



Costly defense traits in structured populations

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Abstract. We propose a model for the dynamics of frequencies of a costly defense trait. More precisely, we consider Lotka–Volterra-type models involving a prey (or host) population consisting of two types and a predator (or parasite) population, where one type of prey individuals – modeling carriers of a defense trait – is more effective in defending against the predators but has a weak reproductive disadvantage. Under certain assumptions we prove that the relative frequency of these defenders in the total prey population converges to spatially structured Wright–Fisher diffusions with frequency-dependent migration rates. For the many-demes limit (mean-field approximation) hereof, we show that the defense trait goes to fixation/extinction if and only if the selective disadvantage is smaller/larger than an explicit function of the ecological model parameters.

1. Introduction

If individuals of a prey population defend their conspecifics when predators arrive, this may be dangerous for the acting individual and thus be costly in the evolutionary sense of decreasing the expected number of their surviving offspring. For example, alarm calling has been described as altruistic behavior ([Tamachi, 1987](#)). Altruism is defined as a behavior that decreases the reproductive success of the actor while increasing the reproductive success of one or more recipients ([Hamilton, 1964](#)). In most natural systems, non-altruistic individuals benefit from altruistic individuals without suffering from the fitness disadvantage and, thus, have a direct reproductive advantage. So how can genetically inherited selfless behavior be explained by natural selection? This problem has bothered biologists since Charles Darwin, who reflected the puzzle of sterile social insects such as the worker castes of ants in his famous book “The Origin of Species” ([Darwin, 1859](#)).

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In behavioral biology and game theory there exist several explanations for the emergence of altruism and cooperation. The central idea of inclusive fitness theory is that helping direct relatives benefits the reproductive success of the altruists' genes. This idea is formalized in Hamilton's rule, which states that traits increase in frequency if $C < B \cdot R$, where C is the reproductive cost to the actor, B is the additional reproductive benefit gained by the recipient, and R is the relatedness of the recipient, that is the probability of sharing the same allele by descent, e.g., 1/2 for two sisters and 1/8 for two cousins (Hamilton, 1964). In other words, genes can spread in a population by kin selection if the inclusive fitness $B \cdot R - C$ is positive. However, general applicability of Hamilton's rule is controversial; e.g., the fundamental criticism of inclusive fitness theory in Nowak et al. (2010) provoked a strong response including a rebuttal from 137 researchers (Abbot et al., 2011).

Another explanation for the emergence of altruistic behavior is the intensively debated theory of group selection; see, e.g., Wade (1978) and Queller (1992). The central idea is that groups of cooperators grow faster and, therefore, split earlier or into more groups than groups of defectors (Traulsen and Nowak, 2006). The importance of group selection (or more generally multilevel selection) in evolution remains controversial (see, e.g. Smith, 1976; Goodnight and Stevens, 1997; Goodnight and Wade, 2000; Nowak, 2006; West et al., 2007; Traulsen, 2010; Gardner, 2015). Many scientists consider group selection (or multi-level selection) and inclusive fitness theory as equivalent (e.g. Marshall, 2011), while others van Veelen et al. (e.g. 2012) argue that group selection is a more general concept. This conflict may be due to disagreement on the precise definition of the variables in Hamilton's rule (Gardner et al., 2011; Birch and Okasha, 2014).

If groups are formed by related individuals, it may seem obvious that group/kin selection can lead to the evolution of altruistic traits. If, however, individuals also compete with their neighbors – which may include relatives – for space, food or other resources, altruistic behavior can be very costly regarding reproductive success. (Wilson et al., 1992; Van Dyken, 2010). In this case, kin selection can still be effective if competition works on a larger spatial scale than altruism or if competition is reduced as the population is growing or sends out migrants to conquer empty demes (Wilson et al., 1992; Taylor, 1992; Alizon and Taylor, 2008; Van Dyken, 2010; Van Dyken and Wade, 2012). The latter can occur in a meta-population – that is, a population that is substructured into demes that are affected by frequent local extinction and recolonization events. Already Smith (1964) proposed that a meta-population dynamic can generate between-deme variation that is required for group selection, and indeed this was subsequently demonstrated for several theoretical models (Levins, 1970; Eshel, 1972; Levin and Kilmer, 1974; Wilson, 1973; Slatkin and Wade, 1978; Taylor, 1992; Alizon and Taylor, 2008; Metzler et al., 2016). The mathematical analyses of Uyenoyama (1979) however showed for a range of possible selection pressures that group selection can also work in island population models without local extinctions, assuming however an extreme migration model in which all surviving offspring are randomly distributed among all islands.

For the eco-evolutionary dynamics in group or kin selection models it is crucial how the benefit of altruism depends on the frequency of altruists and on other factors. Sibly and Curnow (2011) show for some cases of *diminishing returns*, which means that the benefit per altruist decreases with the total number of altruists, that kin selection can maintain the co-existence of altruists and cheaters. Mechanisms that lead to diminishing returns may include feed-back interactions with the abundance of other species, for example predators or parasites if the altruistic trait consist in defending other individuals against these enemies (Boots et al., 2009; Duncan et al., 2011; Débarre et al., 2012; Berngruber et al., 2013; Vitale and Best, 2019).

In this article we focus on the evolution of a trait of defense against parasites or predators. Examples of altruistic or at least costly defense traits include self-sacrificial colony defense in social insects (Heinze and Walter, 2010; Rueppell et al., 2010; Shorter and Rueppell, 2012), costly chemical alarm signaling in aphids (Mondor and Roitberg, 2004; Mondor and Messing, 2007; Wu et al., 2010), suicidal defense of bacteria against pathogen infection (Fukuyo et al., 2012), transmission-blocking immunity in vertebrates against *Plasmodium* (Mendis et al., 1987), and slave rebellion in

ants (Pamminger et al., 2014; Metzler et al., 2016). A number of recent studies propose spatially distributed predator–prey and host–parasite models and study these models via computer simulations (see e.g. Comins et al., 1992; Rand et al., 1995; Haraguchi and Sasaki, 2000; Rauch et al., 2002, 2003; Goodnight et al., 2008; Best et al., 2011; Débarre et al., 2012; Berngruber et al., 2013; Lion and Gandon, 2015; Boëte et al., 2019). We note that our models and results may also be applicable to constitutive resistance or defense traits in host–parasite systems if alternative strategies such as induced resistance (Boëte et al., 2019) and parasite tolerance (Vitale and Best, 2019) are negligible.

We begin with a structured predator–prey model with a prey population consisting of two types, which we denote in the following as defenders and non-defenders. Regarding population structure, we assume that the habitat of prey and predators is subdivided into demes. Prey as well as predators are panmictic within demes and migrate between the demes. Our model for the interaction of predators and prey in each deme is based on a Lotka–Volterra model (Lotka, 1920; Volterra, 1926) allowing for competition among prey individuals of the same deme. Predators in the same deme compete for prey as well as other resources. The defenders in the prey population have a smaller reproduction rate than non-defenders and reduce the growth of the predator population in the same deme. Thus, non-defenders profit from the abundance of defenders in their deme and surpass them in fitness. We note that the behavior of defenders is not exactly altruistic if offspring of defenders benefit from fewer predators and might produce more grandchildren compared to non-defenders. In our view, however, costly defense traits are closely related to altruism and – similarly to altruism – it is a priori not clear whether the behavior of costly defense “pays off”.

We approximate our first model by an asymptotic model of infinitely large deme population sizes. As common in population genetics (Ewens, 2004; Durrett, 2008) we measure time in units of N generations, where N is proportional to effective deme population sizes, and thus obtain Wright–Fisher diffusions maintaining random fluctuations in trait frequencies – so-called genetic drift (Kimura, 1968) – even in the limit of large demes. Moreover we scale migration rates such that the expected number of migrants per generation and deme is constant; which is also common in population genetics (Durrett, 2008). It turns out that, as we let N tend to infinity, the population genetic (or “evolutionary”) time scale of N generations separates from the ecological time scale of predator–prey interactions. In other words, on the population genetic time scale, the deme population sizes of prey and predators instantaneously reach their limits according to the Lotka–Volterra interactions, where these limits depend on the frequency of defenders in the respective deme. Our main result is that the diffusion approximation of the frequencies of defenders is the well-known Kimura’s stepping stone model with negative selection (e.g. Etheridge, 2011, Chapter 6) only with the local population sizes replaced by a function of the local defender frequencies.

Our diffusion approximation of the frequencies of defenders is mathematically hard to analyze due to the lack of a suitable dual process, which is due to variation of local effective population sizes. For this reason we investigate a meta-population setting at which we arrive by considering uniform migration on $D \in \mathbb{N}$ demes and then letting the number D of demes tend to infinity. In this meta-population setting we prove that the defense trait will become fixed in the entire prey population if the ‘cost of the defense trait’ is smaller than the so-called ‘benefit of defense’. This shows that predator–prey dynamics can indeed induce group selection that maintains a defense trait that is under negative selection in each deme, without the need of extinction (and re-colonization) of demes. Group selection on a trait requires that the frequency of the trait in a deme must be correlated with the number of migrants produced by the deme (Queller, 1992). This correlation can only be non-zero if the frequency of the trait varies between the demes. While migration reduces this variation, the only factor in our model that can increase it is genetic drift.

The rest of this article is structured as follows: In section 2 we introduce our model and specify our model assumptions. Moreover we state our main results: first, weak convergence of the frequencies of defenders as the local population sizes converge to infinity (Theorem 2.3) and, second, long-term fixation/extinction of defenders in a meta-population setting depending on whether the ‘cost of

defense' α is smaller than or bigger than the parameter β which we denote as 'benefit of defense' (Theorem 2.4). Sections 3 and 5 are devoted to the proofs of Theorem 2.3 and of Theorem 2.4, respectively. In section 4 we prove that diffusion approximation of the frequencies of defenders with uniform migration on $D \in \mathbb{N}$ demes converges as $D \rightarrow \infty$ to a McKean–Vlasov equation. Finally, in section 6 we consider the many-demes limit of the D -demes equation when initially only a few demes are populated and prove that the total mass process of this many-demes limit converges to infinity or to zero in probability depending on whether the 'cost of defense' α is smaller than or bigger than the parameter β .

1.1. *Notation.* Throughout this article, we will use the following notation. We define $[0, \infty] := [0, \infty) \cup \{\infty\}$. We will use the conventions that $0^0 = 1$, $0 \cdot \infty = 0$, and that for any $x \in (0, \infty)$ we have that $\frac{x}{\infty} = 0$ and $\frac{x}{0} = \infty$. For all $x, y \in \mathbb{R}$ we define $x^+ := \max\{x, 0\}$, $\text{sgn}(x) := \mathbb{1}_{x>0} - \mathbb{1}_{x<0}$, and $x \wedge y := \min\{x, y\}$. We define $\sup(\emptyset) := -\infty$ and $\inf(\emptyset) := \infty$. For a topological space (E, \mathcal{E}) we denote by $\mathcal{B}(E)$ the Borel sigma-algebra of (E, \mathcal{E}) . Moreover we agree on the convention that zero times an undefined expression is set to zero. For every countable set \mathcal{D} and every $\sigma = (\sigma_i)_{i \in \mathcal{D}} \in (0, \infty)^{\mathcal{D}}$ define a function $\|\cdot\|_{\sigma}: \mathbb{R}^{\mathcal{D}} \rightarrow [0, \infty]$ by $\mathbb{R}^{\mathcal{D}} \ni z = (z_i)_{i \in \mathcal{D}} \mapsto \|z\|_{\sigma} := \sum_{i \in \mathcal{D}} \sigma_i |z_i|$ and define $l_{\sigma}^1 := \{z \in \mathbb{R}^{\mathcal{D}}: \|z\|_{\sigma} < \infty\}$.

2. Main results

2.1. *Model.* We assume that predator (or parasite) and prey (or host) individuals populate demes given by a countable set \mathcal{D} , and the prey population consists of defenders and non-defenders. For all $i \in \mathcal{D}$ let $A_i^N(i)$, $C_i^N(i)$, and $P_i^N(i)$ be the total numbers of defenders, non-defenders, and predators in deme i at time t measured in units of $N \in \mathbb{N}$ individuals. We will consider the large population limit $N \rightarrow \infty$. The total number of host/prey individuals in deme $i \in \mathcal{D}$ is denoted as $H_i^N = A_i^N + C_i^N$. Let $\lambda, K, \delta, \nu, \gamma, \eta, \rho \in (0, \infty)$. For every $N \in \mathbb{N}$, let $\kappa_H^N, \kappa_P^N, \alpha^N, \beta_H^N, \beta_P^N, \iota_H^N, \iota_P^N \in [0, \infty)$ satisfy $\alpha^N < \lambda$. We assume that the prey and predator populations interact in each deme according to a Lotka–Volterra model with growth rate λ , carrying capacity K , per-predator death rate δ for the prey, per-prey growth rate η , competition rate γ , and death rate ν for the predator. Furthermore, we assume that being a defender increases its death rate by α^N , and – as effect of the defense behavior – decreases the birth rate of predators in the same deme by ρ . We further assume that prey (resp. predator) individuals migrate at rate $\kappa_H^N m(i, j)$ (resp. $\kappa_P^N m(i, j)$) from deme i to deme j where $m \in [0, \infty)^{\mathcal{D} \times \mathcal{D}}$ is a symmetric stochastic matrix. In addition we assume that additional births happen at rate g_h (resp. g_p) per prey (resp. predator) individual and that additional deaths happen at the same rate g_H (resp. g_P) per prey (resp. predator) individual. Moreover, in order to avoid extinction of the prey populations on the ecological time scale, we assume immigration of prey (resp. predator) individuals at rate ι_H^N (resp. ι_P^N) into each deme. This immigration, however, is only assumed for technical reasons and does not appear in the diffusion approximation of the defender frequencies. To keep the analysis simple, we assume that the probability of an immigrating prey to be defender is equal to the current frequency of defenders. Summarizing, for every $N \in \mathbb{N}$ the process $(N \cdot A^N, N \cdot C^N, N \cdot P^N)$ is a Markov process with state space $(\mathbb{N}_0^3)^{\mathcal{D}}$ and transition rates

(where $i \in \mathcal{D}$ and $\forall k \in \mathcal{D}: a_k, c_k, p_k \in \mathbb{N}_0$):

$$\begin{aligned}
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k + 1_{k=i}, c_k, p_k)_{k \in \mathcal{D}}: & a_i(g_H + \lambda) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k - 1_{k=i}, c_k, p_k)_{k \in \mathcal{D}}: & a_i(g_H + \frac{\lambda}{K} \frac{a_i + c_i}{N} + \delta \frac{p_i}{N} + \alpha^N) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k, c_k + 1_{k=i}, p_k)_{k \in \mathcal{D}}: & c_i(g_H + \lambda) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k, c_k - 1_{k=i}, p_k)_{k \in \mathcal{D}}: & c_i(g_H + \frac{\lambda}{K} \frac{a_i + c_i}{N} + \delta \frac{p_i}{N}) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k, c_k, p_k + 1_{k=i})_{k \in \mathcal{D}}: & p_i(g_P + \eta \frac{c_i}{N} + (\eta - \rho) \frac{a_i}{N}) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k, c_k, p_k - 1_{k=i})_{k \in \mathcal{D}}: & p_i(g_P + \nu + \gamma \frac{p_i}{N}) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k - 1_{k=i} + 1_{k=j}, c_k, p_k)_{k \in \mathcal{D}}: & a_i \kappa_H^N m(i, j) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k, c_k - 1_{k=i} + 1_{k=j}, p_k)_{k \in \mathcal{D}}: & c_i \kappa_H^N m(i, j) \\
 (a_k, c_k, p_k)_{k \in \mathcal{D}} &\rightarrow (a_k, c_k, p_k - 1_{k=i} + 1_{k=j})_{k \in \mathcal{D}}: & p_i \kappa_P^N m(i, j)
 \end{aligned} \tag{2.1}$$

According to a Lotka-Volterra modeling approach and according to the usual diffusion approximation with SDEs (e.g. [Shiga and Shimizu \(1980\)](#)), A^N , C^N and P^N satisfy approximatively the stochastic differential equations (SDEs)

$$\begin{aligned}
 A_t^N(i) &= A_0^N(i) \\
 &+ \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (A_s^N(j) - A_s^N(i)) + A_s^N(i) \left[\lambda \left(1 - \frac{A_s^N(i) + C_s^N(i)}{K} \right) - \delta P_s^N(i) - \alpha^N \right] ds \\
 &+ \int_0^t \iota_H^N \frac{A_s^N(i)}{A_s^N(i) + C_s^N(i)} ds + \int_0^t \sqrt{\beta_H^N A_s^N(i)} dW_s^{A,N}(i), \\
 C_t^N(i) &= C_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (C_s^N(j) - C_s^N(i)) + C_s^N(i) \left[\lambda \left(1 - \frac{A_s^N(i) + C_s^N(i)}{K} \right) - \delta P_s^N(i) \right] ds \\
 &+ \int_0^t \iota_H^N \frac{C_s^N(i)}{A_s^N(i) + C_s^N(i)} ds + \int_0^t \sqrt{\beta_H^N C_s^N(i)} dW_s^{C,N}(i), \\
 P_t^N(i) &= P_0^N(i) + \int_0^t \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) (P_s^N(j) - P_s^N(i)) ds \\
 &+ \int_0^t P_s^N(i) [-\nu - \gamma P_s^N(i) + \eta C_s^N(i) + (\eta - \rho) A_s^N(i)] + \iota_P^N ds + \int_0^t \sqrt{\beta_P^N P_s^N(i)} dW_s^{P,N}(i),
 \end{aligned} \tag{2.2}$$

where $W^{A,N}(i), W^{C,N}(i), W^{P,N}(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}, i \in \mathcal{D}$, are independent standard Brownian motions, $\beta_H^N := 2 \frac{g_H + \lambda}{N}$ and $\beta_P^N := 2 \left(g_P + \left(\nu + \frac{\lambda \gamma \cdot (K \eta - \nu)}{\lambda \gamma + \delta K \nu} \right) \right) / N$. Note that these settings for β_H^N and β_P^N are based on the heuristic assumption that the system is close to the equilibrium of predator-prey interactions and extend the approximations of [Hutzenhaler and Metzler \(2021\)](#) to parameter-ranges in which λ, η and ν are not negligible compared to g_H and g_P .

Existence of solutions to (2.2), which we assume here, can be established in suitable Liggett-Spitzer spaces if \mathcal{D} is an Abelian group and if m is translation invariant and irreducible; cf. Proposition 2.1 in [Hutzenhaler and Wakolbinger \(2007\)](#). Finally, for our analysis we assume that $\rho < \eta$ and we set $a = \frac{\lambda \gamma + \delta K \eta}{\delta K \rho}$ and $b = \frac{\delta \rho}{\delta \nu + \lambda \gamma}$ (which appear in the equilibrium state (2.4) for prey and predators).

2.2. *Computer simulations.* To illustrate the model and the roles of certain model parameters, we present here the results of computer simulations for a setting with finite-size populations in 1000 demes. When nothing else is stated, we used the parameter settings $\lambda = 2, K = 1000, \delta = 0.02,$

$\alpha := \alpha^N = 0.01$, $g_P = 0$, $\eta = 0.005$, $\rho = 0.004$, $\nu = 1$, $\gamma = 0.01$, $\kappa_H^N = \kappa_P^N = 0.01$ and $\iota_{H,\text{defend.}}^N = \iota_{H,\text{non-defend.}}^N = \iota_P^N = 10^{-6}$ without population-size scaling, that is with $N = 1$. For the parameter g_H we used the values 0, 0.5 and 5, resulting in the values of 0.008, 0.01 and 0.028 of $\beta := \beta_H^N N \frac{\delta\rho}{\delta\nu + \lambda\gamma}$, whose analogue in the asymptotic model, the β defined in Theorem 2.3, will turn out to be the benefit-of-defense parameter that is crucial for the fixation probability of the defense allele (Theorem 2.4). We initialized each deme with 1000 prey individuals, of which a uniformly drawn random fraction x were defenders. The number of predators in the deme was then initialized with $\frac{\lambda}{\delta} \cdot (1 - 1/(Kb \cdot (a - x)))$, which is inspired by the equilibrium frequency of the Lotka–Volterra model.

For the simulations we applied a τ -leaping approach (Gillespie, 2001). For this, we chose a time span τ and iterated simulation steps in which we simulated for each deme, each deme sub-population (defenders, non-defenders or predator) of current size n and each type of event (birth, death or emigration) of per-individual rate r a binomial number with parameters $(n, p = \tau \cdot r \cdot n)$ of the corresponding events to take place in the next time span of length τ . We chose τ small enough to make the binomial-distribution parameter $p = \tau \cdot r \cdot n$ smaller than 0.01 under most conditions. In most simulations we performed 50,000,000 τ -leap iterations and read out the total numbers of defenders, non-defenders and predators once every 200,000 iterations.

Figure 2.1 shows results from five simulation runs with $\alpha = 0.01$ and $\beta = 0.028$. In three of these runs, the defenders became way more frequent than the non-defenders, which is in accordance with our asymptotic results as $\alpha < \beta$. At some time points it happened, however, that the predators became rare, which led to a decrease of the number of defenders and an increase of the number of non-defenders. In some cases (Fig. 2.1 top and Fig. 2.1 middle row left) this resulted in an increase of the number of predators, which entailed that the defenders became much more frequent again. In two simulation runs, however, the defenders went extinct before the predators returned (Fig. 2.1 bottom). At the end of the simulated time span, no defenders were present in these simulation runs and non-defenders as well as predators were present in all demes. The final mean numbers of non-defenders per deme were 334.5 and 337.7 in the two simulations and the mean number of predators were 65.63 and 66.04. Note that these values are not far from the limits for the number $\frac{K \cdot (\delta\nu + \gamma\lambda)}{\lambda\gamma + K\delta\eta} \approx 333.3$ of prey individuals and the number $\frac{\lambda K\eta - \lambda\nu}{\lambda\gamma + \delta K\eta} \approx 66.6$ of predators according to the classical Lotka–Volterra model with within-species competition (and without migration).

Figure 2.2 shows simulation results with $g_H = 0.5$ and thus $\beta = 0.01 = \alpha$. In accordance with the asymptotic result for $\alpha = \beta$, the non-defenders co-existed with the defenders over a long time span. In three of the five simulation runs, however, the defenders went extinct and were not able to re-immigrate and spread during the simulated time span.

In five simulations with $\beta = 0.008$ and $\alpha = 0.01$ the defenders quickly went extinct and were not able to re-immigrate and spread in the population (Fig. 2.3). This changed, however, when we reduced the cost of defense to $\alpha = 0.001 < \beta$ (Figure 2.4). In these simulations the defenders became more frequent than the non-defenders and it happened many times that the predators became rare, which resulted in an increase in the frequencies of non-defenders and a decrease in the frequencies of defenders. But the latter happened more slowly than in simulations with a higher defense cost α , such that the defenders were still abundant when the predators returned and were able to take over again.

To study the role of between-deme migration we launched simulation runs without it, that is, with $\kappa_H^N = \kappa_P^N = 0$. In the simulation runs with $\alpha = 0.01$ and $\beta = 0.008$ or $\beta = 0.01$ we observe that demes were in the end of the simulation runs either populated with defenders and neither non-defenders nor predators or with non-defenders and predators and no defenders. The fraction of demes with defenders was 26.14 % in the simulations with $\beta = 0.008$ and it was 30.3 % in the simulations with $\beta = 0.01$. The average numbers of individuals in these demes were 992.59 and 994.15, respectively. In the demes that were populated by non-defenders, the average final

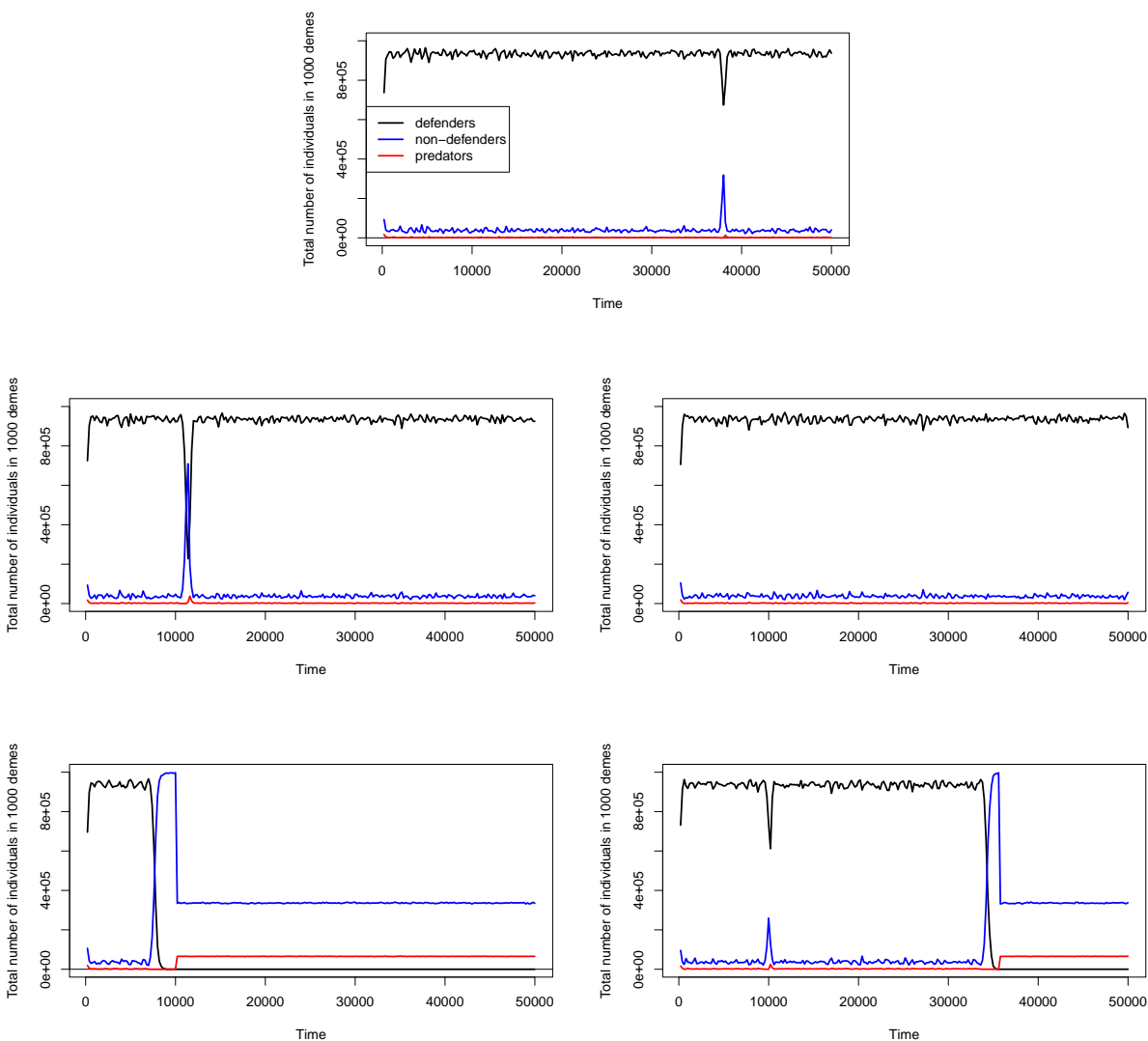


FIGURE 2.1. Results from five computer simulation runs with $\alpha = 0.01$ and $\beta = 0.028$.

prey population sizes were 394.58 and 385.6 per deme, and the average final predator per-deme population sizes were 60.28 and 60.96. Thus, averaged over the five simulation runs with $\beta = 0.008$ the total number of defenders was slightly lower than that of the non-defenders ($2.6 \cdot 10^5$ vs. $2.9 \cdot 10^5$), whereas it was higher than the total number of non-defenders for $\beta = 0.01$ ($3 \cdot 10^5$ vs. $2.7 \cdot 10^5$).

Some of the demes in which one or the other type – defenders or non-defenders – became fixed may have started with very high initial frequencies in their deme. To explore how the fixation chances of the costly defense trait are when *not* starting at high frequencies, we launched additional no-migration simulations in which all demes were initialized with 500 defenders, 500 non-defenders and 50 predators (Fig. 2.5). As we observed quick convergence in the previous no-migration simulations, we only simulated 5,000,000 τ leap iterations in each of the new runs.

In the simulations in which β was larger than the cost α , the final total frequencies of the defenders were still higher than those of the non-defenders (Fig. 2.5, left column). In the cases with $\beta \leq \alpha$ the final total frequencies of the non-defenders were higher than those of the defenders, which, however, did not go extinct (Fig. 2.5, right column). In all those simulations there was at the

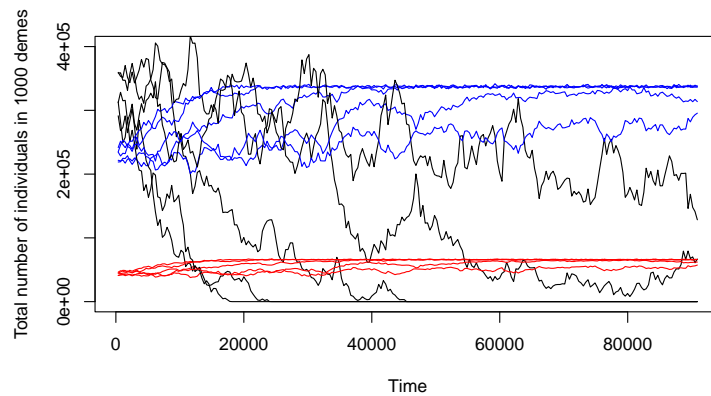


FIGURE 2.2. Total frequencies of defenders (black), non-defenders (blue) and predators (red) in five computer simulation runs with $\alpha = \beta = 0.01$.

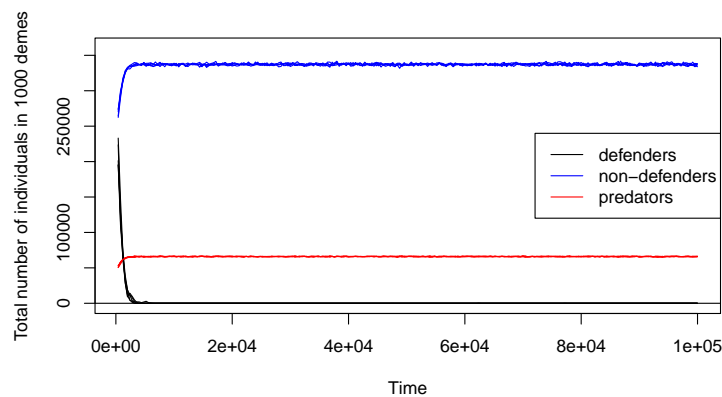


FIGURE 2.3. Five computer simulation runs with $\alpha = 0.01$ and $\beta = 0.008$.

end a strict dichotomy of the demes: A certain fraction of demes was populated by defenders and neither non-defenders nor predators and all other demes contained non-defenders and predators but no defenders. In all cases, there were fewer demes with defenders than without defenders (Table 2.1), but in the simulations with $\alpha \leq \beta$, more than 40 % of the demes contained defenders and the smaller numbers of demes was overcompensated by the larger average number of inhabitants (almost 1000 in defender demes versus 350 to 380 in non-defender demes).

The results of the no-migration simulations illustrate that larger values of β are beneficial for the defenders because the random variation within a deme (“genetic drift”) increases the probability that the defenders can become fixed despite their selective disadvantage compared to the non-defenders. This suggests that also in scenarios with migration between the demes random variation may not only be beneficial for the spread of the defense trait because it leads to variation on the deme level but also because it increases the probability that the locally less fit trait survives in a deme until a deme-level fitness advantage emerges.

In another series of simulations (with migration again) we removed within-species competition from the model, that is we set $K = \infty$ and $\gamma = 0$. In these simulations all demes were initialized with independently drawn numbers between 0 and 300 for each sub-population. The number of τ

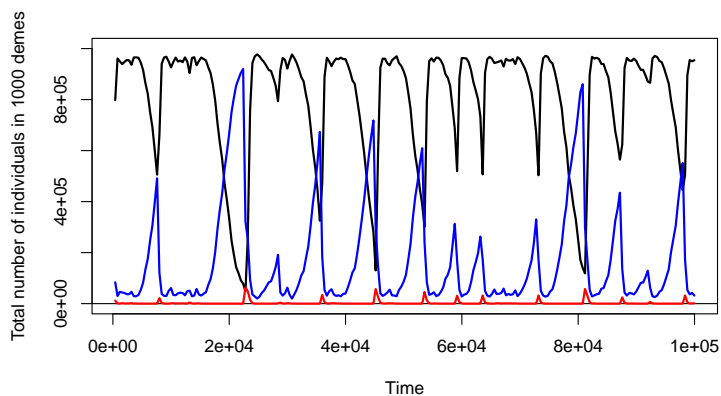


FIGURE 2.4. Total frequencies of defenders (black), non-defenders (blue) and predators (red) in a computer simulation with $\alpha = 0.001$ and $\beta = 0.008$. For visual clarity only one simulation run is shown but the other four gave similar results.

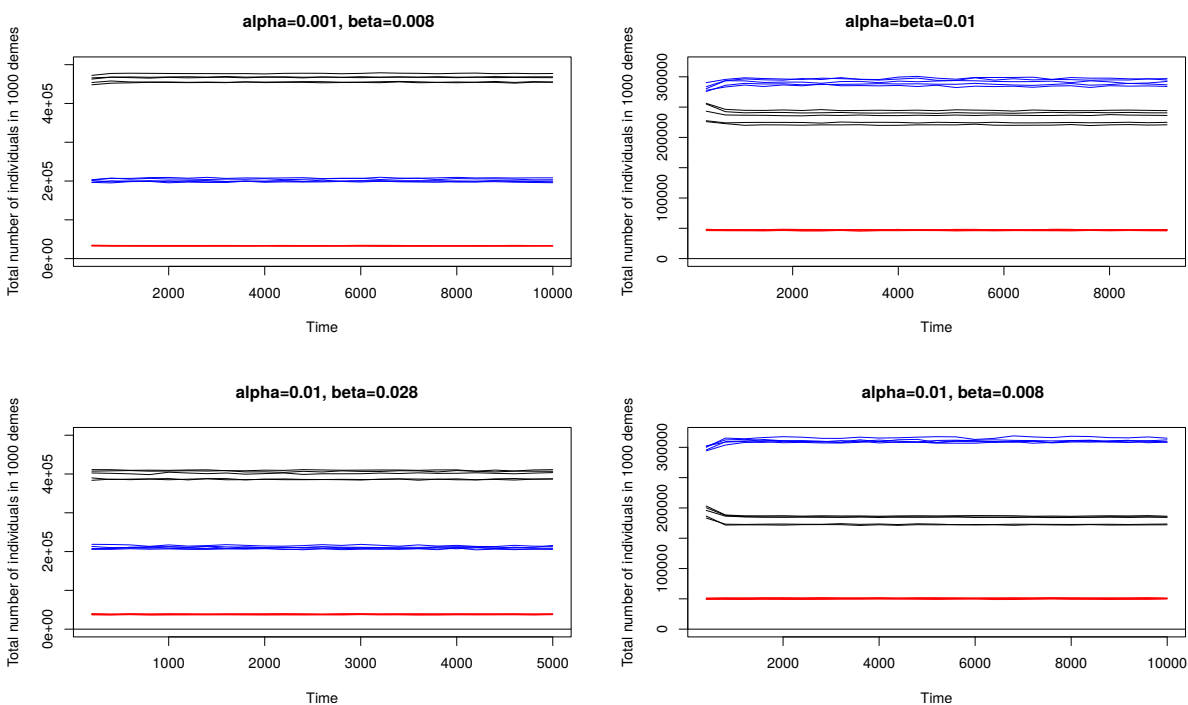


FIGURE 2.5. Total numbers of defenders (black), non-defenders (blue) and predators (red) in computer simulation runs with with no migration and all demes starting with 500 defenders, 500 non-defenders and 50 predators.

leap iterations was 1,000,000 with output every 10,000-th iteration. Even though we used $\alpha = 0.01$ and $\beta = 0.008$ in these simulations, the defenders became more frequent and the non-defenders went extinct (Fig. 2.6). In the classical deterministic Lotka–Volterra model without competition (and without migration) the frequencies of prey and predators oscillate infinitely long around equilibrium

α	β	% demes with defenders	def.	non-def.	pred.
0.001	0.008	46.5	998.5	375.7	62.0
0.01	0.028	40.2	993.2	351.2	64.2
0.01	0.01	23.5	993.6	380.3	61.5
0.01	0.008	18.1	993.9	379.6	61.8

TABLE 2.1. Final states of simulations without migration, starting with 500 defenders, 500 non-defenders and 50 predators in each deme. def.: average number of defenders in demes with defenders, non-def.: average number of non-defenders in demes without defenders, pred.: average number of predators in demes without defenders.

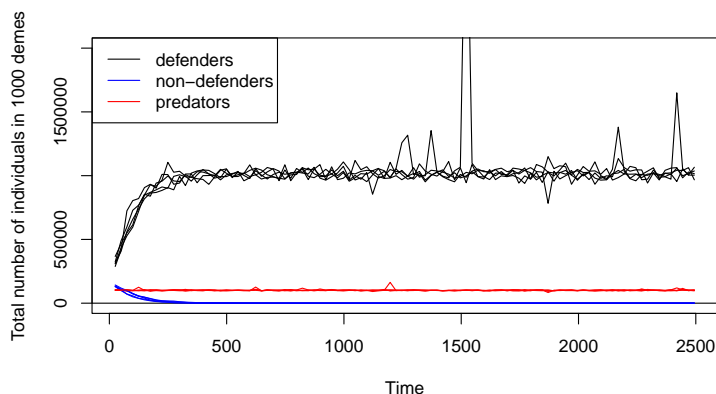


FIGURE 2.6. Five computer simulation runs with $\alpha = 0.01$, $\beta = 0.008$ and no within-species competition. (The hidden peak of the defender numbers in one of the simulation reaches almost 3.64 Mio.)

frequencies. Thus, a possible explanation why removing within-species competition brings an advantage for the defenders over the non-defenders is that oscillations may be maintained over a longer period, increasing between-deme variation of defender frequencies. Another aspect is, however, that removing within-deme kin competition from the model can make kin selection more effective.

In our simulations the final average population sizes per deme were 1017.8 for the prey and 100.98 for the predators. If we for comparison consider a simple Lotka–Volterra model for a single deme without competition and with all prey individuals being defenders, the equilibrium frequencies are $\frac{\nu}{\eta-\rho} = 1000$ for the prey and $\frac{\lambda-\alpha}{\delta} = 99.5$ for the predators.

2.3. *Diffusion equation for frequencies of defenders.* We denote by

$$F_t^N(i) := \frac{A_t^N(i)}{A_t^N(i) + C_t^N(i)} \quad (2.3)$$

the frequency of defenders in the prey population in deme $i \in \mathcal{D}$ at time $t \in [0, \infty)$. The central goal of this article is to prove convergence of the sequence $((F_t^N)_{t \in [0, \infty)})_{N \in \mathbb{N}}$ and to derive the diffusion equation which the limit solves. In other words, we will derive an analog of the Kimura stepping stone model (i.e., spatially structured Wright–Fisher diffusions) for costly defense against predators. Since we measure time in units of N individuals (evolutionary time scale) and the total migrating mass on this time scale should be finite, we need to assume that β_H^N and β_P^N are of order $\frac{1}{N}$ for large $N \in \mathbb{N}$. To get a nontrivial diffusion approximation we additionally assume – as is usual in

the derivation of the Kimura stepping stone model – slow migration and weak selection in the sense that the sequences $(N\kappa_H^N)_{N \in \mathbb{N}}$, $(N\kappa_P^N)_{N \in \mathbb{N}}$ and $(N\alpha^N)_{N \in \mathbb{N}}$ converge. Thus the relative frequency of defenders $\frac{A^N}{A^N + C^N}$ evolves on the time scale of order N as $N \rightarrow \infty$.

In the special case that for some $N \in \mathbb{N}$ it holds that $\kappa_H^N = \kappa_P^N = \iota_H^N = \iota_P^N = \alpha^N = \beta_H^N = \beta_P^N = A_0^N = 0$, then $A^N \equiv 0$ and $(H^N(i), P^N(i))$, $i \in \mathcal{D}$, satisfy classical Lotka–Volterra equations. It is well known that if $K\eta > \nu$, then the solutions of these equations converge to the nontrivial equilibrium $(\frac{K(\delta\nu + \gamma\lambda)}{\lambda\gamma + K\delta\eta}, \frac{\lambda K\eta - \lambda\nu}{\lambda\gamma + \delta K\eta}) \in (0, \infty)^2$ in each deme. Since we assume that $\kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N$ are of order $o(1)$ as $N \rightarrow \infty$ and since the defender frequencies evolve slowly, for every $i \in \mathcal{D}$, the processes $(H^N(i), P^N(i))$ should asymptotically be close to the equilibrium of the classical Lotka–Volterra equations with η being replaced by $\eta - \rho F^N(i)$ as $N \rightarrow \infty$. More precisely, we will prove in Theorem 2.2 below under further assumptions that if the local frequency of defenders is $q \in [0, 1]$, then the equilibrium state for prey and predators should be $(h_\infty(q), p_\infty(q))$ where the functions h_∞ and p_∞ are defined by

$$\begin{aligned} [0, 1] \ni x \mapsto h_\infty(x) &:= \frac{K(\delta\nu + \gamma\lambda)}{\lambda\gamma + \delta K(\eta - \rho x)} = \frac{1}{b(a-x)} \in (0, \infty) \\ [0, 1] \ni x \mapsto p_\infty(x) &:= \frac{\lambda K(\eta - \rho x) - \lambda\nu}{\lambda\gamma + \delta K(\eta - \rho x)} = \frac{\lambda}{\delta} \left(1 - \frac{1}{Kb(a-x)} \right) \in (0, \infty). \end{aligned} \tag{2.4}$$

For these functions to be well defined we will assume that $Kb(a - 1) > 1$ or, equivalently, that $K(\eta - \rho) > \nu$.

The above heuristic is incorrect if all populations go extinct by chance due to stochasticity in the offspring distributions. To avoid this difficulty we will assume that there is sufficient immigration of prey ($2\iota_H^N \geq \beta_H^N$) and predators ($2\iota_P^N \geq \beta_P^N$) in order that both prey populations and predator populations cannot go extinct; see Lemmas 3.2 and 3.3, respectively. However, note that both defenders and non-defenders can locally die out. For our proof, which is based on the Lyapunov function (3.21), we additionally require further restrictions on the parameters and on (inverse) moments of the initial configuration.

Assumption 2.1. *In the setting of the Section 2.1 it holds that $\lambda > \nu$, $\eta - \rho > \frac{\lambda}{K}$, $\gamma \geq 2\delta$, for all $N \in \mathbb{N}$ it holds that $\alpha^N + \kappa_H^N \leq \frac{\lambda}{4}$, $\iota_P^N \leq \frac{\lambda(\nu + \lambda)}{8\delta}$, $\kappa_P^N + \kappa_H^N + \alpha^N \leq \frac{\lambda - \nu}{2}$, $\iota_H^N \geq \frac{4\delta\kappa_P^N}{3(\nu + \lambda)} + \frac{3}{2}\beta_H^N$, $\iota_P^N \geq \beta_P^N$, and there exist $\sigma = (\sigma_i)_{i \in \mathcal{D}} \in (0, \infty)^{\mathcal{D}}$ and $c \in (0, \infty)$ such that $\sum_{i \in \mathcal{D}} \sigma_i < \infty$, such that for every $j \in \mathcal{D}$ it holds that*

$$\sum_{i \in \mathcal{D}} \sigma_i m(i, j) \leq c\sigma_j, \tag{2.5}$$

and such that $\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| \left(H_0^N + P_0^N \right)^4 + \frac{1}{(H_0^N)^2} + \frac{P_0^N}{(H_0^N)^2} + \frac{1}{P_0^N} + \frac{1}{P_0^N H_0^N} \right\|_{\sigma} \right] < \infty$.

The following theorem, which appears to be new even for non-spatial Lotka–Volterra SDEs, implies for every $t \in [0, \infty)$ that the $L^2([0, t] \times l_\sigma^1 \times \Omega; \mathbb{R})$ -distance between $(H_{\cdot N}^N, P_{\cdot N}^N)$ and $(h_\infty(F_{\cdot N}^N), p_\infty(F_{\cdot N}^N))$ converges to 0 as $N \rightarrow \infty$ at least with rate $\frac{1}{2}$. Theorem 2.2 follows immediately from Theorem 3.8 below together with a time substitution. For the proof of Theorem 2.2 below, we exploit uniformly bounded inverse moments and establishing these inverse moments (see Lemmas 3.6 and 3.7 below) is somewhat technical.

Theorem 2.2. *Assume the setting of Section 2.1, let Assumption 2.1 hold, let h_∞ and p_∞ be given by (2.4) and assume that $\sup_{N \in \mathbb{N}} (N \max\{\kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N\}) < \infty$. Then we get for all $t \in [0, \infty)$ that*

$$\sup_{N \in \mathbb{N}} N \int_0^t \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (H_{uN}^N(i) - h_\infty(F_{uN}^N(i)))^2 + \sum_{i \in \mathcal{D}} \sigma_i (P_{uN}^N(i) - p_\infty(F_{uN}^N(i)))^2 \right] du < \infty. \tag{2.6}$$

Knowing the asymptotic behavior of the prey populations, we can formally replace the $(H^N)_{N \in \mathbb{N}}$ in the diffusion equation of the defender frequencies (see (3.4) below) and, thereby, we arrive at the diffusion equation solved by the limit of defender frequencies. Our main result, Theorem 2.3, then proves that the defender frequencies converge to the solution of the diffusion equation (2.8) below. The proof of Theorem 2.3 is deferred to Section 3.4.2 below and is based on a general stochastic averaging result in Kurtz (1992).

Theorem 2.3. *Assume the setting of Section 2.1, let Assumption 2.1 hold, assume that there exist $\kappa, \alpha, \beta \in [0, \infty)$ such that $\lim_{N \rightarrow \infty} \kappa_H^N N = \kappa$, $\lim_{N \rightarrow \infty} \alpha^N N = \alpha$ and $\lim_{N \rightarrow \infty} \beta_H^N N b = \beta$, assume that*

$$\left(\sum_{i \in \mathcal{D}} \sigma_i \sup_{N \in \mathbb{N}} \mathbb{E} \left[H_0^N(i) \right] \right) + \sup_{N \in \mathbb{N}} (N \max \{ \kappa_P^N, \iota_H^N, \iota_P^N, \beta_P^N \}) < \infty \tag{2.7}$$

and assume that $F_0^N \Rightarrow X_0$ as $N \rightarrow \infty$ in l_σ^1 . Then the SDE

$$\begin{aligned} dX_t(i) = & \kappa \sum_{j \in \mathcal{D}} m(i, j) \frac{a - X_t(i)}{a - X_t(j)} \left(X_t(j) - X_t(i) \right) dt - \alpha X_t(i) (1 - X_t(i)) dt \\ & + \sqrt{\beta (a - X_t(i)) X_t(i) (1 - X_t(i))} dW_t(i), \quad t \in (0, \infty), i \in \mathcal{D} \end{aligned} \tag{2.8}$$

(where $\{W(i) : i \in \mathcal{D}\}$ are independent standard Brownian motions) has a unique strong solution and

$$(F_{tN}^N)_{t \in [0, \infty)} \Rightarrow (X_t)_{t \in [0, \infty)} \tag{2.9}$$

as $N \rightarrow \infty$ in $C([0, \infty), l_\sigma^1)$.

2.4. Many-demes limit. An important problem is to derive conditions under which defenders persist, that is, to derive conditions on the parameters of the SDE (2.8) under which the process goes to fixation. Here we simplify this problem and consider the many-demes-limit (also denoted as mean-field approximation) of the SDE (2.8). More precisely, for every $D \in \mathbb{N}$, let $X^D : [0, \infty) \times \{1, \dots, D\} \times \Omega \rightarrow [0, 1]$ be the solution of the SDE (2.8) with \mathcal{D} replaced by $\{1, \dots, D\}$ and with m replaced by $(\frac{1}{D})_{i, j \in \{1, \dots, D\}}$. We will show in Proposition 4.1 together with Lemma 4.2 below that if, for every $D \in \mathbb{N}$, $(X_0^D(i))_{i \in \{1, \dots, D\}}$ are exchangeable $[0, 1]$ -valued random variables, if $\sup_{D \in \mathbb{N}} \mathbb{E}[(X_0^D(1))^2] < \infty$, if $Z : [0, \infty) \times \Omega \rightarrow [0, 1]$ is the solution of the SDE (2.11) below with respect to the Brownian motion $W(1)$ and if $\sup_{D \in \mathbb{N}} \sqrt{D} \mathbb{E}[|X_0^D(i) - Z_0(i)|] < \infty$, then for all $t \in [0, \infty)$ it holds that

$$\sup_{D \in \mathbb{N}} \sqrt{D} \mathbb{E}[|X_t^D(1) - Z_t|] < \infty. \tag{2.10}$$

Thus the solution of the SDE (2.11) is the many-demes limit of the SDE (2.8). For this many-demes limit we derive a simple necessary and sufficient condition ($\alpha < \beta$) under which the costly defense trait goes to fixation when starting with a positive frequency. The proof of Theorem 2.4 is deferred to Section 5.3.

Theorem 2.4. *Let $\alpha, \beta, \kappa \in (0, \infty)$, let $a \in (1, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, \infty)})$ be a filtered probability space, let $W : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t \in [0, \infty)}$ Brownian motion with continuous sample paths, and let $Z_0 : \Omega \rightarrow [0, 1]$ be an $\mathcal{F}_0/\mathcal{B}([0, 1])$ -measurable mapping. Then the SDE*

$$dZ_t = \kappa(a - Z_t) \left((a - Z_t) \mathbb{E} \left[\frac{1}{a - Z_t} \right] - 1 \right) dt - \alpha Z_t (1 - Z_t) dt + \sqrt{\beta(a - Z_t) Z_t (1 - Z_t)} dW_t \tag{2.11}$$

has a unique solution. Furthermore, if $\mathbb{E}[Z_0] = 1$, then $\mathbb{P}[Z_t = 1 \text{ for all } t \in [0, \infty)] = 1$, if $\mathbb{E}[Z_0] = 0$, then $\mathbb{P}[Z_t = 0 \text{ for all } t \in [0, \infty)] = 1$ and if $\mathbb{E}[Z_0] \in (0, 1)$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}[|Z_t - 0|] &= 0, \text{ if } \alpha > \beta, \\ \lim_{t \rightarrow \infty} \mathbb{E}[|Z_t - 1|] &= 0, \text{ if } \alpha < \beta, \\ Z_t &\xrightarrow{t \rightarrow \infty} \int m(z) dz, \text{ if } \alpha = \beta, \end{aligned} \tag{2.12}$$

where $m(z) = \frac{1}{c} z^{\frac{2\kappa}{\beta}(a\theta-1)-1} (1-z)^{\frac{2\kappa}{\beta}(1-\theta(a-1))-1} (a-z)^{\frac{2\alpha}{\beta}-1}$ for $z \in (0, 1)$, where $c \in (0, \infty)$ is a normalizing constant and where $\theta = \mathbb{E}[\frac{1}{a-Z_0}]$.

Informally speaking, Theorem 2.4 asserts that a costly defense allele persists in a space of infinitely many demes if $\alpha < \beta$ and if the mean frequency of defenders over all demes is positive. This does not imply that a new mutation resulting in costly defense behavior can establish itself on one island or even in the total population. Our final result partially closes this gap and considers a process which could be the limit $\lim_{D \rightarrow \infty} \sum_{i=1}^D X^D(i)$ if for all $D \in \mathbb{N}$ and $i \in \{1, \dots, D\}$ it holds that $X_0^D(i) = Y_0 \mathbb{1}_{i=1}$ for some $[0, 1]$ -valued random variable (Hutzenthaler, 2009, 2012). For this limiting process, Proposition 6.1 below shows in the case $\mathbb{P}[Y_0 > 0] = 1$ that the process converges to 0 in probability as time goes to infinity if and only if $\alpha \geq \beta$. Informally speaking, Proposition 6.1 asserts that a costly defense allele has a positive invasion probability in an infinite-dimensional space if and only if $\alpha < \beta$.

3. Convergence of the relative frequency of defenders

The main result of this section, Theorem 3.8 below, proves strong convergence of spatial stochastic Lotka–Volterra processes. Its proof uses a Lyapunov type argument. This argument uses uniformly bounded inverse moments which we establish in Section 3.3. In section 3.4.2, we then prove convergence of relative frequencies as stated in Theorem 2.3.

3.1. *Setting.* Assume the setting of Section 2.1, Define $\bar{\kappa}_H := \sup_{N \in \mathbb{N}} \kappa_H^N$, $\bar{\kappa}_P := \sup_{N \in \mathbb{N}} \kappa_P^N$, $\bar{\beta}_H := \sup_{N \in \mathbb{N}} \beta_H^N$, $\bar{\beta}_P := \sup_{N \in \mathbb{N}} \beta_P^N$, $\bar{\iota}_H := \sup_{N \in \mathbb{N}} \iota_H^N$, and $\bar{\iota}_P := \sup_{N \in \mathbb{N}} \iota_P^N$. For all $z = (z_i)_{i \in \mathcal{D}} \in (0, \infty)^{\mathcal{D}}$ and $p \in \mathbb{R}$ let $z^p = (z_i^p)_{i \in \mathcal{D}}$. Furthermore, let $\underline{1} := (1)_{i \in \mathcal{D}} \in l_\sigma^1$. Define $E_1 := [0, 1]^{\mathcal{D}}$ and $E_2 := l_\sigma^1 \cap [0, \infty)^{\mathcal{D}}$. For all $i \in \mathcal{D}$ and all $N \in \mathbb{N}$ let $W^{H,N}(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$ and $W^{F,N}(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be stochastic processes with continuous sample paths such that for every $t \in [0, \infty)$ it holds \mathbb{P} -a.s. that

$$dW_t^{H,N}(i) = \frac{\sqrt{A_t^N(i)} dW_t^{A,N}(i) + \sqrt{C_t^N(i)} dW_t^{C,N}(i)}{\sqrt{H_t^N(i)}} \tag{3.1}$$

and

$$dW_t^{F,N}(i) = \frac{\sqrt{C_t^N(i)} dW_t^{A,N}(i) - \sqrt{A_t^N(i)} dW_t^{C,N}(i)}{\sqrt{H_t^N(i)}}, \tag{3.2}$$

respectively, with $W_0^{H,N}(i) = W_0^{F,N}(i) = 0$.

3.2. *Preliminaries.* Assume the setting of Section 3.1. In this section we collect some first results that are used in the proofs of the statements in subsequent sections.

Lemma 3.1. *Assume the setting of Section 3.1. Then $W^{H,N}(i)$ and $W^{F,N}(i)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$, are independent Brownian motions and for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ it \mathbb{P} -a.s. holds that*

$$H_t^N(i) = H_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_s^N(j) + (\lambda - \kappa_H^N - \alpha^N F_s^N(i)) H_s^N(i) - \frac{\lambda}{K} (H_s^N(i))^2 \tag{3.3}$$

$$- \delta P_s^N(i) H_s^N(i) + \iota_H^N ds + \int_0^t \sqrt{\beta_H^N H_s^N(i)} dW_s^{H,N}(i),$$

$$F_t^N(i) = F_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_s^N(j) - F_s^N(i)) \frac{H_s^N(j)}{H_s^N(i)} - \alpha^N F_s^N(i) (1 - F_s^N(i)) ds \tag{3.4}$$

$$+ \int_0^t \sqrt{\frac{\beta_H^N F_s^N(i)(1-F_s^N(i))}{H_s^N(i)}} dW_s^{F,N}(i),$$

$$P_t^N(i) = P_0^N(i) + \int_0^t \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_s^N(j) - (\kappa_P^N + \nu) P_s^N(i) - \gamma (P_s^N(i))^2 \tag{3.5}$$

$$+ (\eta - \rho F_s^N(i)) P_s^N(i) H_s^N(i) + \iota_P^N ds + \int_0^t \sqrt{\beta_P^N P_s^N(i)} dW_s^{P,N}(i).$$

Proof: For all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$ we get $\langle W^{H,N}(i) \rangle_t = \langle W^{F,N}(i) \rangle_t = t$ as well as

$$\langle W^{H,N}(i), W^{F,N}(i) \rangle_t = \int_0^t \frac{\sqrt{A_s^N(i)C_s^N(i)} - \sqrt{A_s^N(i)C_s^N(i)}}{H_s^N(i)} ds = 0. \tag{3.6}$$

Hence, we see that $W^{H,N}(i)$ and $W^{F,N}(i)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$, are independent Brownian motions. Equation (3.3) follows from Itô’s lemma (e.g. Klenke, 2008) and rearranging terms. Furthermore, applying Itô’s lemma we see for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ that \mathbb{P} -a.s. it holds that

$$F_t^N(i) = F_0^N(i) + \int_0^t \frac{C_s^N(i)}{(H_s^N(i))^2} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (A_s^N(j) - A_s^N(i)) + A_s^N(i) \left(\lambda \left(1 - \frac{H_s^N(i)}{K} \right) - \delta P_s^N(i) - \alpha^N \right) + \iota_H^N \frac{A_s^N(i)}{H_s^N(i)} \right) ds + \int_0^t \frac{C_s^N(i)}{(H_s^N(i))^2} \sqrt{\beta_H^N A_s^N(i)} dW_s^A(i) - \int_0^t \frac{A_s^N(i)}{(H_s^N(i))^2} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (C_s^N(j) - C_s^N(i)) + C_s^N(i) \left(\lambda \left(1 - \frac{H_s^N(i)}{K} \right) - \delta P_s^N(i) \right) + \iota_H^N \frac{C_s^N(i)}{H_s^N(i)} \right) ds - \int_0^t \frac{A_s^N(i)}{(H_s^N(i))^2} \sqrt{\beta_H^N C_s^N(i)} dW_s^C(i) - \int_0^t \frac{C_s^N(i)}{(H_s^N(i))^3} \beta_H^N A_s^N(i) + \frac{A_s^N(i)}{(H_s^N(i))^3} \beta_H^N C_s^N(i) ds = F_0^N(i) + \int_0^t \frac{\kappa_H^N}{H_s^N(i)} \sum_{j \in \mathcal{D}} m(i, j) \left((1 - F_s^N(i)) F_s^N(j) H_s^N(j) - F_s^N(i) (1 - F_s^N(j)) H_s^N(j) - \alpha^N F_s^N(i) (1 - F_s^N(i)) \right) ds + \int_0^t \sqrt{\frac{\beta_H^N F_s^N(i)(1-F_s^N(i))}{H_s^N(i)}} dW_s^{F,N}(i) \tag{3.7}$$

and (3.4) follows. Finally, we obtain (3.5) from the definition of $(H^N)_{N \in \mathbb{N}}$ and $(F^N)_{N \in \mathbb{N}}$. \square

Lemma 3.2. *Assume the setting of Section 3.1 and assume that for all $N \in \mathbb{N}$ we have $\iota_H^N \geq \frac{1}{2}\beta_H^N$. Furthermore, assume that we have for all $N \in \mathbb{N}$ and all $i \in \mathcal{D}$ that \mathbb{P} -a.s. $H_0^N(i) > 0$. Then we have*

$$\mathbb{P} [H_u^N(i) > 0, \text{ for all } u \in [0, \infty), \text{ all } N \in \mathbb{N}, \text{ and all } i \in \mathcal{D}] = 1. \tag{3.8}$$

Proof: The main problem is to control the contribution of the processes $(P^N)_{N \in \mathbb{N}}$ and resolve this with a cut-off argument. For every $N, M \in \mathbb{N}$ let $\hat{H}^{N,M} : [0, \infty) \times \mathcal{D} \times \Omega \rightarrow [0, \infty)$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \mathcal{D}$ satisfies \mathbb{P} -a.s.

$$\begin{aligned} \hat{H}_t^{N,M}(i) &= \hat{H}_0^{N,M}(i) + \int_0^t \left[\hat{H}_s^{N,M}(i) \left(\lambda - \alpha^N - \kappa_H^N - \frac{\lambda}{K} \hat{H}_s^{N,M}(i) - \delta M \right) + \iota_H^N \right] ds \\ &\quad + \int_0^t \sqrt{\beta_H^N \hat{H}_s^{N,M}(i)} dW_s^{H,N}(i) \end{aligned} \tag{3.9}$$

with $\hat{H}_0^{N,M}(i) = H_0^N(i)$. Due to Feller’s boundary classification (e.g., p. 366 in [Ethier and Kurtz, 1986](#)) with the assumption that for all $N \in \mathbb{N}$ it holds that $\iota_H^N \geq \frac{1}{2}\beta_H^N$ we have for every $N, M \in \mathbb{N}$ and all $i \in \mathcal{D}$ that

$$\mathbb{P} \left[\hat{H}_t^{N,M}(i) > 0, \text{ for all } t \in [0, \infty) \right] = 1. \tag{3.10}$$

For all $N, M \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ consider the event $A_M^N(i) := \left\{ \sup_{s \in [0,t]} P_s^N(i) \leq M \right\}$.

The fact that the processes $P^N(i)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$, have a.s. càdlàg sample paths (which are bounded on closed intervals) implies for all $N, M \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that

$$\begin{aligned} A_M^N(i) &\subseteq A_{M+1}^N(i), \\ \mathbb{P} \left[\bigcup_{K \in \mathbb{N}} A_K^N(i) \right] &= \mathbb{P} \left[\sup_{s \in [0,t]} P_s^N(i) < \infty \right] = 1. \end{aligned} \tag{3.11}$$

Using a comparison result due to Ikeda and Watanabe (see e.g., Theorem V.43.1 in [Rogers and Williams, 2000](#)) we get for all $N, M \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that

$$\mathbb{P} \left[\exists u \in [0, t] : H_u^N(i) < \hat{H}_u^{N,M}(i), \sup_{s \in [0,t]} P_s^N(i) \leq M \right] = 0. \tag{3.12}$$

Thus, combining (3.10), (3.11), and (3.12) we obtain for all $N \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that

$$\begin{aligned} 1 &\geq \mathbb{P} [H_u^N(i) > 0, \text{ for all } u \in [0, t]] = 1 - \mathbb{P} [\exists u \in [0, t] : H_u^N(i) = 0] \\ &\geq 1 - \sum_{M \in \mathbb{N}} \mathbb{P} \left[\exists u \in [0, t] : H_u^N(i) = 0, \sup_{s \in [0,t]} P_s^N(i) \leq M \right] \\ &\geq 1 - \sum_{M \in \mathbb{N}} \mathbb{P} \left[\exists u \in [0, t] : H_u^N(i) < \hat{H}_u^{N,M}(i), \sup_{s \in [0,t]} P_s^N(i) \leq M \right] = 1. \end{aligned} \tag{3.13}$$

This implies for all $N \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that $\mathbb{P} [H_u^N(i) > 0, \text{ for all } u \in [0, t]] = 1$, which in turn implies (3.8). This completes the proof of Lemma 3.2. \square

Lemma 3.3. *Assume the setting of Section 3.1 and assume that for all $N \in \mathbb{N}$ it holds that $\iota_P^N \geq \frac{1}{2}\beta_P^N$. Furthermore, assume that we have for all $N \in \mathbb{N}$ and all $i \in \mathcal{D}$ that \mathbb{P} -a.s. $P_0^N(i) > 0$. Then we have*

$$\mathbb{P} [P_t^N(i) > 0, \text{ for all } t \in [0, \infty), \text{ all } N \in \mathbb{N}, \text{ and all } i \in \mathcal{D}] = 1. \tag{3.14}$$

Proof: Analogous to the proof of Lemma 3.2. \square

Lemma 3.4. *Let \mathcal{D} be a countable set, let $m \in [0, \infty)^{\mathcal{D} \times \mathcal{D}}$ be a stochastic matrix, let $\sigma \in [0, \infty)^{\mathcal{D}}$, $c \in [0, \infty)$ satisfy for all $j \in \mathcal{D}$ that*

$$\sum_{i \in \mathcal{D}} \sigma_i m(i, j) \leq c \sigma_j, \tag{3.15}$$

and let $x = (x_i)_{i \in \mathcal{D}} \in [0, \infty)^{\mathcal{D}}$, $p \in [1, \infty)$, $\mathcal{D}' \subseteq \mathcal{D}$. Then

$$\sum_{i \in \mathcal{D}'} \sigma_i \left(\sum_{j \in \mathcal{D}} m(i, j) x_j \right)^p \leq \sum_{i \in \mathcal{D}} c \sigma_i x_i^p. \tag{3.16}$$

Proof: Jensen’s inequality, the fact that m is a stochastic matrix, the theorem of Fubini–Tonelli and (3.15) imply that

$$\sum_{i \in \mathcal{D}'} \sigma_i \left(\sum_{j \in \mathcal{D}} m(i, j) x_j \right)^p \leq \sum_{i \in \mathcal{D}} \sigma_i \sum_{j \in \mathcal{D}} m(i, j) x_j^p \leq \sum_{j \in \mathcal{D}} c \sigma_j x_j^p. \tag{3.17}$$

This proves the assertion. □

3.3. Strong convergence of the spatial stochastic Lotka–Volterra processes. In this section we will show the convergence of the time-rescaled Lotka–Volterra processes as given in (3.3) and (3.5). In Lemmas 3.5, 3.6, and 3.7 we will provide bounds for the expected value of the sum (over sets of demes) of functionals of the processes weighted by σ . The proofs of these lemmas are technical and are therefore deferred to the appendix. Lemmas 3.5, 3.6, and 3.7 are then used in Theorem 3.8 to study the behavior of a spatial analogue of a well-known Lyapunov function (e.g., Dobrinevski and Frey, 2012).

Lemma 3.5. *Assume the setting of Section 3.1 and let $p \in \{1\} \cup [2, \infty)$. Then we have*

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\left\| (2\eta H_t^N + \delta P_t^N)^p \right\|_{\sigma} \right] &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (2\eta H_0^N + \delta P_0^N)^p \right\|_{\sigma} \right] \\ &+ \|\mathbb{1}\|_{\sigma} \left(\frac{\lambda + (1 - \frac{1}{p} + \frac{c}{p})(\bar{\kappa}_H + \bar{\kappa}_P)}{2 \min\{\frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma\}} \right)^p \left(1 + \sqrt{1 + \frac{4 \min\{\frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma\} [2\eta \bar{\nu}_H + \delta \bar{\nu}_P + (p-1)(2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P)]}{(\lambda + (1 - \frac{1}{p} + \frac{c}{p})(\bar{\kappa}_H + \bar{\kappa}_P))^2}} \right)^p. \end{aligned} \tag{3.18}$$

Lemma 3.6. *Assume the setting of Section 3.1 and assume $\gamma \geq 2\delta$. Furthermore, assume that for all $N \in \mathbb{N}$ we have $\alpha^N + \kappa_H^N \leq \frac{\lambda}{4}$, $\iota_P^N \leq \frac{\lambda(\nu + \lambda)}{8\delta}$, and $\iota_H^N \geq \frac{4\delta \kappa_P^N}{3(\nu + \lambda)} + \frac{3}{2} \beta_H^N$. Let $\hat{\mathcal{D}} \subseteq \mathcal{D}$ be a set. Then we have*

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{2}{\lambda + \nu} \frac{P_t^N(i)}{(H_t^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_t^N(i))^2} \right) \right] \\ \leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{2}{\lambda + \nu} \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{4\bar{\kappa}_P c}{3\lambda(\lambda + \nu)} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (P_t^N(i))^3 \right] \\ + \frac{4}{\lambda(\lambda + \nu)} \left(\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i P_t^N(i) \right] + \frac{2}{K^2 \delta}. \end{aligned} \tag{3.19}$$

Lemma 3.7. *Assume the setting of Section 3.1 and assume $\lambda > \nu$ and $\eta - \rho > \frac{\lambda}{K}$. Furthermore, assume that for all $N \in \mathbb{N}$ we have $\iota_H^N \geq \frac{1}{2} \beta_H^N$, $\kappa_P^N + \kappa_H^N + \alpha^N \leq \frac{\lambda - \nu}{2}$, $\iota_P^N \geq \beta_P^N$, and $\iota_H^N \geq \beta_H^N$. Let*

$\hat{\mathcal{D}} \subseteq \mathcal{D}$ be a set. Then we have

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{(\eta - \rho) - \frac{\lambda}{K}}{2(\bar{\kappa}_P + \nu)} \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i)H_t^N(i)} \right) \right] \\ & \leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{(\eta - \rho) - \frac{\lambda}{K}}{2(\bar{\kappa}_P + \nu)} \frac{1}{P_0^N(i)} + \frac{1}{P_0^N(i)H_0^N(i)} \right) \right] \\ & \quad + \frac{1}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}} \left(\gamma \frac{(\eta - \rho) - \frac{\lambda}{K}}{2(\bar{\kappa}_P + \nu)} + (\gamma + \delta) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \frac{1}{H_t^N(i)} \right] \right). \end{aligned} \tag{3.20}$$

The following theorem implies for every $t \in [0, \infty)$ that the $L^2([0, t] \times l_\sigma^1 \times \Omega; \mathbb{R})$ -distance between (H_t^N, P_t^N) and $(h_\infty(F_t^N), p_\infty(F_t^N))$ converges to 0 as $N \rightarrow \infty$ at least with rate $\frac{1}{2}$; cf. also Theorem 2.2.

Theorem 3.8. Assume the setting of Section 3.1, let (h_∞, p_∞) satisfy (2.4), let Assumption 2.1 hold and let $u: (0, \infty)^2 \times [0, 1] \rightarrow [0, \infty)$ satisfy for all $(x, y, z) \in (0, \infty)^2 \times [0, 1]$ that

$$u(x, y, z) := (\eta - \rho z) \left(x - h_\infty(z) - h_\infty(z) \ln \left(\frac{x}{h_\infty(z)} \right) \right) + \delta \left(y - p_\infty(z) - p_\infty(z) \ln \left(\frac{y}{p_\infty(z)} \right) \right). \tag{3.21}$$

Then u is well-defined and there exists a constant $c_0 \in (0, \infty)$ such that for every set $\hat{\mathcal{D}} \subseteq \mathcal{D}$, for every $N \in \mathbb{N}$, and every $t \in [0, \infty)$ it holds that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i u(H_t^N(i), P_t^N(i), F_t^N(i)) \right] \\ & \quad + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] + \delta \gamma \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \\ & \leq \sup_{M \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i u(H_0^M(i), P_0^M(i), F_0^M(i)) \right] + tc_0 \max\{\kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N\} < \infty. \end{aligned} \tag{3.22}$$

Proof: The main steps of this proof are Ito’s lemma (see (3.40) below) applied to the Lyapunov-type function (3.23), the specific relation (3.43) of the equilibrium state and control of inverse moments appearing in remainder terms by Lemmas 3.5, 3.6, and 3.7. For the rest of this proof fix a set $\hat{\mathcal{D}} \subseteq \mathcal{D}$. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \hat{\mathcal{D}}$ be a set with $|\mathcal{D}_n| = \min\{n, |\hat{\mathcal{D}}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Assume that $\hat{\mathcal{D}} = \cup_{n \in \mathbb{N}} \mathcal{D}_n$. We will first show that u is well-defined. Define for all $x \in (0, \infty)$ the real-valued function $(0, \infty) \ni y \mapsto f_x(y) := x - y - y \ln \left(\frac{x}{y} \right)$. For all $x \in (0, \infty)$ the function f_x has for all $y \in (0, \infty)$ first and second order derivatives $\frac{df_x}{dy}(y) = \ln(y) - \ln(x)$ and $\frac{d^2f_x}{dy^2}(y) = \frac{1}{y} > 0$. Thus, for all $x \in (0, \infty)$ the function f_x has its global minimum at x with $f_x(x) = 0$. Consequently, for any $(x, y) \in (0, \infty)^2$ we have $f_x(y) \geq f_x(x) = 0$. This shows that for all $(x, y, z) \in (0, \infty)^2 \times [0, 1]$ we have that $u(x, y, z) \geq 0$. In order to prove the second part of the claim, we will make use of a Lyapunov function that is defined here analogously to the well-known Lyapunov function in the deterministic setting. Define $D_V := (l_\sigma^1 \cap (0, \infty)^\mathcal{D}) \times (l_\sigma^1 \cap (0, \infty)^\mathcal{D}) \times E_1$. For any subset $\hat{\mathcal{D}}' \subseteq \hat{\mathcal{D}}$ define the function $V_{\hat{\mathcal{D}}'}: D_V \rightarrow [0, \infty]$ for any $(h, p, f) \in D_V$ by

$$V_{\hat{\mathcal{D}}'}((h, p, f)) := \sum_{i \in \hat{\mathcal{D}}'} \sigma_i u(h_i, p_i, f_i). \tag{3.23}$$

Due to the non-negativity of the mapping u , we obtain for any $\hat{\mathcal{D}}' \subseteq \hat{\mathcal{D}}$ and any $z \in D_V$ that $V_{\hat{\mathcal{D}}'}(z) \in [0, \infty]$ is well-defined. The fact that for all $x \in (0, \infty)$ it holds that $-\ln(x) \leq \sqrt{\frac{1}{x}}$ and Young's inequality imply for all $x, y \in (0, \infty)$ that

$$|f_x(y)| = f_x(y) = x - y - y \ln\left(\frac{x}{y}\right) \leq x + y\sqrt{\frac{y}{x}} \leq 2 + x^4 + \frac{y^3}{x}. \tag{3.24}$$

This, the fact that h_∞, p_∞ are bounded and the assumption

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (H_0^N + P_0^N)^4 + \frac{1}{(H_0^N)^2} + \frac{1}{P_0^N} \right\|_\sigma \right] < \infty$$

yield that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \mathbb{E} [V_{\mathcal{D}}(H_0^N, P_0^N, F_0^N)] \\ & \leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(\eta \left(2 + |H_0^N(i)|^4 + \frac{|h_\infty(F_0^N(i))|^3}{H_0^N(i)} \right) + \delta \left(2 + |P_0^N(i)|^4 + \frac{|p_\infty(F_0^N(i))|^3}{P_0^N(i)} \right) \right) \right] \\ & \leq (\eta + \delta) \sup_{N \in \mathbb{N}} \left(5 + \mathbb{E} \left[\|(H_0^N)^4\|_\sigma \right] + \mathbb{E} \left[\left\| \frac{\sup_{z \in [0,1]} h_\infty(z)}{(H_0^N)^2} \right\|_\sigma \right] + \mathbb{E} \left[\|(P_0^N)^4\|_\sigma \right] + \mathbb{E} \left[\left\| \frac{\sup_{z \in [0,1]} p_\infty(z)}{P_0^N} \right\|_\sigma \right] \right) \\ & < \infty. \end{aligned} \tag{3.25}$$

We now calculate the first and second order partial derivatives that we will need in the application of Itô's lemma below. For all $n \in \mathbb{N}$, $z = (h, p, f) \in D_V$, and $i \in \mathcal{D}_n$ we get

$$\begin{aligned} \frac{dV_{\mathcal{D}_n}}{dh_i}(z) &= \sigma_i(\eta - \rho f_i) \left(1 - \frac{h_\infty(f_i)}{h_i} \right), & \frac{d^2V_{\mathcal{D}_n}}{dh_i^2}(z) &= \sigma_i(\eta - \rho f_i) \frac{h_\infty(f_i)}{h_i^2}, \\ \frac{dV_{\mathcal{D}_n}}{dp_i}(z) &= \sigma_i \delta \left(1 - \frac{p_\infty(f_i)}{p_i} \right), & \text{and } \frac{d^2V_{\mathcal{D}_n}}{dp_i^2}(z) &= \sigma_i \delta \frac{p_\infty(f_i)}{p_i^2} \end{aligned}$$

as well as

$$\begin{aligned} \frac{dV_{\mathcal{D}_n}}{df_i}(z) &= \sigma_i \left[-\rho \left(h_i - h_\infty(f_i) - h_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right) + (\eta - \rho f_i) \left(-h'_\infty(f_i) - h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right. \right. \\ & \quad \left. \left. - \frac{(h_\infty(f_i))^2}{h_i} \frac{-h_i}{(h_\infty(f_i))^2} h'_\infty(f_i) \right) + \delta \left(-p'_\infty(f_i) - p'_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) - \frac{(p_\infty(f_i))^2}{p_i} \frac{-p_i}{(p_\infty(f_i))^2} p'_\infty(f_i) \right) \right] \\ &= \sigma_i \left[-\rho \left(h_i - h_\infty(f_i) - h_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right) - (\eta - \rho f_i) h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) - \delta p'_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) \right] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \frac{d^2V_{\mathcal{D}_n}}{df_i^2}(z) &= \sigma_i \left[\rho \left(h'_\infty(f_i) + h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) + \frac{(h_\infty(f_i))^2}{h_i} (-1) \frac{h_i}{(h_\infty(f_i))^2} h'_\infty(f_i) \right) + \rho h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right. \\ & \quad \left. - (\eta - \rho f_i) \left(h''_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) + h'_\infty(f_i) \frac{h_\infty(f_i)}{h_i} \frac{-h_i}{(h_\infty(f_i))^2} h'_\infty(f_i) \right) \right. \\ & \quad \left. - \delta \left(p''_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) + p'_\infty(f_i) \frac{p_\infty(f_i)}{p_i} \frac{-p_i}{(p_\infty(f_i))^2} p'_\infty(f_i) \right) \right] \\ &= \sigma_i \left[2\rho h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) - (\eta - \rho f_i) \left(h''_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) - (h'_\infty(f_i))^2 \frac{1}{h_\infty(f_i)} \right) \right. \\ & \quad \left. - \delta \left(p''_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) - (p'_\infty(f_i))^2 \frac{1}{p_\infty(f_i)} \right) \right]. \end{aligned} \tag{3.27}$$

Next we show that Itô integrals coming from Itô formula have vanishing expectation. Recall that we have for all $x \in [0, 1]$ that $h_\infty(x) = \frac{1}{b(a-x)}$ and $p_\infty(x) = \frac{\lambda}{\delta} \left(1 - \frac{1}{Kb(a-x)}\right)$ and note that the assumption that $\eta - \rho > \frac{\nu}{K}$ implies for all $x \in [0, 1]$ that $p_\infty(x) > 0$. Therefore, we get for all $x \in [0, 1]$,

$$\begin{aligned} h'_\infty(x) &= \frac{1}{b(a-x)^2} > 0, \quad h''_\infty(x) = \frac{2}{b(a-x)^3} > 0, \\ p'_\infty(x) &= -\frac{\lambda}{\delta Kb(a-x)^2} < 0, \quad p''_\infty(x) = -\frac{2\lambda}{\delta Kb(a-x)^3} < 0. \end{aligned} \tag{3.28}$$

So $h_\infty, h'_\infty,$ and h''_∞ are strictly increasing on $[0, 1]$ while $p_\infty, p'_\infty,$ and p''_∞ are strictly decreasing on $[0, 1]$. Also we have that $\max_{x \in [0,1]} \delta p_\infty(x) \leq \lambda$. Observe that for all $x \in (0, \infty)$ we have $|\ln(x)| \leq \sqrt{x} + \frac{1}{\sqrt{x}}$. Together with Young's inequality as well as Lemmas 3.5, 3.6, and 3.7 we get for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \left(\sqrt{\beta_P^N P_u^N(i)} \delta \left(1 - \frac{p_\infty(F_u^N(i))}{P_u^N(i)}\right) \right)^2 du \right] &\leq \bar{\beta}_P \delta^2 \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t P_u^N(i) \left(1 + \frac{(p_\infty(0))^2}{(P_u^N(i))^2}\right) du \right] \\ &\leq \bar{\beta}_P \delta^2 \sup_{u \in [0,t]} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i t \left(P_u^N(i) + \frac{(p_\infty(0))^2}{P_u^N(i)} \right) \right] < \infty \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \left(\sqrt{\beta_H^N H_u^N(i)} (\eta - \rho F_u^N(i)) \left(1 - \frac{h_\infty(F_u^N(i))}{H_u^N(i)}\right) \right)^2 du \right] \\ \leq \bar{\beta}_H \eta^2 \sup_{u \in [0,t]} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i t \left(H_u^N(i) + \frac{(h_\infty(1))^2}{H_u^N(i)} \right) \right] < \infty \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \left(\sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} \left(-\rho \left(H_u^N(i) - h_\infty(F_u^N(i)) - h_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) \right) \right. \right. \right. \\ \left. \left. \left. - (\eta - \rho F_u^N(i)) h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \delta p'_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) \right) \right)^2 du \right] \\ \leq \bar{\beta}_H \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \frac{1}{H_u^N(i)} \left(\rho H_u^N(i) + \rho h_\infty(1) + \rho h_\infty(1) \left(\frac{\sqrt{H_u^N(i)}}{\sqrt{h_\infty(0)}} + \frac{\sqrt{h_\infty(1)}}{\sqrt{H_u^N(i)}} \right) \right. \right. \\ \left. \left. + \eta h'_\infty(1) \left(\frac{\sqrt{H_u^N(i)}}{\sqrt{h_\infty(0)}} + \frac{\sqrt{h_\infty(1)}}{\sqrt{H_u^N(i)}} \right) + \delta |p'_\infty(1)| \left(\frac{\sqrt{P_u^N(i)}}{\sqrt{p_\infty(1)}} + \frac{\sqrt{p_\infty(0)}}{\sqrt{P_u^N(i)}} \right) \right)^2 du \right] \\ \leq \bar{\beta}_H \sup_{u \in [0,t]} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i 2^7 \left(\rho^2 H_u^N(i) + \rho^2 (h_\infty(1))^2 \left(\frac{1}{H_u^N(i)} + \frac{1}{h_\infty(0)} + \frac{h_\infty(1)}{(H_u^N(i))^2} \right) \right. \right. \\ \left. \left. + \eta^2 (h'_\infty(1))^2 \left(\frac{1}{h_\infty(0)} + \frac{h_\infty(1)}{(H_u^N(i))^2} \right) + \delta^2 (p'_\infty(1))^2 \left(\frac{1}{p_\infty(1)} \frac{P_u^N(i)}{H_u^N(i)} + \frac{p_\infty(0)}{P_u^N(i) H_u^N(i)} \right) \right) \right] < \infty. \end{aligned} \tag{3.31}$$

Hence, we obtain for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\beta_P^N P_u^N(i)} \delta \left(1 - \frac{p_\infty(F_u^N(i))}{P_u^N(i)} \right) dW_u^{P,N}(i) \right] = 0, \\ & \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\beta_H^N H_u^N(i)} (\eta - \rho F_u^N(i)) \left(1 - \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \right) dW_u^{H,N}(i) \right] = 0, \\ & \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} \left[-\rho \left(H_u^N(i) - h_\infty(F_u^N(i)) - h_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) \right) \right. \right. \\ & \quad \left. \left. - (\eta - \rho F_u^N(i)) h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \delta p'_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) \right] dW_u^{F,N}(i) \right] = 0. \end{aligned} \tag{3.32}$$

Next we bound certain remainder terms. For all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$ define

$$\begin{aligned} R_t^N(i) &:= \max \left\{ \max \left\{ \eta c, \rho c, \rho, c \eta \frac{h'_\infty(1)}{h_\infty(0)}, \eta \frac{h'_\infty(1)}{h_\infty(0)} \right\} H_t^N(i), \eta h_\infty(1), \eta, \right. \\ & \quad \max \left\{ \frac{\eta}{2} h_\infty(1), \rho (h_\infty(1))^2, \frac{\eta (h'_\infty(1))^2}{2 h_\infty(0)}, \frac{\delta (p'_\infty(1))^2}{2 p_\infty(1)} \right\} \frac{1}{H_t^N(i)}, \delta c P_t^N(i), \delta p_\infty(0), \delta, \frac{\delta p_\infty(0)}{2 P_t^N(i)}, \\ & \quad \max \left\{ \frac{1}{2} \rho^2 (h_\infty(1))^2, \frac{3}{4} \rho^{\frac{4}{3}} (h_\infty(1))^2, \frac{3}{4} \left(\eta h'_\infty(1) \sqrt{h_\infty(1)} \right)^{\frac{4}{3}}, \frac{1}{4}, \frac{\eta}{2} h''_\infty(1) h_\infty(1) \right\} \frac{1}{(H_t^N(i))^2}, \\ & \quad \frac{1}{2} c (H_t^N(i))^2, \frac{1}{4} c (H_t^N(i))^4, \frac{1}{2} \left(\frac{\delta |p'_\infty(1)|}{\sqrt{p_\infty(1)}} \right)^2 \frac{P_t^N(i)}{(H_t^N(i))^2}, \frac{1}{2} \delta^2 |p'_\infty(1)|^2 \frac{p_\infty(0)}{P_t^N(i) H_t^N(i)}, \\ & \quad \left. \delta |p'_\infty(1)| \sqrt{\frac{p_\infty(0)}{P_t^N(i)}}, \rho \frac{h'_\infty(1)}{h_\infty(0)}, \frac{\delta |p''_\infty(1)|}{2 p_\infty(1)} \frac{P_t^N(i)}{H_t^N(i)} \right\}, \\ b^N &:= \max \{ \kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N \}. \end{aligned} \tag{3.33}$$

Note that $\lim_{N \rightarrow \infty} b^N = 0$. Define $c_0 := 32 \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} [\|R_u^M\|_\sigma]$. Observe that due to Lemmas 3.5,

3.6, and 3.7 we have $c_0 \in (0, \infty)$. For all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $a \in \left\{ \eta, \rho, \eta \frac{h'_\infty(1)}{h_\infty(0)} \right\}$ we have that

$$\sum_{i \in \mathcal{D}} \sigma_i a \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \leq \sum_{i \in \mathcal{D}} \sigma_i c a H_t^N(i) \leq \sum_{i \in \mathcal{D}} \sigma_i R_t^N(i). \tag{3.34}$$

Furthermore, we have for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\sum_{i \in \mathcal{D}} \sigma_i \delta \sum_{j \in \mathcal{D}} m(i, j) P_t^N(j) \leq \sum_{i \in \mathcal{D}} \sigma_i \delta c P_t^N(i) \leq \sum_{i \in \mathcal{D}} \sigma_i R_t^N(i). \tag{3.35}$$

Using Young's inequality and Lemma 3.4 we get for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i \rho \frac{h_\infty(F_t^N(i))}{H_t^N(i)} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) &\leq \sum_{i \in \mathcal{D}} \sigma_i \left(\frac{1}{2} \left(\rho \frac{h_\infty(F_t^N(i))}{H_t^N(i)} \right)^2 + \frac{1}{2} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^2 \right) \\ &\leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{2} c (H_t^N(i))^2 \right) \leq \sum_{i \in \mathcal{D}} \sigma_i 2 R_t^N(i), \end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
 & \sum_{i \in \mathcal{D}} \sigma_i \frac{-\delta p'_\infty(F_t^N(i))}{\sqrt{p_\infty(F_t^N(i))}} \frac{\sqrt{P_t^N(i)}}{H_t^N(i)} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i \frac{1}{2} \left(\left(\frac{\delta |p'_\infty(1)|}{\sqrt{p_\infty(1)}} \frac{\sqrt{P_t^N(i)}}{H_t^N(i)} \right)^2 + \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^2 \right) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{2} c (H_t^N(i))^2 \right) \leq \sum_{i \in \mathcal{D}} \sigma_i 2R_t^N(i),
 \end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
 & \sum_{i \in \mathcal{D}} \sigma_i (-1) \delta p'_\infty(F_t^N(i)) \frac{\sqrt{p_\infty(F_t^N(i))}}{\sqrt{P_t^N(i)H_t^N(i)}} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(\frac{1}{2} \delta^2 |p'_\infty(1)|^2 \frac{p_\infty(0)}{P_t^N(i)H_t^N(i)} + \frac{1}{2} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^2 \frac{1}{H_t^N(i)} \right) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{4} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^4 + \frac{1}{4} \frac{1}{(H_t^N(i))^2} \right) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{4} c (H_t^N(i))^4 + R_t^N(i) \right) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i 3R_t^N(i).
 \end{aligned} \tag{3.38}$$

Again using Young’s inequality and Lemma 3.4 we get for all $a \in \left\{ \rho(h_\infty(1))^{\frac{3}{2}}, \eta h'_\infty(1) \sqrt{h_\infty(1)} \right\}$, all $t \in [0, \infty)$, and all $N \in \mathbb{N}$ that

$$\begin{aligned}
 \sum_{i \in \mathcal{D}} \sigma_i a \left(\frac{1}{H_t^N(i)} \right)^{\frac{3}{2}} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(\frac{3}{4} a^{\frac{4}{3}} \left(\frac{1}{H_t^N(i)} \right)^2 + \frac{1}{4} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^4 \right) \\
 & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{4} c (H_t^N(i))^4 \right) \leq \sum_{i \in \mathcal{D}} \sigma_i 2R_t^N(i).
 \end{aligned} \tag{3.39}$$

Due to Lemma 3.1 we have that $W^{H,N}(i), W^{F,N}(i), N \in \mathbb{N}, i \in \mathcal{D}$, are independent Brownian motions and due to Lemmas 3.5, 3.6, and 3.7 we have for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that \mathbb{P} -a.s. $(H_t^N, P_t^N, F_t^N) \in D_V$. Thus, applying Itô’s lemma and using (3.32) we obtain for all $t \in [0, \infty)$

and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\
 &= \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \left((\eta - \rho F_u^N(i)) \left(1 - \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \right) \left\{ \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (H_u^N(j) - H_u^N(i)) \right. \right. \right. \\
 & \quad + H_u^N(i) \left[\lambda \left(1 - \frac{H_u^N(i)}{K} \right) - \delta P_u^N(i) - \alpha^N F_u^N(i) \right] + \iota_H^N \left. \right\} + \frac{(\eta - \rho F_u^N(i)) h_\infty(F_u^N(i))}{2 (H_u^N(i))^2} \beta_H^N H_u^N(i) \\
 & \quad + \delta \left(1 - \frac{p_\infty(F_u^N(i))}{P_u^N(i)} \right) \left\{ \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) (P_u^N(j) - P_u^N(i)) \right. \\
 & \quad + P_u^N(i) [-\nu - \gamma P_u^N(i) + (\eta - \rho F_u^N(i)) H_u^N(i)] + \iota_P^N \left. \right\} + \frac{\delta p_\infty(F_u^N(i))}{2 (P_u^N(i))^2} \beta_P^N P_u^N(i) \\
 & \quad + \left[-\rho \left(H_u^N(i) - h_\infty(F_u^N(i)) - h_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) \right) \right. \\
 & \quad \left. - (\eta - \rho F_u^N(i)) h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \delta p'_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) \right] \\
 & \quad \cdot \left\{ \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_u^N(j) - F_u^N(i)) \frac{H_u^N(j)}{H_u^N(i)} - \alpha^N F_u^N(i) (1 - F_u^N(i)) \right\} \\
 & \quad + \left\{ \rho h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \frac{\eta - \rho F_u^N(i)}{2} \left(h''_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \frac{(h'_\infty(F_u^N(i)))^2}{h_\infty(F_u^N(i))} \right) \right. \\
 & \quad \left. - \frac{\delta}{2} \left(p''_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) - \frac{(p'_\infty(F_u^N(i)))^2}{p_\infty(F_u^N(i))} \right) \right\} \frac{\beta_H^N F_u^N(i) (1 - F_u^N(i))}{H_u^N(i)} du \Big]. \tag{3.40}
 \end{aligned}$$

Next we show that the right-hand side is negative up to small remainder terms. Note that for all $x \in [0, 1]$ it holds that $0 < \eta - \rho x \leq \eta$. Together with the fact that for all $x \in (0, \infty)$ we have $\ln(x) \leq \sqrt{x}$, $\ln(x) \leq x$, $|\ln(x)| \leq \sqrt{x} + \sqrt{\frac{1}{x}}$, and $|\ln(x)| \leq x + \sqrt{\frac{1}{x}}$ and dropping negative terms,

this implies for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\
 & \leq \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \left(\eta \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \eta h_\infty (F_u^N(i)) \kappa_H^N \right. \right. \\
 & \quad + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty (F_u^N(i))) \left[\lambda \left(1 - \frac{H_u^N(i)}{K} \right) - \delta P_u^N(i) \right] + \eta h_\infty (F_u^N(i)) \alpha^N + \eta \iota_H^N \\
 & \quad + \frac{\eta}{2} h_\infty (F_u^N(i)) \beta_H^N \frac{1}{H_u^N(i)} + \delta \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) + \delta p_\infty (F_u^N(i)) \kappa_P^N \\
 & \quad + \delta (P_u^N(i) - p_\infty (F_u^N(i))) [-\nu - \gamma P_u^N(i) + (\eta - \rho F_u^N(i)) H_u^N(i)] + \delta \iota_P^N + \frac{\delta}{2} \frac{p_\infty (F_u^N(i))}{P_u^N(i)} \beta_P^N \\
 & \quad + \rho \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \rho H_u^N(i) \alpha^N + \rho \frac{h_\infty (F_u^N(i))}{H_u^N(i)} \left(1 + \frac{H_u^N(i)}{h_\infty (F_u^N(i))} + \sqrt{\frac{h_\infty (F_u^N(i))}{H_u^N(i)}} \right) \\
 & \quad \cdot \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \rho h_\infty (F_u^N(i)) \frac{h_\infty (F_u^N(i))}{H_u^N(i)} \alpha^N + \left[\eta h'_\infty (F_u^N(i)) \left(\sqrt{\frac{h_\infty (F_u^N(i))}{H_u^N(i)}} \right. \right. \\
 & \quad \left. \left. + \frac{H_u^N(i)}{h_\infty (F_u^N(i))} \right) - \delta p'_\infty (F_u^N(i)) \left(\sqrt{\frac{P_u^N(i)}{p_\infty (F_u^N(i))}} + \sqrt{\frac{p_\infty (F_u^N(i))}{P_u^N(i)}} \right) \right] \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) \frac{H_u^N(j)}{H_u^N(i)} \\
 & \quad + \left[\eta h'_\infty (F_u^N(i)) \frac{H_u^N(i)}{h_\infty (F_u^N(i))} - \delta p'_\infty (F_u^N(i)) \sqrt{\frac{p_\infty (F_u^N(i))}{P_u^N(i)}} \right] \alpha^N \\
 & \quad + \rho h'_\infty (F_u^N(i)) \frac{H_u^N(i)}{h_\infty (F_u^N(i))} \frac{\beta_H^N}{H_u^N(i)} + \frac{\eta}{2} \left(h''_\infty (F_u^N(i)) \frac{h_\infty (F_u^N(i))}{H_u^N(i)} + \frac{(h'_\infty (F_u^N(i)))^2}{h_\infty (F_u^N(i))} \right) \frac{\beta_H^N}{H_u^N(i)} \\
 & \quad \left. + \frac{\delta}{2} \left(-p''_\infty (F_u^N(i)) \frac{P_u^N(i)}{p_\infty (F_u^N(i))} + \frac{(p'_\infty (F_u^N(i)))^2}{p_\infty (F_u^N(i))} \right) \frac{\beta_H^N}{H_u^N(i)} \right] du \Big]. \tag{3.41}
 \end{aligned}$$

Using (3.34), (3.35), (3.36), (3.37), (3.38), and (3.39) we get for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\
 & \leq \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}} \sigma_i \left(b^N 32 R_u^N(i) + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty (F_u^N(i))) \left[\lambda \left(1 - \frac{H_u^N(i)}{K} \right) - \delta P_u^N(i) \right] \right. \right. \\
 & \quad \left. \left. + \delta (P_u^N(i) - p_\infty (F_u^N(i))) \left[-\nu - \gamma P_u^N(i) + (\eta - \rho F_u^N(i)) H_u^N(i) \right] \right) du \right]. \tag{3.42}
 \end{aligned}$$

Note that (2.4) implies for all $x \in [0, 1]$ that

$$\begin{aligned}
 \delta p_\infty(x) + \frac{\lambda}{K} h_\infty(x) - \lambda &= \frac{\delta \lambda K(\eta - \rho x) - \delta \lambda \nu + \lambda \delta \nu + \lambda^2 \gamma}{\lambda \gamma + \delta K(\eta - \rho x)} - \lambda = \frac{\delta K(\eta - \rho x) + \lambda \gamma}{\lambda \gamma + \delta K(\eta - \rho x)} \lambda - \lambda = 0, \\
 \nu - (\eta - \rho x) h_\infty(x) + \gamma p_\infty(x) &= \nu - \frac{(\eta - \rho x) K \delta \nu + (\eta - \rho x) K \gamma \lambda - \gamma \lambda K(\eta - \rho x) + \gamma \lambda \nu}{\lambda \gamma + \delta K(\eta - \rho x)} = \nu - \frac{(\eta - \rho x) K \delta + \gamma \lambda}{\lambda \gamma + \delta K(\eta - \rho x)} \nu = 0. \tag{3.43}
 \end{aligned}$$

From (3.43) we see that for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ it holds that

$$\begin{aligned} & \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\ & \leq \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}} \sigma_i \left(b^N 32 R_u^N(i) + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty(F_u^N(i))) \left[\lambda - \frac{\lambda}{K} (H_u^N(i) - h_\infty(F_u^N(i))) \right] \right. \right. \\ & \quad \left. \left. - \delta (P_u^N(i) - p_\infty(F_u^N(i))) - \lambda \right) + \delta (P_u^N(i) - p_\infty(F_u^N(i))) \left[-\nu - \gamma (P_u^N(i) - p_\infty(F_u^N(i))) \right] \right. \\ & \quad \left. + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty(F_u^N(i))) + \nu \right) du \right]. \end{aligned} \tag{3.44}$$

Hence, we obtain for every $N, n \in \mathbb{N}$ and every $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} [V_{\mathcal{D}_n} (H_t^N, P_t^N, F_t^N)] \\ & + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] + \delta \gamma \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \\ & \leq \mathbb{E} [V_{\hat{\mathcal{D}}} (H_0^N, P_0^N, F_0^N)] + tb^N 32 \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} [\|R_u^M\|_\sigma]. \end{aligned} \tag{3.45}$$

Applying monotone convergence we now see that for every $N \in \mathbb{N}$ and every $t \in [0, \infty)$ we have

$$\begin{aligned} & \mathbb{E} [V_{\hat{\mathcal{D}}} (H_t^N, P_t^N, F_t^N)] + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] \\ & \quad + \delta \gamma \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \\ & = \lim_{n \rightarrow \infty} \left(\mathbb{E} [V_{\mathcal{D}_n} (H_t^N, P_t^N, F_t^N)] + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] \right. \\ & \quad \left. + \delta \gamma \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \right) \leq \mathbb{E} [V_{\hat{\mathcal{D}}} (H_0^N, P_0^N, F_0^N)] + tb^N c_0. \end{aligned} \tag{3.46}$$

The set $\hat{\mathcal{D}} \subseteq \mathcal{D}$ was arbitrarily chosen and thus, this and (3.25) complete the proof of Theorem 3.8. □

3.4. Convergence of relative frequency of defenders.

3.4.1. *A relative compactness condition.* For convenience of the reader, we restate Lemma 3.3 of Klenke and Mytnik (2012).

Lemma 3.9. *Let \mathcal{D} be a countable set, let $\sigma \in (0, \infty)^{\mathcal{D}}$ such that $\sum_{i \in \mathcal{D}} \sigma_i < \infty$, and let $l_\sigma^1 := \{z \in \mathbb{R}^{\mathcal{D}} : \|z\|_\sigma := \sum_{i \in \mathcal{D}} \sigma_i z_i < \infty\}$. A subset $K \subseteq l_\sigma^1$ is relatively compact if and only if*

- (i) $\sup_{x \in K} \|x\|_\sigma < \infty$
- (ii) for every $\varepsilon \in (0, \infty)$ there exists a finite subset $\mathcal{E} \subseteq \mathcal{D}$ such that $\sup_{x \in K} \|x \mathbb{1}_{\mathcal{D} \setminus \mathcal{E}}\|_\sigma < \varepsilon$.

Lemma 3.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let \mathcal{D} be a countable set, let $\sigma \in (0, \infty)^{\mathcal{D}}$ such that $\sum_{i \in \mathcal{D}} \sigma_i < \infty$, let $l_\sigma^1 := \{z \in \mathbb{R}^{\mathcal{D}} : \|z\|_\sigma := \sum_{i \in \mathcal{D}} \sigma_i z_i < \infty\}$, let $E_2 := l_\sigma^1 \cap [0, \infty)^{\mathcal{D}}$, let I be a set, and let $Z^i : \Omega \rightarrow E_2, i \in I$, be a family of random variables. Assume that $\sup_{i \in I} \mathbb{E}[\|Z^i\|_\sigma] < \infty$*

and $\inf_{S \subseteq \mathcal{D}, |S| < \infty} \sup_{i \in I} \sum_{k \in \mathcal{D} \setminus S} \sigma_k \mathbb{E}[Z_k^i] = 0$. Then the family $\{Z^i : i \in I\}$ is relatively compact in E_2 .

Proof: Fix $\varepsilon \in (0, \infty)$. For each $m \in \mathbb{N}$ by assumption there exists a finite set $\mathcal{S}_{m,\varepsilon} \subseteq \mathcal{D}$ such that

$$\sup_{i \in I} \sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m,\varepsilon}} \sigma_k \mathbb{E}[Z_k^i] < \frac{\varepsilon}{2m^2(m+1)}. \tag{3.47}$$

Define the set $K_\varepsilon \subseteq E_2$ by

$$K_\varepsilon := \left\{ x \in E_2 : \|x\|_\sigma \leq \frac{2 \sup_{i \in I} \mathbb{E}[\|Z^i\|_\sigma]}{\varepsilon}, \sup_{m \in \mathbb{N}} \left\{ m \sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m,\varepsilon}} \sigma_k |x_k| \right\} \leq 1 \right\}. \tag{3.48}$$

Due to the Heine–Borel theorem we can apply Lemma 3.9 to obtain relative compactness of K_ε . By Markov’s inequality we get

$$\begin{aligned} \sup_{i \in I} \mathbb{P}\left[Z^i \notin \overline{K_\varepsilon}\right] &\leq \sup_{i \in I} \mathbb{P}\left[Z^i \notin K_\varepsilon\right] \\ &\leq \sup_{i \in I} \mathbb{P}\left[\|Z^i\|_\sigma > \frac{2 \sup_{j \in I} \mathbb{E}[\|Z^j\|_\sigma]}{\varepsilon}\right] + \sup_{i \in I} \sum_{m=1}^\infty \mathbb{P}\left[\sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m,\varepsilon}} \sigma_k Z_k^i > \frac{1}{m}\right] \\ &\leq \frac{\varepsilon}{2 \sup_{j \in I} \mathbb{E}[\|Z^j\|_\sigma]} \sup_{i \in I} \mathbb{E}[\|Z^i\|_\sigma] + \sum_{m=1}^\infty m \sup_{i \in I} \sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m,\varepsilon}} \sigma_k \mathbb{E}[Z_k^i] \leq \frac{\varepsilon}{2} + \sum_{m=1}^\infty m \frac{\varepsilon}{2m^2(m+1)} = \varepsilon. \end{aligned} \tag{3.49}$$

Since ε was arbitrarily chosen it follows that $\{Z^i : i \in I\}$ is tight in E_2 . Due to Prohorov’s theorem (e.g., Theorem 3.2.2 in Ethier and Kurtz, 1986) the claim follows. \square

3.4.2. *Proof of Theorem 2.3.* In our proof of Theorem 2.3 we will apply Lemma 3.10 to show for every $T \in [0, \infty)$ that $\{H_{tN}^N : t \in [0, T], N \in \mathbb{N}\}$ is relatively compact in $l_\sigma^1 \cap [0, \infty)^{\mathcal{D}}$. The difficult part is the last condition of Lemma 3.10 which leads us to show for every $T \in (0, \infty)$ that $\sum_{i \in \mathcal{D}} \sup_{N \in \mathbb{N}, t \in [0, T]} \mathbb{E}[H_{tN}^N(i)] < \infty$. The following lemma will allow us to prove this condition where supremum over N is inside the sum over the demes.

Lemma 3.11. *Assume the setting of Section 3.1 and assume that for all $N \in \mathbb{N}$ we have $\sum_{i \in \mathcal{D}} \sigma_i \mathbb{E}[H_0^N(i)] < \infty$. For all $n \in \mathbb{N}$ denote by m^n the n -fold matrix product of m . Then we get for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ that*

$$\mathbb{E}[H_t^N(i)] \leq \mathbb{E}\left[\sum_{j \in \mathcal{D}} \sum_{n=0}^\infty e^{-t\kappa_H^N} \frac{(t\kappa_H^N)^n}{n!} m^n(i, j) H_0^N(j)\right] + \frac{K}{2} \left(1 + \sqrt{1 + \frac{4L_H}{K\lambda}}\right). \tag{3.50}$$

Proof: If m is the identity matrix, then the assertion follows essentially from Gronwall’s lemma since the drift grows linearly in upward direction. In the general case we average over space with the migration semigroup (3.52) to get rid of the term involving m ; cf. the cancelation in the second step of (3.61). We have for every $n \in \mathbb{N}$ and every $i, j \in \mathcal{D}$ that $m^n(i, j) \in [0, 1]$. Hence, we get for all $T \in [0, \infty)$ and all $i, j \in \mathcal{D}$ that

$$\sum_{n=0}^\infty \sup_{t \in [0, T]} e^{-t\frac{t^n}{n!}} m^n(i, j) < \infty. \tag{3.51}$$

Thereby, for all $t \in [0, \infty)$ and all $i, j \in \mathcal{D}$ we can define

$$m_t(i, j) := \sum_{n=0}^\infty e^{-t\frac{t^n}{n!}} m^n(i, j). \tag{3.52}$$

By (3.51) and using dominated convergence, we can compute for all $t \in [0, \infty)$ and all $i, j \in \mathcal{D}$ that

$$\begin{aligned} \frac{d}{dt} m_t(i, j) &= -m_t(i, j) + \sum_{n=1}^{\infty} e^{-t} \frac{t^{n-1}}{(n-1)!} m^n(i, j) = -m_t(i, j) + \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} m^{n+1}(i, j) \\ &= -m_t(i, j) + \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} \sum_{k \in \mathcal{D}} m^n(i, k) m(k, j) = \sum_{k \in \mathcal{D}} m_t(i, k) (m(k, j) - \mathbb{1}_{j=k}). \end{aligned} \tag{3.53}$$

Furthermore, note that for all $t \in [0, \infty)$ and all $i \in \mathcal{D}$ we have

$$\sum_{j \in \mathcal{D}} m_t(i, j) = \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} m^n(i, j) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} = 1. \tag{3.54}$$

For all $t \in [0, \infty)$, $s \in [0, t]$, $i \in \mathcal{D}$, $N \in \mathbb{N}$ define

$$Y_s^{N,t}(i) := \sum_{j \in \mathcal{D}} m_{(t-s)\kappa_H^N}(i, j) H_s^N(j). \tag{3.55}$$

Observe that since for all $i, j \in \mathcal{D}$ it holds that $m_0(i, j) = \mathbb{1}_{i=j}$ we have for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ that

$$Y_t^{N,t}(i) = H_t^N(i). \tag{3.56}$$

Furthermore, using (2.5) we have for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[Y_0^{N,t}(i) \right] &= \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{t\kappa_H^N}(i, j) H_0^N(j) \right] = \sum_{i \in \mathcal{D}} \sigma_i \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{(t\kappa_H^N)^n}{n!} m^n(i, j) \mathbb{E} [H_0^N(j)] \\ &= \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{(t\kappa_H^N)^n}{n!} \mathbb{E} [H_0^N(j)] \sum_{i \in \mathcal{D}} \sigma_i m^n(i, j) \leq \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{(t\kappa_H^N)^n}{n!} \mathbb{E} [H_0^N(j)] c^n \sigma_j \\ &= \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{(t\kappa_H^N c)^n}{n!} \mathbb{E} [H_0^N(j)] \sigma_j = e^{t\kappa_H^N(c-1)} \mathbb{E} [\|H_0^N\|_{\sigma}]. \end{aligned} \tag{3.57}$$

For all $t \in [0, \infty)$, $s \in [0, t]$, $i, j \in \mathcal{D}$, $N \in \mathbb{N}$ we see from (3.53) that we have

$$\frac{d}{ds} m_{(t-s)\kappa_H^N}(i, j) = -\kappa_H^N \sum_{k \in \mathcal{D}} m_{(t-s)\kappa_H^N}(i, k) (m(k, j) - \mathbb{1}_{j=k}). \tag{3.58}$$

For $t \in [0, \infty)$, $N, l \in \mathbb{N}$, $i \in \mathcal{D}$ define

$$\tau_l^{N,t}(i) := \inf (\{u \in [0, t] : Y_u^{N,t}(i) > l\} \cup \infty). \tag{3.59}$$

Using the fact that for all $t \in [0, \infty)$, all $u \in [0, t]$, all $N \in \mathbb{N}$, and all $i, j \in \mathcal{D}$ we have $m_{(t-u)\kappa_H^N}(i, j) \in [0, 1]$ we get for all $t \in [0, \infty)$, all $s \in [0, t]$, all $N, l \in \mathbb{N}$, and all $i \in \mathcal{D}$ that

$$\begin{aligned} \int_0^{s \wedge \tau_l^{N,t}(i)} \sum_{j \in \mathcal{D}} \left(m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} \right)^2 du &\leq \int_0^{s \wedge \tau_l^{N,t}(i)} \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \beta_H^N H_u^N(j) du \\ &= \int_0^{s \wedge \tau_l^{N,t}(i)} \beta_H^N Y_u^{N,t}(i) du \leq \int_0^s \beta_H^N Y_{u \wedge \tau_l^{N,t}(i)}^{N,t}(i) du \leq t \beta_H^N l. \end{aligned} \tag{3.60}$$

For all $t \in [0, \infty)$, $s \in [0, t]$, $i \in \mathcal{D}$, $N \in \mathbb{N}$ using Itô's lemma with (3.54) and (3.58) we get \mathbb{P} -a.s.

$$\begin{aligned}
 Y_s^{N,t}(i) - Y_0^{N,t}(i) &= \int_0^s \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \left(\kappa_H^N \sum_{k \in \mathcal{D}} m(j, k) H_u^N(k) + (\lambda - \kappa_H^N - \alpha^N F_t^N(j)) H_u^N(j) \right. \\
 &\quad \left. - \frac{\lambda}{K} (H_u^N(j))^2 - \delta H_u^N(j) P_u^N(j) + \iota_H^N \right) - \sum_{j \in \mathcal{D}} \kappa_H^N \sum_{k \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, k) (m(k, j) - \mathbb{1}_{j=k}) H_u^N(j) du \\
 &\quad + \sum_{j \in \mathcal{D}} \int_0^s m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} dW_u^{N,H}(j) \\
 &= \int_0^s \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \left((\lambda - \alpha^N F_t^N(j)) H_u^N(j) - \frac{\lambda}{K} (H_u^N(j))^2 - \delta H_u^N(j) P_u^N(j) \right) + \iota_H^N du \\
 &\quad + \sum_{j \in \mathcal{D}} \int_0^s m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} dW_u^{N,H}(j) \\
 &\leq \int_0^s \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \left(\lambda H_u^N(j) - \frac{\lambda}{K} (H_u^N(j))^2 \right) + \iota_H^N du \\
 &\quad + \sum_{j \in \mathcal{D}} \int_0^s m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} dW_u^{N,H}(j).
 \end{aligned} \tag{3.61}$$

Thus, using (3.60) and (3.61) we get for all $t \in [0, \infty)$, all $s \in [0, t]$, all $i \in \mathcal{D}$, and all $N, l \in \mathbb{N}$ that

$$\begin{aligned}
 \mathbb{E} \left[Y_{s \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] - \mathbb{E} \left[Y_0^{N,t}(i) \right] &\leq \mathbb{E} \left[\int_0^{s \wedge \tau_l^{N,t}(i)} \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \lambda H_u^N(j) + \iota_H^N du \right] \\
 &\leq \mathbb{E} \left[\int_0^s \sum_{j \in \mathcal{D}} m_{(t-u \wedge \tau_l^{N,t}(i))\kappa_H^N}(i, j) \lambda H_{u \wedge \tau_l^{N,t}(i)}^N(j) + \iota_H^N du \right] \\
 &= \int_0^s \lambda \mathbb{E} \left[Y_{u \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] + \iota_H^N du \leq t \iota_H^N + \lambda \int_0^s \mathbb{E} \left[Y_{u \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] du.
 \end{aligned} \tag{3.62}$$

Now, using Gronwall's lemma (e.g., Klenke, 2008), we get for all $t \in [0, \infty)$, all $s \in [0, t]$, all $i \in \mathcal{D}$, and all $N, l \in \mathbb{N}$ that

$$\mathbb{E} \left[Y_{s \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] \leq \left(\mathbb{E} \left[Y_0^{N,t}(i) \right] + t \iota_H^N \right) e^{\lambda s} \leq \left(\mathbb{E} \left[Y_0^{N,t}(i) \right] + t \iota_H^N \right) e^{\lambda t}. \tag{3.63}$$

For all $t \in [0, \infty)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$ the \mathbb{P} -a.s. continuous paths of $\left(Y_u^{N,t}(i) \right)_{u \in [0,t]}$ imply $\mathbb{P} \left[\sup_{u \in [0,t]} Y_u^{N,t}(i) < \infty \right] = 1$. Hence, we get for all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$ that

$$\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,t}(i) = \infty \right] = 1. \tag{3.64}$$

Using the assumption that for all $N \in \mathbb{N}$ we have $\sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} [H_0^N(i)] < \infty$ together with (3.56), (3.57), (3.63), and (3.64) with Fatou’s lemma we obtain for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} [H_t^N(i)] &= \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} [Y_t^{N,t}(i)] = \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[\lim_{l \rightarrow \infty} Y_{t \wedge \tau_l^{N,t}}^{N,t}(i) \right] \leq \sum_{i \in \mathcal{D}} \sigma_i \liminf_{l \rightarrow \infty} \mathbb{E} \left[Y_{t \wedge \tau_l^{N,t}}^{N,t}(i) \right] \\ &\leq \sum_{i \in \mathcal{D}} \sigma_i \liminf_{l \rightarrow \infty} \left(\mathbb{E} [Y_0^{N,t}(i)] + t \iota_H^N \right) e^{\lambda t} = \sum_{i \in \mathcal{D}} \sigma_i \left(\mathbb{E} [Y_0^{N,t}(i)] + t \iota_H^N \right) e^{\lambda t} \\ &\leq \left(e^{t \kappa_H^N (c-1)} \mathbb{E} [\|H_0^N\|_\sigma] + \sum_{i \in \mathcal{D}} \sigma_i t \iota_H^N \right) e^{\lambda t} < \infty. \end{aligned} \tag{3.65}$$

Using the fact that for all $t \in [0, \infty)$, all $u \in [0, t]$, all $N \in \mathbb{N}$, and all $i, j \in \mathcal{D}$ we have $m_{(t-u)\kappa_H^N}(i, j) \in [0, 1]$ this implies for all $t \in [0, \infty)$, all $s \in [0, t]$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$, that

$$\begin{aligned} \mathbb{E} \left[\sum_{j \in \mathcal{D}} \int_0^s \left(m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} \right)^2 du \right] &= \int_0^s \mathbb{E} \left[\sum_{j \in \mathcal{D}} \left(m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} \right)^2 \right] du \\ &\leq \beta_H^N \int_0^s \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) H_u^N(j) \right] du = \beta_H^N \int_0^s \mathbb{E} [Y_u^{N,t}(i)] du < \infty. \end{aligned} \tag{3.66}$$

Thus, taking expectations in (3.61) gives for all $t \in [0, \infty)$, all $s \in [0, t]$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ using Jensen’s inequality

$$\begin{aligned} \mathbb{E} [Y_s^{N,t}(i)] - \mathbb{E} [Y_0^{N,t}(i)] &\leq \int_0^s \left(\lambda \mathbb{E} [Y_u^{N,t}(i)] - \frac{\lambda}{K} \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) (H_u^N(j))^2 \right] + \iota_H^N \right) du \\ &\leq \int_0^s \left(\lambda \mathbb{E} [Y_u^{N,t}(i)] - \frac{\lambda}{K} \mathbb{E} \left[(Y_u^{N,t}(i))^2 \right] + \iota_H^N \right) du \\ &\leq \int_0^s \left(\lambda \mathbb{E} [Y_u^{N,t}(i)] - \frac{\lambda}{K} (\mathbb{E} [Y_u^{N,t}(i)])^2 + \bar{\iota}_H \right) du. \end{aligned} \tag{3.67}$$

For $t \in [0, \infty)$, $i \in \mathcal{D}$, $N \in \mathbb{N}$ let $z^{N,t}(i): [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $s \in [0, \infty)$ satisfies

$$z_s^{N,t}(i) = z_0^{N,t}(i) + \int_0^s \left(\lambda z_u^{N,t}(i) - \frac{\lambda}{K} (z_u^{N,t}(i))^2 + \bar{\iota}_H \right) du \tag{3.68}$$

with $z_0^{N,t}(i) = \mathbb{E} [Y_0^{N,t}(i)]$ where uniqueness follows from local Lipschitz continuity. Define $c_1 := \frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{K \bar{\iota}_H}{\lambda}} \in (0, \infty)$. Using classical comparison results from the theory of ODEs, the above computation shows that for all $N \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ we have

$$\begin{aligned} \mathbb{E} [H_t^N(i)] &= \mathbb{E} [Y_t^{N,t}(i)] \leq z_t^{N,t}(i) \leq \max \left\{ \mathbb{E} [Y_0^{N,t}(i)], \limsup_{s \rightarrow \infty} z_s^{N,t}(i) \right\} = \max \left\{ \mathbb{E} [Y_0^{N,t}(i)], c_1 \right\} \\ &\leq \mathbb{E} [Y_0^{N,t}(i)] + c_1 = \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{t\kappa_H^N}(i, j) H_0^N(j) \right] + c_1 \end{aligned} \tag{3.69}$$

This finishes the proof of Lemma 3.11. □

Proof of Theorem 2.3. The main steps of the proof are to show (see (3.84) below) that the generator of F^N is close to \mathcal{A}_1 (defined in (3.74) below) as $N \rightarrow \infty$ and to prove with Theorem 2.2 that the host population is asymptotically immediately in equilibrium, that is, H^N is close to $h_\infty(F^N)$ for large $N \in \mathbb{N}$. We will use stochastic averaging (see Theorem 2.1 in Kurtz, 1992) to prove the result. So we first check that all conditions of the aforementioned theorem are fulfilled. Define $\beta_H := \frac{\beta}{b} = \lim_{N \rightarrow \infty} N\beta_H^n$. Note that $E_1 = [0, 1]^D$ and $E_2 = l_\sigma^1 \cap [0, \infty)^D$ are complete separable metric spaces. Tychonoff’s theorem implies that E_1 is compact. Since for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ the random variable F_{tN}^N takes values in the compact space E_1 , the compact containment condition holds for $\{(F_{tN}^N)_{t \in [0, \infty)} : N \in \mathbb{N}\}$. We will now use Lemma 3.10 to show for each $T \in [0, \infty)$ that the family $\{H_{tN}^N : t \in [0, T], N \in \mathbb{N}\}$ is relatively compact in E_2 . From Lemma 3.5 and the assumption $\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (H_0^N + P_0^N) \right\|^4 \right] < \infty$ we see that

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\|H_{tN}^N\|_\sigma \right] < \infty. \tag{3.70}$$

Define $\mathcal{D}_0 := \emptyset$ and for all $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \mathcal{D}$ be a set with $|\mathcal{D}_n| = \min\{n, |\mathcal{D}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Assume that $\mathcal{D} = \cup_{n \in \mathbb{N}} \mathcal{D}_n$. Define $c_1 := \frac{K}{2} \left(1 + \sqrt{1 + \frac{4t_H}{K\lambda}} \right)$. From Lemma 3.11 with the assumption that $\sum_{i \in \mathcal{D}} \sup_{N \in \mathbb{N}} \sigma_i \mathbb{E} \left[H_0^N(i) \right] < \infty$ we get for all $T \in [0, \infty)$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[H_{tN}^N(i) \right] &\leq \sum_{i \in \mathcal{D}} \sigma_i \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \left(\sum_{j \in \mathcal{D}} \sum_{n=0}^\infty e^{-tN\kappa_H^n} \frac{(tN\kappa_H^n)^n}{n!} m^n(i, j) \mathbb{E} \left[H_0^N(j) \right] + c_1 \right) \\ &\leq \sum_{j \in \mathcal{D}} \sum_{n=0}^\infty \left(\sum_{i \in \mathcal{D}} \sigma_i m^n(i, j) \right) \sup_{N \in \mathbb{N}} \sup_{t \in [0, TN\kappa_H^n]} e^{-t} \frac{t^n}{n!} \mathbb{E} \left[H_0^N(j) \right] + c_1 \sum_{i \in \mathcal{D}} \sigma_i \\ &\leq \sum_{j \in \mathcal{D}} \sum_{n=0}^\infty c^n \sigma_j \sup_{N \in \mathbb{N}} \frac{(TN\kappa_H^n)^n}{n!} \mathbb{E} \left[H_0^N(j) \right] + c_1 \|\mathbb{1}\|_\sigma \\ &\leq e^{cT} \sup_{M \in \mathbb{N}} M \kappa_H^M \sum_{j \in \mathcal{D}} \sigma_j \sup_{N \in \mathbb{N}} \mathbb{E} \left[H_0^N(j) \right] + c_1 \|\mathbb{1}\|_\sigma < \infty. \end{aligned} \tag{3.71}$$

Now we can use the dominated convergence theorem to obtain for all $T \in [0, \infty)$ that

$$\lim_{n \rightarrow \infty} \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \sum_{k \in \mathcal{D} \setminus \mathcal{D}_n} \sigma_k \mathbb{E} \left[H_{tN}^N(k) \right] \leq \lim_{n \rightarrow \infty} \sum_{k \in \mathcal{D} \setminus \mathcal{D}_n} \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \sigma_k \mathbb{E} \left[H_{tN}^N(k) \right] = 0. \tag{3.72}$$

Hence, for all $T \in [0, \infty)$ we can apply Lemma 3.10 to the family $\{H_{tN}^N : t \in [0, T], N \in \mathbb{N}\}$ and conclude that it is relatively compact in E_2 . Denote by $C_b(E_1, \mathbb{R})$ the set of bounded, continuous real-valued functions on E_1 and by $C_b^2(E_1, \mathbb{R})$ the set of all real-valued functions on E_1 that are twice continuously differentiable and bounded, with bounded first and second order partial derivatives. For $f \in C_b^2(E_1, \mathbb{R})$ let $c_f \in (0, \infty)$ be such that for all $x \in E_1$ and all $i \in \mathcal{D}$ we have $\left| \frac{df}{dx_i}(x) \right| + \left| \frac{d^2f}{dx_i^2}(x) \right| \leq c_f$. Define

$$\text{Dom}(\mathcal{A}) := \{f \in C_b^2(E_1, \mathbb{R}) : f \text{ depends only on finitely many coordinates}\} \tag{3.73}$$

and for any $f \in \text{Dom}(\mathcal{A})$ denote by \mathcal{D}_f the finite set of coordinates that f depends on. Due to the Stone–Weierstrass theorem (see e.g. Klenke, 2008, Theorem 15.2) we see that $\text{Dom}(\mathcal{A})$ is dense in $C_b(E_1, \mathbb{R})$ in the topology of uniform convergence. Denote by $C(E_1 \times E_2, \mathbb{R})$ the set of real-valued

continuous functions on $E_1 \times E_2$ and define the operator $\mathcal{A}_1 : \text{Dom}(\mathcal{A}) \rightarrow C(E_1 \times E_2, \mathbb{R})$ for all $f \in \text{Dom}(\mathcal{A})$, all $x \in E_1$, and all $y \in E_2$ by

$$(\mathcal{A}_1 f)(x, y) := \sum_{i \in \mathcal{D}} \mathbb{1}_{y_i > 0} \left(\left[\kappa_H \sum_{j \in \mathcal{D}} (m(i, j) \frac{y_j}{y_i} (x_j - x_i)) - \alpha x_i (1 - x_i) \right] \frac{df}{dx_i}(x) + \frac{1}{2} \beta_H \frac{x_i(1-x_i)}{y_i} \frac{d^2 f}{dx_i^2}(x) \right). \tag{3.74}$$

For all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ define

$$\begin{aligned} \varepsilon_f^N(t) := & \frac{1}{N} \int_0^t (\mathcal{A}_1 f)(F_u^N, H_u^N) du - \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) \left[\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_u^N(j) - F_u^N(i)) \frac{H_u^N(j)}{H_u^N(i)} \right. \\ & \left. - \alpha^N F_u^N(i) (1 - F_u^N(i)) \right] + \frac{1}{2} \frac{d^2 f}{dx_i^2}(F_u^N) \frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)} du. \end{aligned} \tag{3.75}$$

We now show that this process vanishes on the evolutionary time scale as $N \rightarrow \infty$. From Itô’s lemma and Lemma 3.4 we get for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} f(F_t^N) - f(F_0^N) &= \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) dF_u^N(i) + \frac{1}{2} \sum_{i, j \in \mathcal{D}} \int_0^t \left(\frac{d^2 f}{dx_i dx_j}(F_u^N) \right) d \langle F^N(i), F^N(j) \rangle_u \\ &= \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) \left[\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_u^N(j) - F_u^N(i)) \frac{H_u^N(j)}{H_u^N(i)} - \alpha^N F_u^N(i) (1 - F_u^N(i)) \right] \\ &+ \frac{1}{2} \frac{d^2 f}{dx_i^2}(F_u^N) \frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)} du + \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} dW_u^{F, N}(i). \end{aligned} \tag{3.76}$$

Hence, we get for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} f(F_{tN}^N) - \int_0^t (\mathcal{A}_1 f)(F_{uN}^N, H_{uN}^N) du + \varepsilon_f^N(tN) \\ = f(F_0^N) + \sum_{i \in \mathcal{D}} \int_0^{tN} \frac{df}{dx_i}(F_u^N) \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} dW_u^{F, N}(i). \end{aligned} \tag{3.77}$$

From Tonelli’s theorem and Lemma 3.6 we obtain for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ that

$$\begin{aligned} \mathbb{E} \left[\int_0^{tN} \left(\sum_{i \in \mathcal{D}} \frac{df}{dx_i}(F_u^N) \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} \right)^2 du \right] &\leq tN |\mathcal{D}_f| c_f^2 \beta_H^N \max_{i \in \mathcal{D}_f} \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} \left[\frac{1}{H_u^M(i)} \right] \\ &\leq t |\mathcal{D}_f| c_f^2 N \beta_H^N \max_{i \in \mathcal{D}_f} \frac{1}{\sigma_i} \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} \left[\left\| \frac{1}{H_u^M} \right\|_{\sigma} \right] < \infty. \end{aligned} \tag{3.78}$$

Thus for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ the left-hand side of (3.77) is a martingale. Next, for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ it holds that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T |(\mathcal{A}_1 f)(F_{tN}^N, H_{tN}^N)|^{\frac{4}{3}} dt \right] \\ &= \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \left| \sum_{i \in \mathcal{D}_f} \left(\kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{H_{tN}^N(j)}{H_{tN}^N(i)} (F_{tN}^N(j) - F_{tN}^N(i)) \right) \right. \right. \right. \\ &\quad \left. \left. \left. - \alpha F_{tN}^N(i) (1 - F_{tN}^N(i)) \right) \frac{df}{dx_i} (F_{tN}^N) + \frac{1}{2} \sum_{i \in \mathcal{D}_f} \beta_H \frac{F_{tN}^N(i)(1 - F_{tN}^N(i))}{H_{tN}^N(i)} \frac{d^2 f}{dx_i^2} (F_{tN}^N) \right|^{\frac{4}{3}} dt \right] \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \left(\sum_{i \in \mathcal{D}_f} \left(\left| \kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{H_{tN}^N(j)}{H_{tN}^N(i)} \right) c_f \right| + |\alpha c_f| + \left| \frac{1}{2} \beta_H \frac{1}{H_{tN}^N(i)} c_f \right| \right) \right)^{\frac{4}{3}} dt \right]. \end{aligned} \tag{3.79}$$

Using Young’s inequality and Jensen’s inequality we get for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T |(\mathcal{A}_1 f)(F_{tN}^N, H_{tN}^N)|^{\frac{4}{3}} dt \right] \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \left(\sum_{i \in \mathcal{D}_f} \left(\frac{2}{3} \left(\kappa_H c_f \frac{1}{H_{tN}^N(i)} \right)^{\frac{3}{2}} + \frac{1}{3} \left(\sum_{j \in \mathcal{D}} m(i, j) H_{tN}^N(j) \right)^3 + \alpha c_f + \frac{1}{2} \beta_H \frac{1}{H_{tN}^N(i)} c_f \right) \right)^{\frac{4}{3}} dt \right] \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \frac{(4|\mathcal{D}_f|)^{\frac{1}{3}}}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \sum_{i \in \mathcal{D}_f} \sigma_i \left(\left(\frac{2}{3} \right)^{\frac{4}{3}} \left(\kappa_H c_f \frac{1}{H_{tN}^N(i)} \right)^2 + \left(\frac{1}{3} \right)^{\frac{4}{3}} \left(\sum_{j \in \mathcal{D}} m(i, j) H_{tN}^N(j) \right)^4 \right. \right. \\ &\quad \left. \left. + (\alpha c_f)^{\frac{4}{3}} + \left(\frac{1}{2} \beta_H \frac{1}{H_{tN}^N(i)} c_f \right)^{\frac{4}{3}} \right) dt \right]. \end{aligned} \tag{3.80}$$

Using Lemma 3.4, Tonelli’s theorem, and Lemmas 3.5 and 3.6 we obtain for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T |(\mathcal{A}_1 f)(F_{tN}^N, H_{tN}^N)|^{\frac{4}{3}} dt \right] \\ &\leq \sup_{N \in \mathbb{N}} \frac{(4|\mathcal{D}_f|)^{\frac{2}{3}}}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \int_0^T \left(\frac{4}{9} \right)^{\frac{2}{3}} (\kappa_H c_f)^2 \mathbb{E} \left[\left\| \frac{1}{(H_{tN}^N)^2} \right\|_{\sigma} \right] + \left(\frac{1}{9} \right)^{\frac{2}{3}} c \mathbb{E} \left[\left\| (H_{tN}^N)^4 \right\|_{\sigma} + (\alpha c_f)^{\frac{4}{3}} \|\mathbb{1}\|_{\sigma} \right] \\ &\quad + \left(\frac{1}{4} \right)^{\frac{2}{3}} (\beta_H c_f)^{\frac{4}{3}} \mathbb{E} \left[\left\| \left(\frac{1}{H_{tN}^N} \right)^{\frac{4}{3}} \right\|_{\sigma} \right] dt < \infty. \end{aligned} \tag{3.81}$$

Furthermore, for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $T \in [0, \infty)$ we have that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \varepsilon_f^N(tN) \right| \right] \\ &= \mathbb{E} \left[\sup_{t \in [0, T]} \left| \sum_{i \in \mathcal{D}_f} \int_0^t \frac{df}{dx_i}(F_{uN}^N) \left[(\kappa_H - N\kappa_H^N) \sum_{j \in \mathcal{D}} m(i, j) (F_{uN}^N(j) - F_{uN}^N(i)) \frac{H_{uN}^N(j)}{H_{uN}^N(i)} \right. \right. \right. \\ & \quad \left. \left. - (\alpha - N\alpha^N) F_{uN}^N(i) (1 - F_{uN}^N(i)) \right] + \frac{1}{2} \frac{d^2 f}{dx_i^2}(F_{uN}^N) (\beta_H - N\beta_H^N) \frac{F_{uN}^N(i)(1 - F_{uN}^N(i))}{H_{uN}^N(i)} du \right| \right] \\ &\leq \mathbb{E} \left[\int_0^T \sum_{i \in \mathcal{D}_f} c_f \left(|\kappa_H - N\kappa_H^N| \sum_{j \in \mathcal{D}} m(i, j) \frac{H_{uN}^N(j)}{H_{uN}^N(i)} + |\alpha - N\alpha^N| + \frac{1}{2} |\beta_H - N\beta_H^N| \frac{1}{H_{uN}^N(i)} \right) du \right]. \end{aligned} \tag{3.82}$$

Using Young’s inequality, Lemma 3.4, and Tonelli’s theorem we get for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $T \in [0, \infty)$ that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \varepsilon_f^N(tN) \right| \right] &\leq \mathbb{E} \left[\int_0^T \sum_{i \in \mathcal{D}_f} \frac{\sigma_i c_f}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \left(\frac{|\kappa_H - N\kappa_H^N|}{2} \left(\left(\frac{1}{H_{uN}^N(i)} \right)^2 + \left(\sum_{j \in \mathcal{D}} m(i, j) H_{uN}^N(j) \right)^2 \right) \right. \right. \\ & \quad \left. \left. + |\alpha - N\alpha^N| + \frac{|\beta_H - N\beta_H^N|}{2} \frac{1}{H_{uN}^N(i)} \right) du \right] \\ &\leq \frac{c_f}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \int_0^T \frac{|\kappa_H - N\kappa_H^N|}{2} \left(\mathbb{E} \left[\left\| \frac{1}{(H_{uN}^N)^2} \right\|_{\sigma} \right] + c \mathbb{E} \left[\left\| (H_{uN}^N)^2 \right\|_{\sigma} \right] \right) \\ & \quad + |\alpha - N\alpha^N| \|\mathbb{1}\|_{\sigma} + \frac{|\beta_H - N\beta_H^N|}{2} \mathbb{E} \left[\left\| \frac{1}{H_{uN}^N} \right\|_{\sigma} \right] du. \end{aligned} \tag{3.83}$$

Hence, from Lemmas 3.5 and 3.6 we see for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ that

$$\begin{aligned} 0 &\leq \lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \varepsilon_f^N(tN) \right| \right] \\ &\leq \lim_{N \rightarrow \infty} \frac{T c_f}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \left(\frac{|\kappa_H - N\kappa_H^N|}{2} \sup_{M \in \mathbb{N}} \sup_{t \in [0, \infty)} \left(\mathbb{E} \left[\left\| \frac{1}{(H_t^M)^2} \right\|_{\sigma} \right] + c \mathbb{E} \left[\left\| (H_t^M)^2 \right\|_{\sigma} \right] \right) + |\alpha - N\alpha^N| \|\mathbb{1}\|_{\sigma} \right. \\ & \quad \left. + \frac{|\beta_H - N\beta_H^N|}{2} \sup_{M \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\left\| \frac{1}{H_t^M} \right\|_{\sigma} \right] \right) = 0. \end{aligned} \tag{3.84}$$

Next we show that the host population is asymptotically immediately in the equilibrium state. Define the set $\mathcal{R} := \left\{ \times_{i \in \mathcal{D}} B_i : (B_i)_{i \in \mathcal{D}} \subseteq \mathcal{B}([0, \infty)^{\mathcal{D}}), B_i = [0, \infty) \text{ for all but finitely many } i \in \mathcal{D} \right\}$.

For all $N \in \mathbb{N}$, all $t \in [0, \infty)$, and all $B \in \mathcal{R}$ define the measure-valued random variables

$$\Lambda^N([0, t] \times B) := \int_0^t \mathbb{1}_B(H_{uN}^N) du = \int_0^t \prod_{i \in \mathcal{D}} \mathbb{1}_{B_i}(H_{uN}^N(i)) du, \tag{3.85}$$

Due to Carathéodory’s theorem (see e.g. Klenke, 2008, Theorem 1.41) there is a unique extension of this pre-measure to a measure on $[0, \infty) \times E_2$, which we will denote by the same name. Define

the space

$$\ell(E_2) := \left\{ \mu: \begin{array}{l} \mu \text{ is a measure on } [0, \infty) \times E_2 \text{ such that for all } t \in [0, \infty) \\ \text{it holds that } \mu([0, t] \times E_2) = t \end{array} \right\}$$

and the space $D([0, \infty)) := \{f: [0, \infty) \rightarrow E_1 | f \text{ is càdlàg}\}$. Having checked all assumptions, we can now apply Theorem 2.1 from Kurtz (1992) and conclude that the sequence $\{((F_{tN}^N)_{t \in [0, \infty)}, \Lambda^N) : N \in \mathbb{N}\}$ is relatively compact in $D([0, \infty)) \times \ell(E_2)$. Let (F, Λ) be a $D([0, \infty)) \times \ell(E_2)$ -valued random variable and let $(N_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ be an increasing sequence such that $w\text{-}\lim_{k \rightarrow \infty} ((F_{tN_k}^{N_k})_{t \in [0, \infty)}, \Lambda^{N_k}) = (F, \Lambda)$. Due to Skorohod’s representation theorem (Theorem 3.1.8 of Ethier and Kurtz, 1986) we can assume without loss of generality and for ease of notation that (F, Λ) acts on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Using Hölder’s inequality and Theorem 2.2 we see for all $t \in [0, \infty)$ that

$$\begin{aligned} 0 &\leq \lim_{N \rightarrow \infty} \int_0^t \mathbb{E} \left[\left\| H_{uN}^N - (h_\infty(F_{uN}^N(i)))_{i \in \mathcal{D}} \right\|_\sigma \right] du = \lim_{N \rightarrow \infty} \int_0^t \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i |H_{uN}^N(i) - h_\infty(F_{uN}^N(i))| \right] du \\ &\leq \lim_{N \rightarrow \infty} \sqrt{\int_0^t \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (H_{uN}^N(i) - h_\infty(F_{uN}^N(i)))^2 \right] du} \sqrt{t \sum_{k \in \mathcal{D}} \sigma_k} = 0. \end{aligned} \tag{3.86}$$

For any bounded Lipschitz continuous function $f: l_\sigma^1 \rightarrow \mathbb{R}$, with Lipschitz constant \bar{c}_f , and all $t \in [0, \infty)$, applying (3.86), we then have

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\left| \int_0^t \int_{E_2} f(y) \Lambda(du \times dy) - \int_0^t f(h_\infty(F_u)) du \right| \right] \\ &= \lim_{k \rightarrow \infty} \mathbb{E} \left[\left| \int_0^t f(H_{uN_k}^{N_k}) du - \int_0^t f(h_\infty(F_{uN_k}^{N_k})) du \right| \right] \\ &\leq \bar{c}_f \lim_{k \rightarrow \infty} \mathbb{E} \left[\int_0^t \left\| H_{uN_k}^{N_k} - h_\infty(F_{uN_k}^{N_k}) \right\|_\sigma du \right] = 0. \end{aligned} \tag{3.87}$$

This implies that $\Lambda(du \times dy) = \delta_{h_\infty(F_u)}(dy) du$ where δ_x is the Dirac measure on x . Define the operator $\mathcal{A}_2 : \text{Dom}(\mathcal{A}) \rightarrow C(E_1, \mathbb{R})$ for all $f \in \text{Dom}(\mathcal{A})$ and all $x \in E_1$ by

$$\begin{aligned} (\mathcal{A}_2 f)(x) &:= \sum_{i \in \mathcal{D}} \left(\kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{a-x_i}{a-x_j} (x_j - x_i) \right) - \alpha x_i (1 - x_i) \right) \frac{df}{dx_i}(x) \\ &\quad + \frac{1}{2} \sum_{i \in \mathcal{D}} \beta_H b(a - x_i) x_i (1 - x_i) \frac{d^2 f}{dx_i^2}(x). \end{aligned} \tag{3.88}$$

Note for all $f \in \text{Dom}(\mathcal{A})$ and all $x \in E_1$ that $(\mathcal{A}_2 f)(x) = (\mathcal{A}_1 f)(x, h_\infty(x))$. Therefore, for all $t \in [0, \infty)$, all $f \in \text{Dom}(\mathcal{A})$, and all $x \in E_1$ we have \mathbb{P} -a.s.

$$\int_0^t \int_{E_2} (\mathcal{A}_1 f)(F_s, y) \Lambda(ds \times dy) = \int_0^t (\mathcal{A}_1 f)(F_s, h_\infty(F_s)) ds = \int_0^t (\mathcal{A}_2 f)(F_s) ds. \tag{3.89}$$

Applying Theorem 2.1 of Kurtz (1992) together with (3.89), we see for each $f \in \text{Dom}(\mathcal{A})$ that

$$\left(f(F_t) - \int_0^t (\mathcal{A}_2 f)(F_s) ds \right)_{t \in [0, \infty)} \tag{3.90}$$

is a martingale. Hence, F is a (weak) solution of (2.8). It remains to prove uniqueness. Note that for all $z_1, z_2 \in [0, 1]$ we have that $\frac{a-z_1}{a-z_2}(z_2 - z_1) = (a - z_1) \left(\frac{a-z_1}{a-z_2} - 1 \right)$. Using this and (2.5) we then

have for any subset $\mathcal{S} \subseteq \mathcal{D}$ and any $x, y \in E_2$ that

$$\begin{aligned}
 & \sum_{i \in \mathcal{S}} \sigma_i \mathbb{1}_{x_i \geq y_i} \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \left(\frac{(a-x_i)^2}{a-x_j} - (a-x_i) - \frac{(a-y_i)^2}{a-y_j} + (a-y_i) \right) - \alpha(x_i(1-x_i) - y_i(1-y_i)) \right) \\
 &= \sum_{i \in \mathcal{S}} \sigma_i \mathbb{1}_{x_i \geq y_i} \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \left((x_i - y_i) + \left((a-x_i)^2 - (a-y_i)^2 \right) \frac{1}{a-x_j} - (a-y_i)^2 \left(\frac{1}{a-y_j} - \frac{1}{a-x_j} \right) \right) \right. \\
 &\quad \left. + \alpha(-(x_i - y_i) + x_i^2 - y_i^2) \right) \\
 &\leq \sum_{i \in \mathcal{S}} \sigma_i \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \mathbb{1}_{x_j \geq y_j} (a-y_i)^2 \left(\frac{1}{a-x_j} - \frac{1}{a-y_j} \right) \right) + \sum_{i \in \mathcal{S}} \sigma_i (\kappa_H + 2\alpha) \mathbb{1}_{x_i \geq y_i} (x_i - y_i) \\
 &\leq \sum_{i \in \mathcal{S}} \sigma_i \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \mathbb{1}_{x_j \geq y_j} \frac{a^2}{(a-1)^2} (x_j - y_j) \right) + \sum_{i \in \mathcal{S}} \sigma_i (\kappa_H + 2\alpha) \mathbb{1}_{x_i \geq y_i} (x_i - y_i) \\
 &\leq \sum_{i \in \mathcal{S}} \sigma_i c \kappa_H \mathbb{1}_{x_i \geq y_i} \frac{a^2(x_i - y_i)}{(a-1)^2} + \sum_{i \in \mathcal{S}} \sigma_i (\kappa_H + 2\alpha) \mathbb{1}_{x_i \geq y_i} (x_i - y_i) = \sum_{i \in \mathcal{S}} \sigma_i \left(\frac{c \kappa_H a^2}{(a-1)^2} + \kappa_H + 2\alpha \right) (x_i - y_i)^+.
 \end{aligned} \tag{3.91}$$

This implies that equation (26) of [Hutzenthaler and Wakolbinger \(2007\)](#) is fulfilled. Together with the assumptions on m in Assumption 2.1 we now infer, analogous to Proposition 2.1 of [Hutzenthaler and Wakolbinger \(2007\)](#), that the system (2.8) has a unique strong solution with a.s. continuous paths. We conclude that any limit point of $\{ (F_{tN}^N)_{t \in [0, \infty)} : N \in \mathbb{N} \}$ solves (2.8). Combining this with the fact that $\{ (F_{tN}^N)_{t \in [0, \infty)} : N \in \mathbb{N} \}$ is relatively compact we obtain $(F_{tN}^N)_{t \in [0, \infty)} \implies (X_t)_{t \in [0, \infty)}$, as $N \rightarrow \infty$. This finishes the proof of Theorem 2.3. \square

4. McKean-Vlasov limit

In this section we investigate convergence of a sequence of exchangeable systems of stochastic differential equations and its application to our model.

4.1. *Setting.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $I \subset [0, \infty)$ be an interval of length $|I| \in (0, \infty]$ which is either of the form $[0, |I|]$ if $|I| < \infty$ or of the form $[0, \infty)$ if $|I| = \infty$, let $A \subseteq \mathbb{R}$ be a convex set, and let $\psi: I \rightarrow A$, $\xi: A \times I \rightarrow \mathbb{R}$, and $\sigma^2: I \rightarrow [0, \infty)$ be functions. The function $\sigma^2: I \rightarrow [0, \infty)$ is locally Lipschitz continuous in I and satisfies $\sigma^2(0) = 0$ and if $|I| < \infty$, then $\sigma^2(|I|) = 0$. Furthermore, the function σ^2 is strictly positive on $(0, |I|)$. There exists a constant $L \in (0, \infty)$ such that σ^2 satisfies the growth condition that for all $y \in I$ we have $\sigma^2(y) \leq L(y + y^2)$ and such that ξ satisfies for all $(u, x), (v, y) \in A \times I$ that

$$\mathbb{1}_{x \geq y} (\xi(u, x) - \xi(v, y)) \leq L|u - v| + L(x - y)^+. \tag{4.1}$$

The function $\psi: I \rightarrow [0, \infty)$ satisfies for all $x, y \in I$ that $|\psi(x) - \psi(y)| \leq L|x - y|$. Let $W(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$, $i \in \mathbb{N}$, be independent Brownian motions with continuous sample paths. For all $D \in \mathbb{N}$ let $X^D: [0, \infty) \times \{1, \dots, D\} \times \Omega \rightarrow I$ be an adapted stochastic process with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \{1, \dots, D\}$ \mathbb{P} -a.s. satisfies

$$X_t^D(i) = X_0^D(i) + \int_0^t \xi \left(\frac{1}{D} \sum_{j \in \{1, \dots, D\}} \psi(X_s^D(j)), X_s^D(i) \right) ds + \int_0^t \sqrt{\sigma^2(X_s^D(i))} dW_s(i). \tag{4.2}$$

Let $M: [0, \infty) \times \Omega \rightarrow I$ be an adapted stochastic process with continuous sample paths that for all $t \in [0, \infty)$ \mathbb{P} -a.s. satisfies

$$M_t = M_0 + \int_0^t \xi(\mathbb{E}[\psi(M_s)], M_s) ds + \int_0^t \sqrt{\sigma^2(M_s)} dW_s(1). \tag{4.3}$$

4.2. *McKean–Vlasov limit.* The following proposition, Proposition 4.1, partly generalizes Proposition 4.29 in [Hutzenthaler \(2012\)](#) where ξ depends linearly on its first argument.

Proposition 4.1. *Assume the setting of Section 4.1, let M_0 be an I -valued random variable, for every $D \in \mathbb{N}$ let $(X_0^D(j))_{j \in \{1, \dots, D\}}$ be exchangeable and integrable random variables with values in I . Then, there exists a unique solution M of (4.3) and for all $D \in \mathbb{N}$ and all $t \in [0, \infty)$ we have that*

$$\sqrt{D}\mathbb{E}\left[|X_t^D(1) - M_t|\right] \leq e^{(L^2+L+L\mu)t} \left(\sqrt{D}\mathbb{E}\left[|X_0^D(1) - M_0|\right] + L \int_0^t \left(\text{Var}(\psi(M_s)) \right)^{\frac{1}{2}} ds \right). \tag{4.4}$$

Proof: Existence of a weak solution is straightforward using a tightness argument. Next we show pathwise uniqueness for the SDE (4.3). Let $M, \bar{M}: [0, \infty) \times \Omega \rightarrow I$ be two solutions of the SDE (4.3). Then our assumptions and a standard Yamada-Watanabe argument (cf., e.g., Theorem 1 [Yamada and Watanabe \(1971\)](#)) shows for all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$|M_t - \bar{M}_t| = |M_0 - \bar{M}_0| + \int_0^t \text{sgn}(M_s - \bar{M}_s) d(M_s - \bar{M}_s). \tag{4.5}$$

Let $(\tau_l)_{l \in \mathbb{N}}$ be a localizing sequence for the local martingale

$$\left(\int_0^t \text{sgn}(M_s - \bar{M}_s) (\sigma^2(M_s) - \sigma^2(\bar{M}_s)) dW_s \right)_{t \in [0, \infty)}.$$

Then Fatou’s Lemma and our assumptions imply for all $t \in [0, \infty)$ that

$$\begin{aligned} \mathbb{E}[|M_t - \bar{M}_t|] &\leq \lim_{l \rightarrow \infty} \mathbb{E}[|M_{t \wedge \tau_l} - \bar{M}_{t \wedge \tau_l}|] \\ &\leq \mathbb{E}[|M_0 - \bar{M}_0|] + \mathbb{E}\left[\int_0^t \text{sgn}(M_s - \bar{M}_s) (\xi(\mathbb{E}[\psi(M_s)], M_s) - \xi(\mathbb{E}[\psi(\bar{M}_s)], \bar{M}_s)) ds \right] \\ &\leq \mathbb{E}[|M_0 - \bar{M}_0|] + L \int_0^t |\mathbb{E}[\psi(M_s)] - \mathbb{E}[\psi(\bar{M}_s)]| + \mathbb{E}[|M_s - \bar{M}_s|] ds \\ &\leq \mathbb{E}[|M_0 - \bar{M}_0|] + (L + 1)^2 \int_0^t \mathbb{E}[|M_s - \bar{M}_s|] ds. \end{aligned} \tag{4.6}$$

This together with Gronwall’s lemma implies pathwise uniqueness for the SDE (4.3). Therefore, the theorem of [Yamada and Watanabe \(1971\)](#) implies that the SDE (4.3) is exact. The rest of the proof is analogous to the proof of Proposition 4.29 in [Hutzenthaler \(2012\)](#) and we omit it here. \square

4.3. *Application to costly defense in structured populations.* In this section we verify the applicability of Proposition 4.1 to the case of costly defense in structured populations.

Lemma 4.2. *Let $\alpha, \beta, \kappa \in (0, \infty)$ and $a \in (1, \infty)$, let $I = [0, 1]$ and define the function $\sigma^2: I \rightarrow [0, \infty)$ by $I \ni x \mapsto \sigma^2(x) := \beta(a - x)x(1 - x)$, the function $\psi: I \rightarrow [0, \infty)$ by $I \ni x \mapsto \psi(x) := \frac{1}{a-x}$, and the function $\xi: [0, \infty) \times I \rightarrow \mathbb{R}$ by $[0, \infty) \times I \ni (u, x) \mapsto \xi(u, x) := \kappa(a - x)((a - x)u - 1) - \alpha x(1 - x)$. Then the interval I and the functions σ^2, ψ , and ξ satisfy the setting of Section 4.1 with $L = \max\{\beta a, \kappa a^2, \kappa + \alpha, \frac{1}{(a-1)^2}\}$.*

Proof: For all $(u, x), (v, y) \in [0, \infty) \times [0, 1]$ it holds that

$$\begin{aligned} & \mathbb{1}_{x \geq y}(\xi(u, x) - \xi(v, y)) \\ &= \mathbb{1}_{x \geq y}(\kappa(a - x)((a - x)u - 1) - \kappa(a - y)((a - y)v - 1) - \alpha x(1 - x) + \alpha y(1 - y)) \\ &= \mathbb{1}_{x \geq y}(\kappa[(a - x)^2 u - (a - x) - (a - y)^2 v + (a - y)] - \alpha(1 - (x + y))(x - y)) \tag{4.7} \\ &= \mathbb{1}_{x \geq y}(\kappa[(x - y) + ((a - x)^2 - (a - y)^2)u - (a - y)^2(v - u)] - \alpha(1 - (x + y))(x - y)) \\ &\leq (\kappa + \alpha)(x - y)^+ + \kappa a^2(u - v)^+ \leq L(x - y)^+ + L(u - v)^+. \end{aligned}$$

Moreover, for all $x, y \in I$ it holds that $\sigma^2(x) = \beta(a - x)x(1 - x) \leq \beta a x \leq L(x + x^2)$ and that

$$|\psi(x) - \psi(y)| = \left| \frac{1}{a-x} - \frac{1}{a-y} \right| = \left| \int_y^x \frac{1}{(a-z)^2} dz \right| \leq \frac{1}{(a-1)^2} |x - y| \leq L|x - y|. \tag{4.8}$$

This completes the proof of Lemma 4.2. □

5. Long-term behavior of the average defender frequency

Subject of this section is the proof of Theorem 2.4 which states a necessary and sufficient condition under which the costly defense trait goes to fixation in the many-demes limit (2.11).

5.1. *Setting.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\kappa, \alpha, \beta \in (0, \infty)$, $a \in (1, \infty)$, $c \in (0, 1)$, let $W : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion with continuous sample paths, let $Z : [0, \infty) \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ satisfies \mathbb{P} -a.s.

$$Z_t = Z_0 + \int_0^t (\kappa(a - Z_s)((a - Z_s)\mathbb{E}\left[\frac{1}{a-Z_s}\right] - 1) - \alpha Z_s(1 - Z_s)) ds + \int_0^t \sqrt{\beta(a - Z_s)Z_s(1 - Z_s)} dW_s. \tag{5.1}$$

Moreover, for all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ let $Z^\theta : [0, \infty) \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ satisfies \mathbb{P} -a.s.

$$Z_t^\theta = Z_0^\theta + \int_0^t (\kappa(a - Z_s^\theta) \left((a - Z_s^\theta)\theta - 1 \right) - \alpha Z_s^\theta(1 - Z_s^\theta)) ds + \int_0^t \sqrt{\beta(a - Z_s^\theta)Z_s^\theta(1 - Z_s^\theta)} dW_s. \tag{5.2}$$

For all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ and all $z \in [0, 1]$ define

$$\begin{aligned} m_\theta(z) &:= \beta c^{\frac{2\kappa}{\beta}(a\theta-1)} (1 - c)^{\frac{2\kappa}{\beta}(1-\theta(a-1))} (a - c)^{\frac{2\alpha}{\beta}} \frac{1}{\beta(a-z)z(1-z)} \exp\left(\int_c^z 2 \frac{\kappa(a-y)((a-y)\theta-1) - \alpha y(1-y)}{\beta(a-y)y(1-y)} dy\right) \\ &= z^{\frac{2\kappa}{\beta}(a\theta-1)-1} (1 - z)^{\frac{2\kappa}{\beta}(1-\theta(a-1))-1} (a - z)^{\frac{2\alpha}{\beta}-1}. \end{aligned} \tag{5.3}$$

Note that this defines the speed density (Karlin and Taylor, 1981, p. 95) for (5.2). Furthermore, note that for all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ it holds that

$$\int_0^1 m_\theta(z) dz < \infty. \tag{5.4}$$

For all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ define $c_\theta := \int_0^1 m_\theta(z) dz$, for all $x \in \{0, 1\}$ denote by δ_x the Dirac measure on $[0, 1]$, and for all $\theta \in [\frac{1}{a}, \frac{1}{a-1}]$ define the mapping $\Psi_\theta : \mathcal{B}([0, 1]) \rightarrow [0, 1]$ by

$$\mathcal{B}([0, 1]) \ni A \mapsto \Psi_\theta(A) := \begin{cases} \delta_0(A), & \text{if } \theta = \frac{1}{a}, \\ \delta_1(A), & \text{if } \theta = \frac{1}{a-1}, \\ \int_A \frac{1}{c_\theta} m_\theta(z) dz, & \text{if } \theta \in (\frac{1}{a}, \frac{1}{a-1}). \end{cases} \tag{5.5}$$

5.2. *Results for the equilibrium distribution.* Assume the setting of Section 5.1. Existence and uniqueness of the solution of (5.1) follow from Proposition 4.1. When $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ we have that Ψ_θ defines a probability distribution by (5.4), and we can apply Theorem V.54.5 of Rogers and Williams (2000) to conclude that it is the unique equilibrium distribution for (5.2). The proof of the following lemma, Lemma 5.1, is clear and therefore omitted.

Lemma 5.1. *Assume the setting of Section 5.1. A probability measure $\Phi: \mathcal{B}([0, 1]) \rightarrow [0, 1]$ is an equilibrium distribution of the dynamics (5.1) if and only if there exists a $\theta \in [\frac{1}{a}, \frac{1}{a-1}]$ such that $\Phi = \Psi_\theta$.*

Lemma 5.2. *Assume the setting of Section 5.1 and let $\theta \in (\frac{1}{a}, \frac{1}{a-1})$. Then we have*

$$\int_0^1 \frac{1}{a-z} \Psi_\theta(dz) \begin{cases} < \theta, & \text{if } \alpha > \beta, \\ = \theta, & \text{if } \alpha = \beta, \\ > \theta, & \text{if } \alpha < \beta. \end{cases} \tag{5.6}$$

Proof: Define $u := \frac{2\kappa}{\beta}(a\theta - 1)$ and $v := \frac{2\kappa}{\beta}(1 - \theta(a - 1))$ and note that $u, v \in (0, \infty)$. Let $\Gamma: (0, \infty) \rightarrow (0, \infty)$ be the Gamma function, i.e., for all $x \in (0, \infty)$ let $\Gamma(x) := \int_0^\infty z^{x-1} e^{-z} dz$. It is well-known that for all $x \in (0, \infty)$ the Gamma function satisfies $\Gamma(x + 1) = x\Gamma(x)$ and that for all $x, y \in (0, \infty)$ it holds that $\int_0^1 z^{x-1}(1 - z)^{y-1} dz = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Thus, we obtain

$$\begin{aligned} & \int_0^1 z^{u-1}(1 - z)^{v-1}(a - z) \left(\frac{1}{a-z} - \theta \right) dz \\ &= \int_0^1 z^{u-1}(1 - z)^{v-1} dz - a\theta \int_0^1 z^{u-1}(1 - z)^{v-1} dz + \theta \int_0^1 z^u(1 - z)^{v-1} dz \\ &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} - a\theta \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} + \theta \frac{\Gamma(u+1)\Gamma(v)}{\Gamma(u+v+1)} = \left((1 - a\theta) \frac{(u+v)\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} + \theta \frac{u\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} \right) \\ &= (u(1 - a\theta) + v(1 - a\theta)) \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} \\ &= \frac{2\kappa}{\beta} ((a\theta - 1)(1 - \theta(a - 1)) + (1 - \theta(a - 1))(1 - a\theta)) \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} \\ &= \left(\frac{2\kappa}{\beta} (1 - \theta(a - 1))(a\theta - 1 + 1 - a\theta) \right) \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} = 0. \end{aligned} \tag{5.7}$$

First, consider the case $\alpha = \beta$. Using (5.7) we see that

$$\begin{aligned} \int_0^1 \frac{1}{a-z} \Psi_\theta(dz) - \theta &= \int_0^1 c_\theta z^{\frac{2\kappa}{\beta}(a\theta-1)-1} (1 - z)^{\frac{2\kappa}{\beta}(1-\theta(a-1))-1} (a - z)^{\frac{2\alpha}{\beta}-1} \left(\frac{1}{a-z} - \theta \right) dz \\ &= c_\theta \int_0^1 z^{u-1}(1 - z)^{v-1}(a - z) \left(\frac{1}{a-z} - \theta \right) dz = 0. \end{aligned} \tag{5.8}$$

Now, consider the case $\alpha > \beta$. Let $\hat{\delta} := \alpha - \beta$, $\delta := \frac{2\hat{\delta}}{\beta}$, and $z^* := \sup\{z \in (0, 1) : \frac{1}{a-z} - \theta < 0\}$. Note that $\hat{\delta}, \delta > 0$ and $z^* = a - \frac{1}{\theta} \in (0, 1)$. Also note that for all $z \in (0, z^*)$ we have $\frac{1}{a-z} - \theta < 0$ and $(a - z)^\delta > (a - z^*)^\delta$. Furthermore, for all $z \in (z^*, 1)$ we have $\frac{1}{a-z} - \theta > 0$ and $(a - z)^\delta < (a - z^*)^\delta$.

Together with (5.7) we thereby obtain

$$\begin{aligned}
 \int_0^1 \frac{1}{a-z} \Psi_\theta(dz) - \theta &= \int_0^1 \left(\frac{1}{a-z} - \theta \right) \Psi_\theta(dz) = \int_0^1 c_\theta z^{u-1} (1-z)^{v-1} (a-z)^{\frac{2\alpha}{\beta}-1} \left(\frac{1}{a-z} - \theta \right) dz \\
 &= \int_0^{z^*} c_\theta z^{u-1} (1-z)^{v-1} (a-z)^{1+\delta} \left(\frac{1}{a-z} - \theta \right) dz + \int_{z^*}^1 c_\theta z^{u-1} (1-z)^{v-1} (a-z)^{1+\delta} \left(\frac{1}{a-z} - \theta \right) dz \\
 &< c_\theta (a-z^*)^\delta \left(\int_0^{z^*} z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz + \int_{z^*}^1 z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz \right) \\
 &= c_\theta (a-z^*)^\delta \int_0^1 z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz = 0.
 \end{aligned}
 \tag{5.9}$$

The case $\alpha < \beta$ can be proved analogously and thereby, we omit it here. This completes the proof. \square

5.3. Proof of Theorem 2.4.

Proof of Theorem 2.4. Applying Itô’s lemma, we get for all $t \in [0, \infty)$ that

$$\begin{aligned}
 \frac{1}{a-Z_t} - \frac{1}{a-Z_0} &= \int_0^t \frac{1}{(a-Z_s)^2} \left(\kappa(a-Z_s) \left((a-Z_s) \mathbb{E} \left[\frac{1}{a-Z_s} \right] - 1 \right) - \alpha Z_s(1-Z_s) \right) \\
 &\quad + \frac{1}{2} \frac{2(a-Z_s)}{(a-Z_s)^4} \beta(a-Z_s) Z_s(1-Z_s) ds + \int_0^t \frac{1}{(a-Z_s)^2} \sqrt{\beta(a-Z_s) Z_s(1-Z_s)} dW_s \\
 &= \int_0^t \kappa \left(\mathbb{E} \left[\frac{1}{a-Z_s} \right] - \frac{1}{a-Z_s} \right) - \frac{\alpha Z_s(1-Z_s)}{(a-Z_s)^2} + \frac{\beta Z_s(1-Z_s)}{(a-Z_s)^2} ds + \int_0^t \frac{1}{(a-Z_s)^2} \sqrt{\beta(a-Z_s) Z_s(1-Z_s)} dW_s.
 \end{aligned}
 \tag{5.10}$$

After taking expectations we can apply Fubini’s theorem to obtain for all $t \in [0, \infty)$ that

$$\begin{aligned}
 \mathbb{E} \left[\frac{1}{a-Z_t} \right] - \mathbb{E} \left[\frac{1}{a-Z_0} \right] &= \int_0^t \kappa \left(\mathbb{E} \left[\frac{1}{a-Z_s} \right] - \mathbb{E} \left[\frac{1}{a-Z_s} \right] \right) - \alpha \mathbb{E} \left[\frac{Z_s(1-Z_s)}{(a-Z_s)^2} \right] + \beta \mathbb{E} \left[\frac{Z_s(1-Z_s)}{(a-Z_s)^2} \right] ds \\
 &= (\beta - \alpha) \int_0^t \mathbb{E} \left[\frac{Z_s(1-Z_s)}{(a-Z_s)^2} \right] ds.
 \end{aligned}
 \tag{5.11}$$

Since for all $s \in [0, \infty)$ it holds that $\mathbb{E} \left[\frac{Z_s(1-Z_s)}{(a-Z_s)^2} \right] \geq 0$ we conclude that the function $[0, \infty) \ni t \mapsto \mathbb{E} \left[\frac{1}{a-Z_t} \right] \in \left[\frac{1}{a}, \frac{1}{a-1} \right]$ converges monotonically non-increasing as $t \rightarrow \infty$ if $\alpha > \beta$, monotonically non-decreasing if $\alpha < \beta$, or is constant if $\alpha = \beta$.

First, assume $\alpha > \beta$. From (5.1) we see that δ_1 is an invariant measure for Z . So if $\mathbb{P}[Z_0 = 1] = 1$, then for all $t \in [0, \infty)$ it holds that $\mathbb{P}[Z_t = 1] = 1$. Now let $\mathbb{P}[Z_0 = 1] < 1$, implying $\mathbb{E} \left[\frac{1}{a-Z_0} \right] \in \left[\frac{1}{a}, \frac{1}{a-1} \right)$. Define $\theta := \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{a-Z_t} \right]$ and fix it for the rest of the paragraph. Note that due to the monotonicity stated above we have $\theta \in \left[\frac{1}{a}, \frac{1}{a-1} \right)$. Aiming at a contradiction, we assume that $\theta \in \left(\frac{1}{a}, \frac{1}{a-1} \right)$. Choose any $\varepsilon \in \left(0, \frac{1}{a-1} - \theta \right)$ and fix it for the rest of the proof. By definition of θ there exists an $s_\varepsilon \in (0, \infty)$, such that for all $t \in [s_\varepsilon, \infty)$ it holds that $\mathbb{E} \left[\frac{1}{a-Z_t} \right] < \theta + \varepsilon$. Let $\tilde{W} : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion with continuous sample paths, let $\tilde{Z} : [0, \infty) \times \Omega \rightarrow [0, 1]$ and $\tilde{Z}^{\theta+\varepsilon} : [0, \infty) \times \Omega \rightarrow [0, 1]$ be adapted processes with continuous sample paths that satisfy for

all $t \in [0, \infty)$ \mathbb{P} -a.s.

$$\begin{aligned} \tilde{Z}_t &= \tilde{Z}_0 + \int_0^t \left(\kappa(a - \tilde{Z}_s) \left((a - \tilde{Z}_s) \mathbb{E} \left[\frac{1}{a - \tilde{Z}_s} \right] - 1 \right) - \alpha \tilde{Z}_s (1 - \tilde{Z}_s) \right) ds \\ &\quad + \int_0^t \sqrt{\beta(a - \tilde{Z}_s) \tilde{Z}_s (1 - \tilde{Z}_s)} d\tilde{W}_s, \\ \tilde{Z}_t^{\theta+\varepsilon} &= \tilde{Z}_0^{\theta+\varepsilon} + \int_0^t \left(\kappa(a - \tilde{Z}_s^{\theta+\varepsilon}) \left((a - \tilde{Z}_s^{\theta+\varepsilon})(\theta + \varepsilon) - 1 \right) - \alpha \tilde{Z}_s^{\theta+\varepsilon} (1 - \tilde{Z}_s^{\theta+\varepsilon}) \right) ds \\ &\quad + \int_0^t \sqrt{\beta(a - \tilde{Z}_s^{\theta+\varepsilon}) \tilde{Z}_s^{\theta+\varepsilon} (1 - \tilde{Z}_s^{\theta+\varepsilon})} d\tilde{W}_s, \end{aligned} \tag{5.12}$$

such that $\tilde{Z}_0^{\theta+\varepsilon} = \tilde{Z}_0$ and such that \tilde{Z}_0 and Z_{s_ε} are equal in distribution. Then for each $t \in [s_\varepsilon, \infty)$ we have that Z_t and $\tilde{Z}_{t-s_\varepsilon}$ are equal in distribution and the drift term of $\tilde{Z}_{t-s_\varepsilon}$ is lower than that of $\tilde{Z}_{t-s_\varepsilon}^{\theta+\varepsilon}$. Together with the fact that the mapping $[0, 1] \ni z \mapsto \frac{1}{a-z}$ is strictly monotonically increasing this implies for all $t \in [s_\varepsilon, \infty)$ that

$$\mathbb{E} \left[\frac{1}{a - Z_t} \right] = \mathbb{E} \left[\frac{1}{a - \tilde{Z}_{t-s_\varepsilon}} \right] \leq \mathbb{E} \left[\frac{1}{a - \tilde{Z}_{t-s_\varepsilon}^{\theta+\varepsilon}} \right]. \tag{5.13}$$

Recall from Section 5.2 that for any $\eta \in (\frac{1}{a}, \frac{1}{a-1})$ we have that Ψ_η is the unique equilibrium distribution of \tilde{Z}^η . Combining this with (5.13) we obtain (see, e.g., Theorem V.54.5 Rogers and Williams (2000))

$$\theta = \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{a - Z_t} \right] \leq \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{a - \tilde{Z}_{t-s_\varepsilon}^{\theta+\varepsilon}} \right] = \int_0^1 \frac{1}{a-z} \Psi_{\theta+\varepsilon}(dz). \tag{5.14}$$

The dominated convergence theorem yields that the mapping $(\frac{1}{a}, \frac{1}{a-1}) \ni \eta \mapsto \Psi_\eta$ is continuous with respect to the weak topology. Applying this, (5.14) together with the fact that $\varepsilon \in (0, \frac{1}{a-1} - \theta)$ was arbitrarily chosen, and Lemma 5.2, we obtain the contradiction

$$\theta \leq \lim_{\delta \rightarrow 0} \int_0^1 \frac{1}{a-z} \Psi_{\theta+\delta}(dz) = \int_0^1 \frac{1}{a-z} \Psi_\theta(dz) < \theta. \tag{5.15}$$

Hence, we have $\theta = \frac{1}{a}$, implying

$$0 \leq \lim_{t \rightarrow \infty} \mathbb{E} [Z_t] \leq \lim_{t \rightarrow \infty} a^2 \mathbb{E} \left[\frac{Z_t}{a(a - Z_t)} \right] = \lim_{t \rightarrow \infty} a^2 \mathbb{E} \left[\frac{1}{a - Z_t} \right] - a^2 \frac{1}{a} = 0. \tag{5.16}$$

The case $\alpha < \beta$ can be proved analogously and we omit it here.

Finally, assume $\alpha = \beta$, define $\theta := \mathbb{E} \left[\frac{1}{a - Z_0} \right]$, and fix it for the rest of the proof. We see from (5.11) that $\mathbb{E} \left[\frac{1}{a - Z_t} \right]$ is constant in $t \in [0, \infty)$. Thus, assuming that Z_0 and Z_0^θ are equal in distribution we see from (5.1) and (5.2) that for all $t \in [0, \infty)$ it holds that Z_t and Z_t^θ are equal in distribution. Recall from Section 5.2 that Ψ_θ is the unique equilibrium distribution of Z^θ . Consequently, Ψ_θ is the unique equilibrium distribution of Z . This completes the proof of Theorem 2.4. \square

6. Invasion of a costly defense allele

In this section we investigate a tree of excursions which could be the many-demes limit of the total mass process when there is only one island occupied initially. For this tree of excursions, Proposition 6.1 establishes a necessary and sufficient condition for extinction. Thus, informally speaking, we get an explicit condition when the invasion probability is positive in an infinite-dimensional space.

6.1. *Setting.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\kappa, \alpha, \beta \in (0, \infty)$, $a \in (1, \infty)$, and let $W(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$, $i \in \mathbb{N}$, be independent Brownian motions with continuous sample paths. For all $D \in \mathbb{N}$ let $X^D: [0, \infty) \times \{1, \dots, D\} \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \{1, \dots, D\}$ \mathbb{P} -a.s. satisfies

$$X_t^D(i) = X_0^D(i) + \int_0^t \kappa(a - X_s^D(i)) \left((a - X_s^D(i)) \frac{1}{D} \sum_{j=1}^D \frac{1}{a - X_s^D(j)} - 1 \right) - \alpha X_s^D(i)(1 - X_s^D(i)) ds + \int_0^t \sqrt{\beta(a - X_s^D(i))X_s^D(i)(1 - X_s^D(i))} dW_s(i). \tag{6.1}$$

Let $\tilde{a}: [0, 1] \rightarrow [0, \infty)$ be a function defined by

$$[0, 1] \ni x \mapsto \tilde{a}(x) := \kappa a \frac{x}{a-x}. \tag{6.2}$$

Then, assuming there is positive mass only in deme 1, the dynamics in deme 1 follows asymptotically the following process Y . Let $Y: [0, \infty) \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths such that for all $t \in [0, \infty)$ it \mathbb{P} -a.s. holds that

$$Y_t = Y_0 - \int_0^t \frac{\kappa}{a} Y_s(a - Y_s) + \alpha Y_s(1 - Y_s) ds + \int_0^t \sqrt{\beta(a - Y_s)Y_s(1 - Y_s)} dW_s(1). \tag{6.3}$$

In addition, let Q_Y be the excursion measure which satisfies $Q_Y = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{P}[Y \in \cdot | Y_0 = \varepsilon]$ in a suitable sense; see [Pitman and Yor \(1982\)](#) and [Hutzenthaler \(2009\)](#) for details. Asymptotically in the many-demes limit, every deme with population path $\chi \in C([0, \infty), [0, 1])$ populates demes through migration and these new populations are given by a Poisson point process with intensity measure $\tilde{a}(\chi_t)dt \times Q_Y(d\psi)$. Now let $(V_t)_{t \in [0, \infty)}$ be the total mass process of the associated tree of excursions with initial island measure that equals the distribution of Y in (6.3) and excursion measure Q_Y .

6.2. *Survival or extinction of an invading costly defense allele.*

Proposition 6.1. *Assume the setting of Section 6.1. Let $x \in (0, 1]$ and assume $Y_0 = x = V_0$. Then the total mass process dies out (i.e., converges in probability to zero as $t \rightarrow \infty$) if and only if*

$$\alpha \geq \beta. \tag{6.4}$$

Proof: Define the functions $s: [0, 1] \rightarrow [0, \infty)$ and $S: [0, 1] \rightarrow [0, \infty)$ for all $y \in [0, 1]$ by $s(y) := \exp\left(-\int_0^y \frac{-\frac{\kappa}{a}x(a-x) - \alpha x(1-x)}{\frac{1}{2}\beta(a-x)x(1-x)} dx\right)$ and by $S(y) := \int_0^y s(z) dz$. Note that for all $z \in [0, 1]$ it holds that

$$s(z) = \exp\left(\int_0^z \frac{2\kappa}{a\beta} \frac{1}{1-x} + \frac{2\alpha}{\beta} \frac{1}{a-x} dx\right) = (1-z)^{\frac{-2\kappa}{a\beta}} \left(\frac{a-z}{a}\right)^{\frac{-2\alpha}{\beta}} \tag{6.5}$$

and

$$S(z) = \int_0^z s(x) dx \leq zs(z). \tag{6.6}$$

We will apply Theorem 5 from [Hutzenthaler \(2009\)](#) to show the result. First, we verify that the assumptions of the aforementioned theorem are satisfied. Using (6.6), we see that

$$\int_0^{\frac{1}{2}} S(y) \frac{2}{\beta(a-y)y(1-y)s(y)} dy \leq \int_0^{\frac{1}{2}} \frac{2}{\beta(a-y)(1-y)} dy \leq \frac{1}{2} \frac{2}{\beta(a-\frac{1}{2})(1-\frac{1}{2})} < \infty. \tag{6.7}$$

Furthermore, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\frac{1}{2}} \frac{-\frac{\kappa}{a}(a-y)y - \alpha y(1-y)}{\frac{1}{2}\beta(a-y)y(1-y)} dy &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\frac{1}{2}} \frac{-2\kappa}{a\beta(1-y)} - \frac{2\alpha}{\beta(a-y)} dy \\ &= \lim_{\varepsilon \rightarrow 0} \left(\frac{2\kappa}{a\beta} (\ln(1 - \frac{1}{2}) - \ln(1 - \varepsilon)) + \frac{2\alpha}{\beta} (\ln(a - \frac{1}{2}) - \ln(a - \varepsilon)) \right) \\ &= \frac{2\kappa}{a\beta} \ln(1 - \frac{1}{2}) + \frac{2\alpha}{\beta} (\ln(a - \frac{1}{2}) - \ln(a)) \in (-\infty, \infty). \end{aligned} \tag{6.8}$$

From (6.5) as well as the fact that $\frac{2\kappa}{a\beta} > 0$ we see that

$$\begin{aligned} \int_{\frac{1}{2}}^1 \frac{\tilde{a}(y)}{\frac{1}{2}\beta(a-y)y(1-y)s(y)} dy &= \int_{\frac{1}{2}}^1 \frac{\kappa a \frac{y}{a-y}}{\frac{1}{2}\beta(a-y)y(1-y)} (1-y)^{\frac{2\kappa}{a\beta}} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}} dy \\ &= 2 \frac{\kappa a}{\beta} a^{-\frac{2\alpha}{\beta}} \int_{\frac{1}{2}}^1 (1-y)^{\frac{2\kappa}{a\beta}-1} (a-y)^{\frac{2\alpha}{\beta}-2} dy \\ &\leq 2 \frac{\kappa a}{\beta} a^{-\frac{2\alpha}{\beta}} \left((a - \frac{1}{2})^{\frac{2\alpha}{\beta}-2} + (a-1)^{\frac{2\alpha}{\beta}-2} \right) \int_{\frac{1}{2}}^1 (1-y)^{\frac{2\kappa}{a\beta}-1} dy < \infty. \end{aligned} \tag{6.9}$$

We obtain from (6.7), (6.8), and (6.9) together with a straightforward adaptation of Lemmas 9.6, 9.9, and 9.10 in [Hutzenthaler \(2009\)](#) to the state space $[0, 1]$ that the assumptions of Theorem 5 in [Hutzenthaler \(2009\)](#) are satisfied. Applying the aforementioned theorem shows that the total mass process dies out if and only if

$$\int \int_0^\infty \tilde{a}(\chi_t) dt Q_Y(d\chi) \leq 1. \tag{6.10}$$

Moreover, a straight forward adaptation of Lemma 9.8 in [Hutzenthaler \(2009\)](#) to the state space $[0, 1]$ together with (6.7) shows that

$$\int \int_0^\infty \tilde{a}(\chi_t) dt Q_Y(d\chi) = \int_0^1 \frac{\kappa a \frac{y}{a-y}}{\frac{1}{2}\beta(a-y)y(1-y)} (1-y)^{\frac{2\kappa}{a\beta}} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}} dy. \tag{6.11}$$

Observe that we have $\frac{2\kappa}{a\beta} \int_0^1 (1-y)^{\frac{2\kappa}{a\beta}-1} dy = 1$. Combining this with (6.10) and (6.11) we see that the total mass process dies out if and only if

$$\begin{aligned} 0 &\geq \int_0^1 \frac{\kappa a \frac{y}{a-y}}{\frac{1}{2}\beta(a-y)y(1-y)} (1-y)^{\frac{2\kappa}{a\beta}} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}} dy - 1 \\ &= \frac{2\kappa}{a\beta} \int_0^1 (1-y)^{\frac{2\kappa}{a\beta}-1} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}-2} dy - 1 = \frac{2\kappa}{a\beta} \int_0^1 (1-y)^{\frac{2\kappa}{a\beta}-1} \left(\left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}-2} - 1 \right) dy. \end{aligned} \tag{6.12}$$

Consequently, the total mass process dies out if and only if $\alpha \geq \beta$. This completes the proof of Proposition 6.1. \square

Appendix

In this appendix we prove Lemmas 3.5, 3.6, and 3.7. The main step in the proofs of these lemmas is a comparison with the solution of an ordinary differential equation whose derivative is negative near infinity. Solutions of such ordinary differential equations are bounded uniformly in time.

Proof of Lemma 3.5: If we assume $\sup_{N \in \mathbb{N}} \mathbb{E} [\| (H_0^N + P_0^N)^p \|_\sigma] = \infty$, then the claim trivially holds. For the remainder of the proof assume $\sup_{N \in \mathbb{N}} \mathbb{E} [\| (H_0^N + P_0^N)^p \|_\sigma] < \infty$. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \mathcal{D}$ be a set with $|\mathcal{D}_n| = \min\{n, |\mathcal{D}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define real numbers $c_0 := \min\left\{\frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma\right\}$, $c_1 := p [2\eta \bar{l}_H + \delta \bar{l}_P + (p-1) (2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P)] \in (0, \infty)$, $c_2 := \lambda p + (p-1+c)(\bar{\kappa}_H + \bar{\kappa}_P) \in (0, \infty)$, $c_3 := c_0 p (\sum_{k \in \mathcal{D}} \sigma_k)^{-\frac{1}{p}} \in (0, \infty)$, and $c_4 := c_1 (\sum_{k \in \mathcal{D}} \sigma_k)^{\frac{1}{p}} \in (0, \infty)$. For

all $N \in \mathbb{N}$, $t \in [0, \infty)$ define $Y_t^N := 2\eta H_t^N + \delta P_t^N$ and for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ let $M_t^{N,n}$ be a real-valued random variable such that \mathbb{P} -a.s. it holds that

$$M_t^{N,n} = \sum_{i \in \mathcal{D}_n} \sigma_i \left(\int_0^t 2\eta p(Y_u^N(i))^{p-1} \sqrt{\beta_H^N H_u^N(i)} dW_u^{H,N}(i) + \int_0^t \delta p(Y_u^N(i))^{p-1} \sqrt{\beta_P^N P_u^N(i)} dW_u^{P,N}(i) \right). \tag{6.13}$$

Applying Itô's lemma we get for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} & \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p - \sum_{i \in \mathcal{D}_n} \sigma_i (Y_0^N(i))^p \\ &= \sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t 2\eta p(Y_u^N(i))^{p-1} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + (\lambda - \kappa_H^N - \alpha^N F_u^N(i)) H_u^N(i) \right. \\ & \quad \left. - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) + \delta p(Y_u^N(i))^{p-1} \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) \right. \\ & \quad \left. - (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 + (\eta - \rho F_u^N(i)) P_u^N(i) H_u^N(i) + \iota_P^N \right) \\ & \quad + \frac{1}{2} 4\eta^2 p(p-1) (Y_u^N(i))^{p-2} \beta_H^N H_u^N(i) + \frac{1}{2} \delta^2 p(p-1) (Y_u^N(i))^{p-2} \beta_P^N P_u^N(i) du + M_t^{N,n}. \end{aligned} \tag{6.14}$$

Because $1 \geq c_0 4$, $\frac{\lambda}{K} \geq c_0 2\eta$, and $\gamma \geq \delta c_0$ we get for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} & \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p - \sum_{i \in \mathcal{D}_n} \sigma_i (Y_0^N(i))^p \\ & \leq \sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t p(Y_u^N(i))^{p-1} \left(2\eta \bar{\kappa}_H \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + 2\eta \lambda H_u^N(i) - c_0 (2\eta H_u^N(i))^2 \right. \\ & \quad \left. - [\eta \delta + c_0 4\eta \delta] P_u^N(i) H_u^N(i) + 2\eta \bar{\iota}_H \right) \\ & \quad + p(Y_u^N(i))^{p-1} \left(\delta \bar{\kappa}_P \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) + \lambda \delta P_u^N(i) - c_0 (\delta P_u^N(i))^2 + \eta \delta P_u^N(i) H_u^N(i) + \delta \bar{\iota}_P \right) \\ & \quad + p(p-1) (Y_u^N(i))^{p-2} \left((2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P) 2\eta H_u^N(i) + (\frac{1}{2} \delta \bar{\beta}_P + 2\eta \bar{\beta}_H) \delta P_u^N(i) \right) du + M_t^{N,n} \\ & = \int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i p(Y_u^N(i))^{p-1} 2\eta \bar{\kappa}_H \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \sum_{i \in \mathcal{D}_n} \sigma_i p(Y_u^N(i))^{p-1} \\ & \quad \left(\lambda (Y_u^N(i)) - c_0 (Y_u^N(i))^2 + (2\eta \bar{\iota}_H + \delta \bar{\iota}_P) \right) + \sum_{i \in \mathcal{D}_n} \sigma_i p(Y_u^N(i))^{p-1} \delta \bar{\kappa}_P \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) \\ & \quad + \sum_{i \in \mathcal{D}_n} \sigma_i p(p-1) (2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P) (Y_u^N(i))^{p-1} du + M_t^{N,n}. \end{aligned} \tag{6.15}$$

Using Young’s inequality and Lemma 3.4 we get for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} & \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p - \sum_{i \in \mathcal{D}_n} \sigma_i (Y_0^N(i))^p \\ & \leq \int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \frac{p-1}{p} p (Y_u^N(i))^p \bar{\kappa}_H + \sum_{i \in \mathcal{D}} \sigma_i \frac{1}{p} p \bar{\kappa}_{HC} (2\eta H_u^N(i))^p + \sum_{i \in \mathcal{D}_n} \sigma_i \lambda p (Y_u^N(i))^p \\ & \quad + \sum_{i \in \mathcal{D}_n} \sigma_i c_1 (Y_u^N(i))^{p-1} - \sum_{i \in \mathcal{D}_n} \sigma_i c_0 p (Y_u^N(i))^{p+1} + \sum_{i \in \mathcal{D}_n} \sigma_i \frac{p-1}{p} p (Y_u^N(i))^p \bar{\kappa}_P \\ & \quad + \sum_{i \in \mathcal{D}} \sigma_i \frac{1}{p} p \bar{\kappa}_{PC} (\delta P_u^N(i))^p du + M_t^{N,n} \\ & \leq \int_0^t \sum_{i \in \mathcal{D}} \sigma_i c_2 (Y_u^N(i))^p + \sum_{i \in \mathcal{D}_n} \sigma_i c_1 (Y_u^N(i))^{p-1} - \sum_{i \in \mathcal{D}_n} \sigma_i c_0 p (Y_u^N(i))^{p+1} du + M_t^{N,n}. \end{aligned} \tag{6.16}$$

For $N, n, l \in \mathbb{N}$ define $[0, \infty]$ -valued stopping times

$$\tau_l^{N,n} := \inf \left(\left\{ t \in [0, \infty) : \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p > l \right\} \cup \infty \right). \tag{6.17}$$

We now get for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \left[\left(2\eta p (Y_u^N(i))^{p-1} \sqrt{\beta_H^N H_u^N(i)} \right)^2 + \left(\delta p (Y_u^N(i))^{p-1} \sqrt{\beta_P^N P_u^N(i)} \right)^2 \right] du \right] \\ & = \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i p^2 \left[2\eta \beta_H^N \left((Y_u^N(i))^{p-1} \sqrt{2\eta H_u^N(i)} \right)^2 + \delta \beta_P^N \left((Y_u^N(i))^{p-1} \sqrt{\delta P_u^N(i)} \right)^2 \right] du \right] \\ & \leq (2\eta \beta_H^N + \delta \beta_P^N) \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i p^2 \left[\left(2 (Y_u^N(i))^{\frac{2p-1}{2}} \right)^2 \right] du \right] \end{aligned} \tag{6.18}$$

Using Young’s inequality, we obtain for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \left[\left(2\eta p (Y_u^N(i))^{p-1} \sqrt{\beta_H^N H_u^N(i)} \right)^2 + \left(\delta p (Y_u^N(i))^{p-1} \sqrt{\beta_P^N P_u^N(i)} \right)^2 \right] du \right] \\ & \leq (2\eta \beta_H^N + \delta \beta_P^N) \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \left[4p^2 \left(\frac{2p-1}{2p} (Y_u^N(i))^p + \frac{1}{2p} \right)^2 \right] du \right] \\ & \leq (2\eta \beta_H^N + \delta \beta_P^N) \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \frac{\sigma_i^2}{\min_{k \in \mathcal{D}_n} \sigma_k} \left[\left((2p-1) (Y_u^N(i))^p + 1 \right)^2 \right] du \right] \\ & \leq \frac{2\eta \beta_H^N + \delta \beta_P^N}{\min_{k \in \mathcal{D}_n} \sigma_k} \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \left[\left(\sum_{i \in \mathcal{D}_n} \sigma_i \left((2p-1) (Y_u^N(i))^p + 1 \right) \right)^2 \right] du \right] \\ & \leq \frac{2\eta \beta_H^N + \delta \beta_P^N}{\min_{k \in \mathcal{D}_n} \sigma_k} \mathbb{E} \left[\int_0^t \left[\left((2p-1) \sum_{i \in \mathcal{D}_n} \sigma_i (Y_{u \wedge \tau_l^{N,n}}(i))^p + \|\mathbb{1}\|_\sigma \right)^2 \right] du \right] \\ & \leq \frac{2\eta \beta_H^N + \delta \beta_P^N}{\min_{k \in \mathcal{D}_n} \sigma_k} t \left[((2p-1)l + \|\mathbb{1}\|_\sigma)^2 \right] < \infty. \end{aligned} \tag{6.19}$$

Hence, we get for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that $\mathbb{E} \left[M_{t \wedge \tau_l^{N,n}}^{N,n} \right] = 0$. From this and (6.16) and using Tonelli’s theorem we see for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_{t \wedge \tau_l^{N,n}}^N(i) \right)^p + \int_0^{t \wedge \tau_l^{N,n}} c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_u^N(i) \right)^{p+1} du \right] \\ & \leq \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} c_2 \left\| (Y_u^N)^p \right\|_\sigma + c_1 \left\| (Y_u^N)^{p-1} \right\|_\sigma du \right] \\ & \leq \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \int_0^t c_2 \mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] + c_1 \mathbb{E} \left[\left\| (Y_u^N)^{p-1} \right\|_\sigma \right] du. \end{aligned} \tag{6.20}$$

For every $N, n \in \mathbb{N}$ the map $[0, \infty) \ni t \mapsto \sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_t^N(i) \right)^p \in \mathbb{R}$ is \mathbb{P} -a.s. continuous which implies for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that $\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,n} < t \right] = 0$. From Tonelli’s theorem and monotone convergence, then using Fatou’s lemma, and finally applying (6.20) we see for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_t^N(i) \right)^p \right] + \int_0^t c_0 p \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^{p+1} \right] du \\ & = \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_t^N(i) \right)^p + \int_0^t c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_u^N(i) \right)^{p+1} du \right] \\ & = \lim_{n \rightarrow \infty} \mathbb{E} \left[\lim_{l \rightarrow \infty} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_{t \wedge \tau_l^{N,n}}^N(i) \right)^p + \int_0^{t \wedge \tau_l^{N,n}} c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_u^N(i) \right)^{p+1} du \right) \right] \\ & \leq \lim_{n \rightarrow \infty} \liminf_{l \rightarrow \infty} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_{t \wedge \tau_l^{N,n}}^N(i) \right)^p + \int_0^{t \wedge \tau_l^{N,n}} c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i \left(Y_u^N(i) \right)^{p+1} du \right] \\ & \leq \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \int_0^t c_2 \mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] + c_1 \mathbb{E} \left[\left\| (Y_u^N)^{p-1} \right\|_\sigma \right] du. \end{aligned} \tag{6.21}$$

This implies using Jensen’s inequality for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ that we get

$$\begin{aligned} & \mathbb{E} \left[\left\| (Y_t^N)^p \right\|_\sigma \right] - \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] \\ & \leq \int_0^t c_2 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^p \right] + c_1 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^{p-1} \right] - c_0 p \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^{p+1} \right] du \\ & = \int_0^t c_2 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^p \right] + \frac{\sum_{k \in \mathcal{D}} \sigma_k}{\sum_{i \in \mathcal{D}} \sigma_i} \left(c_1 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^{p-1} \right] - c_0 p \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^{p+1} \right] \right) du \\ & \leq \int_0^t c_2 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^p \right] + c_4 \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^p \right] \right)^{\frac{p-1}{p}} - c_3 \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(Y_u^N(i) \right)^p \right] \right)^{\frac{p+1}{p}} du \\ & = \int_0^t \left(\mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] \right)^{\frac{p-1}{p}} \left\{ c_4 + c_2 \left(\mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] \right)^{\frac{1}{p}} - c_3 \left(\mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] \right)^{\frac{2}{p}} \right\} du. \end{aligned} \tag{6.22}$$

For every $N \in \mathbb{N}$ let $z^N : [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $t \in [0, \infty)$ satisfies

$$z_t^N = z_0^N + \int_0^t (z_s^N)^{\frac{p-1}{p}} \left\{ c_4 + c_2 (z_s^N)^{\frac{1}{p}} - c_3 (z_s^N)^{\frac{2}{p}} \right\} ds \tag{6.23}$$

with $z_0^N = \mathbb{E} [\| (Y_0^N)^p \|_\sigma]$, where uniqueness follows from local Lipschitz continuity. Using classical comparison results from the theory of ODEs, the above computation shows that for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ we have $\mathbb{E} [\| (Y_t^N)^p \|_\sigma] \leq z_t^N$ and for all $N \in \mathbb{N}$ we have $\sup_{t \in [0, \infty)} z_t^N = \max \left\{ \mathbb{E} [\| (Y_0^N)^p \|_\sigma], \left(\frac{c_2}{2c_3} + \sqrt{\frac{(c_2)^2}{4c_3^2} + \frac{c_4}{c_3}} \right)^p \right\}$. We thereby conclude that

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\left\| (2\eta H_t^N + \delta P_t^N)^p \right\|_\sigma \right] &\leq \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} z_t^N \leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \left(\frac{c_2}{2c_3} + \sqrt{\frac{c_2^2}{4c_3^2} + \frac{c_4}{c_3}} \right)^p \\ &= \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \frac{c_2}{2c_3} \left(1 + \sqrt{1 + \frac{c_4 c_3}{c_2^2}} \right)^p = \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (2\eta H_0^N + \delta P_0^N)^p \right\|_\sigma \right] \\ &\quad + \|\mathbb{1}\|_\sigma \left(\frac{\lambda + \left(1 - \frac{1}{p} + \frac{c}{p}\right)(\bar{\kappa}_H + \bar{\kappa}_P)}{2 \min\left\{\frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma\right\}} \right)^p \left(1 + \sqrt{1 + \frac{4 \min\left\{\frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma\right\} [2\eta \bar{l}_H + \delta \bar{l}_P + (p-1)(2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P)]}{\left(\lambda + \left(1 - \frac{1}{p} + \frac{c}{p}\right)(\bar{\kappa}_H + \bar{\kappa}_P)\right)^2}} \right)^p. \end{aligned} \tag{6.24}$$

This completes the proof of Lemma 3.5. □

Proof of Lemma 3.6: If the right-hand side of (3.19) is infinite, then the claim trivially holds. For the remainder of the proof assume the right-hand side of (3.19) to be finite. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \hat{\mathcal{D}}$ be a set with $|\mathcal{D}_n| = \min\{n, |\hat{\mathcal{D}}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define $c_1 := \frac{1}{\lambda + \nu}$ and for all $n \in \mathbb{N}$ let

$$c_0^n := \frac{2c_1 \bar{\kappa}_{PC}}{3} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (P_t^N(i))^3 \right] + 2c_1 \left[\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right] \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i P_t^N(i) \right] + \frac{\lambda}{K^2 \delta}. \tag{6.25}$$

For $N, n, l \in \mathbb{N}$ define $[0, \infty]$ -valued stopping times

$$\tau_l^{N,n} := \inf \left(\left\{ t \in [0, \infty) : \sum_{i \in \mathcal{D}_n} \sigma_i \left(P_t^N(i) + (H_t^N(i))^{-1} \right) > l \right\} \cup \infty \right). \tag{6.26}$$

We infer from Lemma 3.2 that for all $N, n \in \mathbb{N}$ the map $[0, \infty) \ni t \mapsto \sum_{i \in \mathcal{D}_n} \sigma_i \left(P_t^N(i) + (H_t^N(i))^{-1} \right) \in \mathbb{R}$ is \mathbb{P} -a.s. continuous. Thereby, we have for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,n} < t \right] = 0. \tag{6.27}$$

For all $t \in [0, \infty)$, $N, n, l \in \mathbb{N}$ applying Young's inequality we get

$$\begin{aligned} &\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(2c_1 \frac{\sqrt{\beta_P^N P_u^N(i)}}{(H_u^N(i))^2} \right)^2 du \right] \\ &\leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \frac{\sigma_i^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} t \sup_{u \in [0, t]} 4c_1^2 \bar{\beta}_P \left(\frac{1}{5} \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^5 + \frac{4}{5} \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} \right) \right] \\ &\leq \frac{t 4c_1^2 \bar{\beta}_P}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \mathbb{E} \left[\sup_{u \in [0, t]} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(P_{u \wedge \tau_l^{N,n}}^N(i) + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right) \right)^5 \right] \leq \frac{t 4c_1^2 \bar{\beta}_P}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} l^5 < \infty \end{aligned} \tag{6.28}$$

and

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left((4c_1 P_u^N(i) + \frac{1}{\delta}) \frac{\sqrt{\beta_H^N H_u^N(i)}}{(H_u^N(i))^3} \right)^2 du \right] \\
 & \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \frac{\sigma_i^7}{\min_{k \in \mathcal{D}_n} \{\sigma_k^6\}} t \sup_{u \in [0,t]} (4c_1 + \frac{1}{\delta})^2 \bar{\beta}_H \left(\frac{2}{7} \left(P_{u \wedge \tau_l^{N,n}}^N(i) + 1 \right)^7 + \frac{5}{7} \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-7} \right) \right] \\
 & \leq \frac{t(4c_1 + \frac{1}{\delta})^2 \bar{\beta}_H}{\min_{k \in \mathcal{D}_n} \{\sigma_k^6\}} \mathbb{E} \left[\sup_{u \in [0,t]} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(P_{u \wedge \tau_l^{N,n}}^N(i) + 1 + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right) \right)^7 \right] \\
 & \leq \frac{t(4c_1 + \frac{1}{\delta})^2 \bar{\beta}_H}{\min_{k \in \mathcal{D}_n} \{\sigma_k^6\}} (l + \|\underline{1}\|_\sigma)^7 < \infty.
 \end{aligned} \tag{6.29}$$

Hence, we obtain for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} 2c_1 \frac{\sqrt{\beta_P^N P_u^N(i)}}{(H_u^N(i))^2} dW_u^{P,N}(i) \right] = 0, \\
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} (4c_1 P_u^N(i) + \frac{1}{\delta}) \frac{\sqrt{\beta_H^N H_u^N(i)}}{(H_u^N(i))^3} dW_u^{H,N}(i) \right] = 0.
 \end{aligned} \tag{6.30}$$

Define the function $y: \mathbb{N} \times \mathbb{N} \times [0, \infty) \rightarrow [0, \infty]$ by

$$\mathbb{N} \times \mathbb{N} \times [0, \infty) \ni (N, n, t) \mapsto y_t^{N,n} := \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_t^N(i)}{(H_t^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_t^N(i))^2} \right) \right]. \tag{6.31}$$

Recall from the beginning of the proof that we assume for all $N, n \in \mathbb{N}$ that $y_0^{N,n} < \infty$. Now, applying Itô's lemma and using (6.30), we obtain for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_{t \wedge \tau_l^{N,n}}^N(i)}{(H_{t \wedge \tau_l^{N,n}}^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_{t \wedge \tau_l^{N,n}}^N(i))^2} \right) \right] - y_0^{N,n} \\
 & = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} 2c_1 \frac{1}{(H_u^N(i))^2} \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) - (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 \right. \right. \\
 & \quad \left. \left. + (\eta - \rho F_u^N(i)) P_u^N(i) H_u^N(i) + \iota_P^N \right) - \left(2c_1 \frac{2P_u^N(i)}{(H_u^N(i))^3} + \frac{1}{2\delta} \frac{2}{(H_u^N(i))^3} \right) \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) \right. \right. \\
 & \quad \left. \left. + (-\kappa_H^N + \lambda - \alpha^N F_u^N(i)) H_u^N(i) - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) \right. \\
 & \quad \left. + \frac{1}{2} 2c_1 \frac{6P_u^N(i)}{(H_u^N(i))^4} \beta_H^N H_u^N(i) + \frac{1}{2} \frac{1}{2\delta} \frac{6}{(H_u^N(i))^4} \beta_H^N H_u^N(i) du \right].
 \end{aligned} \tag{6.32}$$

Dropping some negative terms, we now get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^{N,n}(i)}{(H_u^{N,n}(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^{N,n}(i))^2} \right) \right] - y_0^{N,n} \\
 & \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N,n}} \frac{2c_1}{(H_u^N(i))^2} \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) - \nu P_u^N(i) - \gamma (P_u^N(i))^2 + \eta P_u^N(i) H_u^N(i) + \iota_P^N \right) \right. \\
 & \quad - \left(4c_1 \frac{P_u^N(i)}{(H_u^N(i))^3} + \frac{1}{\delta} \frac{1}{(H_u^N(i))^3} \right) \left((-\kappa_H^N + \lambda - \alpha^N) H_u^N(i) - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) \\
 & \quad \left. + 6c_1 \frac{P_u^N(i)}{(H_u^N(i))^3} \beta_H^N + \frac{3}{2\delta} \frac{1}{(H_u^N(i))^3} \beta_H^N du \right] \\
 & = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N,n}} 2c_1 \left(\kappa_P^N \frac{1}{(H_u^N(i))^2} \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) - \nu \frac{P_u^N(i)}{(H_u^N(i))^2} - \gamma \frac{(P_u^N(i))^2}{(H_u^N(i))^2} + \eta \frac{P_u^N(i)}{H_u^N(i)} \right. \right. \\
 & \quad + \iota_P^N \frac{1}{(H_u^N(i))^2} - 2(-\kappa_H^N + \lambda - \alpha^N) \frac{P_u^N(i)}{(H_u^N(i))^2} + 2\frac{\lambda}{K} \frac{P_u^N(i)}{H_u^N(i)} + 2\delta \frac{(P_u^N(i))^2}{(H_u^N(i))^2} - 2\iota_H^N \frac{P_u^N(i)}{(H_u^N(i))^3} \\
 & \quad \left. \left. + 3\frac{P_u^N(i)}{(H_u^N(i))^3} \beta_H^N \right) + \frac{\kappa_H^N - \lambda + \alpha^N}{\delta} \frac{1}{(H_u^N(i))^2} + \frac{\lambda}{K\delta} \frac{1}{H_u^N(i)} + \frac{P_u^N(i)}{(H_u^N(i))^2} - \frac{\iota_H^N}{\delta} \frac{1}{(H_u^N(i))^3} + \frac{3\beta_H^N}{2\delta} \frac{1}{(H_u^N(i))^3} du \right]. \tag{6.33}
 \end{aligned}$$

Using Young’s inequality as well as Lemma 3.4 we get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^{N,n}(i)}{(H_u^{N,n}(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^{N,n}(i))^2} \right) \right] - y_0^{N,n} \\
 & \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N,n}} 2c_1 \left(\frac{2}{3} \kappa_P^N \frac{1}{(H_u^N(i))^3} + \frac{1}{3} \kappa_P^N c (P_u^N(i))^3 - \nu \frac{P_u^N(i)}{(H_u^N(i))^2} - \gamma \frac{(P_u^N(i))^2}{(H_u^N(i))^2} \right. \right. \\
 & \quad + \frac{1}{2} \frac{\lambda}{2\eta} \eta \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2} \frac{2\eta}{\lambda} \eta P_u^N(i) + \iota_P^N \frac{1}{(H_u^N(i))^2} - 2(-\kappa_H^N + \lambda - \alpha^N) \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2} \frac{K}{4} 2\frac{\lambda}{K} \frac{P_u^N(i)}{(H_u^N(i))^2} \\
 & \quad \left. \left. + \frac{1}{2} \frac{4}{K} 2\frac{\lambda}{K} P_u^N(i) + 2\delta \frac{(P_u^N(i))^2}{(H_u^N(i))^2} - 2\iota_H^N \frac{P_u^N(i)}{(H_u^N(i))^3} + 3\frac{P_u^N(i)}{(H_u^N(i))^3} \beta_H^N \right) + \frac{\kappa_H^N - \lambda + \alpha^N}{\delta} \frac{1}{(H_u^N(i))^2} \right. \\
 & \quad \left. + \frac{1}{2} \frac{K}{2} \frac{\lambda}{K\delta} \frac{1}{(H_u^N(i))^2} + \frac{1}{2} \frac{2}{K} \frac{\lambda}{K\delta} + \frac{P_u^N(i)}{(H_u^N(i))^2} - \frac{\iota_H^N}{\delta} \frac{1}{(H_u^N(i))^3} + \frac{3\beta_H^N}{2\delta} \frac{1}{(H_u^N(i))^3} du \right] \\
 & = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N,n}} \left[\frac{4c_1}{3} \kappa_P^N - \frac{1}{\delta} \iota_H^N + \frac{3}{2\delta} \beta_H^N \right] \frac{1}{(H_u^N(i))^3} + \frac{2c_1 \kappa_P^N c}{3} (P_u^N(i))^3 + \left[c_1 \left(-2\nu + \frac{\lambda}{2} \right. \right. \right. \\
 & \quad \left. \left. + 4(\kappa_H^N - \lambda + \alpha^N) + \frac{\lambda}{2} \right) + 1 \right] \frac{P_u^N(i)}{(H_u^N(i))^2} + 2c_1 [-\gamma + 2\delta] \frac{(P_u^N(i))^2}{(H_u^N(i))^2} + 2c_1 \left[\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right] P_u^N(i) \\
 & \quad \left. + \left[2c_1 \iota_P^N + \frac{\kappa_H^N - \lambda + \alpha^N}{\delta} + \frac{\lambda}{4\delta} \right] \frac{1}{(H_u^N(i))^2} + 2c_1 [-2\iota_H^N + 3\beta_H^N] \frac{P_u^N(i)}{(H_u^N(i))^3} + \frac{\lambda}{K^2\delta} du \right]. \tag{6.34}
 \end{aligned}$$

Recall $\bar{\kappa}_P = \sup_{N \in \mathbb{N}} \kappa_P^N$ and that for all $N \in \mathbb{N}$ we have $\alpha^N + \kappa_H^N \leq \frac{\lambda}{4}$, $\iota_P^N \leq \frac{\lambda(\nu + \lambda)}{8\delta}$, and $\iota_H^N \geq \frac{4\delta\kappa_P^N}{3(\nu + \lambda)} + \frac{3\beta_H^N}{2}$. Furthermore, note that $\frac{\lambda}{2} \leq \frac{1}{2c_1}$. Together with the assumption that $\gamma \geq 2\delta$ we see

for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^{N, n}(i)}{(H_u^{N, n}(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^{N, n}(i))^2} \right) \right] - y_0^{N, n} \\ & \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N, n}} \frac{2c_1}{3} \bar{\kappa}_{PC} (P_u^N(i))^3 - \frac{P_u^N(i)}{(H_u^N(i))^2} + 2c_1 \left[\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right] P_u^N(i) - \frac{\lambda}{4\delta} \frac{1}{(H_u^N(i))^2} + \frac{\lambda}{K^2 \delta} du \right] \\ & \leq \int_0^t c_0^n du - \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N, n}} \frac{\lambda}{2} \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) du \right]. \end{aligned} \tag{6.35}$$

Using Tonelli’s theorem, Fatou’s lemma, and (6.27) this implies for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned} y_t^{N, n} + \int_0^t \frac{\lambda}{2} y_u^{N, n} du &= y_t^{N, n} + \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \frac{\lambda}{2} \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) du \right] \\ &\leq \liminf_{l \rightarrow \infty} \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^{N, n}(i)}{(H_u^{N, n}(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^{N, n}(i))^2} \right) \right] \right. \\ &\quad \left. + \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N, n}} \frac{\lambda}{2} \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) du \right] \right) \leq y_0^{N, n} + \int_0^t c_0^n du. \end{aligned} \tag{6.36}$$

For every $N, n \in \mathbb{N}$ let $z^{N, n} : [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $t \in [0, \infty)$ satisfies $z_t^{N, n} = z_0^{N, n} + \int_0^t (c_0^n - \frac{\lambda}{2} z_s^{N, n}) ds$ with $z_0^{N, n} = y_0^{N, n}$, where uniqueness follows from local Lipschitz continuity. Due to classical comparison results of the theory of ODEs, the above computation yields for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that $y_t^{N, n} \leq z_t^{N, n}$ and for all $N, n \in \mathbb{N}$ that $\sup_{t \in [0, \infty)} z_t^{N, n} = \max \left\{ z_0^{N, n}, \frac{2c_0^n}{\lambda} \right\}$. We obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} y_t^{N, n} &\leq \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} z_t^{N, n} = \max \left\{ \sup_{N \in \mathbb{N}} z_0^{N, n}, \frac{2c_0^n}{\lambda} \right\} \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{2c_0^n}{\lambda}. \end{aligned} \tag{6.37}$$

Using monotone convergence we thereby conclude

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(\frac{2}{\lambda + \nu} \frac{P_t^N(i)}{(H_t^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_t^N(i))^2} \right) \right] &\leq \lim_{n \rightarrow \infty} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} y_t^{N, n} \\ &\leq \lim_{n \rightarrow \infty} \left(\sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{2c_0^n}{\lambda} \right) \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(\frac{2}{\lambda + \nu} \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{4\bar{\kappa}_{PC}}{3\lambda(\lambda + \nu)} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (P_t^N(i))^3 \right] \\ &\quad + \frac{4}{\lambda(\lambda + \nu)} \left(\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i P_t^N(i) \right] + \frac{2}{K^2 \delta}, \end{aligned} \tag{6.38}$$

completes the proof of Lemma 3.6. □

Proof of Lemma 3.7: If the right-hand side of (3.20) is infinite, then the claim trivially holds. For the remainder of the proof assume the right-hand side of (3.20) to be finite. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \hat{\mathcal{D}}$ be a set with $|\mathcal{D}_n| = \min\{n, |\hat{\mathcal{D}}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define $c_0 := \frac{1}{2(\bar{\kappa}_P + \nu)} [(\eta - \rho) - \frac{\lambda}{K}]$ and for every $n \in \mathbb{N}$ let

$$C^n := \gamma c_0 + \left[\gamma + \delta \right] \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \frac{1}{H_t^N(i)} \right]. \tag{6.39}$$

Note that due to the assumption $\eta - \rho > \frac{\lambda}{K}$ we have $c_0 \in (0, \infty)$. For all $N, n, l \in \mathbb{N}$ define $[0, \infty]$ -valued stopping times

$$\tau_l^{N,n} := \inf \left(\left\{ t \in [0, \infty) : \sum_{i \in \mathcal{D}_n} \sigma_i \left((P_t^N(i))^{-1} + (H_t^N(i))^{-1} \right) > l \right\} \cup \infty \right). \tag{6.40}$$

We infer from Lemmas 3.2 and 3.3 that for all $N, n \in \mathbb{N}$ the map

$$[0, \infty) \ni t \mapsto \sum_{i \in \mathcal{D}_n} \sigma_i \left((P_t^N(i))^{-1} + (H_t^N(i))^{-1} \right) \in \mathbb{R}$$

is \mathbb{P} -a.s. continuous which implies that we have for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,n} < t \right] = 0. \tag{6.41}$$

For all $t \in [0, \infty)$, $N, n, l \in \mathbb{N}$ applying Young’s inequality we see that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(\frac{\sqrt{\beta_P^N P_u^N(i)}}{(P_u^N(i))^2} \left(c_0 + \frac{1}{H_u^N(i)} \right) \right)^2 du \right] \\ & \leq \bar{\beta}_P \mathbb{E} \left[t \sup_{u \in [0, t]} \sum_{i \in \mathcal{D}_n} \frac{\sigma_i^5}{\min\{\sigma_k^4\}} \left(\frac{3}{5} \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} + \frac{2}{5} \left(c_0 + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right)^5 \right) du \right] \\ & \leq \frac{\bar{\beta}_P}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \mathbb{E} \left[t \sup_{u \in [0, t]} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(\left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} + c_0 \right) \right)^5 \right] \\ & \leq \frac{\bar{\beta}_P t (l + c_0 \|\mathbb{1}\|_\sigma)^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} < \infty \end{aligned} \tag{6.42}$$

and

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(\frac{\sqrt{\beta_H^N H_u^N(i)}}{(H_u^N(i))^2 P_u^N(i)} \right)^2 du \right] \\ & \leq \bar{\beta}_H \mathbb{E} \left[t \sup_{u \in [0, t]} \sum_{i \in \mathcal{D}_n} \frac{\sigma_i^5}{\min\{\sigma_k^4\}} \left(\frac{3}{5} \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} + \frac{2}{5} \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} \right) du \right] \\ & \leq t \sup_{u \in [0, t]} \frac{\bar{\beta}_H}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \mathbb{E} \left[\left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(\left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} + \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right) \right)^5 \right] \leq \frac{t \bar{\beta}_H l^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} < \infty. \end{aligned} \tag{6.43}$$

Hence, we obtain for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\beta_P^N P_t^N(i)} \frac{1}{(P_t^N(i))^2} \left(c_0 + \frac{1}{H_t^N(i)} \right) dW_u^{P,N}(i) \right] = 0, \\ & \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \frac{\sqrt{\beta_H^N H_t^N(i)}}{(H_t^N(i))^2 P_t^N(i)} dW_u^{H,N}(i) \right] = 0. \end{aligned} \tag{6.44}$$

Define the function $y: \mathbb{N} \times \mathbb{N} \times [0, \infty) \rightarrow [0, \infty]$ by

$$\mathbb{N} \times \mathbb{N} \times [0, \infty) \ni (N, n, t) \mapsto y_t^{N,n} := \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i) H_t^N(i)} \right) \right]. \tag{6.45}$$

Recall from the beginning of the proof that we assume for all $N, n \in \mathbb{N}$ that $y_0^{N,n} < \infty$. Applying Itô's lemma and using (6.44), we get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i)} + \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i) H_{t \wedge \tau_l^{N,n}}^N(i)} \right) \right] - y_0^{N,n} \\ &= \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} - \left(c_0 \frac{1}{(P_u^N(i))^2} + \frac{1}{(P_u^N(i))^2 H_u^N(i)} \right) \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) \right. \right. \\ & \quad \left. \left. - (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 + (\eta - \rho F_u^N(i)) P_u^N(i) H_u^N(i) + \iota_P^N \right) + \frac{1}{2} c_0 \frac{2}{(P_u^N(i))^3} \beta_P^N P_u^N(i) \right. \\ & \quad \left. + \frac{1}{2} \frac{2}{(P_u^N(i))^3 H_u^N(i)} \beta_P^N P_u^N(i) - \frac{1}{P_u^N(i) (H_u^N(i))^2} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + (-\kappa_H^N + \lambda - \alpha^N F_u^N(i)) H_u^N(i) \right. \right. \\ & \quad \left. \left. - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) + \frac{1}{2} \frac{2}{P_u^N(i) (H_u^N(i))^3} \beta_H^N H_u^N(i) du \right]. \end{aligned} \tag{6.46}$$

Dropping some negative terms, we now get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^{N, n}(i)} + \frac{1}{P_u^{N, n}(i) H_u^{N, n}(i)} \right) \right] - y_0^{N, n} \\
 & \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N, n}} - \left(c_0 \frac{1}{(P_u^N(i))^2} + \frac{1}{(P_u^N(i))^2 H_u^N(i)} \right) \left(- (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 \right. \right. \\
 & \quad \left. \left. + (\eta - \rho) P_u^N(i) H_u^N(i) + \iota_P^N \right) + c_0 \beta_P^N \frac{1}{(P_u^N(i))^2} + \beta_P^N \frac{1}{(P_u^N(i))^2 H_u^N(i)} + \beta_H^N \frac{1}{P_u^N(i) (H_u^N(i))^2} \right. \\
 & \quad \left. - \frac{1}{P_u^N(i) (H_u^N(i))^2} \left((-\kappa_H^N + \lambda - \alpha^N) H_u^N(i) - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) du \right] \\
 & = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N, n}} (\kappa_P^N + \nu) c_0 \frac{1}{P_u^N(i)} + \gamma c_0 - (\eta - \rho) c_0 \frac{H_u^N(i)}{P_u^N(i)} - \iota_P^N c_0 \frac{1}{(P_u^N(i))^2} + (\kappa_P^N + \nu) \frac{1}{P_u^N(i) H_u^N(i)} \right. \\
 & \quad \left. + \gamma \frac{1}{H_u^N(i)} - (\eta - \rho) \frac{1}{P_u^N(i)} - \iota_P^N \frac{1}{(P_u^N(i))^2 H_u^N(i)} + c_0 \beta_P^N \frac{1}{(P_u^N(i))^2} + \beta_P^N \frac{1}{(P_u^N(i))^2 H_u^N(i)} + \beta_H^N \frac{1}{P_u^N(i) (H_u^N(i))^2} \right. \\
 & \quad \left. - (-\kappa_H^N + \lambda - \alpha^N) \frac{1}{P_u^N(i) H_u^N(i)} + \frac{\lambda}{K} \frac{1}{P_u^N(i)} + \delta \frac{1}{H_u^N(i)} - \iota_H^N \frac{1}{P_u^N(i) (H_u^N(i))^2} du \right] \\
 & = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N, n}} \left[(\kappa_P^N + \nu) c_0 - (\eta - \rho) + \frac{\lambda}{K} \right] \frac{1}{P_u^N(i)} + \gamma c_0 - (\eta - \rho) c_0 \frac{H_u^N(i)}{P_u^N(i)} \right. \\
 & \quad \left. + [-\iota_P^N c_0 + c_0 \beta_P^N] \frac{1}{(P_u^N(i))^2} + [(\kappa_P^N + \nu) - (-\kappa_H^N + \lambda - \alpha^N)] \frac{1}{P_u^N(i) H_u^N(i)} + [\gamma + \delta] \frac{1}{H_u^N(i)} \right. \\
 & \quad \left. + [-\iota_P^N + \beta_P^N] \frac{1}{(P_u^N(i))^2 H_u^N(i)} + [\beta_H^N - \iota_H^N] \frac{1}{P_u^N(i) (H_u^N(i))^2} du \right]. \tag{6.47}
 \end{aligned}$$

Recall from Section 3.1 that $\bar{\kappa}_P = \sup_{N \in \mathbb{N}} \kappa_P^N$, and from Assumption 2.1 that $\lambda > \nu$, $\eta - \rho > \frac{\lambda}{K}$ and that for all $N \in \mathbb{N}$ we have $\kappa_P^N + \kappa_H^N + \alpha^N \leq \frac{\lambda - \nu}{2}$, $\iota_P^N \geq \beta_P^N$, and $\iota_H^N \geq \beta_H^N$. Hence, we get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^{N, n}(i)} + \frac{1}{P_u^{N, n}(i) H_u^{N, n}(i)} \right) \right] - y_0^{N, n} \\
 & \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_i^{N, n}} - (\bar{\kappa}_P + \nu) c_0 \frac{1}{P_u^N(i)} + \gamma c_0 - \frac{\lambda - \nu}{2} \frac{1}{P_u^N(i) H_u^N(i)} + [\gamma + \delta] \frac{1}{H_u^N(i)} du \right] \tag{6.48} \\
 & \leq \int_0^t C^n du - \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N, n}} \min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \} \sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^N(i)} + \frac{1}{P_u^N(i) H_u^N(i)} \right) du \right].
 \end{aligned}$$

Applying Tonelli’s theorem, Fatou’s lemma, and (6.41) we obtain for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
 & y_t^{N,n} + \int_0^t \min \left\{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \right\} y_u^{N,n} du \\
 &= y_t^{N,n} + \mathbb{E} \left[\int_0^t \min \left\{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \right\} \sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^{N,i}} + \frac{1}{P_u^{N,i} H_u^{N,i}} \right) du \right] \\
 &\leq \liminf_{l \rightarrow \infty} \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_{t \wedge \tau_l^{N,n}}^{N,i}} + \frac{1}{P_{t \wedge \tau_l^{N,n}}^{N,i} H_{t \wedge \tau_l^{N,n}}^{N,i}} \right) \right] \right. \\
 &\quad \left. + \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \min \left\{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \right\} \sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^{N,i}} + \frac{1}{P_u^{N,i} H_u^{N,i}} \right) du \right] \right) \leq y_0^{N,n} + \int_0^t C^n du.
 \end{aligned}
 \tag{6.49}$$

For every $N, n \in \mathbb{N}$, let $z^{N,n} : [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $t \in [0, \infty)$ satisfies $z_t^{N,n} = z_0^{N,n} + \int_0^t (C^n - \min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \}) z_s^{N,n} ds$, with $z_0^{N,n} = y_0^{N,n}$, where uniqueness follows from local Lipschitz continuity. Using classical comparison results from the theory of ODEs, the above computation yields for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that $y_t^{N,n} \leq z_t^{N,n}$ and for all $N, n \in \mathbb{N}$ that $\sup_{t \in [0, \infty)} z_t^{N,n} = \max \left\{ z_0^{N,n}, \frac{C^n}{\min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \}} \right\}$. Hence, we obtain for every $n \in \mathbb{N}$ that

$$\begin{aligned}
 & \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_t^{N,i}} + \frac{1}{P_t^{N,i} H_t^{N,i}} \right) \right] = \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} y_t^{N,n} \leq \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} z_t^{N,n} \\
 &= \max \left\{ \sup_{N \in \mathbb{N}} z_0^{N,n}, \frac{C^n}{\min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \}} \right\} \\
 &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_0^{N,i}} + \frac{1}{P_0^{N,i} H_0^{N,i}} \right) \right] + \frac{C^n}{\min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \}}.
 \end{aligned}
 \tag{6.50}$$

Using monotone convergence, we thereby conclude that

$$\begin{aligned}
 & \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(c_0 \frac{1}{P_t^{N,i}} + \frac{1}{P_t^{N,i} H_t^{N,i}} \right) \right] = \lim_{n \rightarrow \infty} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_t^{N,i}} + \frac{1}{P_t^{N,i} H_t^{N,i}} \right) \right] \\
 &\leq \lim_{n \rightarrow \infty} \left(\sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_0^{N,i}} + \frac{1}{P_0^{N,i} H_0^{N,i}} \right) \right] + \frac{C^n}{\min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \}} \right) \\
 &= \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(c_0 \frac{1}{P_0^{N,i}} + \frac{1}{P_0^{N,i} H_0^{N,i}} \right) \right] + \frac{\gamma c_0 + (\gamma + \delta) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \frac{1}{H_t^{N,i}} \right]}{\min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \}},
 \end{aligned}
 \tag{6.51}$$

finishing the proof of Lemma 3.7. □

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