

XX^T matrices with independent entries

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Abstract. Let $S = XX^T$ be the (unscaled) sample covariance matrix where X is a real $p \times n$ matrix with independent entries. It is well known that if the entries of X are independent and identically distributed (i.i.d.) with enough moments and $p/n \rightarrow y \neq 0$, then the limiting spectral distribution (LSD) of $\frac{1}{n}S$ converges to a Marčenko-Pastur law. Several extensions of this result are also known. We prove a general result on the existence of the LSD of S in probability or almost surely, and in particular, many of the above results follow as special cases. At the same time several new LSD results also follow from our general result.

The moments of the LSD are quite involved but can be described via a set of partitions. Unlike in the i.i.d. entries case, these partitions are not necessarily non-crossing, but are related to the special symmetric partitions which are known to appear in the LSD of (generalised) Wigner matrices with independent entries.

We also investigate the existence of the LSD of $S_A = AA^T$ when A is the $p \times n$ symmetric or the asymmetric version of any of the following four random matrices: reverse circulant, circulant, Toeplitz and Hankel. The LSD of $\frac{1}{n}S_A$ for the above four cases have been studied in (Bose et al., 2010) when the entries are i.i.d. We show that under some general assumptions on the entries of A , the LSD of S_A exists and this result generalises the existing results of (Bose et al., 2010) significantly.

1. Introduction

Suppose M_n is an $n \times n$ real symmetric random matrix (i.e., a matrix whose entries are random variables) with (real) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Its *empirical spectral measure* is the random

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probability measure:

$$\mu_{M_n} = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i},$$

where δ_x is the Dirac measure at x . The random probability distribution function, F^{M_n} , known as the *empirical spectral distribution* (ESD) of M_n is given by

$$F^{M_n}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(\lambda_i \leq x).$$

The expectation of the above distribution function, denoted by $\mathbb{E}[F^{M_n}]$, is a non-random distribution function, and is known as the *expected empirical spectral distribution* (EESD). The corresponding probability measure will be denoted by $\mathbb{E}\mu_{M_n}$. The notions of convergence that are used in this article are: (i) the *(weak) convergence of the EESD* $\mathbb{E}\mu_{M_n}$, and (ii) the *(weak) convergence of the ESD* μ_{M_n} (either in *probability* or *almost surely (a.s.)*). The limits in (i) and (ii) are identical when the latter limits are non-random. In any case, any of these limits will be referred to as the *limiting spectral distribution* (LSD) of $\{M_n\}$.

Now let X_p be a $p \times n$ matrix with real independent entries $\{x_{ij,n} : 1 \leq i \leq p, 1 \leq j \leq n\}$, where $p = p(n), p/n \rightarrow y \in (0, \infty)$. The matrix $S = X_p X_p^T$ will be called the *Sample covariance matrix* (without scaling). Note that the entries of X_p are not necessarily identically distributed, and they do not necessarily have identical variances. We will also be interested in the matrix $S_A = A_p A_p^T$ where A_p is any one of the $p \times n$ patterned matrices, namely reverse circulant, circulant, Toeplitz and Hankel, with entries that are real and independent.

We explore the existence of the LSD of S and S_A under suitable conditions on the entries of X_p and A_p . The motivation to work on these problems, along with a brief discussion to relate our results with the models and results that already exist in the literature are given below. Our two main theorems, namely Theorems 2.1 and 2.4, are given in Section 2.

(a) The S matrix is arguably one of the most important matrices in random matrix theory with varied applications in physics, statistics and other areas. There have been several works regarding its LSD. When the entries of X_p are i.i.d. with mean zero and finite fourth moment, (Marčenko and Pastur, 1967) first established the LSD of $\frac{1}{n}S$ and this LSD has been named the Marčenko-Pastur (MP) law. Subsequent works by (Grenander and Silverstein, 1977), (Wachter, 1978), (Yin, 1986), (Belinschi et al., 2009), investigated the existence and properties of the LSD under varied assumptions on the entries. In these works the distribution of the entries of X_p remain unaltered for every n .

It is natural to ask what happens when the distribution of the entries depend on n and/or the entries are not identically distributed. The convergence of the ESD of $\frac{1}{a_n}S$, when the entries of X_p are i.i.d. with heavy tails, and a_n is a sequence of constants related to the tail probability of the entry distribution, was proved in (Belinschi et al., 2009). There, an appropriate truncation of the variables at levels that depend on n was crucial in the arguments. Thus it becomes relevant to probe the case where the distribution of the entries of X_p is allowed to depend on n , not just due to a scaling constant that depends on n , but where a genuine triangular sequence of entries is used.

Such a model was already considered by Zakharevich (Zakharevich, 2006) for the (symmetric) Wigner matrix. For any distribution F , let $\beta_k(F)$ be the k th moment of F . Consider a *generalized Wigner matrix* W_n whose entries are $\{x_{ij,n}; 1 \leq i \leq j \leq n\}$ (with $x_{ij,n} = x_{ji,n}$) with distribution G_n for every fixed n . Assume that,

$$\lim_{n \rightarrow \infty} \frac{\beta_k(G_n)}{n^{k/2-1} \beta_2(G_n)^{k/2}} = g_k, \text{ say, exists for all } k \geq 1. \quad (1.1)$$

Then she proved that the ESD of $\frac{W_n}{\sqrt{n\mu_n(2)}}$ converges in probability to a distribution μ_{zak} that depends only on the sequence $\{g_{2k}\}$. LSD of Wigner matrices with general independent triangular array of entries were explored in (Bose et al., 2022). They found that a class of partitions, the *special symmetric partitions*, play a crucial role in the moments of the LSD.

Matrices whose entries satisfy conditions like (1.1), are referred to as *matrices with exploding moments* and have been considered by several authors ((Benaych-Georges and Cabanal-Duvillard, 2012), (Male, 2017), (Noiry, 2018), (Zitelli, 2022)). In particular, the S matrix with exploding moments have been studied in Theorem 3.2 of (Benaych-Georges and Cabanal-Duvillard, 2012), Proposition 3.1 of (Noiry, 2018), and Theorem 1 of (Zitelli, 2022). Moreover, formulae for the moments of the LSD have been provided, using free probability theory and graph theory respectively.

We establish LSD results for the S matrix (see Theorem 2.1) where the distribution of any entry is allowed to be dependent not only on n but also on its position in the matrix. We describe a formula for the moments of the LSD using certain partitions. We relate these moments not only to the ones that have appeared in (Benaych-Georges and Cabanal-Duvillard, 2012) and (Noiry, 2018), but also to the limiting moments in the (generalised) Wigner case ((Zakharevich, 2006), (Bose et al., 2022))—under our assumptions, only the class of special symmetric partitions contribute to the moments. Incidentally, the moment method that we use, cannot yield the LSD for non-symmetric matrices, see for example, (Bordenave and Chafaï, 2012).

In Section 3, we provide some simulations to show a glimpse of the various distributions that can appear as the LSD. In Section 4.2, we discuss how Theorem 2.1 brings the various results such as (Marčenko and Pastur, 1967), (Belinschi et al., 2009), (Benaych-Georges and Cabanal-Duvillard, 2012), (Noiry, 2018), (Dykema and Haagerup, 2004) etc., under one umbrella, as well as generates some new results. As a special case of Theorem 2.1, the ESD of S with sparse entries converges a.s. (see Section 4.2.4), and we relate this LSD to the free Poisson and Poisson distributions. *Matrices with variance profile* also come under our purview (see Section 4.2.5) and again, under suitable assumptions, the a.s. convergence of the ESD of S holds.

(b) Let us now consider the matrix $S_A = A_p A_p^T$ where A_p has one of the following patterns:

$$T^{(s)} = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_0 & x_1 & \cdots & x_{n-2} \\ x_2 & x_1 & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{p-1} & x_{p-2} & x_{p-3} & \cdots & x_{|p-n|} \end{bmatrix}, \quad T = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_0 & x_1 & \cdots & x_{n-2} \\ x_2 & x_1 & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{p-1} & x_{p-2} & x_{n-3} & \cdots & x_{p-n} \end{bmatrix},$$

$$H^{(s)} = \begin{bmatrix} x_2 & x_3 & x_4 & \cdots & x_{n+1} \\ x_3 & x_4 & x_5 & \cdots & x_{n+2} \\ x_4 & x_5 & x_6 & \cdots & x_{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{p+1} & x_{p+2} & x_{n+3} & \cdots & x_{p+n} \end{bmatrix}, \quad H = \begin{bmatrix} x_2 & x_{-3} & x_{-4} & \cdots & x_{-(n+1)} \\ x_3 & x_4 & x_{-5} & \cdots & x_{-(n+2)} \\ x_4 & x_5 & x_6 & \cdots & x_{-(n+3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{p+1} & x_{p+2} & x_{p+3} & \cdots & x_{-(p+n)} \end{bmatrix},$$

$$R^{(s)} = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_0 \\ x_2 & x_3 & x_4 & \cdots & x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{(p-1) \bmod n} & \cdots & \cdots & \cdots & x_{(p-2) \bmod n} \end{bmatrix},$$

$$R = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_{-1} & x_2 & x_3 & \cdots & x_0 \\ x_{-2} & x_{-3} & x_4 & \cdots & x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{-(p-1) \bmod n} & \cdots & \cdots & \cdots & x_{(p-2) \bmod n} \end{bmatrix},$$

$$C^{(s)} = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_1 \\ x_1 & x_0 & x_1 & \cdots & x_2 \\ x_2 & x_1 & x_0 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n/2-|n/2-p-1|} & \cdots & \cdots & \cdots & x_{n/2-|n/2-p-n|} \end{bmatrix},$$

$$C = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_0 & x_1 & \cdots & x_{n-2} \\ x_2 & x_1 & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{(p-1) \bmod n} & \cdots & \cdots & \cdots & x_{(p-n) \bmod n} \end{bmatrix}.$$

We have dropped the suffix p here for ease of reading. The matrices $T^{(s)}$, $H^{(s)}$, $R^{(s)}$ and $C^{(s)}$ are the rectangular versions of the symmetric Toeplitz, Hankel, reverse circulant and circulant matrices where the (i, j) th entry is equal to the (j, i) th entry whenever $1 \leq i, j \leq \min(p, n)$. The matrices T, H, R and C are the *asymmetric* versions of these matrices. These matrices can also be described via *link functions* (see Section 5.1). (Bose et al., 2010) showed that when the entries of A_p come from a single i.i.d. sequence with mean zero and variance 1, the ESD of $\frac{1}{n}S_A$ converges a.s. to a non-random probability distribution. We generalise this result by allowing the distribution of the entries to vary with n as well as with their positions in the matrix.

Such a model for *symmetric patterned matrices* such as reverse circulant, circulant, Toeplitz and Hankel was considered in (Bose et al., 2021).

Under assumptions on A_p similar to those used for X , the EESD of S_A converges. In particular, this convergence holds for the special cases when A_p is a triangular, a band or a block matrix, has a variance profile or is a matrix with exploding moments. Illustration of the variety of distributions that appear as the LSD of S_A is given in Section 3.

Theorem 2.4 claims the convergence only of the EESD. The a.s. or in probability convergence of the ESD to a non-random probability measure is not true in general. Simulation given in Figures 6.2 and 6.3 confirm this. This is very different from the case of the S matrix. In particular for the sparse case, the ESD does not converge to a non-random limit. This is similar to the phenomenon observed by Banerjee and Bose (2017) for certain symmetric patterned sparse random matrices. Of course, as mentioned above, in special cases the a.s. convergence can hold, see (Bose et al., 2010).

We find some relationships between the LSDs of $S_{R^{(s)}}$, S_C , S_T and $S_{H^{(s)}}$. For instance, when the entries are i.i.d. for every n and have exploding moments, the LSDs of S_T and $S_{H^{(s)}}$ are identical; so are the LSDs of S_C and $S_{R^{(s)}}$. In Section 5.3, we discuss the connection of our theorem to some existing results.

2. Main results

The notion of *multiplicative extension* is required to describe our results. Let $[k] := \{1, 2, \dots, k\}$ and let $\mathcal{P}(k)$ denote the set of all partitions of $[k]$. Let $\mathcal{P}_2(2k)$ be the set of *pair-partitions* of $[2k]$. Suppose $\{c_k, k \geq 1\}$ is any sequence of numbers. Its multiplicative extension is defined on $\mathcal{P}(k)$,

$k \geq 1$ as follows. For any $\sigma \in \mathcal{P}_k$, define

$$c_\sigma = \prod_{V \text{ is a block of } \sigma} c_{|V|}.$$

Consider the matrix $S = X_p X_p^T$, where the entries of X_p are given by the bi-sequence $\{x_{ij,n}\}$. We drop the suffix n and p for convenience wherever there is no scope for confusion. For any real-valued function g on $[0, 1]$, $\|g\| := \sup_{0 \leq x \leq 1} |g(x)|$ will denote its sup norm. We introduce the following assumptions on the entries $\{x_{ij}\}$.

Assumption A. There exists a sequence $\{t_n\}$ with $t_n \in [0, \infty]$ such that

(i) For each $k \in \mathbb{N}$,

$$n \mathbb{E} \left[x_{ij}^{2k} \mathbf{1}_{\{|x_{ij}| \leq t_n\}} \right] = g_{2k,n} \left(\frac{i}{p}, \frac{j}{n} \right) \quad \text{for } 1 \leq i \leq p, 1 \leq j \leq n, \quad (2.1)$$

$$\lim_{n \rightarrow \infty} n^\alpha \sup_{1 \leq i \leq p, 1 \leq j \leq n} \mathbb{E} \left[x_{ij}^{2k-1} \mathbf{1}_{\{|x_{ij}| \leq t_n\}} \right] = 0 \quad \text{for all } \alpha < 1, \quad (2.2)$$

where $\{g_{2k,n}\}$ is a sequence of bounded Riemann integrable functions on $[0, 1]^2$.

(ii) The functions $g_{2k,n}(\cdot), n \geq 1$ converge uniformly to $g_{2k}(\cdot)$ for each $k \geq 1$.

(iii) With $M_{2k} = \|g_{2k}\|$, $M_{2k-1} = 0$ for all $k \geq 1$, the sequence $\alpha_k = \sum_{\sigma \in \mathcal{P}(2k)} M_\sigma$ satisfies *Carleman's condition*,

$$\sum_{k=1}^{\infty} \alpha_{2k}^{-\frac{1}{2k}} = \infty.$$

All of these conditions are naturally satisfied by well-known models such as, the *standard i.i.d.* case where the entries of X are $\frac{x_{ij}}{\sqrt{n}}$ with $\{x_{ij}\}$ being i.i.d. with zero mean and finite variance, and the *sparse* case where entries of X are i.i.d, $Ber(p_n)$ with $np_n \rightarrow \lambda > 0$, for every n . We will discuss this in more details in Section 2.1. Now we state our first result.

Theorem 2.1. *Let X be a $p \times n$ real matrix with independent entries $\{x_{ij,n}; 1 \leq i \leq p, 1 \leq j \leq n\}$ that satisfy Assumption A and $p/n \rightarrow y \in (0, \infty)$ as $n \rightarrow \infty$. Suppose Z is the $p \times n$ real matrix whose entries are $y_{ij} = x_{ij} \mathbf{1}_{\{|x_{ij}| \leq t_n\}}$. Then*

(a) *The ESD of $S_Z = ZZ^T$ converges a.s. to a probability measure μ say, whose moments are determined by the functions $\{g_{2k}\}_{k \geq 1}$ (as described in (4.35)).*

(b) *Moreover, if*

$$\frac{1}{n} \sum_{i=1}^p \sum_{j=1}^n x_{ij}^2 \mathbf{1}_{\{|x_{ij}| > t_n\}} \rightarrow 0, \quad \text{a.s. (or in probability),} \quad (2.3)$$

then the ESD of $S = XX^T$ converges a.s. (or in probability) to the probability measure μ given in (a).

Since the distribution of the (i, j) th entry of X can depend on n as well as i, j , the limiting moments are more involved than the i.i.d. case. These moment formulae are thus developed during the proof and are given in (4.35).

Remark 2.2. While the MP_y law has bounded support, that is not necessarily the case for μ in Theorem 2.1. Suppose the entries of X satisfy Assumption A. Let for every $m \geq 1$, $f_{2m}(x) = \int_{[0,1]} g_{2m}(x, y) dy$. Now suppose that there exist an $m > 1$ such that $\inf_{t \geq 1} \int_{[0,1]} \left(\frac{f_{2m}(x)}{m!} \right)^t dx = c > 0$.

Then the LSD μ in Theorem 2.1 has unbounded support.

This has implications on the partition description of moments. As is known, the moments of MP_y can be described via the set of non-crossing pair partitions. In the present case, these partitions

are not enough to describe μ and we need a much bigger set of partitions. This will be discussed in details later in Section 4.3.

Remark 2.3. It is known that if Y follows the MP_1 law and Y' follows the semi-circle law, then $Y \stackrel{\mathcal{D}}{=} Y'^2$. A similar result holds for μ . Suppose $p/n \rightarrow 1$, and the entries of X satisfy Assumption A and (2.3). Then the ESD of S converges a.s. to a probability distribution μ as given in Theorem 2.1. At the same time, consider the (generalised) Wigner matrix (i.e., a symmetric matrix) with independent entries $\{x_{ij,n}; 1 \leq i \leq j \leq n\}$ that satisfy the conditions (similar to Assumption A here) of Theorem 2.1 in (Bose et al., 2022). Then its ESD converges a.s. surely to a symmetric probability measure μ' . The two measures μ and μ' are connected. Suppose Y and Y' are two random variables such that $Y \sim \mu$ and $Y' \sim \mu'$. If $\{g_{2k}\}_{k \geq 1}$ are symmetric functions, i.e., $g_{2k}(x, y) = g_{2k}(y, x)$ for all $x, y \in [0, 1]$, then, $Y \stackrel{\mathcal{D}}{=} Y'^2$. This is proved in Section 4.3.1.

Now, we shall consider the matrices S_A , where the entries of A_p are constructed from the sequence of random variables $\{x_{i,n}; -(n+p) \leq i \leq (n+p)\}$. We will denote A_p by A and write x_i for $x_{i,n}$. Recall that as $p, n \rightarrow \infty$, $p/n \rightarrow y \in (0, \infty)$.

Assumption B. Suppose there exists a sequence $\{t_n\}$ with $t_n \in [0, \infty]$ such that

(i) for each $k \in \mathbb{N}$,

$$n \mathbb{E} \left[x_i^{2k} \mathbf{1}_{\{|x_i| \leq t_n\}} \right] = f_{2k,n} \left(\frac{i}{n} \right) \quad \text{for } -(n+p) \leq i \leq n+p, \quad (2.4)$$

$$\lim_{n \rightarrow \infty} n^\alpha \sup_{0 \leq i \leq n-1} \mathbb{E} \left[x_i^{2k-1} \mathbf{1}_{\{|x_i| \leq t_n\}} \right] = 0 \quad \text{for all } \alpha < 1, \quad (2.5)$$

where $\{f_{k,n}; 0 \leq k \leq n\}$ is a sequence of bounded and integrable functions on $[-(1+y), 1+y]$.

(ii) For each $k \geq 1$, $f_{2k,n}, n \geq 1$ converge uniformly to a function f_{2k} .

(iii) Let $M_{2k} = \|f_{2k}\|$ (where $\|\cdot\|$ denotes the sup norm) and $M_{2k-1} = 0$ for all $k \geq 1$. Then $\alpha_k = \sum_{\sigma \in \mathcal{P}(2k)} M_\sigma$ satisfy Carleman's condition,

$$\sum_{k=1}^{\infty} \alpha_{2k}^{-\frac{1}{2k}} = \infty.$$

As we will see in Section 5.3, these assumptions are naturally satisfied by various well-known models. Now we state our second result.

Theorem 2.4. Suppose A is one of the eight rectangular matrices $T^{(s)}, T, H^{(s)}, H, R^{(s)}, R, C^{(s)}, C$ described in Section 1, with entries $\{x_i\}$ which are independent and satisfy Assumption B. Let Z_A be the $p \times n$ (truncated) matrix with entries $y_l = x_l \mathbf{1}_{\{|x_l| \leq t_n\}}$. Then the EESD of $S_{Z_A} = Z_A Z_A^T$ converges weakly to a probability measure μ_A say, whose moment sequence is determined by the functions f_{2k} , $k \geq 1$, in each of the eight cases. Further if

$$\sum_l \mathbb{E}[x_l^2 \mathbf{1}_{\{|x_l| > t_n\}}] \rightarrow 0, \quad (2.6)$$

then the EESD of $S_A = AA^T$ also converges weakly to μ_A .

Remark 2.5. The a.s. or in probability convergence of the ESD to the limit μ_A does not hold in general. The reason is that, unlike X for the S matrix, where each entry appears exactly once, there is a significant structural dependence among the entries of A , and each entry appears $\mathcal{O}(n)$ times. Simulations given in Figures 6.2 and 6.3 also confirm the non-convergence in the a.s. sense. In particular there is no a.s. convergence in the sparse case. A similar phenomenon occurs for the sparse symmetric patterned matrices (see (Banerjee and Bose, 2017)). However, a.s. convergence can hold in special cases, for example in the fully i.i.d. case.

Remark 2.6. In the fully i.i.d. case (both for S and S_A matrices), if the entries are complex random variables, many of the arguments in the real case continue to hold since only the first two moments are involved. Indeed, it is known that due to cancellations, some of the arguments are actually easier. However, in the present case, due to Assumptions A and B, the higher moments are involved too and that leads to substantial additional difficulties, and we decided to restrict attention to only the real case. The complex case can be pursued in future.

3. Simulations

The LSDs cannot be universal and a variety of limit distributions are possible. In Figure 6.1, we see the diversity of the LSDs for the S matrix. Moreover, even though μ_S converges a.s. to μ , as noted in Remark 2.5, μ_{S_A} does not converge a.s. to μ_A in general. However, the a.s. convergence can hold in special cases. For instance when the entries of A are $\frac{x_i}{\sqrt{n}}$ with x_i s i.i.d. with mean zero finite variance, then it is well-known that the ESDs of S_A do converge a.s. to non-random probability measures (see (Bose et al., 2010)). Figure 6.2 and 6.3 illustrates that a.s. convergence of μ_{S_A} holds when the entries of A are $N(0, 1)/\sqrt{n}$, and fails when the entries of A are $Ber(3/n)$ for every fixed n . Since the entries of A need to be only independent, matrices with variance profile serve as natural examples in demonstrating the diversity of the limit distributions. In Figure 6.4 we give some simulated μ_{S_A} when A obeys a variance profile.

4. Details for the S matrix

We begin with some preliminaries that are required in the proofs in Section 4.1. Then in Section 4.2, we discuss how Theorem 2.1 is applicable when specific features such as i.i.d, heavy-tails, triangular (size dependent entries), sparsity, variance profile are present in the model. We give a detailed proof of Theorem 2.1 in Section 4.3. Finally, in Section 4.4 we connect our moment formula for the LSD with the moment formulae known in the literature that are based on hypergraphs and words.

4.1. *Preliminaries.* We first briefly introduce the language of link functions, circuits, words etc. in the context of the S matrix that we shall use heavily. For more details of these concepts, please refer to Section 3 of (Bose et al., 2022) and Section 4 in (Bose et al., 2021).

Multiplicative extension: Let $[k]$ denote the set $\{1, 2, \dots, k\}$ and $\mathcal{P}(k)$ denote the set of all partitions of $[k]$. Suppose $\{c_n\}_{n \geq 1}$ is any sequence of real numbers. Its *multiplicative extension* is defined on the set of all partitions $\mathcal{P}(k)$ of $[k]$, $k \geq 1$ as follows. For any partition $\sigma \in \mathcal{P}(k)$, define

$$c_\sigma = \prod c_{|V|},$$

where the product is taken over all blocks V of the partition σ and $|V|$ denotes the cardinality of the set V .

Link function: The link function for S is given by a pair of functions as follows.

$$L1(i, j) = (i, j) \quad \text{and} \quad L2(i, j) = (j, i).$$

Circuits and Words: In case of the S matrix, a **circuit** π is a function $\pi : \{0, 1, 2, \dots, 2m\} \rightarrow \{1, 2, 3, \dots, \max(p, n)\}$ with $\pi(0) = \pi(2m)$ and $1 \leq \pi(2i) \leq p, 1 \leq \pi(2i - 1) \leq n$ for $1 \leq i \leq m$. We say that the *length* of π is $2m$ and denote it by $\ell(\pi)$. Next, let

$$\begin{aligned} \xi_\pi(2i - 1) &= L1(\pi(2i - 2), \pi(2i - 1)), 1 \leq i \leq k, \\ \xi_\pi(2i) &= L2(\pi(2i - 1), \pi(2i)), 1 \leq i \leq k. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}[\mathrm{Tr}(S^k)] &= \mathbb{E}[\mathrm{Tr}(XX^*)^k] \\ &= \sum_{\pi: \ell(\pi)=k} x_{L1(\pi(0), \pi(1))} x_{L2(\pi(1), \pi(2))} \cdots x_{L2(\pi(2k-1), \pi(2k))} = \sum_{\pi: \ell(\pi)=2k} \mathbb{E}[Y_\pi], \end{aligned} \quad (4.1)$$

where $Y_\pi = \prod_{i=1}^k y_{\xi_{\pi(2i-1)}} y_{\xi_{\pi(2i)}}$.

From (4.1), observe that the k th moment of an entry of S involves the $2k$ th moment of the entries of X . Hence the circuits that are required to deal with the k th moment of the S matrix are of length $2k$.

For any π , the values $Lt(\pi(i-1), \pi(i)), t = 1, 2$ will be called *edges* or *L-values*. When an edge appears more than once in a circuit π , then it is called *matched*. Any m circuits $\pi_1, \pi_2, \dots, \pi_m$ are said to be *jointly-matched* if each edge occurs at least twice across all circuits. They are said to be *cross-matched* if each circuit has an edge which occurs in at least one of the other circuits. Circuits π_1 and π_2 are said to be *equivalent* if

$$Lt(\pi_1(i-1), \pi_1(i)) = Lt(\pi_1(j-1), \pi_1(j)) \iff Lt(\pi_2(i-1), \pi_2(i)) = Lt(\pi_2(j-1), \pi_2(j)), t = 1, 2.$$

The above is an equivalence relation on $\{\pi : \ell(\pi) = k\}$. Any equivalence class of circuits can be indexed by an element of $\mathcal{P}(k)$. The positions where the edges match are identified by each block of a partition of $[k]$. Also, an element of $\mathcal{P}(k)$ can be identified with a *word* of length k of letters. Given a partition, we represent the integers of the same partition block by the same letter, and the first occurrence of each letter is in alphabetical order and vice versa. For example, the partition $\{\{1, 5\}, \{2, 3, 4\}\}$ of $[5]$ corresponds to the word *abbba*. On the other hand, the word *abcccaa* represents the partition $\{\{1, 6, 7\}, \{2\}, \{3, 4, 5\}\}$ of $[7]$. A typical word will be denoted by ω and its i -th letter as $\omega[i]$.

The class $\Pi_S(\omega)$: For a given word ω , this is the set of all circuits which correspond to ω . For any word ω , $\omega[i] = \omega[j] \iff \xi_\pi(i) = \xi_\pi(j)$. This implies

$$\begin{aligned} Lt(\pi(i-1), \pi(i)) &= Lt(\pi(j-1), \pi(j)) \quad \text{if } i \text{ and } j \text{ are of same parity, } t = 1, 2, \\ Lt(\pi(i-1), \pi(i)) &= Lt'(\pi(j-1), \pi(j)) \quad \text{if } i \text{ and } j \text{ are of different parity, } t, t' \in \{1, 2\}, t \neq t'. \end{aligned}$$

Therefore the class $\Pi_S(\omega)$ is given as follows:

$$\begin{aligned} \Pi_S(\omega) &= \{\pi; \omega[i] = \omega[j] \iff \xi_\pi(i) = \xi_\pi(j)\} \\ &= \left\{ \pi : \omega[i] = \omega[j] \iff \begin{array}{l} (\pi(i-1), \pi(i)) = (\pi(j-1), \pi(j)) \quad \text{or} \\ (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1)) \end{array} \right\}. \end{aligned} \quad (4.2)$$

From (4.1) observe that,

$$\lim_{p \rightarrow \infty} \frac{1}{n} \mathbb{E}[\mathrm{Tr}(S^k)] = \lim_{n \rightarrow \infty} \frac{1}{p} \sum_{\pi: \ell(\pi)=2k} \mathbb{E}[Y_\pi] = \lim_{p \rightarrow \infty} \sum_{b=1}^k \sum_{\omega \text{ matched of length } 2k \text{ with } b \text{ distinct letters}} \frac{1}{p} \sum_{\pi \in \Pi_S(\omega)} \mathbb{E}(Y_\pi). \quad (4.3)$$

Note that all words that appear above are of length $2k$. For every $k \geq 1$, the words of length $2k$ corresponding to the circuits of S and generalised Wigner matrix, W , are related (see Observation 1 below). We will find a connection between the k th moments of the LSD of S and $2k$ th moments of the LSD of the generalised Wigner matrix. We will also discover that the partitions that contribute to the latter plays a crucial role in the former.

Let $L_W(i, j) = (\min(i, j), \max(i, j))$ denote the link function of the Wigner matrix. For words corresponding to L_W , the class $\Pi_W(\omega)$ is given by

$$\begin{aligned} \Pi_W(\omega) &= \left\{ \pi : \omega[i] = \omega[j] \Leftrightarrow L_W(\pi(i-1), \pi(i)) = L_W(\pi(j-1), \pi(j)) \right\} \\ &= \left\{ \pi : \omega[i] = \omega[j] \Leftrightarrow \begin{array}{l} (\pi(i-1), \pi(i)) = (\pi(j-1), \pi(j)) \text{ or} \\ (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1)) \end{array} \right\}. \end{aligned} \quad (4.4)$$

Next, we make a key observation about $\Pi_S(\omega)$ and $\Pi_W(\omega)$.

Observation 1: Let $\tilde{\Pi}_W(\omega)$ be the possibly larger class of circuits for the Wigner Link function with range $1 \leq \pi(i) \leq \max(p, n), 0 \leq i \leq 2k$. Then for a word ω of length $2k$,

$$\Pi_S(\omega) \subset \tilde{\Pi}_W(\omega). \quad (4.5)$$

Now we recall the definition of special symmetric words from (Bose et al., 2022). Towards that, we first define the following:

Pure block of a word: Any string of length m ($m > 1$) of same letter in a word will be called a pure block of size m . For example, in the word $aabcddeeeeb$, a , d and e appear in pure blocks of sizes 2, 4 and 3 respectively.

Special Symmetric word: A word ω is *special symmetric* if between two successive appearances of any letter, each of the other letters can appear an even number of times.

For example, $abbccaabba$ is a special symmetric word of length 10 with 3 distinct letters. Notice that between successive appearances of a , b and c appear even number of times. Again, between successive appearances of b , a and c appear even number of times. It is to be note that this is a crossing word.

$abcabc$ is not a special symmetric word as between two successive appearances of a , b and c appear odd number of times.

This definition first appeared in (Bose et al., 2022) with the following additional conditions:

- (a) between successive appearances of any letter, each of the other letters can appear even number of times,
- (b) the last new letter appears in pure blocks of even sizes, and
- (c) each of the other letters appears an equal number of times in the odd and even positions.

Later in (Pernici, 2021), the author showed that Condition (a) actually implies both Conditions (b) and (c).

We denote the set of all special symmetric partitions of $[k]$ by $SS(k)$, and its subset where each partition has b distinct blocks by $SS_b(k)$. Clearly $SS(k) = \emptyset$ when k is odd. Let $\mathcal{P}_2(2k)$ and $NC_2(2k)$ be respectively the set of pair-partitions and non-crossing pair-partitions of $[2k]$. Then it is easy to check that

$$(SS(2k) \cap \mathcal{P}_2(2k)) = NC_2(2k) \subset SS(2k). \quad (4.6)$$

Next, recall the definition of *generating and non-generating vertices* from (Bose et al., 2022) and (Bose et al., 2021).

Definition 4.1. If π is a circuit then any $\pi(i)$ will be called a *vertex*. This vertex is *generating* if $i = 0$ or $\omega[i]$ is the first occurrence of a letter in the word ω corresponding to π . All other vertices are *non-generating*.

For example, for the word $abbc$, $\pi(0), \pi(1), \pi(2)$ and $\pi(4) = \pi(0)$ are generating. For the word $aaaa$, $\pi(0)$ and $\pi(1)$ are generating.

Even and odd generating vertices: A generating vertex $\pi(i)$ is called even (odd) if i is even (odd). Any word has at least one of each, namely $\pi(0)$ and $\pi(1)$. So for a matched word with $b(\leq k/2)$ distinct letters, there can be $(r+1)$ even generating vertices where $0 \leq r \leq b-1$.

Observe that

$$|\Pi_S(\omega)| = \left| \left\{ (\pi(0), \pi(1), \dots, \pi(2k)) : 1 \leq \pi(2i) \leq p, 1 \leq \pi(2i-1) \leq n \text{ for } i = 0, 1, \dots, k, \right. \right. \\ \left. \left. \pi(0) = \pi(2k), \quad \xi_\pi(i) = \xi_\pi(j) \text{ if and only if } \omega[i] = \omega[j] \right\} \right|, \quad (4.7)$$

where $|A|$ denotes the cardinality of A . Circuits corresponding to a word ω are completely determined by the generating vertices. $\pi(0)$ is always generating, and there is one generating vertex for each new letter in ω . So, if ω has b distinct letters then it has $(b+1)$ generating vertices. Hence the growth of $|\Pi_S(\omega)|$ is determined by the number of generating vertices that can be chosen freely. For some words, depending on the link function and the nature of the word, some of these vertices may not have a free choice, that is, some of the generating vertices might be a linear combination of the other generating vertices. In any case, as $p/n \rightarrow y > 0$,

$$|\Pi_S(\omega)| = \mathcal{O}(p^{r+1}n^{b-r}) \text{ whenever } \omega \text{ has } b \text{ distinct letters and } (r+1) \text{ even generating vertices.} \quad (4.8)$$

The existence of

$$\lim_{p,n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{p^{r+1}n^{b-r}} \quad (4.9)$$

for every word ω is tied very intimately to the LSD of S . To see this observe that if the variables are centered (see (4.1)),

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{1}{n} \mathbb{E}[\text{Tr}(S_Z)^k] &= \lim_{n \rightarrow \infty} \frac{1}{p} \sum_{\pi: \ell(\pi)=2k} \mathbb{E}[Y_\pi] \\ &= \lim_{p \rightarrow \infty} \sum_{b=1}^k \sum_{\substack{\omega \text{ matched} \\ \text{with } b \text{ distinct letters}}} \frac{1}{p} \sum_{\pi \in \Pi_S(\omega)} \mathbb{E}(Y_\pi) \\ &= \lim_{p \rightarrow \infty} \sum_{b=1}^k \sum_{r=0}^b \sum_{\substack{\omega \text{ matched} \\ \text{with } b \text{ distinct letters} \\ \text{and } (r+1) \text{ even generating vertices}}} \frac{1}{p} \sum_{\pi \in \Pi_S(\omega)} \mathbb{E}(Y_\pi). \end{aligned} \quad (4.10)$$

As the entries x_{ij} are independent, $\mathbb{E}[Y_\pi] = \prod_{l=1}^b \mathbb{E}[x_{(\pi(l-1), \pi(l))}^{k_l}]$, where $(\pi(l-1), \pi(l))$ denotes the distinct L -values corresponding to each distinct letter in ω and $k_l, 1 \leq l \leq b$ denotes the number of times the L -value $(\pi(l-1), \pi(l))$ appears in ω . Now from (4.8), it can be seen that $\lim_{p,n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{p^{r+1}n^{b-r}}$ determines whether or not all generating vertices of ω have free choice. The words that have free choice for all of its generating vertices will contribute positively to the limiting moments. Therefore, from (4.10), it is easy to see that $\lim_{n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{p^{r+1}n^{b-r}}$ will determine when ω can have a positive contribution to the limiting moments.

In the next section, we identify for which words the above limit is positive for S .

We shall use the the *Lévy metric*. Let F and G be two distribution functions. Then the *Lévy distance* between F and G is given by

$$L(F, G) = \inf\{\epsilon : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon\}.$$

It is well-known that if $\{\mu_n\}$ and μ are probability measures, then as $n \rightarrow \infty$, $L(\mu_n, \mu) \rightarrow 0$ implies μ_n converges to μ .

The next lemma is a well-known result that is useful in the proof of Theorem 2.1. For a proof see Corollary A.42 in (Bai and Silverstein, 2010).

Lemma 4.2. *Suppose A and B are real $p \times n$ matrices and F^{S_A} and F^{S_B} denote the ESDs of AA^T and BB^T respectively. Then the Lévy distance, L between the distributions F^{S_A} and F^{S_B} satisfy the following inequality:*

$$L^4(F^A, F^B) \leq \frac{2}{p^2} (\text{Tr}(AA^T + BB^T)) (\text{Tr}[(A - B)(A - B)^T]). \quad (4.11)$$

Next, we state the following elementary result that helps us conclude the a.s. convergence of the ESD of matrices. See Section 1.2 of (Bose, 2018) for a proof.

Lemma 4.3. *Suppose A_n is any sequence of symmetric $n \times n$ random matrices such that the following conditions hold:*

- (i) *For every $k \geq 1$, $\frac{1}{n} \mathbb{E}[\text{Tr}(A_n)^k] \rightarrow \alpha_k$ as $n \rightarrow \infty$.*
- (ii) *$\sum_{n=1}^{\infty} \frac{1}{n^4} \mathbb{E}[\text{Tr}(A_n^k) - \mathbb{E}(\text{Tr}(A_n^k))]^4 < \infty$ for every $k \geq 1$.*
- (iii) *The sequence $\{\alpha_k\}$ is the moment sequence of a unique probability measure μ .*

Then μ_{A_n} converges to μ weakly a.s.

Condition (i) and (ii) of Lemma 4.3 will be referred to as the *first moment condition* and the *fourth moment condition*, respectively.

4.2. Relation to existing results on the S Matrix. In this section we discuss S matrices where the entries of the matrix X are– (a) fully i.i.d. with finite mean and variance (Section 4.2.1), (b) fully i.i.d. with heavy tails (Section 4.2.2), (c) triangular i.i.d. (Section 4.2.3), (d) sparse i.e., i.i.d. $Ber(p_n)$ (Section 4.2.4) and (e) have non-trivial variance structure (Section 4.2.5).

4.2.1. I.I.D. entries with finite mean and variance. Suppose $X = ((x_{ij}/\sqrt{n}))$ where $\{x_{ij}\}$ are i.i.d. with distribution F which has mean zero and variance 1. It is known that μ_S converges a.s. to MP_y . For a brief history and precursors of this result, see (Bose, 2018) and (Bai and Silverstein, 2010). Here, we show how this result follows as a special case of Theorem 2.1.

First, let us verify that the conditions of Assumption A are satisfied in this case. Towards that, let $t_n = n^{-1/3}$. Using the same line of reasoning as in Section 5.1 (a) in (Bose et al., 2022), it follows that $g_2 \equiv 1$ and $g_{2k} \equiv 0, k > 1$. Thus $M_2 = 1, M_{2k} = 0, k \geq 2$ (see (iii) in Assumption A) and $\alpha_k = \sum_{\sigma \in \mathcal{P}(2k)} 1$ clearly satisfies Carleman’s condition. Now for any $t > 0$,

$$\begin{aligned} \frac{1}{p} \sum_{i,j} (x_{ij}/\sqrt{n})^2 [\mathbf{1}_{\{|x_{ij}/\sqrt{n}| > t_n\}}] &= \frac{1}{np} \sum_{i,j} x_{ij}^2 [\mathbf{1}_{\{|x_{ij}| > t_n \sqrt{n}\}}] \\ &\leq \frac{1}{np} \sum_{i,j} x_{ij}^2 [\mathbf{1}_{\{|x_{ij}| > t\}}] \text{ for all large } n, \\ &\xrightarrow{\text{a.s.}} \mathbb{E}[x_{11}^2 [\mathbf{1}_{\{|x_{11}| > t\}}]]. \end{aligned}$$

As $\mathbb{E}[x_{11}^2] = 1$, taking t to infinity, the above limit is 0 a.s. Hence applying Theorem 2.1 and using formula (4.35), the ESD of S converges a.s. to μ whose k -th moment is given by

$$\beta_k(\mu) = \sum_{r=0}^{k-1} \sum_{\substack{\pi \in SS_k(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r = \sum_{r=0}^{k-1} \sum_{\substack{\pi \in NC_2(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r. \quad (4.12)$$

The last equality follows from (4.6). Now the number of pair-matched words of length $2k$ with $r + 1$ even generating vertices is shown to be $\frac{1}{r+1} \binom{k}{r} \binom{k-1}{r}$ in Theorem 5(a) of (Bose and Sen, 2008). Hence the rhs of the above equation reduces to the k th moment of the MP_y law. Hence μ_S converges to

the MP_y law a.s. More details about how $NC_2(2k)$ and $NC(k)$ play crucial parts in describing the moments of the MP_y law can be found in Lemma 8.2.2 and Remark 8.2.1 in (Bose, 2021).

4.2.2. Heavy-tailed entries. Suppose $\{x_{ij}, 1 \leq i \leq p, 1 \leq j \leq n\}$ are i.i.d. with an α -stable distribution ($0 < \alpha < 2$) and $n/p \rightarrow \gamma \in (0, 1]$. Let $X = ((x_{ij}/a_p))$ where

$$a_p = \inf \left\{ u : \mathbb{P}[|x_{ij}| \geq u] \leq \frac{1}{p} \right\}.$$

The existence of LSD of S has been proved in (Belinschi et al., 2009) using Stieltjes transform. Theorem 2.1 may be used to give an alternative proof. We recall that for the Wigner matrix with heavy tailed entries, a proof using truncation and moments is available in (Bose et al., 2022). That proof can be easily adapted here. For a fixed constant B , let $X^B = ((\frac{x_{ij}}{a} \mathbf{1}_{|x_{ij}| \leq Ba}))$. Then X^B satisfies Assumption A. Hence from Theorem 2.1, the ESD of $S^B = X^B (X^B)^T$, converges a.s., to say μ_B . The rest of the arguments are as in Section 5.2 of (Bose et al., 2022). Thus μ_S converges to $\tilde{\mu}$ in probability and yields the convergence in Theorem 1.10 of (Belinschi et al., 2009).

4.2.3. Triangular i.i.d. (size dependent matrices). Suppose $\{x_{ij,n}; 1 \leq i \leq p, 1 \leq j \leq n\}$ is a sequence of i.i.d. random variables with distribution F_n that has finite moments of all orders, for every n . Also assume that for every $k \geq 1$,

$$n\beta_k(F_n) \rightarrow C_k \tag{4.13}$$

where $\beta_k(F_n)$ denotes the k th moment of F_n . Suppose $\{0, C_2, 0, C_4, \dots\}$ is the cumulant sequence of a random variable T such that its moment sequence satisfies Carleman's condition.

Remark 4.4. The condition (4.13) is equivalent to the statement that $\sum_{i=1}^n a_{i,n}$, where $a_{i,n}$ are i.i.d. F_n converges to some limit distribution F , whose cumulants are $\{C_k\}$. In particular, if F is infinitely divisible, then the existence of such variables $\{a_{i,n}\}$ are guaranteed. See, p.766 (Characterization 1) in (Bose et al., 2002). *Sample Lévy Covariance ensemble* in (Zitelli, 2022) serves as a specific example of such matrices.

Let $X = ((x_{ij,n}))$ and $p/n \rightarrow y > 0$. Condition (4.13) implies that Assumption A holds with $t_n = \infty$ and $g_{2k} \equiv C_{2k}, k \geq 1$. Therefore by Theorem 2.1, μ_S converges a.s. to μ with moments

$$\beta_k(\mu) = \sum_{r=0}^{k-1} \sum_{\substack{\pi \in SS(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r C_\pi. \tag{4.14}$$

We now show how Theorem 3.2 in (Benaych-Georges and Cabanal-Duvillard, 2012) and Proposition 3.1 in (Noiry, 2018) can be deduced from Theorem 2.1. So, suppose the entries of X are i.i.d. with distribution μ_n that has mean zero and all moments finite, and

$$\lim_{n \rightarrow \infty} \frac{\beta_k(\mu_n)}{n^{k/2-1} \beta_2(\mu_n)^{k/2}} = c_k, \quad \text{say, exists for all } k \geq 1.$$

The LSD of $S = XX^T$ when $c_k^{1/k}$ is bounded, was considered in Theorem 3.2 of (Benaych-Georges and Cabanal-Duvillard, 2012) and Proposition 3.1 of (Noiry, 2018). Clearly Assumption A is satisfied with $t_n = \infty$, $g_2 \equiv 1$ and $g_{2k} \equiv C_{2k}, k \geq 2$. Hence Theorem 2.1 can be applied, and the resulting LSD, say, μ has moments as in (4.14). In Section 4.4, we shall verify that this LSD is the same as those obtained in the above references.

Connection to the limiting moments of the generalised Wigner matrices: Suppose the entries are triangular i.i.d. that satisfy (4.13). As observed above, $g_{2k} \equiv C_{2k}, k \geq 1$. Hence Remark

2.3, applies and hence $X \stackrel{\mathcal{D}}{=} Y^2$ where $X \sim \mu$, $Y \sim \mu'$, μ and μ' are the LSDs of the S matrix and the generalised Wigner matrix respectively.

4.2.4. **Sparse S .** A well-studied sparse matrix model is where the entries of X have *Bernoulli* distribution with parameter p_n such that $np_n \rightarrow \lambda > 0$. Thus, (4.13) holds with $C_k \equiv \lambda$ for all $k \geq 1$. Hence by Theorem 2.1 μ_S converges a.s. to μ whose moments are (see (4.14)):

$$\beta_k(\mu) = \sum_{r=0}^{k-1} \sum_{\substack{\pi \in SS(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r \lambda^{|\pi|}. \quad (4.15)$$

Explicit description of μ is not available. However, we can say the following. Let $E(2k)$ (and $NCE(2k)$) be the set of partitions (and non-crossing partitions) whose blocks are all of even sizes. Then it is easily seen that $NCE(2k) \subset SS(2k) \subset E(2k)$. Therefore we have the following:

Case 1: $y \leq 1$. Then from (4.15)

$$\sum_{\pi \in NCE(2k)} (\lambda y)^{|\pi|} < \beta_k(\mu) < \sum_{\pi \in E(2k)} \lambda^{|\pi|}. \quad (4.16)$$

Case 2: $y > 1$. Then from (4.15)

$$\sum_{\pi \in NCE(2k)} \lambda^{|\pi|} < \beta_k(\mu) < \sum_{\pi \in E(2k)} (\lambda y)^{|\pi|}. \quad (4.17)$$

Now suppose $P_1(\gamma)$ is a free Poisson variable with mean γ and $P_2(\gamma)$ is a Poisson variable with mean γ . Let Y be a random variable which takes value 1 and -1 with probability $\frac{1}{2}$ each. Suppose Y is independent of $P_1(\gamma)$ and $P_2(\gamma)$. Consider $Q_1(\gamma) = P_1(\gamma)Y$ and $Q_2(\gamma) = P_2(\gamma)Y$. Then the moments of $Q_1(\gamma)$ and $Q_2(\gamma)$ are give as follows:

$$\mathbb{E}[Q_1^k(\gamma)] = \begin{cases} 0 & \text{if } k \text{ is odd ,} \\ \sum_{\pi \in NCE(k)} \gamma^{|\pi|} & \text{if } k \text{ is even .} \end{cases} \quad (4.18)$$

$$\mathbb{E}[Q_2^k(\gamma)] = \begin{cases} 0 & \text{if } k \text{ is odd ,} \\ \sum_{\pi \in E(k)} \gamma^{|\pi|} & \text{if } k \text{ is even .} \end{cases} \quad (4.19)$$

Hence (4.16) and (4.17), can be rewritten as

$$\mathbb{E}[(Q_1(\lambda y))^{2k}] < \beta_k(\mu) < \mathbb{E}[(Q_2(\lambda))^{2k}] \quad \text{for every } k \geq 1, y \leq 1, \quad (4.20)$$

$$\mathbb{E}[(Q_1(\lambda))^{2k}] < \beta_k(\mu) < \mathbb{E}[(Q_2(\lambda y))^{2k}] \quad \text{for every } k \geq 1, y > 1. \quad (4.21)$$

Thus the LSD of S lies between the squares of compound free Poisson and compound Poisson distributions in the above sense.

4.2.5. **Matrices with a variance profile.** The S matrix, where the entries of X are independent but not necessarily identically distributed have been considered in (Yin, 1986), (Lytova and Pastur, 2009) and (Bai and Silverstein, 2010). A common theme has been to assume that the entries have equal variances. Works such as (Hachem et al., 2006), (Zhu, 2020), (Jin and Xie, 2020) drop this assumption and study the bulk distribution of the eigenvalues. Recently the local laws of such matrices with unequal variance structure was studied in (Alt et al., 2017). We now show that if X has a suitable variance profile, then μ_S converges a.s. We consider two profiles.

In the first, X has a *discrete variance profile* so that $X = ((y_{ij} = \sigma_{ij}x_{ij})); 1 \leq i \leq p, 1 \leq j \leq n\}$ where $\{x_{ij}\}$ are i.i.d. random variables and $\{\sigma_{ij}\}$ satisfy certain conditions. A similar model for the Wigner matrix was considered in Result 5.1 of (Bose et al., 2022). We state a similar result for S whose proof uses arguments similar to the proof of Theorem 2.1 and Result 5.1 in (Bose et al., 2022), and we omit the details.

Result 4.1. Consider the matrix X with entries $\{\frac{y_{ij}}{\sqrt{n}} = \frac{\sigma_{ij}x_{ij}}{\sqrt{n}} : 1 \leq i \leq p, 1 \leq j \leq n\}$ that are independent and satisfy the following conditions:

- (i) $\mathbb{E}x_{ij} = 0$ and $\mathbb{E}[x_{ij}^2] = 1$.
- (ii) σ_{ij} satisfy the following:

$$\sup_{1 \leq i \leq p} \left| \frac{1}{n} \sum_{j=1}^n \sigma^2_{ij} - 1 \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.22)$$

- (iii) $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j} \mathbb{E}[x_{ij}^2] \mathbf{1}_{[|x_{ij}| > \eta\sqrt{n}]} = 0$ for every $\eta > 0$.

Then the ESD of S converges a.s. to the MP_y law, where $0 < y = \lim p/n$.

Remark 4.5. Theorem 1.2 in (Jin and Xie, 2020) states a similar result where (4.22) is replaced by $\frac{1}{n} \sum_i \left| \frac{1}{n} \sum_{j=1}^n \sigma^2_{ij} - 1 \right| \rightarrow 0$. However, the proof of equation (2.6) there is not very clear.

Next, we consider a *continuous variance profile*. Suppose $X = ((y_{ij,n} = \sigma(i/p, j/n)x_{ij,n}))$ where $\{x_{ij,n}\}$ are i.i.d. for every fixed n and satisfy the conditions given in (4.13), and σ is a bounded piecewise continuous function on $[0, 1]^2$. Then μ_S converges a.s. to a symmetric probability measure ν whose k th moment is determined by σ and $\{C_{2m}\}_{1 \leq m \leq 2k}$.

To see this, note that $\{y_{ij,n}\}$ satisfy Assumption A with $g_{2k} \equiv \sigma^{2k} C_{2k}$. By Theorem 2.1, μ_S converges a.s. to a probability measure ν . The expressions for the moments of ν are quite involved and shall be given in Section 4.3 after the proof of Theorem 2.1. Incidentally, from those expressions, it is evident that the contribution to the moment from distinct special symmetric partitions may be different even when they have the same number of blocks and block sizes.

In (Hachem et al., 2006), the authors studied the ESD of YY^T where the entries of Y are $\{\frac{\sigma(i/p, j/n)x_{ij}}{\sqrt{n}}, 1 \leq i \leq p, 1 \leq j \leq n\}$ with $\{x_{ij}\}$ being centered i.i.d. variables that have variance 1 and $\mathbb{E}[x_{ij}^{4+\epsilon}] < \infty$ for some $\epsilon > 0$. The variance profile σ^2 is a continuous function on $[0, 1]^2$. They prove that when $p/n \rightarrow y > 0$, the ESD of YY^T converges weakly a.s. to a non-random probability distribution. Now, we will show that we can deduce this result from Theorem 2.1.

As σ^2 is a continuous function on $[0, 1]^2$, we have $\|\sigma\| \leq c$, where c is a constant. Now using this fact and the same arguments in Section 4.2.1, we have that the variables $\{\frac{\sigma(i/p, j/n)x_{ij}}{\sqrt{n}}\}$ satisfy Assumption A with $t_n = n^{-1/3}$, $g_2 \equiv \sigma^2$ and $g_{2k} \equiv 0, k \geq 2$. Also (2.3) is satisfied similarly. Hence from Theorem 2.1, the ESD of YY^T converges weakly a.s. to a non-random probability measure μ whose moments are determined by σ .

Consider the special case $p = n$, with $X = ((y_{ij} = \sigma(i/n, j/n)x_{ij}))$ with

$$\sigma(x, y) = \begin{cases} 1, & x \leq y, \\ 0 & \text{otherwise,} \end{cases}$$

and $\{x_{ij}; 1 \leq i, j \leq n\}$ are i.i.d. with mean zero and variance 1. Then X_p equals

$$X_p = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{22} & x_{23} & \cdots & x_{2n} \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & x_{nn} \end{bmatrix}. \quad (4.23)$$

The spectral distribution of $n^{-1}X_p X_p^*$ was studied in (Dykema and Haagerup, 2004) in the Gaussian case. Later, (Basu et al., 2012) studied the LSD of this and similar other models where the entries are i.i.d. with mean zero and variance 1. If all moments of $\{x_{ij}\}$ are finite, then the a.s. convergence of μ_S is immediate from Theorem 2.1. When only the variance is known to be finite, a truncation argument similar to that given in Section 4.2.1, can be used. As a consequence, μ_S converges a.s. to a non-random probability measure.

4.3. *Proofs for the S matrix.* As discussed in Section 4.1, the existence of $\lim_{n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{p^{r+1}n^{b-r}}$ is crucial in finding the LSD of the S matrix. So we look into it first.

Lemma 4.6. *Suppose $\omega \in SS_b(2k)$ with $(r + 1)$ even generating vertices $(0 \leq r \leq k)$. Then, $|\Pi_S(\omega)| \approx p^{r+1}n^{b-r}$.*

Proof: We argue by induction on b , the number of distinct letters. If $b = 1$, then $r = 0$ and $\omega = aa \cdots aa$. Therefore $\pi(0)$ and $\pi(1)$ are the generating vertices and both can be chosen freely. Thus, $|\Pi_S(\omega)| \approx pn$. Now assume that the result is true upto $b - 1$. Then it is enough to prove that if ω has b distinct letters with $(r + 1)$, $(0 \leq r \leq b - 1)$ even generating vertices, then $|\Pi_S(\omega)| \approx p^{r+1}n^{b-r}$.

First let $0 \leq r \leq b - 2$. Suppose the last distinct letter of ω , say, z appears for the first time at the i th position, that is at $(\pi(i - 1), \pi(i))$ or $(\pi(i), \pi(i - 1))$ (depending on whether i is odd or even). Then z appears in pure even blocks. Let m (m even) be the length of the first pure block. Then we have the following two cases:

Case 1: i is odd. Then we have

$$\begin{aligned} \pi(i - 1) &= \pi(i + 1) = \cdots = \pi(i + m - 1), \\ \pi(i) &= \pi(i + 2) = \cdots = \pi(i + m - 2). \end{aligned} \quad (4.24)$$

Similar identities can be shown for all other pure blocks of z . Hence $\pi(i)$ can be chosen freely with $1 \leq \pi(i) \leq n$ as it does not appear elsewhere in ω other than the letter z . Let ω' be the word with $(b - 1)$ distinct letters and $(r + 1)$ even generating vertices, obtained by dropping all z s from ω . It is easy to see that ω' is a special symmetric word with $(b - 1)$ distinct letters. Therefore, by induction hypothesis, $|\Pi_S(\omega')| \approx p^{r+1}n^{b-(r+1)}$. Now as $\pi(i)$ is another odd vertex that can be chosen freely, we have $|\Pi_S(\omega)| \approx p^{r+1}n^{b-(r+1)}n = p^{r+1}n^{b-r}$.

Case 2: i is even. Then we have

$$\begin{aligned} \pi(i - 1) &= \pi(i + 1) = \cdots = \pi(i + m - 1), \\ \pi(i) &= \pi(i + 2) = \cdots = \pi(i + m - 2). \end{aligned}$$

As in Case 1, the generating vertex $\pi(i)$ can be chosen freely with $1 \leq \pi(i) \leq p$. As before, dropping all z s from ω leads to a special symmetric word ω' with $(b - 1)$ distinct letters and r even generating vertices. Therefore, by induction hypothesis, $|\Pi_S(\omega')| \approx p^r n^{b-r}$. Now as $\pi(i)$ is another even vertex that can be chosen freely, we have $|\Pi_S(\omega)| \approx p^r p n^{b-r} = p^{r+1}n^{b-r}$.

Now let $r = b - 1$. Then there are $r + 1 = b$ even generating vertices (one of them being $\pi(0)$) and b distinct letters in ω . Therefore all letters except the first appear for the first time at even positions in ω . So, if z is the last distinct letter of ω , then z appears for the first time at $(\pi(i - 1), \pi(i))$ where

i is even. Thus similar to Case 1, (4.24) holds and $\pi(i)$ can be chosen freely with $1 \leq \pi(i) \leq n$. If we drop all z s as before from ω , then we get a special symmetric word ω' with $(b-1)$ distinct letters and $(b-2)$ even generating vertices. Therefore, $|\Pi_S(\omega')| \approx p^{b-1} n^{b-(b-1)}$. As $\pi(i)$ is another even vertex that can be chosen freely, we have $|\Pi_S(\omega)| \approx p^{b-1} n p = p^b n = p^{r+1} n^{b-r}$, $r = b-1$. This completes the proof of the lemma. \square

Lemma 4.7. *Let ω be a word with b distinct letters and $(r+1)$ even generating vertices ($0 \leq r \leq b-1$). Then*

$$\lim_{n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{n^{b+1}} = \begin{cases} y^{r+1}, & \omega \in SS_b(2k) \\ 0, & \omega \notin SS_b(2k). \end{cases} \quad (4.25)$$

Thus, $\lim_{n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{p^{r+1} n^{b-r}} = 1$ if and only if ω is a special symmetric word.

Proof: First suppose $\omega \in \mathcal{P}(2k) \setminus SS_b(2k)$. Then from (4.5) and Lemma 3.1 and 3.3 in (Bose et al., 2022), it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{n^{b+1}} = 0.$$

If $\omega \in SS_b(2k)$, then from Lemma 4.6, it immediately follows that $\lim_{n \rightarrow \infty} \frac{|\Pi_S(\omega)|}{n^{b+1}} = y^{r+1}$. This completes the proof of the lemma. \square

Lemma 4.8. *Let*

$$Q_{k,4}^b = |\{(\pi_1, \pi_2, \pi_3, \pi_4) : \ell(\pi_i) = 2k; \pi_i, 1 \leq i \leq 4 \text{ jointly- and cross-matched with} \quad (4.26)$$

$$b \text{ distinct edges or } b \text{ distinct letters across all } (\pi_i)_{1 \leq i \leq 4}\}|. \quad (4.27)$$

Then there exists a constant C , such that,

$$Q_{k,4}^b \leq C n^{b+2}. \quad (4.28)$$

This was proved for the Wigner link function in Lemma 4.2 in (Bose et al., 2022). The arguments in that proof can be used for the S link function here as $1 \leq \pi(2i) \leq p$ and $1 \leq \pi(2i-1) \leq n$, and p and n are comparable for large n . We omit the details.

4.3.1. Proof of Theorem 2.1 and Remark 2.3.

Proof of Theorem 2.1 : (a) We make use of Lemma 4.3 and use the notion of words and circuits in order to calculate the moments. We break the proof into a few steps.

Step 1: (Reduction to mean zero) Consider the zero mean matrix $\tilde{Z} = ((y_{ij} - \mathbb{E}y_{ij}))$. Now

$$n \mathbb{E}[(y_{ij} - \mathbb{E}y_{ij})^{2k}] = n \mathbb{E}[y_{ij}^{2k}] + n \sum_{t=0}^{2k-1} \binom{2k}{t} \mathbb{E}[y_{ij}^t] (\mathbb{E}y_{ij})^{2k-t}. \quad (4.29)$$

The first term of the r.h.s. equals $g_{2k}(i/p, j/n)$ by (2.1). The second term is tackled as follows:

$$\begin{aligned} \text{For } t \neq 2k-1, \quad n \mathbb{E}[y_{ij}^t] (\mathbb{E}y_{ij})^{2k-t} &= (n^{\frac{1}{2k-t}} \mathbb{E}y_{ij})^{2k-t} \mathbb{E}[y_{ij}^t] \\ &\xrightarrow{n \rightarrow \infty} 0, \quad \text{by condition (2.2)}. \end{aligned}$$

$$\begin{aligned} \text{For } t = 2k-1, \quad n \mathbb{E}[y_{ij}^{2k-1}] \mathbb{E}y_{ij} &= (\sqrt{n} \mathbb{E}[y_{ij}^{2k-1}]) (\sqrt{n} \mathbb{E}y_{ij}) \\ &\xrightarrow{n \rightarrow \infty} 0, \quad \text{by condition (2.2)}. \end{aligned}$$

Hence from (4.29), we see condition (2.1) is true for the matrix \tilde{Z} . Similarly we can show that (2.2) is true for \tilde{Z} . Hence, Assumption A holds for the matrix \tilde{Z} .

Now from Lemma 4.2,

$$\begin{aligned} L^4(F^{S_Z}, F^{S_{\tilde{Z}}}) &\leq \frac{2}{p^2} (\text{Tr}(ZZ^T + \tilde{Z}\tilde{Z}^T)) (\text{Tr}[(Z - \tilde{Z})(Z - \tilde{Z})^T]) \\ &\leq \frac{2}{p} \left(\sum_{i,j} (2y_{ij}^2 + (\mathbb{E}y_{ij})^2 - 2y_{ij}\mathbb{E}y_{ij}) \right) \left(\frac{1}{p} \sum_{i,j} (\mathbb{E}y_{ij})^2 \right). \end{aligned} \quad (4.30)$$

The second factor of the rhs in (4.30) is bounded by

$$n(\sup_{i,j} \mathbb{E}y_{ij})^2 = (\sup_{i,j} \sqrt{n} \mathbb{E}y_{ij})^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{by (2.2)}.$$

Now by Borel-Cantelli lemma it can be seen that $\frac{1}{p} \sum_{i,j} (y_{ij}^2 - \mathbb{E}[y_{ij}^2]) \rightarrow 0$ a.s. as $p \rightarrow \infty$ (proof is given in Section 6). Also $\mathbb{E}[\frac{1}{p} \sum_{i,j} y_{ij}^2] \rightarrow \int_{[0,1]^2} g_2(x, y) dx dy$. Hence,

$$\mathbb{P}[\{\omega; \limsup_p \frac{1}{p} \sum_{i,j} y_{ij}^2(\omega) = \infty\}] = 0.$$

Therefore the first term of the rhs in (4.30) also tends to zero a.s. Hence, the LSD of S_Z and $S_{\tilde{Z}}$ are same a.s. Thus we can assume that the entries of Z have mean 0.

We will now verify the conditions (i), (ii) and (iii) of Lemma 4.3.

Step 2: (Verification of the fourth moment condition for S_Z) Observe that

$$\frac{1}{p^4} \mathbb{E}[\text{Tr}(ZZ^T)^k - \mathbb{E}(\text{Tr}(ZZ^T)^k)]^4 = \frac{1}{p^4} \sum_{\pi_1, \pi_2, \pi_3, \pi_4} \mathbb{E}[\Pi_{i=1}^4 (Y_{\pi_i} - \mathbb{E}Y_{\pi_i})]. \quad (4.31)$$

If $(\pi_1, \pi_2, \pi_3, \pi_4)$ are not jointly-matched, then one of the circuits has a letter that does not appear elsewhere. Hence by independence and mean zero assumption, $\mathbb{E}[\Pi_{i=1}^4 (Y_{\pi_i} - \mathbb{E}Y_{\pi_i})] = 0$. If $(\pi_1, \pi_2, \pi_3, \pi_4)$ are not cross-matched, then one of the circuits say π_j is only self-matched. Then we have $\mathbb{E}[Y_{\pi_j} - \mathbb{E}Y_{\pi_j}] = 0$. So again we have $\mathbb{E}[\Pi_{i=1}^4 (Y_{\pi_i} - \mathbb{E}Y_{\pi_i})] = 0$.

So we consider only circuits $(\pi_1, \pi_2, \pi_3, \pi_4)$ that are jointly- and cross-matched. Here each circuit is of length $2k$, so the total number of edges (L - values) is $8k$. As the circuits are at least pair-matched, the number of distinct edges is at most $4k$.

Suppose π_i has k_i distinct letters, $1 \leq i \leq 4$ with $k_1 + k_2 + k_3 + k_4 = b$. Suppose the j th distinct letter appears s_j times across $\pi_1, \pi_2, \pi_3, \pi_4$ and first at the i_j -th position. Let b_1 and b_2 ($b_1 + b_2 = b$) be respectively the number of even and odd s_i 's, denoted by $s_{i_1}, s_{i_2}, \dots, s_{i_{b_1}}$ and $s_{i_{b_1+1}}, s_{i_{b_1+2}}, \dots, s_{i_{b_2}}$. Each term can then be written as

$$\frac{1}{p^4} \sum_{b=1}^{4k} p^{-b_1} p^{-(b_2 - \frac{1}{2})} \prod_{j=1}^{b_1} g_{s_{i_j}, n}(\pi(i_j - 1)/p, \pi(i_j)/n) \prod_{m=b_1+1}^{b_1+b_2} n^{\frac{b_2 - (1-1/2)}{b_2}} \mathbb{E}[y_{\pi(i_m-1)\pi(i_m)}^{s_{i_m}}].$$

We note that $g_{s_{i_j}, n} \rightarrow g_{s_{i_j}}$ for all $1 \leq j \leq b_1$. Therefore, the sequence $\|g_{s_{i_j}, n}\|$ is bounded by a constant M_j . Also as $\frac{b_2 - (1-1/2)}{b_2} < 1$, by (2.2), we have $n^{\frac{b_2 - (1-1/2)}{b_2}} \mathbb{E}[y_{\pi(i_m-1)\pi(i_m)}^{s_{i_m}}]$ is bounded by 1 for n large when $b_1 + 1 \leq m \leq b_1 + b_2$. Let

$$M' = \max_{b_1+b_2=b} \{M_t, 1 : 1 \leq t \leq b_1\} \quad \text{and} \quad M'_0 = \max\{M'^b : 1 \leq b \leq 2k\}.$$

By (4.5) and Lemma 4.8, the number of such circuits that have b distinct letters ($b = 1, \dots, k$) is bounded by $C_1 n^{b+2}$ for some constant $C_1 > 0$. Therefore with $C_2 = y^{b+2} C_1$,

$$\frac{1}{p^4} \mathbb{E}[\text{Tr}(ZZ^T)^k - \mathbb{E}(\text{Tr}(ZZ^T)^k)]^4 \leq C_2 M'_0 \sum_{b=1}^{4k} \frac{1}{p^{b+3\frac{1}{2}}} p^{b+2} = \mathcal{O}(p^{-\frac{3}{2}}).$$

This completes the proof of Step 2.

Step 3: (Verification of the first moment condition) By Lemma 4.3 and the previous step, it is now enough to show that for every $k \geq 1$, $\lim_{n \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(ZZ^T)^k]$ exists, and is given by $\beta_k(\mu')$ for each $k \geq 1$. First note that, we can write (4.1) as

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(ZZ^T)^k] &= \lim_{n \rightarrow \infty} \sum_{b=1}^k \left[\frac{1}{p} \sum_{\omega \in SS_b(2k)} \sum_{\pi \in \Pi(\omega)} \mathbb{E}(Y_\pi) + \frac{1}{n} \sum_{\substack{\omega \notin SS(2k) \\ \omega \text{ with } b \text{ letters}}} \sum_{\pi \in \Pi(\omega)} \mathbb{E}(Y_\pi) \right]. \\ &= T_1 + T_2. \end{aligned} \quad (4.32)$$

Suppose that ω has b distinct letters and let $\pi \in \Pi_S(\omega)$. Suppose the j th new letter appears at the $(\pi(i_j - 1), \pi(i_j))$ -th position for the first time, $1 \leq j \leq b$. Let D denote the set of all distinct generating vertices. Thus $|D| \leq (b + 1)$.

Suppose ω has b distinct letters but does not belong to $SS(2k)$. Then from Lemma 4.7, $|D| \leq b$. Hence ω , and as a consequence, T_2 has no contribution to (4.32).

Now suppose $\omega \in SS_b(2k)$ with $(r + 1)$ even generating vertices. By Lemma 4.7, ω has $(b + 1)$ distinct generating vertices. For each $j \in \{1, 2, \dots, b\}$ denote $(\pi(i_j - 1), \pi(i_j))$ as (t_j, l_j) . Then $t_1 = \pi(0)$ and $l_1 = \pi(1)$. It is easy to see that any distinct (t_j, l_j) corresponds to a distinct letter in ω . Suppose the j th new letter appears s_j times in ω . Clearly all the s_j are even. So the total contribution of this ω to T_1 in (4.32) is:

$$\frac{1}{pn^b} \sum_D \prod_{j=1}^b g_{s_j, n}(t_j/p, l_j/n). \quad (4.33)$$

Recall that there are $(r + 1)$ even generating vertices in D with range between 1 and p , and $(b - r)$ vertices (odd generating) with range between 1 and n . So as $n \rightarrow \infty$, (4.33) converges to

$$y^r \int_{[0,1]^{b+1}} \prod_{j=1}^b g_{s_j}(x_{t_j}, x_{l_j}) \prod_{i \in D} dx_i. \quad (4.34)$$

Hence we obtain

$$\lim_{p \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr} S^k] = \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\pi \in SS_b(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r \int_{[0,1]^{b+1}} \prod_{j=1}^b g_{s_j}(x_{t_j}, x_{l_j}) \prod_{i \in D} dx_i. \quad (4.35)$$

This completes the verification of the first moment condition. *Note that also gives the expression for the moments of the LSD.*

Step 4: (Uniqueness of the measure) We have obtained

$$\gamma_k = \lim_{p \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(S)^k] \leq \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in SS_b(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r M_\sigma.$$

Let $c = \max(y, 1)$. Then

$$\gamma_k \leq \sum_{\sigma \in SS(2k)} c^k M_\sigma \leq \sum_{\sigma \in \mathcal{P}(2k)} c^k M_\sigma = c^k \alpha_k.$$

As $\{\alpha_k\}$ satisfies Carleman's condition, $\{\gamma_k\}$ also does so. By Lemma 4.3, we see that there exists a measure μ with moments $\{\gamma_k\}_{k \geq 1}$ such that μ_{S_Z} converges a.s. to μ . This completes the proof of part (a).

(b) From Lemma 4.2, we have

$$\begin{aligned} L^4(F^S, F^{S_Z}) &\leq \frac{2}{p^2} (\text{Tr}(XX^T + ZZ^T)) (\text{Tr}[(X - Z)(X - Z)^T]) \\ &= \frac{2}{p} \left(2 \sum_{i,j} y_{ij}^2 + \sum_{ij} x_{ij}^2 \mathbf{1}_{\{|x_{ij}| > t_n\}} \right) \left(\frac{1}{p} \sum_{ij} x_{ij}^2 \mathbf{1}_{\{|x_{ij}| > t_n\}} \right). \end{aligned} \quad (4.36)$$

The second factor in the above equation tends to zero a.s. (or in probability) as $n \rightarrow \infty$ due to the condition (2.3). Now $\frac{1}{p} \sum_{i,j} (y_{ij}^2 - \mathbb{E}[y_{ij}^2]) \rightarrow 0$ a.s. (see Section 6) and $\mathbb{E}[\frac{1}{p} \sum_{i,j} y_{ij}^2] \rightarrow$

$\int_{[0,1]^2} g_2(x, y) dx dy$, and hence is finite. This implies that $\frac{1}{p} \sum_{i,j} y_{ij}^2$ is bounded a.s. Therefore the first factor in (4.36) is bounded a.s., and thus the rhs of (4.36) tends to 0 as $p, n \rightarrow \infty$.

From the discussion above, we infer that the ESD of XX^T converges to the probability measure μ a.s. (or in probability). This completes the proof of the theorem. \square

Proof of Remark 2.3: In the case $p = n$, if $\{g_{2k,n}\}$ are symmetric functions, then the assumption on the entries of X_n are no different from that on the entries of W_n in Theorem 2.1 of (Bose et al., 2022). Now from (4.35), and equation (4.11) in (Bose et al., 2022) we see that $\mathbb{E}[Y^k] = \mathbb{E}[Y'^{2k}]$, $k \geq 1$.

However observe that even though $\{g_{2k,n}\}$ are not symmetric for every n , the functions $\{g_{2k}\}$ are symmetric and hence $\mathbb{E}[Y^k] = \mathbb{E}[Y'^{2k}]$, $k \geq 1$ still holds, (see (4.35)) as the limiting moments depend only on $\{g_{2k}\}$. Therefore, by the uniqueness criterion of a probability distribution via moments, we have $Y \stackrel{D}{=} Y'^2$. As the limiting moments ((4.35)) depend on $\{g_{2k}\}$ and $\lim p/n$, if $p/n \rightarrow 1$, and $\{g_{2k}\}$ are symmetric, we have $Y \stackrel{D}{=} Y'^2$. \square

Proof of Remark 2.2: Consider $k = mt$ for some $t \geq 1$. Then from (4.35), we have

$$\beta_k(\mu) = \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\pi \in SS_b(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r \int_{[0,1]^{b+1}} \prod_{j=1}^b g_{k_j}(x_{t_j}, x_{l_j}) \prod_{i \in D} dx_i. \quad (4.37)$$

Recall that π in the above expression could be described as a word in $SS_b(2k)$ with $(r + 1)$ even generating vertices. Let us focus on words $\omega \in SS_t(2k)$ with t distinct letters and where each letter appears $2m$ times in pure even blocks. Clearly ω has only one even generating vertex $\pi(0)$. Therefore as $n \rightarrow \infty$, the contribution of ω in the limiting moment is (see (4.34)):

$$\int_{[0,1]^{t+1}} g_{2m}(x_0, x_1) g_{2m}(x_0, x_2) \cdots g_{2m}(x_0, x_t) dx_0 dx_1 \cdots dx_t = \int_{[0,1]} (f_{2m}(x_0))^t dx_0. \quad (4.38)$$

Next, observe that the number of such words is

$$\frac{1}{t!} \binom{mt}{m} \binom{mt-t}{m} \cdots \binom{m}{m} = \frac{1}{t!} \frac{(mt)!}{(m!)^t}. \quad (4.39)$$

Since the integrand in (4.37) is non-negative, using (4.38) and (4.39), we have

$$\begin{aligned} \beta_k(\mu) &> \frac{1}{t!} \frac{(mt)!}{(m!)^t} \int_{[0,1]} (f_{2m}(x_0))^t dx_0 \\ &= \frac{(mt)!}{t!} \int_{[0,1]} \left(\frac{f_{2m}(x_0)}{m!} \right)^t dx_0 > c \frac{(mt)!}{t!}, \quad k = mt. \end{aligned}$$

Therefore for t sufficiently large (with $k = mt$),

$$(\beta_k(\mu))^{1/k} > K t^\eta \quad \text{for some constant } K > 0 \text{ and } \eta > 0.$$

Therefore $(\beta_k(\mu))^{1/k} \rightarrow \infty$ as $k = mt \rightarrow \infty$. Hence μ has unbounded support. \square

Moments for the variance profile matrices: Now we give a description of the moments of LSD ν for S matrices with variance profile. From Step 3 in the proof of Theorem 2.1, for each word in $SS_b(2k)$ with $(r+1)$ even generating vertices and where the distinct letters appear s_1, s_2, \dots, s_b times, its contribution to the limiting moments is (see (4.34))

$$y^r \int_{[0,1]^{b+1}} \prod_{j=1}^b \sigma^{s_j}(x_{t_j}, x_{l_j}) \prod_{i \in D} dx_i \prod_{j=1}^b C_{s_j},$$

where (t_j, l_j) denotes the position of the first appearance of the j th distinct letter in the word. Hence the k th moment of ν is

$$\beta_k(\nu) = \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\pi \in SS_b(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r \int_{[0,1]^{b+1}} \prod_{j=1}^b \sigma^{s_j}(x_{t_j}, x_{l_j}) \prod_{i \in D} dx_i \prod_{j=1}^b C_{s_j}.$$

4.4. *Hypergraphs, Noiry-words and $SS(2k)$.* The distribution of the S matrix with triangular i.i.d. entries, was studied in (Benaych-Georges and Cabanal-Duvillard, 2012) and (Noiry, 2018), where the authors used the concepts of *Hypergraphs* and *words*, which we call *Noiry words* here. In Section 4.2.3, we discussed the triangular i.i.d. cases and showed how Theorem 2.1 is used in this situation. We also described the limiting moments via $SS(2k)$ partitions. Now we verify that the moments that we have obtained are identical with those obtained in (Benaych-Georges and Cabanal-Duvillard, 2012) and (Noiry, 2018).

Definition 4.9. Let G be a graph with vertex set V . Let π and τ be partitions, respectively, of V and the edge set. Then the hypergraph $H(\pi, \tau)$ is a graph with vertex set G_π (i.e. π) and edges $\{E_W; W \in \tau\}$, where each edge E_W is the set of blocks $J \in \pi$ such that at least one edge of G_π starting or ending at J belongs to W . Further if no two of the edges can have more than one common vertex, then $H(\pi, \tau)$ is said to be a *hypergraph with no cycles*.

For details on Hypergraphs, see Sections 5.3 and 12.3.2 in (Benaych-Georges and Cabanal-Duvillard, 2012) and (Berge, 1989). In (Benaych-Georges and Cabanal-Duvillard, 2012), their equation (22) describes the moments of the LSD of S as a sum on *Hypergraphs with no cycles*.

Lemma 4.10. *For every word $\omega \in SS_b(2k)$, there exists partitions $\pi, \tau \in \mathcal{P}(k)$ such that there is a unique hypergraph $H(\pi, \tau)$ which has no cycle with $|\pi| + |\tau| = b + 1$. The converse is also true.*

Proof: Let $\omega \in SS_b(2k)$ with $(r+1)$ and $(b-r)$ even and odd generating vertices respectively. Suppose the even and the odd generating vertices are respectively $\pi(i_{t_0}) = \pi(0), \pi(i_{t_1}), \dots, \pi(i_{t_r})$ and $\pi(i_{m_1}) = \pi(1), \pi(i_{m_2}), \dots, \pi(i_{m_{b-r}})$. Let $V_j = \{\pi(2i) : \pi(2i) = \pi(i_{t_j}), 1 \leq i \leq k\}$, $0 \leq j \leq r$ and $W_j = \{\pi(2i-1) : \pi(2i-1) = \pi(i_{m_j}), 1 \leq i \leq k\}$, $1 \leq j \leq (b-r)$. Clearly, $\sigma = \{V_j; 0 \leq j \leq r\}$ and $\tau = \{W_j; 1 \leq j \leq (b-r)\}$ are two partitions of $\{1, 2, \dots, k\}$. Therefore, we can construct a hypergraph $H(\sigma, \tau)$ where σ is the vertex set and $\{E_W; W \in \tau\}$ is the edge set (see (4.10)).

Now suppose if possible, $H(\sigma, \tau)$ has a cycle. That means by construction, there exists a, b ($a \neq b$) $\in \{1, 2, \dots, (b-r)\}$ and q, l ($q \neq l$) $\in \{0, 1, \dots, r\}$ such that $V_q, V_l \in W_a \cap W_b$. That is, there are edges $(\pi(k_1-1), \pi(k_1)), (\pi(k_2-1), \pi(k_2)), (\pi(k_3-1), \pi(k_3)), (\pi(k_4-1), \pi(k_4))$ with $k_i, 1 \leq i \leq 4$ odd such that $\pi(k_1-1) \in V_q, \pi(k_1) \in W_a, \pi(k_2-1) \in V_q, \pi(k_2) \in W_b, \pi(k_3-1) \in V_l, \pi(k_3) \in W_a$ and $\pi(k_4-1) \in V_l, \pi(k_4) \in W_b$. As the positions $(\pi(k_i-1), \pi(k_i)), i = 1, 2, 3, 4$ are all distinct, there are four distinct letters that appear at these four positions in ω . Without loss of generality suppose, from left to right $(\pi(k_4-1), \pi(k_4))$ is the rightmost (among the four positions mentioned above) in ω . Since $\pi(k_4-1) \in V_l$ and $\pi(t_l)$ comes before $\pi(k_4-1)$, it cannot be chosen freely. Using a similar argument, $\pi(k_4)$ also cannot be chosen freely. Also they have been chosen as generating vertices of three different letters that have appeared at the positions $(\pi(k_i-1), \pi(k_i)), 1 \leq i \leq 3$.

Using Lemma 4.7, this is not possible as the letter at $(\pi(k_4 - 1), \pi(k_4))$ is different from the previous three letters. Thus $H(\sigma, \tau)$ does not have a cycle. Moreover, it is evident by construction that for every special symmetric word we get a unique $H(\sigma, \tau)$ without any cycles.

Conversely, suppose $H(\sigma, \tau)$ is a hypergraph with no cycle and $|\sigma| + |\tau| = b + 1$. We form a word of length $2k$ from it in the following manner. Now $\sigma, \tau \in \mathcal{P}(k)$. Let $\sigma = \{V_0, V_1, \dots, V_r\}$ and $\tau = \{W_1, \dots, W_{b-r}\}$ (as $|\sigma| + |\tau| = b + 1$). Then we choose the even vertices $\pi(2i), 0 \leq i \leq k - 1$ from σ and odd vertices $\pi(2i - 1), 1 \leq i \leq k$ from τ , and $\pi(i) = \pi(j)$ if i and j belong to the same block of σ or τ (depending on i and j both being even or odd respectively). Thus we get a word ω of length $2k$ whose even and odd generating vertices are $\{\pi(\min\{V_s\})\}_{0 \leq s \leq r}$ and $\{\pi(\min\{W_t\})\}_{1 \leq t \leq (b-r)}$ respectively. Thus there are b distinct letters in ω . Now as $H(\sigma, \tau)$ does not have a cycle, using the same arguments as the previous paragraph, it can be shown that all the generating vertices can be chosen freely. This can happen only if the word is special symmetric. Thus we obtain $\omega \in SS_b(2k)$ with $(r + 1)$ even generating vertices. It is easy to see that, two hypergraphs with no cycles cannot give rise to the same special symmetric word.

Hence there is a one-one correspondence between special symmetric words and hypergraphs with no cycles. This completes the proof of the lemma. \square

Thus we see that (4.14) can be written as

$$\begin{aligned} \beta_k(\mu) &= \sum_{r=0}^{k-1} \sum_{\substack{\pi \in SS(2k) \\ \text{with } (r+1) \\ \text{even generating vertices}}} y^r C_\pi \\ &= \sum_{r=0}^{k-1} \sum_{\substack{\sigma \in \mathcal{P}(k) \\ \text{with } r+1 \text{ blocks } H(\sigma, \tau) \text{ has no cycle}}} \sum_{\tau \in \mathcal{P}(k)} \prod_{i=1}^{b-r} y^{b-r} f(W_i), \end{aligned} \tag{4.40}$$

where W_i are the blocks of τ and f is some function determined by $(C_{2k})_{k \geq 1}$ (which is not necessarily multiplicative in the sense of partitions.) Therefore, when the entries satisfy (4.13), using (4.40) and Remark 2.2, we obtain the conclusions of Theorem 3.2 of (Benaych-Georges and Cabanal-Duvillard, 2012). Since this is a special case (as the entries are iid for every n) of Theorem 2.1, it indeed generalises Theorem 3.2 of (Benaych-Georges and Cabanal-Duvillard, 2012).

In Proposition 3.1 in (Noiry, 2018), the author describes the limiting moments via equivalence class of words. His notion of words is different from ours and so we call the former *Noiry words*.

Noiry words: Suppose $G = (V, E)$ is a graph with labelled vertices. A word of length $k \geq 1$ on G is a sequence of labels i_1, i_2, \dots, i_k such that for each $j \in \{1, 2, \dots, k - 1\}$, $\{i_j, i_{j+1}\}$ is a pair of adjacent labels, i.e., the associated vertices are neighbours in G . A word of length k is closed if $i_1 = i_k$. Such closed words will be called Noiry words. See Section 3 in (Noiry, 2018) for more details.

Equivalence of Noiry words: Let $\mathbf{i} = i_1, i_2, \dots, i_k$ and $\mathbf{i}' = i'_1, i'_2, \dots, i'_k$ be two Noiry words on two labeled graphs G and G' with vertex set V . These words are said to be equivalent if there is a bijection σ of $\{1, 2, \dots, |V|\}$ such that $\sigma(i_j) = i'_j, 1 \leq j \leq k$. This defines an equivalence relation on the set of all Noiry words, thereby giving rise to equivalence classes of Noiry words.

Using the developments in Section 3, and equation (3.2) of (Noiry, 2018). $\mathbf{W}_k(a, a + 1, l, \mathbf{b}), \mathbf{b} = (b_1, b_2, \dots, b_a) \in \mathbb{N}^a, b_i \geq 2, \sum_{i=1}^a b_i = 2k$, denotes an equivalence class of Noiry words on a labeled rooted planar tree with a edges, of which l are odd and each edge is traversed b_i times, $1 \leq i \leq a$.

Then the k th moment of the LSD is given in equation (3.2) of (Noiry, 2018) as

$$\beta_k(\mu) = \sum_{a=1}^k \sum_{l=1}^a \alpha^l \sum_{\substack{\mathbf{b}=(b_1, b_2, \dots, b_a) \\ b_i \geq 2, b_1 + \dots + b_a = 2k}} |\mathbf{W}_k(a, a+1, l, \mathbf{b})| \prod_{i=1}^a C_{b_i}.$$

In the next lemma we show how each of these equivalence classes of words correspond to special symmetric words.

Lemma 4.11. *Each equivalence class $\mathbf{W}_k(a, a+1, l, \mathbf{b})$, $\mathbf{b} = (b_1, b_2, \dots, b_a) \in \mathbb{N}^a$, $b_i \geq 2$, $\sum_{i=1}^a b_i = 2k$ is a word $\omega \in SS_a(2k)$ with l odd generating vertices and where each letter appears b_i , $1 \leq i \leq a$ times in ω .*

Proof: Recall from Section 4.1 that we have defined words to be equivalence classes of circuits with the relation arising from the link functions (see (4.2)). Now Noiry words are not equivalence classes to begin with, they form equivalence classes if they are relabeled in a certain way as described above. From this and how we have defined equivalence of circuits, observe that an equivalence class of Noiry words is nothing but a word in our case. Now the only words with a distinct letters for which $a+1$ generating vertices can be chosen freely are the special symmetric words with a distinct letters (see Lemma 4.7). Thus $\mathbf{W}_k(a, a+1, l, \mathbf{b})$, $\mathbf{b} = (b_1, b_2, \dots, b_a) \in \mathbb{N}^a$, $b_i \geq 2$, $\sum_{i=1}^a b_i = 2k$ is a word $\omega \in SS_a(2k)$ with l odd generating vertices where each letter appears b_i , $1 \leq i \leq a$ times in ω . \square

Using this lemma it readily follows that

$$\sum_{a=1}^k \sum_{l=1}^a \sum_{\substack{\mathbf{b}=(b_1, b_2, \dots, b_a) \\ b_i \geq 2, b_1 + \dots + b_a = 2k}} |\mathbf{W}_k(a, a+1, l, \mathbf{b})| = \sum_{a=1}^k \sum_{l=1}^a \sum_{\substack{\pi \in SS_a(2k) \\ \text{with } l \text{ odd generating vertices} \\ \text{with block sizes } b_1, \dots, b_a}} 1 = \sum_{l=1}^k \sum_{\pi \in SS(2k)} 1.$$

Hence it follows that (4.14) is same as the moment expression in equation (3.2) of (Noiry, 2018).

5. Details for the S_A matrices

This section deals with the details for S_A . We first describe some notions and definitions followed by a detailed proof of Theorem 2.4. Finally, we discuss how this theorem can deal with triangular (size dependent) entries, sparsity, i.i.d., variance profile, band and block matrices.

5.1. *Preliminaries.* We recall link functions, circuits and words from Section 4 in (Bose et al., 2021), as applied to S_A .

Link function: The link functions for the eight choices of A are (here $1 \leq i \leq p$, $1 \leq j \leq n$):

- (i) Symmetric reverse circulant, $R^{(s)}$: $L_{R^{(s)}}(i, j) = (i + j - 2)(\text{mod } n)$.
- (ii) Asymmetric reverse circulant, R : $L_R(i, j) = \begin{cases} (i + j - 2)(\text{mod } n) & i \leq j, \\ -[(i + j - 2)(\text{mod } n)] & i > j. \end{cases}$
- (iii) Symmetric circulant, $C^{(s)}$: $L_{C^{(s)}}(i, j) = n/2 - |n/2 - |i - j||$.
- (iv) Circulant, C : $L_C(i, j) = (j - i)(\text{mod } n)$.
- (v) Symmetric Toeplitz, $T^{(s)}$: $L_{T^{(s)}}(i, j) = |i - j|$.
- (vi) Asymmetric Toeplitz, T : $L_T(i, j) = i - j$.
- (vii) Symmetric Hankel, $H^{(s)}$: $L_{H^{(s)}}(i, j) = i + j$.

(viii) Asymmetric Hankel, $H: L_H(i, j) = \begin{cases} (i + j) & i \geq j, \\ -(i + j) & i < j. \end{cases}$

Circuits and Words: Recall the definition of circuits and words from Section 4.1 for the S matrix. In this case those definitions remain unaltered. Suppose A is one of the eight mentioned patterned matrices. Then as before, notice that circuits π with $\ell(\pi) = 2k$ are required to deal with the k th moment of S_A . For any choice of the link L ,

$$\begin{aligned} \xi_\pi(2i - 1) &= L(\pi(2i - 2), \pi(2i - 1)), 1 \leq i \leq k, \\ \xi_\pi(2i) &= L(\pi(2i), \pi(2i - 1)), 1 \leq i \leq k, \end{aligned}$$

$$\mathbb{E}[\text{Tr}(S_A^k)] = \mathbb{E}[\text{Tr}(AA^T)^k] = \sum_{\pi: \ell(\pi)=2k} \mathbb{E}[Y_\pi], \quad (5.1)$$

where $Y_\pi = \prod_{i=1}^k x_{\xi_\pi(2i-1)} x_{\xi_\pi(2i)}$.

The class $\Pi(\omega)$: For ω ,

$$\Pi_{S_A}(\omega) = \{\pi : \omega[i] = \omega[j] \Leftrightarrow \xi_\pi(i) = \xi_\pi(j) \text{ for all } i, j\}. \quad (5.2)$$

Now,

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{1}{n} \mathbb{E}[\text{Tr}(S_A)^k] &= \lim_{n \rightarrow \infty} \frac{1}{p} \sum_{\pi: \ell(\pi)=2k} \mathbb{E}[Y_\pi] \\ &= \lim_{p \rightarrow \infty} \sum_{b=1}^k \sum_{\omega \text{ matched of length } 2k} \frac{1}{p} \sum_{\pi \in \Pi_{S_A}(\omega)} \mathbb{E}(Y_\pi). \end{aligned} \quad (5.3)$$

with b distinct letters

Note that all words that appear above are of length $2k$. For every $k \geq 1$, the words of length $2k$ corresponding to the circuits of A and S_A , are related. Here we make a key observation in that regard.

Observation (i): Let $A^{(s)}$ stand for any of the symmetric matrices $R^{(s)}, H^{(s)}, C^{(s)}$ or $T^{(s)}$ and let $\Pi_{A^{(s)}}(\omega)$ be the possibly larger class of circuits for $A^{(s)}$ with range $1 \leq \pi(i) \leq \max(p, n), 0 \leq i \leq 2k$. Let $\Pi_{S_{A^{(s)}}}(\omega)$ and $\Pi_{A^{(s)}}(\omega)$ denote the set of all circuits corresponding to a word ω arising from the circuits corresponding to $A^{(s)}$ and $S_{A^{(s)}}$, respectively. Then, for every $k \geq 1$ and any word ω of length $2k$,

$$\Pi_{S_A}(\omega) \subset \Pi_{S_{A^{(s)}}}(\omega) \subset \Pi_{A^{(s)}}(\omega). \quad (5.4)$$

Now we recall the definition of even and symmetric words from (Bose et al., 2021).

Even word: A word ω is called *even* if each distinct letter in ω appears an even number of times. We shall denote the set of all even words of length $2k$ as $E(2k)$, and the set of all even words of length $2k$ with b distinct letters as $E_b(2k)$. For example, $ababcc$ is an even word of length 6 with 3 distinct letters. The corresponding partition of $[6]$ is $\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$.

Symmetric word: A word ω is *symmetric* if each distinct letter appears equal number of times in odd and even positions. We shall denote the set of all symmetric words of length $2k$ as $S(2k)$, and the set of all symmetric words of length $2k$ with b distinct letters as $E_b(2k)$.

Even and odd generating vertices are defined exactly as before. Observe that

$$\begin{aligned} |\Pi_{S_A}(\omega)| &= |\{(\pi(0), \pi(1), \dots, \pi(2k)) : 1 \leq \pi(2i) \leq p, 1 \leq \pi(2i - 1) \leq n \text{ for } i = 0, 1, \dots, k, \\ &\quad \pi(0) = \pi(2k), \quad \xi_\pi(i) = \xi_\pi(j) \text{ if and only if } \omega[i] = \omega[j], 1 \leq i, j \leq 2k\}|. \end{aligned} \quad (5.5)$$

As $p/n \rightarrow y > 0$,

$$|\Pi_{S_A}(\omega)| = \mathcal{O}(p^{r+1}n^{b-r}) \text{ whenever } \omega \text{ has } b \text{ distinct letters and } (r+1) \text{ even generating vertices.} \quad (5.6)$$

As before, the existence of the following limit is tied very intimately to the LSD of S_A .

$$\lim_{p,n \rightarrow \infty} \frac{|\Pi_{S_A}(\omega)|}{p^{r+1}n^{b-r}} \text{ as } p/n \rightarrow y. \quad (5.7)$$

In Section 5.2, we identify the words for which the above limit is positive.

As before we will use the moment method. The Lévy metric defined in Section 4.1 will be helpful in dealing with non-centered variables and truncation inequalities.

Lemma 5.1 (Theorem A.38 in (Bai and Silverstein, 2010)). *Let λ_k and δ_k , $1 \leq k \leq n$ be two sets of complex numbers and F and \bar{F} denote their empirical distributions. Then for any $\alpha > 0$,*

$$L^{\alpha+1}(F, \bar{F}) \leq \min_{\pi} \frac{1}{n} \sum_{k=1}^n |\lambda_k - \delta_{\pi(k)}|^{\alpha}, \quad (5.8)$$

where L is the Lévy distance and $\pi = (\pi(1), \dots, \pi(n))$ is any permutation of $1, 2, \dots, n$.

Lemma 5.2 (Theorem A.37 in (Bai and Silverstein, 2010)). *Suppose A and B are real $p \times n$ matrices and λ_k and δ_k , $1 \leq k \leq p$ are the singular values of A and B arranged in descending order. Then,*

$$\sum_{k=1}^{\min(p,n)} |\lambda_k - \delta_k|^2 \leq \text{Tr}[(A - B)(A - B)^T]. \quad (5.9)$$

The following inequality on the Lévy distance between the EESDs of two matrices can be easily deduced from the above lemmas.

Lemma 5.3. *Suppose A and B are real $p \times n$ matrices and $\mathbb{E}F^{S_A}$ and $\mathbb{E}F^{S_B}$ denote the EESDs of AA^T and BB^T respectively. Then the Lévy distance, L between these distributions satisfies the following inequality:*

$$L^4(\mathbb{E}F^{S_A}, \mathbb{E}F^{S_B}) \leq \frac{2}{p^2} (\mathbb{E} \text{Tr}(AA^T + BB^T)) (\mathbb{E} \text{Tr}[(A - B)(A - B)^T]). \quad (5.10)$$

Proof: Let $\sigma_i(A), \sigma_i(B), i = 1, 2, \dots, p$ denote respectively the singular values of A and B , each set arranged in descending order. Then with $\alpha = 1$, from similar arguments as Lemma 5.1 and Cauchy-Schwarz inequality, we have,

$$\begin{aligned} L^2(\mathbb{E}F^{S_A}, \mathbb{E}F^{S_B}) &\leq \frac{1}{p} \mathbb{E} \left[\sum_{i=1}^p |\lambda_i^2 - \delta_i^2| \right] \\ &\leq \frac{1}{p} \mathbb{E} \left[\left(\sum_{i=1}^p (\lambda_i + \delta_i)^2 \right)^{1/2} \left(\sum_{i=1}^p |\lambda_i - \delta_i|^2 \right)^{1/2} \right] \\ &\leq \frac{1}{p} \left(\mathbb{E} \left[\sum_{i=1}^p (\lambda_i + \delta_i)^2 \right] \right)^{1/2} \left(\mathbb{E} \left[\sum_{i=1}^p |\lambda_i - \delta_i|^2 \right] \right)^{1/2} \\ &\leq \frac{1}{p} \left(2 \mathbb{E} \left[\sum_{i=1}^p (\lambda_i^2 + \delta_i^2) \right] \right)^{1/2} \left(\mathbb{E} \left[\sum_{i=1}^p |\lambda_i - \delta_i|^2 \right] \right)^{1/2}. \end{aligned}$$

Now using Lemma 5.2 on the second factor of the above inequality, we get (5.10). \square

5.2. *Proofs for S_A matrices.* We look at the words that contribute to the limiting moments and determine their contribution for each of the matrices.

5.2.1. $\lim_{n \rightarrow \infty} \frac{|\Pi(\boldsymbol{\omega})|}{p^{r+1}n^{b-r}}$ for S_A matrices.

Lemma 5.4. *Suppose $\boldsymbol{\omega}$ is a word with b distinct letters and $(r+1)$ even generating vertices. Then $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{T(s)}}(\boldsymbol{\omega})| = \alpha_{T(s)}(\boldsymbol{\omega}) > 0$ if and only if $\boldsymbol{\omega}$ is an even word.*

Proof: First suppose $\boldsymbol{\omega} \in \mathcal{P}(2k) \setminus E_b(2k)$. Then from (5.4) and Lemma 5.3 in (Bose et al., 2021), and using the fact that $p/n \rightarrow y > 0$ as $n \rightarrow \infty$, it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{T(s)}}(\boldsymbol{\omega})| = 0.$$

Now suppose $\boldsymbol{\omega}$ is an even word with b distinct letter and $(r+1)$ even generating vertices. Let i_1, i_2, \dots, i_b be the positions where new letters made their first appearances. First we fix the generating vertices $\pi(i_j), 0 \leq j \leq b$ where $\pi(i_0) = \pi(0)$. Let

$$s_i = \pi(i) - \pi(i-1) \quad \text{for } 1 \leq i \leq 2k.$$

Clearly, from (5.2), $\boldsymbol{\omega}[i] = \boldsymbol{\omega}[j]$ if and only if $\xi_\pi(i) = \xi_\pi(j)$. That is, $|s_i| = |s_j|$, that is, $s_i - s_j = 0$ or $s_i + s_j = 0$. Clearly $s_{i_1} = s_1$. If the first letter appears in the j -th position, then $s_j = s_1$ or $s_j = -s_1$. Similarly, for every i , $1 \leq i \leq 2k$, for some $j \in \{1, 2, \dots, b\}$,

$$s_i = s_{i_j} \text{ or } s_i = -s_{i_j}. \quad (5.11)$$

Thus, we have

$$\pi(i) = \pm(\pi(i_j) - \pi(i_j - 1)) + \pi(i - 1) = \pm s_{i_j} + \pi(i - 1) \quad \text{for some } j. \quad (5.12)$$

Let

$$v_{2i} = \frac{\pi(2i)}{p} \quad \text{for } 0 \leq i \leq k \quad \text{and} \quad v_{2i-1} = \frac{\pi(2i-1)}{n} \quad \text{for } 1 \leq i \leq k,$$

$$u_i = \frac{s_i}{n} \quad \text{for } 1 \leq i \leq 2k.$$

Let $y_n = p/n$. Now, $\pi(i) = \pi(i-1) \pm s_{i_j}$ whenever the i -th letter in $\boldsymbol{\omega}$ is same as the j -th distinct letter that appeared first at the i_j -th position. Therefore $v_i = \frac{1}{y_n}(v_{i-1} \pm u_{i_j})$ when i is even and $v_i = y_n v_{i-1} \pm u_{i_j}$ when i is odd.

Let

$$S = \{\pi(i_j) : 0 \leq j \leq b\} \quad \text{and} \quad S' = \{i : \pi(i) \notin S\}.$$

That is, S is the set of all distinct generating vertices and S' is the set of all indices of the non-generating vertices. We have the following claim.

Claim: For any $1 \leq i \leq 2k$,

$$v_i = \begin{cases} v_0 + \frac{1}{y_n} \sum_{j=1}^i \alpha_{ij} u_{i_j} & \text{if } i \text{ is even,} \\ y_n v_0 + \sum_{j=1}^i \alpha_{ij} u_{i_j} & \text{if } i \text{ is odd,} \end{cases} \quad (5.13)$$

where α_{ij} depends on the choice of sign in (5.12).

We prove this by induction on i . We know that $\pi(1) \in S$. Clearly, $v_1 = u_1 + y_n v_0$. Now either $\pi(2) \in S$ or $2 \in S'$. If $\pi(2) \in S$, then $v_2 = \frac{1}{y_n}(u_2 - v_1)$ and $v_1 = u_1 - y_n v_0$. Therefore $v_2 = \frac{1}{y_n}(u_2 - u_1) + v_0$. If $2 \in S'$, then $u_2 = \pm u_1$ and $v_2 = \frac{1}{y_n}(v_1 \pm u_1)$. So either $v_2 = \frac{1}{y_n}(u_1 + y_n v_0 + u_1)$ or $v_2 = \frac{1}{y_n}(u_1 + y_n v_0 - u_1) = v_0$. Hence the claim is true for $i = 2$.

Now we assume that the claim is true for all $j < i$ and prove it for i . There are two cases:

Case 1: i is even. Now either $\pi(i) \in S$ or $i \in S'$. If $\pi(i) \in S$, then

$$\begin{aligned} v_i &= \frac{1}{y_n}(u_i + v_{i-1}) \\ &= \frac{1}{y_n}(u_i + y_n v_0 + \sum_{j=1}^{i-1} \alpha_{(i-1)j} u_{i_j}) \quad (\text{by induction hypothesis as } i-1 \text{ is odd}) \\ &= v_0 + \frac{1}{y_n} \sum_{j=1}^i \alpha_{ij} u_{i_j}, \end{aligned}$$

where $\alpha_{ii} = 1$. If $i \in S'$, then there exists j such that $i_j < i$ and $u_i = \pm u_{i_j}$. Then either $v_i = \frac{1}{y_n}(v_{i-1} + u_{i_j})$, or $v_i = \frac{1}{y_n}(v_{i-1} - u_{i_j})$. Hence either

$$v_i = v_0 + \frac{1}{y_n} \left(\sum_{j=1}^{i-1} \alpha_{(i-1)j} u_{i_j} + u_{i_j} \right), \quad \text{or} \quad v_i = v_0 + \frac{1}{y_n} \left(\sum_{j=1}^{i-1} \alpha_{(i-1)j} u_{i_j} - u_{i_j} \right).$$

Therefore $v_i = v_0 + \frac{1}{y_n} \sum_{j=1}^i \alpha_{ij} u_{i_j}$ where $\alpha_{ij} = \alpha_{(i-1)j} + 1$ or $\alpha_{(i-1)j} - 1$ (depending on the sign of the equation).

Case 2: i is odd. Then using similar argument as above we can show that, $v_i = y_n v_0 + \sum_{j=1}^i \alpha_{ij} u_{i_j}$.

Thus the claim is proved.

Let $u_S = \{u_i : \pi(i) \in S\}$ and $v_S = \{v_i : \pi(i) \in S\}$. From the previous claim, we have, for $1 \leq i \leq 2k$,

$$v_i = \begin{cases} v_0 + \frac{1}{y_n} L_{i,u,n}^T(u_S), & \text{if } i \text{ is even,} \\ y_n v_0 + L_{i,u,n}^T(u_S) & \text{if } i \text{ is odd,} \end{cases}$$

where $L_{i,u,n}^T(u_S)$ denotes a linear combination of $\{u_i : \pi(i) \in S\}$.

Also, for $1 \leq i \leq 2k$, we have

$$v_i = L_{i,n}^T(v_S), \quad (5.14)$$

where $L_{i,n}^T(v_S)$ denotes a linear combination of $\{v_i : \pi(i) \in S\}$ arising from (5.11). Now, the linear combinations vary depending on the sign chosen for each s_i . As we know, for each block of an even word, the number of positive and negative signs in the relations among the s_i 's (i.e., the equations like (5.11)) are equal. Therefore there are $\prod_{i=1}^b \binom{k_i - 1}{\frac{k_i}{2}}$ different sets of linear combinations corresponding to each word ω , where k_1, \dots, k_b are the block sizes of ω .

Let $U_p = \{0, 1/p, \dots, (p-1)/p\}$, $U_n = \{0, 1/n, \dots, (n-1)/n\}$. Then it is easy to see that for a word ω of length $2k$ (L is the link function for $T^{(s)}$),

$$\begin{aligned} |\Pi_{S_{T^{(s)}}}(\omega)| &= \left| \left\{ (v_0, v_1, \dots, v_{2k}) : v_{2i} \in U_p, v_{2i-1} \in U_n \text{ for } 0 \leq i \leq k, v_0 = v_{2k}, \right. \right. \\ &\quad \left. \left. L(v_{i-1}, v_i) = L(v_{j-1}, v_j) \text{ whenever } \omega[i] = \omega[j] \right\} \right|. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_{T^{(s)}}}(\omega)| = \sum_{L_{\omega}^T} \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S, \quad (5.15)$$

where $dx_S = \prod_{j=0}^b dx_{i_j}$ denotes the $(b + 1)$ -dimensional Lebesgue measure, $x_{i_0} = x_0$, and L_i^T is the

limit of $L_{i,n}^T$ as $p/n \rightarrow y$ and $\sum_{L_\omega^T}$ is the sum over all the $\prod_{i=1}^b \binom{k_i - 1}{\frac{k_i}{2}}$ such different sets of linear combinations corresponding to ω .

As observed in Step 1 in Lemma 5.3 of (Bose et al., 2021), choosing $v_{i_j}, 0 \leq j \leq b$ freely is equivalent to choosing v_0 and $u_{i_j}, 1 \leq j \leq b$ freely. Note that $-1 \leq u_{2i} \leq y_n$ and $-y_n \leq u_{2i-1} \leq 1$ for $0 \leq i \leq k$. Also by abuse of notation we denote the variables in the limit as u_{i_j} . So,

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_T(s)}(\omega)| = \sum_{L_\omega^T} \int_0^1 \int_{-1}^y \cdots \int_{-y}^1 \mathbf{1}(0 \leq x_0 + \frac{1}{y} L_{i,u}^T(u_S) \leq 1, \forall 2i \in S') \mathbf{1}(0 \leq yx_0 + L_{i,u}^T(u_S) \leq 1, \forall (2i - 1) \in S') dx_0 du_S, \quad (5.16)$$

where $du_S = \prod_{j=1}^b du_{i_j}$ denotes the $(b + 1)$ -dimensional Lebesgue measure on $[-1, y]^r \times [-y, 1]^{b-r}$.

Suppose, a particular set of linear combinations $L_{i,u}^T$ is given, i.e., for $i \in S'$, (5.13) holds and the values of $\alpha_{im}, 1 \leq i \leq 2k, 1 \leq j \leq b$ are known. Here we show that the integral in (5.16) is positive on a certain region in $[0, 1] \times [-1, y]^r \times [-y, 1]^{b-r}$. We divide this proof into two cases.

Case 1: $y > 1$. First let

$$C = \max\{|\alpha_{ij}| : 1 \leq j \leq b \text{ and } i \in S'\}. \quad (5.17)$$

Next we choose ϵ such that $Cb\epsilon < 1/2$. Now, let $|u_{i_j}| < \epsilon$ for $1 \leq j \leq b$ and $\frac{Cb\epsilon}{y} \leq v_0 \leq \frac{1-Cb\epsilon}{y}$. Then, for all $2i \in S'$, $0 \leq v_0 + \frac{1}{y} L_{2i,u}^T(u_S) \leq 1$ and for all $2i - 1 \in S'$, $0 \leq yv_0 + L_{2i-1,u}^T(u_S) \leq 1$. Also the circuit condition is automatically satisfied. Note that as observed before, we cannot choose the s_{i_j} 's freely in case of words that are not even.

Case 2: $y \leq 1$. First let C be as in (5.17). Next we choose ϵ such that $\frac{Cb\epsilon}{y} < 1/2$. Now, let $|u_{i_j}| < \epsilon$ for $1 \leq j \leq b$ and $\frac{Cb\epsilon}{y} \leq v_0 \leq 1 - \frac{Cb\epsilon}{y}$. Then, for all $2i \in S'$, $0 \leq v_0 + \frac{1}{y} L_{2i,u}^T(u_S) \leq 1$ and for all $2i - 1 \in S'$, $0 \leq yv_0 + L_{2i-1,u}^T(u_S) \leq 1$.

Thus

$$\lim_{n \rightarrow \infty} \frac{1}{n^{b+1}} |\Pi_{S_T(s)}(\omega)| = \alpha(\omega) > 0 \text{ for any even word } \omega,$$

where $\alpha_{T(s)}(\omega)$ is the sum of the integrals defined in (5.16).

This completes the proof of the lemma. □

Lemma 5.5. *Suppose ω is a word with b distinct letters and $(r + 1)$ even generating vertices. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_T}(\omega)| = \alpha_T(\omega) > 0 \text{ if and only if } \omega \text{ is symmetric.}$$

Proof: Let

$$s_i = \pi(i) - \pi(i - 1), \quad \text{for } 1 \leq i \leq 2k.$$

From (5.2), we know that $\omega[i] = \omega[j]$ if and only if $\xi_\pi(i) = \xi_\pi(j)$. This implies

$$\begin{aligned} s_i &= s_j & \text{when } i \text{ and } j \text{ are of same parity,} \\ s_i &= -s_j & \text{when } i \text{ and } j \text{ are of opposite parity.} \end{aligned} \quad (5.18)$$

Now we fix an ω with b distinct letters which appear at i_1, i_2, \dots, i_b positions for the first time. Also let ω have $(r + 1)$ even generating vertices. Using the same arguments as in Step 1 of Lemma 5.3 of

(Bose et al., 2021), being able to choose $\pi(i_j), 0 \leq j \leq b$ is equivalent to choosing $\pi(0), s_{i_j}, 1 \leq j \leq b$ freely. Next we show that if $\pi(0)$ and $s_{i_j}, 1 \leq j \leq b$ can be chosen freely, then the word is symmetric.

To see this, observe that the circuit condition gives

$$\sum_{i=1}^{2k} s_i = \pi(0) - \pi(2k) = 0. \quad (5.19)$$

Using (5.18), we see that there exists $\alpha_j, 1 \leq j \leq b$ such that

$$\sum_{i=1}^b \alpha_j s_{i_j} = 0.$$

Since we need the s_{i_j} 's to have free choice, we must have $\alpha_j = 0$ for all $j \in \{1, 2, \dots, b\}$. Therefore for each j ,

$$|\{l : s_l = s_{i_j}\}| = |\{l : s_l = -s_{i_j}\}|. \quad (5.20)$$

Now from the definition of ξ_π , $\xi_\pi(2i) = s_{2i}, 0 \leq i \leq k$ and $\xi_\pi(2i-1) = -s_{2i-1}, 0 \leq i \leq k$. Therefore for each $j \in \{1, 2, \dots, b\}$, to satisfy (5.20), we must have

$$|\{l : l \text{ even and } \xi_\pi(l) = \xi_\pi(i_j)\}| = |\{l : l \text{ odd and } \xi_\pi(l) = \xi_\pi(i_j)\}|.$$

That is, each letter appears equal number of times at odd and even places. Hence the word is symmetric.

Now if ω is not symmetric, at least one of the generating vertices is a linear combination of the others, and hence cannot be chosen freely. So,

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_T}(\omega)| = 0 \text{ if } \omega \text{ is not symmetric.}$$

Next, suppose ω is a symmetric word with b distinct letters and $(r+1)$ even generating vertices. We shall show that $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_T}(\omega)| = \alpha_T(\omega) > 0$.

Suppose the letters make their first appearances at i_1, i_2, \dots, i_b positions in ω . First we fix the generating vertices $\pi(i_j), 0 \leq j \leq b$. Suppose $S = \{\pi(i_j) : 0 \leq j \leq b\}$ and $S' = \{i : \pi(i) \notin S\}$. For $i \in S'$, $\xi_\pi(i) = \xi_\pi(i_j)$ for some $j \in \{1, 2, \dots, b\}$. Then

$$\begin{aligned} \pi(i) &= s_{i_j} + \pi(i-1) \quad \text{if } i \text{ and } i_j \text{ are of same parity,} \\ \pi(i) &= -s_{i_j} + \pi(i-1) \quad \text{if } i \text{ and } i_j \text{ are of opposite parity.} \end{aligned} \quad (5.21)$$

Thus, (5.21) is nothing but (5.12) where the sign has been determined depending on the parity of i and i_j . Hence, for $1 \leq i \leq 2k$, $v_i = L_{i,n}^T(v_S)$, (the notations are the same as in the proof of Lemma 5.4) where $L_{i,n}^T(\cdot)$ is a particular set of linear combinations that has been determined by (5.21). As a result, the rest of the proof is same as that in Lemma 5.4. Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_T}(\omega)| = \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S, \quad (5.22)$$

where $dx_S = \prod_{j=0}^b dx_{i_j}$ denotes the $(b+1)$ -dimensional Lebesgue measure, $x_{i_0} = x_0$ and L_i^T is the limit of $L_{i,n}^T$ as $n \rightarrow \infty$.

That the integral is positive now follows from the proof of the same fact in Lemma 5.4. Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_T}(\omega)| = \alpha_T(\omega) > 0 \text{ for every symmetric word } \omega,$$

where α_T is the value of an individual integral in the rhs of (5.16). \square

Lemma 5.6. *Suppose ω is a word with b distinct letters and $(r+1)$ even generating vertices. Then*

- (i) $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{H(s)}}(\boldsymbol{\omega})| = \alpha_{H(s)}(\boldsymbol{\omega}) > 0$ if and only if $\boldsymbol{\omega}$ is a symmetric word.
- (ii) $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_H}(\boldsymbol{\omega})| = \alpha_H(\boldsymbol{\omega})$ can only be positive if $\boldsymbol{\omega}$ is a symmetric word. In case of a symmetric word, the value of $\alpha_H(\boldsymbol{\omega})$ is determined by an integral given in (5.33).

Proof: First suppose $\boldsymbol{\omega} \in \mathcal{P}(2k) \setminus S_b(2k)$. Then from (5.4) and Lemma 5.4 in (Bose et al., 2021), it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{H(s)}}(\boldsymbol{\omega})| = \lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_H}(\boldsymbol{\omega})| = 0.$$

Now suppose $\boldsymbol{\omega}$ is a symmetric word with b distinct letters and $(r + 1)$ even generating vertices. Suppose i_1, i_2, \dots, i_b are the positions where new letters made their first appearances. First we fix the generating vertices $\pi(i_j), 0 \leq j \leq b$ where $\pi(i_0) = \pi(0)$. Let

$$t_i = \pi(i) + \pi(i - 1) \text{ for } 1 \leq i \leq 2k.$$

Now let us first consider the symmetric Hankel link function. Clearly from (5.2), $\boldsymbol{\omega}[i] = \boldsymbol{\omega}[j]$ if and only if $t_i - t_j = 0$. Further $t_{i_1} = t_1$. If the first letter again appears at the j -th position, then $t_j = t_1$. Similarly, for every i , $1 \leq i \leq 2k$,

$$t_i = t_{i_j} \text{ for some } j \in \{1, 2, \dots, b\}. \tag{5.23}$$

First we fix the generating vertices $\pi(i_j), j = 0, 1, 2, \dots, b$. Let

$$v_{2i} = \frac{\pi(i)}{p}, \quad v_{2i-1} = \frac{\pi(i)}{n} \text{ for } 0 \leq i \leq k, S = \{\pi(i_j) : 0 \leq j \leq b\} \text{ and } S' = \{i : \pi(i) \notin S\}.$$

For $1 \leq i \leq 2k$, from the link function and the formula for t_i we have

$$v_i = L_{i,n}^H(v_S), \tag{5.24}$$

where $L_{i,n}^H(v_S)$ denotes a linear combination of $\{v_i : \pi(i) \in S\}$.

Let $U_p = \{0, 1/p, \dots, (p - 1)/p\}$ and $U_n = \{0, 1/n, \dots, (n - 1)/n\}$. From (5.5), it is easy to see that for a word $\boldsymbol{\omega}$ of length $2k$,

$$|\Pi_{S_{H(s)}}(\boldsymbol{\omega})| = \left| \left\{ (v_0, v_1, \dots, v_{2k}) : v_{2i} \in U_p, v_{2i-1} \in U_n \text{ for } 0 \leq i \leq k, v_0 = v_{2k}, v_i = L_{i,n}^H(v_S) \right\} \right|.$$

Transforming $v_i \mapsto x_i = v_i - \frac{1}{2}$, we get that

$$\begin{aligned} |\Pi_{S_{H(s)}}(\boldsymbol{\omega})| &= \left| \left\{ (x_0, x_1, \dots, x_{2k}) : x_{2i} \in \{-1/2, -1/2 + 1/p, \dots, -1/2 + (p - 1)/p\}, \right. \right. \\ &\quad \left. \left. x_{2i-1} \in \{-1/2, -1/2 + 1/n, \dots, -1/2 + (n - 1)/n\} \text{ for } 0 \leq i \leq k, \right. \right. \\ &\quad \left. \left. x_0 = x_{2k} \text{ and } x_i = L_{i,n}^H(y_S) \right\} \right|. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} \Pi_{S_{H(s)}}(\boldsymbol{\omega}) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \dots \int_{-1/2}^{1/2} \mathbf{1}(-1/2 \leq L_i^H(x_S) \leq 1/2, \forall i \in S') dx_S, \tag{5.25}$$

where $dx_S = \prod_{j=0}^b dx_{i_j}$ denotes the $(b + 1)$ -dimensional Lebesgue measure on $[-\frac{1}{2}, \frac{1}{2}]^{b+1}$ and L_i^H is the limit of $L_{i,n}^H$ as $n \rightarrow \infty$.

Let $y_n = p/n$ and for $1 \leq i \leq k$,

$$p_{2i} = x_{2i-1} + y_n x_{2i}, \quad p_{2i-1} = y_n x_{2i-2} + x_{2i-1}, \tag{5.26}$$

and

$$q_{2i} = x_{2i-1} - y_n x_{2i}, \quad q_{2i-1} = y_n x_{2i-2} - x_{2i-1}. \tag{5.27}$$

Now we have the following claim.

Claim: For any $1 \leq i \leq 2k$,

$$x_i = \begin{cases} x_0 + \frac{1}{y_n} \sum_{j=1}^i \alpha_{ij} p_{i_j} & \text{if } i \text{ is even,} \\ -y_n x_0 + \sum_{j=1}^i \alpha_{ij} p_{i_j} & \text{if } i \text{ is odd.} \end{cases}$$

We prove this by induction in i . We know that $\pi(1) \in S$. Clearly, $x_1 = p_1 - y_n x_0$. Now either $\pi(2) \in S$ or $2 \in S'$. If $\pi(2) \in S$, then $x_2 = \frac{1}{y_n}(p_2 - x_1)$. Therefore $x_2 = \frac{1}{y_n}(p_2 - p_1) + x_0$. If $2 \in S'$, then $p_2 = p_1$ and $x_2 = \frac{1}{y_n}(p_1 - y_1) = y_0$. So the claim is true for $i = 2$.

Now we assume that the claim is true for all $j < i$ and try to prove it for i . Then either $\pi(i) \in S$ or $i \in S'$.

(a) If $\pi(i) \in S$ and i is even, then $y_i = \frac{1}{y_n}(p_i - y_{i-1})$. Now, $i - 1$ is odd and hence $y_{i-1} = -y_n y_0 + \sum_{j=1}^{i-1} \alpha_{(i-1)j} p_{i_j}$ by induction hypothesis. Therefore $y_i = y_0 + \frac{1}{y_n} \sum_{j=1}^i \alpha_{ij} p_{i_j}$ where $\alpha_{ii} = 1$.

The case where i is odd can be tackled similarly.

(b) If $i \in S'$, then there exists m such that $i_m < i$ and $p_i = p_{i_m}$. Now if i is even, $y_i = \frac{1}{y_n}(p_{i_m} - y_{i-1})$.

As $i - 1$ is odd, $y_{i-1} = -y_n y_0 + \sum_{j=1}^{i-1} \alpha_{(i-1)j} p_{i_j}$ and therefore $y_i = y_0 + \frac{1}{y_n} \sum_{j=1}^{i-1} \alpha_{ij} p_{i_j}$ where $\alpha_{im} = \alpha_{(i-1)m} + 1$. The case where i is odd can be tackled similarly.

Thus the claim is proved.

Now we perform the following change of variables in (5.25):

$$(x_0, x_1, x_2, x_3, \dots, x_{2k}) \longrightarrow (x_0, -x_1, x_2, -x_3, \dots, x_{2k}) = (z_0, z_1, z_2, z_3, \dots, z_{2k}) \text{ (say).}$$

Observe that this transformation does not alter the value of the integral in (5.25). Also observe that using (5.26) and (5.27), under this transformation,

$$(p_1, p_2, p_3, \dots, p_{2k}) \longrightarrow (q_1, -q_2, q_3, \dots, -q_{2k}).$$

Then from the claim it follows that

$$z_i = \begin{cases} z_0 + \frac{1}{y_n} \sum_{j=1}^i \beta_{ij} q_{i_j} & \text{if } i \text{ is even,} \\ y_n z_0 + \sum_{j=1}^i \beta_{ij} q_{i_j} & \text{if } i \text{ is odd,} \end{cases}$$

where $\beta_{ij} = \pm \alpha_{ij}$ according as i_j is odd or even. We shall use the notation $z_i = L_{i,q,n}^H(z_S)$ to denote this linear relation.

Also note that choosing $x_{i_j}, 0 \leq j \leq b$ freely is equivalent to choosing $p_{i_j}, 0 \leq j \leq b$ (where $p_{i_0} = x_0$). Further, $-\frac{y_n+1}{2} \leq q_{i_j} \leq \frac{y_n+1}{2}$. Therefore we can write (5.25) as,

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} \Pi_{S_{H(s)}}(\omega) = \int_{-1/2}^{1/2} \int_{-\frac{y+1}{2}}^{\frac{y+1}{2}} \int_{-\frac{y+1}{2}}^{\frac{y+1}{2}} \cdots \int_{-\frac{y+1}{2}}^{\frac{y+1}{2}} \mathbf{1}(-1/2 \leq L_{i,q}^H(z_S) \leq 1/2, \forall i \in S') dq_S,$$

where $dq_S = \prod_{j=0}^b dq_{i_j}$ denotes the $(b+1)$ -dimensional Lebesgue measure on $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{y+1}{2}, \frac{y+1}{2}]^b$,

and $L_{i,q}^H$ denotes the limit of the linear combination $L_{i,q,n}^H$ as $y_n \rightarrow y > 0$.

Now it can be proved that the above integrand is positive on a region of positive measure on $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{y+1}{2}, \frac{y+1}{2}]^b$ —the proof is similar to the proof that the integral in the rhs of (5.16) is positive. So we omit the details.

This prove part (i) of the lemma.

To prove part (ii), note that for the asymmetric Hankel link function,

$$\begin{aligned} \xi_\pi(i) &= \xi_\pi(j) \quad \text{if and only if } t_i = t_j \quad \text{and} \\ \text{sgn}(\pi(i) - \pi(i-1)) &= \text{sgn}(\pi(j) - \pi(j-1)) \quad \text{if } i \text{ and } j \text{ are of same parity, or} \\ \text{sgn}(\pi(i) - \pi(i-1)) &= \text{sgn}(\pi(j-1) - \pi(j)) \quad \text{if } i \text{ and } j \text{ are of opposite parity.} \end{aligned}$$

Let

$$\mathcal{E}_\omega = \{0, i_j; i_j \text{ is even}, 1 \leq j \leq b\} \quad (5.28)$$

and

$$\mathcal{O}_\omega = \{i_j; i_j \text{ is odd}, 1 \leq j \leq b\}. \quad (5.29)$$

For every $j \in \{1, 2, \dots, b\}$, let

$$C_{i_j}^o = \{i; \xi_\pi(i) = \xi_\pi(i_j), i, i_j \text{ are of opposite parity}, 0 \leq i \leq 2k\} \quad (5.30)$$

$$C_{i_j}^e = \{i; \xi_\pi(i) = \xi_\pi(i_j), i, i_j \text{ are of same parity}, 0 \leq i \leq 2k\}. \quad (5.31)$$

Using the notations as in the proof of part (i), we now have that

$$\begin{aligned} |\Pi_{S_H}(\omega)| &= \\ & \left| \left\{ (v_0, v_1, \dots, v_{2k}) : v_{2i} \in U_p, v_{2i-1} \in U_n \text{ for } 0 \leq i \leq k, v_0 = v_{2k}, v_i = L_{i,n}^H(v_S), \right. \right. \\ & \quad \text{sgn}(y_n L_{i,n}^H(v_S) - L_{i-1,n}^H(v_S)) = \text{sgn}(y_n v_{i_j} - L_{i-1,n}^H(v_S)) \text{ when } i_j \in \mathcal{E}_\omega \text{ and } i \in C_{i_j}^e \\ & \quad \text{or } \text{sgn}(y_n L_{i-1,n}^H(v_S) - L_{i,n}^H(v_S)) = \text{sgn}(y_n v_{i_j} - L_{i-1,n}^H(v_S)) \text{ when } i_j \in \mathcal{E}_\omega \text{ and } i \in C_{i_j}^o, \\ & \quad \text{sgn}(y_n L_{i,n}^H(v_S) - L_{i-1,n}^H(v_S)) = \text{sgn}(y_n L_{i_j-1,n}^H(v_S) - v_{i_j-1}) \text{ when } i_j \in \mathcal{O}_\omega \text{ and } i \in C_{i_j}^e \\ & \quad \left. \left. \text{or } \text{sgn}(y_n L_{i-1,n}^H(v_S) - L_{i,n}^H(v_S)) = \text{sgn}(y_n v_{i_j} - L_{i-1,n}^H(v_S)) \text{ when } i_j \in \mathcal{O}_\omega \text{ and } i \in C_{i_j}^o \right\} \right|. \end{aligned}$$

Now, $\Pi_{S_H}(\omega) \subset \Pi_{S_{H(s)}}(\omega)$. If ω is a word with b distinct letters but not symmetric, by part (i), $\frac{1}{p^{r+1}b^{n-r}} |\Pi_{S_H}(\omega)| \rightarrow 0$ as $n \rightarrow \infty$.

Next let $\omega \in S_b(2k)$ with $(r+1)$ even generating vertices. Clearly for ω , $|\mathcal{E}_\omega| = r+1$ and $|\mathcal{O}_\omega| = b-r$. Now suppose,

$$\begin{aligned} f_n^H(v_S) &= \prod_{j=1}^b \left[\prod_{i_j \in \mathcal{E}_\omega} \left(\prod_{i \in C_{i_j}^e} \mathbf{1}(\text{sgn}(y_n L_{i,n}^H(v_S) - L_{i-1,n}^H(v_S)) = \text{sgn}(y_n v_{i_j} - L_{i-1,n}^H(v_S))) \right. \right. \\ & \quad \prod_{i \in C_{i_j}^o} \mathbf{1}(\text{sgn}(y_n L_{i-1,n}^H(v_S) - L_{i,n}^H(v_S)) = \text{sgn}(y_n v_{i_j} - L_{i-1,n}^H(v_S))) \\ & \quad \left. \prod_{i_j \in \mathcal{O}_\omega} \left(\prod_{i \in C_{i_j}^e} \mathbf{1}(\text{sgn}(y_n L_{i,n}^H(v_S) - L_{i-1,n}^H(v_S)) = \text{sgn}(y_n L_{i_j-1,n}^H(v_S) - v_{i_j-1})) \right) \right. \\ & \quad \left. \prod_{i \in C_{i_j}^o} \text{sgn}(y_n L_{i-1,n}^H(v_S) - L_{i,n}^H(v_S)) = \text{sgn}(y_n v_{i_j} - L_{i-1,n}^H(v_S)) \right), \quad (5.32) \end{aligned}$$

and let f^H be the limit of f_n^H as $y_n \rightarrow y > 0$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} \Pi_{S_H}(\omega) = \int_0^1 \int_0^1 \dots \int_0^1 \mathbf{1}(0 \leq L_i^H(v_S) \leq 1) f^H(v_S) dv_S \quad (5.33)$$

where $dv_S = \prod_{j=0}^b dv_{i_j}$ is the $(b+1)$ -dimensional Lebesgue integral on $[0, 1]^{b+1}$, L_i^H is the limit of the linear combination $L_{i,n}^H$ as $y_n \rightarrow y$.

This completes the proof of part (ii). □

Let $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Lemma 5.7. *Suppose ω is a word of length $2k$ with b distinct letters and $(r+1)$ even generating vertices. Then*

- (i) $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{R(s)}}(\omega)| = [y]^{k-(r+1)} + \alpha_{R(s)}(\omega) > 0$ if and only if ω is a symmetric word.
- (ii) $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_R}(\omega)| = [y]^{k-(r+1)} + \alpha_R(\omega)$ if and only if ω is a symmetric word.

The proof of this lemma borrows the main ideas from the proof of Lemma 5.6, and is given in details in Section 6.

Recall the sequence $a_{2n} = \frac{1}{2} \binom{2n}{n}$ from Lemma 5.2 in (Bose et al., 2021).

Lemma 5.8. *Suppose ω is a word of length $2k$ with b distinct letters and $(r+1)$ even generating vertices. Then*

- (i) $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{C(s)}}(\omega)| = a_\omega [y]^{k-(r+1)} + \alpha_{C(s)}(\omega) > 0$ if and only if ω is an even word and a_ω is the multiplicative extension of the sequence a_{2n} when ω is considered as a partition in $\{1, 2, \dots, 2k\}$.
- (ii) $\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_C}(\omega)| = [y]^{k-(r+1)} + \alpha_C(\omega) > 0$ if and only if ω is a symmetric word.

The proof of this lemma borrows a lot of ideas from the proofs of Lemmas 5.4 and Lemma 5.5, and is given in details in Section 6.

5.2.2. Proof of Theorem 2.4.

Lemma 5.9. *Recall the matrix Z_A from Theorem 2.4. Suppose, \widetilde{Z}_A is the $p \times n$ matrix whose entries are $(y_i - \mathbb{E}y_i)$ and thus have mean 0. Under the assumptions of Theorem 2.4, the LSD of S_{Z_A} and $S_{\widetilde{Z}_A}$ are same.*

Proof: In Step 1 of the proof of Theorem 2.1, we dealt with the same problem but for a bi-sequence of random variables $\{x_{ij,n}\}$. The same proof can be adapted in this case replacing it by the sequence $\{x_{i,n}\}$. Therefore, condition (2.4) is true for \widetilde{Z}_A . Similarly we can show that (2.5) is true for \widetilde{Z}_A . Hence, Assumption B holds for \widetilde{Z}_A .

Now from Lemma 5.3,

$$\begin{aligned} L^4(\mathbb{E}F^{S_{Z_A}}, \mathbb{E}F^{S_{\widetilde{Z}_A}}) &\leq \frac{2}{p^2} (\mathbb{E} \text{Tr}(Z_A Z_A^T + \widetilde{Z}_A \widetilde{Z}_A^T)) (\mathbb{E} \text{Tr}[(Z_A - \widetilde{Z}_A)(Z_A - \widetilde{Z}_A)^T]) \\ &\leq \frac{2}{p} \left(\sum_{i=-(n+p)}^{n+p} cn \mathbb{E}(2y_i^2 + (\mathbb{E}y_i)^2 - 2y_{ij} \mathbb{E}y_{ij}) \right) \frac{1}{p} \left(\sum_{i=-(n+p)}^{n+p} cn (\mathbb{E}y_i)^2 \right), \end{aligned}$$

where c is a constant depending on the link function of the matrix. Observe that for all matrices with link functions (i)-(viii), the second inequality is true due to the structure of the link functions. The second factor of the rhs in the above inequality is bounded by

$$\frac{2}{y_n} (n+p) (\sup_i \mathbb{E}y_i)^2 = \frac{2}{y_n} (\sup_i \sqrt{n} \mathbb{E}y_i)^2 + \frac{2}{y_n} (\sup_i \sqrt{p} \mathbb{E}y_i)^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty, p/n \rightarrow y > 0 \quad \text{by (2.5).}$$

Again, $\mathbb{E} \left[\frac{1}{p} \sum_i y_i^2 \right] \rightarrow \int_{[0,1]^2} f_2(x, y) dx dy$. Therefore the first term of the rhs in the inequality is bounded uniformly, and hence $L^4(\mathbb{E}F^{S_{Z_A}}, \mathbb{E}F^{S_{\widetilde{Z}_A}}) \rightarrow 0$ as $p \rightarrow \infty$. Thus we can assume that the entries of Z_A have mean 0. \square

Lemma 5.10. *Under the conditions of Theorem 2.4, if the EESD of the matrices S_{Z_A} converges weakly to μ_A , then, with the assumption (2.6), the EESD of S_A (where A is the non-truncated version of Z) converges weakly to μ_A .*

Proof: Observe that from Lemma 5.3, we have

$$\begin{aligned} L^4(\mathbb{E}F^S, \mathbb{E}F^{S_{Z_A}}) &\leq \frac{2}{p^2}(\mathbb{E} \operatorname{Tr}(AA^T + Z_A Z_A^T))(\mathbb{E} \operatorname{Tr}[(A - Z_A)(A - Z_A)^T]) \\ &\leq \frac{2}{p} \left(2cn \sum_{i=-(n+p)}^{n+p} \mathbb{E}[y_i^2] + cn \sum_{i=-(n+p)}^{n+p} \mathbb{E}[x_i^2 \mathbf{1}_{\{|x_i| > t_n\}}] \right) \left(\frac{1}{p} \sum_{i=-(n+p)}^{n+p} cn \mathbb{E}[x_i^2 \mathbf{1}_{\{|x_i| > t_n\}}] \right). \end{aligned} \quad (5.34)$$

The second factor in the above inequality tends to zero a.s. (or in probability) as $n \rightarrow \infty$ from (2.6). Again, the first factor is uniformly bounded as in the proof of Lemma 5.9. Thus $L^4(\mathbb{E}F^{S_A}, \mathbb{E}F^{S_{Z_A}}) \rightarrow 0$ as $p \rightarrow \infty$.

This completes the proof of the lemma. □

Now we will prove Theorem 2.4. The arguments in the proof of the different parts are often repetitive. So we prove Part (i) in details and omit the elaborate arguments for the other parts.

Proof of Theorem 2.4: We shall prove the theorem in different parts.

(i): Let $A = T^{(s)}$. First observe that from Lemma 5.10, it is enough to prove that the EESD of S_{Z_A} converges to $\mu_{T^{(s)}}$. Further, from Lemma 5.9 we may assume that $\mathbb{E}(y_i) = 0$. Therefore it suffices to verify the first moment condition and the Carleman's condition for S_{Z_A} .

As $\mathbb{E}(y_i) = 0$, from (5.1), if $\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{\pi \in \Pi_{S_{T^{(s)}}}(\omega)} \mathbb{E}(Y_\pi)$ exists for every matched word ω of length

$2k$ with b distinct letters and $(r + 1)$ even generating vertices ($k \geq 1, 1 \leq b \leq b, 0 \leq r \leq (b - 1)$), then the first moment condition would follow.

Suppose ω is a word with b distinct letters, $(r + 1)$ even generating vertices and the distinct letters appear k_1, k_2, \dots, k_b times. Let the j th distinct letter appear at $(\pi(i_j - 1), \pi(i_j))$ th position for the first time. Denote $(\pi(i_j - 1), \pi(i_j))$ as (m_j, l_j) . Let us now recall $v_i, 1 \leq i \leq 2k$ and $s_i, u_i, 1 \leq i \leq 2k$ as defined in Lemma 5.4.

First, let $\omega \notin E(2k)$. Suppose ω contains b_1 distinct letters that appear even number of times and b_2 number of distinct letters that appear odd number of times and $b = b_1 + b_2$. So we assume that for each $\pi \in \Pi(\omega)$, $k_{j_p}, 1 \leq p \leq b_1$ are even and $k_{j_q}, b_1 + 1 \leq q \leq b_1 + b_2$ are odd. Hence the contribution of this ω to (4.10) is as follows:

$$\frac{1}{pn^{b_1+b_2-\frac{1}{2}}} \sum_S \prod_{p=1}^{b_1} f_{k_{j_p}}(|s_{j_p}|) \prod_{q=b_1+1}^{b_1+b_2} n^{\frac{b_2-1/2}{b_2}} \mathbb{E} \left[y_{|s_{j_q}|}^{k_{j_q}} \right]. \quad (5.35)$$

For n large, $n^{\frac{b_2-1/2}{b_2}} \mathbb{E} \left[y_{(s_{j_q}-2) \pmod{n}}^{k_{j_q}} \right] < 1$ for any $b_1 + 1 \leq q \leq b_1 + b_2$ and $\prod_{p=1}^{b_1} f_{k_{j_p}}(|s_{j_p}|) \leq M$ (independent of n). Now as $\omega \notin E_b(2k)$ and $p/n \rightarrow y > 0$, from Lemma 5.4 we have, $|S| \leq b$. Hence, as $p, n \rightarrow \infty$ and $p/n \rightarrow y > 0$, (5.35) goes to 0. Thus any word that is not even, contributes 0 to the limiting moments.

Now let $\omega \in E_b(2k)$. Let \mathcal{E}_ω and \mathcal{O}_ω be as in (5.28) and (5.29). Clearly, as observed in Lemma 5.5, there are $\prod_{i=1}^b \binom{k_i-1}{2}$ combination of equations for the s_j 's (and hence v_j 's) for determining the non-generating vertices, once the generating vertices are chosen. Let us denote a generic combination of the v_j 's by L_ω^T (see (5.15)). For each of the combination of equations we get positive (possibly different) contribution (see Lemma 5.5). Then the contribution of each combination L_ω^T

corresponding to the word ω can be written as

$$y_n^r \frac{1}{p^{r+1} n^{b-r}} \sum_S \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j, n}(|v_{m_j} - y_n v_{l_j}|) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j, n}(|y_n v_{m_j} - v_{l_j}|) \right) \mathbf{1}(0 \leq L_i^T(v_S) \leq 1, \forall i \in S'), \quad (5.36)$$

where S is the set of distinct generating vertices, and S' is the set of indices of the non-generating vertices of ω . By abuse of notation let m_1 and $l_j, 1 \leq j \leq b$ denote the indices of the generating vertices. Therefore as $p \rightarrow \infty$, the contribution of ω in (4.10) is given by

$$y^r \sum_{L_\omega^T} \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(|x_{m_j} - y x_{l_j}|) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(|y x_{m_j} - x_{l_j}|) \right) \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S, \quad (5.37)$$

where $dx_S = dx_{m_1} dx_{l_1} \cdots dx_{l_b}$ denotes the $(b+1)$ -dimensional Lebesgue measure on $[0, 1]^{b+1}$ and $0 < y = \lim p/n$. As for each $k \geq 1$, there are finitely many even words, the first moment condition is established. Hence we have,

$$\lim_{n \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(S_{Z_A})^k] = \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in E_b(2k) \\ (r+1) \text{ even generating vertices}}} y^r \sum_{L_\sigma^T} \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(|x_{m_j} - y x_{l_j}|) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(|y x_{m_j} - x_{l_j}|) \right) \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S. \quad (5.38)$$

Now we show that the limits, $\lim_{n \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(S_{Z_A})^k] = \gamma_k, k \geq 1$ determines a unique distribution. If $y \leq 1$,

$$\gamma_k = \lim_{n \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(S_{Z_A})^k] \leq \sum_{\sigma \in E(2k)} M_\sigma \leq \sum_{\sigma \in \mathcal{P}(2k)} M_\sigma = \alpha_k.$$

As $\{\alpha_k\}$ satisfies Carleman's condition, $\{\gamma_k\}$ does so. Hence the sequence of moments $\{\gamma_k\}$ determines a unique distribution.

If $y > 1$, $y^r \leq y^b, 0 \leq r \leq b, 1 \leq b \leq k$ and hence

$$\gamma_k = \lim_{n \rightarrow \infty} \frac{1}{p} \mathbb{E}[\text{Tr}(S_Z)^k] \leq \sum_{\sigma \in E(2k)} y_\sigma M_\sigma \leq \sum_{\sigma \in \mathcal{P}(2k)} y^{2k} M_\sigma = y^{2k} \alpha_k.$$

As, $y \in (1, \infty)$ and α_k satisfies Carleman's condition, $\{\gamma_k\}$ does so. Hence the sequence of moments $\{\gamma_k\}$ determines a unique distribution.

Therefore, there exists a measure $\mu_{T^{(s)}}$ with moment sequence $\{\gamma_k\}$ such that $\mathbb{E} \mu_{S_{Z_A}}$ converges to $\mu_{T^{(s)}}$, whose moments are given as in (5.38).

This completes the proof of part (i).

(ii) Let $A = T$. Just as in Part (i), it suffices to verify the first moment condition and the Carleman's condition for S_Z .

As $\mathbb{E}(y_i) = 0$, from (5.1), if $\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{\pi \in \Pi_{S_T}(\omega)} \mathbb{E}(Y_\pi)$ exists for every matched word ω of length $2k$

with b distinct letters and $(r+1)$ even generating vertices ($k \geq 1, 1 \leq b \leq b, 0 \leq r \leq (b-1)$), then the first moment condition follows.

Now suppose ω is a word with b distinct letters each letter appearing k_1, k_2, \dots, k_b times. From this point, we borrow all notations from Part (i).

Let $v_i = \pi(i)/n$ as defined in Lemma 5.5 and $U_n = \{0, 1/n, 2/n, \dots, (n-1)/n\}$.

If ω is not an even word, then its contribution to the limiting moments is 0. This follows using the same argument as Part (i).

Now suppose $\omega \in E_b(2k) \setminus S_b(2k)$. Then the contribution of this ω can be written as

$$y_n^r \frac{1}{p^{r+1}n^{b-r}} \sum_S \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j, n}(y_n v_{l_j} - v_{m_j}) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j, n}(y_n v_{m_j} - v_{l_j}) \right) \mathbf{1}(0 \leq L_i^T(v_S) \leq 1, \forall i \in S'), \quad (5.39)$$

where S is the set of distinct generating vertices, and S' is the set of indices of the non-generating vertices of ω and $y_n = p/n$. From Lemma 5.5, observe that for $j \neq 1$, m_j can be written as a linear combination of $\{l_i; 1 \leq i \leq j-1\}$ and m_1 . By abuse of notation, let m_1 and $l_j, 1 \leq j \leq b$ denote the indices of the generating vertices. Then, as $p \rightarrow \infty$, the above sum goes to

$$y^r \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(yx_{l_j} - x_{m_j}) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(yx_{m_j} - x_{l_j}) \right) \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S, \quad (5.40)$$

where $dx_S = dx_{m_1} dx_{l_1} dx_{l_2} \cdots dx_{l_b}$.

By Lemma 5.5, it follows that the above integral reduces to a c dimensional integral where $c \leq b$, if $\omega \notin S_b(2k)$ as x_{m_1} and the x_{l_j} 's then satisfy a linear equation (see proof of Lemma 5.5). As a result, the contribution of ω as described in (5.41) is equal to 0 if $\omega \in E_b(2k) \setminus S_b(2k)$.

Now let $\omega \in S_b(2k)$. Then the contribution the word ω to the limiting moments can be written as

$$y^r \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(yx_{l_j} - x_{m_j}) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(yx_{m_j} - x_{l_j}) \right) \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S, \quad (5.41)$$

where $dx_S = dx_{m_1} dx_{l_1} dx_{l_2} \cdots dx_{l_b}$ is the $(b+1)$ -dimensional Lebesgue measure on $[0, 1]^{(b+1)}$ and $0 < y = \lim y_n$. As for each $k \geq 1$, there are finitely many symmetric words, each of which contributes (5.41) to the k th limiting moment, the first moment condition holds true.

Therefore for $k \geq 1$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\text{Tr}(S_{Z_A})^k] \\ &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \\ & \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(yx_{l_j} - x_{m_j}) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(yx_{m_j} - x_{l_j}) \right) \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S. \end{aligned} \quad (5.42)$$

As $S(2k) \subset E(2k)$ and the integrand in (5.42) is bounded, the same arguments as in Part (i) are applicable. Thus the Carleman's condition for S_Z is satisfied. Therefore, there exists a measure μ_T with moment sequence $\{\gamma_k\}$ such that $\mathbb{E}\mu_{S_Z}$ converges to μ_T . This proves Part (ii).

(iii) Let $A = H^{(s)}$. Since the arguments are similar to the previous parts, we resolve to describe only the limiting moments from this point onward.

For $k \geq 1$,

$$\begin{aligned} \beta_k(\mu_{H^{(s)}}) &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \\ &\quad \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(x_{m_j} + yx_{l_j}) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(yx_{m_j} + x_{l_j}) \right) \mathbf{1}(0 \leq L_i^H(x_S) \leq 1, \forall i \in S') dx_S. \end{aligned} \quad (5.43)$$

(iv) Let $A = H$. Then the EESD of S_H converges to μ_H , whose moments are as follows:

$$\begin{aligned} \beta_k(\mu_H) &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \\ &\quad \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(\text{sgn}(yx_{l_j} - x_{m_j})(x_{m_j} + yx_{l_j})) \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(\text{sgn}(yx_{m_j} - x_{l_j})(yx_{m_j} + x_{l_j})) \right) \\ &\quad \mathbf{1}(0 \leq L_i^H(x_S) \leq 1, \forall i \in S') f^H(x_S) dx_S. \end{aligned} \quad (5.44)$$

(v) Let $A = R^{(s)}$.

For any $m \geq 1$, let

$$\begin{aligned} h_{2m,n}(x_1, x_2) &= f_{2m,n}(x_1 + x_2) \mathbf{1}(0 \leq x_1 + x_2 \leq 1) + f_{2m,n}(x_1 + x_2 - 1) \mathbf{1}(x_1 + x_2 > 1), \\ h_{2m}(x_1, x_2) &= f_{2m}(x_1 + x_2) \mathbf{1}(0 \leq x_1 + x_2 \leq 1) + f_{2m}(x_1 + x_2 - 1) \mathbf{1}(x_1 + x_2 > 1) \end{aligned} \quad (5.45)$$

Then for $k \geq 1$ (see Lemma 5.7),

$$\begin{aligned} \beta_k(\mu_{R^{(s)}}) &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r \\ &\quad \left[[y]^{k-(r+1)} \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} h_{k_j}(x_{m_j}, yx_{l_j}) \prod_{i_j \in \mathcal{O}_\omega} h_{k_j}(yx_{m_j}, x_{l_j}) \right) dx_S \right. \\ &\quad + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- - S_0|} \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} h_{k_j}(x_{m_j}, yx_{l_j}) \prod_{i_j \in \mathcal{O}_\omega} h_{k_j}(yx_{m_j}, x_{l_j}) \right) \\ &\quad \left. \mathbf{1}(F(yL_{2i}^H(x_S)) \leq y - [y], \forall 2i \in S_0) dx_S \right]. \end{aligned} \quad (5.46)$$

(vi) Let $A = R$. Suppose

$$\begin{aligned} \tilde{h}_{2m}(x, x_2) &= f_{2m}(\text{sgn}(x_2 - x_1)(x_1 + x_2)) \mathbf{1}(0 \leq x_1 + x_2 \leq 1) + \\ &\quad f_{2m}(\text{sgn}(x_2 - x_1)(x_1 + x_2 - 1)) \mathbf{1}(x_1 + x_2 > 1) \end{aligned} \quad (5.47)$$

The moments of μ_R are as follows (see (6.8)):

$$\begin{aligned}
 \beta_k(\mu_R) &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r \\
 &\left[[y]^{k-(r+1)} \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} \tilde{h}_{k_j}(x_{m_j}, yx_{l_j}) \prod_{i_j \in \mathcal{O}_\omega} \tilde{h}_{k_j}(yx_{m_j}, x_{l_j}) \right) f^H(x_S) dx_S \right. \\
 &+ \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- - S_0|} \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} \tilde{h}_{k_j}(x_{m_j}, yx_{l_j}) \prod_{i_j \in \mathcal{O}_\omega} \tilde{h}_{k_j}(yx_{m_j}, x_{l_j}) \right) \\
 &\left. \mathbf{1}(F(yL_{2i}^H(x_S)) \leq y - [y], \forall 2i \in S_0) f^H(x_S) dx_S \right]. \tag{5.48}
 \end{aligned}$$

(vii) Let $A = C^{(s)}$. In this case, we have,

$$\begin{aligned}
 \beta_k(\mu_{C^{(s)}}) &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in E_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r a_\sigma \\
 &\left[[y]^{k-(r+1)} \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(|1/2 - |1/2 - |x_{m_j} - yx_{l_j}|||) \right. \right. \\
 &\quad \left. \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(|1/2 - |1/2 - |yx_{m_j} - x_{l_j}|||) \right) + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- - S_0|} \\
 &\quad \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} f_{k_j}(|1/2 - |1/2 - |x_{m_j} - yx_{l_j}|||) \right. \\
 &\quad \left. \left. \prod_{i_j \in \mathcal{O}_\omega} f_{k_j}(|1/2 - |1/2 - |yx_{m_j} - x_{l_j}|||) \right) \mathbf{1}(F(yL_{2i}^T(x_S)) \leq y - [y], \forall 2i \in S_0) \right]. \tag{5.49}
 \end{aligned}$$

(viii) Let $A = C$. Suppose

$$\eta_{2m}(yx_1, x_2) = f_{2m}(x_2 - yx_1) \mathbf{1}(0 \leq x_2 - yx_1 \leq 1) + f_{2m}(1 - x_2 + yx_1) \mathbf{1}(x_2 - yx_1 < 0) \tag{5.50}$$

where $(x_1, x_2) \in [0, 1]^2$. Then the limiting moments are given by the following formula:

$$\begin{aligned} \beta_k(\mu_C) &= \sum_{b=1}^k \sum_{r=0}^{b-1} \sum_{\substack{\sigma \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} y^r a_\sigma \\ &\quad \left[[y]^{k-(r+1)} \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} \eta_{k_j}(x_{m_j} - yx_{l_j}) \right. \right. \\ &\quad \left. \left. \prod_{i_j \in \mathcal{O}_\omega} \eta_{k_j}(x_{l_j} - yx_{m_j}) \right) + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- - S_0|} \right. \\ &\quad \left. \int_0^1 \cdots \int_0^1 \prod_{j=1}^b \left(\prod_{i_j \in \mathcal{E}_\omega} \eta_{k_j}(x_{m_j} - yx_{l_j}) \prod_{i_j \in \mathcal{O}_\omega} \eta_{k_j}(x_{l_j} - yx_{m_j}) \right) \right. \\ &\quad \left. \mathbf{1}(F(yL_{2i}^T(x_S)) \leq y - [y], \forall 2i \in S_0) \right]. \end{aligned} \quad (5.51)$$

□

5.3. *Applications of Theorem 2.4.* As the entries are dependent on i, j, n , the formula for the limiting moments, as seen in the proof of Theorem 2.4, can often be very complicated. Here we discuss a few special cases where the limiting moment formulae are relatively simple. These special cases would be when the entries of the matrix A are– (a) triangular i.i.d. (Section 5.3.1), (b) sparse triangular i.i.d. (Section 5.3.2), (c) fully i.i.d. with finite mean and variance (Section 5.3.3), (d) have a non-trivial variance structure (Section 5.3.4), (e) triangular, i.e., only lower triangular entries are non-zero (Section 5.3.5) and (f) have a band structure (Section 5.3.6).

Theorem 2.4 concludes the convergence of the EESD of S_A . However, as we will see in the upcoming sections, a.s. convergence of the ESD can be obtained in some cases. To establish the a.s. convergence of the ESD in such cases, we will use Lemma 4.3, just as we did in case of the S matrix. Recall the set $Q_{k,4}^b$ from (4.26) that was used to establish the fourth moment condition for S . Analogous version of Lemma 4.8 is not true for S_A (see Erratum of (Bose et al., 2021)). However, it can be shown that

$$|Q_{k,4}^b| \leq n^{2k+2} \quad \text{for any } 1 \leq b \leq 2k. \quad (5.52)$$

For a proof of this fact, see Lemma 1.4.3 (a) in (Bose, 2018). Even though the proof given there is for the case where the entries are i.i.d., the arguments can be used to prove the same for the S_A –link function, as $1 \leq \pi(2i) \leq p$ and $1 \leq \pi(2i-1) \leq n$ and p and n are comparable for large n .

5.3.1. *General triangular i.i.d. entries.* Let A be one of the $p \times n$ patterned matrices mentioned in Section 1. Suppose for each fixed n , the input sequence $\{x_{i,n} : i \geq 0\}$ are i.i.d. for every fixed n , with all moments finite. Assume that for all $k \geq 1$,

$$n\mathbb{E}[x_{0,n}^k] \rightarrow C_k \quad \text{as } n \rightarrow \infty. \quad (5.53)$$

Also assume that the moments of the random variable whose cumulants are $\{C_2, C_4, \dots\}$ satisfy Carleman’s condition. We can actually find such variables $\{x_{i,n}\}$ as discussed in Remark 4.4 of Section 4.2.3. Now observe that Assumption B (i), (ii) and (iii) are satisfied with $t_n = \infty$ and $f_{2k} \equiv C_{2k}$ for $k \geq 1$. Thus Theorem 2.4 can be applied to conclude that the EESD of S_A converges to a probability distribution, μ_A . A brief description of the limiting moments is given below.

(i) Suppose $A = T^{(s)}$ whose entries satisfy (5.53). Thus by Part (i) of Theorem 2.4, the EESD of $S_{T_p^{(s)}}$ converges to $\mu_{T^{(s)}}$ whose moment sequence is given as follows (see (5.38)):

$$\beta_k(\mu_{T^{(s)}}) = \sum_{b=1}^k \sum_{r=0}^b y^r \sum_{\substack{\pi \in E_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} \sum_{L_\pi^T} C_\pi \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S'). \quad (5.54)$$

Note that since an even word can be identified as an even partition, for every $\pi \in E_b(2k)$, $L_\pi^T = L_\omega^T$ for the corresponding even word with b distinct letters.

(ii) By Part (ii) of Theorem 2.4 (see (5.42)):

$$\beta_k(\mu_T) = \sum_{b=1}^k \sum_{r=0}^b y^r \sum_{\substack{\pi \in S_b(2k) \text{ with} \\ (r+1) \text{ even generating vertices}}} C_\pi \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(0 \leq L_i^T(x_S) \leq 1, \forall i \in S') dx_S. \quad (5.55)$$

(iii) By Part (iii) of Theorem 2.4, the EESD of $S_{H^{(s)}}$ converges to $\mu_{H^{(s)}}$ whose moment sequence is as in (5.55), where the integrand is replaced by $\mathbf{1}(0 \leq L_i^H(x_S) \leq 1, \forall i \in S')$ (see (5.43)).

(iv) By Part (iv) of Theorem 2.4, the EESD of S_{H_p} converges to μ_H whose moment sequence is as in (5.55), where the integrand is replaced by $\mathbf{1}(0 \leq L_i^H(x_S) \leq 1, \forall i \in S') f^H(s)$ (see (5.44)).

(v) By Part (v) of Theorem 2.4, the EESD of $S_{R_p^{(s)}}$ converges to $\mu_{R^{(s)}}$ whose moment sequence is given as in (5.55), where the function inside the square brackets is (see (5.46))

$$C_\pi [[y]^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- \setminus S_0|} \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(yL_{2i}^H(x_S)) \leq y - [y], \forall 2i \in S_0) dx_S].$$

(vi) By Part (vi) of Theorem 2.4, $\beta_k(\mu_R)$ is same as $\beta_k(\mu_{R^{(s)}})$, with an extra factor $f^H(s)$ in the integrand.

(vii) By Part (vii) of Theorem 2.4, the EESD of $S_{C^{(s)}}$ converges to $\mu_{C^{(s)}}$ whose moment sequence is as in (5.54), where the function inside the square brackets is (see (5.49))

$$C_\pi [[y]^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- \setminus S_0|} \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(yL_{2i}^T(x_S)) \leq y - [y], \forall 2i \in S_0) f^H(x_S) dx_S].$$

(viii) By Part (viii) of Theorem 2.4, $\beta_k(\mu_C)$ is as in (5.55), where the function inside the square brackets is (see (5.51))

$$C_\pi [[y]^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- \setminus S_0|} \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(yL_{2i}^T(x_S)) \leq y - [y], \forall 2i \in S_0) f^H(x_S) dx_S].$$

Remark 5.11. (a) The linear combinations L_i^T and L_i^H from Lemma 5.4, 5.5 and 5.6 play a crucial role in the moments of the LSD of S_A . Observe that for S_T and $S_{H^{(s)}}$ (see Lemmas 5.5 and 5.6), only symmetric words can contribute positively to the limiting moments. Now from the proof of Part (i) of Lemma 5.6 and Lemma 5.5, it follows that when the word is symmetric, after having chosen the generating vertices $\{v_S\}$, for every $i \in S'$, $L_i^T(v_S) = L_{i,q}^H(z_S)$. As the z_i s are derived by elementary transformations that do not alter the integral, we have $L_i^T(v_S) = L_i^H(v_S)$, for symmetric word. This will be useful in finding relations between $\mu_T, \mu_{H^{(s)}}, \mu_{R^{(s)}}, \mu_C$, that we discuss next.

(b) Suppose $y \in \mathbb{N}$. Then, observe that the integrals in Parts (v) and (viii) above are zero and thus

$$\beta_k(\mu_{R^{(s)}}) = \beta_k(\mu_C) = \sum_{\pi \in S(2k)} y^{k-1} C_\pi.$$

Further, using Part (a), we can say more about these limits even when $y \notin \mathbb{N}$. Recall that the only contributing words for $S_T, S_{H^{(s)}}, S_{R^{(s)}}$ and S_C are symmetric. Now, from (a) and uniqueness of the limit, it is easy to see that when the variables are triangular i.i.d. and satisfy (5.53), we have $\mu_T = \mu_{H^{(s)}}$ and $\mu_{R^{(s)}} = \mu_C$.

Remark 5.12. However, in general the LSDs of S_A for symmetric and the asymmetric cases are not identical. For instance, for the Toeplitz and the circulant matrices, this is evident from the moment formula, as the set of partitions that contribute positively to the limiting moments are different in the two cases. For the Hankel and the reverse circulant, there is an extra factor in the integrand for the asymmetric versions and that gives rise to the difference in the limit. We illustrate this for $S_{H^{(s)}}$ and S_H below.

For all words that are special symmetric, the contributions for the symmetric and asymmetric Hankel are same as there are no further restrictions for the signs arising from (5.32). However, if $\omega \in S_b(2k) \setminus SS_b(2k)$, some additional conditions do appear in case of asymmetric Hankel.

For instance, let us consider the word $abcabc \in S_3(6) \setminus SS_3(6)$. In case of symmetric Hankel, its contribution to $\mu_{H^{(s)}}$ is

$$C_2^3 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbf{1}(0 \leq x_0 + x_1 - x_3, x_2 - x_0 + x_3 \leq 1) dx_0 dx_1 dx_2 dx_3. \quad (5.56)$$

On the other hand, the contribution of $abcabc$ (in case of asymmetric Hankel) to μ_H is

$$\begin{aligned} C_2^3 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbf{1}(0 \leq x_0 + x_1 - x_3, x_2 - x_0 + x_3 \leq 1) \mathbf{1}(\operatorname{sgn}(x_1 - x_0) = \\ \operatorname{sgn}(2x_3 - x_0 - x_1), \operatorname{sgn}(x_1 - x_2) = \operatorname{sgn}(x_2 - 2x_0 - x_1 + 2x_3), \\ \operatorname{sgn}(x_3 - x_2) = \operatorname{sgn}(x_2 - 2x_0 + x_3)) dx_0 dx_1 dx_2 dx_3. \end{aligned} \quad (5.57)$$

The integrand in (5.57) is less than that in (5.56) due to the extra restrictions arising from the sign functions. Thus, the k th moment of μ_H is in general smaller than that of $\mu_{H^{(s)}}$. A very similar thing occurs in case of $\mu_{R^{(s)}}$ and μ_R .

5.3.2. Sparse triangular i.i.d. entries. Suppose the input sequence $\{x_{i,n} : i \geq 0\}$ are $\operatorname{Ber}(p_n)$ where $np_n \rightarrow \lambda > 0$. Then (5.53) is satisfied with $C_k = \lambda$ for all $k \geq 1$. Therefore from the discussion in Section 5.3.1, the EESD of S_A converges to say μ_A whose moments are as in (i)-(viii) in Section 5.3.1, where $C_\pi = \lambda^{|\pi|}$ for all $\pi \in \mathcal{P}(2k)$.

5.3.3. I.i.d. Entries. (Bose et al., 2010) established the LSD of S_A when A is the asymmetric or symmetric versions of Toeplitz, Hankel, circulant and reverse circulant matrices with entries $\{\frac{1}{\sqrt{n}}x_i\}$, where x_i are independent and identically distributed with mean 0 and variance 1. Here we show how these LSD results of (Bose et al., 2010) can be obtained as special cases of Theorem 2.4.

First, observe that just like the S matrix, but now dealing with a single sequence of random variables, $\{x_{i,n}\}$, we can show that conditions (i), (ii) and (iii) of Assumption B hold with $f_2 \equiv 1$ and $f_{2k} \equiv 0$ for all $k \geq 2$. Then from Theorem 2.4, we obtain the convergence of the EESD.

The moment formulae are given as in (i)-(viii) in Section 5.3.1, where $C_2 = 1$ and $C_{2k} = 0$ for all $k \geq 2$. Thus the words that contribute to the limiting moments are now pair matched. Hence the moments are indeed equal to the ones in (Bose et al., 2010).

Now, as S_A satisfies (5.52), we have

$$\begin{aligned} \frac{1}{p^4} \mathbb{E}[\text{Tr}(S_A^k) - \mathbb{E}(\text{Tr}(S_A^k))]^4 &= \mathcal{O}(p^{-2}) \quad \text{and therefore,} \\ \sum_{p=1}^{\infty} \frac{1}{p^4} \mathbb{E}[\text{Tr}(S_A^k) - \mathbb{E}(\text{Tr}(S_A^k))]^4 &< \infty \quad \text{for every } k \geq 1. \end{aligned} \tag{5.58}$$

Then using Lemma 4.3, we can conclude that μ_{S_A} converges a.s.

5.3.4. *Matrices with variance profile.* Suppose the input sequence is $\{\sigma(i/n)x_{i,n}; i \geq 0\}$, where $\sigma : [0, 1] \rightarrow \mathbb{R}$ is a bounded and Riemann integrable function and $\{x_{i,n}; i \in \mathbb{Z}\}$ are i.i.d. random variables with mean zero and all moments finite. Assume that $\{x_{i,n}; i \geq 0\}$ satisfy (5.53). Then the EESD of S_A for each of the eight patterns of A , converges to a probability distribution whose moments are determined by σ and $\{C_{2k}, k \geq 1\}$. This follows from Theorems 2.4 as argued below:

First observe that the entries of A satisfy Assumption B (i) and (ii) with $t_n = \infty$, $f_{2k} = \sigma^{2k}C_{2k}$, $k \geq 1$. Since σ is bounded, Assumption B (iii) is also true. Hence from Theorem 2.4, we can conclude that the EESD of S_A converges.

Note that in the i.i.d. situation where each $x_{i,n}$ has the same distribution F for all i and n , $C_{2k} = 0$ for all $k \geq 2$. Hence the EESD of S_A converges. As σ is bounded, (5.52) and hence (5.58) hold true. Thus we can conclude that μ_{S_A} converges a.s. to the respective limits.

5.3.5. *Triangular Matrices.* As discussed in Section 4.2.5, the LSD of triangular matrices have been studied in (Dykema and Haagerup, 2004), where the entries of the matrix are i.i.d. Gaussian. Later LSD results were proved in (Basu et al., 2012) for triangular matrices with other patterns such as Hankel, Toeplitz and symmetric circulant, and with i.i.d. input. The matrices that the authors of (Basu et al., 2012) considered are symmetric, and hence the entries $y_{L(i,j),n}$ are of the form $y_{L(i,j),n} = x_{L(i,j),n} \mathbf{1}(i + j \leq n + 1)$. However, the matrix considered in (Dykema and Haagerup, 2004) is upper triangular, as in (4.23). It is natural to ask what happens to such matrices when there are other patterns involved.

Let A be any of the eight matrices that are being discussed in this article. Let A^U be the matrix whose entries $y_{L(i,j),n}$ are as follows:

$$y_{L(i,j),n} = \begin{cases} x_{L(i,j),n} & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.59}$$

Then we have the following result.

Result 5.1. Consider the matrices A^U . Assume that the variables $\{x_{i,n}; i \geq 0\}$ in (5.59) are i.i.d. random variables with all moments finite, for every fixed n . Also assume that $\{x_{i,n}; i \geq 0\}$ satisfy (5.53). Then, for each of the eight matrices mentioned above, the EESD of S_{A^U} converges weakly to some probability measure μ_{A^U} that depends on $\{C_{2k}\}_{k \geq 1}$.

Proof: Define the function σ on $[0, 1]^2$ as

$$\sigma(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that the entries $y_{L(i,j),n}$ of the matrix A_n^U can be written as $\sigma(i/p, j/n)x_{L(i,j),n}$.

Following the proofs in Theorem 2.4, it is easy to see that the first moment condition holds for S_{A^U} . As $\|\sigma\| \leq 1$, the Carleman's condition also holds for the limiting moment sequence. Hence the EESD of S_{A^U} converges to a probability measure μ_{A^U} .

□

Remark 5.13. (i) If the entries of A^U are $\frac{y_{i,n}}{\sqrt{n}}$ where $\{y_{i,n}; i \geq 0\}$ are as in (5.59) and $\{x_{i,n}; i \geq 0\}_{n \geq 1}$ are i.i.d. random variables with mean 0 and variance 1, then using familiar truncation arguments (as in Theorem 8.1.2 in (Bose, 2018)), the variables $\{y_{i,n}; i \geq 0\}$ can be assumed to be uniformly bounded and hence satisfy (5.53) with $C_2 = 1$ and $C_{2k} = 0$ for $k \geq 2$. Hence from Result 5.1, we obtain the convergence of the EESD. Again it can be verified that (5.52) and (5.58) are true in this case. Thus the ESD of S_{A^U} converges a.s. to a non-random probability measure.

5.3.6. *Band matrices.* Band matrices had been discussed previously in (Basak and Bose, 2011), (Popescu, 2009), (Liu and Wang, 2011) and others. In Section 7.4 of (Bose et al., 2021), the LSD of band matrices where the non-zero entries satisfy (5.53) had been studied. So it was natural to ask what happens to the LSD of $A^b A^{bT}, A^B A^{BT}$, where A^b and A^B are matrices with entries $y_{L(i,j)} = x_{L(i,j)} \mathbf{1}(L(i,j) \leq m_n)$ and $y_{L(i,j)} = x_{L(i,j)} [\mathbf{1}(L(i,j) \leq m_n) + \mathbf{1}(L(i,j) \geq n - m_n)]$ (see Section 7.4 in (Bose et al., 2021)). Here we provide an answer to that question.

Result 5.2. Consider the matrices A^b and A^B . Assume that the variables $\{x_{i,n}; i \geq 0\}$ associated with the matrices A^U (as in (5.59)) are i.i.d. random variables with all moments finite for every fixed n , and satisfy (5.53). Suppose $\alpha = \lim_{n \rightarrow \infty} \frac{m_n}{n} > 0$. Then, for each of the eight matrices, the EESD of S_{A^b} and S_{A^B} converge to some probability measures μ_α^b and μ_α^B that depend on $\{C_{2k}\}_{k \geq 1}$.

For A^b , the entries y_i can be written as $\sigma_n(i/n)x_i$, where $\sigma_n(x) = \mathbf{1}(x \leq \frac{m_n}{n})$. Observe that for $k \geq 1$, $\int \sigma_n^k(x) dx \rightarrow \int \sigma_0^k(x) dx$ as $n \rightarrow \infty$, where $\sigma_0(x) = \mathbf{1}(x \leq \alpha)$.

For the Type II band versions $R^{(s)B}$ of $R^{(s)}$ and T^B of $T^{(s)}$, the entries y_i can be written as $\sigma_n(i/n)x_i$, where $\sigma_n(x) = \mathbf{1}(\frac{m_n}{n} \leq x \leq 1 - \frac{m_n}{n})$. Clearly, $\int \sigma_n^k(x) dx \rightarrow \int \sigma_1^k(x) dx$ as $n \rightarrow \infty$ for each $k \geq 1$ where $\sigma_1 = \mathbf{1}_{[0,\alpha] \cup [1-\alpha,1]}$. For the Type II band versions H^B of $H^{(s)}$, the entries y_i can be written as $\sigma_n(i/n)x_i$, where $\sigma_n(x) = \mathbf{1}(1 - m_n/n \leq x \leq 1 + m_n/n)$. Clearly, σ_n converges to $\sigma_2 = \mathbf{1}_{[1-\alpha,1+\alpha]}$.

Thus in all of the above cases, Assumption B is true with $t_n = \infty$ and $g_{2k} = \sigma_t^{2k} C_{2k}$, $t = 0, 1$, or 2 . Thus from Theorem 2.4 the result follows.

In this case too if x_i are all i.i.d. and the entries of the matrices are $\{\frac{y_i}{\sqrt{m_n}}\}$, then additionally (5.52) and thereby (5.58) holds. Thus the a.s. convergence of the ESDs can be concluded.

6. Appendix

In this section, we will first prove a result (Lemma 6.1) that helps us to conclude the a.s. convergence of the ESD of the S matrix. Next, we give the detailed proofs of Lemma 5.7 and 5.8.

Lemma 6.1. *Suppose $\{x_{ij}; 1 \leq i \leq p, 1 \leq j \leq j \leq n\}$ are independent variables that satisfy Assumption A and $y_{ij} = x_{ij} \mathbf{1}_{[|x_{ij}| \leq t_n]}$. Then*

$$(i) \quad \frac{1}{p} \sum_{i,j} (y_{ij}^2 - \mathbb{E}[y_{ij}^2]) \rightarrow 0 \quad \text{a.s. as } p \rightarrow \infty.$$

$$(ii) \quad \text{Additionally if, } \frac{1}{p} \sum_{i,j} x_{ij}^2 \mathbf{1}_{[|x_{ij}| > t_n]} \rightarrow 0 \text{ a.s. (or in probability), then } \limsup_p \frac{1}{p} \sum_{i,j} x_{ij}^2 < \infty \text{ a.s. (or in probability).}$$

Proof: (i) Let $\epsilon > 0$ be fixed. Then

$$\begin{aligned} \mathbb{P}\left[\left|\frac{1}{p}\sum_{i,j}(y_{ij}^2 - \mathbb{E}[y_{ij}^2])\right| > \epsilon\right] &\leq \frac{1}{\epsilon^4 p^4} \mathbb{E}\left[\left(\sum_{i,j} y_{ij}^2 - \mathbb{E}[y_{ij}^2]\right)^4\right] \\ &= \frac{1}{\epsilon^4 p^4} \mathbb{E}\left[\sum_{\substack{i_1, i_2, i_3, i_4 \\ j_1, j_2, j_3, j_4}} \prod_{l=1}^4 (y_{i_l j_l}^2 - \mathbb{E}[y_{i_l j_l}^2])\right]. \end{aligned}$$

As $\{y_{ij}\}$ are independent, the above inequality becomes

$$\begin{aligned} \mathbb{P}\left[\left|\frac{1}{p}\sum_{i,j}(y_{ij}^2 - \mathbb{E}[y_{ij}^2])\right| > \epsilon\right] &\leq \frac{1}{\epsilon^4 p^4} \sum_{ij} \mathbb{E}[(y_{ij}^2 - \mathbb{E}[y_{ij}^2])^4] \\ &\quad + 6 \frac{1}{\epsilon^4 p^4} \sum_{\substack{i_1, i_2 \\ j_1, j_2}} \mathbb{E}[(y_{i_1 j_1}^2 - \mathbb{E}[y_{i_1 j_1}^2])^2 (y_{i_2 j_2}^2 - \mathbb{E}[y_{i_2 j_2}^2])^2]. \end{aligned}$$

Now from (2.1), as $\{g_{2k,n}\}$ are bounded integrable, the first term in the rhs of the above inequality is $\mathcal{O}(\frac{1}{p^3})$ and the second term is $\mathcal{O}(\frac{1}{p^2})$. Therefore,

$$\sum_p \mathbb{P}\left[\left|\frac{1}{p}\sum_{i,j}(y_{ij}^2 - \mathbb{E}[y_{ij}^2])\right| > \epsilon\right] < \infty.$$

Hence by Borel-Cantelli lemma, $\frac{1}{p}\sum_{i,j}(y_{ij}^2 - \mathbb{E}[y_{ij}^2]) \rightarrow 0$ a.s. as $p \rightarrow \infty$.

(ii) Observe that $\sum_{i,j} x_{ij}^2 = \sum_{i,j} (y_{ij}^2 + x_{ij}^2 \mathbf{1}_{[|x_{ij}| > t_n]})$. Also note that $\frac{1}{p}\sum_{i,j} \mathbb{E}[y_{ij}^2] \rightarrow \int g_2(x, y) dx dy$ as $n, p \rightarrow \infty$. Then by the condition $\frac{1}{p}\sum_{i,j} x_{ij}^2 \mathbf{1}_{[|x_{ij}| > t_n]} \rightarrow 0$ a.s. (or in probability) and (i), (ii) holds true. \square

Proof of Lemma 5.7: First suppose $\omega \in \mathcal{P}(2k) \setminus S_b(2k)$. Then from (5.4) and Lemma 5.1 in (Bose et al., 2021), and using the fact that $p/n \rightarrow y > 0$ as $n \rightarrow \infty$, it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_R(s)}(\omega)| = \lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_R}(\omega)| = 0.$$

Now suppose ω is a symmetric word with b distinct letters and $(r + 1)$ even generating vertices. Suppose i_1, i_2, \dots, i_b are the positions where new letters made their first appearances. First we fix the generating vertices $\pi(i_j), 0 \leq j \leq b$ where $\pi(i_0) = \pi(0)$. Let

$$t_i = \pi(i) + \pi(i - 1) \text{ for } 1 \leq i \leq 2k.$$

Now let us first consider the symmetric Reverse circulant link function.

From (5.2), $\omega[i] = \omega[j]$ if and only if $t_i = t_j \pmod{n}$. Clearly, for ω , $t_{i_1} = t_1$ and for every $1 \leq i \leq 2k$,

$$t_i = t_{i_j} \pmod{n} \text{ for some } j \in \{1, 2, \dots, b\}. \tag{6.1}$$

First we fix the generating vertices $\pi(i_j), j = 0, 1, 2, \dots, b$. Let $S = \{\pi(i_j) : 0 \leq j \leq b\}$ and $S' = \{i : \pi(i) \notin S\}$. For every $i \in S'$,

$$\begin{aligned} \pi(i) &= (t_{i_j} + \pi(i - 1)) \pmod{n} \text{ for some } i_j, j \in \{1, 2, \dots, b\} \\ \text{i.e., } \pi(i) &= \sum_{j < i} \alpha_{ij} \pi(i_j) \pmod{n} \text{ for some } \alpha_{ij} \in \mathbb{Z}. \end{aligned} \tag{6.2}$$

Thus for every $i \in S' \setminus \{2k\}$, there exists unique integer $m_{i,n}$ such that

$$1 \leq \sum_{j < i} \alpha_{ij} \pi(i_j) + m_{i,n} \leq n. \quad (6.3)$$

As we have already fixed the generating vertices, from (6.2) and (6.3) it follows that there is a unique choice for $\pi(2i-1)$, $1 \leq i \leq k$ such that $2i-1 \in S'$. For all $2i \in S'$, $1 \leq i < k$, we can have $\lfloor y_n \rfloor$ choices as $1 \leq \pi(2i) \leq p$ and $y_n = p/n$. Moreover, there is an additional choice if

$$\sum_{j < 2i} \alpha_{2ij} \pi(i_j) + m_{2i,n} \leq p - \lfloor y_n \rfloor n. \quad (6.4)$$

Next let

$$v_{2i} = \frac{\pi(i)}{p}, \quad v_{2i-1} = \frac{\pi(i)}{n} \text{ for } 0 \leq i \leq k, L(a) = \max\{m \in \mathbb{Z}\}, F(a) = a - L(a).$$

Also let

$$S^- = \{2i : 2i \notin S' \text{ and (6.4) holds true}\}. \quad (6.5)$$

Now observe that from (6.3) and (6.4) it follows that for every $i \in S^-$,

$$F(y_n L_{2i,n}^H(v_S)) \leq y_n - \lfloor y_n \rfloor, \quad (6.6)$$

where $L_{2i,n}^H$ is the set of linear combinations defined in (5.24).

From (5.5) and the discussion above, it is easy to see that for a word ω of length $2k$,

$$\begin{aligned} & \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_{R(s)}}(\omega)| \\ &= \lfloor y_n \rfloor^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} \lfloor y_n \rfloor^{|S^- - S_0|} |\{v_S : F(y_n L_{2i,n}^H(v_S)) \leq y_n - \lfloor y_n \rfloor, \mathbf{1}(2i \in S_0)\}|. \end{aligned}$$

Therefore as $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_{R(s)}}(\omega)| &= \lfloor y \rfloor^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} \lfloor y \rfloor^{|S^- - S_0|} \\ & \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^H(v_S)) \leq y - \lfloor y \rfloor) \forall 2i \in S_0 \, dv_S, \end{aligned} \quad (6.7)$$

where $dv_S = \prod_{j=0}^b dv_{i_j}$ is the $(b+1)$ -dimensional Lebesgue integral on $[0, 1]^{b+1}$ and S^- is as in (6.5).

When $y \geq 1$, the rhs of (6.7) is positive. We next show that when $y < 1$, the value of the integral $\int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^H(v_S)) \leq y - \lfloor y \rfloor) \forall 2i \in S^- \, dv_S$ is positive.

First note that as $y < 1$, $\lfloor y \rfloor = 0$. Now note that we had previously established in the proof of Part (i) of Lemma 5.6 that for certain values of $v_S \in [0, 1]^{b+1}$, $\mathbf{1}(0 \leq L_i^H(v_S) \leq 1, \forall i \in S') = 1$. As, $\{v_S : 0 \leq L_{2i}^H(v_S) \leq 1, \forall i \in S^-\} \subset \{v_S : 0 \leq L_i^H(v_S) \leq 1, \forall i \in S'\}$, for these chosen values of v_S , we have $\mathbf{1}(0 \leq L_{2i}^H(v_S) \leq 1, \forall 2i \in S^-) = 1$. Therefore, with this choice of $v_S \in [0, 1]^{b+1}$,

$$y L_{2i}^H(v_S) \leq y < 1 \implies F(y L_{2i}^H(v_S)) \leq y.$$

That is, the integral $\int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^H(v_S)) \leq y - \lfloor y \rfloor) \forall 2i \in S^- \, dv_S$ is positive.

Hence the proof of Part (i) is complete.

To prove Part (ii), observe that

$$\begin{aligned} \xi_\pi(i) &= \xi_\pi(j) \quad \text{if and only if } t_i = t_j \pmod{n} \quad \text{and} \\ \text{sgn}(\pi(i) - \pi(i-1)) &= \text{sgn}(\pi(j) - \pi(j-1)) \quad \text{if } i \text{ and } j \text{ are of same parity, or} \\ \text{sgn}(\pi(i) - \pi(i-1)) &= \text{sgn}(\pi(j-1) - \pi(j)) \quad \text{if } i \text{ and } j \text{ are of opposite parity.} \end{aligned}$$

As $\Pi_{S_R}(\omega) \subset \Pi_{S_{R(s)}}(\omega)$, if ω is a word with b distinct letters but not symmetric, by Part (i), $\frac{1}{p^{r+1}b^{n-r}} |\Pi_{S_R}(\omega)| \rightarrow 0$ as $n \rightarrow \infty$.

Next let $\omega \in S_b(2k)$ with $(r+1)$ even generating vertices. Clearly for $|\mathcal{E}_\omega| = r+1$ and $|\mathcal{O}_\omega| = b-r$.

Recall the sets $\mathcal{E}_\omega, \mathcal{O}_\omega, C_{i_j}^e, C_{i_j}^o$ from (5.28), (5.29) and (5.30). Similarly we can define the functions f_n^H and f^H . Thus we can conclude

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_R}(\omega)| = [y]^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} [y]^{|S^- - S_0|} \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(yL_{2i}^H(v_S)) \leq y - [y]) \forall 2i \in S_0) f^H(v_S) dv_S, \quad (6.8)$$

where $dv_S = \prod_{j=0}^b dv_{i_j}$ is the $(b+1)$ -dimensional Lebesgue integral on $[0, 1]^{b+1}$ and S^- is as in (6.5).

This completes the proof of Part (ii). □

Proof of Lemma 5.8: (i) First suppose $\omega \in \mathcal{P}(2k) \setminus E_b(2k)$. Then from (5.4) and Lemma 5.2 in (Bose et al., 2021), and using the fact that $p/n \rightarrow y > 0$ as $n \rightarrow \infty$, it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1}n^{b-r}} |\Pi_{S_{C(s)}}(\omega)| = 0 \text{ if } \omega.$$

Now suppose ω is an even word of length $2k$ with b distinct letter and $(r+1)$ even generating vertices. Suppose i_1, i_2, \dots, i_b are the positions where new letters made their first appearances. First we fix the generating vertices $\pi(i_j), 0 \leq j \leq b$ where $\pi(i_0) = \pi(0)$. Let

$$s_i = \pi(i) - \pi(i-1) \text{ for } 1 \leq i \leq 2k.$$

Clearly, from (5.2), $\omega[i] = \omega[j]$ if and only if $\xi_\pi(i) = \xi_\pi(j)$. That is, $|s_i| = |s_j| \pmod{n}$, that is, $s_i = s_j \pmod{n}$ or $s_i = -s_j \pmod{n}$. Thus for every $i, 1 \leq i \leq 2k$,

$$s_i = s_{i_j} \pmod{n} \text{ or } s_i = -s_{i_j} \pmod{n}, \quad (6.9)$$

for some $j \in \{1, 2, \dots, b\}$. Thus, we have

$$\pi(i) = \pm(\pi(i_j) - \pi(i_j - 1)) + \pi(i - 1) = \pm s_{i_j} + \pi(i - 1) \pmod{n} \text{ for some } j. \quad (6.10)$$

First we fix the generating vertices $\pi(i_j), j = 0, 1, 2, \dots, b$. Let $S = \{\pi(i_j) : 0 \leq j \leq b\}$ and $S' = \{i : \pi(i) \notin S\}$. Having chosen a particular sign in (6.10), for every $i \in S'$, there is some $i_j, 1 \leq j \leq b$,

$$\begin{aligned} \pi(i) &= (s_{i_j} + \pi(i - 1)) \pmod{n} \text{ or } \pi(i) = (-s_{i_j} + \pi(i - 1)) \pmod{n} \\ \text{i.e., } \pi(i) &= \sum_{j < i} \beta_{ij} \pi(i_j) \pmod{n} \text{ for some } \{\beta_{ij}\} \subset \mathbb{Z}. \end{aligned} \quad (6.11)$$

Thus for every $i \in S' \setminus \{2k\}$, there exists a unique integer $m_{i,n}$ such that

$$1 \leq \sum_{j < i} \beta_{ij} \pi(i_j) + m_{i,n} \leq n. \quad (6.12)$$

As we have already fixed the generating vertices, from (6.11) and (6.12) it follows that, there is a unique choice for all $\pi(2i-1), 1 \leq i \leq k$ such that $2i-1 \in S'$. For all $2i \in S', 1 \leq i < k$, we can have $[y_n]$ choices as $1 \leq \pi(2i) \leq p$ and $y_n = p/n$. Moreover, there is an additional choice if

$$\sum_{j < 2i} \beta_{2ij} \pi(i_j) + m_{2i,n} \leq p - [y_n]n. \quad (6.13)$$

Next let

$$v_{2i} = \frac{\pi(i)}{p}, \quad v_{2i-1} = \frac{\pi(i)}{n} \text{ for } 0 \leq i \leq k, \quad L(a) = \max\{m \in \mathbb{Z}\}, \quad F(a) = a - L(a).$$

Also let

$$S^- = \{2i : 2i \notin S' \text{ and (6.13) holds true}\}. \quad (6.14)$$

Now observe that from (6.12) and (6.13) it follows that for every $i \in S^-$,

$$F(y_n L_{2i,n}^T(v_S)) \leq y_n - \lfloor y_n \rfloor, \quad (6.15)$$

where $L_{2i,n}^T$ is the linear combination defined in (5.14). Now, this linear combinations vary depending on the sign chosen for each s_i . As we know for each block of an even word, the number of positive and negative signs in the relations among the s_i 's (i.e., the equation like (6.10)) are equal. Therefore

there are $\prod_{i=1}^b \binom{k_i - 1}{\frac{k_i}{2}} = a_\omega$ different linear combinations corresponding to each word ω , where k_1, \dots, k_b are the block sizes of ω .

From (5.5) and the discussion above, it is easy to see that for a word ω of length $2k$,

$$\begin{aligned} & \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_C^s}(\omega)| = \\ & a_\omega \left[\lfloor y_n \rfloor^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} \lfloor y_n \rfloor^{|S^- - S_0|} |\{v_S : F(y_n L_{2i,n}^T(v_S)) \leq y_n - \lfloor y_n \rfloor, \mathbf{1}(2i \in S_0)\}| \right]. \end{aligned}$$

Therefore as $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_C^s}(\omega)| &= a_\omega \left[\lfloor y \rfloor^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} \lfloor y \rfloor^{|S^- - S_0|} \right. \\ & \left. \int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^T(v_S)) \leq y - \lfloor y \rfloor \forall 2i \in S_0) dv_S \right]. \quad (6.16) \end{aligned}$$

When $y \geq 1$, the rhs of (6.16) is obviously positive. We next show that

$$\int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^T(v_S)) \leq y - \lfloor y \rfloor \forall 2i \in S^-) dv_S > 0 \text{ when } y < 1.$$

First note that as $y < 1$, $\lfloor y \rfloor = 0$. Now note that we had previously concluded in the proof of part (i) of Lemma 5.4 that for certain values of $v_S \in [0, 1]^{b+1}$, $\mathbf{1}(0 \leq L_i^T(v_S) \leq 1, \forall i \in S') = 1$. As, $\{v_S : 0 \leq L_{2i}^T(v_S) \leq 1, \forall i \in S^-\} \subset \{v_S : 0 \leq L_i^T(v_S) \leq 1, \forall i \in S'\}$, for these chosen values of v_S , we have $\mathbf{1}(0 \leq L_{2i}^T(v_S) \leq 1, \forall 2i \in S^-) = 1$. Therefore, with this choice of $v_S \in [0, 1]^{b+1}$,

$$y L_{2i}^T(v_S) \leq y < 1 \implies F(y L_{2i}^T(v_S)) \leq y.$$

That is, the integral $\int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^T(v_S)) \leq y - \lfloor y \rfloor \forall 2i \in S^-) dv_S$ is positive.

Hence the proof of part (i) is complete.

(ii) Using the same arguments as in the proof of Lemma 5.5, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_C}(\omega)| = 0 \text{ if } \omega \text{ is not symmetric.}$$

Next, suppose ω is a symmetric word with b distinct letters and $(r+1)$ even generating vertices. Also suppose the letters make their first appearances at i_1, i_2, \dots, i_b positions in ω .

Using similar arguments as (5.21) in Lemma 5.5 and (6.13) in the proof of Part (i), we have for $2i - 1 \in S'$, there is a unique choice of $\pi(2i - 1)$, once the generating vertices have been chosen. For all $2i \in S'$, $1 \leq i < k$, we can have $\lfloor y_n \rfloor$ choices for $\pi(2i)$ as $1 \leq \pi(2i) \leq p$ and $y_n = p/n$. Moreover, there is an additional choice for if

$$\sum_{j < 2i} \beta_{2ij} \pi(i_j) + m_{2i,n} \leq p - \lfloor y_n \rfloor n. \quad (6.17)$$

Using the same notations and arguments as in Part (i) we have

$$F(y_n L_{2i,n}^T(v_S)) \leq y_n - \lfloor y_n \rfloor, \tag{6.18}$$

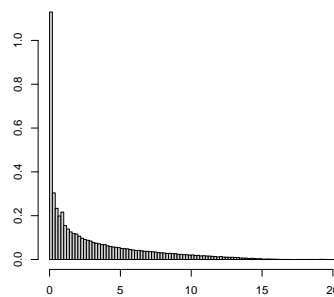
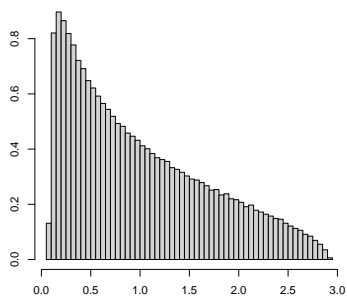
where $L_{2i,n}^T$ is a particular set of linear combinations defined whose sign has been chosen as in (5.22), (see proof of Lemma 5.5). Thus we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{p^{r+1} n^{b-r}} |\Pi_{S_C}(\omega)| &= \lfloor y \rfloor^{k-(r+1)} + \sum_{\phi \neq S_0 \subset S^-} \lfloor y \rfloor^{|S^- - S_0|} \\ &\int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^T(v_S)) \leq y - \lfloor y \rfloor) \forall 2i \in S_0) dv_S. \end{aligned} \tag{6.19}$$

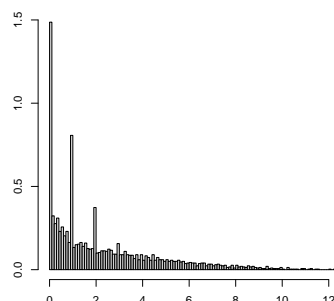
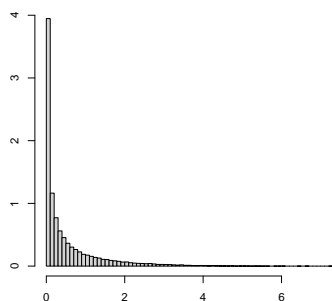
As in the proof of Part(i), we can show that when $y < 1$,

$$\int_0^1 \int_0^1 \cdots \int_0^1 \mathbf{1}(F(y L_{2i}^T(v_S)) \leq y - \lfloor y \rfloor) \forall 2i \in S^-) dv_S > 0.$$

This completes the prof of part (ii). □



(A) Input is i.i.d $x_{ij} \sim N(0, 1)/\sqrt{n}$ for every n . (B) Input is i.i.d $x_{ij} \sim Ber(3/n)$ for every n .



(C) Input is i.i.d $x_{ij} \sim \frac{(i+j)^2}{2n^2} Ber(3/n)$ for every n . (D) Input is i.i.d $x_{ij} \sim Ber(3/n), i \leq j$.

FIGURE 6.1. Histogram of the eigenvalues of S for $p = 1000, n = 2000, 30$ replications.

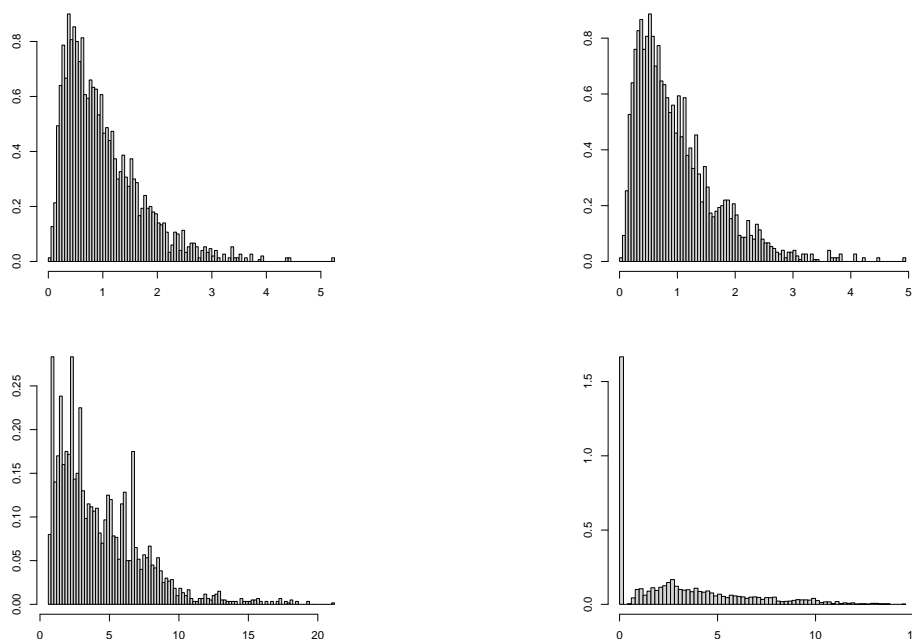


FIGURE 6.2. Histogram of the eigenvalues of $S_{R(s)}$ with entries i.i.d. $N(0, 1)/\sqrt{n}$ (top row) and i.i.d. $\text{Ber}(3/n)$ for every n (bottom row), $p = 1000, n = 2000, 2$ replications.

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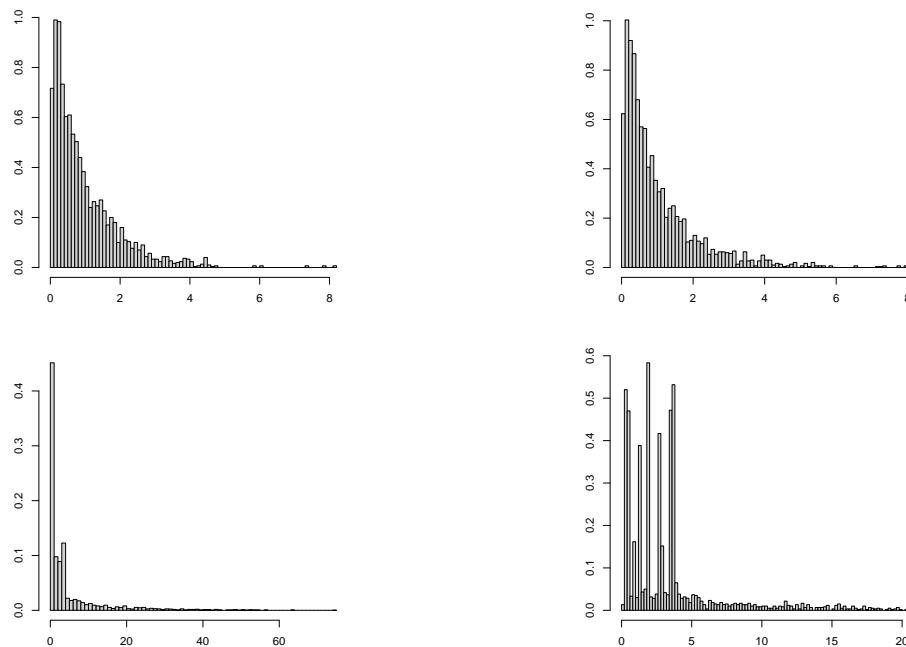


FIGURE 6.3. Histogram of the eigenvalues of $S_{C(s)}$ with entries i.i.d. $N(0, 1)/\sqrt{n}$ (top row) and i.i.d. $\text{Ber}(3/n)$ for every n (bottom row), $p = 1000, n = 2000, 2$ replications.

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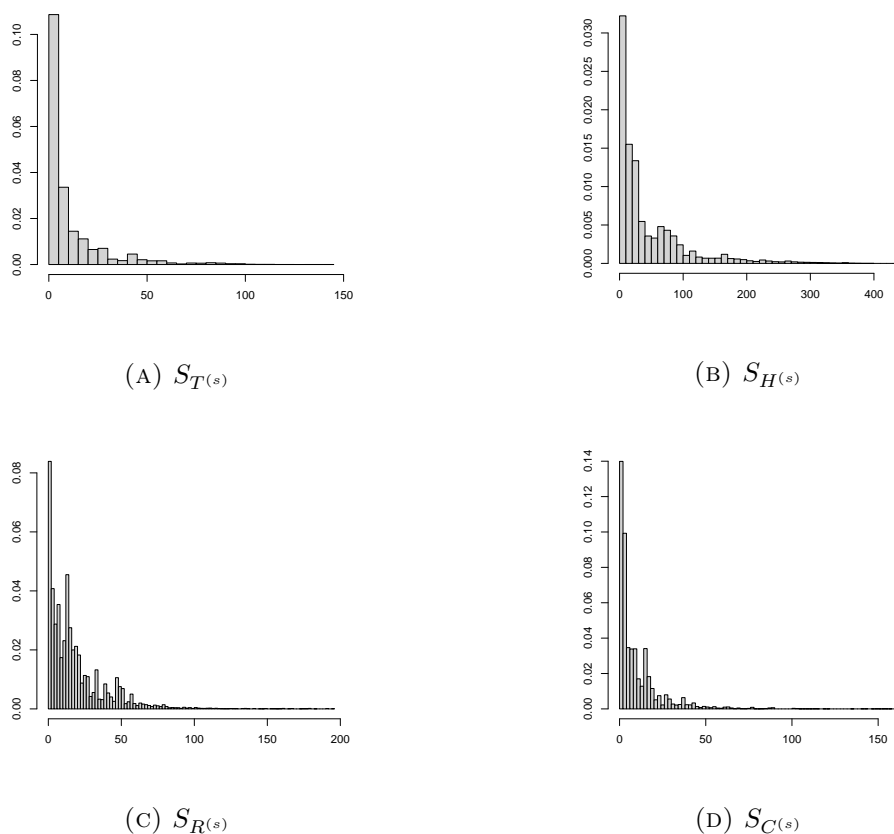


FIGURE 6.4. Histogram of the eigenvalues of S_A for $p = 1000, n = 2000$, 30 replications with variance profile $\sigma(x) = x^2 + 4x$ and $x_{ij,n} \sim Ber(3/n)$.

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