ALEA, Lat. Am. J. Probab. Math. Stat. 20, 33–51 (2023)
DOI: 10.30757/ALEA.v20-03

Recurrence and transience of Rademacher series

Satyaki Bhattacharya and Stanislav Volkov

Centre for Mathematical Sciences, Lund University, Box 118 SE-22100, Lund, Sweden
E-mail address: satyaki.bhattacharya@matstat.lu.se

Centre for Mathematical Sciences, Lund University, Box 118 SE-22100, Lund, Sweden
E-mail address: stanislav.volkov@matstat.lu.se

Abstract. We introduce the notion of $\mathbf{a}$-walk $S(n) = a_1 X_1 + \cdots + a_n X_n$, based on a sequence of positive numbers $\mathbf{a} = (a_1, a_2, \dots)$ and a Rademacher sequence $X_1, X_2, \dots$. We study recurrence/transience (properly defined) of such walks for various sequences of $\mathbf{a}$. In particular, we establish the classification in the cases where $a_k = \lfloor k^\beta \rfloor$, $\beta > 0$, as well as in the case $a_k = \lceil \log_\gamma k \rceil$ or $a_k = \log_\gamma k$ for $\gamma > 1$.

1. Introduction

We will say that a random variable X has a Rademacher distribution and write $X \sim \mathcal{Rademacher}$, if $\mathbb{P}(X = +1) = \mathbb{P}(X = -1) = \frac{1}{2}$. Let $X_i \sim \mathcal{Rademacher}$, $i = 1, 2, \dots$, be i.i.d., and $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ be the sigma-algebra generated by the first n members of this sequence. Let $\mathbf{a} = (a_1, a_2, \dots)$ be a non-random sequence of positive numbers. Define the $\mathbf{a}$-walk as

$$S(n) = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n = \sum_{k=1}^{n} a_k X_k$$

with the convention $S(0) = 0$.

Definition 1. Let $C \geq 0$. We call the $\mathbf{a}$-walk S defined above $C-$*recurrent*, if the event $\{|S(n)| \leq C\}$ occurs for infinitely many n. (In case when $C = 0$, this is equivalent to the usual recurrence, i.e., $S(n) = 0$ for infinitely many n, so we will call the walk just *recurrent*.)

We call the $\mathbf{a}$-walk *transient*, if it is not C-recurrent for any $C \geq 0$.

Our aim is to determine the probability that the $\mathbf{a}$-walk for given $\mathbf{a}$ and C is recurrent; in principle, this probability may be different from 0 and 1 (for example, if $\mathbf{a} = (1, 1, 3, 3, 3, 3 \dots)$ then the $\mathbf{a}$-walk is recurrent with probability $1/2$). A simplest example of an $\mathbf{a}$-walk is when all $a_i \equiv a \in \mathbb{R}_+$. Such a random walk is obviously a.s. recurrent since it is equivalent to the one-dimensional simple random walk.

Received by the editors May 30th, 2022; accepted November 28th, 2022.

2010 *Mathematics Subject Classification.* 60G50, 60J10.

Key words and phrases. Recurrence, transience, Rademacher distribution, non-homogeneous Markov chains.

The research is partially supported by Swedish Science Foundation grants VR 2019-04173 and Crafoord foundation grant no. 20190667.

The question of recurrence is naturally related to the Littlewood-Offord problem which deals with the maximization of probability $\mathbb{P}(S(n)=v)$ over all v, subject to various hypotheses on $\mathbf{a}$. In particular, in Tao and Vu (2009) the authors develop an inverse Littlewood-Offord theory, using which they show that this probability is large only when the elements of $\mathbf{a}$ are contained in a generalized arithmetic progression; see also Nguyen (2012).

The study of $\mathbf{a}$-walk is also somewhat relevant to the conjecture by Boguslaw Tomaszewski (1986), which says that $\mathbb{P}\left(|S(n)| \le \sqrt{a_1^2+\dots+a_n^2}\right) \ge \frac{1}{2}$ for all sequences $\mathbf{a}$ and all n. The conjecture was recently proved in Keller and Klein (2022).

Let us first start with some general statements. First, we show that the choice of $C>0$ is sometimes unimportant for the definition of C-recurrence.

Theorem 1. Suppose that $a_n \to \infty$ and at the same time $|a_n - a_{n-1}| \to 0$ as $n \to \infty$. Then if an $\mathbf{a}$-walk is C-recurrent with a *positive* probability for some $C>0$ then it is $\tilde{C}$-recurrent with a *positive* probability for *all* $\tilde{C}>0$.

Proof: Since the notion of C-recurrence is monotone in C, i.e. if an $\mathbf{a}$-walk is C_1-recurrent for $C_1>0$ then it is C_2-recurrent for all $C_2 \ge C_1$, it suffices to prove that $C-$recurrence implies $\frac{2C}{3}-$recurrence.

Indeed, suppose the $\mathbf{a}$-walk is C-recurrent; formally, if we define the events

$$E = \{S(n) \in [-C, C] \text{ for infinitely many } n\},$$
$$\tilde{E} = \{S(n) \in [-2C/3, 2C/3] \text{ for infinitely many } n\}$$

then $\mathbb{P}(E)>0$. We want to show that $\mathbb{P}(\tilde{E})>0$ as well.

Let n_1 be so large that $|a_i - a_{i-1}| < C/6$ for all $i \ge n_1$. Define the sequence n_k, $k \ge 2$, by setting

$$n_k = \min\{i \ge n_{k-1}+1 : \ a_i \ge a_{n_{k-1}} + C/6\}$$

(which is well-defined since $a_i \to \infty$), then trivially

$$\frac{C}{6} \le a_{n_{k+1}} - a_{n_k} \le \frac{C}{3} \qquad \text{for each } k = 1, 2, \dots \tag{1.1}$$

Fix a positive integer K and for $y = (y_1, y_2, \dots, y_K) \in \Omega_K := \{-1, +1\}^K$ define

$$\bar{X}_K = \{X_1, X_2, \dots, X_K\};$$
$$s_y = a_1 y_1 + a_2 y_2 + \dots a_k y_K.$$

Let $y \in \Omega_K$ be such that $\mathbb{P}(\{\bar{X}_K = y\} \cap E) > 0$. Observe that

$$\{\bar{X}_K = y\} \cap E = \{\bar{X}_K = y\} \cap B_K(s_y)$$

where

$$B_K^+(u) = \{\text{there exist } m_1 < m_2 < \dots \text{ such that } u + \sum_{i=K+1}^{m_j} a_i X_i \in [0, C]\};$$
$$B_K^-(u) = \{\text{there exist } m_1' < m_2' < \dots \text{ such that } u + \sum_{i=K+1}^{m_j'} a_i X_i \in [-C, 0]\};$$
$$B_K(u) = B_K(u)^+ \cup B_K(u)^-.$$

Since $\{\bar{X}_K = y\}$ and $B_K(u)$ are independent, we have

$$\mathbb{P}(\{\bar{X}_K = y\} \cap B_K(s_y)) = \mathbb{P}(\bar{X}_K = y)\, \mathbb{P}(B_K(s_y)).$$

Consequently, $\mathbb{P}(B_K(s_y)) > 0$, and as a result, $\mathbb{P}(B_K^+(s_y)) > 0$ or $\mathbb{P}(B_K^-(s_y)) > 0$ (or both).

Let $\Omega_K^* \subseteq \Omega_K$ contain those ys for which there is an index k such that $n_{k+2} \le K$ and $y_{n_k} = -1$, $y_{n_{k+1}} = +1$, $y_{n_{k+2}} = -1$; let k be the smallest such index. For $y \in \Omega_K^*$ define the mappings $\sigma^+, \sigma^- : \Omega_K^* \to \Omega_K$ by

$$\sigma^+(y) = \begin{cases} -y_i, & \text{if } i = n_k \text{ or } i = n_{k+1}; \\ y_i, & \text{otherwise;} \end{cases}$$

$$\sigma^-(y) = \begin{cases} -y_i, & \text{if } i = n_{k+1} \text{ or } i = n_{k+2}; \\ y_i, & \text{otherwise.} \end{cases}$$

Then for $y \in \Omega_K^*$

$$s_{\sigma^+(y)} = s_y + 2a_{n_k} - 2a_{n_k+1} \in [s_y - 2C/3, s_y - C/3],$$
$$s_{\sigma^-(y)} = s_y - 2a_{n_k} + 2a_{n_k+1} \in [s_y + C/3, s_y + 2C/3].$$

As a result, it is not hard to see that

$$\{\bar{X}_K = \sigma^+(y)\} \cap B_K^+(s_y) \subseteq \left\{ \sum_{i=1}^m a_i X_i \in [-2C/3, 2C/3] \text{ for infinitely many } ms \right\} = \tilde{E},$$
$$\{\bar{X}_K = \sigma^-(y)\} \cap B_K^-(s_y) \subseteq \left\{ \sum_{i=1}^m a_i X_i \in [-2C/3, 2C/3] \text{ for infinitely many } ms \right\} = \tilde{E}.$$

Since at least one of $B_K^+(s_y)$ and $B_K^-(s_y)$ has a positive probability, $\mathbb{P}\left(\bar{X}_K = \sigma^\pm(y)\right) = 2^{-K}$ and the events on the LHS are independent, we conclude that $\mathbb{P}(\tilde{E}) > 0$.

Now it only remains to show that there exists $y \in \Omega_K^*$ such that $\mathbb{P}(\{\bar{X}_K = y\} \cap E) > 0$. Let $\kappa := \kappa(K) = \max\{k \in \mathbb{Z}_+ : \ n_k \le K\}$; obviously, $\kappa(K) \to \infty$ as $K \to \infty$. If we choose y from Ω_K uniformly, we can trivially bound the probability that $y \notin \Omega_K^*$ by[1]

$$\left(1 - \frac{1}{8}\right)^{\lfloor \kappa/3 \rfloor} \to 0 \quad \text{as } \kappa \to \infty$$

by grouping together triples $(X_{n_1}, X_{n_2}, X_{n_3})$, $(X_{n_4}, X_{n_5}, X_{n_6})$, etc.; in each such a triple

$$\mathbb{P}\left((X_{n_k}, X_{n_{k+1}}, X_{n_{k+2}}) = (-1, +1, -1)\right) = 1/8.$$

Hence

$$\mathbb{P}(E) = \sum_{y \in \Omega_K} \mathbb{P}(\{\bar{X}_K = y\} \cap E) = \sum_{y \in \Omega_K^*} \mathbb{P}(\{\bar{X}_K = y\} \cap E) + \mathbb{P}\left(\left\{\bar{X}_K \in \Omega_K \setminus \Omega_K^*\right\} \cap E\right). \tag{1.2}$$

Since $\mathbb{P}\left(\left\{\bar{X}_K \in \Omega_K \setminus \Omega_K^*\right\} \cap E\right) \le \mathbb{P}\left(\bar{X}_K \in \Omega_K \setminus \Omega_K^*\right)$, by making K sufficiently large, we can ensure that the second term on the RHS of (1.2) is less than $\mathbb{P}(E)$, implying that there exist some $y \in \Omega_K^*$ such that $\mathbb{P}(\{\bar{X}_K = y\} \cap E) > 0$, as required. □

Our next result shows that if the sequence **a** is non-decreasing, then the walk will always "jump" over 0 infinitely many times, even if the walk is not C-recurrent.

Theorem 2. Suppose that a_i is a non-decreasing positive sequence. Then the event $\{S(n) > 0\}$ holds for infinitely many n a.s. The same is true for the event $\{S(n) < 0\}$.

The theorem immediately follows from symmetry and the more general

[1]exact: see the sequence A005251 in the online encyclopedia of integer sequences (https://oeis.org/A005251), $\mathbb{P}\left(\bar{X}_K \notin \Omega_K^*\right) \approx \lambda^\kappa$, $\lambda = \frac{\sqrt[3]{100+12\sqrt{69}}}{6} + \frac{2}{3\sqrt[3]{100+12\sqrt{69}}} + \frac{2}{3} = 0.877...$, $\kappa = \kappa(K)$

Proposition 1. Suppose that a_i is a non-decreasing sequence, m is an integer such that $a_{m+1} > 0$, and $S(m) = A > 0$. Define

$$\tau = \inf\{k \geq 0 : \ S(m+k) \leq 0\}.$$

Let $Y_j \sim \mathcal{R}ademacher$ be i.i.d., and

$$\tilde{\tau} = \inf\{k \geq 0 : \ Y_1 + Y_2 + \dots + Y_k \leq -r\}$$

where $r = \lceil A/a_{m+1} \rceil$; note that $\tilde{\tau} < \infty$ a.s. and that, in fact, $Y_1 + \dots + Y_{\tilde{\tau}} = -r$. Then τ is stochastically smaller than $\tilde{\tau}$, that is,

$$\mathbb{P}(\tau > m) \leq \mathbb{P}(\tilde{\tau} > m), \quad m = 0, 1, 2, \dots$$

Proof: We will use coupling. Indeed, we can write

$$S(m+j) = A + a_{m+1}Y_1 + a_{m+2}Y_2 + \dots + a_{m+j}Y_j, \quad j = 1, 2, \dots$$

Suppose that $\tilde{\tau} = k$, that is

$$\begin{aligned} Y_1 > -r, \ Y_1 + Y_2 > -r, \ \dots, \ Y_1 + Y_2 + \dots + Y_{k-1} > -r; \\ Y_1 + Y_2 + \dots + Y_{k-1} + Y_k = -r. \end{aligned}$$

Then, recalling that a_i is a non-decreasing sequence,

$$\begin{aligned} S(m+k) &= A + a_{m+1}Y_1 + \dots + a_{m+k-1}Y_{k-1} + a_{m+k}Y_k \\ &\leq A + a_{m+1}Y_1 + \dots + a_{m+k-2}Y_{k-2} + a_{m+k-1}Y_{k-1} + a_{m+k-1}Y_k \quad (\text{since } Y_k = -1) \\ &= A + a_{m+1}Y_1 + \dots + a_{m+k-2}Y_{k-2} + a_{m+k-1}[Y_{k-1} + Y_k] \\ &\leq A + a_{m+1}Y_1 + \dots + a_{m+k-2}Y_{k-2} + a_{m+k-2}[Y_{k-1} + Y_k] \\ &= A + a_{m+1}Y_1 + \dots + a_{m+k-2}[Y_{k-2} + Y_{k-1} + Y_k] \\ &\leq \dots \leq A + a_{m+1}[Y_1 + \dots + Y_k] = A - ra_{m+1} \leq 0, \end{aligned}$$

since Y_k, $Y_{k-1} + Y_k$, $Y_{k-2} + Y_{k-1} + Y_k$, $\dots$, $Y_1 + \dots + Y_k$ are all negative. Therefore, $\tau \leq \tilde{\tau}$. □

Throughout the paper we will use a version of the Azuma-Hoeffding inequality; compare with the results of Montgomery-Smith (1990).

Lemma 1.1. *Suppose that $b_1, b_2, \dots, b_m$ is a sequence of non-negative numbers and $\mathcal{S} = b_1Y_1 + b_2Y_2 + \dots + b_mY_m$, where $Y_j \sim \mathcal{R}ademacher$ are i.i.d. Then*

$$\mathbb{P}(|\mathcal{S}| \geq A) \leq 2\exp\left(-\frac{A^2}{2(b_1^2 + \dots + b_m^2)}\right) \text{ for all } A > 0. \tag{1.3}$$

We also state the following fairly standard result.

Lemma 1.2. *Let $T_i = Y_1 + \dots + Y_i$ be a simple random walk. Suppose that L_k and y_k, $k = 1, 2, \dots$, are two sequences such that $L_k \to \infty$, $y_k \to \infty$ and $y_k/\sqrt{L_k} \to r > 0$. Then*

$$\lim_{k\to\infty} \mathbb{P}\left(\max_{1\leq i\leq L_k} T_i \geq y_k\right) = 2\,\mathbb{P}(\eta \geq r) = 2 - 2\,\Phi(r)$$

where $\eta \sim \mathcal{N}(0,1)$ and $\Phi(\cdot)$ is its CDF.

Proof: Let $\tilde{y}_k = \lceil y_k \rceil \in \mathbb{Z}_+$. By the reflection principle,

$$\begin{aligned} \mathbb{P}\left(\max_{1\leq i\leq L_k} T_i \geq y_k\right) = \mathbb{P}\left(\max_{1\leq i\leq L_k} T_i \geq \tilde{y}_k\right) &= 2\,\mathbb{P}(T_{L_k} \geq \tilde{y}_k) - \mathbb{P}(T_{L_k} = \tilde{y}_k) \\ &= 2\,\mathbb{P}\left(\frac{T_{L_k}}{\sqrt{L_k}} \geq \frac{\tilde{y}_k}{\sqrt{L_k}}\right) + O\left(\frac{1}{\sqrt{L_k}}\right) \to 2\mathbb{P}(\eta \geq r) \end{aligned}$$

by the Central Limit Theorem, using also the fact that $\tilde{y}_k/y_k \to 1$. □

2. Integer-valued a-walks

Suppose that the sequence $\mathbf{a}$ contains only integers.

Proposition 2. Let $z \in \mathbb{Z}$. Suppose that the sequence

$$\int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k)\mathrm{d}t, \qquad n = 1, 2, \ldots$$

is summable. Then the events $\{S(n) = z\}$ occur for finitely many n a.s.

Proof: The result follows from standard Fourier analysis. Indeed,

$$\mathbb{E}e^{itS(n)} = \sum_{k\in\mathbb{Z}} e^{itk}\mathbb{P}(S(n) = k)$$

where the sum above goes, in fact, effectively over a finite number of ks (as $|S(n)| \le a_1 + \cdots + a_n$). At the same time,

$$\int_{-\pi}^\pi e^{it(k-z)}\mathrm{d}t = \begin{cases} 2\pi, & \text{if } k = z; \\ 0, & \text{if } k \in \mathbb{Z} \setminus \{z\}. \end{cases}$$

By changing the order of summation and integration, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^\pi \mathbb{E}e^{it(S(n)-z)}\mathrm{d}t = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^\pi e^{it(k-z)}\, \mathbb{P}(S(n) = k)\, \mathrm{d}t = \mathbb{P}(S(n) = z).$$

On the other hand,

$$\frac{1}{2\pi}\int_{-\pi}^\pi \mathbb{E}e^{it(S(n)-z)}\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^\pi e^{-itz}\prod_{k=1}^n \mathbb{E}e^{ita_kX_k}\mathrm{d}t = \frac{1}{\pi}\int_0^\pi \cos(tz)\prod_{k=1}^n \cos(ta_k)\mathrm{d}t$$

by the symmetry of cos and the fact that the imaginary part must equal zero. Now the result follows from the Borel-Cantelli lemma, since $\sum_n \mathbb{P}(S(n) = z) < \infty$. □

Corollary 2.1. *Suppose that the sequence*

$$\int_0^\pi \left|\prod_{k=1}^n \cos(ta_k)\right| \mathrm{d}t, \qquad n = 1, 2, \ldots$$

is summable. Then the $\mathbf{a}$*-walk is transient a.s.*

Proof: From Proposition 2 we know that for each $z \in \mathbb{Z}$

$$\pi\mathbb{P}(S(n) = z) = \int_0^\pi \cos(tz)\prod_{k=1}^n \cos(ta_k)\mathrm{d}t \le \int_0^\pi \left|\cos(tz)\prod_{k=1}^n \cos(ta_k)\right| \mathrm{d}t \le \int_0^\pi \left|\prod_{k=1}^n \cos(ta_k)\right| \mathrm{d}t.$$

Hence the event $\{S(n) = z\}$ occurs finitely often a.s. for each z. Since for each $C > 0$ there are only finitely many integers in $[-C, C]$ we conclude that the walk is not C-recurrent a.s. for every C. □

An interesting and quite natural example is when $\mathbf{a} = (1, 2, 3, \ldots)$, i.e., $a_i = i$. It was previously published in the IMS Bulletin, **51**(2), in the Student Puzzle Corner no. 37.

Theorem 3. The $\mathbf{a}$-walk with $\mathbf{a} = (1, 2, 3, \ldots)$ is a.s. transient.

This statement follows from a much stronger Theorem 4, but for the sake of completeness, we present its self-contained proof.

Proof of Theorem 3: Let $A_n = \{S(n) = 0\} = \{X_1 + 2X_2 + \cdots + nX_n = 0\}$. Then $\mathbb{P}(A_n) = Q_n/2^n$, where

$$Q_n = \text{number of ways to put } \pm \text{ in the sequence } *1*2*3*\cdots*n \text{ such that the sum equals } 0.$$

For example, $Q_1 = Q_2 = 0$, $Q_3 = Q_4 = 2$, $Q_5 = Q_6 = 0$, $Q_7 = 8$, $Q_8 = 14$, etc. It was essentially shown in Sullivan (2013) that

$$Q_n \sim \sqrt{\frac{6}{\pi}} \frac{2^n}{n^{3/2}} \qquad \text{when } n \bmod 4 \in \{0, 3\}$$

(and zero otherwise) as $n \to \infty$, meaning that the ratio of the RHS and the LHS converges to one. Consequently, $\sum_n \mathbb{P}(A_n) \sim \sum_n \frac{\text{const}}{n^{3/2}} < \infty$ and the events A_n occur a.s. finitely often by the Borel-Cantelli lemma. Hence the walk is a.s. not recurrent.

Moreover, since for any $m \in \mathbb{Z}$

$$\mathbb{P}(S(n + 2|m|) = S(n) - m \mid \mathcal{F}_n) \geq \frac{1}{2^{2m}}$$

(by making the signs of $X_{n+1}, X_{n+2}, \ldots, X_{n+2|m|}$ alternate), we conclude that if the event $\{S(n) = m\}$ occurs infinitely often, then A_n shall also occur infinitely often a.s., leading to contradiction. As a result, $\mathbb{P}(\{S(n) = m\} \text{ i.o.}) = 0$ for all integer ms, and thus the walk is a.s. not C-recurrent for any non-negative C. □

Remark 1. Though the $(1, 2, 3, \ldots)$-walk is transient, it still can jump over zero infinitely many times, as it was shown by Theorem 2.

In fact, Theorem 3 can be generalized greatly, using the result from Sárközi and Szemerédi (1965), or even a weaker result of Erdős (1965), which provide the estimates for the maximum number of solutions of the equation $\sum_{i=1}^n \varepsilon_i a_i = t$ where $\epsilon_i \in \{0, 1\}$ while a_i's and t are all integers.

Theorem 4. Let $\mathbf{a}$ be such that all a_i's are distinct integers. Then $\mathbf{a}$-walk is a.s. transient.

Proof: The main result of Sárközi and Szemerédi (1965) implies that for any $\epsilon > 0$

$$\text{card}(\{(x_1, x_2, \ldots, x_n) : \text{ all } x_i = \pm 1,\ a_1 x_1 + \cdots + a_n x_n = m\}) \leq \frac{(1 + \epsilon) 2^{n+3}}{n^{3/2}\sqrt{\pi}}$$

for all $n \geq n_0(\epsilon)$ and all m. Setting $\epsilon = 1$, and fixing $m \in \mathbb{Z}$, we obtain that

$$\sum_{n=n_0(1)}^{\infty} \mathbb{P}(S(n) = m) \leq \sum_{n=n_0(1)}^{\infty} \frac{2 \cdot 2^{n+3}}{n^{3/2}\sqrt{\pi}} \times \frac{1}{2^n} = \frac{16}{\sqrt{\pi}} \sum_{n=n_0(1)}^{\infty} \frac{1}{n^{3/2}} < \infty.$$

Therefore, by the Borel-Cantelli lemma, only finitely many events $\{S(n) = m\}$ occur a.s. Since $S(n)$ takes only integer values, this implies that $\{|S(n)| \leq C\}$ happens finitely often a.s. for any $C > 0$. □

Remark 2.
(a) It is not difficult to see that under the condition of Theorem 4 it suffices that all a_k's are distinct only starting from some $k_0 \geq 1$.
(b) If $a_k = \lfloor k^\beta \rfloor$ with $\beta \geq 1$, then we immediately have a.s. transience by Theorem 4.
(c) In the proof of Theorem 4 we use the result of Sarkozy and Szemeredi from 1965. The constant in their bound can, in fact, be replaced by the constant $\sqrt{6/\pi}$ from Sullivan's result. Even though the value of the constant does not matter for out proof, it is worth mentioning the (1980) result of Stanley (1980) that the set $\{1, 2, 3, \ldots, n\}$ is extremal among sets of n distinct integers for maximizing the maximum concentration probability of its Rademacher sum. This fact was proved by Stanley using some high-powered algebraic geometry but was then proved again soon afterwards in a simpler way using Lie algebras in Proctor (1982).

3. A non-trivial recurrent example

We assume here that $\mathbf{a} = (B_1, B_2, B_3, \dots)$ where each B_k is a consecutive block of k's of length precisely $L_k \geq 1$. Denote also by $i_k = 1 + L_1 + L_2 + \dots + L_{k-1}$ the index of the first element of the $k-$th block. For example, if $L_k = 2^k$, then $i_k = 2^k - 1$ and

$$\mathbf{a} = (1, 1, \underbrace{2, 2, 2, 2}_{L_2 \text{ times}}, \underbrace{3, 3, 3, 3, 3, 3, 3, 3}_{L_3 \text{ times}}, \underbrace{4, \dots, 4}_{L_4 \text{ times}}, \dots),$$

one can also notice that $a_i = \lfloor \log_2(i+1) \rfloor = \lceil \log_2(i+2) \rceil - 1$.

Theorem 5. Suppose that for some $\varepsilon > 0$, $r > 0$, and k_0 we have

$$\begin{aligned} \frac{L_k}{L_1 + L_2 + \dots + L_{k'}} &\geq (2+\varepsilon) \ln k; \\ \frac{L_k}{L_{k'+1} + L_{k'+2} + \dots + L_{k-1}} &\geq 2r; \\ L_k &\geq k^4, \end{aligned} \tag{3.1}$$

whenever $k - k' \geq \frac{k}{\ln k} - 2$ and $k, k' \geq k_0$. Then the $\mathbf{a}$-walk described above is a.s. recurrent.

Remark 3. One can easily check that the conditions of the theorem are satisfied if $a_k = \left\lfloor (\log_\gamma k)^\beta \right\rfloor$, where $\gamma > 1$ and $\beta \in (0, 1]$.

Proof of Theorem 5: We will proceed in FIVE steps.

Step 1: Preliminaries

First, we need the following lemma, which is probably known.

Lemma 3.1. *Let $m \in \mathbb{Z}_+$ and T_m be a simple symmetric random walk on $\mathbb{Z}^1$, that is, $T_m = Y_1 + \dots + Y_m$, where $Y_i \sim$ Rademacher are i.i.d. There exists a universal constant $c_1 > 0$ such that for all integers z such that $|z| \leq 2\sqrt{m}$, assuming that m is sufficiently large and $m + z$ is even,*

$$\mathbb{P}(T_m = z) \geq \frac{c_1}{\sqrt{m}}.$$

Proof: W.l.o.g. assume $z \geq 0$. We have

$$\mathbb{P}(T_m = z) = \mathbb{P}\left(\frac{T_m + m}{2} = \frac{z + m}{2}\right) = \mathbb{P}(\tilde{T} = w)$$

where $\tilde{T} \sim \text{Bin}(m, 1/2)$ and $w = \frac{z+m}{2} \in \mathbb{Z}_+$. Note that $\tilde{m} \leq w \leq \tilde{m} + \sqrt{m}$ where $\tilde{m} = m/2$. So

$$\begin{aligned} \mathbb{P}(\tilde{T} = w) &= \binom{m}{w} 2^{-m} = \binom{2\tilde{m}}{\tilde{m}} \frac{1}{2^{2\tilde{m}}} \frac{\tilde{m}!\,\tilde{m}!}{w!(m-w)!} = \frac{1+o(1)}{\sqrt{\pi \tilde{m}}} \frac{(2\tilde{m} - w + 1)(2\tilde{m} - w + 2)\dots\tilde{m}}{(\tilde{m}+1)(\tilde{m}+2)\dots w} \\ &= \frac{1+o(1)}{\sqrt{\pi \tilde{m}}} \left(1 - \frac{w - \tilde{m}}{\tilde{m}+1}\right)\left(1 - \frac{w - \tilde{m}}{\tilde{m}+2}\right)\dots\left(1 - \frac{w - \tilde{m}}{w}\right) \\ &\geq \frac{1+o(1)}{\sqrt{\pi \tilde{m}}} \left(1 - \frac{\sqrt{m}}{\tilde{m}+1}\right)^{w - \tilde{m}} \geq \frac{1+o(1)}{\sqrt{\pi \tilde{m}}} \left(1 - \frac{\sqrt{2} + o(1)}{\sqrt{\tilde{m}}}\right)^{\sqrt{2\tilde{m}}} = \frac{e^{-2} + o(1)}{\sqrt{\pi m/2}} \geq \frac{0.1}{\sqrt{m}} \end{aligned}$$

for large enough m. □

Corollary 3.2. *Let T_m, $m = 0, 1, 2, \dots,$ be as simple symmetric random walk as in Lemma 3.1. Assume that m and k are positive integers such that $k^2 \leq m$. Let $u \in \mathbb{Z}$, and either k is odd, or both k and $m - u$ are even. Then for large ks*

$$\mathbb{P}(T_m - u \bmod k = 0) \geq \frac{c_1}{2k}$$

where c_1 is the constant from Lemma 3.1.

Proof of Corollary 3.2: First, assume that m, and hence u, are both even. Since $(T_m - u) \bmod k = 0 \iff T_m = \tilde{u} \bmod k$, where $\tilde{u} = (u \bmod k) \in \{0, 1, 2, \dots, k-1\}$, it suffices to show the statement for $\tilde{u}$.

Let $M = \lfloor 2\sqrt{m} \rfloor \in (2\sqrt{m} - 1, 2\sqrt{m}]$ and define

$$\mathbb{I} = [-M, -M+1, \dots, -1, 0, 1, \dots, M] = \mathbb{I}_0 \cup \mathbb{I}_1;$$
$$\mathbb{I}_0 = \{z \in \mathbb{I} : \ z \text{ is even}\}; \quad \mathbb{I}_1 = \{z \in \mathbb{I} : \ z \text{ is odd}\}.$$

There are at least M elements in each $\mathbb{I}_0$ and $\mathbb{I}_1$.

If k is odd, then each of these two sets contains at least $\lfloor \frac{M}{k} \rfloor$ elements z such that $z = \tilde{u} \bmod k$. If m is even (odd, resp.) for all z either in $\mathbb{I}_0$ (in $\mathbb{I}_1$, resp.) by Lemma 3.1 for large ks (and hence large m) we have $\mathbb{P}(T_m = z) \ge c_1/\sqrt{m}$. Consequently,

$$\begin{aligned}\mathbb{P}(T_m = \tilde{u} \bmod k) &\ge \sum_{z\in\mathbb{I},\ z=\tilde{u} \bmod k} \mathbb{P}(T_m = z) \ge \left\lfloor \frac{M}{k} \right\rfloor \times \frac{c_1}{\sqrt{m}} \ge \left(\frac{M}{k} - 1\right) \times \frac{c_1}{\sqrt{m}} \\ &\ge \left(\frac{2\sqrt{m}-1}{k} - 1\right) \times \frac{c_1}{\sqrt{m}} \ge \left(1 - \frac{1}{k}\right) \times \frac{c_1}{\sqrt{m}} = \frac{c_1}{k} - O(k^{-2})\end{aligned}$$

since $m \ge k^2$.

If k is even, then if m is even (and thus a is also even) then $\mathbb{I}_0$ contains at least $\lfloor \frac{M}{k} \rfloor$ elements z such that $z = \tilde{u} \bmod k$ and at the same time Lemma 3.1 is applicable for $z \in \mathbb{I}_0$. On the other hand, if m (and so u) is odd then $\mathbb{I}_1$ contains at least $\lfloor \frac{M}{k} \rfloor$ elements z such that $z = \tilde{u} \bmod k$ and Lemma 3.1 is applicable for $z \in \mathbb{I}_1$. The rest of the proof is the same as for the case when k is odd. □

Step 2: Splitting $S(n)$

Recall that that i_k denotes the first index of block k and note that the sum of all the steps within block k can be represented as

$$S(i_{k+1} - 1) - S(i_k - 1) = k \cdot T_k, \qquad T_k = X_1^{(k)} + \dots + X_{L_k}^{(k)}$$

where $X_j^{(k)}$'s are i.i.d. $\mathcal{R}ademacher$ random variables.

For $m = 2, \dots,$ let

$$k_m = \begin{cases} \lfloor m \ln m \rfloor & \text{if } \lfloor m \ln m \rfloor \text{ is odd;} \\ \lfloor m \ln m \rfloor + 1 & \text{if } \lfloor m \ln m \rfloor \text{ is even.} \end{cases} \tag{3.2}$$

Thus k_m is *always* odd; k_m, $m = 2, 3, \dots$ equal $1, 3, 5, 7, 9, 13, 15, 19, 23$, etc. Define also

$$A_m = \{S(j) = 0 \text{ for some } i_{k_m} \le j < i_{k_m+1}\},$$

the event that $S(j)$ hits zero for the steps within block B_{k_m}, and the sequence of sigma-algebras

$$\mathcal{G}_m = \mathcal{F}_{i_{k_{m+1}}-1} = \sigma\left(\bigcup_{\ell=1}^{k_m} \sigma\left(X_1^{(\ell)}, X_2^{(\ell)}, \dots, X_{L_\ell}^{(\ell)}\right)\right).$$

Intuitively, $\mathcal{G}_m$ contains all the information about the walk during its steps corresponding to the first k_m blocks.

To simplify notations, let us now write $k = k_m$ and $k' = k_{m-1}$, and observe that

$$k - k' = k_m - k_{m-1} \ge m \ln m - (m-1)\ln(m-1) - 2 = \ln m - 1 + O\left(\frac{1}{m}\right) \ge \ln m - 2 \tag{3.3}$$

for large m.

Let us split $S(j)$ where $j \in [i_k, i_{k+1})$, as follows:

$$\begin{aligned}
S(j) &= S(i_{k'}) + \sum_{n=k'+1}^{k-1} \left(X_1^{(n)} + \dots + X_{L_n}^{(n)} \right) + k \cdot \left(X_1^{(k)} + \dots + X_{j-i_k}^{(k)} \right) \\
&= S(i_{k'}) + \left[\sum_{n=k'+1}^{k-2} \left(X_1^{(n)} + \dots + X_{L_n}^{(n)} \right) + (k-1) \sum_{\ell=1}^{i_k - 2k^2 - 1} X_\ell^{(k-1)} \right] \\
&+ (k-1) \cdot \Sigma_3 + k \cdot \left(X_1^{(k)} + \dots + X_{j-i_k}^{(k)} \right) \\
&= \Sigma_1 + \Sigma_2 + (k-1) \cdot \Sigma_3 + k \cdot \Sigma_4
\end{aligned}$$

where $\Sigma_1 = S(i_{k'})$ and

$$\begin{aligned}
\Sigma_2 &= \sum_{n=k'+1}^{k-2} nT_n + (k-1)T'_{k-1}, \quad T'_{k-1} = \sum_{\ell=i_{k-1}}^{i_k - 1 - 2k^2} X_\ell^{(k-1)}; \\
\Sigma_3 &= X_{i_k - 2k^2}^{(k-1)} + X_{i_k - 2k^2 + 1}^{(k-1)} + \dots + X_{i_k - 2}^{(k-1)} + X_{i_k - 1}^{(k-1)}; \\
\Sigma_4 &= X_1^{(k)} + X_2^{(k)} + \dots + X_{j-i_k}^{(k)}.
\end{aligned}$$

Note that Σ_i, $i = 1, 2, 3, 4$ are independent, and Σ_3 has precisely $2k^2$ terms.

Step 3: Estimating Σ_1

Recall that $k = k_m$, $k' = k_{m-1}$ and let

$$E_{m-1} = \left\{ |\Sigma_1| < k\sqrt{L_k} \right\} \in \mathcal{G}_{m-1}.$$

By Lemma 1.1 and (3.1), assuming k is large,

$$\begin{aligned}
\mathbb{P}(E_{m-1}^c) &\le \mathbb{P}(|S(k')| \ge k'\sqrt{L_k}) \le 2\exp\left(-\frac{k'^2 \cdot L_k}{2(L_1 + 2^2 \cdot L_2 + 3^2 \cdot L_3 + \dots + k'^2 \cdot L_{k'})} \right) \\
&\le 2\exp\left(-\frac{L_k}{2(L_1 + L_2 + L_3 + \dots + L_{k'})} \right) \le 2\exp\left(-(1+\varepsilon/2)\ln k\right) = \frac{2}{k^{1+\varepsilon/2}} =: \varepsilon_m.
\end{aligned} \tag{3.4}$$

Step 4: Estimating Σ_2

Again, by Lemma 1.1 and (3.1), assuming that k is sufficiently large,

$$\begin{aligned}
\mathbb{P}\left(|\Sigma_2| \ge k\sqrt{\frac{L_k}{r}} \right) &\le 2\exp\left(-\frac{k^2 r^{-1} L_k}{2[(k'+1)^2 L_{k'+1} + \dots + (k-2)^2 L_{k-2} + (k-1)^2 (L_{k-1} - 2k^2)]} \right) \\
&\le 2\exp\left(-\frac{r^{-1} L_k}{2[L_{k'+1} + \dots + L_{k-1} - 2k^2]} \right) \le 2\exp(-1) = 0.7357588824\dots.
\end{aligned}$$

Consequently,

$$\mathbb{P}\left(|\Sigma_2| < k\sqrt{L_k/r} \right) \ge 0.2 \qquad \text{for large } k. \tag{3.5}$$

Step 5: Finishing the proof

We have a trivial lower bound

$$\mathbb{P}\left(A_m \mid E_{m-1},\mathcal{G}_{m-1}\right) \ge \mathbb{P}\left(A_m \mid |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1},\mathcal{G}_{m-1}\right) \times \mathbb{P}\left(|\Sigma_2| < k\sqrt{\frac{L_k}{r}} \mid E_{m-1},\mathcal{G}_{m-1}\right)$$
$$=: (*) \times 0.2 \qquad \text{for large } k \tag{3.6}$$

by (3.5), since the second multiplier equals $\mathbb{P}\left(|\Sigma_2| < k\sqrt{L_k/r}\right)$ by independence.

Let

$$\mathsf{Div}_k = \{\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3 = 0 \bmod k\} = \{\Sigma_1 + \Sigma_2 - \Sigma_3 = 0 \bmod k\}.$$

Since only on the event Div_k, it is possible that $S(j) = 0$ for some j (since the step sizes are $\pm k$ in the block B_k), we conclude that for large k

$$(*) = \mathbb{P}\left(A_m \mid \mathsf{Div}_k, |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1},\mathcal{G}_{m-1}\right) \times \mathbb{P}\left(\mathsf{Div}_k \mid |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1},\mathcal{G}_{m-1}\right)$$
$$\ge \mathbb{P}\left(A_m \mid \mathsf{Div}_k, |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1},\mathcal{G}_{m-1}\right) \times \frac{c_1}{2k} \tag{3.7}$$

due to the fact that by Corollary 3.2, $\mathbb{P}\left(\mathsf{Div}_k \mid \mathcal{F}_{i_k-2k^2-1}\right) \ge c_1/(2k)$. On the other hand,

$$\mathbb{P}\left(A_m \mid \mathsf{Div}_k, |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1},\mathcal{G}_{m-1}\right) \ge \min_{z\in Z_k} \mathbb{P}(z + T_m = 0 \text{ for some } m \in [0, L_k])$$
$$\ge \beta := 1 - \Phi\left(r^{-1/2} + 3\right) > 0 \tag{3.8}$$

where $z + T_m$ is a simple random walk starting at z (see Lemma 3.1), and

$$Z_k = \left\{z \in \mathbb{Z} : \ |z| \le (r^{-1/2} + 3)\sqrt{L_k}\right\}.$$

Indeed, using the last part of (3.1), and the conditions we imposed, we have

$$|\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3| \le k\sqrt{L_k} + k\sqrt{L_k/r} + 2(k-1)k^2 < (1 + r^{-1/2} + 2)k\sqrt{L_k}$$

for large k, $S(j) = [\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3] + k\cdot\Sigma_4$, and by Lemma 1.2

$$\liminf_{k\to\infty} \min_{z\in Z_k} \mathbb{P}(z + T_m = 0 \text{ for some } m \in [0, L_k]) \ge 2\,\mathbb{P}(\eta > r^{-1/2} + 3) = 2\beta,$$

so the minimum in (3.8) is $\ge \beta$ for all sufficiently large k.

Finally, from (3.6), (3.7), and (3.8) we get that

$$\sum_m \mathbb{P}\left(A_m \mid E_{m-1},\mathcal{G}_{m-1}\right) \ge \sum_m \frac{0.2\, c_1\beta}{2m\log m} = +\infty$$

and $\mathbb{P}(E_m^c)$ is summable by (3.4), so we can apply Lemma 5.1 of Appendix 1 to conclude that events A_m occur infinitely often and thus our **a**-walk is recurrent. □

4. Continuous example

The example of **a**-walk described in Theorem 5 roughly corresponds to the case $a_k = \lceil \log_\gamma k\rceil$, $k = 1, 2, \dots$. But what if a_k's take non-integer values, but, for example, equal

$$a_k = \log_\gamma k \equiv c\ln k, \qquad k = 1, 2, \dots,$$

where $\gamma = e^{1/c} > 1$? In this Section we will study this example. It is unreasonable to assume that such **a**-walk is recurrent, because of the irrationality of the step sizes, however, we might want to investigate if this walk is C-recurrent for *some* $C > 0$. Our main result is the following

Theorem 6. Let $c > 0$ and $a_k = c \ln k$. Then the **a**-walk is a.s. C-recurrent for every $C > 0$.

To prove this theorem, it is sufficient to show that whatever the value $c > 0$ is, $\{|S(n)| \le 3\}$ happens i.o. almost surely. Indeed, take any $C > 0$. Then the statement that $\mathbf{a}'$-walk with $a'_k = \frac{3c}{C} \ln k$, $k = 1, 2, \dots$, is 3-recurrent is equivalent to the statement that **a**-walk with $a_k = c \ln k$ is C-recurrent.

The proof will proceed similarly to that of Theorem 5. Let us define k_m slightly differently from (3.2); namely, let

$$k_m = \begin{cases} \lfloor m \ln m \rfloor & \text{if } \lfloor m \ln m \rfloor \text{ is even;} \\ \lfloor m \ln m \rfloor - 1 & \text{if } \lfloor m \ln m \rfloor \text{ is odd.} \end{cases}$$

Thus now k_m are always *even*. As before, set $k = k_m$, and $k' = k_{m-1}$, and define

$$i_k = \lceil \gamma^k \rceil = \max\{i \ge 1 : \ a_i < k\} + 1 = \min\{i \ge 1 : \ a_i \ge k\} \in [\gamma^k, \gamma^k + 1),$$

i.e., the first index when a_i starts exceeding k. For $i \in J_k := [i_k, i_{k+1})$ write

$$\begin{aligned} S(i) &= S(i_{k'} - 1) && + [S(i_k - 1) - S(i_{k'} - 1) - \Sigma_3] && + \Sigma_3 && + [S(i) - S(i_k - 1)] \\ &= \Sigma_1 && + \Sigma_2 && + \Sigma_3 && + \Sigma_4(i) \end{aligned} \tag{4.1}$$

where

$$\Sigma_3 = \left[S(i_{k-1} + k^2 - 1) - S(i_{k-1} - 1)\right] + \left[S(i_k) - S(i_k - k^2)\right].$$

Note that Σ_i, $i = 1, 2, 3, 4$ are independent, and Σ_3 has $2 \cdot k^2$ terms, and contains the first k^2 and the last k^2 steps of the walk, when the step sizes lie in $[k, k+1)$.

Let

$$E_{m-1} = \left\{|\Sigma_1| < k\sqrt{i_k}\right\} = \left\{S(i_{k_{m-1}}) < k_m \sqrt{i_{k_m}}\right\} \tag{4.2}$$

By Lemma 1.1, since $a_i < k' < k$ for $i < i_{k'}$,

$$\begin{aligned} \mathbb{P}(E_{m-1}^c) &= \mathbb{P}\left(|\Sigma_1| \ge k\sqrt{i_k}\right) \le 2\exp\left(-\frac{i_k k^2}{2\sum_{j=1}^{i_{k'}} a_j^2}\right) \le 2\exp\left(-\frac{i_k}{2 i_{k'}}\right) \le 2\exp\left(-\frac{\gamma^k - 1}{2\gamma^{k'}}\right) \\ &= 2\exp\left(-\frac{\gamma^{k-k'}(1 + o(1))}{2}\right) = 2\exp\left(-\frac{\gamma^{\ln m - 2}}{2 + o(1)}\right) = 2\exp\left(-\frac{m^{\ln \gamma}}{2\gamma^2 + o(1)}\right) =: \varepsilon_{m-1} \end{aligned} \tag{4.3}$$

using (3.3) for k sufficiently large[2]. Observe that ε_m is summable.

Similarly, by Lemma 1.1

$$\mathbb{P}\left(|\Sigma_2| \ge 2k\sqrt{i_k}\right) \le 2\exp\left(-\frac{4k^2 i_k}{2k^2 (i_k - i_{k'} - 2k^2)}\right) < 2\, e^{-2} = 0.27\dots$$

Hence,

$$\mathbb{P}(F_m) \ge 0.72 \quad \text{where } F_m = \left\{|\Sigma_2| < 2k\sqrt{i_k}\right\}. \tag{4.4}$$

[2]Note that (3.3) was stated for k_m defined slightly differently, however, it holds here as well.

Lemma 4.1. *Let $n = k^2$ where k is an even positive integer, and assume also that k is sufficiently large. Suppose that X_i, Y_i, $i = 1, 2, \dots, n$, are i.i.d. $\mathcal{R}ademacher$. Let*

$$T = (k-1)\,(X_1 + X_2 + \cdots + X_n) + k\,(Y_1 + Y_2 + \cdots + Y_n). \tag{4.5}$$

Then

$$\mathbb{P}(T = j) \geq \frac{c_1^2}{4n} \quad \text{for each } j = 0, \pm 2, \pm 4, \dots, \pm n$$

where c_1 is the constant from Lemma 3.1.

Proof: It follows from Corollary 3.2 that

$$\mathbb{P}(X_1 + \cdots + X_n = \ell) \geq \frac{c_1}{2k}, \quad \mathbb{P}(Y_1 + \cdots + Y_n = \ell) \geq \frac{c_1}{2k} \quad \text{for all even } \ell \text{ such that } |\ell| \leq 2k. \tag{4.6}$$

Let $j = 2\tilde{j} \in \{0, 2, 4, \dots, n-2, n\}$. Consider the sequence of $k-1$ numbers

$$\tilde{j},\ \tilde{j} - k,\ \tilde{j} - 2k,\ \tilde{j} - 3k, \dots, \tilde{j} - (k-2)k;$$

they all give different remainders when divided by $k-1$. Hence there must be an $m \in \{0, 1, \dots, k-2\}$ such that $\tilde{j} - mk = b(k-1)$ and b is an integer; moreover, since $0 \leq \tilde{j} \leq n/2$, we have $b \in \left[-\frac{k(k-2)}{k-1}, \frac{n}{2(k-1)}\right]$. For such m and b we have $j = 2\tilde{j} = (2m)k + (2b)(k-1)$, and, since both $|2m|$ and $|2b| \leq 2k$,

$$\mathbb{P}(T = j) \geq \mathbb{P}(X_1 + \cdots + X_n = 2b) \cdot \mathbb{P}(Y_1 + \cdots + Y_n = 2m) \geq \left(\frac{c_1}{2k}\right)^2 = \frac{c_1^2}{4n}$$

by (4.6). The result for negative j follows by symmetry. □

Corollary 4.2. *Let $\varepsilon = \frac{2ck^4}{\gamma^{k-1}}$. Then for large even k*

$$\mathbb{P}\left(\Sigma_3 \in [j - \varepsilon, j + \varepsilon]\right) \geq \frac{c_1^2}{4k^2} \quad \text{for each } j = 0, \pm 2, \pm 4, \dots, \pm k^2.$$

Proof: Σ_3 has the same distribution as

$$\sum_{\ell=1}^{k^2} c\ln(i_{k-1} - 1 + \ell)\,X_\ell + \sum_{\ell=1}^{k^2} c\ln(i_k - \ell)\,Y_\ell$$

for some i.i.d. $X_\ell, Y_\ell \sim \mathcal{R}ademacher$. At the same time, for $\ell \geq 1$,

$$\begin{aligned} |c\ln(i_{k-1} - 1 + \ell) - (k-1)| &= |c\ln(\lceil \gamma^{k-1} \rceil + \ell - 1) - (k-1)| \\ &\leq |c\ln(\gamma^{k-1} + \ell) - (k-1)| = c\ln\left(1 + \frac{\ell}{\gamma^{k-1}}\right) \leq \frac{c\ell}{\gamma^{k-1}} \end{aligned}$$

Similarly,

$$k - c\ln(i_k - \ell) = k - c\ln(\lceil \gamma^k \rceil - \ell) = k - c\ln(\gamma^k - \ell') = -c\ln\left(1 - \frac{\ell'}{\gamma^k}\right) \in \left[0, \frac{c\ell}{\gamma^{k-1}}\right]$$

for some $\ell' \in [\ell - 1, \ell]$, assuming $\ell = o(\gamma^k)$. As a result, for T defined by (4.5),

$$|\Sigma_3 - T| \leq \sum_{\ell=1}^{k^2} \frac{2c\ell}{\gamma^{k-1}} = \frac{ck^2(k^2+1)}{\gamma^{k-1}} \leq \frac{2ck^4}{\gamma^{k-1}}.$$

Now the result follows from Lemma 4.1. □

Proof of Theorem 6: Recall that $J_k = [i_k, i_{k+1})$ and define

$$A_m = \{S(i) = 0 \text{ for some } i \in J_{k_m}\}.$$

Then

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq 0.72 \times \mathbb{P}(A_m \mid F_m, E_{m-1}, \mathcal{G}_{m-1}) \tag{4.7}$$

(please see the definition of F_m in (4.4)). Recall formula (4.1) and write

$$\tilde{S}(i) = S(i) - \Sigma_3 = \Sigma_1 + \Sigma_2 + \Sigma_4(i).$$

From now on assume that $|\Sigma_1| < k\sqrt{i_k}$ and $|\Sigma_2| < 2k\sqrt{i_k}$, that is, E_{m-1} and F_m occur. Also assume w.l.o.g. that $\Sigma_1 + \Sigma_2 \geq 0$. Let

$$L_k = i_{k+1} - i_k - k^2 = (\gamma - 1)\gamma^k + o(\gamma^k).$$

Consider a simple random walk with steps $Y_i \sim \mathcal{R}ademacher$ during its first L_k steps. The probability that its minimum will be equal to or below the level $-3\sqrt{i_k} = -\frac{3+o(1)}{\sqrt{\gamma-1}}\sqrt{L_k}$ converges by Lemma 1.2 to

$$2\,\mathbb{P}\left(\eta > \frac{3}{\sqrt{\gamma - 1}}\right) = 2 - 2\,\Phi\left(\frac{3}{\sqrt{\gamma - 1}}\right) =: 2c_2 \in (0, 1)$$

as $k \to \infty$ (recall that $\eta \sim \mathcal{N}(0, 1)$). As a result, by Proposition 1, the probability that for some $j_0 \in \{i_k, i_k + 1, i_k + 2, \dots, i_k + L_k - 1\}$ we have the down-crossing, that is,

$$\tilde{S}(j_0 - 1) \geq 0 > \tilde{S}(j_0)$$

is bounded below by c_2 for k sufficiently large. Formally, let

$$j_0 = \inf\{j > i_k : \ \tilde{S}(j) < 0\};$$
$$\mathcal{C}_0 = \{i_k \leq j_0 \leq i_k + L_k - 1\},$$

so we have showed that on $E_{m-1} \cap F_m \cap \{\Sigma_1 + \Sigma_2 > 0\}$ we have $\mathbb{P}(\mathcal{C}_0) > c_2$ for large k.

Now assume that event $\mathcal{C}_0$ occurred and define additionally

$$b_0 = \tilde{S}(j_0) \in (-k - 1, 0];$$
$$\mathcal{C} = \left\{ \max_{0 \leq h \leq k^2} \sum_{g=1}^{h} X_{j_0+g} \geq k \right\}.$$

Again, from Lemma 1.2,

$$\mathbb{P}(\mathcal{C}) = 2\,\mathbb{P}(X_{j_0+1} + X_{j_0+2} + \cdots + X_{j_0+k^2} \geq k) \to 2(1 - \Phi(1)) = 0.3173\ldots \quad \text{as } k \to \infty.$$

From now on assume that k is so large that $\mathbb{P}(\mathcal{C}) > 0.2$. On the event $\mathcal{C}$ there exists an increasing sequence $j_1, j_2, \dots, j_k$ such that

$$j_0 < j_1 < j_2 < \cdots < j_k \leq j_0 + k^2 < i_{k+1}$$

such that $X_{j_0+1} + X_{j_0+2} + \cdots + X_{j_\ell} = \ell$ for each $\ell = 1, 2, \dots, k$, since the random walk must pass through each integer in $\{1, 2, \dots, k\}$ in order to reach level k.

For $\ell = 1, 2, \dots, k$, define

$$\begin{aligned} b_\ell := \tilde{S}(j_\ell) &= b_0 + \sum_{h=j_0+1}^{j_\ell} a_h X_h = b_0 + a_{j_0}\left[\sum_{h=j_0+1}^{j_\ell} X_h\right] + \sum_{h=j_0+1}^{j_\ell} (a_h - a_{j_0})\, X_h \\ &= b_0 + a_{j_0}\ell + O\left(\frac{k^4}{\gamma^k}\right) \end{aligned}$$

since for $h \in [j_0, j_0 + k^2] \subseteq [i_k, i_{k+1})$ we have

$$|a_h - a_{j_0}| = c\left|\ln \frac{a_h}{a_{j_0}}\right| \le c\left|\ln \frac{i_k + k^2}{j_k}\right| = O\left(\frac{k^2}{\gamma^k}\right).$$

As a result,

$$-(k+1) < b_0 < b_1 < b_2 < \cdots < b_{k-1} < (k-1)(k+1) < k^2$$

and moreover the distance between consecutive b_g's is at least two (provided k is large). For $\ell = 0, 1, \dots, k-1$ define

$$\tilde{b}_\ell = \sup\left\{x \in 2\mathbb{Z} : x + b_\ell \in \left[\frac{1}{2}, 3 - \frac{1}{2}\right)\right\} \equiv -2\left\lfloor \frac{b_\ell}{2} - \frac{1}{4}\right\rfloor.$$

Then $\tilde{b}_\ell$'s are all distinct even integers satisfying $|\tilde{b}_\ell| \le k^2$.

As a result,

$$\begin{aligned}
\mathbb{P}(A_m \mid F_m, E_{m-1}, \mathcal{G}_{m-1}) &\ge c_2 \times 0.2 \times \mathbb{P}(S(i) \in [0,3] \text{ for some } i \in J_k \mid \mathcal{C}, \mathcal{C}_0) \\
&\ge c_2 \times 0.2 \times \mathbb{P}(\tilde{S}(i_\ell) + \Sigma_3 \in [0,3] \text{ for some } \ell = 0,1,\dots,k-1 \mid \mathcal{C}, \mathcal{C}_0) \\
&= c_2 \times 0.2 \times \mathbb{P}(b_\ell + \Sigma_3 \in [0,3] \text{ for some } \ell = 0,1,\dots,k-1 \mid \mathcal{C}, \mathcal{C}_0) \\
&\ge c_2 \times 0.2 \times \mathbb{P}(\tilde{\Sigma}_3 \in [\tilde{b}_\ell - \varepsilon, \tilde{b}_\ell + \varepsilon] \text{ for some } \ell = 0,1,\dots,k-1 \mid \mathcal{C}, \mathcal{C}_0) \\
&= c_2 \times 0.2 \times \sum_{\ell=0}^{k-1} \mathbb{P}(\Sigma_3 \in [\tilde{b}_\ell - \varepsilon, \tilde{b}_\ell + \varepsilon] \mid \mathcal{C}, \mathcal{C}_0) \ge c_2 \times 0.2 \times k \times \frac{c_1^2}{4k^2} = \frac{c_1^2 c_2}{20k}
\end{aligned}$$

assuming that ε in Corollary 4.2 is sufficiently small. Finally,

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \ge 0.72 \times \mathbb{P}(A_m \mid F_m, E_{m-1}, \mathcal{G}_{m-1}) \ge \frac{0.72\, c_1^2 c_2}{20 k_m} \ge \frac{0.036\, c_1^2 c_2}{m \ln m}$$

the sum of which diverges. Hence, recalling (4.3), we can again apply Lemma 5.1. □

5. Sublinear growth of step sizes

Throughout this Section we assume

$$a_k = \lfloor k^\beta \rfloor, \qquad 0 < \beta < 1.$$

Proposition 3. Let $S(n) = a_1 X_1 + \cdots + a_n X_n$ where $a_k = \lfloor k^\beta \rfloor$, $0 < \beta < 1$. Then

$$\mathbb{P}(|S(n)| = z) \le \frac{\nu}{n^{1/2+\beta}} \qquad \text{for all large } n,$$

for some $\nu > 0$.

Proof: Let $F_n(t) = \prod_{k=1}^n |\cos(t a_k)|$. For all $z \in \mathbb{Z}$ we have

$$\mathbb{P}(S(n) = z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itz}\mathbb{E}e^{itS(n)}\mathrm{d}t \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathbb{E}e^{itS(n)}\right|\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(t)\mathrm{d}t \le \frac{\nu}{n^{1/2+\beta}}$$

by Lemma 5.2 for some $\nu > 0$, for all large n. □

Theorem 7. Suppose that $a_k = \lfloor k^\beta \rfloor$, $0 < \beta < 1$. Then the **a**-walk is a.s. transient.

Proof: In the case $\beta > 1/2$ the a.s. transience follows immediately from Borel-Cantelli lemma and Proposition 3, as $\sum_n \mathbb{P}(|S(n)| \le C) < \infty$ for each $C \ge 1$.

Assume from now on that $0 < \beta \le 1/2$. Define k_m, Δ_m, m_n as in Case 3 of the proof of Lemma 5.2. Fix a positive integer M and consider now only those n for which $m_n = M$. Let $I_M = \{k_M, k_M + 1, \dots, k_{M+1} - 1\}$. Note that the elements of I_M are precisely those n for which $a_n = M$, and that the cardinality of I_M is of order $M^{1/\beta - 1}$. Next, fix some $z \in \mathbb{Z}$ and define

$$E_M = E_M(z) = \{S(n) = z \text{ for some } n \in I_M\}.$$

For each z we will show that $\sum_M \mathbb{P}(E_M) < \infty$, and so by the Borel-Cantelli lemma a.s. only finitely many events E_M occur. Since $S(n)$ takes only integer values, this will imply that the walk is not C-recurrent for any $C \geq 0$.

So, fix z from now on, write $S(n) = S(k_M) + R(n)$ where

$$R(n) = \sum_{i=k_M}^{n} a_i X_i = M \sum_{i=k_M}^{n} X_i,$$

Observe also that $S(k_M)$ and $R(n)$ are independent. In order $S(n) = z$ for some $n \in I_M$, we need that $S(k_M) = z \mod M$. Let $Q = M^{\frac{1}{2\beta}+1-\varepsilon}$ for an $\varepsilon \in (0, 1/2)$. Assuming M is so large that $Q \geq 2|z|$,

$$\begin{aligned}\mathbb{P}(|R(n)| \geq Q - |z|) &\leq 2 \exp\left(-\frac{(Q/2)^2}{2M^2 \cdot (k_{M+1} - k_M)}\right) = 2 \exp\left(-\frac{\beta Q^2}{(8 + o(1))M^2 \cdot M^{\frac{1}{\beta}-1}}\right)\\ &= 2 \exp\left(-\frac{\beta M^{1-2\varepsilon}}{8 + o(1)}\right) \quad \text{for all } n \in I_M\end{aligned}$$

by Lemma 1.1; hence

$$\mathbb{P}(\max_{n \in I_M} |R(n)| \geq Q - |z|) \leq |I_M| \times 2 \exp\left(-\frac{\beta M^{1-2\varepsilon}}{8 + o(1)}\right) =: \alpha_M = O\left(M^{\frac{1}{\beta}-1} e^{-\frac{\beta M^{1-2\varepsilon}}{8+o(1)}}\right),$$

which is summable in M. So,

$$\begin{aligned}\mathbb{P}(E_M) &\leq \mathbb{P}\left(E_M,\ \max_{n \in I_M} |R(n)| < Q - |z|\right) + \mathbb{P}\left(\max_{n \in I_M} |R(n)| \geq Q - |z|\right)\\ &= \mathbb{P}\left(E_M,\ S(k_M) = z \mod M,\ \max_{n \in I_M} |R(n)| < Q - |z|\right) + \alpha_M = (*) + \alpha_M\end{aligned}$$

where the term α_M is summable since $1 - 2\varepsilon > 0$. Since E_M implies implies $-S(k_M) = R(n) - z$ for some $n \in I_M$,

$$\begin{aligned}(*) &\leq \mathbb{P}(|S(k_M)| < Q,\ S(k_M) = z \mod M) = \sum_{j:\ |j|<Q,\ j=z \mod M} \mathbb{P}(|S(k_M)| = j)\\ &\leq \frac{\nu}{k_M^{1/2+\beta}} \times |\{j :\ |j| < Q,\ j = z \mod M\}| \leq \frac{\nu + o(1)}{M^{1+\frac{1}{2\beta}}} \times \left[\frac{2Q + 1}{M} + 1\right] = \frac{(\nu + o(1))}{\pi\, M^{1+\varepsilon}}\end{aligned}$$

by Proposition 3. The RHS is summable in M, which concludes the proof. □

Remark 4. By setting $\varepsilon = 1/2 - \delta/2$, where $\delta > 0$ is very close to zero, we can ensure that

$$\begin{aligned}\mathbb{P}(|S(n)| < M^{1/2-\delta} \text{ for some } n \in I_M) &\leq \sum_{z:|z|<M^{1/2-\delta}} \mathbb{P}(E_M(z)) \leq \left[\frac{\text{const}}{M^{1+\varepsilon}} + \alpha_M\right] \times 2M^{1/2-\delta}\\ &= \frac{2\,\text{const}}{M^{1+\delta/2}} + 2M^{1/2-\delta}\alpha_M\end{aligned}$$

is summable. Hence, a.s. eventually $|S(n)|$ will be larger that $n^{\beta/2-\delta}$ for any $\delta > 0$.

Appendix 1: Modified conditional Borel-Cantelli lemma

Lemma 5.1. *Suppose that we have an increasing sequence of sigma-algebras $\mathcal{G}_m$ and a sequence of $\mathcal{G}_m$-measurable events A_m and E_m such that*

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq \alpha_m, \quad \mathbb{P}(E_m^c) \leq \varepsilon_m \quad a.s.$$

where the non-negative α_n *and* ε_n *satisfy*

$$\sum_m \alpha_m = \infty, \quad \sum_m \varepsilon_m < \infty. \tag{5.1}$$

Then $\mathbb{P}(A_m \text{ i.o.}) = 1$.

Proof: Let $m > \ell \geq 1$ and $B_{\ell,m} = \bigcap_{i=\ell}^m A_i^c$. We need to show that for any $\ell \geq 1$

$$\mathbb{P}(B_{\ell,\infty}) = \mathbb{P}\left(A_\ell^c \cap A_{\ell+1}^c \cap A_{\ell+2}^c \cap \dots\right) = 0. \tag{5.2}$$

We have for $m \geq \ell + 1$

$$\begin{aligned}\mathbb{P}(B_{\ell,m}) &= \mathbb{P}\left(A_m^c \cap B_{\ell,m-1}\right) \leq \mathbb{P}\left(A_m^c \cap B_{\ell,m-1} \cap E_{m-1}\right) + \mathbb{P}\left(E_{m-1}^c\right) \\ &= \mathbb{P}\left(A_m^c \mid B_{\ell,m-1} \cap E_{m-1}\right) \mathbb{P}\left(B_{\ell,n-1} \cap E_{m-1}\right) + \mathbb{P}\left(E_{m-1}^c\right) \\ &\leq \mathbb{P}\left(A_m^c \mid B_{\ell,m-1} \cap E_{m-1}\right) \mathbb{P}\left(B_{\ell,m-1}\right) + \varepsilon_{m-1} \leq (1-\alpha_m)\mathbb{P}\left(B_{\ell,m-1}\right) + \varepsilon_{m-1}.\end{aligned} \tag{5.3}$$

By induction over $m, m-1, m-2, \dots, \ell+1$ in (5.3), we get that

$$\begin{aligned}\mathbb{P}(B_{\ell,m}) &\leq \varepsilon_{m-1} + (1-\alpha_m)\left[(1-\alpha_{m-1})\mathbb{P}\left(B_{\ell,m-2} \mid \mathcal{G}_m\right) + \varepsilon_{m-2}\right] \leq \dots \\ &\leq \varepsilon_{m-1} + (1-\alpha_m)\varepsilon_{m-2} + (1-\alpha_m)(1-\alpha_{m-1})\varepsilon_{m-3} + \dots \\ &+ (1-\alpha_m)(1-\alpha_{m-1})\dots(1-\alpha_{\ell+2})\varepsilon_\ell + (1-\alpha_m)(1-\alpha_{m-1})\dots(1-\alpha_{\ell+1}).\end{aligned}$$

Hence, for any integer $M \in (\ell, m)$

$$\begin{aligned}\mathbb{P}(B_{\ell,m}) \leq\ & \varepsilon_{m-1} + \varepsilon_{m-2} + \dots + \varepsilon_M \\ &+ (1-\alpha_m)(1-\alpha_{m-1})\cdots(1-\alpha_{M+1})\varepsilon_{M-1} + \dots + (1-a_m)(1-\alpha_{m-1})\dots(1-\alpha_{\ell+2})\varepsilon_\ell \\ &+ (1-\alpha_m)(1-\alpha_{m-1})\dots(1-\alpha_{\ell+1}) \\ &\leq [\varepsilon_{m-1} + \varepsilon_{m-2} + \dots + \varepsilon_M] \\ &+ (1-\alpha_m)(1-\alpha_{m-1})\cdots(1-\alpha_{M+1})[\varepsilon_{M-1} + \varepsilon_{M-2} + \dots + \varepsilon_\ell + 1]\end{aligned}$$

Fix any $\delta > 0$. By (5.1) we can find an M be so large that $\sum_{i=M}^\infty \varepsilon_i < \delta/2$. Then, again by (5.1), there exists an $m_0 > M$ such that $\displaystyle\prod_{i=M+1}^{m_0}(1-\alpha_i) < \frac{\delta}{2\left(1 + \sum_{i=\ell}^{M-1}\varepsilon_i\right)}$. Hence, for all $m \geq m_0$ we have $\mathbb{P}(B_{\ell,m}) \leq \delta/2 + \delta/2 = \delta$. Since $\delta > 0$ is arbitrary, and $B_{\ell,m}$ is a decreasing sequence of events in m, we conclude that $\mathbb{P}(B_{\ell,\infty}) = 0$, as required. ☐

Appendix 2: Generalization of Blair Sullivan's results

Let $a_k = \lfloor k^\beta \rfloor$, where $0 < \beta < 1$.

Lemma 5.2. *Let* $F_n(t) = \prod_{k=1}^n |\cos(ta_k)|$. *Then*

$$\int_{-\pi}^{\pi} F_n(t)\mathrm{d}t = \frac{\sqrt{8\pi(1+2\beta)} + o(1)}{n^{\beta+1/2}} \quad \text{as } n \to \infty.$$

Remark 5. Note that for $\beta = 1$ we would have obtained the same result as in Sullivan (2013).

Proof: We will proceed in the spirit of Sullivan (2013). Note that by symmetry

$$\int_{-\pi}^{\pi} F_n(t)\mathrm{d}t = 2\int_0^\pi F_n(t)\mathrm{d}t = 2\int_0^{\pi/2} F_n(t)\mathrm{d}t + 2\int_0^{\pi/2} F_n(\pi - t)\mathrm{d}t = 4\int_0^{\pi/2} F_n(t)\mathrm{d}t$$

since $|\cos((\pi - t)a_k)| = |\cos(\pi a_k - ta_k)| = |\cos(ta_k)|$ as a_k is an integer. Let $\varepsilon > 0$ be very small and define

$$I_0 = \left[0, \frac{1}{n^{\beta+1/2-\varepsilon}}\right], \quad I_1 = \left[\frac{1}{n^{\beta+1/2-\varepsilon}}, \frac{1}{n^\beta}\right], \quad I_2 = \left[\frac{1}{n^\beta}, \frac{c_1}{n^\beta}\right], \quad I_3 = \left[\frac{c_1}{n^\beta}, \frac{\pi}{2}\right].$$

for some $c_1 > 1$ to be determined later. Then

$$\int_0^{\pi/2} F_n(t)\mathrm{d}t = \int_{I_0} F_n(t)\mathrm{d}t + \int_{I_1} F_n(t)\mathrm{d}t + \int_{I_2} F_n(t)\mathrm{d}t + \int_{I_3} F_n(t)\mathrm{d}t.$$

We will show that the contribution of all the integrals, except the first one, is negligible, and estimate the value of the first one.

First, observe that when $0 \le ta_k \le \pi/2$ for all $k \le n$, by the elementary inequality $|\cos u| \le e^{-u^2/2}$ valid for $|u| \le \pi/2$, we have

$$F_n(t) \le \prod_{k=1}^n \exp\left(-\frac{t^2 a_k^2}{2}\right) = \exp\left(-\frac{t^2}{2}\sum_{k=1}^n a_k^2\right) = \exp\left(-\frac{t^2\, n^{2\beta+1}(1+o(1))}{2(1+2\beta)}\right) \tag{5.4}$$

since $a_k^2 = k^{2\beta}(1+o(1))$.

Case 0: $t \in I_0$

Here $ta_k \le \frac{1}{n^{1/2-\varepsilon}} \ll 1$, hence for n large enough

$$\frac{(ta_k)^2}{2} \le -\ln\cos(ta_k) = \frac{(ta_k)^2}{2} + O\left((ta_k)^4\right) \le (1+o(1))\frac{(ta_k)^2}{2}$$

yielding $F_n(t) = \exp\left(-\frac{t^2\, n^{2\beta+1}\rho_{n,t}}{2(1+2\beta)}\right)$ where $\rho_{n,t} = 1+o(1)$ for large n (compare with (5.4)). Since for any $r > 0$ we have

$$\int_0^{n^{-\beta-1/2+\varepsilon}} \exp\left(-\frac{t^2\, n^{2\beta+1} r}{2(1+2\beta)}\right)\mathrm{d}t = \frac{1}{n^{1/2+\beta}}\int_0^{n^\varepsilon} \exp\left(-\frac{s^2\, r}{2(1+2\beta)}\right)\mathrm{d}s$$
$$= \frac{1}{n^{1/2+\beta}}\left[\sqrt{\frac{\pi(1+2\beta)}{2r}} + o(1)\right]$$

where the main term is monotone in r, by substituting $r = \rho_{n,t} = 1+o(1)$ we conclude that

$$\int_{I_0} F_n(t)\mathrm{d}t = \frac{1}{n^{1/2+\beta}}\left[\sqrt{\pi(1/2+\beta)} + o(1)\right].$$

Case 1: $t \in I_1$

Since $ta_k \le 1 < \pi/2$, by (5.4) for some $C_2 > 0$ we have $F_n(t) \le \exp\left(-\frac{n^{2\varepsilon}(1+o(1))}{2(1+2\beta)}\right) \le e^{-C_2 n^{2\varepsilon}}$, so $\int_{I_1} F_n(t)\mathrm{d}t \le e^{-C_2 n^{2\varepsilon}}$, which decays faster than polynomially.

Case 2: $t \in I_2$

As in Case 2 in Sullivan (2013), we will use monotonicity of $F_n(t)$ in n. Let $r = c_1^{-1/\beta} \in (0,1)$ then $\lfloor rn \rfloor^\beta \le (rn)^\beta = \frac{n^\beta}{c_1}$, consequently by (5.4), since $t \le \frac{c_1}{n^\beta}$,

$$F_{\lfloor rn \rfloor}(t) \le \exp\left(-\frac{t^2\,(rn)^{2\beta+1}(1+o(1))}{2(1+2\beta)}\right)$$

and since $F_n(t) \le F_{\lfloor rn \rfloor}(t)$, we get a similar bound as in Case 1.

Case 3: $t \in I_3$

Let

$$k_m = \inf\{k \in \mathbb{Z}_+ :\ k^\beta \ge m\} = \lceil m^{1/\beta} \rceil, \quad m = 1, 2, \dots;$$
$$\Delta_m = k_{m+1} - k_m = \beta^{-1}\, m^\gamma + O(m^{1/\beta-2}) + \rho_0, \qquad \gamma := \frac{1-\beta}{\beta},$$

where $|\rho_0| \le 1$. Then

$$a_k = m \quad \text{if and only if} \quad k \in \{k_m, k_m+1, \dots, k_m+\Delta_m-1\,(\equiv k_{m+1}-1)\}.$$

For $n \in \mathbb{Z}_+$ let

$$m_n = \max\{m:\ k_m \le n\} = n^\beta\,(1+o(1)), \qquad n \in [k_{m_n}, k_{m_n}+\Delta_{m_n}-1].$$

By the inequality between the mean geometric and the mean arithmetic,

$$F_n(t) = \sqrt{\prod_{k=1}^n \cos^2(ta_k)} = \left(\sqrt[n]{\prod_{k=1}^n \cos^2(ta_k)}\right)^{n/2} \le \left(\frac{\sum_{k=1}^n \cos^2(ta_k)}{n}\right)^{n/2}$$
$$= \left(\frac{1}{2} + \frac{U_n(t)}{2n}\right)^{n/2} \qquad \text{where } U_n(t) = \sum_{k=1}^n \cos(2ta_k).$$

We will show that if t is not too small, then for some $0 \le c < 1$ we have $U_n(t) \le cn$ and hence $F_n(t) \le \left(\frac{1+c}{2}\right)^{n/2}$. In order to do that, first note that

$$U_n(t) \le \sum_{k=1}^{k_{m_n}} \cos(2ta_k) + (n-k_{m_n}) = \sum_{m=1}^{m_n} \Delta_m \cos(2tm) + (n-k_{m_n}).$$

Let $r \in (0,1)$ and assume w.l.o.g. that rm_n is an integer. For $m \in [rm_n+1, m_n]$ we have

$$A \le \Delta_m \le \bar{A} \text{ where } A = \beta^{-1}(rm_n)^\gamma + O(1), \quad \bar{A} = \beta^{-1}m_n^\gamma + O(1).$$

Consequently,

$$\sum_{m=rm_n+1}^{m_n} \Delta_m \cos(2tm) \le \sum_{m=rm_n+1}^{m_n} \left[\bar{A}\cdot \mathbf{1}_{\cos(2tm)\ge 0} + A \cdot \mathbf{1}_{\cos(2tm)<0}\right]\cos(2tm)$$
$$= \sum_{m=rm_n+1}^{m_n} \left[\bar{A}-A\right]\cos(2tm)\,\mathbf{1}_{\cos(2tm)\ge 0} + \sum_{m=rm_n+1}^{m_n} A\cos(2tm)$$
$$\le (1-r)m_n\left(\bar{A}-A\right) + A\sum_{m=rm_n+1}^{m_n}\cos(2tm) = (1-r)m_n\left(\bar{A}-A\right)$$
$$+ A\left(\cos^2(rtm_n+t) - \cos^2(tm_n+t) + \frac{\cos t}{2\sin t}\left[\sin(2t(m_n+1)) - \sin(2t(rm_n+1))\right]\right)$$
$$\le \frac{m_n^{1/\beta}}{\beta}(1-r)\left[1-r^\gamma + O(m_n^{-\gamma})\right] + m_n^\gamma\left(\frac{r^\gamma}{\beta} + O(m_n^{-\gamma})\right)\left(1 + \frac{1}{|\sin t|}\right)$$

Hence, since $m_n^{1/\beta} = n + o(n)$,

$$U_n(t) \le \sum_{m=1}^{rm_n}\Delta_m + \sum_{m=rm_n+1}^{m_n}\Delta_m\cos(2tm) + (n-k_{m_n}) = k_{rm_n+1} + \sum_{m=rm_n+1}^{m_n}\Delta_m\cos(2tm) + O(\Delta_{m_n})$$
$$\le r^{1/\beta}n + \frac{m_n^{1/\beta}}{\beta}(1-r)\left[1-r^\gamma+O(m_n^{-\gamma})\right] + m_n^\gamma\left(\frac{r^\gamma}{\beta}+O(m_n^{-\gamma})\right)\left(1+\frac{1}{|\sin t|}\right) + O(m_n^{\gamma\beta})$$
$$\le n\left(r^{1/\beta} + \frac{(1-r)(1-r^\gamma)}{\beta}\right) + \frac{4n^{1-\beta}\beta^{-1}r^\gamma}{|\sin t|} + o(n)$$

Consider now the the function

$$h(r,\beta) := r^{1/\beta} + \frac{(1-r)(1-r^\gamma)}{\beta} = r^{1/\beta} + \frac{(1-r)\left(1-r^{1/\beta-1}\right)}{\beta}$$

and note that

$$h(1-\beta,\beta) = (1-\beta)^{1/\beta} + 1 - (1-\beta)^{1/\beta-1} = 1 - \beta(1-\beta)^{1/\beta-1} \le 1 - e^{-1}\beta < 1 - \beta/3$$

since $\sup_{\beta\in(0,1)}(1-\beta)^{1/\beta-1} = e^{-1}$ by elementary calculus. So if we set $r = 1-\beta \in (0,1)$, by noting $t \le 2\sin t$ for $t \in [0,\pi/2]$, we conclude that $U_n(t) \le \left(1 - \frac{\beta}{4}\right) n$ provided that $t \ge \frac{c_1}{n^\beta}$ for some $c_1 > 0$. Consequently, $\int_{I_3} F_n(t)\mathrm{d}t \le \left(1 - \frac{\beta}{8}\right)^{n/2}$ for large n, which converges to zero exponentially fast. □

Acknowledgment

We thank Edward Crane and Andrew Wade for providing us with useful references.

References

Erdős, P. Extremal problems in number theory. In *Proc. Sympos. Pure Math., Vol. VIII*, pp. 181–189. Amer. Math. Soc., Providence, R.I. (1965). MR0174539.

IMS Bulletin, **51**(2) (March 2022). Available at https://imstat.org/archive-vol-51-2022/.

Keller, N. and Klein, O. Proof of Tomaszewski's conjecture on randomly signed sums. *Adv. Math.*, **407**, Paper No. 108558, 39 (2022). MR4452672.

Montgomery-Smith, S. J. The distribution of Rademacher sums. *Proc. Amer. Math. Soc.*, **109** (2), 517–522 (1990). MR1013975.

Nguyen, H. H. A new approach to an old problem of Erdős and Moser. *J. Combin. Theory Ser. A*, **119** (5), 977–993 (2012). MR2891377.

Proctor, R. A. Solution of two difficult combinatorial problems with linear algebra. *Amer. Math. Monthly*, **89** (10), 721–734 (1982). MR683197.

Sárközi, A. and Szemerédi, E. Über ein Problem von Erdős und Moser. *Acta Arith.*, **11**, 205–208 (1965). MR182619.

Stanley, R. P. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods*, **1** (2), 168–184 (1980). MR578321.

Sullivan, B. D. On a conjecture of Andrica and Tomescu. *J. Integer Seq.*, **16** (3), Article 13.3.1, 6 (2013). MR3033734.

Tao, T. and Vu, V. H. Inverse Littlewood-Offord theorems and the condition number of random discrete matrices. *Ann. of Math. (2)*, **169** (2), 595–632 (2009). MR2480613.