{fi}}_{\hat\pi} \ge \frac{0.77}{KK'}n$ and $d^{\mathrm{la}}_{\hat\pi} \ge \frac{0.77}{KK'}n$: both vertices being in the fuzzy zone, we proceed exactly as in (7.139) to obtain

$$\begin{aligned} l_{\hat\pi}(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}) &\le 2\frac{1.46}{KK'}n + 2\frac{0.46}{KK'}n \\ &\le 4\frac{0.77}{KK'}n + 4\frac{0.77}{KK'}n \\ &\overset{(\mathbf{T1})}{\le} 4d(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}\,. \end{aligned} \tag{7.150}$$

Claim (7.124) thus holds true for all possible sub-scenarios of the third (and last) H-case: this finishes the proof of Lemma 7.3. □

Proof of Lemma 6.10: We want now to estimate $f^{(s)}_{\pi}(n,k)$: Let $f^{(s)}_{l,\pi}(n,k)$ (respectively $f^{(s)}_{r,\pi}(n,k)$) the number of paths which are sharing k edges with π with at least one common edge betweeen H_m and the middle of the hypercube (respectively between the middle of the hypercube and H_{K-m}) but without considering first and last edge. It holds

$$f^{(s)}_{\pi}(n,k) = f^{(s)}_{l,\pi}(n,k) + f^{(s)}_{r,\pi}(n,k) = 2f^{(s)}_{l,\pi}(n,k), \tag{7.151}$$

the last equality by symmetry (see (7.44)). Using (7.81), (7.82) and(7.96), it clearly holds

$$\begin{aligned} f^{(s)}_{l,\pi}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^{\alpha}} \sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}}\sum_{\boldsymbol{\sigma}}\sum_{\boldsymbol{l'}}\prod_{i=1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \\ \times \cosh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{n} \left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \\ \times \prod_{i=0}^{b} \tanh\left(\frac{\hat l_i \mathsf{E}}{\mathsf{L}_{opt}n}\right)^{d\left(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}\right)} \cosh\left(\frac{\hat l_i \mathsf{E}}{\mathsf{L}_{opt}n}\right)^{n} \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\hat l_i}. \end{aligned} \tag{7.152}$$

Using the monotonicity of the cosh-function (7.100), and the fact that all paths in $\mathcal{J}$ have the same length[11], in (7.152) yields

$$\begin{aligned} f^{(s)}_{l,\pi}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^{\alpha}} \sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}}\sum_{\boldsymbol{\sigma}}\sum_{\boldsymbol{l'}} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^{n} \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n - k} \\ \prod_{i=0}^{b} \tanh\left(\frac{\hat l_i \mathsf{E}}{\mathsf{L}_{opt}n}\right)^{d\left(\hat v^{\mathrm{la}}_{h(i)}, \hat v^{\mathrm{fi}}_{h(i+1)}\right)} \prod_{i=1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}. \end{aligned} \tag{7.153}$$

[11] Recall from (7.107) that $\sum_{i=0}^{b} \hat l_i + \sum_{i=1}^{b} d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)} = \mathsf{L}_{\mathsf{opt}}\mathsf{n} - \mathsf{k}$.

Let $q \equiv \min\{h(i) > m, k_{h(i)} > 0\}$, splitting the product of the tanh-terms according to q, we obtain

$$\begin{aligned}
&\prod_{i=0}^{b} \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\mathrm{la}}_{h(i)},\hat{v}^{\mathrm{fi}}_{h(i+1)}\right)} \prod_{i=1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\mathrm{fi}}_{h(i)},v^{\mathrm{la}}_{h(i)})-k_{h(i)}} \\
&= \prod_{i=0}^{q-1} \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\mathrm{la}}_{h(i)},\hat{v}^{\mathrm{fi}}_{h(i+1)}\right)} \prod_{i=1}^{q} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\mathrm{fi}}_{h(i)},v^{\mathrm{la}}_{h(i)})-k_{h(i)}} \\
&\times \prod_{i=q}^{b} \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\mathrm{la}}_{h(i)},\hat{v}^{\mathrm{fi}}_{h(i+1)}\right)} \prod_{i=q+1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\mathrm{fi}}_{h(i)},v^{\mathrm{la}}_{h(i)})-k_{h(i)}} \\
&\leq \tanh\left(\mathsf{E}\frac{\widehat{L}_{q-1} + \mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_q} \times \tanh\left(\mathsf{E}\frac{\widehat{L}_b - \widehat{L}_{q-1} + \mathcal{D}_b - \mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_b-\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_b-\mathcal{D}_q},
\end{aligned} \tag{7.154}$$

the last r.h.s using the monotonicity of the tanh-terms (7.99) two times: one time for the first line and a second time for the second line of the second equality. Putting (7.154) into (7.153) yields

$$\begin{aligned}
f^{(s)}_{l,\pi}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k} \\
\tanh\left(\mathsf{E}\frac{\widehat{L}_{q-1} + \mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_q} \times \tanh\left(\mathsf{E}\frac{\widehat{L}_b - \widehat{L}_{q-1} + \mathcal{D}_b - \mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_b-\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_b-\mathcal{D}_q}.
\end{aligned} \tag{7.155}$$

We now claim that for $0 < x \leq y \leq \mathsf{E}$,

$$\tanh(x) \leq \frac{3}{4}\tanh(x+y). \tag{7.156}$$

Indeed, using the addition formula for the tanh function, it holds

$$\frac{\tanh(x)}{\tanh(x+y)} = \frac{\tanh(x)\,(1+\tanh(x)\tanh(y))}{\tanh(x)+\tanh(y)} = \frac{1+\tanh(x)\tanh(y)}{1+\frac{\tanh(y)}{\tanh(x)}} \leq \frac{1+\tanh(\mathsf{E})^2}{2} = \frac{3}{4}, \tag{7.157}$$

the last inequality because the function tanh is increasing and the claim (7.156) is settled.

Again using that tanh is increasing we also have that

$$\tanh(y) \leq \tanh(x+y). \tag{7.158}$$

Using in (7.155) the estimates (7.156) and (7.158) with

$$x \equiv \min\{\widehat{L}_{q-1} + \mathcal{D}_q, \widehat{L}_b - \widehat{L}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\}, \tag{7.159}$$

and

$$y \equiv \max\{\widehat{L}_{q-1} + \mathcal{D}_q, \widehat{L}_b - \widehat{L}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\}, \tag{7.160}$$

we obtain

$$\begin{aligned}
f^{(s)}_{l,\pi}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \left(\frac{3}{4}\right)^{\min\{\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_q, \widehat{\mathcal{D}}_b-\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_b-\mathcal{D}_q\}} \\
\tanh\left(\mathsf{E}\frac{\mathcal{D}_b + \widehat{L}_b}{\mathsf{L}_{opt}n}\right)^{\mathcal{D}_b+\widehat{\mathcal{D}}_b} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}.
\end{aligned} \tag{7.161}$$

With the same line of reasoning as in (7.121), we clearly have that

$$\widehat{\mathcal{D}}_{q-1} + \mathcal{D}_q \geq m\hat{n}_K - k, \tag{7.162}$$

and

$$\widehat{\mathcal{D}}_b - \widehat{\mathcal{D}}_{q-1} + \mathcal{D}_b - \mathcal{D}_q \geq \frac{n}{2} - k. \tag{7.163}$$

Thus, it follows from (7.162) and (7.163) that

$$\min\{\widehat{\mathcal{D}}_{q-1} + \mathcal{D}_q, \widehat{\mathcal{D}}_b - \widehat{\mathcal{D}}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\} \geq m\hat{n}_K - k. \tag{7.164}$$

Plugging (7.164) into (7.161) and recalling that paths in $\mathcal{J}$ have the same, prescribed length (recall once more (7.107) or, which is the same, footnote 11), it holds

$$\begin{aligned} f^{(s)}_{l,\pi}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \Bigg[\left(\frac{3}{4}\right)^{m\hat{n}_K - k} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^{\mathcal{D}_b + \widehat{\mathcal{D}}_b} \\ \times \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n - k} \Bigg]. \end{aligned} \tag{7.165}$$

We follow *exactly* the same steps which from (7.106) lead to (7.119), this time of course with the factor $\left(\frac{3}{4}\right)^{m\hat{n}_K - k}$. Omitting the details, we obtain

$$\begin{aligned} f^{(s)}_{l,\pi}(n,k) \leq P_n n^{Kn^\alpha} \left(\frac{3}{4}\right)^{m\hat{n}_K - k} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k, \frac{\mathsf{L}_{opt}n-k}{4}\right)} \\ \times \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n - k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n - k}, \end{aligned} \tag{7.166}$$

where P_n is a finite degree polynomial. Combining (7.151) and (7.166) and the fact that for $k \leq 200\hat{n}_K$, $\left(\frac{3}{4}\right)^{m\hat{n}_K - k} \leq \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K}$ finishes the proof of Lemma 6.10.

□

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