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Monotonicity result for the simple exclusion process on finite connected graphs

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Abstract. The simple exclusion process studied in this paper consists of a system of K particles moving on the vertex set of a finite undirected connected graph. If there is one, the particle at x chooses one of the $\deg(x)$ neighbors of its current location uniformly at random at rate $\rho_x > 0$, and jumps to that vertex if and only if it is empty. Though this model is a natural mathematical object, it is also motivated by applications in the control of robotic swarms. After expressing the stationary distribution and the limiting occupation times of the process, we study how the total number of particles affects the occupation times ratios at different vertices. Using a novel qualitative argument based on combinatorial techniques, we prove that, while the occupation time at x increases with both $D(x) = \deg(x)/\rho_x$ and the total number of particles, the limiting occupation time increases faster with the total number of particles at vertices with a low $D(x)$.

1. Introduction

The simple exclusion process introduced by Spitzer (1970) is one of the most popular interacting particle systems with the voter model (Clifford and Sudbury (1973); Holley and Liggett (1975)) and the contact process (Harris (1974)). These three models can be viewed as the fundamental spatial stochastic models of diffusion, competition, and invasion, respectively. More precisely, the simple exclusion process consists of a system of random walks that move independently on a connected graph except that jumps onto already occupied vertices are suppressed (exclusion rule) so that each vertex is occupied by at most one particle.

All three models have been extensively studied on the infinite integer lattices. The literature on this topic is copious and we refer the reader to Kipnis and Landim (1999) and Liggett (1999, 2005) for reviews of these three models and references therein for more details. These models have also been studied on other graphs. Focusing on the simple exclusion process, the mixing time

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of the process on the d-dimensional torus has been explored by Morris (2006). Another example is the paper of Gantert and Schmid (2020) who studied the simple exclusion process on Galton-Watson trees. Similarly, various aspects of the simple exclusion process such as hydrodynamic limits, rates of convergence to the stationary distribution, and fluctuations around the equilibrium have been studied on finite graphs. The first instance chronologically is probably the asymmetric simple exclusion processes (ASEP) on the finite segment with fixed densities at the two boundaries which was introduced by Liggett (1975) and is used to understand the current and the motion of shocks of its infinite counterpart on the one-dimensional lattice. A combinatorial formula for the stationary distribution of the ASEP was provided by Corteel and Williams (2011). The literature on this topic is also copious and we refer the reader to the works of Liggett (1999, Section III.3) and Derrida and Evans (1997) for reviews, and to Bahadoran (2007); Derrida et al. (2004, 1993, 2002, 2007); Gonçalves (2019) for more recent results. Other works are concerned with the simple exclusion process on the torus in one (Franco and Landim (2010); Gonçalves and Jara (2019)) and higher (Gonçalves et al. (2009)) dimensions in which case the main objective is to investigate the hydrodynamic limit and the fluctuations around the limit, and the complete graph (Forsström and Jonasson (2017); Lacoin and Leblond (2011); Mendonça (2013)) where the focus is on studying the spectrum and the rate of convergence to the stationary distribution. Few works have considered the simple exclusion process on general finite connected graphs. One example is the seminal work of Caputo et al. (2010) establishing Aldous' spectral gap conjecture about the rate of convergence of the interchange model (and by projection the simple exclusion process) on general finite weighted connected graphs. Hermon and Pymar (2020); Lacoin (2016); Oliveira (2013); Salez (2022) also studied the mixing times for the process on finite graphs. Our work is also concerned with the simple exclusion process on general finite weighted connected graphs but the weights take the form of jump rates on the vertices rather than exchange rates along the edges like in the exchange model, and our main focus/objective is quite different from Caputo et al. (2010).

More precisely, assume that each of the vertices of a finite undirected connected graph is either empty or occupied by exactly one particle. There are two natural versions of the symmetric exclusion process: the constant speed model in which vertices become active at rate one, which causes a particle at that vertex to choose one of its nearest neighbors at random and to jump to that neighbor if it is empty, and the variable speed model in which edges become active at rate one, which results in an exchange of the states at the extremities of the edge. Note that the two models are mathematically equivalent on regular graphs. Our model assumes more generally that a particle at vertex x becomes active at rate $\rho_x > 0$, chooses one of its $\deg(x)$ nearest neighbors uniformly at random, and jumps to that neighbor if it is empty. The constant speed model is the particular case $\rho_x \equiv 1$ while the variable speed model is the particular case $\rho_x = \deg(x)$. The main objective is to study the limiting occupation times and how the total number of particles in the system affects the occupation times ratios at different vertices.

Although this stochastic process is a natural mathematical object, our motivation also comes from applications in the control of robotic swarms, and more particularly, controlling the distribution of robots at equilibrium. The algorithm presented here is particularly suited to formation control problem in multi-agent systems theory. The goal of this problem is to have the agents settle to a target configuration. Assuming that each robot evolves according to a Markov process, the distribution of the entire swarm over the domain can be determined from the Kolmogorov forward equation. Moreover, a specific target distribution can be made invariant by using the transition probabilities or transition rates as the control parameters (Açıkmeşe and Bayard (2015); Berman et al. (2009); Biswal et al. (2022); Elamvazhuthi et al. (2017); Lerman et al. (2005); Mather and Hsieh (2014)). A common shortcoming of these models, however, is that robots can occupy the same vertex so avoidance of agent collisions is not accounted for. The inclusion of collision avoidance would also make the forward equation nonlinear. In contrast, the simple exclusion process accounts for the physical constraint that robots must avoid collisions with one another. In addition, by including vertex-dependent jump rates as control parameters, we can address the problem of redistributing a swarm of robots over a finite state space toward a desired target distribution. For example, it was proved by Açıkmeşe and Bayard (2015) and Elamvazhuthi et al. (2017) that by designing state-dependent jump rates, the density of a population of robots evolving according to a continuous-time Markov chain converges asymptotically to a desired distribution. Similar results have been proved in the setting of discrete-time Markov chains evolving on compact subsets of $\mathbb{R}^n$ by Biswal et al. (2022, 2021). In this latter case, the transition kernel is designed to be state-dependent.

2. Main results

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a finite undirected connected graph on N vertices. The set $\mathscr{V}$ refers to the vertex set of the graph and the set $\mathscr{E}$ refers to the edge set, and we write $x \sim y$ to indicate that the two vertices are connected by an edge. The process considered in this paper is a continuous-time Markov chain whose state at time t is a spatial configuration

$$\eta_t : \mathscr{V} \to \{0, 1\} \quad \text{where} \quad \eta_t(x) = \begin{cases} 0 & \text{if vertex } x \text{ is empty} \\ 1 & \text{if vertex } x \text{ is occupied by a particle.} \end{cases}$$

It will be convenient later to identify η with the subset of occupied vertices:

$$\eta \equiv \{x \in \mathscr{V} : \eta(x) = 1\} \subset \mathscr{V}.$$

This defines a natural bijection between the set of configurations and the subsets of the vertex set, and it will be obvious from the context whether η refers to a configuration or a subset. Motivated by applications in robotic swarms, we assume that particles may jump at a rate that depends on their location, and denote by $\rho_x > 0$ the rate attached to vertex x. To describe the dynamics, for each pair of vertices x, y, we define the configuration $\tau_{x,y}\,\eta$ as

$$(\tau_{x,y}\,\eta)(z) = \eta(x)\,\mathbf{1}\{z = y\} + \eta(y)\,\mathbf{1}\{z = x\} + \eta(z)\,\mathbf{1}\{z \notin \{x, y\}\}$$

obtained from η by exchanging the states at x and y. Then, for all $\eta, \xi \in \{0, 1\}^{\mathscr{V}}$, the process jumps from configuration η to configuration ξ at rate

$$q(\eta, \xi) = \frac{\rho_x}{\deg(x)}\,\mathbf{1}\{\eta(x) > \eta(y) \text{ and } \xi = \tau_{x,y}\,\eta \text{ for some } y \sim x\}. \tag{2.1}$$

In words, whenever there is a particle at vertex x, this particle chooses a neighbor uniformly at random at rate ρ_x, and jumps to this vertex if and only if it is empty.

The main objective of this paper is to study the so-called occupation times of the process, i.e., the fraction of time each vertex is occupied by a particle. We can prove that, in the long run, the occupation times converge almost surely to limits that only depend on the initial configuration through its number of particles, so we will write from now on

$$\mathscr{O}_K(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \eta_s(x)\,ds \quad \text{for all} \quad 0 < K \leq N \text{ and } x \in \mathscr{V} \tag{2.2}$$

where K refers to the (initial) number of particles. To state our results, we let Λ_K be the collection of all the possible subsets of $\mathscr{V}$ of size K and, for all $B \subset \mathscr{V}$, define

$$\begin{aligned} \Lambda_K^+(B) &= \{\eta \in \Lambda_K : B \cap \eta = B\} = \text{the subsets of size } K \text{ that contain } B, \\ \Lambda_K^-(B) &= \{\eta \in \Lambda_K : B \cap \eta = \varnothing\} = \text{the subsets of size } K \text{ that exclude } B. \end{aligned}$$

In addition, for each $z \in \mathscr{V}$, each $\eta \subset \mathscr{V}$, and each collection $\mathscr{C}$ of subsets of $\mathscr{V}$, we let

$$D(z) = \frac{\deg(z)}{\rho_z}, \quad D(\eta) = \prod_{z \in \eta} D(z) \quad \text{and} \quad \Sigma(\mathscr{C}) = \sum_{\eta \in \mathscr{C}} D(\eta). \tag{2.3}$$

Using irreducibility and reversibility, standard techniques reviewed for instance by Liggett (2005, Section VIII) imply that the process has a unique stationary distribution π_K given by

$$\pi_K(\eta) = \frac{D(\eta)}{\Sigma(\Lambda_K)} \quad \text{for all} \quad \eta \subset \Lambda_K.$$

In particular, by the ergodic theorem for Markov chains, (2.2) becomes

$$\mathscr{O}_K(x) = \sum_{\eta:\eta(x)=1} \pi_K(\eta) = \frac{\Sigma(\Lambda_K^+(x))}{\Sigma(\Lambda_K)} \quad \text{for all} \quad 0 < K \leq N \text{ and } x \in \mathscr{V}. \tag{2.4}$$

It follows that the ratio of the occupation times is characterized by

$$\frac{\mathscr{O}_K(x)}{\mathscr{O}_K(y)} = \frac{\Sigma(\Lambda_K^+(x))}{\Sigma(\Lambda_K^+(y))} \quad \text{and} \quad \sum_{z \in \mathscr{V}} \mathscr{O}_K(z) = K. \tag{2.5}$$

It is intuitively clear that

- the graph $\mathscr{G}$ and the rates $\rho_x > 0$ being fixed, $\mathscr{O}_K(x)$ increases with K,
- the graph $\mathscr{G}$ and the number of particles K being fixed, $\mathscr{O}_K(x)$ decreases with ρ_x.

The first statement follows from a standard coupling argument to compare simple exclusion processes with the same jump rates but a different number of particles while the second statement follows from the expression of the occupation time in (2.4) using also (2.3). In contrast, how the ratio in the left-hand side of (2.5) behaves with the total number of particles in the system is unclear, and our main result shows that, when $D(x) < D(y)$, the occupation time at x is always smaller but increases faster than the occupation time at y as the number of particles increases:

Theorem 2.1 (monotonicity)**.** *For all $K = 2, 3, \ldots, N-2$,*

$$\frac{D(x)}{D(y)} = \frac{\mathscr{O}_1(x)}{\mathscr{O}_1(y)} < \frac{\mathscr{O}_K(x)}{\mathscr{O}_K(y)} < \frac{\mathscr{O}_{K+1}(x)}{\mathscr{O}_{K+1}(y)} < \frac{\mathscr{O}_N(x)}{\mathscr{O}_N(y)} = 1 \quad \textit{when} \quad D(x) < D(y).$$

Because, for general graphs, the occupation times $\mathscr{O}_K(x)$ do not have a simple closed form, the monotonicity result cannot be proved from a calculation. Indeed, the expression of the occupation times become quite complicated when the vertices have different $D(x)$. In particular, our proof relies on a novel qualitative argument involving combinatorial techniques.

To conclude, we give examples where computing the limiting occupation times is simplified by the fact that $\rho_x \equiv 1$ and most of the vertices have the same degree. To begin with, we look at the star in which all the vertices have degree one except for the center. In this case, the limiting occupation times are given by the following expressions.

Example 2.2 (star). For the star graph with N vertices and center 0,

$$\begin{aligned} \mathscr{O}_K(0) &= \frac{(N-1)K}{(N-1)K + (N-K)} \\ \mathscr{O}_K(x) &= \left(\frac{K}{N-1}\right) \frac{(N-1)K - (K-1)}{(N-1)K + (N-K)} \quad \text{for} \quad x \neq 0. \end{aligned}$$

Note that taking the ratio in the example gives

$$\frac{\mathscr{O}_K(x)}{\mathscr{O}_K(0)} = \frac{(N-1)(K-1) + (N-K)}{(N-1)^2} \quad \text{for} \quad x \neq 0.$$

The right-hand side $= 1/(N-1)$ when $K = 1$ and one when $K = N$, and is increasing with respect to the number of particles, in accordance with Theorem 2.1. Next, we look at the path in which all the vertices have degree two except for the two endpoints that have degree one. The simple exclusion process on this graph is usually defined by slowing down the dynamics at the two endpoints by a

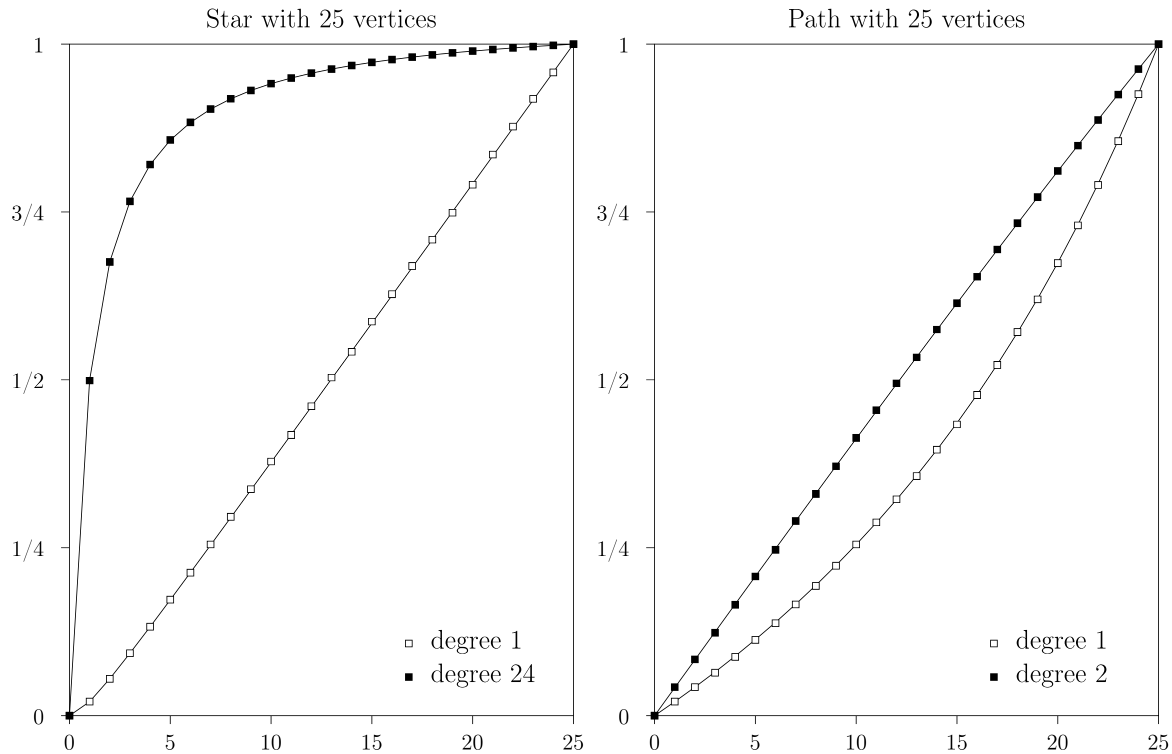

Figure 2.1. Limiting occupation times on the star and the path with 25 vertices and $\rho_x \equiv 1$. The horizontal axis represents the number of particles K, and the vertical axis the fraction of time $\mathcal{O}_K(x)$ vertices with degree d are occupied. The squares are obtained from simulating the process for 10^8 units of time, while the curves show the analytical results from Examples 2.2 and 2.3.

factor one-half, which results in the uniform distribution being stationary. However, when $\rho_x \equiv 1$, the algebra becomes more complicated than for the star and we have

Example 2.3 (path). For the path graph with vertex set $\mathscr{V} = \{0, 1, \dots, N-1\}$,

$$\mathcal{O}_K(x) = \frac{(2N-K-1)K}{(K-1)K + 4(N-K)(N-1)} \quad \text{for} \quad x = 0, N-1,$$

$$\mathcal{O}_K(x) = \left(\frac{K}{N-2}\right) \frac{(K-2)(K-1) + 4(N-K)(N-2)}{(K-1)K + 4(N-K)(N-1)} \quad \text{for} \quad 0 < x < N-1.$$

Taking the ratio in the example now gives

$$\frac{\mathcal{O}_K(0)}{\mathcal{O}_K(x)} = \frac{(N-2)(2N-K-1)}{(K-2)(K-1) + 4(N-K)(N-2)} \quad \text{for} \quad 0 < x < N-1.$$

The right-hand side is equal to one-half when $K = 1$ and one when $K = N$, and is increasing with respect to the number of particles, again in accordance with Theorem 2.1. Figure 2.1 shows the limiting occupation times obtained from simulations of the process run for 10^8 units of time on the star and the path with 25 vertices, along with our analytical results.

3. Proof of Theorem 2.1

According to (2.4), we have

$$\frac{\mathscr{O}_1(x)}{\mathscr{O}_1(y)} = \frac{\Sigma(\Lambda_1^+(x))}{\Sigma(\Lambda_1^+(y))} = \frac{D(x)}{D(y)} \quad \text{and} \quad \frac{\mathscr{O}_N(x)}{\mathscr{O}_N(y)} = \frac{\Sigma(\Lambda_N^+(x))}{\Sigma(\Lambda_N^+(y))} = \frac{D(\mathscr{V})}{D(\mathscr{V})} = 1. \tag{3.1}$$

In all the other cases, however, the ratios above become much more complicated. In particular, the proof of Theorem 2.1 relies on a qualitative approach, and the main ingredient is Lemma 3.1 below whose proof relies on an elaborate construction. To understand the intuition behind this result, note that the inequality in the lemma is reminiscent of the following basic result: the rectangle with the largest area and fixed perimeter $4K$ is the square with length side K. In particular,

$$(K+1)(K-1) < K^2.$$

Although this is more difficult to visualize, the same holds looking at the number of possible choices of two subsets with total size equal to $2K$. This implies that, in the special case of the variable speed simple exclusion process, in which case $\rho_x = \deg(x)$ and so $D(z) \equiv 1$, we have

$$\Sigma(\Lambda_{K+1})\,\Sigma(\Lambda_{K-1}) = \binom{N}{K+1}\binom{N}{K-1} < \binom{N}{K}^2 = (\Sigma(\Lambda_K))^2.$$

Lemma 3.1 shows that the result still holds when the stationary distribution is not uniform in the general case where the jump rates $\rho_x > 0$ are arbitrary.

Lemma 3.1. *For all $K = 1, 2, \ldots, N-1$, we have $\Sigma(\Lambda_{K+1})\,\Sigma(\Lambda_{K-1}) < (\Sigma(\Lambda_K))^2$.*

Proof: We refer to Figure 3.2 for an illustration of the construction introduced in this proof. The key is to find partitions $\mathscr{P}$ of $\Lambda_{K+1} \times \Lambda_{K-1}$ and $\mathscr{Q}$ of $\Lambda_K \times \Lambda_K$ such that

(1) partition $\mathscr{P}$ has less elements than partition $\mathscr{Q}$,

(2) each $A \in \mathscr{P}$ can be paired with a $B \in \mathscr{Q}$ such that $\operatorname{card}(A) < \operatorname{card}(B)$,

(3) for all $(\eta', \eta'') \in A_i$ and $(\xi', \xi'') \in B_i$, we have $D(\eta')\,D(\eta'') = D(\xi')\,D(\xi'')$.

Let $x_1, x_2, \ldots, x_N$ denote the N vertices. To construct the partitions, let

$$S_{2K} = \{(u_1, u_2, \ldots, u_N) \in \{0,1,2\}^N : u_1 + \cdots + u_N = 2K\}.$$

Then, define $\phi : \Lambda_{K+1} \times \Lambda_{K-1} \to S_{2K}$ and $\psi : \Lambda_K \times \Lambda_K \to S_{2K}$ as

$$\begin{aligned} \phi(\eta', \eta'') &= u = (u_1, u_2, \ldots, u_N) \quad \text{where} \quad u_i = \mathbf{1}\{x_i \in \eta'\} + \mathbf{1}\{x_i \in \eta''\} \\ \psi(\xi', \xi'') &= u = (u_1, u_2, \ldots, u_N) \quad \text{where} \quad u_i = \mathbf{1}\{x_i \in \xi'\} + \mathbf{1}\{x_i \in \xi''\}. \end{aligned} \tag{3.2}$$

The two functions have the same expression but note that they differ in that they are not defined on the same sets of configurations. The two partitions are defined by setting

$$\begin{aligned} \mathscr{P} &= \{\phi^{-1}(u) : u \in S_{2K} \text{ and } \phi(\eta', \eta'') = u \text{ for some } (\eta', \eta'') \in \Lambda_{K+1} \times \Lambda_{K-1}\} \\ \mathscr{Q} &= \{\psi^{-1}(u) : u \in S_{2K} \text{ and } \psi(\xi', \xi'') = u \text{ for some } (\xi', \xi'') \in \Lambda_K \times \Lambda_K\}. \end{aligned}$$

Now that the two partitions are defined, we can prove the three items.

Proof of (1). The function ϕ is not surjective because $\phi(\eta', \eta'')$ always has $\operatorname{card}(\eta' \setminus \eta'') \geq 2$ coordinates equal to one. In particular, for all $u \in S_{2K}$ with K coordinates equal to two,

$$\phi(\eta', \eta'') \neq u \quad \text{for all} \quad (\eta', \eta'') \in \Lambda_{K+1} \times \Lambda_{K-1}.$$

In contrast, the function ψ is surjective. Indeed, fix $u \in S_{2K}$ and let

$$\mathscr{C}_j = \{i : u_i = j\} \quad \text{and} \quad K_j = \operatorname{card}(\mathscr{C}_j) \quad \text{for} \quad j = 1, 2. \tag{3.3}$$

Then $K_1 + 2K_2 = 2K$ therefore K_1 is even. In particular, the set $\mathscr{C}_1$ can be partitioned into two subsets of equal size, say $\mathscr{C}_{11}$ and $\mathscr{C}_{12}$. Then, defining ξ' and ξ'' as

$$\xi' = \{x_i : i \in \mathscr{C}_{11} \cup \mathscr{C}_2\} \quad \text{and} \quad \xi'' = \{x_i : i \in \mathscr{C}_{12} \cup \mathscr{C}_2\},$$

we have $\psi(\xi', \xi'') = u$ so ψ is surjective. In conclusion,

$$\begin{aligned} \text{card}(\mathscr{P}) &= \text{card}\{u \in S_{2K} : \phi(\eta', \eta'') = u \text{ for some } (\eta', \eta'') \in \Lambda_{K+1} \times \Lambda_{K-1}\} \\ &< \text{card}(S_{2K}) \\ &= \text{card}\{u \in S_{2K} : \psi(\xi', \xi'') = u \text{ for some } (\xi', \xi'') \in \Lambda_K \times \Lambda_K\} \\ &= \text{card}(\mathscr{Q}), \end{aligned} \tag{3.4}$$

which proves the first item (1).

Proof of (2). Let $u \in S_{2K}$ and K_1, K_2 as in (3.3), with $K_1 \geq 2$, i.e.,

$$K_1 = \text{card}\{i : u_i = 1\} \quad \text{and} \quad K_2 = \text{card}\{i : u_i = 2\}.$$

To count the number of preimages (η', η'') and (ξ', ξ'') of the vector u, note that the vertices that are either empty in all four configurations or occupied in all four configurations are fixed by the vector u. This leaves K_1 vertices that are occupied in two of the configurations and

- the number of choices for (η', η'') is the number of choices of $K_1/2 + 1 = K + 1 - K_2$ vertices among K_1 vertices to be occupied in η' but not η'',
- the number of choices for (ξ', ξ'') is the number of choices of $K_1/2 = K - K_2$ vertices among K_1 vertices to be occupied in ξ' but not ξ''.

Recalling that $K_1 + 2K_2 = 2K$, we get

$$\begin{aligned} \text{card}(\phi^{-1}(u)) &= \binom{K_1}{K + 1 - K_2} = \frac{K_1 - K + K_2}{K + 1 - K_2} \binom{K_1}{K - K_2} \\ &= \frac{K_1/2}{K_1/2 + 1} \binom{K_1}{K - K_2} < \binom{K_1}{K - K_2} = \text{card}(\psi^{-1}(u)), \end{aligned} \tag{3.5}$$

which shows the second item (2).

Proof of (3). Note that, for all $(\eta', \eta'') \in \phi^{-1}(u)$,

$$\begin{aligned} D(\eta')\, D(\eta'') &= D(\eta' \,\Delta\, \eta'')(D(\eta' \cap \eta''))^2 \\ &= \Big(\prod_{z \in \eta' \Delta \eta''} D(z)\Big)\Big(\prod_{z \in \eta' \cap \eta''} D(z)\Big)^2 = \prod_{i=1}^{N} (D(x_i))^{u_i} \end{aligned}$$

is a function $\widehat{D}(u)$ of the vector u that does not depend on the particular choice of the pair of configurations. In the previous expression, Δ refers to the symmetric difference:

$$\eta' \,\Delta\, \eta'' = (\eta' \cup \eta'') \setminus (\eta' \cap \eta'') \quad \text{for all} \quad \eta', \eta'' \subset \mathscr{V}.$$

The same holds for $(\xi', \xi'') \in \psi^{-1}(u)$. This implies that

$$D(\eta')\, D(\eta'') = D(\xi')\, D(\xi'') = \widehat{D}(u) \quad \text{for all} \quad (\eta', \eta'') \in \phi^{-1}(u), (\xi', \xi'') \in \psi^{-1}(u), \tag{3.6}$$

which proves the third item (3).

To deduce the lemma from the three items, recall that the function $\psi : \Lambda_K \times \Lambda_K \to S_{2K}$ is surjective. The function $\phi : \Lambda_{K+1} \times \Lambda_{K-1} \to S_{2K}$ is not surjective but letting

$$S_{2K}^* = \{u \in S_{2K} : \phi(\eta', \eta'') = u \text{ for some } (\eta', \eta'') \in \Lambda_{K+1} \times \Lambda_{K-1}\},$$

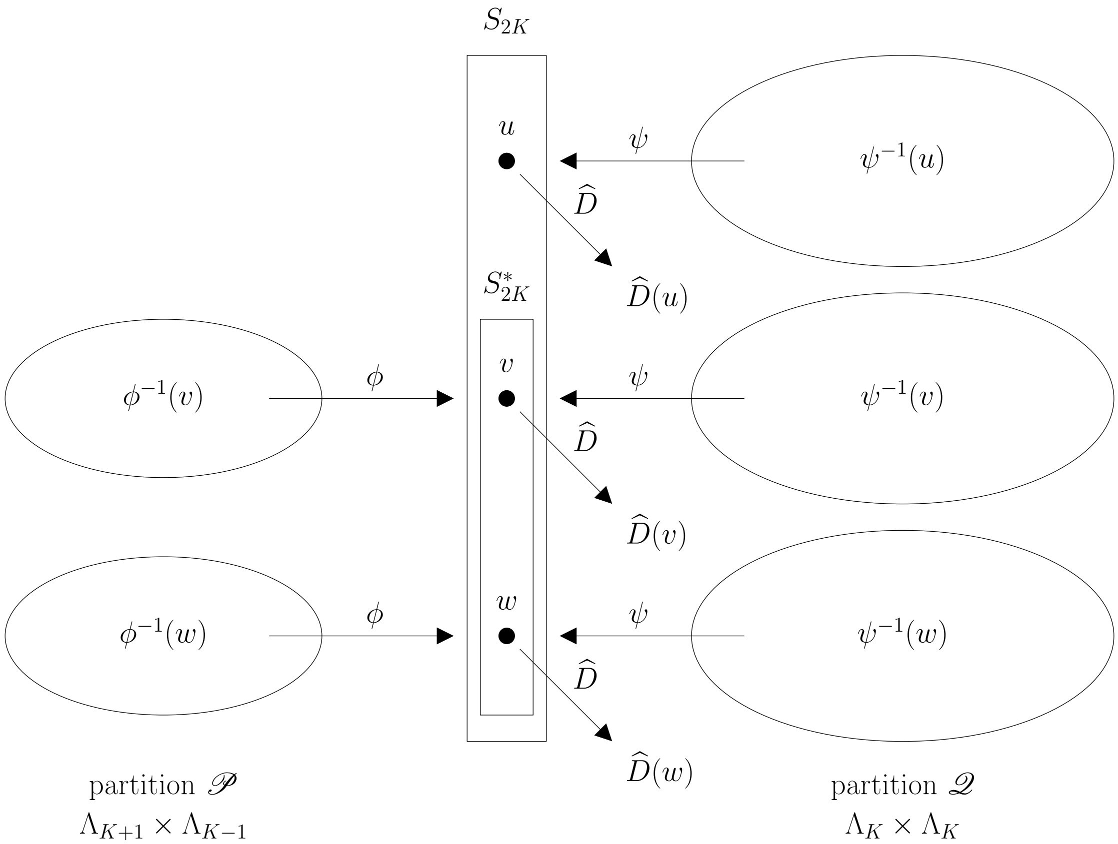

FIGURE 3.2. Construction in the proof of Lemma 3.1.

the function $\phi : \Lambda_{K+1} \times \Lambda_{K-1} \to S^*_{2K}$ becomes surjective. In particular,

$$\begin{aligned} \Sigma(\Lambda_K)\,\Sigma(\Lambda_K) &= \sum_{u \in S_{2K}} \ \sum_{(\xi', \xi'') \in \psi^{-1}(u)} D(\xi')\,D(\xi''), \\ \Sigma(\Lambda_{K+1})\,\Sigma(\Lambda_{K-1}) &= \sum_{u \in S^*_{2K}} \ \sum_{(\eta', \eta'') \in \phi^{-1}(u)} D(\eta')\,D(\eta''). \end{aligned} \tag{3.7}$$

Combining (3.4)–(3.7), we conclude that

$$\begin{aligned} \Sigma(\Lambda_{K+1})\,\Sigma(\Lambda_{K-1}) &\overset{(3.7)}{=} \sum_{u \in S^*_{2K}} \ \sum_{(\eta', \eta'') \in \phi^{-1}(u)} D(\eta')\,D(\eta'') \overset{(3.6)}{=} \sum_{u \in S^*_{2K}} \ \sum_{(\eta', \eta'') \in \phi^{-1}(u)} \widehat{D}(u) \\ &\overset{(3.5)}{<} \sum_{u \in S^*_{2K}} \ \sum_{(\xi', \xi'') \in \psi^{-1}(u)} \widehat{D}(u) \overset{(3.6)}{=} \sum_{u \in S^*_{2K}} \ \sum_{(\xi', \xi'') \in \psi^{-1}(u)} D(\xi')\,D(\xi'') \\ &\overset{(3.4)}{<} \sum_{u \in S_{2K}} \ \sum_{(\xi', \xi'') \in \psi^{-1}(u)} D(\xi')\,D(\xi'') \overset{(3.7)}{=} (\Sigma(\Lambda_K))^2. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.1. To simplify the notations, we write

$$A_K = \Sigma(\Lambda^+_K(\{x, y\})) \quad \text{and} \quad B_K = \Sigma(\Lambda^-_{K-1}(\{x, y\})).$$

Then, we can rewrite

$$\begin{aligned}\Sigma(\Lambda_K^+(x)) &= \Sigma(\Lambda_K^+(x)\cap\Lambda_K^+(y)) + \Sigma(\Lambda_K^+(x)\setminus\Lambda_K^+(y))\\ &= \Sigma(\Lambda_K^+(\{x,y\})) + D(x)\,\Sigma(\Lambda_{K-1}^-(\{x,y\})) = A_K + D(x)\,B_K.\end{aligned}$$

Using some obvious symmetry, we deduce that

$$\frac{\mathscr{O}_K(x)}{\mathscr{O}_K(y)} = \frac{\Sigma(\Lambda_K^+(x))}{\Sigma(\Lambda_K^+(y))} = \frac{A_K + D(x)\,B_K}{A_K + D(y)\,B_K}. \tag{3.8}$$

Now, applying Lemma 3.1 to the configurations on $\mathscr{V}\setminus\{x,y\}$, we get

$$\begin{aligned}A_K B_{K+1} &= \Sigma(\Lambda_K^+(\{x,y\}))\,\Sigma(\Lambda_K^-(\{x,y\}))\\ &= D(\{x,y\})\,\Sigma(\Lambda_{K-2}^-(\{x,y\}))\,\Sigma(\Lambda_K^-(\{x,y\}))\\ &< D(\{x,y\})\,\Sigma(\Lambda_{K-1}^-(\{x,y\}))\,\Sigma(\Lambda_{K-1}^-(\{x,y\}))\\ &= \Sigma(\Lambda_{K+1}^+(\{x,y\}))\,\Sigma(\Lambda_{K-1}^-(\{x,y\})) = A_{K+1}B_K.\end{aligned}$$

This, together with $D(x) < D(y)$, implies that

$$D(x)(A_{K+1}B_K - A_K B_{K+1}) < D(y)(A_{K+1}B_K - A_K B_{K+1}).$$

Rearranging the terms, we deduce that

$$D(x)\,A_{K+1}B_K + D(y)\,A_K B_{K+1} < D(x)\,A_K B_{K+1} + D(y)\,A_{K+1}B_K$$

which is equivalent to

$$(A_K + D(x)\,B_K)(A_{K+1} + D(y)\,B_{K+1}) < (A_{K+1} + D(x)\,B_{K+1})(A_K + D(y)\,B_K). \tag{3.9}$$

Combining (3.8) and (3.9) gives

$$\frac{\mathscr{O}_K(x)}{\mathscr{O}_K(y)} = \frac{A_K + D(x)\,B_K}{A_K + D(y)\,B_K} < \frac{A_{K+1} + D(x)\,B_{K+1}}{A_{K+1} + D(y)\,B_{K+1}} = \frac{\mathscr{O}_{K+1}(x)}{\mathscr{O}_{K+1}(y)}. \tag{3.10}$$

The theorem follows from (3.1) and (3.10). □

4. Star and path graphs

We now use (2.4) to find the limiting occupation times for the constant speed ($\rho_x \equiv 1$) simple exclusion process on the star and the path graphs. In these cases, the occupation times have a simple closed form because most of the vertices have the same degree.

Proof of Example 2.2. The center 0 is connected to all the other $N-1$ vertices therefore its degree is $N-1$ while the other vertices $1,2,\ldots,N-1$, called leaves, are only connected to the center and therefore have degree one. For the center, we compute

$$\Sigma(\Lambda_K^+(0)) = \deg(0)\ \Sigma(\Lambda_{K-1}^-(0)) = (N-1)\binom{N-1}{K-1}, \tag{4.1}$$

while for all $x = 1,2,\ldots,N-1$, we have

$$\begin{aligned}\Sigma(\Lambda_K^+(x)) &= \deg(x)\ \Sigma(\Lambda_{K-1}^-(x)) = \Sigma(\Lambda_{K-1}^-(x))\\ &= \Sigma(\Lambda_{K-1}^-(x)\cap\Lambda_{K-1}^+(0)) + \Sigma(\Lambda_{K-1}^-(x)\cap\Lambda_{K-1}^-(0))\\ &= \deg(0)\ \Sigma(\Lambda_{K-2}^-(0,x)) + \Sigma(\Lambda_{K-1}^-(0,x))\end{aligned}$$

from which it follows that

$$\Sigma(\Lambda_K^+(x)) = (N-1)\binom{N-2}{K-2} + \binom{N-2}{K-1}. \tag{4.2}$$

Combining (4.1)–(4.2) and simplying, we deduce that

$$\mathcal{O}_K(0) = \frac{\Sigma(\Lambda_K^+(0))}{\Sigma(\Lambda_K)} = \frac{\Sigma(\Lambda_K^+(0))}{\Sigma(\Lambda_K^+(0)) + \Sigma(\Lambda_K^-(0))} = \frac{(N-1)K}{(N-1)K + (N-K)}$$

$$\mathcal{O}_K(x) = \frac{\Sigma(\Lambda_K^+(x))}{\Sigma(\Lambda_K)} = \frac{\Sigma(\Lambda_K^+(x))}{\Sigma(\Lambda_K^+(0)) + \Sigma(\Lambda_K^-(0))} = \left(\frac{K}{N-1}\right)\frac{(N-1)K - (K-1)}{(N-1)K + (N-K)}.$$

This completes the proof. □

Proof of Example 2.3. The end vertices 0 and $N-1$ each have degree one while the other vertices $1, 2, \ldots, N-2$, each have degree two. For the end nodes, we compute

$$\begin{aligned}\Sigma(\Lambda_K^+(0)) &= \deg(0)\ \Sigma(\Lambda_{K-1}^-(0)) = \Sigma(\Lambda_{K-1}^-(0)) \\ &= \Sigma(\Lambda_{K-1}^-(0) \cap \Lambda_{K-1}^+(N-1)) + \Sigma(\Lambda_{K-1}^-(0) \cap \Lambda_{K-1}^-(N-1)) \\ &= \deg(N-1)\ \Sigma(\Lambda_{K-2}^-(0, N-1)) + \Sigma(\Lambda_{K-1}^-(0, N-1))\end{aligned}$$

from which it follows that

$$\Sigma(\Lambda_K^+(0)) = \Sigma(\Lambda_K^+(N-1)) = 2^{K-2}\binom{N-2}{K-2} + 2^{K-1}\binom{N-2}{K-1}. \tag{4.3}$$

Similarly, for all $0 < x < N-1$,

$$\Sigma(\Lambda_K^+(x)) = \deg(x)\ \Sigma(\Lambda_{K-1}^-(x)) = 2\,\Sigma(\Lambda_{K-1}^-(x)).$$

Including and/or excluding 0 and/or $N-1$, and simplifying, we get

$$\begin{aligned}\Sigma(\Lambda_K^+(x)) &= 2^{K-2}\binom{2}{2}\binom{N-3}{K-3} + 2^{K-1}\binom{2}{1}\binom{N-3}{K-2} + 2^K\binom{2}{0}\binom{N-3}{K-1} \\ &= 2^{K-2}\binom{N-3}{K-3} + 2^K\binom{N-2}{K-1}.\end{aligned} \tag{4.4}$$

The total mass in this case is

$$\begin{aligned}\Sigma(\Lambda_K) &= 2^{K-2}\binom{2}{2}\binom{N-2}{K-2} + 2^{K-1}\binom{2}{1}\binom{N-2}{K-1} + 2^K\binom{2}{0}\binom{N-2}{K} \\ &= 2^{K-2}\binom{N-2}{K-2} + 2^K\binom{N-1}{K}.\end{aligned} \tag{4.5}$$

Combining (4.3)–(4.5), applying (2.4), and simplifying, we conclude that

$$\begin{aligned}\mathcal{O}_K(0) = \frac{\Sigma(\Lambda_K^+(0))}{\Sigma(\Lambda_K)} &= \frac{(2N-K-1)K}{(K-1)K + 4(N-K)(N-1)} \\ \mathcal{O}_K(x) = \frac{\Sigma(\Lambda_K^+(x))}{\Sigma(\Lambda_K)} &= \left(\frac{K}{N-2}\right)\frac{(K-2)(K-1) + 4(N-K)(N-2)}{(K-1)K + 4(N-K)(N-1)}.\end{aligned} \tag{4.6}$$

This completes the proof. □

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