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Balls-in-bins models with asymmetric feedback and reflection

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Abstract. Balls-in-bins models describe a random sequential allocation of infinitely many balls into a finite number of bins. In these models a ball is placed into a bin with probability proportional to a given function (feedback function), which depends on the number of existing balls in the bin. Typically, the feedback function is the same for all bins (symmetric feedback), and there are no constraints on the number of balls in the bins. In this paper we study versions of BB models with two bins, in which the above assumptions are violated. In the first model of interest the feedback functions can depend on a bin (BB model with asymmetric feedback). In the case when both feedback functions are power law and superlinear, a single bin receives all but finitely many balls almost surely, and we study the probability that this happens for a given bin. In particular, under certain initial conditions we derive the normal approximation for this probability. This generalizes the result in Mitzenmacher et al. (2004) obtained in the case of the symmetric feedback. The main part of the paper concerns the BB model with asymmetric feedback evolving subject to certain constraints on the numbers of allocated balls. The model can be interpreted as a transient reflecting random walk in a curvilinear wedge, and we obtain a complete classification of its long term behavior.

1. Introduction

A balls-in-bins (BB) model is a classic probabilistic model describing a random sequential allocation of infinitely many balls into a finite number of bins. The probability of allocating a ball into a bin is proportional to a function, which depends on the number of existing balls in the bin. In the standard setup, the feedback function is the same for all bins (i.e. the feedback is symmetric), and there are no constraints on the number of allocated balls.

This paper concerns two models of random sequential allocation of balls into bins, which can be regarded as versions of commonly studied BB models with a power law feedback function $f(x) = x^{\beta}$, where $\beta > 0$. The following results for such BB model are known (e.g. see Mitzenmacher et al.

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(2004), Oliveira (2009) and references therein). If $\beta \leq 1$, then, with probability one, all bins receive infinitely many balls. For example, if $\beta = 1$ (the classical Pólya urn model), then the distribution of fractions of balls in the bins converges to a Dirichlet distribution. On the other hand, if $\beta > 1$, then, with probability one, a single random bin receives all but finitely many balls.

Our first model of interest goes as follows. Suppose that there are two bins labeled 1 and 2 respectively. If x_i is the number of existing balls in the bin $i = 1, 2$, then the next ball is placed in the bin i with probability proportional to $x_i^{\beta_i}$, where β_i, $i = 1, 2$ are given positive constants. If $\beta_1 = \beta_2$ (the symmetric case), then this is the model studied in Mitzenmacher et al. (2004). Here we focus on the asymmetric case $\beta_1 \neq \beta_2$, in which both $\beta_1 > 1$ and $\beta_2 > 1$. In this case, similarly to the symmetric one, all but finitely many balls are placed in a single random bin almost surely, and we study the probability that this will happen in a given bin. The probability depends on initial conditions, and is trivially equal to 1 or 0 (depending on a bin) in most cases. There is, however, a subset of initial conditions, for which this probability is non-trivial, and we derive the normal approximation for it. This generalizes the result in Mitzenmacher et al. (2004), where the normal approximation for this probability is obtained in the symmetric case.

The main results of this paper concern the other model. This model can be thought of as the BB model with asymmetric feedback, where the balls are allocated subject to certain constraints. The model can be naturally interpreted as a reflecting random walk in a curvilinear wedge, and in what follows we call it the BB model with reflection.

We obtain a complete classification of the long term behavior of the BB model with reflection in the case when the boundaries of the wedge are given by power law functions. Briefly this can be described as follows. Let $\zeta_i(t)$ be the number of balls in the bin $i = 1, 2$ at time t. Then, with probability one, the process $(\zeta_1(t), \zeta_2(t))$ eventually confines to a strip of a finite width along the boundary of the wedge. Moreover, we exactly calculate the width of the strip. The width is an integer that depends on the model parameters. This effect is somewhat similar to boundary effects detected in competition processes in Menshikov and Shcherbakov (2018) and Shcherbakov and Volkov (2019).

Our motivation for the models in this paper has arisen from previous studies of growth models that can be thought of reinforced urn models with graph based interaction. In these models bins are labeled by vertices of a graph, and allocation probabilities at a bin depend on the configuration of allocated balls in the neighbourhood of that bin. Several such growth models have been studied in the case where the allocation probability at a bin is proportional to a log-linear function of the local configuration (see Costa et al. (2018), Menshikov and Shcherbakov (2020) and Shcherbakov and Volkov (2010)). A strong reinforcement mechanism in those models allowed to characterize precisely their limit behavior. In contrast to the growth models with the log-linear interaction, the growth models with interaction given by power law functions are harder to analyze. The BB model with the power law asymmetric feedback can be regarded as such a growth model *without* interaction. Secondly, the BB model with reflection can be regarded as a toy model mimicking the behavior of one of these growth models in a very particular situation.

The rest of the paper is organized as follows. In Section 2 we formally define both BB models of interest and state the main results. In Section 3 we collect some known facts about pure birth processes with continuous time. These facts are used in Section 4 to study the long term behavior of the BB model with asymmetric feedback. In Sections 5.1-5.2 we prove a series of lemmas that describe the most likely long term behavior of the BB model with reflection. These statements are used in Section 5.3 to prove the main results (Theorems 2.9 and 2.10 below) for the BB model with reflection. Finally, in Section 6 we state an open problem.

2. Models and results

Let $\mathbb{Z}_+ = \{1, 2....\}$ and $\mathbb{Z}_+ = \{0, 1, 2, ...\}$ be sets of natural numbers and non-negative numbers respectively. We assume that all random variables and processes are realised on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation with respect to the probability $\mathbb{P}$ is denoted by $\mathbb{E}$. The variance of a random variable X is denoted by $\mathsf{Var}(X)$. Given a Markov process with values in $\mathbb{Z}_+^2$ we denote by $\mathbb{P}_{\mathbf{z}}$ the distribution of the process started at $\mathbf{z} \in \mathbb{Z}_+^2$. Given quantities $a(m)$ and $b(m)$ depending on $m \in \mathbb{Z}_+$, we write $a(m) \sim b(m)$ to denote the fact that $a(m)/b(m) \to 1$, as $m \to \infty$.

Throughout we deal with many analytical equations involving integer valued quantities. This formally requires using the floor function and the ceiling function. However, most of the equations are of interest for us only asymptotically, when the effect of applying floor/ceiling functions can be neglected. Therefore, to simplify notations, we skip writing these functions.

2.1. *The model with asymmetric feedback.* In this section we define the BB model with asymmetric feedback and state the corresponding results. The model is as follows. Infinitely many balls are sequentially placed in two available bins according to the following rule. If x_i is the number of existing balls in the bin $i = 1, 2$, then the next arriving ball is placed in this bin with probability proportional to $x_i^{\beta_i}$, where $\beta_1 > 0$ and $\beta_2 > 0$ are given constants. Let $X_i(t)$ be the number of balls in the bin $i = 1, 2$ at time $t \in \mathbb{Z}_+$. Then the random process $X(t) = (X_1(t), X_2(t)) \in \mathbb{Z}_+^2$, $t \in \mathbb{Z}_+$, is a discrete time Markov chain (DTMC) with the transition probabilities

$$\begin{aligned} \mathbb{P}_{\mathbf{x}}(X_1(1) = x + 1, X_2(1) = y) &= \frac{x^{\beta_1}}{x^{\beta_1} + y^{\beta_2}}, \\ \mathbb{P}_{\mathbf{x}}(X_1(1) = x, X_2(1) = y + 1) &= \frac{y^{\beta_2}}{x^{\beta_1} + y^{\beta_2}}, \end{aligned} \tag{2.1}$$

for $\mathbf{x} = (x, y)$.

Define events

$$A_i = \{\textit{all but finitely many balls are placed in the bin } i\}, \quad i = 1, 2. \tag{2.2}$$

Theorem 2.1. *1) If both $\beta_1 > 1$ and $\beta_2 > 1$, then $\mathbb{P}_{\mathbf{x}}(A_1) + \mathbb{P}_{\mathbf{x}}(A_2) = 1$ for any $\mathbf{x} \in \mathbb{Z}_+^2$.*
2) If $\beta_1 \le 1$ and $\beta_2 > 1$, then $\mathbb{P}_{\mathbf{x}}(A_2) = 1$ for any $\mathbf{x} \in \mathbb{Z}_+^2$.

Given $1 < \beta_1 \le \beta_2$, define

$$\alpha_{cr} = \frac{\beta_1 - 1}{\beta_2 - 1} \quad \text{and} \quad \nu_{cr} = \alpha_{cr}^{\frac{1}{\beta_2 - 1}} = \left(\frac{\beta_1 - 1}{\beta_2 - 1}\right)^{\frac{1}{\beta_2 - 1}}. \tag{2.3}$$

Theorem 2.2. *Let $X(t) = (X_1(t), X_2(t))$ be the DTMC with transition probabilities (2.1), where $1 < \beta_1 \le \beta_2$. Assume that $\mathbf{x} = (x, y) \in \mathbb{Z}_+^2$, where*

$$y = y(x) = \nu x^{\alpha} + o(x^{\alpha}), \quad \textit{as} \quad x \to \infty,$$

for some $\alpha \in (0, 1)$ and $\nu > 0$.

1) *If either $\alpha < \alpha_{cr}$, or $\alpha = \alpha_{cr}$ and $\nu < \nu_{cr}$, then $\lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_1) = 1$.*
2) *If either $\alpha > \alpha_{cr}$, or $\alpha = \alpha_{cr}$ and $\nu > \nu_{cr}$, then $\lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_2) = 1$.*

Theorem 2.3 below deals with the critical case $\alpha = \alpha_{cr}, \nu = \nu_{cr}$ not covered by Theorem 2.2.

Theorem 2.3. *Let $X(t) = (X_1(t), X_2(t))$ be the DTMC with transition probabilities (2.1), where $1 < \beta_1 \le \beta_2$. Assume that $\mathbf{x} = (x, y) \in \mathbb{Z}_+^2$ is such that $y = y(x) = \nu_{cr} x^{\alpha_{cr}} + \mu x^{\delta} + o(x^{\delta})$, as $x \to \infty$, for some $\mu \in \mathbb{R}$ and $\delta \in [0, \alpha_{cr})$, and let*

$$\rho = -\mu \left(\frac{2\beta_2 - 1}{\nu_{cr}}\right)^{\frac{1}{2}}. \tag{2.4}$$

1) *If* $\delta = \frac{\alpha_{cr}}{2}$, *then*

$$\lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_1) = \Phi(\rho) \quad \text{and} \quad \lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_2) = 1 - \Phi(\rho),$$

where Φ *is the cumulative distribution function of the standard normal distribution.*

2) *If* $\delta \in \left[0, \frac{\alpha_{cr}}{2}\right)$, *then* $\lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_1) = \lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_2) = \frac{1}{2}$.

3) *If* $\delta \in \left(\frac{\alpha_{cr}}{2}, \alpha_{cr}\right)$ *and* $\mu \neq 0$, *then*
   (a) $\lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_1) = 0$ *for* $\mu > 0$,
   (b) $\lim_{x\to\infty} \mathbb{P}_{\mathbf{x}}(A_1) = 1$ *for* $\mu < 0$.

Remark 2.4. Note that a curve defined by the equation

$$y = \nu_{cr} x^{\alpha_{cr}}, \tag{2.5}$$

where α_{cr} and ν_{cr} are defined in (2.3), plays a special role in our analysis of the model in the case $1 < \beta_1 \leq \beta_2$. Namely, this curve divides the quarter plane into two parts, such that the limiting behavior of the process depends on which of these parts the process starts in. In other words, it determines a phase transition in the long time behavior of the process. In what follows we call this curve the critical curve. The critical curve has a natural probabilistic interpretation. Namely, the DTMC $X(t) = (X_1(t), X_2(t))$ can be embedded into a pair of independent continuous time pure birth processes, say $Z_1(t)$ and $Z_2(t)$, with appropriate power law birth rates determined by parameters β_1 and β_2 respectively. If both $\beta_1 > 1$ and $\beta_2 > 1$, then these birth processes are explosive. Suppose that the process $Z_1(t)$ starts at $x \in Z_+$ and the process $Z_2(t)$ starts at y. Then the ratio of their mean explosion times tends to 1, as $x \to \infty$, provided that $y \sim \nu_{cr} x^{\alpha_{cr}}$ (see Section 3 for more details). Note also that, if $1 < \beta_1 < \beta_2$, then the critical curve lies strictly below the curve $y = x^{\frac{\beta_1}{\beta_2}}$. The latter is obtained by equating (similarly to the case of explosion times) the probability of the jump up to the probability of the jump to the right in the discrete time process. Finally, note that in the symmetric case $\beta_1 = \beta_2$ we have that $\alpha_{cr} = 1$ and $\gamma_{cr} = 1$. Therefore both the critical curve and the curve $y = x^{\frac{\beta_1}{\beta_2}}$ coincide with the bisector $y = x$.

Remark 2.5. If $\beta_1 = \beta_2 = \beta > 1$, and $X_1(0) = a$ and $X_2(0) = a - \frac{\rho}{\sqrt{2\beta-1}}\sqrt{a}$, then, by Part 1) of Theorem 2.3, the probability of event A_1 is approximated by $\Phi(\rho)$ for sufficiently large a. This recovers the result of Mitzenmacher et al. (2004), where they obtained the normal approximation $\Phi(\rho)$ for the same probability in the symmetric case assuming that the process starts at (x, y), where $x = a + \frac{\rho}{\sqrt{4\beta-2}}\sqrt{a}$ and $y = a - \frac{\rho}{\sqrt{4\beta-2}}\sqrt{a}$. The minor difference in initial conditions is due to our attempt to somehow accommodate both the symmetric and the asymmetric case in a single statement (clearly, other variants of the initial condition are possible).

2.2. *The model with reflection.* In this section we define the BB model with reflection and state the corresponding results.

Given two monotonically increasing and sufficiently smooth functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\varphi(x) < \psi(x)$ for all $x \geq 0$, define the domain

$$Q_{\varphi,\psi} := \{(x, y) \in \mathbb{Z}_+^2 : \varphi(x) \leq y \leq \psi(x)\}. \tag{2.6}$$

Given constants $\beta_1 > 0$ and $\beta_2 > 0$ consider a DTMC $\zeta(t) = (\zeta_1(t), \zeta_2(t)) \in Q_{\varphi,\psi}$, $t \in \mathbb{Z}_+$ with the following transition probabilities

$$\begin{aligned} \mathbb{P}_\zeta(\zeta_1(1) = x + 1, \zeta_2(1) = y) &= \frac{x^{\beta_1}}{x^{\beta_1} + y^{\beta_2}} \\ \mathbb{P}_\zeta(\zeta_1(1) = x, \zeta_2(1) = y + 1) &= \frac{y^{\beta_2}}{x^{\beta_1} + y^{\beta_2}} \\ \text{for} \quad \zeta = (x, y) : \varphi(x + 1) &\leq y \leq \psi(x) - 1; \end{aligned} \tag{2.7}$$

and

$$\mathbb{P}_\zeta(\zeta_1(1) = x, \zeta_2(1) = y+1) = 1 \quad \text{for} \quad \zeta = (x,y) : \varphi(x) \le y < \varphi(x+1), \tag{2.8}$$

$$\mathbb{P}_\zeta(\zeta_1(1) = x+1, \zeta_2(1) = y) = 1 \quad \text{for} \quad \zeta = (x,y) : \psi(x) - 1 < y \le \psi(x). \tag{2.9}$$

Definition 2.6.

- We call the Markov chain $\zeta(t)$ with transition probabilities given in equations (2.7)-(2.9) the BB model with reflection in the domain $Q_{\varphi,\psi}$. Curves $y = \varphi(x)$ and $y = \psi(x)$ are called the lower boundary and the upper boundary respectively.
- The one-step transitions of the DTMC $\zeta(t)$ described by equations (2.7) are called jumps (jumps to the right and jumps up), and the one-step transitions described by equations (2.8)–(2.9) are called reflections (by the lower and by the upper boundary respectively).
- We say that the process is reflected by the lower (upper) boundary, when the transition described by (2.8) (by (2.9)) occurs.

Remark 2.7. Note that, by construction, in the interior of the domain $Q_{\varphi,\psi}$ the DTMC $\zeta(t)$ evolves in the same way as the DTMC $X(t)$ with transition probabilities (2.1). Equations (2.8)–(2.9) describe a reflection mechanism preventing the process from leaving the domain.

Remark 2.8. It should be also noted that our results for the DTMC $\zeta(t)$ concern only the case, when the boundaries are given by power law functions, that is $\varphi(x) = x^\alpha$ and $\psi(x) = x^\gamma$ for some $0 < \alpha < \gamma$. However, it is convenient to have the model domain defined in the general case. First, it is convenient for future references (e.g. Section 6). Also, conditions in both (2.8) and (2.9) look more transparent (at least, as it seems to us) in the general case.

Theorems 2.9 and 2.10 below are the main results concerning the long time behavior of the DTMC $\zeta(t)$ with transition probabilities (2.7)-(2.9) (i.e. the BB model with reflection). To state these theorems we need some definitions.

Define events

$$A_{\alpha,k} = \{\limsup_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = k\} \cap \{\liminf_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = 0\} \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad k \in \mathbb{Z}_+. \tag{2.10}$$

$$\widetilde{A}_{\gamma,k} = \{\limsup_{t\to\infty}(\zeta_1(t) - \zeta_2^{1/\gamma}(t)) = k\} \cap \{\liminf_{t\to\infty}(\zeta_1(t) - \zeta_2^{1/\gamma}(t)) = 0\} \quad \text{for} \quad \gamma > 1 \quad \text{and} \quad k \in \mathbb{Z}_+. \tag{2.11}$$

$$E_k = \{\limsup_{t\to\infty}(\zeta_1(t) - \zeta_2(t)) = k\} \cap \{\zeta_1(t) - \zeta_2(t) = 0 \text{ i.o.}\} \cap \{\zeta_1(t) - \zeta_2(t) = k \text{ i.o.}\} \quad \text{for} \quad k \in \mathbb{Z}_+, \tag{2.12}$$

$$B_\gamma = \{\limsup_{t\to\infty}(\zeta_1^\gamma(t) - \zeta_2(t)) = 1\} \cap \{\liminf_{t\to\infty}(\zeta_1^\gamma(t) - \zeta_2(t)) = 0\} \quad \text{for} \quad \gamma \in (0,1). \tag{2.13}$$

Theorem 2.9. *Let $\beta_1 = \beta_2 = \beta > 1$, and let $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ be the BB model with reflection in the domain $Q_{\varphi,\psi}$, where $\varphi(x) = x^\alpha$ and $\psi(x) = x^\gamma$ for some $\alpha, \gamma : 0 < \alpha < 1 < \gamma$. Then for any $\zeta(0) = \zeta \in Q_{\varphi,\psi}$ the following holds*

$$\mathbb{P}_\zeta(A_{\alpha,k_1}) + \mathbb{P}_\zeta\left(\widetilde{A}_{\gamma,k_2}\right) = 1, \tag{2.14}$$

where events A_{α,k_1} and $\widetilde{A}_{\gamma,k_2}$ are defined in (2.10) and (2.11) respectively, $k_1 = k_1(\alpha,\beta)$ and $k_2 = k_2(\gamma,\beta)$ are unique integer solutions of the following inequalities

$$\frac{\alpha}{(\beta-1)(1-\alpha)} < k_1 \le 1 + \frac{\alpha}{(\beta-1)(1-\alpha)} \tag{2.15}$$

and

$$\frac{1}{(\beta-1)(\gamma-1)} < k_2 \leq 1 + \frac{1}{(\beta-1)(\gamma-1)}. \tag{2.16}$$

Theorem 2.10. *Let $1 < \beta_1 < \beta_2$, and let $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ be the BB model with reflection in the domain $Q_{\varphi,\psi}$, where $\varphi(x) = x^\alpha$ and $\psi(x) = x^\gamma$ for some $\alpha, \gamma : 0 < \alpha < \alpha_{cr} < \gamma$, where*

$$\alpha_{cr} = \frac{\beta_1 - 1}{\beta_2 - 1}.$$

Then, for any $\zeta(0) = \zeta \in Q_{\varphi,\psi}$ the following hold

$$\mathbb{P}_\zeta\left(A_{\alpha,k_1}\right) + \mathbb{P}_\zeta\left(B_\gamma\right) = 1, \quad \text{if} \quad \alpha_{cr} < \gamma < 1, \tag{2.17}$$

$$\mathbb{P}_\zeta\left(A_{\alpha,k_1}\right) + \mathbb{P}_\zeta\left(E_{k_2}\right) = 1, \quad \text{if} \quad \gamma = 1, \tag{2.18}$$

$$\mathbb{P}_\zeta\left(A_{\alpha,k_1}\right) + \mathbb{P}_\zeta\left(\widetilde{A}_{\gamma,k_2}\right) = 1, \quad \text{if} \quad \gamma > 1, \tag{2.19}$$

where events $A_{\alpha,k}$, $\widetilde{A}_{\gamma,k}$, E_k and B_γ are defined in (2.10)-(2.13), $k_1 = k_1(\alpha, \beta_1, \beta_2)$ and $k_2 = k_2(\gamma, \beta_1, \beta_2)$ are unique integer solutions of the following inequalities

$$\frac{\alpha}{(\beta_2-1)(\alpha_{cr}-\alpha)} < k_1 \leq 1 + \frac{\alpha}{(\beta_2-1)(\alpha_{cr}-\alpha)} \tag{2.20}$$

$$\frac{1}{(\beta_2-1)(\gamma-\alpha_{cr})} < k_2 \leq 1 + \frac{1}{(\beta_2-1)(\gamma-\alpha_{cr})}. \tag{2.21}$$

Remark 2.11. All events in (2.10)-(2.13) can be interpreted geometrically. For example, the event $A_{\alpha,k}$ means that given any arbitrary small $\varepsilon > 0$ the process eventually confines to the strip $\{(\zeta_1, \zeta_2) : \zeta_1^\alpha \leq \zeta_2 \leq \zeta_1^\alpha + k + \varepsilon\}$, where it fluctuates by approaching both the lower boundary $y = x^\alpha$ and the curve $y = x^\alpha + k$ at any arbitrarily small distance. Similarly, the event E_k in (2.12) (i.e. in the case when the upper boundary is the bisector) means that the process eventually confines to the strip between the bisector $y = x$ and the straight line $y = x - k$, and visits both of these lines infinitely often. In particular, the event E_1 simply means, that the process eventually follows a "zigzag" trajectory located between the bisector and the line $y = x - 1$ (see Remark 5.17).

Remark 2.12. Note that the inequality (2.21) for determining k_2 in (2.18) can be simplified. Indeed, if $\gamma = 1$, then

$$\frac{1}{(\beta_2-1)(\gamma-\alpha_{cr})} = \frac{1}{(\beta_2-1)(1-\alpha_{cr})} = \frac{1}{\beta_2-\beta_1},$$

and we get the inequality $\frac{1}{\beta_2-\beta_1} < k_2 \leq 1 + \frac{1}{\beta_2-\beta_1}$ for determining k_2. For example, if $\beta_2 - \beta_1 > 1$, then $k_2 = 1$; if $\beta_2 - \beta_1 = 1$, then $k_2 = 2$, and so on.

3. Pure birth processes with power law rates

For the reader's convenience we provide in this section some facts about pure birth processes with continuous time. The material of the section will be used to obtain Theorems 2.1, 2.2 and 2.3.

We say that a random variable ξ has exponential distribution with parameter $\lambda > 0$ and write $\xi \sim Exp(\lambda)$, if $\mathbb{P}(\xi > x) = e^{-\lambda x}$ for $x \geq 0$. Let $Z(t) = (Z(t),\, t \in \mathbb{R}_+)$ be a continuous time pure birth process on $\mathbb{Z}_+$ with birth rates $\{\lambda_i > 0,\, i \geq 0\}$, that is, $Z(t)$ is a continuous time Markov chain (CTMC) on $\mathbb{Z}_+$ evolving as follows. The process stays at state i for a period of time given by a random variable $\xi_i \sim Exp(\lambda_i)$ and then jumps to state $i+1$ with probability one. Random variables $(\xi_i,\, i \geq 0)$ are assumed to be independent. The random variable $T_k = \sum_{i=k}^{\infty} \xi_i$ is called the time to explosion of the birth process started at $Z(0) = k$. It is known (e.g. Feller (1971)) that

$\mathbb{P}(T_0<\infty)=1$ if and only if $\sum_{i=0}^{\infty}\lambda_i^{-1}<\infty$. When $\mathbb{P}(T_0<\infty)=1$ (and, hence, $\mathbb{P}(T_k<\infty)=1$ for $k\in\mathbb{Z}_+$) we say that the process is explosive with probability 1. For the explosion time T_k we have that

$$\mathbb{E}(T_k)=\sum_{i=k}^{\infty}\mathbb{E}(\xi_i)=\sum_{i=k}^{\infty}\lambda_i^{-1}\quad\text{and}\quad\mathsf{Var}(T_k)=\sum_{i=k}^{\infty}\mathsf{Var}(\xi_i)=\sum_{i=k}^{\infty}\lambda_i^{-2}.\tag{3.1}$$

Consider a pure birth process birth rates $\lambda_k=k^{\beta}$, $k\geq 1$, where $\beta>1$. Given the above, such a birth process is explosive with probability 1, and the mean and the variance of the explosion time T_k in this case are as follows

$$m_k:=\mathbb{E}(T_k)\sim\frac{k^{-\beta+1}}{\beta-1}\quad\text{and}\quad s_k^2:=\mathsf{Var}(T_k)\sim\frac{k^{-2\beta+1}}{2\beta-1}.\tag{3.2}$$

Therefore,

$$\frac{s_k^2}{m_k^2}=\frac{O(1)}{k}\to 0,\quad\text{as}\quad k\to\infty,\tag{3.3}$$

and, by Chebyshev's inequality,

$$\mathbb{P}\left(|T_k-\mathbb{E}(T_k)|\geq\delta\cdot\mathbb{E}(T_k)\right)\leq\frac{O(1)}{\delta^2 k}\quad\text{for all}\quad\delta>0.\tag{3.4}$$

Rewriting $T_k=\sum_{i=k}^{k^2-1}\xi_i+\eta_k$, where $\eta_k=\sum_{i=k^2}^{\infty}\xi_i$, gives the scheme of series $\left(\tilde{\xi}_{k,i},\ i=1,...,k^2\right)$, $k\geq 2$, where $\tilde{\xi}_{k,i}=\xi_i$ for $i=k,...,k^2-1$ and $\tilde{\xi}_{k,k^2}=\eta_k$, that satisfies the Lyapunov's conditions of the Central Limit Theorem. Therefore, the random variable $\frac{T_k-m_k}{s_k}$ converges in distribution to the standard normal distribution, as $k\to\infty$. Moreover, since

$$\mathbb{E}(\xi^3)=6\lambda^{-3}\quad\text{and}\quad\mathbb{E}\left(|\xi-\mathbb{E}(\xi)|^3\right)\leq\frac{7}{\lambda^3}\quad\text{for}\quad\xi\sim Exp(\lambda),$$

we obtain the following for the pure birth process with polynomial rates

$$r_k:=\sum_{i=k}^{\infty}\mathbb{E}\left(|\xi_i-\mathbb{E}(\xi_i)|^3\right)\leq 7\sum_{i=k}^{\infty}i^{-3\beta}\leq C_3\frac{k^{-3\beta+1}}{3\beta-1}.\tag{3.5}$$

Combining this with (3.2) gives $\frac{r_k}{s_k^3}\leq\frac{O(1)}{\sqrt{k}}$, as $k\to\infty$. Therefore, by Berry-Esseen theorem (e.g. Theorem 2 in Feller (1971), Section 5, Chapter XVI), we have that

$$\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left(\frac{T_k-m_k}{s_k}\leq t\right)-\Phi(t)\right|\leq\frac{O(1)}{\sqrt{k}},\tag{3.6}$$

where Φ is the cumulative distribution function of the standard normal distribution.

Remark 3.1. For comparison, consider a pure birth process with exponential transition rates $\lambda_k=e^{\beta k}$, where $\beta>0$. Then, a direct computation gives that $s_k^2=\mathsf{Var}(T_k)\sim C_2e^{-2\beta k}$ and $r_k\leq C_3e^{-3\beta k}$. Therefore, $\frac{r_k}{s_k^3}\sim C$, and, hence, the normal approximation does not apply in this case.

4. Proofs of Theorems 2.1, 2.2 and 2.3

Proofs of Theorems 2.1, 2.2 and 2.3 are based on the well known embedding argument known as Rubin's construction (Davis (1990)). The idea of this construction applied to our model is as follows. The DTMC $X(t)=(X_1(t),X_2(t))$ with transition probabilities (2.1) can be regarded as the embedded Markov chain for a pair of independent continuous time pure birth processes $Z_1(t)$ and $Z_2(t)$ with the power law birth rates $k^{\beta_1},\ k\in\mathbb{Z}_+$, and $k^{\beta_2},\ k\in\mathbb{Z}_+$, respectively. Then events

$$A_i = \{\textit{all but finitely many balls are placed in the bin } i\},\quad i=1,2,$$

(initially defined in (2.2)) can be interpreted in terms of explosion times of these pure birth processes, which allows to estimate their probabilities by using the asymptotic results for the explosion times (provided in Section 3). For example, Theorem 2.1 immediately follows from the embedding argument, and we include its proof just for the sake of completeness. The two other theorems require some additional computations, and we provide necessary details.

Throughout this section we denote by $T_{z,i},\ i=1,2$ the explosion time of a pure birth process $Z_i(t)$ (with the power law birth rates $k^{\beta_i},\ k\in\mathbb{Z}_+$), started at $z\in\mathbb{Z}_+$.

4.1. *Proof of Theorem 2.1.* Observe that $\mathbb{P}_{\mathbf{x}}(A_1)=\mathbb{P}(T_{x,1}<T_{y,2})$ and $\mathbb{P}_{\mathbf{x}}(A_2)=\mathbb{P}(T_{x,1}>T_{y,2})$ for $\mathbf{x}=\in\mathbb{Z}_+^2$. If both $\beta_1>1$ and $\beta_2>1$, then $\mathbb{P}(T_{x,1}<\infty)=\mathbb{P}(T_{y,2}<\infty)=1$. Random variables $T_{x,1}$ and $T_{y,2}$ are independent and have absolutely continuous distributions. Therefore,

$$\mathbb{P}(T_{x,1}<T_{y,2})+\mathbb{P}(T_{x,1}>T_{y,2})=1,$$

and, hence, $\mathbb{P}_{\mathbf{x}}(A_1)+\mathbb{P}_{\mathbf{x}}(A_2)=1$, as claimed in Part 1) of Theorem 2.1. If $\beta_1\le 1$ and $\beta_2>1$, then $\mathbb{P}(T_{x,1}=\infty)=\mathbb{P}(T_{y,2}<\infty)=1$, and, hence, $\mathbb{P}_{\mathbf{x}}(A_2)=1$, as claimed in Part 2) of Theorem 2.1.

4.2. *Proof of Theorem 2.2.* The proof of Theorem 2.2 is based on the fact that the explosion times can be approximated by their expectations, as it follows from (3.4). We provide main details of the proof only for Part 1) of the theorem, as Part 2) can be proved similarly.

Recall that the process starts at $(x,y(x))$, where $y(x)=\nu x^{\alpha_{cr}}+o(x^{\alpha_{cr}})$ and, in addition,

$$\nu<\nu_{cr}=\alpha_{cr}^{\frac{1}{\beta_2-1}}=\left(\frac{\beta_1-1}{\beta_2-1}\right)^{\frac{1}{\beta_2-1}}. \tag{4.1}$$

By (3.2), we have that

$$\mathbb{E}(T_{x,1})\sim\frac{x^{1-\beta_1}}{\beta_1-1} \tag{4.2}$$

and

$$\mathbb{E}(T_{y(x),2})\sim\frac{(\nu x^{\alpha_{cr}}+o(x^{\alpha_{cr}}))^{1-\beta_2}}{\beta_2-1}\sim\frac{x^{\alpha_{cr}(1-\beta_2)}}{(\beta_2-1)\nu^{\beta_2-1}}=\frac{x^{\beta_1-1}}{(\beta_2-1)\nu^{\beta_2-1}}. \tag{4.3}$$

Combining (4.2) and (4.3) with (4.1) gives that

$$\frac{\mathbb{E}(T_{y(x),2})}{\mathbb{E}(T_{x,1})}\sim\left(\frac{\nu_{cr}}{\nu}\right)^{\beta_2-1}>1. \tag{4.4}$$

Therefore, $(1+\delta)\mathbb{E}(T_{x,1})<(1-\delta)\mathbb{E}(T_{y(x),2})$ for some $\delta>0$ and sufficiently large x, and, hence, the event

$$\{T_{x,1}<(1+\delta)\mathbb{E}(T_{x,1})\}\bigcap\{T_{y(x),2}>(1-\delta)\mathbb{E}(T_{y(x),2}\}$$

implies the event $\{T_{x,1}<T_{y(x),2}\}$. By (3.4),

$$\mathbb{P}(T_{x,1}<(1+\delta)\mathbb{E}(T_{x,1}))\ge 1-\mathbb{P}(|T_{x,1}-\mathbb{E}(T_{x,1})|\ge\delta\mathbb{E}(T_{x,1}))\ge 1-Cx^{-1}\to 1,$$

as $x\to\infty$. Similarly, $\mathbb{P}\left((1-\delta)\mathbb{E}(T_{y(x),2})<T_{y(x),2}\right)\to 1$, as $x\to\infty$. Then, by independence of explosion times,

$$\mathbb{P}\left(T_{x,1}<T_{y(x),2}\right)\ge[\mathbb{P}(T_{x,1}<(1+\delta)\mathbb{E}(T_{x,1}))][\mathbb{P}\left(T_{y(x),2}>(1-\delta)\mathbb{E}(T_{y(x),2})\right)]\to 1,$$

as $x\to\infty$, and, hence, $\lim_{\zeta\to\infty}\mathbb{P}_{(x,y(x))}(A_1)=1$, as claimed.

If $y(x) = \nu x^{\alpha} + o(x^{\alpha})$ for $\alpha < \alpha_{cr}$, then, $\alpha(1-\beta_2) - (1-\beta_1) > 0$. Using equation (3.2) we get (similarly to how we got equation (4.4)) that

$$\frac{\mathbb{E}(T_{y(x),2})}{\mathbb{E}(T_{x,1})} \sim O(1)x^{\alpha(1-\beta_2)-(1-\beta_1)} \to \infty,$$

as $x \to \infty$. Therefore, $(1+\delta)\mathbb{E}(T_{x,1}) < (1-\delta)\mathbb{E}(T_{y(x),2})$ for some $\delta > 0$ and sufficiently large x, and the claim of the theorem in this case follows again from independence of the explosion times and bound (3.4).

4.3. *Proof of Theorem 2.3.* Recall that $T_{z,i}$ is the explosion time of the continuous time pure birth process $Z_i(t)$ started at $z_i \in \mathbb{Z}_+$ and define $V_{x,y} = T_{x,1} - T_{y,2}$ for $x, y \in \mathbb{Z}_+$.

Proposition 4.1. *If $X(0) = (x, y(x))$, where $y(x) = \nu_{cr}x^{\alpha_{cr}} + \mu x^{\delta} + o(x^{\delta})$ for $\delta \in [0, \alpha_{cr})$, then*

$$\frac{\mathbb{E}(V_{x,y(x)})}{\sqrt{\mathsf{Var}(V_{x,y(x)})}} \sim -\rho x^{\delta - \frac{\alpha_{cr}}{2}},$$

where $\rho = -\mu\left(\frac{2\beta_2 - 1}{\nu_{cr}}\right)^{\frac{1}{2}}$.

Proof: Using equation (3.2) we obtain that

$$\begin{aligned}
\mathbb{E}(V_{x,y(x)}) = \mathbb{E}(T_{x,1}) - \mathbb{E}(T_{y(x),2}) &\sim \frac{x^{1-\beta_1}}{\beta_1 - 1} - \frac{\left(\nu_{cr}x^{\alpha_{cr}} + \mu x^{\delta} + o(x^{\delta})\right)^{1-\beta_2}}{\beta_2 - 1} \\
&\sim \frac{x^{1-\beta_1}}{\beta_1 - 1} - \frac{\nu_{cr}^{1-\beta_2}x^{\alpha_{cr}(1-\beta_2)}}{\beta_2 - 1}\left(1 + \frac{\mu x^{\delta-\alpha_{cr}}}{\nu_{cr}}\right)^{1-\beta_2} \\
&\sim \frac{x^{1-\beta_1}}{\beta_1 - 1} - \frac{\nu_{cr}^{1-\beta_2}x^{\alpha_{cr}(1-\beta_2)}}{\beta_2 - 1}\left(1 + \frac{\mu(1-\beta_2)x^{\delta-\alpha_{cr}}}{\nu_{cr}}\right) \\
&\sim \frac{x^{1-\beta_1}}{\beta_1 - 1} - \frac{\nu_{cr}^{1-\beta_2}x^{\alpha_{cr}(1-\beta_2)}}{\beta_2 - 1} + \frac{\mu}{\nu_{cr}^{\beta_2}}x^{\delta-\alpha_{cr}\beta_2}.
\end{aligned}$$

Noting that

$$\frac{x^{1-\beta_1}}{\beta_1 - 1} - \frac{\nu_{cr}^{1-\beta_2}x^{\alpha_{cr}(1-\beta_2)}}{\beta_2 - 1} = 0$$

we get the following approximation

$$\mathbb{E}(V_{x,y(x)}) = \frac{\mu}{\nu_{cr}^{\beta_2}}x^{\delta-\alpha_{cr}\beta_2}. \tag{4.5}$$

Further, by equation (3.2) we have that

$$\mathsf{Var}(T_{x,1}) \sim \frac{x^{-2\beta_1+1}}{2\beta_1 - 1} \quad \text{and} \quad \mathsf{Var}(T_{y(x),2}) \sim \frac{(\nu_{cr}x^{\alpha_{cr}})^{-2\beta_2+1}}{2\beta_2 - 1}\left(1 + \frac{\mu}{\nu_{cr}}x^{\delta-\alpha_{cr}}\right)^{-2\beta_2+1}.$$

Observe that

$$\alpha_{cr} = \frac{\beta_1 - 1}{\beta_2 - 1} < \frac{m\beta_1 - 1}{m\beta_2 - 1} \iff (-m\beta_1 + 1) - \alpha_{cr}(-m\beta_2 + 1) < 0 \quad \text{for} \quad m > 1.$$

Therefore, $\mathsf{Var}(V_{x,y(x)}) = \mathsf{Var}(T_{x,1}) + \mathsf{Var}(T_{y(x),2}) \sim \mathsf{Var}(T_{y(x),2})$, and

$$\sqrt{\mathsf{Var}(V_{x,y(x)})} \sim \sqrt{\mathsf{Var}(T_{y(x),2})} \sim \frac{\nu_{cr}^{\frac{1}{2}-\beta_2}x^{\alpha_{cr}\left(\frac{1}{2}-\beta_2\right)}}{\sqrt{2\beta_2 - 1}}. \tag{4.6}$$

Combining equations (4.5) and (4.6) gives the proposition. □

Further, observe that $\lim_{x\to\infty}\mathbb{P}\left(A^{(1)}_{x,y(x)}\right)=\lim_{x\to\infty}\mathbb{P}(V_{x,y(x)}<0)$ and

$$\mathbb{P}(V_{x,y(x)}<0)=\mathbb{P}\left(\widetilde{V}_{x,y(x)}<-\frac{\mathbb{E}(V_{x,y(x)})}{\sqrt{\mathsf{Var}(V_{x,y(x)})}}\right)\sim\mathbb{P}\left(\widetilde{V}_{x,y(x)}<\rho x^{\alpha-\frac{\alpha_{cr}}{2}}\right), \tag{4.7}$$

where $\widetilde{V}_{x,y(x)}=\frac{V_{x,y(x)}-\mathbb{E}(V_{x,y(x)})}{\sqrt{\mathsf{Var}(V_{x,y(x)})}}$. The normal approximation for explosion times discussed above yields the normal approximation for the random variable $V_{x,y(x)}$. In particular, we have (similarly to (3.6)), that

$$|\mathbb{P}(\widetilde{V}_{x,y(x)}\le z)-\Phi(z)|\le O(1)x^{-\frac{\alpha_{cr}}{2}}\quad\text{for}\quad z\in\mathbb{R}. \tag{4.8}$$

Summarizing the above, we obtain the results listed below.

i) If $\alpha=\frac{\alpha_{cr}}{2}$, then $\lim_{x\to\infty}\mathbb{P}\left(A^{(1)}_{x,y(x)}\right)=\Phi(\rho)$.

ii) If $\delta<\frac{\alpha_{cr}}{2}$, then $\lim_{x\to\infty}\mathbb{P}\left(A^{(1)}_{x,y(x)}\right)=\lim_{x\to\infty}\Phi\left(\rho x^{\delta-\frac{\alpha_{cr}}{2}}\right)=\Phi(0)=\frac{1}{2}$.

iii) If $\frac{\alpha_{cr}}{2}<\delta<\alpha_{cr}$

(a) and $\gamma<0$, then $\rho x^{\delta-\frac{\alpha_{cr}}{2}}\to\infty$, as $x\to\infty$, and, hence,

$$\lim_{x\to\infty}\mathbb{P}\left(A^{(1)}_{x,y(x)}\right)=\lim_{x\to\infty}\Phi\left(\rho x^{\delta-\frac{\alpha_{cr}}{2}}\right)=1;$$

(b) and $\gamma>0$, then $\rho x^{\delta-\frac{\alpha_{cr}}{2}}\to-\infty$, as $x\to\infty$, and, hence,

$$\lim_{x\to\infty}\mathbb{P}\left(A^{(1)}_{x,y(x)}\right)=\lim_{x\to\infty}\Phi\left(\rho x^{\delta-\frac{\alpha_{cr}}{2}}\right)=0.$$

Theorem 2.3 now follows from items i)-iii).

5. Proofs of results for the model with reflection

In this section we prove a series of lemmas that describe the long term behavior of the DTMC $\zeta(t)$ in the case, when the process starts near either the lower, or the upper boundary. The case of the lower boundary is treated in detail in Section 5.1. The case of the upper boundary is considered in Section 5.2. The main results for the model with reflection, i.e. Theorems 2.9 and 2.10, will follow from the series of lemmas and explosiveness of the BB model with asymmetric feedback, see Section 5.3 for details.

5.1. *The case of the lower boundary.* Start with some definitions and auxiliary facts.

Definition 5.1. Given $\beta_1,\beta_2:1<\beta_1\le\beta_2$ define

$$\alpha_k=\frac{\beta_1-1}{\beta_2-1+\frac{1}{k}}\quad\text{for}\quad k\ge1\quad\text{and}\quad\alpha_0=0. \tag{5.1}$$

Note that

$$\alpha_k<\alpha_{k+1}\quad\text{and}\quad\lim_{k\to\infty}\alpha_k=\alpha_{cr}, \tag{5.2}$$

where α_{cr} is defined in (2.3).

Definition 5.2. Let the lower boundary be given by the curve $y=x^\alpha$, for some $0<\alpha<\alpha_{cr}$. Given $r>0$ define the set (stripe along the lower boundary)

$$U_{\alpha,r}=\{(x,y)\in\mathbb{Z}^2_+:x^\alpha\le y\le x^\alpha+r\}. \tag{5.3}$$

If $0 < \alpha < \alpha_{cr}$, then $\alpha\beta_2 - \beta_1 < 0$, which implies the following approximation

$$\mathbb{P}_\zeta(\zeta_1(1) = x,\, \zeta_2(1) = y+1) \sim \frac{x^{\alpha\beta_2}}{x^{\beta_1} + x^{\alpha\beta_2}} \sim x^{\alpha\beta_2-\beta_1} \quad \text{for} \quad \zeta = (x,y) : y \sim x^\alpha. \tag{5.4}$$

Proposition 5.3 below concerns a pair of simple geometric facts, that basically follow from sublinearity of the lower boundary $y = x^\alpha$ and the reflection rule defined in (2.8). However, these facts are repeatedly used in subsequent proofs, therefore it is convenient to arrange them in a separate statement.

Proposition 5.3. *Consider a trajectory of the DTMC $\zeta(t)$ that is reflected by the lower boundary infinitely many times.*

1) *Then, for such a trajectory the following holds*

$$\liminf_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = 0.$$

In other words, such a trajectory approaches the lower boundary at any arbitrarily small distance.

2) *If, in addition, there are at most $m \geq 0$ jumps up between consecutive reflections, then such a trajectory eventually confines to the strip $U_{\alpha,m+1+\varepsilon}$ for any given $\varepsilon > 0$.*

Proof of Proposition 5.3: Let (ζ_1, ζ_2) be a state, at which the DTMC $\zeta(t)$ is reflected by the lower boundary $y = x^\alpha$, i.e. the state satisfies (2.8) with $\varphi(x) = x^\alpha$. Then,

$$\zeta_2 - \zeta_1^\alpha \leq \delta(\zeta_1) := (\zeta_1+1)^\alpha - \zeta_1^\alpha \sim \alpha(\zeta_1)^{\alpha-1} \to 0, \quad \text{as} \quad \zeta \to \infty. \tag{5.5}$$

Let τ_n be the time, when the n-th reflection takes place. It is obvious that $\tau_n \to \infty$, and, hence, $\zeta_1(\tau_n) \to \infty$, as $n \to \infty$. Therefore, by (5.5)

$$\zeta_2(\tau_n) - \zeta_1^\alpha(\tau_n) = \delta(\zeta_1(\tau_n)) \to 0, \quad \text{as} \quad n \to \infty, \tag{5.6}$$

which implies Part 1) of the proposition.

If, in addition, there are at most $m \geq 0$ jumps up between consecutive reflections, then, by (5.5), the process does not exit the strip $U_{\alpha,m+1+\delta(\zeta_1(\tau_n))}$ after n-th reflection. Combining this fact with (5.6) gives Part 2) of the proposition. □

Lemmas 5.4 and 5.5 below describe in detail the long term behavior of the DTMC $\zeta(t)$ in the case, when the parameter α (determining the lower boundary $y = x^\alpha$) is smaller than $\alpha_1 = \frac{\beta_1 - 1}{\beta_2}$, and the process starts near the lower boundary.

Lemma 5.4. *Let $\alpha \in (0, \alpha_1)$, and let $\zeta = (x,y) \in U_{\alpha,C}$ for some $C > 0$. Then*

$$\mathbb{P}_\zeta(\{\textit{the DTMC } \zeta(t) \textit{ jumps to the right, whenever possible}\}) \to 1, \quad \textit{as} \quad x \to \infty.$$

Proof of Lemma 5.4: Denote for short

$$A = \{\text{the DTMC } \zeta(t) \text{ jumps to the right, whenever possible}\}.$$

Observe that, given an initial state ζ, the event A consists of a single trajectory, such that the component $\zeta_2(t)$ increases only at those moments in time, when the process is reflected by the lower boundary. Since the function $y = x^\alpha$ is monotonically increasing to infinity, the trajectory does not leave the strip $U_{\alpha,C}$ before the first reflection, and then, by Proposition 5.3, it confines to the strip $U_{\alpha,2}$. By (5.4)

$$\mathbb{P}_\zeta(A) \sim \prod_{k=x}^{\infty}\left(1 - k^{\alpha\beta_2-\beta_1}\right) \sim e^{-\sum_{k=x}^{\infty} k^{\alpha\beta_2-\beta_1}}, \quad \text{as} \quad x \to \infty, \quad \text{for} \quad \zeta = (x,y) \in U_{\alpha,2}.$$

Since $\alpha < \alpha_1 = \frac{\beta_1-1}{\beta_2} \iff \alpha\beta_2 - \beta_1 < -1$, we get that $\sum_{k=x}^{\infty} k^{\alpha\beta_2-\beta_1} \to 0$, as $x \to \infty$. Therefore, $\mathbb{P}_\zeta(A) \to 1$, as $x \to \infty$, and the lemma is proved. □

Lemma 5.5. *Let $\alpha \in (0, \alpha_1)$, and let $\zeta = (x, y) \in U_{\alpha,C}$ for some $C > 0$. Then*

$$\mathbb{P}_\zeta(A_{\alpha,1}) \to 1, \quad as \quad x \to \infty,$$

where

$$A_{\alpha,1} = \{\limsup_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = 1\} \cap \{\liminf_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = 0\}$$

is a special case of the event defined in (2.10).

Proof of Lemma 5.5: First recall (see Remark 2.11) that the event $A_{\alpha,1}$ means the following. Given any arbitrary small $\varepsilon > 0$ the process eventually confines to the strip $U_{\alpha,1+\varepsilon} = \{(\zeta_1, \zeta_2) : \zeta_1^\alpha \leq \zeta_2 \leq \zeta_1^\alpha + 1 + \varepsilon\}$, where it fluctuates by approaching both the lower boundary $y = x^\alpha$ and the curve $y = x^\alpha + 1$ at any arbitrarily small distance. To show this consider the same trajectory as in the proof of Lemma 5.4, i.e. where the component $\zeta_2(t)$ increases only due to reflections at the lower boundary. It is easy to see that this trajectory is reflected by the lower boundary infinitely many times. Consequently, by Part 1) of Proposition 5.3, we have that $\liminf_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = 0$. The component $\zeta_2(t)$ increases by 1 at the moment of reflection, therefore, by Part 2) of Proposition 5.3, the trajectory eventually confines to the strip $U_{\alpha,1+\varepsilon}$ for any given $\varepsilon > 0$, so that $\limsup_{t\to\infty}(\zeta_2(t) - \zeta_1^\alpha(t)) = 1$. Lemma 5.5 follows now from Lemma 5.4. □

To proceed further, recall the second item in Definition 2.6 that distinguishes between jumps and reflections of the DTMC $\zeta(t)$.

Lemma 5.6. *Given $\alpha \in (0, \alpha_{cr})$, let $\zeta = (x, y) \in U_{\alpha,C_1}$ and let $N = N(x) = C_2 x^{1-\alpha}$ for some $C_1 > 0$ and $C_2 > 0$. Let ξ_N be the number of jumps up of the DTMC $\zeta(t)$ on the interval $[0, N]$. Suppose that x is sufficiently large. Then for any fixed $m \geq 1$ the following bounds hold*

$$C_3\lambda_x^m \leq \mathbb{P}_\zeta(\xi_N \geq m) \leq C_4\lambda_x^m, \tag{5.7}$$

where

$$\lambda_x := x^{\alpha\beta_2-\beta_1+1-\alpha}, \tag{5.8}$$

and constants $C_3 > 0$ and $C_4 > 0$ do not depend on x.

Proof of Lemma 5.6: Start with noting that the condition $\alpha < \alpha_{cr}$ is equivalent to the condition $\alpha\beta_2 - \beta_1 + 1 - \alpha < 0$, which, in particular, implies that $\lambda_x \to 0$, as $x \to \infty$. Also, in this case we have that $\alpha\beta_2 - \beta_1 < 0$. Therefore, by equation (5.4),

$$\mathbb{P}_\zeta(\zeta_1(1) = x,\, \zeta_2(1) = y + 1) \sim x^{\alpha\beta_2-\beta_1}. \tag{5.9}$$

Next, assuming that the process does not jump up more than a fixed number of times (not depending on x) on the interval $[0, N]$, we can approximate the probability of the jump up at any time $t \in [0, N]$ by the probability of the jump up at time 0, that is, $x^{\alpha\beta_2-\beta_1}$ (see (5.9)) for sufficiently large x. This allows to approximate the distribution of the random variable ξ_N by the binomial distribution with the mean λ_x, which, in turn, can be approximated by the Poisson distribution with the same mean. More precisely, noting first that

$$(1 - x^{\alpha\beta_2-\beta_1})^N \leq \mathbb{P}_\zeta(\xi_N = 0) \leq (1 - (x + N)^{\alpha\beta_2-\beta_1})^N,$$

and

$$(1 - x^{\alpha\beta_2-\beta_1})^N \sim (1 - (x + N)^{\alpha\beta_2-\beta_1})^N \sim e^{-\lambda_x},$$

we obtain that

$$\mathbb{P}_\zeta(\xi_N = 0) \sim e^{-\lambda_x}, \quad \text{as} \quad x \to \infty.$$

Combining this with equation (5.9) gives that

$$\mathbb{P}_\zeta(\xi_N = i) \sim e^{-\lambda_x}\frac{\lambda_x^i}{i!}, \quad \text{as} \quad x \to \infty,$$

for any fixed i, which, in turn, implies the bound (5.7), as claimed. □

Definition 5.7. Let $\alpha \in (0, \alpha_{cr})$ be the parameter determining the lower boundary $y = x^\alpha$. Given $r > 0$ and $\zeta = (x, y) : x^\alpha \le y \le x^\gamma$, define n_x as an integer such that

$$(n_x r)^{\frac{1}{\alpha}} \le x < ((n_x + 1)r)^{\frac{1}{\alpha}} \tag{5.10}$$

and define the sequence of hitting times

$$t_{r,n} = \min(t : \zeta_1(t_{r,n}) \ge (nr)^{\frac{1}{\alpha}}) \quad \text{for} \quad n > n_x.$$

Remark 5.8. Note that due to reflection both $\zeta_1(t) \to \infty$, and $\zeta_2(t) \to \infty$, as $t \to \infty$. Therefore, $t_{r,n} < \infty$ and $t_{r,n} < t_{r,n+1}$ for all n. In addition, note that

$$((n+1)r)^{\frac{1}{\alpha}} - (nr)^{\frac{1}{\alpha}} \sim \frac{r}{\alpha}(nr)^{\frac{1-\alpha}{\alpha}}, \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Definition 5.9. Let $(t_{r,n},\ n > n_x)$ be the sequence of hitting times defined in Definition 5.7 for some $\zeta = (x, y)$. Define $S_{r,n}$ as the random variable that is equal to the number of jumps up of the DTMC $\zeta(t)$ on the interval $[t_{r,n}, t_{r,n+1})$.

Lemma 5.10. *Let $\alpha \in [\alpha_{k-1}, \alpha_k)$ for some $k \ge 2$, and let $\zeta = (x, y) \in U_{\alpha,C}$ for some $C > 0$. Let $(S_{r,n},\ n > n_x)$ be the sequence of random variables defined in Definition 5.9. Then,*

$$\mathbb{P}_\zeta\left(S_{r,n} < k \text{ for all } n > n_x\right) \to 1, \quad \text{as} \quad x \to \infty, \tag{5.12}$$

and

$$\mathbb{P}_\zeta\left(S_{r,n} = k-1 \text{ for infinitely many } n\right) = 1. \tag{5.13}$$

Proof of Lemma 5.10: Start with showing (5.12) in the case, when r is sufficiently large. Recall that n_0 is a positive integer, such that (5.10) holds. It is easy to see that the lower boundary $y = x^\alpha$ increases by r on the interval $[(rn)^{\frac{1}{\alpha}}, (r(n+1))^{\frac{1}{\alpha}})$. Therefore, if r is sufficiently large, and $S_{r,n} < k$ for all $n > n_x$, then

$$\{S_{r,n} < k \ \text{ for all } \ n\} \subseteq \{\zeta(t) \in U_{\alpha,2k-1+C} \ \text{ for all } \ t \ge 0\}, \tag{5.14}$$

i.e. the process cannot escape the strip $U_{\alpha,2k-1+C}$. Indeed, it is easy to see that the difference $\zeta_2(t) - \zeta_1^\alpha(t)$ cannot be more than $2k - 1 + C$. On the other hand, the difference $\zeta_2(t) - \zeta_1^\alpha(t)$ can be equal to $2k - 1 + C$. Indeed, this can happen, when the process is reflected near the end of the interval $[t_{r,n}, t_{r,n+1})$, immediately jumps up $k - 1$ times, and, jumps up again $k - 1$ times at the beginning of the next interval.

Further, by Definition 5.7,

$$\zeta_1(t_{r,n}) - (rn)^{\frac{1}{\alpha}} \le 1 \quad \text{for all} \quad n > n_x. \tag{5.15}$$

Without loss of generality, we can also assume that the initial position $\zeta = (x, y)$ is such that

$$x - (rn_x)^{\frac{1}{\alpha}} \le 1. \tag{5.16}$$

Then, it follows from (5.14)-(5.16) and Lemma 5.6 that

$$\mathbb{P}_\zeta(S_{r,n} \ge k | S_{r,n_x} < k, ..., S_{r,n-1} < k) \le C_3 \lambda_n^k, \tag{5.17}$$

and, hence,

$$\mathbb{P}_\zeta(\cap_{i=n_x}^n \{S_{r,i} < k\}) \ge \prod_{i=n_x}^n (1 - C_3 \lambda_i^k), \tag{5.18}$$

where

$$\lambda_i := (ir)^{\frac{\alpha(\beta_2-1)-(\beta_1-1)}{\alpha}} \quad \text{for} \quad i \ge 0. \tag{5.19}$$

Note that the assumption $\alpha \in [\alpha_{k-1}, \alpha_k)$ can be rewritten as follows

$$\frac{\alpha(\beta_2 - 1) - (\beta_1 - 1)}{\alpha} k < -1 \le \frac{\alpha(\beta_2 - 1) - (\beta_1 - 1)}{\alpha}(k - 1). \tag{5.20}$$

The left inequality in the preceding display implies that $\sum_{n=n_x}^{\infty}\lambda_n^k \to 0$, as $n_x\to\infty$. Combining this with (5.18) gives that

$$\mathbb{P}_{\zeta}\left(S_{r,n}<k \text{ for all } n\right) \geq \prod_{n=n_x}^{\infty}\left(1-C_3\lambda_n^k\right) \geq e^{-O(1)\sum\limits_{n=n_x}^{\infty}\lambda_n^k} \to 1, \quad \text{as} \quad x\to\infty. \tag{5.21}$$

Next, consider the case of an arbitrary r. It is easy to see that in this case time intervals $[t_{r,n}, t_{r,n+1})$ (r-intervals) can be covered by similar intervals with a sufficiently large $r'>r$ (r'-intervals) in such a manner that each r-interval is covered by an r'-interval. Clearly, if it occurs that $S_{r,n}\geq k$ for infinitely many n, then there are infinitely many r'-intervals, such that $S_{r',n}\geq k$. The probability of the latter tends to zero, as we have proved in the case of large r, and, hence, (5.12) holds for an arbitrary r.

Next, let us show (5.13). To this end observe that

$$\mathbb{P}_{\zeta}(S_{r,n}\geq k-1|\mathcal{F}_{n-1}) \geq C_2\lambda_n^{k-1}, \tag{5.22}$$

where $\mathcal{F}_{n-1}$ is the σ-field generated by $\zeta(t)$, $t<t_{r,n}$. Indeed, it suffices to consider the case when the process starts near the lower boundary (i.e. $\zeta\in U_{\alpha,C}$ for some $C>0$). By Lemma 5.6 and (5.20), we have that $\sum_{n=1}^{\infty}\lambda_n^{k-1}=\infty$, and (5.13) follows from the conditional variant of the second Borel-Cantelli lemma (e.g. see Theorem 5.3.2 in Durrett (2019)). □

Lemma 5.11. *Let $\alpha\in[\alpha_{k-1},\alpha_k)$ for some $k\geq 2$, and let $\zeta=(x,y)\in U_{\alpha,C}$ for some $C>0$. Then $\mathbb{P}_{\zeta}\left(A_{\alpha,k}\right)\to 1$, as $x\to\infty$, where $A_{\alpha,k}$ is the event defined in (2.10).*

Proof of Lemma 5.11: We use notations of Lemma 5.10.

First, given a sufficiently large $r>0$ consider a trajectory of the process, such that $S_{r,n}<k$ for all n. Comparing the growth of the trajectory with that of the lower boundary (similarly to the proof of Lemma 5.10) gives that such a trajectory is reflected by the lower boundary at least once on each time interval $[t_{r,n}, t_{r,n+1})$. Consequently, the trajectory is reflected infinitely many times. Combining this with Part 1) of Proposition 5.3 and Lemma 5.10 gives that

$$\lim_{x\to\infty}\mathbb{P}_{\zeta}\left(\liminf_{t\to\infty}(\zeta_2(t)-\zeta_1^{\alpha}(t))=0\right)=1 \quad \text{for} \quad \zeta=(x,y)\in U_{\alpha,C}. \tag{5.23}$$

Further, given $n\geq 1$ let τ_n be the time moment, when the process is reflected for the n-th time, and let $\tau_0=0$. Let S_n be the number of jumps up on the interval $[\tau_n,\tau_{n+1})$ for $n\geq 0$. It is easy to see that, if $S_n<k$ for all n, then differences $\tau_{n+1}-\tau_n$ are uniformly bounded, provided that $S_n<k$ for all n. Therefore, any of these intervals is covered by some deterministic interval $[t_{r,n}, t_{r,n+1})$ for a sufficiently large r. Consequently, with a little of extra work, it follows from Part 1) of Lemma 5.10 that

$$\lim_{x\to\infty}\mathbb{P}_{\zeta}(S_n<k \text{ for all } n)=1, \quad \text{for} \quad \zeta=(x,y)\in U_{\alpha,C}.$$

Combining this result with Part 2) of Proposition 5.3, gives that, with $\mathbb{P}_{\zeta}$-probability tending to 1, as $x\to\infty$,

a) the DTMC $\zeta(t)$ eventually confines to the strip $U_{\alpha,k+\varepsilon}$ for any given $\varepsilon>0$.

On the other hand,

b) for any given $\varepsilon>0$ the DTMC $\zeta(t)$ almost surely exits the strip $U_{\alpha,k-\varepsilon}$ infinitely many times.

Indeed, choose $r>0$ in Part 2) of Lemma 5.10 in such a way that the lower boundary $y=x^{\alpha}$ grows between points $(nr)^{\frac{1}{\alpha}}$ and $((n+1)r)^{\frac{1}{\alpha}}$ by not more than an arbitrarily small $\varepsilon'>0$. By Part 2) of Lemma 5.10, almost surely there are infinitely many intervals $t_{r,n}, t_{r,n+1})$, where the process jumps up exactly $k-1$ times. This gives b).

Items a) and b) imply that

$$\lim_{x\to\infty}\mathbb{P}_\zeta\left(\limsup_{t\to\infty}(\zeta_2(t)-\zeta_1^\alpha(t))=k\right)=1\quad\text{for}\quad\zeta=(x,y)\in U_{\alpha,C}. \tag{5.24}$$

The lemma now follows from (5.23) and (5.24). □

5.2. *The case of the upper boundary.* In this section we study the long term behavior of the DTMC $\zeta(t)$ in the case, when the process starts near the upper boundary. Recall that $1<\beta_1\le\beta_2$, and the upper boundary is given by the curve $y=x^\gamma$, where $\gamma>\alpha_{cr}=\frac{\beta_1-1}{\beta_2-1}$. We distinguish three cases, namely, $\gamma>1$ (superlinear upper boundary), $\gamma=1$ (the case of the bisector) and $\alpha_{cr}<\gamma<1$ (sublinear upper boundary). Note that the last two cases are possible only in the asymmetric case $1<\beta_1<\beta_2$. The case of the super-linear upper boundary and the case of the bisector are considered in Section 5.2.1, and the case of the sublinear upper boundary is considered in Section 5.2.2.

5.2.1. *Upper boundary: the bisector and above.* In this section we study the most probable long term behavior of the DTMC $\zeta(t)$ in the case when it starts near the upper boundary given by $y=x^\gamma$, where $\gamma\ge 1$. In this case the corresponding results are similar to the results obtained in Section 5.1 in the case of the lower boundary. Therefore, we only state the results and relate them to their analogues in Section 5.1. However, in the case of a bisector (which is possible only in the asymmetric case $1<\beta_1<\beta_2$), the corresponding results can be refined (see Remark 5.17 and Lemma 5.24 below).

We start with some definitions.

Definition 5.12. Given $\beta_1,\beta_2: 1<\beta_1\le\beta_2$ define

$$\gamma_k=\frac{\beta_1-1+\frac{1}{k}}{\beta_2-1}=\alpha_{cr}+\frac{1}{(\beta_2-1)k}\quad\text{for}\quad k\in\mathbb{Z}_+. \tag{5.25}$$

Remark 5.13. Note that quantities $(\gamma_k,\ k\ge1)$ are similar to quantities $(\alpha_k,\ k\ge1)$ (defined in (5.1)). Note that $\gamma_{k+1}<\gamma_k$ for any $k\ge1$. If $\beta_1=\beta_2$, then $\gamma_k=\alpha_k^{-1}$ for any $k\ge1$.

Definition 5.14. Let the upper boundary given by the curve $y=x^\gamma$, where $\gamma>1$. Given $r>0$ define the set (stripe along the upper boundary)

$$\begin{aligned}\widetilde{U}_{\gamma,r}&=\{(x,y)\in\mathbb{Z}_+^2: x\ge r,\ (x-r)^\gamma<y\le x^\gamma\}\\&=\{(x,y)\in\mathbb{Z}_+^2: x\ge r,\ y^{\frac{1}{\gamma}}\le x<y^{\frac{1}{\gamma}}+r\}.\end{aligned} \tag{5.26}$$

Remark 5.15. Note that the stripes $\widetilde{U}_{\gamma,r}$ are analogous of stripes $U_{\alpha,r}$ along the lower boundary defined in Definition 5.2.

Observe, that in the case, when $\frac{\beta_1}{\gamma}-\beta_2<0$, we have the following approximation for the probability of the jump to the right (see (2.7))

$$\mathbb{P}_\zeta(\zeta_1(1)=x+1,\,\zeta_2(1)=y)\sim\frac{y^{\frac{\beta_1}{\gamma}}}{y^{\frac{\beta_1}{\gamma}}+y^{\beta_2}}\sim y^{\frac{\beta_1}{\gamma}-\beta_2}\sim x^{\beta_1-\gamma\beta_2}\quad\text{for}\quad\zeta=(x,y): y\sim x^\gamma. \tag{5.27}$$

Lemma 5.16. *Recall that $1<\beta_1\le\beta_2$ and suppose one of the following two conditions is fulfilled*

a) $\beta_2-\beta_1\le1$ *(i.e. $\gamma_1=\frac{\beta_1}{\beta_2-1}\ge1$) and $\gamma>\max(1,\gamma_1)$;*

b) $\beta_2-\beta_1>1$ *(i.e. $\gamma_1=\frac{\beta_1}{\beta_2-1}<1$) and $\gamma\ge1$*

and $\zeta=(x,y)\in\widetilde{U}_{\gamma,C}$ for some $C>0$. Then,

1) $\lim_{y\to\infty}\mathbb{P}_\zeta$*(the DTMC $\zeta(t)$ jumps up, whenever possible)* $=1$,

2) $\lim_{y\to\infty}\mathbb{P}_\zeta(\widetilde{A}_{\gamma,1})=1$, *where the event $\widetilde{A}_{\gamma,1}$ is defined in (2.11).*

Proof: Part 1) of Lemma 5.16 is an analogue of Lemma 5.4. By (5.27), the probability that the process jumps up whenever possible can be estimated by $e^{-\sum_{k=y}^{\infty} k^{\frac{\beta_1}{\gamma}-\beta_2}}$. Both a) and b) yield, that $\frac{\beta_1}{\gamma} - \beta_2 < -1$. Therefore, the expression in the exponent above is a tail of a convergent series, so that it tends to 0, and, hence, the exponent tends to 1, as $y \to \infty$.

Part 2) of Lemma 5.16 is the analogue of Lemma 5.5 and can be proven similarly. □

Remark 5.17. Note a special case of Lemma 5.16, when $\beta_2 - \beta_1 > 1$ and $\gamma = 1$ (i.e. the upper boundary is given by the bisector). In this case, the process started at a state (x, x) follows the "zigzag" trajectory

$$\zeta = (x, x) \to (x+1, x) \to (x+1, x+1) \to (x+2, x+1) \to (x+2, x+2) \to ....$$

with $\mathbb{P}_\zeta$ probability tending to 1, as $x \to \infty$.

Definition 5.18. Let $\gamma \geq 1$ be the parameter determining the lower boundary $y = x^\gamma$. Given $r > 0$ and $\zeta = (x, y)$ (belonging to the state space of the process) define n_y as an integer such that

$$(n_y r)^\gamma \leq y < ((n_y+1)r)^\gamma \tag{5.28}$$

and

$$\tilde{t}_{r,n} = \min(t : \zeta_2(t_{r,n}) \geq (nr)^\gamma) \quad \text{for} \quad n > n_y.$$

Definition 5.19. Let $(\tilde{t}_{r,n},\, n > n_y)$ be the sequence of time moments defined in Definition 5.18 for some $\zeta = (x, y)$. Define $\widetilde{S}_{r,n}$ as the random variable that is equal to the number of jumps to the right of the DTMC $\zeta(t)$ on the interval $[\tilde{t}_{r,n}, \tilde{t}_{r,n+1})$.

Remark 5.20. Note that if $\beta_2 - \beta_1 < 1$, then $\gamma_k \geq 1$ only for $k = 1, ..., k_{max}$, where k_{max} is the maximal integer, such that $k \leq \frac{1}{\beta_2 - \beta_1}$.

Lemma 5.21. *Let $\beta_2 - \beta_1 < 1$ and $\max(1, \gamma_k) < \gamma \leq \gamma_{k-1}$ for some integer $k \geq 2$, and let $\zeta = (x, y) \in \widetilde{U}_{\gamma,C}$ for some $C > 0$. Then, for any given $r > 0$,*

$$\lim_{y\to\infty} \mathbb{P}_\zeta\left(\widetilde{S}_{r,n} < k \textit{ for all } n\right) = 1 \tag{5.29}$$

and

$$\mathbb{P}_\zeta\left(\widetilde{S}_{r,n} = k-1 \textit{ for infinitely many } n\right) = 1. \tag{5.30}$$

Lemma 5.21 is an analogue of Lemma 5.10 and can be proven in a similar way (by using an analogue of Lemma 5.6).

Remark 5.22. Note that in the special case $\gamma = 1$ in Lemma 5.21 we have that $((n+1)r)^\gamma - (nr)^\gamma = (n+1)r - nr = r$, which implies that the intervals $[t_{r,n}, t_{r,n+1})$ are uniformly bounded by a constant. In turn, this implies that

$$\mathbb{P}_\zeta\left(\widetilde{S}_{r,n} \geq k | \widetilde{S}_{r,n_y} < k, ..., \widetilde{S}_{r,n-1} < k\right) \sim O(1) n^{(\beta_1-\beta_2)k}$$

for any possible integer $k \geq 0$. Therefore, if $\frac{1}{k} < \beta_2 - \beta_1 \leq \frac{1}{k-1}$ (i.e. $k - 1 \leq \frac{1}{\beta_2-\beta_1} < k$), then $\sum_n n^{(\beta_1-\beta_2)k} < \infty$ and $\sum_n n^{(\beta_1-\beta_2)(k-1)} = \infty$, which implies (5.29) and (5.30) respectively.

Lemma 5.23. *Let $\beta_2 - \beta_1 < 1$ and $\max(1, \gamma_k) < \gamma \leq \gamma_{k-1}$ for some $k \geq 2$ and let $\zeta = (x, y) \in \widetilde{U}_{\gamma,C}$ for some $C > 0$. Then $\lim_{y\to\infty} \mathbb{P}_\zeta(\widetilde{A}_{\gamma,k}) = 1$, where the event $\widetilde{A}_{\gamma,k}$ is defined in (2.11).*

Lemma 5.23 is an analogue of Lemma 5.11 and can be proven similarly.

Lemma 5.24 below is a refinement (implied by Remark 5.22) of Lemma 5.23 in the case, when the upper boundary is the bisector.

Lemma 5.24. *Let $\gamma = 1$ and let $\zeta = (x,x)$ for some $x \in \mathbb{Z}_+$. If $\frac{1}{k} < \beta_2 - \beta_1 \le \frac{1}{k-1}$ (i.e. $k-1 \le \frac{1}{\beta_2-\beta_1} < k$) for some integer $k \ge 2$, then*

$$\lim_{x\to\infty} \mathbb{P}_\zeta(D_1 \cap D_2 \cap D_3) = 1,$$

where events D_i, $i = 1,2,3$ are defined below

$$\begin{aligned} D_1 &= \{\zeta_1(t) - \zeta_2(t) \le k \text{ for all } t \ge 0\},\\ D_2 &= \{\zeta_1(t) - \zeta_2(t) = k \text{ for infinitely many } t\},\\ D_3 &= \{\zeta_1(t) = \zeta_2(t) \text{ for infinitely many } t\}.\end{aligned}$$

5.2.2. *Upper boundary: below the bisector and above the critical curve.* Define

$$W_{\gamma,C} = \{\zeta = (x,y) \in \mathbb{Z}_+^2 : x^\gamma - C < y \le x^\gamma\} \quad \text{for} \quad C > 0.$$

Lemma 5.25. *Let $1 < \beta_1 < \beta_2$ and $\gamma \in (\alpha_{cr}, 1)$, and let $\zeta = (x,y) \in W_{\gamma,C}$ for some $C > 0$. Then, $\lim_{x\to\infty} \mathbb{P}_\zeta(B_\gamma) = 1$, where the event B_γ is defined in (2.13).*

Proof of Lemma 5.25: Without loss of generality, assume that the initial state $\zeta = (x,y) \in W_{\gamma,1}$, in which case $y = x^\gamma - 1 + \varepsilon_0$ for some $0 < \varepsilon_0 < 1$. Given $0 < \varepsilon < 1$, define the sequence of stopping times

$$\begin{aligned} t_0 &= \min(t : \zeta_1^\gamma(t) \ge x^\gamma + \varepsilon_0),\\ t_n &= \min(t : \zeta_1^\gamma(t) \ge x^\gamma + \varepsilon + n) \text{ for } n \ge 1,\\ t_n' &= \min(\zeta_1^\gamma(t) \ge x^\gamma + 2\varepsilon + n) \text{ for } n \ge 1.\end{aligned}$$

Note that $0 \le t_0 < t_1' < t_1 < ... < t_n' < t_n < t_{n+1}' < t_{n+1} < ....$ It follows from equation (2.9) and the choice of the initial condition, that the DTMC $\zeta(t)$ is reflected by the upper boundary all the time in the interval $t \in [0, t_0)$. Also, the process can jump up only *once* in the interval $[t_0, t_1')$. We show that, with $\mathbb{P}_\zeta$ probability tending to 1, as $x \to \infty$, the process behaves in the same manner during the subsequent time intervals. Namely, it jumps up on each time interval $[t_n, t_{n+1}')$, and, as a result, it is reflected by the upper boundary all the time on each time interval $t \in [t_n', t_n)$. Clearly, such a behavior implies that the process stays inside the strip $W_{\gamma,1+\varepsilon}$ for all $t \ge t_1'$. It is convenient to introduce random variables S_n, $n \ge 0$, where S_n is the number of jumps up of the DTMC $\zeta(t)$ in the interval $[t_n, t_{n+1}')$. In terms of these random variables it suffices to show that

$$\lim_{\zeta\to\infty} \mathbb{P}_\zeta(S_n = 1 \text{ for all } n \ge 0) = 1, \tag{5.31}$$

to guarantee the described above behavior of the process.

To proceed, observe first that, under our assumptions on the initial condition, if $S_0 = 1, ..., S_{n-1} = 1$, then

$$N_n := t_{n+1}' - t_n \sim O(1)\,(x^\gamma + n)^{(1-\gamma)/\gamma}, \tag{5.32}$$

$$\zeta_{1,n} := \zeta_1(t_n) \sim t_n + x \sim O(1)\,(x^\gamma + n)^{1/\gamma}, \tag{5.33}$$

as $x, n \to \infty$.

Next, consider two cases.

Case 1): $\gamma \in (\alpha_{cr}, \frac{\beta_1}{\beta_2})$.. In this case $\gamma\beta_2 - \beta_1 < 0$, so that

$$\mathbb{P}_\zeta(\zeta_1(1) = x+1,\, \zeta_2(1) = y) \sim 1 - x^{\gamma\beta_2-\beta_1} \text{ for } \zeta = (x,y) : y \sim x^\gamma, \text{ as } x \to \infty.$$

The above gives that

$$\mathbb{P}_\zeta(S_n = 0 | S_0 = 1, ..., S_{n-1} = 1) \sim \left(1 - \zeta_{1,n}^{\gamma\beta_2-\beta_1}\right)^{N_n} \sim e^{-\zeta_{1,n}^{\gamma\beta_2-\beta_1} N_n}, \tag{5.34}$$

for $n \ge 0$.

By (5.32), (5.33)and (5.34),

$$\mathbb{P}_{\zeta}(S_n = 1 | S_0 = 1, ..., S_{n-1} = 1) \sim 1 - e^{-a(x^{\gamma}+\varepsilon+n)^b}, \tag{5.35}$$

where $a = 2\varepsilon/\gamma$ and $b = \frac{1-\gamma+\gamma\beta_2-\beta_1}{\gamma} > 0$, and, hence,

$$\lim_{\zeta\to\infty} \mathbb{P}_{\zeta}\left(S_n = 1 \text{ for all } n \geq 0\right) = \lim_{x\to\infty} \prod_{n=0}^{\infty} \left(1 - e^{-a(x^{\gamma}+\varepsilon+n)^b}\right) = 1,$$

as required.
Case 2): $\gamma \in [\frac{\beta_1}{\beta_2}, 1)$.. In this case $\beta_1 - \gamma\beta_2 < 0$ and, hence, if $\zeta = (x, y) : y \sim x^{\gamma}$, then

$$\mathbb{P}_{\zeta}(\zeta_1(1) = x + 1, \zeta_2(1) = y) \sim \begin{cases} x^{\beta_1-\gamma\beta_2}, & \text{if } \frac{\beta_1}{\beta_2} < \gamma < 1, \\ \frac{1}{2}, & \text{if } \gamma = \frac{\beta_1}{\beta_2}, \end{cases}$$

as $\zeta \to \infty$. Therefore,

$$\mathbb{P}_{\zeta}(S_n = 0 | S_0 = 1, ..., S_{n-1} = 1) \sim \zeta_{1,n}^{\frac{\beta_1-\gamma\beta_2}{\gamma} N_n} = e^{\frac{\varepsilon(\beta_1-\gamma\beta_2)}{\gamma} \log(\zeta_{1,n})\zeta_{1,n}^{1-\gamma}}, \quad \text{if} \quad \frac{\beta_1}{\beta_2} < \gamma < 1, \tag{5.36}$$

and

$$\mathbb{P}_{\zeta}(S_n = 0 | S_0 = 1, ..., S_{n-1} = 1) \sim \left(\frac{1}{2}\right)^{\zeta_{1,n}^{1-\gamma}}, \quad \text{if} \quad \gamma = \frac{\beta_1}{\beta_2}. \tag{5.37}$$

Consequently, $\lim_{\zeta\to\infty} \mathbb{P}_{\zeta}\left(S_n = 1 \text{ for all } n \geq 0\right) = 1$ in both $\frac{\beta_1}{\beta_2} < \gamma < 1$ and $\gamma = \frac{\beta_1}{\beta_2}$ cases, as required. The lemma is proved. □

5.3. *Proofs of Theorems 2.9 and 2.10.*

Proof of Theorem 2.9: Recall quantities α_k, $k \geq 0$ defined in (5.1). It follows from (5.2), that for a given $\alpha \in (0, \alpha_{cr})$ there exists $k \geq 1$, such that $\alpha \in [\alpha_{k-1}, \alpha_k)$. It is easy to verify that $\alpha \in [\alpha_{k-1}, \alpha_k)$ is equivalent to (2.15). Similarly, given $\gamma > 1$, one can determine a unique k, such that $\gamma \in (\max(1, \gamma_k), \gamma_{k-1}]$, or, equivalently, k satisfying (2.16). It follows from Theorem 2.2 (for the BB model without reflection) that, with probability one, the DTMC $\zeta(t)$ hits either the lower, or the upper boundary. If follows then from Lemmas 5.5, 5.11 and 5.23, that, with probability one, either the event A_{α,k_1}, or the event $\widetilde{A}_{\gamma,k_2}$ occurs. It is left to note that these events are disjoint, which finishes the proof. □

Proof of Theorem 2.10: The proof of Theorem 2.10 is similar to the proof of Theorem 2.9. It only needs to be complemented by using Lemma 5.24, Remark 5.17 and Lemma 5.25, where appropriate. For example, Lemma 5.25 is needed to deal with the case of the upper boundary given by $y = x^{\gamma}$ for $\alpha_{cr} < \gamma \leq 1$. We skip further details. □

6. Open problem

In this paper we have given a complete classification of the long term behavior of the BB model with reflection in the domain $Q_{\varphi,\psi}$ (defined in (2.6)), where the lower and the upper boundaries are given by $\varphi(x) = x^{\alpha}$ and $\psi(x) = x^{\gamma}$ respectively for some $0 < \alpha < \alpha_{cr} < \gamma$. Specifically, we have shown that, for any initial state $\zeta \in Q_{\varphi,\psi}$, with probability one, the DTMC $\zeta(t)$ eventually (i.e. after a finite time depending on the initial condition) confines either to the strip U_{α,k_1} (along the lower boundary), or to the strip $\widetilde{U}_{\gamma,k_2}$ (along the upper boundary), where integers k_1 and k_2 (interpreted as widths of the strips) can be calculated for a given set of the model parameters. We would like to stress that widths of both strips are *finite*.

We believe in a rather different behavior of the process in the case, when $\varphi(x) = \nu_{cr}x^{\alpha_{cr}} - b_1x^{\delta_1}$ and $\psi(x) = \nu_{cr}x^{\alpha_{cr}} + b_2x^{\delta_2}$ for some $b_1, b_2 > 0$ and $0 < \delta_1, \delta_2 < \alpha_{cr}$. We conjecture that if both

$0 < \delta_1 \leq \alpha_{cr}/2$ and $0 < \delta_1 \leq \alpha_{cr}/2$, then, with probability one, the process will be reflected infinitely many times by both the lower and the upper boundary, i.e. the process "visits" *both* boundaries infinitely many times. On the other hand, if $\alpha_{cr}/2 < \delta_1 \leq \alpha_{cr}$ and $\alpha_{cr}/2 < \delta_2 \leq \alpha_{cr}$, then, with probability one, the process eventually sticks to one of the boundaries. At the same time, deviations of the process from the "final" boundary are not uniformly bounded, so that the process eventually confines to an indefinitely expanding strip along one of the boundaries. The open problem of interest is to determine the asymptotic width of the strip.

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