



The non-linear supersymmetric hyperbolic sigma model on a complete graph with hierarchical interactions

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Abstract. We study the non-linear supersymmetric hyperbolic sigma model $H^{2|2}$ on a complete graph with hierarchical interactions. For interactions which do not decrease too fast in the hierarchical distance, we prove tightness of certain spin variables in horospherical coordinates, uniformly in the pinning and in the size of the graph. The proof relies on a reduction to an effective $H^{2|2}$ -model; its size is logarithmic in the size of the original model.

1. Introduction

The supersymmetric hyperbolic sigma model, $H^{2|2}$ -model for short, was introduced by Zirnbauer (1991) as a “toy model for studying diffusion and localization in disordered one-electron systems”. It is a supersymmetric variant of a statistical mechanics model on a finite graph. The model deals with spin variables taking values in the supermanifold $H^{2|2}$ with two Grassmann components over the hyperboloid $H^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, z > 0\}$. The model has supersymmetries, which extend the generators of the Lorentz group acting on H^2 . In Disertori et al. (2010), Disertori, Spencer, and Zirnbauer examine this model over boxes in \mathbb{Z}^d , $d \geq 3$. For sufficiently small temperature, they derive moment estimates and conclude that the spins are aligned with high probability. For high temperature in any dimension d , Disertori and Spencer (2010) show exponential decay

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of the y - y -two-point function for the $H^{2|2}$ -model. The result holds for any temperature in dimension one. In two dimensions, polynomial decay of the correlations was shown for all temperatures independently by [Kozma and Peled \(2021\)](#) and [Sabot \(2021\)](#), using different tools.

Surprisingly, although the model was originally designed to describe condensed matter systems, there is a close connection with vertex-reinforced jump processes as was observed by [Sabot and Tarrès \(2015\)](#). All spin variables of the $H^{2|2}$ -model written in horospherical coordinates have a probabilistic interpretation in terms of the vertex-reinforced jump process as is worked out by Merkl, Rolles, and Tarrès in [Merkl et al. \(2019\)](#).

There is another surprising connection of the $H^{2|2}$ -model to certain random Schrödinger operators discovered by Sabot, Tarrès, and Zeng in [Sabot et al. \(2017\)](#) and [Sabot and Zeng \(2019\)](#). For the analysis of these random Schrödinger operators, certain martingales derived from the spin variables of the $H^{2|2}$ -model play an essential role. The authors generalized these martingales in [Disertori et al. \(2017\)](#) and described supersymmetric variants of them in [Disertori et al. \(2019\)](#).

The classical isomorphism theorems relating the local times of a continuous-time random walk to the square of a Gaussian free field are generalized by Bauerschmidt, Helmuth, and Swan in [Bauerschmidt et al. \(2021c\)](#) to spin systems taking values in hyperbolic and spherical geometries.

There are generalizations of the $H^{2|2}$ -model, called $H^{p|2q}$ -models, which deal with p real-valued spin variables and $2q$ Grassmann variables per vertex. For $p = 2$, these models are examined first by [Crawford \(2021\)](#). The case of the $H^{0|2}$ -model, which describes the arboreal gas (configurations of unrooted spanning forests), and the closely related $H^{2|4}$ -model are examined by Bauerschmidt, Crawford, Helmuth, and Swan in [Bauerschmidt et al. \(2021b\)](#) and by Bauerschmidt, Crawford, and Helmuth in [Bauerschmidt et al. \(2021a\)](#).

In particular, the $H^{0|2}$ -model can be represented by a purely Fermionic model with a Gaussian measure perturbed by a short range interaction. This allows to apply a rigorous renormalization group analysis in [Bauerschmidt et al. \(2021a\)](#). It is not clear how to extend such an analysis to the $H^{2|2}$ -model, the main obstruction being the hyperbolic structure and the presence of long range interactions. If one considers the corresponding vertex-reinforced jump process on hierarchical lattices, informal computations by Abdesselam, Brydges, Helmuth, Lupu, Sabot, Swan, and Zeng done during an AIM workshop¹ suggest, for a special choice of the pinning and the interactions, the existence of a renormalization semigroup operation, which leaves the process invariant under appropriate scaling.

A recent survey on the subject is given by [Bauerschmidt and Helmuth \(2021\)](#).

Overview of the paper. In this paper, we deal with the $H^{2|2}$ -model on a complete graph of size 2^N with interactions depending on the hierarchical distance in a binary tree. The precise definition is given in Section 2.2. For some observables this model can be reduced to an effective $H^{2|2}$ -model with only $O(N)$ vertices. This reduction is described in Section 2.3, Theorem 2.2. As an application of this reduction, we prove tightness of differences of certain spin variables in the original model, provided that the interactions do not decay too fast with the hierarchical distance. This is made precise in Section 2.2, Theorem 2.1.

2. Model and main results

2.1. The $H^{2|2}$ -model on general graphs.

Introduction of the model. Let Λ be a finite non-empty set. We view it as the vertex set of a complete undirected graph with edge set $\mathcal{E}_\Lambda := \{\{i, j\} : i, j \in \Lambda\}$. To construct the supermanifold $H^{2|2}$, we introduce a Grassmann algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with the even (commuting) subalgebra \mathcal{A}_0 and the odd (anticommuting) subspace \mathcal{A}_1 ; see [Swan \(2020\)](#) for more details. To keep the

¹AIM Workshop on Self-interacting processes, supersymmetry, and Bayesian statistics organized by P. Diaconis, M. Disertori, C. Sabot, and P. Tarrès, 2019, <https://aimath.org/pastworkshops/selfsuperbayes.html>

exposition self-contained, we also give some details in the appendix. We assume that Λ is large enough so that we can associate to each vertex $i \in \Lambda$ three variables x_i, y_i, z_i taking values in \mathcal{A}_0 and linearly independently two odd (Grassmann) variables $\xi_i, \eta_i \in \mathcal{A}_1$. Each even element $x \in \mathcal{A}_0$ can be decomposed in a unique way in a real part $\text{body}(x) \in \mathbb{R}$ and a nilpotent part $x - \text{body}(x)$. We define $H^{3|2} := \mathcal{A}_0^3 \times \mathcal{A}_1^2$ and endow it with the inner product

$$\langle v, v' \rangle := xx' + yy' - zz' + \xi\eta' - \eta\xi' \tag{2.1}$$

for $v = (x, y, z, \xi, \eta), v' = (x', y', z', \xi', \eta') \in H^{3|2}$. The supermanifold $H^{2|2}$ can now be defined by

$$\begin{aligned} H^{2|2} &= \{v = (x, y, z, \xi, \eta) \in H^{3|2} : \langle v, v \rangle = -1, \text{body}(z) > 0\} \\ &= \{(x, y, z, \xi, \eta) \in H^{3|2} : x^2 + y^2 - z^2 + 2\xi\eta = -1, \text{body}(z) > 0\}. \end{aligned}$$

For any smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ the corresponding extension to a superfunction $f : \mathcal{A}_0^k \rightarrow \mathcal{A}_0$ is denoted by the same symbol f and is constructed by a Taylor expansion in the nilpotent parts.

The superintegration form $\mathcal{D}v$ on $H^{2|2}$ is defined by

$$f \mapsto \int_{H^{2|2}} \mathcal{D}v f(v) := \frac{1}{2\pi} \int_{\mathbb{R}^2} dx dy \partial_\xi \partial_\eta \left(\frac{1}{z} f(x, y, z, \xi, \eta) \right),$$

where

$$z = \sqrt{1 + x^2 + y^2 + 2\xi\eta} = \sqrt{1 + x^2 + y^2} + \frac{\xi\eta}{\sqrt{1 + x^2 + y^2}}$$

according to the constraint $v \in H^{2|2}$. Here, f is any superfunction decaying sufficiently fast to make the integral well-defined.

We abbreviate

$$o := (0, 0, 1, 0, 0) \in H^{2|2}. \tag{2.2}$$

We take a weight function $W : \mathcal{E}_\Lambda \rightarrow [0, \infty), \{i, j\} \mapsto W_{ij} = W_{ji}$ with $W_{ii} = 0$ for all $i \in \Lambda$ and consider the set of edges

$$\mathcal{E}_{\Lambda, W} := \{\{i, j\} \in \mathcal{E}_\Lambda : W_{ij} > 0\}. \tag{2.3}$$

Furthermore, we take a pinning strength function $h : \Lambda \rightarrow [0, \infty), i \mapsto h_i$. Throughout the paper, we assume that it fulfills $h_i > 0$ for at least one i in every connected component of the graph $(\Lambda, \mathcal{E}_{\Lambda, W})$. This connectivity assumption guarantees that the following $H^{2|2}$ -model is well-defined: its action with couplings W and pinning h is defined for $v_\Lambda := (v_j)_{j \in \Lambda} \in (H^{2|2})^\Lambda$ by

$$\begin{aligned} S_{W, h}^\Lambda(v_\Lambda) &:= -\frac{1}{2} \sum_{i, j \in \Lambda} W_{ij} \langle v_i - v_j, v_i - v_j \rangle + \sum_{i \in \Lambda} h_i (1 + \langle v_i, o \rangle) \\ &= \sum_{i, j \in \Lambda} W_{ij} (1 + \langle v_i, v_j \rangle) + \sum_{i \in \Lambda} h_i (1 + \langle v_i, o \rangle). \end{aligned} \tag{2.4}$$

The $H^{2|2}$ -model over Λ is given by the superintegration form

$$f \mapsto E_{W, h}^\Lambda[f(v_\Lambda)] := \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda (e^{S_{W, h}^\Lambda(v_\Lambda)} f(v_\Lambda)) \tag{2.5}$$

with the abbreviation

$$\mathcal{D}v_\Lambda := \prod_{j \in \Lambda} \mathcal{D}v_j. \tag{2.6}$$

By supersymmetry, $E_{W, h}^\Lambda[1] = 1$ for the constant function $f(v_\Lambda) \equiv 1$, see formula (5.1) in the paper [Disertori et al. \(2010\)](#) by Disertori, Spencer, and Zirnbauer.

Horospherical coordinates. For any vertex $i \in \Lambda$ and $v_i = (x_i, y_i, z_i, \xi_i, \eta_i) \in H^{2|2}$, we introduce horospherical coordinates consisting of even components u_i, s_i and Grassmann variables $\bar{\psi}_i, \psi_i$ as follows, cf. Section 2.2 in Disertori et al. (2010):

$$u_i = \log(x_i + z_i), \quad s_i = \frac{y_i}{x_i + z_i}, \quad \bar{\psi}_i = \frac{\xi_i}{x_i + z_i}, \quad \psi_i = \frac{\eta_i}{x_i + z_i}.$$

The inverse transformation is given by

$$\begin{aligned} x_i &= \sinh u_i - \left(\frac{1}{2}s_i^2 + \bar{\psi}_i\psi_i\right)e^{u_i}, & y_i &= s_i e^{u_i}, & \xi_i &= \bar{\psi}_i e^{u_i}, & \eta_i &= \psi_i e^{u_i}, \\ z_i &= \sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i} = \cosh u_i + \left(\frac{1}{2}s_i^2 + \bar{\psi}_i\psi_i\right)e^{u_i}. \end{aligned}$$

In these coordinates, the even components (u_Λ, s_Λ) restricted to $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ have the following probabilistic interpretation: there is a unique probability measure $P^\Lambda = P_{W,h}^\Lambda$ on $(\mathbb{R}^\Lambda)^2$ such that for any smooth bounded test function $f : (\mathbb{R}^\Lambda)^2 \rightarrow \mathbb{R}$ one has

$$\int_{(\mathbb{R}^\Lambda)^2} f dP_{W,h}^\Lambda = E_{W,h}^\Lambda[f(u_\Lambda(v_\Lambda), s_\Lambda(v_\Lambda))].$$

Using the abbreviations

$$B_{ij} := \cosh(u_i - u_j) + \frac{1}{2}(s_i - s_j)^2 e^{u_i+u_j}, \quad B_j := \cosh u_j + \frac{1}{2}s_j^2 e^{u_j}, \tag{2.7}$$

$$M_{ij} := 1_{\{i=j\}} \left(h_i e^{u_i} + \sum_{k \in \Lambda} W_{ik} e^{u_i+u_k} \right) - 1_{\{i \neq j\}} W_{ij} e^{u_i+u_j} \tag{2.8}$$

for $i, j \in \Lambda$, the corresponding probability density equals

$$P_{W,h}^\Lambda(du_\Lambda ds_\Lambda) = e^{-\frac{1}{2} \sum_{i,j \in \Lambda} W_{ij}(B_{ij}-1)} e^{-\sum_{i \in \Lambda} h_i(B_i-1)} \det(M_{ij})_{i,j \in \Lambda} \prod_{i \in \Lambda} \frac{e^{-u_i} du_i ds_i}{2\pi}, \tag{2.9}$$

where $du_i ds_i$ is the Lebesgue measure on \mathbb{R}^2 , cf. Disertori et al. (2010). In the rest of the paper we mostly work with the v -variables (x, y, z, ξ, η) , but sometimes also with the u -variables.

2.2. Hierarchical $H^{2|2}$ -model and first main result. We now consider the special case that Λ consists of 2^N vertices with some given $N \in \mathbb{Z}_+^0 = \{0, 1, 2, \dots\}$, where every vertex i interacts with every other vertex j in a hierarchical manner. The hierarchical structure is best visualized by the binary tree with set of leaves equal to the vertex set Λ . More precisely, for any $n \in \mathbb{Z}_+^0$, let $\Lambda_n = \{0, 1\}^n$ be the set of binary words of length n . In particular, $\Lambda_0 = \{\emptyset\}$ and $\Lambda = \Lambda_N$. The reader may visualize the words as binary numbers $0, \dots, 2^N - 1$ with reversed order of digits. Let $\mathcal{T}^N := \bigcup_{n=0}^N \Lambda_n$ be the vertex set of a binary tree of height N with root \emptyset . For $j \in \Lambda_n$, let $|j| = n$ be the length of the word j , which is the distance of j from the root. The elements of Λ_N are the leaves of the tree. An illustration of the tree \mathcal{T}^3 is given in Figure 2.1.

For $j = (j_0, \dots, j_{|j|-1}) \in \mathcal{T}^N \setminus \{\emptyset\}$, let $\text{cut}(j) := (j_1, \dots, j_{|j|-1})$ denote the vertex directly below j . The *hierarchical distance* between $i, j \in \Lambda_N$ is defined as

$$d(i, j) := \min\{l \in \mathbb{Z}_+^0 : \text{cut}^l(i) = \text{cut}^l(j)\}.$$

Here, cut^l denotes the l -fold iteration of cut and cut^0 is the identity map. The hierarchical distance is the distance to the first point in \mathcal{T}^N where the shortest paths from i and j to the root meet. It is also half the graph distance between i and j in the tree graph corresponding to \mathcal{T}^N . The *minimum* of $i, j \in \Lambda_N$ is defined by

$$i \wedge j := \text{cut}^{d(i,j)}(i) = \text{cut}^{d(i,j)}(j).$$

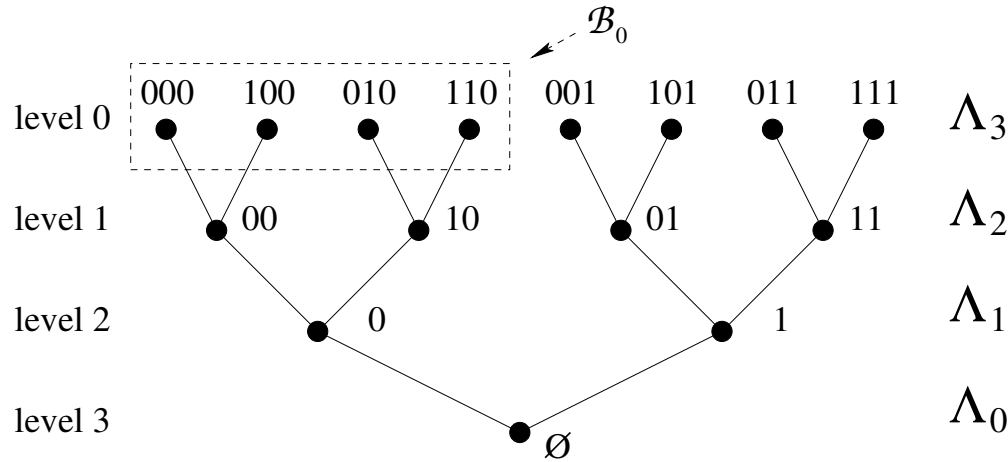


FIGURE 2.1. The tree $\mathcal{T}^3 = \Lambda_3 \cup \Lambda_2 \cup \Lambda_1 \cup \Lambda_0$.

More generally, for $i, j \in \mathcal{T}^N$, we define $i \wedge j$ to be the least common ancestor of i and j , which equals $\text{cut}^m(i)$ with the smallest $m \in \mathbb{Z}_+^0$ such that $\text{cut}^m(i) = \text{cut}^{m'}(j)$ for some $m' \in \mathbb{Z}_+^0$.

Take a weight function $w : \mathbb{Z}_+^0 \rightarrow [0, \infty)$. It encodes the weight function $W : \mathcal{E}_{\Lambda_N} \rightarrow [0, \infty)$ on the set of edges \mathcal{E}_{Λ_N} as follows: for $i, j \in \Lambda_N$,

$$W_{ij} := w(d(i, j)). \tag{2.10}$$

Thus, it depends only on the hierarchical distance of i and j . In this paper we study the $H^{2|2}$ -model on $\Lambda = \Lambda_N$ with hierarchical weight function W given by (2.10) and a pinning function $h = (h_i)_{i \in \Lambda_N}$. According to formula (2.4), the action is given by

$$S_{W,h}^{\Lambda_N}(v_{\Lambda_N}) = \sum_{i,j \in \Lambda_N} w(d(i, j))(1 + \langle v_i, v_j \rangle) + \sum_{i \in \Lambda_N} h_i(1 + \langle v_i, o \rangle).$$

Important particular cases are constant pinning $h = \epsilon 1 = (\epsilon)_{i \in \Lambda_N}$ and pinning at one point $i_0 \in \Lambda_N$, i.e. $h = \epsilon \delta_{i_0} = (\epsilon 1_{\{i=i_0\}})_{i \in \Lambda_N}$, with some pinning strength $\epsilon > 0$. We show that for this particular choice of the pinning the horospherical coordinates $(u_i)_{i \in \Lambda_N}$ are tight with respect to $P_{W,h}^{\Lambda_N}$, uniformly in N and the pinning strength $\epsilon > 0$ if the weight function w does not decrease too fast.

Theorem 2.1 (Tightness). *Let $w : \mathbb{Z}_+^0 \rightarrow (0, \infty)$ and $W : \mathcal{E}_{\Lambda_N} \rightarrow (0, \infty)$ be weight functions related by (2.10) such that the numbers $\beta_l := 2^{2l+1}w(l+2)$, $l \in \mathbb{Z}_+^0$, fulfill*

$$\sum_{l=0}^{\infty} \sqrt{\frac{\log \max\{\beta_l, e\}}{\beta_l}} < \infty. \tag{2.11}$$

Then, using the notation $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$, one has

$$\lim_{M \rightarrow \infty} \sup_{N \in \mathbb{Z}_+} \sup_h \sup_{p, q \in \Lambda_N} P_{W,h}^{\Lambda_N}(|u_p - u_q| \geq M) = 0,$$

where the supremum over h is taken over all uniform pinning functions $h = \epsilon 1$ and all pinning functions at one point $h = \epsilon \delta_{i_0}$ with $\epsilon > 0$ and $i_0 \in \Lambda_N$.

Remark. Intuitively, one might expect another regime without tightness when β_l does not grow fast enough since in this case the original graph looks almost like a collection of disconnected pieces. However, we do not expect criterion (2.11) to precisely characterize the phase transition between the two regimes.

Interpretation in terms of the vertex-reinforced jump process. Let us briefly and informally discuss the meaning of tightness in terms of the vertex-reinforced jump process (VRJP). Proposition 1 and Theorem 2 in Sabot and Tarrès (2015) imply that the asymptotic ratio R_{pq} of local times in two vertices p and q of VRJP (suitably time-changed) is described in law by $e^{2(u_p - u_q)}$. Thus, the tightness result means that this asymptotic ratio remains bounded with high probability with a bound e^{2M} which is uniform in p, q , the pinning strength in the starting point i_0 , and the size of the graph, precisely:

$$e^{-2M} \leq R_{pq} \leq e^{2M} \quad \text{with high probability.}$$

A key tool to prove Theorem 2.1 is a representation of expectations in terms of an effective reduced model.

2.3. Effective model and second main result. It turns out that the value $E_{W,h}^{\Lambda_N}[e^{\sum_{j \in \Lambda} \langle a_j, v_j \rangle}]$ of the superintegration form agrees, for any suitable a_j and h , with the corresponding value of the superintegration form for an $H^{2|2}$ -model with a smaller vertex set, rescaled weights and rescaled pinning. While the original model has $|\Lambda_N| = 2^N$ vertices, the effective model used in the proof of Theorem 2.1 has only $O(N)$ vertices, resulting in an exponential reduction of complexity; see Remark 4.2 for more details. To define this effective model we need to introduce some notation.

Block spin variables. For $i, j \in \mathcal{T}^N$, we write $i \preceq j$ if $i = \text{cut}^l(j)$ for some $l \in \mathbb{Z}_+^0$, which means that j is an extension of i . Let $\mathcal{B}_i := \{j \in \Lambda_N : i \preceq j\}$ denote the set of leaves above i in Λ_N . In particular, $\mathcal{B}_\emptyset = \Lambda_N$. For any vertex $j \in \mathcal{T}^N$, let

$$\ell(j) := N - |j|$$

denote its level. All vertices in Λ_N are at level 0 and the root \emptyset is at level N ; see Figure 2.1 for an illustration. Note that for $i \in \Lambda_n$, one has $|\mathcal{B}_i| = 2^{N-n} = 2^{\ell(i)}$, and for $i, j \in \Lambda_N$ we have $\ell(i \wedge j) = d(i, j)$. Starting with the variables v_j with $j \in \Lambda_N$ at level 0, we attach *block spin variables* to all vertices of the binary tree \mathcal{T}^N as follows. For $j \in \mathcal{T}^N$, let

$$v_j = (x_j, y_j, z_j, \xi_j, \eta_j) := \frac{1}{|\mathcal{B}_j|} \sum_{i \in \mathcal{B}_j} v_i = \frac{1}{2^{\ell(j)}} \sum_{i \in \mathcal{B}_j} v_i. \quad (2.12)$$

For $j \in \Lambda_N$, this redefinition preserves the original meaning of v_j .

Antichains. For the effective model, we choose the vertex set $A \subseteq \mathcal{T}^N$ to be an antichain, meaning that for all $i, j \in A$ with $i \neq j$ one has $i \not\preceq j$ and $j \not\preceq i$. An antichain A is called maximal if for any antichain $A' \supseteq A$ one has $A' = A$. Equivalently, $A \subseteq \mathcal{T}^N$ is a maximal antichain if and only if for every leaf $i \in \Lambda_N$ there is precisely one $j \in A$ with $j \preceq i$. For a maximal antichain A , the set $A^\uparrow := \{i \in \mathcal{T}^N : j \preceq i \text{ for some } j \in A\}$ consists of all vertices above A . Given two maximal antichains A and A' , we write $A \preceq A'$ if $A' \subseteq A^\uparrow$. The notation $A \prec A'$ means that $A \preceq A'$ and $A \neq A'$.

Rescaled weights and pinning. We extend the weight function $W : \mathcal{E}_{\Lambda_N} \rightarrow [0, \infty)$ to $W : \{\{i, j\} : i, j \in \mathcal{T}^N\} \rightarrow [0, \infty)$ by

$$W_{ij} := 2^{\ell(i) + \ell(j)} w(\ell(i \wedge j)). \quad (2.13)$$

Note that, for $i, j \in \Lambda_N$ this definition coincides with the original weight function $W_{ij} = w(d(i, j))$.

Assume now $h : \Lambda_N \rightarrow [0, \infty)$ is a pinning function such that $h_j > 0$ for at least one point in each connected component of the graph $(\Lambda_N, \mathcal{E}_{\Lambda_N, W})$ (cf. remarks below (2.3)). We call a maximal antichain A *compatible with h* , if for all $j \in A$ the restriction $h|_{\mathcal{B}_j}$ of h to \mathcal{B}_j is constant; see Figure 2.2 for an illustration.

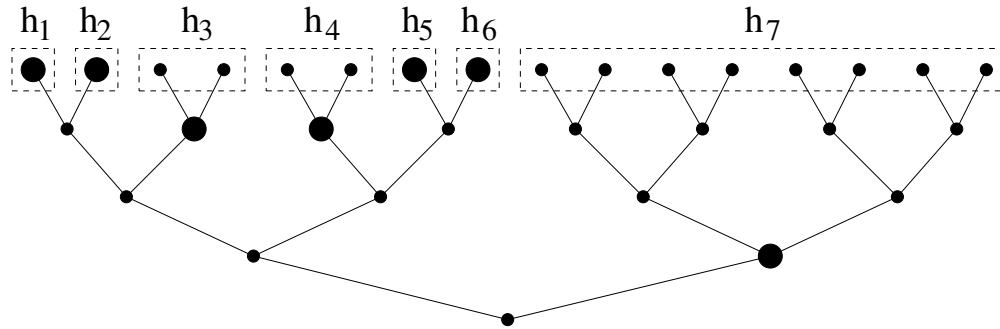


FIGURE 2.2. The antichain consisting of the big dots is compatible with pinning functions given by h_i , $1 \leq i \leq 7$.

Given a maximal antichain A compatible with a pinning function h , we define an extended pinning function $H : A^\uparrow \rightarrow [0, \infty)$ by setting

$$H_j = \sum_{i \in \mathcal{B}_j} h_i \text{ for } j \in A^\uparrow. \tag{2.14}$$

Note that all summands in the last sum are equal and hence $H_j = 2^{\ell(j)} h_i$ for any $i \in \mathcal{B}_j$, $j \in A^\uparrow$.

For any $\Lambda \subseteq A$ the notation $S_{W,H}^\Lambda$, $E_{W,H}^\Lambda$ means the weight function W is restricted to the underlying edge set \mathcal{E}_Λ and the pinning H is restricted to the vertex set Λ .

Reduction of the model. We now compare the hierarchical $H^{2|2}$ -model over the set Λ_N of leaves of the tree \mathcal{T}^N with arbitrary pinning h with the $H^{2|2}$ -model over a maximal antichain $A \subseteq \mathcal{T}^N$ compatible with h .

Recall the even and odd Grassmann subspaces \mathcal{A}_0 and \mathcal{A}_1 , respectively, and $H^{3|2} = \mathcal{A}_0^3 \times \mathcal{A}_1^2$ endowed with the inner product $\langle \cdot, \cdot \rangle$ defined in equation (2.1). We extend the comparison relations $<, \leq, >, \geq$ on \mathbb{R} to even elements $a, b \in \mathcal{A}_0$ by defining $a < b$ to mean $\text{body}(a) < \text{body}(b)$, and similarly for $\leq, >$, and \geq . We abbreviate

$$H_+^{3|2} := \{v = (x, y, z, \xi, \eta) \in H^{3|2} : \langle v, v \rangle < 0, z > 0\},$$

$$H_{+0}^{3|2} := \{v = (x, y, z, \xi, \eta) \in H^{3|2} : \langle v, v \rangle \leq 0, z \geq 0\}.$$

For the next theorem and its corollary, consider the extended weight function W from (2.13), a pinning function $h : \Lambda_N \rightarrow [0, \infty)$, a maximal antichain $A \subseteq \mathcal{T}^N$ being compatible with h , and the extension H of h given in (2.14). In analogy to (2.3), we consider the graph $(A, \mathcal{E}_{A,W})$ with edge set $\mathcal{E}_{A,W} := \{\{i, j\} : i, j \in A, W_{ij} > 0\}$.

Theorem 2.2 (Reduction to the effective model). *For all $(a_j)_{j \in A} \in (H^{3|2})^A$ such that $H_j o + a_j \in H_{+0}^{3|2}$ for all $j \in A$ and $H_j o + a_j \in H_+^{3|2}$ for at least one j in every connected component of the graph $(A, \mathcal{E}_{A,W})$, one has*

$$E_{W,h}^{\Lambda_N} [e^{\sum_{j \in A} \langle a_j, v_j \rangle}] = E_{W,H}^A [e^{\sum_{j \in A} \langle a_j, v_j \rangle}]. \tag{2.15}$$

Note that in formula (2.15) the variables v_j have different meanings on the left-hand side and on the right-hand side. More precisely, on the left-hand side, they are understood as block spin variables as in formula (2.12), while on the right-hand side they are the original variables $v_j \in H^{2|2}$ of the $H^{2|2}$ -model with vertex set A .

We remark that taking derivatives with respect to the components of the parameters a_j in formula (2.15), one can treat also polynomials in the $x_j, y_j, z_j, \xi_j, \eta_j$. We expect that this can be extended to general observables $f((v_j)_{j \in A})$ rather than only $e^{\sum_{j \in A} \langle a_j, v_j \rangle}$ by working out a theory for super Fourier Laplace transforms of super measures in the spirit of Fresta (2021). However, for our

application, we need only a very special case which is treated in the following corollary in an elementary way.

Let $\mathcal{L}_P((X_i)_{i \in I})$ denote the joint law of the random variables $(X_i)_{i \in I}$ with respect to some probability law P . Recall the assumptions specified right above Theorem 2.2.

Corollary 2.3 (Coincidence of distributions). *Let $B := A \cap \Lambda_N$ denote the set of leaves of the maximal antichain A . Then, the joint laws of $u_B := (u_j)_{j \in B}$ and $s_B := (s_j)_{j \in B}$ with respect to $P_{W,h}^{\Lambda_N}$ and $P_{W,H}^A$ coincide. More generally, the following joint laws agree*

$$\mathcal{L}_{P_{W,h}^{\Lambda_N}} \left(\left(|\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} e^{u_i}, |\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} s_i e^{u_i} \right)_{j \in A} \right) = \mathcal{L}_{P_{W,H}^A} \left((e^{u_j}, s_j e^{u_j})_{j \in A} \right). \tag{2.16}$$

3. Derivation of the effective model

3.1. *Reduction of certain integrals.* The (super-)symmetries of the $H^{2|2}$ -model allow to calculate certain integrals. Lemma 3.2 below provides such a result. It is crucially used in the proof of Theorem 2.2.

Definition 3.1 (Fast decaying superfunctions). For any finite set Λ , a function $f : (H^{3|2})^\Lambda \rightarrow \mathcal{A}_0$, possibly depending on parameters, is called fast decaying on $(H^{2|2})^\Lambda$ if it is a smooth superfunction and all coefficients of $(\mathbb{R}^2)^\Lambda \ni (x_j, y_j)_{j \in \Lambda} \mapsto f((x_j, y_j, \sqrt{1 + x_j^2 + y_j^2} + 2\xi_j \eta_j, \xi_j, \eta_j)_{j \in \Lambda})$ in the sense of formula (A.1) in the appendix are Schwartz functions.

For $a \in H_+^{3|2}$, we set $\|a\| := \sqrt{-\langle a, a \rangle}$. Note that for $a \in H_{+0}^{3|2}$ with body $\langle a, a \rangle = 0$ and non zero Grassmann components the norm cannot be defined. Recall the abbreviation $\mathcal{D}v_\Lambda$ introduced in formula (2.6).

Lemma 3.2 (Reduction to single spins). *For any finite set Λ , $c_j \in [0, \infty)$, $j \in \Lambda$, $a \in H_+^{3|2}$, and any function $f : (H_+^{3|2})^\Lambda \rightarrow \mathcal{A}_0$ such that $v_\Lambda \mapsto f((\langle v_i, v_j \rangle)_{i,j \in \Lambda}) e^{\sum_{j \in \Lambda} c_j \langle v_j, a \rangle}$ is fast decaying on $(H^{2|2})^\Lambda$, one has*

$$\begin{aligned} & \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda f((\langle v_i, v_j \rangle)_{i,j \in \Lambda}) e^{\sum_{j \in \Lambda} c_j \langle v_j, a \rangle} \\ &= f((-1)_{i,j \in \Lambda}) e^{-\sum_{j \in \Lambda} c_j \|a\|} = \int_{H^{2|2}} \mathcal{D}v f((\langle v, v \rangle)_{i,j \in \Lambda}) e^{\sum_{j \in \Lambda} c_j \langle v, a \rangle}. \end{aligned} \tag{3.1}$$

This lemma has the following simple but interesting consequence.

Corollary 3.3 (Super Laplace transform of a block spin). *Let $\epsilon > 0$ and consider the $H^{2|2}$ -model on an arbitrary finite set Λ with arbitrary weights W and uniform pinning given by $h_j = |\Lambda|^{-1} \epsilon$ for all $j \in \Lambda$. For all $a \in H^{3|2}$ with $a + \epsilon o \in H_+^{3|2}$, the super Laplace transform of $|\Lambda|^{-1} \sum_{j \in \Lambda} v_j$ is given by*

$$E_{W,h}^\Lambda \left[e^{\langle a, |\Lambda|^{-1} \sum_{j \in \Lambda} v_j \rangle} \right] = e^{\epsilon - \|a + \epsilon o\|} = e^{\epsilon - \sqrt{\epsilon^2 - 2\epsilon \langle a, o \rangle - \langle a, a \rangle}}.$$

In particular, the mean $|\Lambda|^{-1} \sum_{j \in \Lambda} (x_j + z_j) = |\Lambda|^{-1} \sum_{j \in \Lambda} e^{u_j}$ has the probability density

$$f_{1,\epsilon}(t) = \sqrt{\frac{\epsilon}{2\pi t^3}} \exp\left(-\frac{\epsilon(t-1)^2}{2t}\right), \quad t > 0,$$

i.e. it is an inverse Gaussian distribution with parameters 1 and ϵ with respect to $P_{W,h}^\Lambda$.

We now give the proofs of the two results.

Proof of Lemma 3.2: It suffices to prove the first equality in (3.1), because the second one is just the special case of Λ being a singleton. Let $F : H_+^{3|2} \times (H_+^{3|2})^\Lambda \rightarrow \mathcal{A}_0$ be defined by

$$F(a, v_\Lambda) := f(\langle\langle v_i, v_j \rangle\rangle_{i,j \in \Lambda}) e^{\sum_{j \in \Lambda} c_j \langle v_j, a \rangle}.$$

Recall o from equation (2.2).

The proof uses the supersymmetry operators L_{ij} (cf. Def. A.1 in the appendix), and is organized in four steps, according to the form of the vector a . Precisely $a = (0, 0, z_a, 0, 0)$ in case 1, $a = (x_a, 0, z_a, 0, 0)$ in case 2, $a = (x_a, y_a, z_a, 0, 0)$ in case 3 and finally a has the most general form $a = (x_a, y_a, z_a, \xi_a, \eta_a)$ in case 4.

Case 1. Let $a = \alpha o$, where the parameter α is an even element of the Grassmann algebra with $\alpha > 0$, and must not depend on the integration variables v_j . In this case, by formula (A.5) (from Fact A.2 in the appendix), the integrand F is annihilated by the odd supersymmetry operator $Q := \sum_{j \in \Lambda} Q^{v_j}$, where Q^{v_j} is defined in (A.2). The hypothesis that $F(a, \cdot)$ is fast decaying allows us to apply the localization result from Appendix C in Disertori et al. (2010), in particular Proposition 2, as follows

$$\begin{aligned} \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda f(\langle\langle v_i, v_j \rangle\rangle_{i,j \in \Lambda}) e^{\sum_{j \in \Lambda} c_j \langle v_j, a \rangle} \\ = f(\langle\langle o, o \rangle\rangle_{i,j \in \Lambda}) e^{\sum_{j \in \Lambda} c_j \langle o, a \rangle} = f((-1)_{i,j \in \Lambda}) e^{-\sum_{j \in \Lambda} c_j \|a\|}. \end{aligned}$$

Case 2. Let $a = (x_a, 0, z_a, 0, 0)$ with x_a, z_a even elements, $z_a > 0$, $\langle a, a \rangle < 0$. We define $\varphi := \operatorname{artanh} \frac{x_a}{z_a}$, and set, for $0 \leq t \leq 1$, $t \in \mathbb{R}$,

$$a(t) = (x(t), 0, z(t), 0, 0) := \sqrt{z_a^2 - x_a^2} (\sinh(t\varphi), 0, \cosh(t\varphi), 0, 0),$$

where $\sqrt{z_a^2 - x_a^2} = \sqrt{-\langle a, a \rangle}$ is well defined because $\langle a, a \rangle < 0$. Note that $\langle a(t), a(t) \rangle = \langle a, a \rangle$ for all t . Moreover,

$$a(0) = \sqrt{z_a^2 - x_a^2} (0, 0, 1, 0, 0) = \sqrt{z_a^2 - x_a^2} o \quad \text{and} \quad a(1) = a. \tag{3.2}$$

By direct computation one has

$$\frac{d}{dt} F(a(t), v_\Lambda) = -\varphi L_{13}^{a(t)} F(a(t), v_\Lambda),$$

where $L_{13}^{a(t)}$ acts on the first component of F , which is then substituted by $a(t)$. By formula (A.3) in Fact A.2, one has

$$\left(L_{13}^{a(t)} + \sum_{j \in \Lambda} L_{13}^{v_j} \right) F(a(t), v_\Lambda) = 0$$

because the variables appear only inside the inner product. We conclude

$$\begin{aligned} \frac{d}{dt} \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda F(a(t), v_\Lambda) &= \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda \frac{d}{dt} F(a(t), v_\Lambda) \\ &= - \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda \varphi L_{13}^{a(t)} F(a(t), v_\Lambda) = \varphi \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda \sum_{j \in \Lambda} L_{13}^{v_j} F(a(t), v_\Lambda) = 0, \end{aligned} \tag{3.3}$$

where in the last step we have used the Ward identity (A.4). Consequently, the function

$$[0, 1] \ni t \mapsto \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda F(a(t), v_\Lambda)$$

is constant and in particular, its values at $t = 0$ and $t = 1$ agree. Recall the expressions (3.2) for $a(0)$ and $a(1)$. Using that Case 1 is applicable to $a(0)$, we conclude

$$\begin{aligned} \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda F(a, v_\Lambda) &= \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda F(a(1), v_\Lambda) \\ &= \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda F(a(0), v_\Lambda) = F((-1)_{i,j \in \Lambda}) e^{-\sum_{j \in \Lambda} c_j \|a(0)\|} = F((-1)_{i,j \in \Lambda}) e^{-\sum_{j \in \Lambda} c_j \|a\|}. \end{aligned}$$

Case 3: Let $a = (x_a, y_a, z_a, 0, 0)$ with x_a, y_a, z_a even elements, $z_a > 0$, $\langle a, a \rangle < 0$. Set $\varphi = \operatorname{artanh} \frac{y_a}{z_a}$ and, for $0 \leq t \leq 1$, $t \in \mathbb{R}$,

$$\begin{aligned} a(t) &= (x_a, y(t), z(t), 0, 0) \\ \text{with } y(t) &= \sqrt{z_a^2 - y_a^2} \sinh(t\varphi), \quad z(t) = \sqrt{z_a^2 - y_a^2} \cosh(t\varphi), \end{aligned}$$

where $\sqrt{z_a^2 - y_a^2}$ is well defined because $z_a^2 - y_a^2 \geq -\langle a, a \rangle > 0$. Note that $\langle a(t), a(t) \rangle = \langle a, a \rangle$ for all t . Moreover,

$$a(0) = (x_a, 0, \sqrt{z_a^2 - y_a^2}, 0, 0) \quad \text{and} \quad a(1) = a.$$

As in Case 2, by direct computation one has

$$\frac{d}{dt} F(a(t), v_\Lambda) = -\varphi L_{23}^{a(t)} F(a(t), v_\Lambda)$$

and again formula (A.3) in Fact A.2 implies

$$\left(L_{23}^{a(t)} + \sum_{j \in \Lambda} L_{23}^{v_j} \right) F(a(t), v_\Lambda) = 0.$$

Using the same argument as in (3.3), with L_{13} replaced by L_{23} , we conclude

$$\frac{d}{dt} \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda F(a(t), v_\Lambda) = 0.$$

Using that Case 2 is applicable to $a(0)$, the claim follows.

Case 4: Finally, we consider the most general case $a = (x_a, y_a, z_a, \xi_a, \eta_a)$. As in the previous cases, we define for $0 \leq t \leq 1$,

$$\begin{aligned} a(t) &= (x_a, y_a, z(t), \xi(t), \eta(t)) \quad \text{with} \\ z(t) &= z_a - \frac{1-t^2}{z_a} \xi_a \eta_a = \sqrt{z_a^2 - (1-t^2)2\xi_a \eta_a}, \quad \xi(t) = t\xi_a, \quad \eta(t) = t\eta_a. \end{aligned}$$

Note that $\langle a(t), a(t) \rangle = \langle a, a \rangle$ for all t . Moreover,

$$a(0) = (x_a, y_a, z_a - \frac{1}{z_a} \xi_a \eta_a, 0, 0) = (x_a, y_a, \sqrt{z_a^2 - 2\xi_a \eta_a}, 0, 0) \quad \text{and} \quad a(1) = a.$$

Below we denote by $\partial_{z(t)} F(a(t), v_\Lambda)$ the derivative with respect to the z -component of $a(t)$, which is then substituted by $z(t)$. The same notational convention holds for $\partial_{\xi(t)}$ and $\partial_{\eta(t)}$. Analogously to Cases 2 and 3, using $\xi_a z(t) = \xi_a z_a$ and $\eta_a z(t) = \eta_a z_a$, we get

$$\begin{aligned} \frac{d}{dt} F(a(t), v_\Lambda) &= [z'(t)\partial_{z(t)} + \xi'(t)\partial_{\xi(t)} + \eta'(t)\partial_{\eta(t)}] F(a(t), v_\Lambda) \\ &= \left[\frac{2t}{z_a} \xi_a \eta_a \partial_{z(t)} + \xi_a \partial_{\xi(t)} + \eta_a \partial_{\eta(t)} \right] F(a(t), v_\Lambda) \\ &= \frac{1}{z_a} [\xi_a (z(t)\partial_{\xi(t)} + \eta(t)\partial_{z(t)}) + \eta_a (z(t)\partial_{\eta(t)} - \xi(t)\partial_{z(t)})] F(a(t), v_\Lambda) \\ &= -\frac{1}{z_a} [\xi_a L_{34}^{a(t)} + \eta_a L_{35}^{a(t)}] F(a(t), v_\Lambda). \end{aligned}$$

Note that $\frac{\xi_a}{z_a}$ and $\frac{\eta_a}{z_a}$ are the analogue of φ in cases 2 and 3. Formula (A.3) implies

$$\left(L_{ik}^{a(t)} + \sum_{j \in \Lambda} L_{ik}^{v_j} \right) F(a(t), v_\Lambda) = 0$$

for $ik = 34, 35$. Using again the Ward identity from Fact A.2 and the fact that Case 3 is applicable to $a(0)$ we conclude. \square

As a first application of Lemma 3.2, we determine the super Laplace transform of $|\Lambda|^{-1} \sum_{j \in \Lambda} v_j$.

Proof of Corollary 3.3: Using first the definition (2.5) of the superintegration form and then Lemma 3.2 with the function $f(\langle v_i, v_j \rangle_{i,j \in \Lambda}) = e^{\sum_{i,j \in \Lambda} W_{ij}(1 + \langle v_i, v_j \rangle) + \epsilon}$, $c_j = |\Lambda|^{-1}$ for all j , and a replaced by $a + \epsilon o \in H_+^{3|2}$, we obtain

$$\begin{aligned} E_{W,h}^\Lambda \left[e^{\langle a, |\Lambda|^{-1} \sum_{j \in \Lambda} v_j \rangle} \right] &= \int_{(H^{2|2})^\Lambda} \mathcal{D}v_\Lambda e^{\sum_{i,j \in \Lambda} W_{ij}(1 + \langle v_i, v_j \rangle) + |\Lambda|^{-1} \epsilon \sum_{i \in \Lambda} (1 + \langle v_i, o \rangle)} e^{\langle a, |\Lambda|^{-1} \sum_{j \in \Lambda} v_j \rangle} \\ &= e^{\epsilon - \|a + \epsilon o\|} = \exp(\epsilon - \sqrt{\epsilon^2 - 2\epsilon \langle a, o \rangle - \langle a, a \rangle}). \end{aligned}$$

We consider $a = (b, 0, -b, 0, 0)$ with $b < 0$. Then, we find $\langle a, a \rangle = 0$, $\langle a, o \rangle = b$, $a + \epsilon o \in H_+^{3|2}$ and $be^{u_j} = b(x_j + z_j) = \langle a, v_j \rangle$. Consequently, the Laplace transform of $|\Lambda|^{-1} \sum_{j \in \Lambda} e^{u_j} = |\Lambda|^{-1} \sum_{j \in \Lambda} (x_j + z_j)$ is given by

$$E_{W,h}^\Lambda \left[e^{b|\Lambda|^{-1} \sum_{j \in \Lambda} e^{u_j}} \right] = E_{W,h}^\Lambda \left[e^{\langle a, |\Lambda|^{-1} \sum_{j \in \Lambda} v_j \rangle} \right] = e^{\epsilon - \sqrt{\epsilon^2 - 2\epsilon b}}.$$

This is the Laplace transform of an inverse Gaussian distribution with parameters 1 and ϵ and the claim follows. \square

3.2. *Reduction to the effective model.* Using the reduction of integrals given in Lemma 3.2 and a representation of the action of the hierarchical $H^{2|2}$ -model, we prove Theorem 2.2.

Proof of Theorem 2.2: The conditions on $H_j o + a_j$ guarantee that both integrals in (2.15) are well-defined and finite. Let \mathcal{A}_h denote the set of maximal antichains which are compatible with h and recall the partial order \prec for antichains defined in Section 2.3. Since every non-empty subset of \mathcal{A}_h contains a maximal element with respect to the partial order \prec , the general induction principle applies and we can prove the claim by induction over \mathcal{A}_h with respect to the reversed partial order \succ of \prec .

Let $A \in \mathcal{A}_h$. As induction hypothesis, assume that the claim holds for all maximal antichains $A' \in \mathcal{A}_h$ with $A \prec A'$. If $A = \Lambda_N$, there is nothing to prove. Otherwise, we find $i \in A \setminus \Lambda_N$ and set $A' = (A \setminus \{i\}) \cup \mathcal{B}_i$. Note that A' is a maximal antichain compatible with h and fulfilling $A \prec A'$. Set $a'_j = a_j$ for $j \in A \setminus \{i\}$ and $a'_j = |\mathcal{B}_i|^{-1} a_i$ for $j \in \mathcal{B}_i$. Then, $\sum_{j \in \mathcal{B}_i} \langle a'_j, v_j \rangle = \langle a_i, v_i \rangle$ by the definition (2.12) of the block spin variable v_i , and hence $\sum_{j \in A} \langle a_j, v_j \rangle = \sum_{j \in A'} \langle a'_j, v_j \rangle$. Moreover, for $j \in \mathcal{B}_i$, the equality $H_j = h_j$ and $a_i + H_i o \in H_{+0}^{3|2}, H_+^{3|2}$ imply $a'_j + h_j o \in H_{+0}^{3|2}, H_+^{3|2}$, respectively. Consequently, the induction hypothesis is applicable to A' and the a'_j . This yields

$$E_{W,h}^{\Lambda_N} \left[e^{\sum_{j \in A} \langle a_j, v_j \rangle} \right] = E_{W,h}^{\Lambda_N} \left[e^{\sum_{j \in A'} \langle a'_j, v_j \rangle} \right] = E_{W,H}^{A'} \left[e^{\sum_{j \in A'} \langle a'_j, v_j \rangle} \right]. \tag{3.4}$$

By the definition (2.4) of the action $S_{W,H}^{A'}(v_{A'})$ with vertex set A' , weights W_{ij} and pinning H_j given in (2.13) and (2.14), respectively, we obtain

$$\begin{aligned}
 S_{W,H}^{A'}(v_{A'}) &= \sum_{i,j \in A'} W_{ij}(1 + \langle v_i, v_j \rangle) + \sum_{j \in A'} H_j(1 + \langle v_j, o \rangle) \\
 &= S_{W,H}^{A \setminus \{i\}}(v_{A \setminus \{i\}}) + \sum_{j,k \in \mathcal{B}_i} W_{jk}(1 + \langle v_j, v_k \rangle) + 2 \sum_{\substack{j \in A \setminus \{i\} \\ k \in \mathcal{B}_i}} W_{jk}(1 + \langle v_j, v_k \rangle) + \sum_{k \in \mathcal{B}_i} H_k(1 + \langle v_k, o \rangle).
 \end{aligned} \tag{3.5}$$

Note that for $j \in A \setminus \{i\}$ and $k \in \mathcal{B}_i$, one has $j \wedge k = j \wedge i$ because $j \in A$ and A is an antichain. It follows that $W_{jk} = 2^{\ell(j)+\ell(k)} w(\ell(j \wedge k)) = 2^{\ell(j)} w(\ell(j \wedge i)) = 2^{-\ell(i)} W_{ij}$, which is independent of k . Furthermore, since A is compatible with h , one has the same value $H_k = 2^{-\ell(i)} H_i$ for all $k \in \mathcal{B}_i$. Hence, the last two summands in (3.5) can be written as follows

$$2 \sum_{\substack{j \in A \setminus \{i\} \\ k \in \mathcal{B}_i}} W_{jk}(1 + \langle v_j, v_k \rangle) + \sum_{k \in \mathcal{B}_i} H_k(1 + \langle v_k, o \rangle) = C_i + \sum_{k \in \mathcal{B}_i} 2^{-\ell(i)} \langle v_k, V_i \rangle, \tag{3.6}$$

where

$$C_i = H_i + \sum_{j \in A \setminus \{i\}} 2W_{ij}, \quad V_i = H_i o + \sum_{j \in A \setminus \{i\}} 2W_{ij} v_j,$$

and we used, for each $k \in \mathcal{B}_i$,

$$\sum_{j \in A \setminus \{i\}} 2W_{jk} v_j + H_k o = 2^{-\ell(i)} \left(H_i o + \sum_{j \in A \setminus \{i\}} 2W_{ij} v_j \right) = 2^{-\ell(i)} V_i.$$

Substituting this in (3.5) yields

$$S_{W,H}^{A'}(v_{A'}) = S_{W,H}^{A \setminus \{i\}}(v_{A \setminus \{i\}}) + C_i + \sum_{j,k \in \mathcal{B}_i} W_{jk}(1 + \langle v_j, v_k \rangle) + \sum_{k \in \mathcal{B}_i} 2^{-\ell(i)} \langle v_k, V_i \rangle.$$

Using first the definition of the superintegration form (2.5) and then this representation and the fact $a'_j = 2^{-\ell(i)} a_i$ for $j \in \mathcal{B}_i$, we obtain

$$\begin{aligned}
 E_{W,H}^{A'} \left[e^{\sum_{j \in A'} \langle a'_j, v_j \rangle} \right] &= \int_{(H^{2|2})^{A'}} \mathcal{D}v_{A'} e^{S_{W,H}^{A'}(v_{A'})} e^{\sum_{j \in A'} \langle a'_j, v_j \rangle} \\
 &= \int_{(H^{2|2})^{A \setminus \{i\}}} \mathcal{D}v_{A \setminus \{i\}} e^{S_{W,H}^{A \setminus \{i\}}(v_{A \setminus \{i\}})} e^{\sum_{j \in A \setminus \{i\}} \langle a'_j, v_j \rangle} e^{C_i} \\
 &\quad \cdot \int_{(H^{2|2})^{\mathcal{B}_i}} \mathcal{D}v_{\mathcal{B}_i} e^{\sum_{j,k \in \mathcal{B}_i} W_{jk}(1 + \langle v_j, v_k \rangle)} e^{\sum_{k \in \mathcal{B}_i} 2^{-\ell(i)} \langle v_k, V_i + a_i \rangle}.
 \end{aligned}$$

We apply Lemma 3.2 to the inner integral with $\Lambda = \mathcal{B}_i$, $c_j = 2^{-\ell(i)}$, $a = V_i + a_i$, and the function $f(\langle v_j, v_k \rangle)_{j,k \in \mathcal{B}_i} = e^{\sum_{j,k \in \mathcal{B}_i} W_{jk}(1 + \langle v_j, v_k \rangle)}$. To see that $a \in H_+^{3|2}$ we distinguish two cases. In the case $W_{ij} = 0$ for all $j \in A \setminus \{i\}$, all these points are in different connected components; hence the assumption on the pinning guarantees $a = H_i o + a_i \in H_+^{3|2}$. Otherwise, we have $\sum_{j \in A \setminus \{i\}} W_{ij} v_j \in H_+^{3|2}$, which implies $a \in H_+^{3|2}$ because of $H_i o + a_i \in H_{+0}^{3|2}$. Lemma 3.2 allows us to replace the multidimensional integration with respect to $\mathcal{D}v_{\mathcal{B}_i}$ by a single integral with respect to $\mathcal{D}v_i$ over

$H^{2|2}$. We obtain

$$\begin{aligned} & E_{W,H}^{A'} \left[e^{\sum_{j \in A'} \langle a'_j, v_j \rangle} \right] \\ &= \int_{(H^{2|2})^{A \setminus \{i\}}} \mathcal{D}v_{A \setminus \{i\}} e^{S_{W,H}^{A \setminus \{i\}}(v_{A \setminus \{i\}})} e^{\sum_{j \in A \setminus \{i\}} \langle a_j, v_j \rangle} e^{C_i} \int_{H^{2|2}} \mathcal{D}v_i e^{\sum_{j,k \in \mathcal{B}_i} W_{jk}(1 + \langle v_i, v_i \rangle)} e^{\langle v_i, V_i + a_i \rangle} \\ &= \int_{(H^{2|2})^{A \setminus \{i\}}} \mathcal{D}v_{A \setminus \{i\}} e^{S_{W,H}^{A \setminus \{i\}}(v_{A \setminus \{i\}})} \int_{H^{2|2}} \mathcal{D}v_i e^{2 \sum_{j \in A \setminus \{i\}} W_{ij}(1 + \langle v_i, v_j \rangle)} e^{H_i(1 + \langle v_i, o \rangle)} e^{\sum_{j \in A} \langle a_j, v_j \rangle}, \end{aligned}$$

where we used $\langle v_i, v_i \rangle = -1$ for $v_i \in H^{2|2}$ and the definitions of C_i and V_i . To merge the exponents, we observe that

$$S_{W,H}^A(v_A) = S_{W,H}^{A \setminus \{i\}}(v_{A \setminus \{i\}}) + 2 \sum_{j \in A \setminus \{i\}} W_{ij}(1 + \langle v_i, v_j \rangle) + H_i(1 + \langle v_i, o \rangle).$$

Together with (3.4) this concludes the induction step:

$$E_{W,H}^{A'} \left[e^{\sum_{j \in A'} \langle a'_j, v_j \rangle} \right] = \int_{(H^{2|2})^A} \mathcal{D}v_A e^{S_{W,H}^A(v_A)} e^{\sum_{j \in A} \langle a_j, v_j \rangle} = E_{W,H}^A \left[e^{\sum_{j \in A} \langle a_j, v_j \rangle} \right].$$

□

Using the reduction to the effective model, we can prove the identities in distribution for certain horospherical coordinates as stated in Corollary 2.3.

Proof of Corollary 2.3: First we prove claim (2.16) under the extra assumption that $h_i > 0$ for all $i \in \Lambda_N$. It suffices to show that the joint Laplace transforms of $(|\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} e^{u_i}, |\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} s_i e^{u_i})_{j \in A}$ and $(e^{u_j}, s_j e^{u_j})_{j \in A}$ with respect to $P_{W,h}^{\Lambda_N}$ and $P_{W,H}^A$, respectively, agree on a non-empty open set and are finite there. Thus, we will show the following equality for all $(c_j)_{j \in A} \in (0, \infty)^A$ and all $c'_j \in (-H_j, H_j)$, $j \in A$:

$$E_{W,h}^{\Lambda_N} \left[e^{-\sum_{j \in A} |\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} (c_j e^{u_i} + c'_j s_i e^{u_i})} \right] = E_{W,H}^A \left[e^{-\sum_{j \in A} (c_j e^{u_j} + c'_j s_j e^{u_j})} \right].$$

We rewrite this claim in cartesian coordinates. Using $-c_j e^{u_i} - c'_j s_i e^{u_i} = -c_j(x_i + z_i) - c'_j y_i = \langle v_i, a_j \rangle$ with $a_j = (-c_j, -c'_j, c_j, 0, 0)$ for $j \in A$, $i \in \mathcal{B}_j$ on the left-hand side and $-c_j e^{u_j} - c'_j s_j e^{u_j} = \langle v_j, a_j \rangle$ for $j \in A$ on the right-hand side, it takes the following form:

$$E_{W,h}^{\Lambda_N} \left[e^{\sum_{j \in A} \langle v_j, a_j \rangle} \right] = E_{W,h}^{\Lambda_N} \left[e^{\sum_{j \in A} |\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} \langle v_i, a_j \rangle} \right] = E_{W,H}^A \left[e^{\sum_{j \in A} \langle v_j, a_j \rangle} \right]. \tag{3.7}$$

Since $\langle H_j o + a_j, H_j o + a_j \rangle = c_j^2 - 2c_j H_j - H_j^2 < 0$, $H_j + c_j > 0$, and thus $H_j o + a_j \in H_+^{3|2}$, the claim, including finiteness of the expectations in (3.7), follows from Theorem 2.2.

Now we drop the extra assumption $h_i > 0$ for all $i \in \Lambda_N$. Let $\mathcal{N} := \{i \in \Lambda_N : h_i > 0\}$,

$$\mathcal{H} := \{(h'_i)_{i \in \Lambda_N} : h'_i = h_i \text{ for } i \in \mathcal{N}, 0 < h'_i \leq 1 \text{ for } i \notin \mathcal{N}, h' \text{ compatible with } A\}.$$

The probability density of $P_{W,h'}^{\Lambda_N}(du_{\Lambda_N} ds_{\Lambda_N})$ specified in (2.9) is dominated by

$$c e^{-\frac{1}{2} \sum_{i,j \in \Lambda_N} W_{ij}(B_{ij}-1)} e^{-\sum_{i \in \Lambda_N} h_i(B_i-1)} p((e^{u_i})_{i \in \Lambda_N}) \prod_{i \in \Lambda_N} \frac{e^{-u_i}}{2\pi}$$

with some numerical polynomial p with positive coefficients and a constant $c = c((h_i)_{i \in \mathcal{N}})$, uniformly in $h' \in \mathcal{H}$. This dominating function is integrable by our assumption on the pinning strength function h , cf. remarks below formula (2.3). Hence, by dominated convergence, we obtain

$$\lim_{h' \downarrow h} E_{W,h'}^{\Lambda_N} [f(u_{\Lambda_N}, s_{\Lambda_N})] = E_{W,h}^{\Lambda_N} [f(u_{\Lambda_N}, s_{\Lambda_N})] \tag{3.8}$$

for every bounded measurable function $f : (\mathbb{R}^2)^{\Lambda_N} \rightarrow \mathbb{R}$. The same argument applies when we replace Λ_N by A and h by $H|_A$. Note that $H|_A$ inherits the assumption on the pinning strength function phrased below formula (2.3). This yields

$$\lim_{H' \downarrow H} E_{W,H'}^A [f(u_A, s_A)] = E_{W,H}^A [f(u_A, s_A)] \tag{3.9}$$

for every bounded measurable function $f : (\mathbb{R}^2)^A \rightarrow \mathbb{R}$. The left-hand sides of (3.8) and (3.9) agree for $h' \in \mathcal{H}$ and the induced H' , which allows us to conclude.

The first claim in Corollary 2.3 is an immediate consequence since $(e^{u_j}, s_j e^{u_j})_{j \in B}$ uniquely determines $(u_j, s_j)_{j \in B}$ and $\mathcal{B}_j = \{j\}$ holds for $j \in B = A \cap \Lambda_N$. □

4. Tightness

To prove Theorem 2.1 we use a slight reformulation of a bound from Lemma 4 in Disertori et al. (2010). This will be an essential ingredient of the argument. Recall the abbreviation B_{ij} introduced in (2.7).

Lemma 4.1 (Alignment of two spins). *Consider the $H^{2|2}$ -model on a finite vertex set Λ with arbitrary weights W and pinning h . For $W_{ij} > 0$ and $\delta > W_{ij}^{-1}$, the following bound holds*

$$P_{W,h}^\Lambda(B_{ij} \geq 1 + \delta) \leq W_{ij} \delta e^{-(W_{ij} \delta - 1)}.$$

Proof: The proof is based on Lemmas 3 and 4 in Disertori et al. (2010), which generalize to non-constant weights W_{ij} in a straightforward way. Using first the exponential Chebyshev inequality and then Lemma 3 in Disertori et al. (2010), we obtain for all $\delta > 0$ and all $\gamma \in (0, 1)$

$$P_{W,h}^\Lambda(B_{ij} \geq 1 + \delta) = E_{W,h}^\Lambda[1_{\{B_{ij} \geq 1 + \delta\}}] \leq E_{W,h}^\Lambda[e^{W_{ij} \gamma (B_{ij} - 1 - \delta)}] \leq (1 - \gamma)^{-1} e^{-W_{ij} \gamma \delta}.$$

The upper bound is of the form $e^{-f(\gamma)}$ with $f(\gamma) = \log(1 - \gamma) + \alpha \gamma$ and $\alpha = W_{ij} \delta$. In order to optimize this upper bound, we maximize f . Observe that $f'(\gamma) = \alpha - (1 - \gamma)^{-1}$, which equals zero iff $\gamma = \gamma_c = 1 - \alpha^{-1}$. By assumption $\alpha > 1$, and hence the critical point γ_c belongs to $(0, 1)$. Using $f(1 - \alpha^{-1}) = -\log \alpha + \alpha - 1$, the claim follows. □

Using Lemma 4.1 and the reduction to the effective model, we prove our main result.

Proof of Theorem 2.1: We treat the two cases, uniform pinning (“U”) and pinning at one point (“1P”) simultaneously.

Let $\rho > 0$. It suffices to find $M = M(\rho) > 0$ such that for all $N \in \mathbb{Z}_+$, $\epsilon > 0$, and $p, q \in \Lambda_N$ with $p \neq q$, one has

$$\text{case U: } P_{W,\epsilon 1}^{\Lambda_N}(|u_p - u_q| \geq M) \leq \rho, \tag{4.1}$$

$$\text{case 1P: } P_{W,\epsilon \delta_p}^{\Lambda_N}(|u_p - u_q| \geq M) \leq \rho. \tag{4.2}$$

Note that in (4.2) the vertex p appears twice, as pinning point and in the variable u_p . This suffices indeed because for $i_0, p, q \in \Lambda_N$ we have the estimate

$$P_{W,\epsilon \delta_{i_0}}^{\Lambda_N}(|u_p - u_q| \geq M) \leq P_{W,\epsilon \delta_{i_0}}^{\Lambda_N}(|u_{i_0} - u_p| \geq \frac{M}{2}) + P_{W,\epsilon \delta_{i_0}}^{\Lambda_N}(|u_{i_0} - u_q| \geq \frac{M}{2}).$$

Let $N \in \mathbb{Z}_+$, $\epsilon > 0$, and $p, q \in \Lambda_N$ with $p \neq q$. Recall the notation introduced at the beginning of Section 2.2. We abbreviate $d = d(p, q) \geq 1$. Let $\bar{0} := 1, \bar{1} := 0$. For $j = (j_0, \dots, j_{|j|-1}) \in \mathcal{T}^N \setminus \{\emptyset\}$,

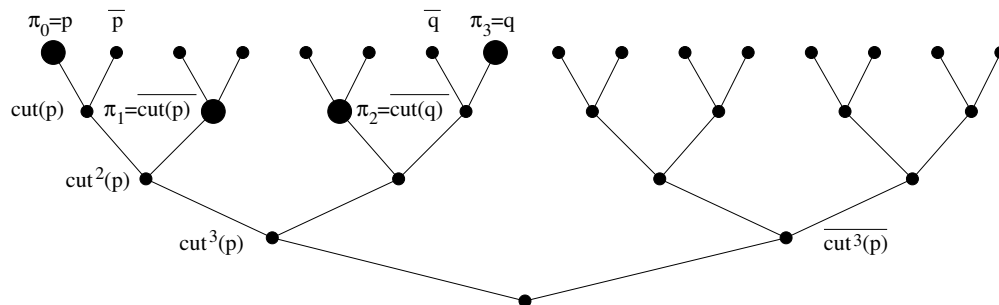


FIGURE 4.3. The vertices $p, q \in \Lambda_4$ have hierarchical distance $d = 3$. The maximal antichain from (4.3) equals $A := \{p, q, \bar{p}, \bar{q}, \overline{\text{cut}(p)}, \overline{\text{cut}(q)}, \overline{\text{cut}^3(p)}\}$. The path π in (4.4) is given by $\pi = (p, \overline{\text{cut}(p)}, \overline{\text{cut}(q)}, q)$.

let $\bar{j} := (\bar{j}_0, j_1, j_2, \dots, j_{|j|-1})$ be the nearest neighbor of j on the same level. We consider the set $A \subseteq \mathcal{T}^N$ given by

$$A := \{p, q\} \cup \{j \in \mathcal{T}^N \setminus \{\emptyset\} : j \not\leq p, j \not\leq q, (\text{cut}(j) \leq p \text{ or } \text{cut}(j) \leq q)\} \tag{4.3}$$

$$= \{p, q\} \cup \{\overline{\text{cut}^l(p)} : 0 \leq l \leq N - 1, l \neq d - 1\} \cup \{\overline{\text{cut}^l(q)} : 0 \leq l \leq d - 2\}.$$

An illustration is given in Figure 4.3.

Note that A is a maximal antichain. Furthermore, it is compatible with both pinning functions, $h = \epsilon 1$ (case U) and $h = \epsilon \delta_p$ (case 1P). We remark that this is the reason why we need p as a pinning point in formula (4.2).

We consider the path $\pi = (\pi_l)_{0 \leq l \leq m}$ in the antichain A from $\pi_0 = p$ to $\pi_m = q$ given by the concatenation of $p, (\overline{\text{cut}^l(p)})_{1 \leq l \leq d-2}$, the reversed path of $(\overline{\text{cut}^l(q)})_{1 \leq l \leq d-2}$, and q . In other words, for $m = d = 1$, we have $\pi = (p, q)$ and for $d \geq 2$, we have $m = 2d - 3$, and

$$\pi_l = \begin{cases} p & \text{for } l = 0, \\ \overline{\text{cut}^l(p)} & \text{for } 1 \leq l \leq d - 2, \\ \overline{\text{cut}^{m-l}(q)} & \text{for } d - 1 \leq l \leq m - 1, \\ q & \text{for } l = m. \end{cases} \tag{4.4}$$

Note that we exclude on purpose the points $\bar{p} = \overline{\text{cut}^0(p)}$ and $\bar{q} = \overline{\text{cut}^0(q)}$ in the second and third line of (4.4) (though they belong to the antichain) since they do not help in the estimates below.

In the case $d \geq 2$, the edges $\{\pi_l, \pi_{l+1}\}$ (with $l = 0, \dots, m - 1$) of the path π have the weights

$$\beta'_l := W_{\pi_l \pi_{l+1}} = 2^{\ell(\pi_l) + \ell(\pi_{l+1})} w(\ell(\pi_l \wedge \pi_{l+1}))$$

$$= \begin{cases} 2^{2l+1} w(l + 2) = \beta_l & \text{if } 0 \leq l \leq d - 3, \\ 2^{2d-4} w(d) = \frac{1}{2} \beta_{d-2} & \text{if } l = d - 2, \\ 2^{2(m-l)-1} w(m - l + 1) = \beta_{m-l-1} & \text{if } d - 1 \leq l \leq m - 1. \end{cases} \tag{4.5}$$

In the case $d = 1$, the single edge $\{p, q\}$ of the path π has the weight $\beta''_0 := W_{pq} = w(1)$. We choose $\delta''_0 > 0$ large enough such that $\beta''_0 \delta''_0 > 1$ and $\beta''_0 \delta''_0 e^{-(\beta''_0 \delta''_0 - 1)} < \rho$. We show below the following claim.

Claim: There is a sequence $(\delta_l)_{l \in \mathbb{Z}^0_+}$ of positive numbers (independent of p and q) such that

$$\beta_l \delta_l > 1 \text{ for all } l, \quad \sum_{l=0}^{\infty} \text{arcosh}(1 + \delta_l) < \infty \quad \text{and} \quad \sum_{l=0}^{\infty} \beta_l \delta_l e^{-\beta_l \delta_l + 1} < \frac{\rho}{2}. \tag{4.6}$$

Using this claim, we show first that the estimates (4.1) and (4.2) hold. We define

$$M := \max \left\{ \sup_{d' \geq 2} \left(2 \sum_{l=0}^{d'-3} \operatorname{arcosh}(1 + \delta_l) + \operatorname{arcosh}(1 + 2\delta_{d'-2}) \right), \operatorname{arcosh}(1 + \delta''_0) \right\}. \tag{4.7}$$

As a consequence of (4.6), the sequence $(\delta_l)_l$ is bounded and M is finite.

We consider now the case $d \geq 2$. We introduce

$$\delta'_l = \begin{cases} \delta_l & \text{for } 0 \leq l \leq d-3, \\ 2\delta_{d-2} & \text{for } l = d-2, \\ \delta_{m-l-1} & \text{for } d-1 \leq l \leq m-1. \end{cases} \tag{4.8}$$

Note that

$$\sum_{l=0}^{m-1} \operatorname{arcosh}(1 + \delta'_l) = 2 \sum_{l=0}^{d-3} \operatorname{arcosh}(1 + \delta_l) + \operatorname{arcosh}(1 + 2\delta_{d-2}) \leq M. \tag{4.9}$$

Since A is a maximal antichain containing p and q , the distributions of $u_p - u_q$ with respect to $P_{W,h}^{A,N}$ and $P_{W,H}^A$ coincide by Corollary 2.3; here H is the extension of the pinning function h given in (2.14). Using (4.9) in the first inequality, we estimate

$$\begin{aligned} P_{W,h}^{A,N}(|u_p - u_q| \geq M) &= P_{W,H}^A(|u_{\pi_0} - u_{\pi_m}| \geq M) \\ &\leq P_{W,H}^A\left(|u_{\pi_0} - u_{\pi_m}| \geq \sum_{l=0}^{m-1} \operatorname{arcosh}(1 + \delta'_l)\right) \\ &\leq P_{W,H}^A\left(\sum_{l=0}^{m-1} |u_{\pi_l} - u_{\pi_{l+1}}| \geq \sum_{l=0}^{m-1} \operatorname{arcosh}(1 + \delta'_l)\right) \\ &\leq \sum_{l=0}^{m-1} P_{W,H}^A(|u_{\pi_l} - u_{\pi_{l+1}}| \geq \operatorname{arcosh}(1 + \delta'_l)) \\ &= \sum_{l=0}^{m-1} P_{W,H}^A(\cosh(u_{\pi_l} - u_{\pi_{l+1}}) \geq 1 + \delta'_l). \end{aligned} \tag{4.10}$$

Recall the notation B_{ij} from formula (2.7). In particular, $\cosh(u_i - u_j) \leq B_{ij}$ for $i, j \in A$. To estimate the l -th summand on the right-hand side of formula (4.10) we observe that $W_{\pi_l \pi_{l+1}} = \beta'_l$ and $\beta'_l \delta'_l = \beta_l \delta_l > 1$ as a consequence of (4.5), (4.8), and Claim (4.6) so that Lemma 4.1 is applicable and yields

$$P_{W,H}^A(\cosh(u_{\pi_l} - u_{\pi_{l+1}}) \geq 1 + \delta'_l) \leq P_{W,H}^A(B_{\pi_l \pi_{l+1}} \geq 1 + \delta'_l) \leq \beta'_l \delta'_l e^{-(\beta'_l \delta'_l - 1)}.$$

Summing over l and using Claim (4.6) in the last step, we conclude

$$\begin{aligned} P_{W,h}^{A,N}(|u_p - u_q| \geq M) &\leq \sum_{l=0}^{m-1} \beta'_l \delta'_l e^{-(\beta'_l \delta'_l - 1)} \\ &= 2 \sum_{l=0}^{d-3} \beta_l \delta_l e^{-(\beta_l \delta_l - 1)} + \beta_{d-2} \delta_{d-2} e^{-(\beta_{d-2} \delta_{d-2} - 1)} \leq 2 \sum_{l=0}^{\infty} \beta_l \delta_l e^{-(\beta_l \delta_l - 1)} < \rho. \end{aligned}$$

For $d = 1$, using $\operatorname{arcosh}(1 + \delta''_0) \leq M$ from (4.7) and again Lemma 4.1, it also holds

$$P_{W,h}^{A,N}(|u_p - u_q| \geq M) \leq P_{W,h}^{A,N}(\cosh(u_p - u_q) \geq 1 + \delta''_0) \leq \beta''_0 \delta''_0 e^{-(\beta''_0 \delta''_0 - 1)} < \rho.$$

It remains to show Claim (4.6). Assume that the hypothesis (2.11) of Theorem 2.1 holds. To ensure $\sum_{l=0}^{\infty} \operatorname{arcosh}(1 + \delta_l) < \infty$, it is necessary that $\delta_l \rightarrow 0$ as $l \rightarrow \infty$. On the other hand, to ensure

$\sum_{l=0}^{\infty} \beta_l \delta_l e^{-\beta_l \delta_l + 1} < \frac{\rho}{2}$, we will choose δ_l such that $\beta_l \delta_l \rightarrow \infty$ fast enough as $l \rightarrow \infty$ and $\beta_l \delta_l$ is large enough for a sufficiently large initial piece $l = 0, \dots, l_1$. More precisely, we proceed as follows.

From assumption (2.11) we know $\lim_{l \rightarrow \infty} \beta_l = \infty$. Take $l_0 \in \mathbb{Z}_+$ so large that $\log \log \sqrt{\beta_l}$ is well-defined and positive for all $l \geq l_0$. For these l , we set

$$\tilde{\delta}_l := \frac{1}{\beta_l} \left(1 + \log \sqrt{\beta_l} + \frac{3}{2} \log \log \sqrt{\beta_l} \right) > \frac{1}{\beta_l}. \tag{4.11}$$

We observe that

$$\begin{aligned} \sum_{l=l_0}^{\infty} \beta_l \tilde{\delta}_l e^{-\beta_l \tilde{\delta}_l + 1} &= \sum_{l=l_0}^{\infty} \left(1 + \log \sqrt{\beta_l} + \frac{3}{2} \log \log \sqrt{\beta_l} \right) \exp \left(-\log \sqrt{\beta_l} - \frac{3}{2} \log \log \sqrt{\beta_l} \right) \\ &= \sum_{l=l_0}^{\infty} \left(1 + \log \sqrt{\beta_l} + \frac{3}{2} \log \log \sqrt{\beta_l} \right) \frac{1}{\sqrt{\beta_l}} (\log \sqrt{\beta_l})^{-\frac{3}{2}} < \infty. \end{aligned} \tag{4.12}$$

The last estimate used assumption (2.11). However, we remark that its full strength is not used here, but only below. The estimate (4.12) implies that there exists $l_1 \geq l_0$ such that

$$\sum_{l=l_1}^{\infty} \beta_l \tilde{\delta}_l e^{-\beta_l \tilde{\delta}_l + 1} < \frac{\rho}{4}.$$

We choose $\delta_l > \beta_l^{-1}$ for $0 \leq l \leq l_1 - 1$ large enough, such that

$$\sum_{l=0}^{l_1-1} \beta_l \delta_l e^{-\beta_l \delta_l + 1} < \frac{\rho}{4}.$$

For $l \geq l_1$, we set $\delta_l := \tilde{\delta}_l$. Then, the condition for the second series in (4.6) follows. Observe that $\beta_l \delta_l > 1$ for all $l \in \mathbb{Z}_+^0$ by (4.11) and our choice of δ_l for small l .

It remains to prove the inequality for the first series in (4.6), which is equivalent to the claim

$$\sum_{l=l_1}^{\infty} \operatorname{arcosh}(1 + \delta_l) < \infty.$$

We estimate the l -th summand in the last series:

$$\operatorname{arcosh}(1 + \delta_l) \leq \sqrt{2\delta_l} = \sqrt{\frac{2}{\beta_l}} \sqrt{1 + \log \sqrt{\beta_l} + \frac{3}{2} \log \log \sqrt{\beta_l}} = O\left(\sqrt{\frac{\log \beta_l}{\beta_l}}\right)$$

as $l \rightarrow \infty$, which is summable over l by hypothesis (2.11). This completes the proof of Claim (4.6) and hence the proof of the theorem. \square

Remark 4.2. In the proof of Theorem 2.1 it is shown that for $p, q \in \Lambda_N$ with $d(p, q) = d$, the set

$$A := \{p, q\} \cup \overline{\operatorname{cut}^l(p)} : 0 \leq l \leq N - 1, l \neq d - 1 \cup \overline{\operatorname{cut}^l(q)} : 0 \leq l \leq d - 2$$

is a maximal antichain containing p and q ; see Figure 4.3 for an illustration. In order to study the distribution of (u_p, u_q) with respect to $P_{W,h}^{\Lambda_N}$ with a pinning function h such that A is compatible with it, cf. Figure 2.2, one can equally well study its distribution with respect to the reduced model $P_{W,H}^A$. Note that $|\Lambda_N| = 2^N$, whereas $|A| = N + d$, resulting in an exponential decrease of complexity. More generally, one can construct a maximal antichain $A(B)$ containing a finite set $B \subseteq \Lambda_N$ as follows. The set $A(B)$ consists of B and all vertices which are minimal elements with respect to \leq in the set $\{i \in \mathcal{T}^N : i \not\leq j \text{ for all } j \in B\}$. For fixed $|B|$, as $N \rightarrow \infty$, the size of $A(B)$ is again bounded by $O(N)$.

Appendix A. Appendix: Grassmann algebras and supersymmetry operators

Grassmann algebras. Here we give some more details on the Grassmann algebra \mathcal{A} underlying the $H^{2|2}$ -model with vertex set Λ . We consider a real vector space V of finite dimension K . We assume that V has a basis $\mathcal{B} = \{\chi_1, \dots, \chi_K\}$ consisting of ξ_i, η_i for $i \in \Lambda$ and possibly additional Grassmann parameters. Then the Grassmann algebra \mathcal{A} is defined as the direct sum of exterior product spaces $\Lambda^k V$ of V : $\mathcal{A} = \bigoplus_{k=0}^K \Lambda^k V = \mathcal{A}_0 \oplus \mathcal{A}_1$ with the even (commuting) subalgebra $\mathcal{A}_0 = \bigoplus_{k=0}^{\lfloor K/2 \rfloor} \Lambda^{2k} V$ and the odd (anticommuting) subspace $\mathcal{A}_1 = \bigoplus_{k=0}^{\lfloor (K-1)/2 \rfloor} \Lambda^{2k+1} V$. Note that the body of an even element is the projection to the 0th-component: $\text{body} : \mathcal{A}_0 \rightarrow \Lambda^0 V = \mathbb{R}$. Every element ω in the Grassmann algebra \mathcal{A} can be uniquely written as follows:

$$\omega = \sum_{I \subseteq \{1, \dots, K\}} c_I \chi^I, \tag{A.1}$$

with coefficients $c_I \in \mathbb{R}$ and basis elements $\chi^I := \chi_{i_1} \dots \chi_{i_l}$ for $I = \{i_1, \dots, i_l\}$ with $i_1 < \dots < i_l$ and $\chi^\emptyset = 1$.

Grassmann derivatives. For every basis element $\chi \in \mathcal{B}$ and every element $\omega \in \mathcal{A}$, we have a unique decomposition $\omega = \omega_0 + \chi \omega_1$, where ω_0 and ω_1 belong to the subalgebra of \mathcal{A} generated by the elements of $\mathcal{B} \setminus \{\chi\}$. We define the Grassmann derivative

$$\partial_\chi \omega := \omega_1.$$

Note that this derivative depends not only on χ and ω , but also on the choice of the basis \mathcal{B} . This is in analogy to the ordinary partial derivative, which also depends on the choice of other variables which are kept fixed.

Supersymmetry operators. We review now the (super)symmetry operators of the $H^{2|2}$ -model. For $n \in \mathbb{Z}_+$, let $[n] := \{1, 2, \dots, n\}$. We define the parity function

$$p : [5] \rightarrow \{0, 1\}, \quad p(i) = \begin{cases} 0 & \text{for } i \in [3], \\ 1 & \text{for } i \in \{4, 5\}. \end{cases}$$

For $i, j \in [5]$, we set

$$\sigma_{ij} := (-1)^{p(i)+p(j)+p(i)p(j)} = \begin{cases} 1 & \text{for } i, j \in [3], \\ -1 & \text{otherwise.} \end{cases}$$

Using the matrix

$$g = (g_{ij})_{i,j \in [5]} := \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array} \right)$$

the inner product of $v = (x, y, z, \xi, \eta)$ and $v' = (x', y', z', \xi', \eta')$ reads

$$\langle v', v \rangle = v' g v^t.$$

We write $v = (v^i)_{i \in [5]}$ and $\partial_{v^i} = \frac{\partial}{\partial v^i}$.

Definition A.1 (Supersymmetry operators). For $i, j \in [5]$, we define the supersymmetry operators

$$L_{ij} = L_{ij}^v := \sum_{k \in [5]} [v^k g_{kj} \partial_{v^i} - v^k \sigma_{ij} g_{ki} \partial_{v^j}]$$

We observe that $L_{ij} = -\sigma_{ji} L_{ji}$ for all i, j . Furthermore, $L_{11} = L_{22} = L_{33} = 0$,

- $L_{12} = y \partial_x - x \partial_y$ generates Euclidean rotations in the x - y -plane,

- $-L_{13} = z\partial_x + x\partial_z$ and $-L_{23} = z\partial_y + y\partial_z$ generate Lorentz boosts in the x - z - and y - z -plane, respectively,
- $L_{44} = -2\eta\partial_\xi$, $L_{45} = \xi\partial_\xi - \eta\partial_\eta$, and $L_{55} = 2\xi\partial_\eta$ are even operators, and
- $L_{14} = x\partial_\xi - \eta\partial_x$, $L_{24} = y\partial_\xi - \eta\partial_y$, $L_{15} = \xi\partial_x + x\partial_\eta$, $L_{25} = \xi\partial_y + y\partial_\eta$, $L_{34} = -(\eta\partial_z + z\partial_\xi)$, and $L_{35} = \xi\partial_z - z\partial_\eta$ are odd operators.

We define

$$Q^v := L_{15} - L_{24} = x\partial_\eta - y\partial_\xi + \xi\partial_x + \eta\partial_y. \tag{A.2}$$

We collect some basic facts on these supersymmetry operators. For more details see Section 4 and Appendices B and C of [Disertori et al. \(2010\)](#).

Fact A.2 (Consequences of supersymmetry).

(1) For $i, j \in [5]$, the following holds.

(a) The operator L_{ij} annihilates the inner product as follows

$$(L_{ij}^v + L_{ij}^{v'})\langle v, v' \rangle = 0 \quad \text{for all } v, v' \in H^{3|2}.$$

Moreover, for $k \in \mathbb{Z}_+$ and any smooth superfunction $f : \mathcal{A}_0^{k^2} \rightarrow \mathcal{A}_0$, as v_1, \dots, v_k run over $H^{3|2}$, one has

$$\sum_{l=1}^k L_{ij}^{v_l} f(\langle (v_j, v_{j'})_{1 \leq j, j' \leq k} \rangle) = 0. \tag{A.3}$$

(b) *Ward identity.* For any smooth superfunction $f : (H_+^{3|2})^k \rightarrow \mathcal{A}_0$ with $k \in \mathbb{Z}_+$ such that f is fast decaying on $(H^{2|2})^k$, one has

$$\int_{(H^{2|2})^k} \mathcal{D}v_1 \dots \mathcal{D}v_k \sum_{l=1}^k L_{ij}^{v_l} f(v_1, \dots, v_k) = 0. \tag{A.4}$$

(2) The operator Q annihilates the following inner products

$$(Q^v + Q^{v'})\langle v, v' \rangle = 0, \quad Q^v \langle v, o \rangle = 0, \quad \text{for all } v, v' \in H^{3|2}. \tag{A.5}$$

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