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Optimizing stakes in simultaneous bets

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Abstract. We want to find the convex combination S of iid Bernoulli random variables that maximizes $\mathbb{P}(S \geq t)$ for a given threshold t. Endre Csóka conjectured that such an S is an average if $t \geq p$, where p is the success probability of the Bernoulli random variables. We prove this conjecture for a range of p and t.

1. Introduction

We study tail probabilities of convex combinations of iid Bernoulli random variables. More specifically, let $\beta_1, \beta_2, \ldots$ be an infinite sequence of independent Bernoulli random variables with success probability p, and let $t \geq p$ be a real number. We consider the problem of maximizing $\mathbb{P}(\sum c_i\beta_i \geq t)$ over all sequences $c_1, c_2, \ldots$ of non-negative reals such that $\sum c_i = 1$. By the weak law of large numbers, the supremum of $\mathbb{P}(\sum c_i\beta_i \geq t)$ is equal to 1 if $t < p$. That is why we restrict our attention to $t \geq p$. As a motivating example, consider a venture capitalist who has a certain fortune f to invest in any number of startup companies. Each startup has an (independent) probability p of succeeding, in which case it yields a return r on investment. If the capitalist divides f into a (possibly infinite!) sequence f_i of investments, then the total return is $\sum r f_i \beta_i$. Suppose the capitalist wants to maximize the probability that the total return reaches a threshold d. Then we get our problem with $t = \frac{d}{rf}$.

The problem has a how-to-gamble-if-you-must flavor as in Dubins and Savage (1965): the capitalist places stakes c_i on a sequence of simultaneous bets. There is no need to place stakes higher than t. The way to go all out, i.e., *bold play*, is to wager t on $\lfloor \frac{1}{t} \rfloor$ bets, but this is not a convex combination. That is why we say that placing stakes $\frac{1}{k}$ on k bets with $k = \lfloor \frac{1}{t} \rfloor$ is bold play. In

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a convex combination $\sum c_i\beta_i$ we order $c_1 \geq c_2 \geq c_3 \geq \dots$. We denote the sequence (c_i) by γ and write $S_\gamma = \sum c_i\beta_i$. We study the function

$$\pi(p,t) = \sup\left\{\mathbb{P}\left(S_\gamma \geq t\right) \mid \gamma\right\} \tag{1.1}$$

for $0 \leq p \leq t \leq 1$. It is non-decreasing in p and non-increasing in t. The following has been conjectured by Csóka, in analogy of some well-known open problems in combinatorics:

Conjecture 1.1 (Csóka (2015))**.** *For every p and t there exists a $k \in \mathbb{N}$ such that $\pi(p,t)$ is realized by $c_i = \frac{1}{k}$ if $i \leq k$ and $c_i = 0$ if $i > k$ for some $k \in \mathbb{N}$. In other words, the maximal probability is realized by an average.*

If the conjecture is true, then $\pi(p,t)$ is a binomial tail probability and we still need to determine the optimal k. Numerical results of Csóka suggest that bold play is optimal for most parameter values. We are able to settle the conjecture for certain parameter values, as illustrated in Figure 1.1 below. It is natural to expect, though we are unable to prove this, that a gambler becomes bolder

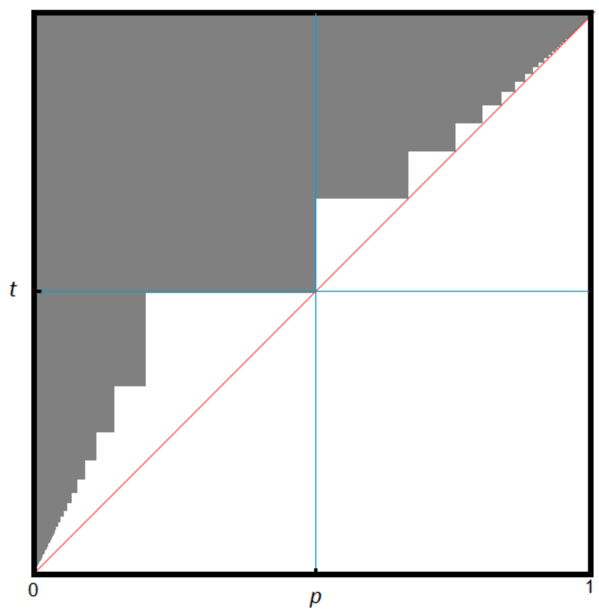

FIGURE 1.1. The shaded region represents all (p,t) for which we are able to settle the conjecture. In all these cases bold play is optimal. Our results can be divided into three parts: favorable odds $p > \frac{1}{2}$, high threshold $t \geq \frac{1}{2}$, and unfavorable odds $p < \frac{1}{2}$.

if the threshold goes up or if the odds go down. In particular, if $p' \leq p$ and $t' \geq t$ and if bold play is optimal for (p,t), then it is natural to expect that bold play is optimal for (p',t') as well. This is clearly visible in Figure 1.1, in which the shaded area represents points at which bold play is optimal. The shaded area is a union of rectangles with lower right vertices $(\frac{k}{k+1}, \frac{k}{k+1})$ and $(\frac{1}{2k+1}, \frac{1}{k+1})$ for $k \in \mathbb{N}$, which are all points of bold play.

Our paper is organized as follows. We first lay the groundwork by analyzing properties of $\pi(p,t)$ and prove that the supremum in equation 1.1 is a maximum. Our analysis of the shaded region in Figure 1.1 is divided into three parts: odds greater than one, threshold greater than half, odds smaller than one. In the final section we recall an old result on binomial probabilities which would imply that (assuming Csóka's conjecture holds and bold play is stable in the sense that we just explained) bold play is optimal if $p \leq \frac{1}{n} \leq t$ for all $n \in \mathbb{N}$.

2. Related conjectures and results

According to Csóka's conjecture, if the coin is fixed and the stakes vary, then the maximum tail probability is attained by a (scaled) binomial. If the stake is fixed and the coins vary, then Chebyshev already showed that the maximum probability is attained by a binomial:

Theorem 2.1 (Tchebichef (1846)). *For a given s and l, let $Z = \beta_1 + \cdots + \beta_l$ be any sum of l independent Bernoullis such that $\mathbb{E}[Z] = s$. Then $\mathbb{P}(Z \geq t)$ is maximized by Bernoullis for which the success probabilities assume at most three different values, only one of which is distinct from* 0 *and* 1. *In particular, the maximum $\mathbb{P}(Z \geq t)$ is a binomial tail probability.*

Samuels considered a situation in which the gambler does not necessarily select coins, but may choose any non-negative random variable. He conjectured that coins remain a gambler's best bet.

Conjecture 2.2 (Samuels (1966)). *Let $0 \leq c_1 \leq \cdots \leq c_l$ have sum $\sum_{i=1}^{l} c_i < 1$. Consider $\sup \mathbb{P}(X_1 + \cdots + X_l \geq 1)$ over all collections of l independent non-negative random variables such that $\mathbb{E}[X_i] = c_i$. This supremum is a maximum which is attained by $X_j = c_j$ for $j \leq k$ and $X_j = (1-b)\beta_j$ for $j > k$, where k is an integer, the β_j are Bernoulli random variables, and $b = \sum_{i=1}^{k} c_i$. In other words, the gambler accumulates b from the small constants before switching to bold play.*

If one assumes that Conjecture 2.2 holds, then one still needs to determine the optimal switching index k. If $c_1 = \ldots = c_l = \frac{1}{l+1}$ then the optimal k is equal to zero Alon et al. (2012) and the supremum is a binomial tail probability. This implies that another well-known conjecture is a consequence of Samuels' conjecture, see also Paulin (2017).

Conjecture 2.3 (Feige (2006)). *For all collections of l independent non-negative random variables such that $\mathbb{E}[X_i] \leq 1$ it is true that*

$$\mathbb{P}(X_1 + \cdots + X_l < l + 1) \geq \frac{1}{e}.$$

As a step towards solving this conjecture, Feige proved the remarkable theorem that there exists a $\delta > 0$ such that $\mathbb{P}(X_1 + \cdots + X_l < \mathbb{E}[S_l] + 1) \geq \delta$, where $S_l = X_1 + \ldots + X_l$. His original value of $\delta = \frac{1}{13}$ has been improved gradually. The current best result is 0.1798 by Guo et al Guo et al. (2020).

3. Properties of $\pi(p, t)$

We say that a sequence γ is *finite* if $c_i = 0$ for all but finitely many i, and infinite otherwise.

Proposition 3.1. $\pi(p, t) = \sup \{\mathbb{P}(S_\gamma \geq t) \mid \gamma \text{ is finite}\}$

Proof: According to Jessen and Wintner's law of pure type Breiman (1968, Theorem 3.5), either $\mathbb{P}(S_\gamma = s) = 0$ for each $s \in \mathbb{R}$ or there exists a countable set $\mathcal{C}$ such that $\mathbb{P}(S_\gamma \in \mathcal{C}) = 1$. In other words, the random variable S_γ is either non-atomic or discrete. If X and Y are independent, and if X is non-atomic, then the convolution formula implies that $X + Y$ is non-atomic.

Suppose that γ is infinite. We prove that S_γ is non-atomic. Let (c_{i_j}) be a subsequence such that $c_{i_j} > 2\sum_{k=j+1}^{\infty} c_{i_k}$. Let I be the set of all i_j and let J be its complement. Then both $S_I = \sum_I c_i\beta_i$ and $S_J = \sum_J c_i\beta_i$ are either discrete or non-atomic. By our choice of c_{i_j}, S_I has the property that its value determines the values of all β_i for $i \in I$. This implies that S_I is non-atomic. Therefore, $S_\gamma = S_I + S_J$ is non-atomic. In particular $\mathbb{P}(S_\gamma \geq t) = \mathbb{P}(S_\gamma > t)$ if γ is infinite.

Denote a truncated sum by $S_{\gamma,n} = \sum_{i \leq n} c_i\beta_i$. By monotonic convergence we have that $\mathbb{P}(S_\gamma > t) = \lim_{n\to\infty} \mathbb{P}(S_{\gamma,n} > t)$. Therefore, for any infinite γ, $\mathbb{P}(S_\gamma \geq t)$ can be approximated by tail probabilities of finite convex combinations. □

The function $\pi(p,t)$ is defined on a region bounded by a rectangular triangle. It is easy to compute its value on the legs of the triangle: $\pi(0,t)=0$ and $\pi(p,1)=p$. It is much harder to compute the value on the hypothenuse.

Proposition 3.2. *$\frac{1}{2}<\pi(p,p)<1$ if $0<p<1$.*

Proof: By Proposition 3.1 we may restrict our attention to finite γ. We follow the proof of Arieli et al. (2020, Lemma 1). The following Paley-Zygmund type inequality for random variables of zero mean was proved in He et al. (2008, Lemma 2.2) and extended in He et al. (2010):

$$\mathbb{P}(X<0)\geq\left(2\sqrt{3}-3\right)\frac{\mathbb{E}[X^2]^2}{\mathbb{E}[X^4]}.$$

Applying this to $S_\gamma-p$ we have

$$\mathbb{P}(S_\gamma<p)\geq\left(2\sqrt{3}-3\right)\frac{\mathbb{E}[(S_\gamma-p)^2]^2}{\mathbb{E}[(S_\gamma-p)^4]}.$$

The second moment of $S_\gamma-p$ is equal to $p(1-p)\sum c_i^2$ and the fourth moment is equal to

$$3p^2(1-p)^2\sum_{i\neq j}c_i^2c_j^2+(p(1-p)^4+p^4(1-p))\sum c_i^4$$

This can be bounded by

$$\max\left(3,\frac{1}{p(1-p)}-3\right)p^2(1-p)^2\left(\sum c_i^2\right)^2.$$

The Paley-Zygmund type inequality produces a lower bound on $\mathbb{P}(S_\gamma<p)$. Its complementary probability $\pi(p,p)$ is bounded by:

$$\pi(p,p)\leq 1-\frac{2\sqrt{3}-3}{\max\left(3,\frac{1}{p(1-p)}-3\right)}. \tag{3.1}$$

It is possible to improve on this bound for small p. We write $S_\gamma=c_1\beta_1+S$ so that

$$\mathbb{P}(S_\gamma<p)\geq\mathbb{P}(\beta_1=0)\mathbb{P}(S<p)=(1-p)\mathbb{P}(S<p).$$

Since γ is finite, S is a finite sum and we can apply Feige's theorem.

$$\mathbb{P}(S<\mathbb{E}[S]+pc_1)=\mathbb{P}\left(S/pc_1<\mathbb{E}[S/pc_1]+1\right)\geq\delta$$

. Applying the best current bound 0.1798 of Guo et al. (2020) we find

$$\pi(p,p)\leq 0.8202+0.1798p. \tag{3.2}$$

We now have two upper bounds 3.1 and 3.2 on $\pi(p,p)$. The first is more restrictive for large p and the second is more restrictive for small p.

A lower bound follows from bold play. Let $k\in\mathbb{N}$ be such that $\frac{1}{k+1}<p\leq\frac{1}{k}$. If $\bar{S}_k$ is the average of $k\geq 1$ Bernoullis, then $\mathbb{P}(\bar{S}_k\geq p)=\mathbb{P}(\bar{S}_k\geq\frac{1}{k})=1-(1-p)^k>1-(1-\frac{1}{k+1})^k$. This is minimal and equal to $\frac{1}{2}$ if $k=1$. □

Csóka conjectures that the tail probability $\mathbb{P}(S_\gamma\geq t)$ is maximized by an average of n Bernoulli random variables for an optimal n. In particular, the conjecture implies that the supremum in equation 1.1 is a maximum. We are able to prove this if $p<t$.

Theorem 3.3. *If $p<t$ then $\pi(p,t)=\mathbb{P}(S_\gamma\geq t)$ for some γ. Furthermore, $\pi(p,t)$ is left-continuous in t.*

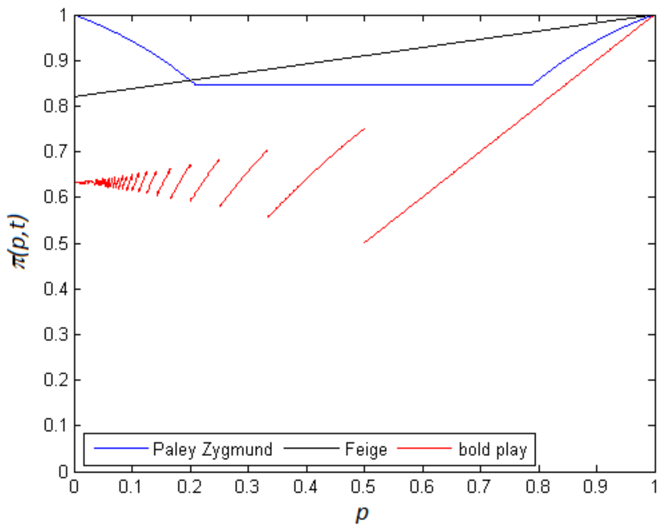

FIGURE 3.2. Bounds on $\pi(p,p)$ for $0 \leq p \leq 1$. The upper bounds follow from the Paley-Zygmund inequality and Feige's theorem. The lower bound (red) follows from bold play. If Feige's conjecture holds, then the (black) upper bound would meet the lower bound at $p = 0$ in this figure. In corollary 5.2 we find $\pi(\frac{1}{2},\frac{1}{2}) = \frac{3}{4}$, which is on the zigzag graph for the lower bound. If Conjecture 1.1 is correct, then Theorem 7.1 implies that bold play is optimal for $p = t = \frac{1}{n}$ for all n. These are the tops of the zigzag.

Proof: We write $\pi(p,t^-) = \lim_{s\uparrow t}\pi(p,s)$. Since $\pi(p,t)$ is decreasing in t, it suffices to show that there exists an S_γ such that $\mathbb{P}(S_\gamma \geq t) \geq \pi(p,t^-)$. Let $\gamma_n = (c_{n,i})_i$ be such that $\mathbb{P}(S_{\gamma_n} \geq t_n)$ converges to $\pi(p,t^-)$ for an increasing sequence $t_n \uparrow t$. Since we may replace the sequence (γ_n) by any subsequence, we may assume that $(c_{n,i})_n$ is convergent for each fixed i, by the standard diagonal argument. Let $c_i = \lim_{n\to\infty} c_{n,i}$ and let $\gamma = (c_i)_i$. Then γ is a non-increasing sequence which adds up to $\sum c_i = 1 - c \leq 1$ for some c. Observe that γ cannot be the all zero sequence, since this would imply that $c_{n,1} \to 0$ and $\mathrm{Var}(S_{\gamma_n}) = p(1-p)\sum c_{n,i}^2 \leq p(1-p)c_{n,1} \to 0$, so S_{γ_n} converges to p in distribution. Since we limit ourselves to $p < t$, this means that $\mathbb{P}(S_{\gamma_n} \geq t) \to 0$ which is nonsense. Therefore, $1 - c > 0$.

We first prove that $\pi(p,t^-) \leq \mathbb{P}(S_\gamma \geq t - cp)$. Fix an arbitrary $\epsilon > 0$. Let i_0 be such that $\sum_{j\geq i_0} c_j < \frac{\epsilon}{4}$ and $c_{i_0} < \epsilon^4$. Let n_0 be such that $\sum_{j\leq i_0} |c_{n,j} - c_j| < \frac{\epsilon}{4}$ and $c_{n,i_0} < \epsilon^4$ for all $n \geq n_0$. Now

$$\{S_{\gamma_n} \geq t_n\} \quad\subset\quad \left\{\sum_{j\leq i_0} c_{n,j}\beta_j \geq t_n - cp - \epsilon\right\} \;\bigcup\; \left\{\sum_{j\geq i_0} c_{n,j}\beta_j \geq cp + \epsilon\right\}$$

so by our assumptions

$$\begin{aligned}\{S_{\gamma_n} \geq t_n\} &\subset \left\{\sum_{j\leq i_0} c_j\beta_j \geq t_n - cp - 2\epsilon\right\} \;\bigcup\; \left\{\sum_{j\geq i_0} c_{n,j}\beta_j \geq cp + \epsilon\right\} \\ &\subset \{S_\gamma \geq t_n - cp - 2\epsilon\} \;\bigcup\; \{T_n \geq cp + \epsilon\}\end{aligned}$$

where we write $T_n = \sum_{j\geq i_0} c_{n,j}\beta_j$. Observe that

$$\mathbb{E}[T_n] = \mathbb{E}[S_{\gamma_n}] - p\sum_{j<i_0} c_{n,j} < p - p\left(\sum_{j<i_0} c_j - \tfrac{\epsilon}{4}\right) < p - p\left(1 - c - \tfrac{\epsilon}{2}\right) < pc + \tfrac{\epsilon}{2}$$

and

$$\mathrm{Var}(T_n) = p(1-p)\sum_{j\geq i_0} c_{n,j}^2 \leq c_{n,i_0}\sum_{j\geq i_0} c_{n,j} < \epsilon^4.$$

By Chebyshev's inequality, we conclude that $\mathbb{P}(T_n \geq cp+\epsilon) < \epsilon$ for sufficiently small ϵ. It follows that

$$\mathbb{P}(S_{\gamma_n} \geq t_n) \leq \mathbb{P}(S_\gamma \geq t_n - cp - 2\epsilon) + \epsilon.$$

By taking limits $n \to \infty$ and $\epsilon \to 0$ we conclude that

$$\pi(p, t^-) \leq \mathbb{P}(S_\gamma \geq t - cp).$$

Let $\bar\gamma = \frac{1}{1-c}\gamma$. Then $S_{\bar\gamma} = \frac{1}{1-c}S_\gamma$ is a convex combination such that

$$\mathbb{P}(S_{\bar\gamma} \geq t) = \mathbb{P}(S_\gamma \geq (1-c)t) \geq \mathbb{P}(S_\gamma \geq t - cp) \geq \pi(p, t^-).$$

Therefore, $\pi(p,t) = \mathbb{P}(S_{\bar\gamma} \geq t)$ and these inequalities are equalities. In particular, $\pi(p,t^-) = \pi(p,t)$ so that this function is left-continuous in t. $\square$

We now more or less repeat this proof to show that $\pi(p,t)$ is continuous in p. Since we need to vary p, we write β^p for a Bernoulli with success probability p and $S^p_\gamma = \sum c_i\beta^p_i$.

Theorem 3.4. *$\pi(p,t)$ is continuous in p.*

Proof: For any $\epsilon > 0$ choose a finite γ such that $\mathbb{P}(S^p_\gamma \geq t) \geq \pi(p,t) - \epsilon$. If p_n converges to p then β^{p_n} converges to β^p in probability. Since γ is finite

$$\limsup_{n\to\infty} \pi(p_n, t) \geq \lim_{n\to\infty} \mathbb{P}(S^{p_n}_\gamma \geq t) = \mathbb{P}(S^p_\gamma \geq t) \geq \pi(p,t) - \epsilon.$$

It follows that $\limsup_{n\to\infty} \pi(p_n,t) \geq \pi(p,t)$ for any sequence $p_n \to p$. Since $\pi(p,t)$ is increasing in p, it follows that $\pi(p,t)$ is left-continuous in p.

We need to prove right continuity, i.e., $\pi(p^+,t) = \pi(p,t)$. This is trivially true on the hypothenuse, because this is the right-hand boundary of the domain. Consider $p < t$. Let $p_n \downarrow p$ and γ_n be such that $\lim_{n\to\infty} \mathbb{P}(S^{p_n}_{\gamma_n} \geq t) = \pi(p^+,t)$. By the standard diagonal argument we may again assume that the γ_n converge coordinatewise to a sequence γ, which may not sum up to one. It cannot be the all zero sequence, i.e., not all entries in γ can be zero, by the same argument as in the proof of Theorem 3.3. The sequence γ therefore sums up to $1-c$ for some $0 \leq c < 1$. Again, we split $S^p_\gamma = H + T$ where $H = \sum_{j\leq i_0} c_j\beta^p_j$ and $T = \sum_{j>i_0} c_j\beta^p_j$. We choose i_0 such that $\mathbb{E}[T] < \frac{\epsilon}{4}$ and $c_{i_0} < \epsilon^4$. Similarly, $S_{\gamma_n} = H_n + T_n$ where H_n converges to H in probability, $\mathbb{E}[T_n] < pc + \frac{\epsilon}{2}$ and $\mathrm{Var}(T_n) < \epsilon^4$ for sufficiently large n. As in the previous proof, Chebyshev's inequality and convergence in probability imply that

$$\mathbb{P}(S^{p_n}_{\gamma_n} \geq t) \leq \mathbb{P}(H_n \geq t - cp - \epsilon) + \epsilon \leq \mathbb{P}(H \geq t - cp - 2\epsilon) + \epsilon.$$

for sufficiently large n. By taking limits $n \to \infty$ and $\epsilon \to 0$ it follows that $\pi(p^+,t) \leq \mathbb{P}(S^p_\gamma \geq t - cp)$. If we standardize γ to a sequence $\bar\gamma$ so that we get a convex combination, we again find that $\pi(p^+,t) \leq \mathbb{P}(S^p_{\bar\gamma} \geq t)$. $\square$

4. Favorable odds

We consider $\frac{1}{2} \leq p < t$. In this case, bold play comes down to a single stake $c_1 = 1$. The approach from this point on is combinatorial and we introduce some notation and notions from combinatorial set theory. We say that $I \subset \mathbb{Z}/n\mathbb{Z}$ is an *interval of length* $a < n$ if $I = [b, b+a) = \{b, b+1, \ldots, b+a-1\}$, where addition is modulo n. We say that b is the initial element of I and we say that two intervals I and J are *separate* if $I \cup J$ is not an interval. Note that if I and J are disjoint, then they need not be separate. We denote the cardinality of a set by $|S|$. In particular, $|I| = a$ if I is an interval of length a. If $\mathcal{F}$ is a family of sets, then we write $\bigcup\mathcal{F}$ for the union of all these sets. Two families $\mathcal{F}$ and $\mathcal{G}$ are *cross-intersecting* if $I \cap J \neq \emptyset$ for all $I \in \mathcal{F}$ and $J \in \mathcal{G}$.

Lemma 4.1. *Let $\mathcal{F}$ be a family of k intervals of length a in $\mathbb{Z}/n\mathbb{Z}$ such that $\bigcup\mathcal{F}$ is a proper subset of $\mathbb{Z}/n\mathbb{Z}$. Then $|\bigcup\mathcal{F}| \geq k + a - 1$.*

Proof: By our assumption $\bigcup\mathcal{F}$ is a union of, say, $c \geq 1$ separate intervals, all of lengths $\geq a$. Any interval of length $b \geq a$ contains $b-(a-1)$ intervals of length a. Since $\bigcup\mathcal{F}$ consists of c separate intervals, it follows that it contains $|\bigcup\mathcal{F}|-c(a-1)$ intervals of length a. The k elements of $\mathcal{F}$ must be among them, and it follows that $|\bigcup\mathcal{F}|-c(a-1)\geq k$. □

Lemma 4.2. *Let $\mathcal{F}$ be a family of k intervals of length k in $\mathbb{Z}/n\mathbb{Z}$. Let $\mathcal{G}$ be a family of intervals of length $a \leq n-k$ such that $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting. Then $|\mathcal{G}| \leq a$.*

Proof: Let $I=[b,b+k)$ be any element in $\mathcal{F}$. An interval $[c,c+a)$ intersects I if and only if $c\in[b-a+1,b+k)$, which is an interval of length $k+a-1$. Therefore, the set $\mathcal{I}$ of initial elements c of intervals in $\mathcal{G}$ is contained in an intersection of k intervals of length $k+a-1$. The complement of $\mathcal{I}$ thus contains a union of k intervals of length $n-k-a+1$. By the previous lemma, this union has cardinality $\geq n-a$. Therefore, $\mathcal{I}$ contains at most a elements. □

Lemma 4.3. *Let (V,μ) be a finite measure space such that $\mu(V)=b$ and let $V_i\subset V$ for $i=1,\ldots,k$ be such that $\mu(V_i)\geq t$. Then $\mu\left(\bigcap V_i\right)\geq kt-(k-1)b$.*

Proof:
$$\mu\left(\bigcap V_i\right)=b-\mu\left(\bigcup V_i^c\right)\geq b-\sum(b-\mu(V_i))\geq kt-(k-1)b.$$
□

Theorem 4.4. *If $\frac{k}{k+1}<p\leq\frac{k+1}{k+2}<t$ for some positive integer k, then bold play is optimal.*

Proof: Bold play has success probability p. Therefore, we need to prove that $\mathbb{P}(S_\gamma\geq t)\leq p$ for arbitrary γ. By Proposition 3.1 we may assume that γ is finite. It suffices to prove that $\mathbb{P}(S_\gamma\geq t)\leq p$ for rational p, since $\pi(p,t)$ is monotonic in p.

Let n be the number of non-zero c_i in γ and let $p=\frac{a}{b}$. Let X_i be a sequence of n independent discrete uniform $U\{0,b-1\}$ random variables, i.e, $X_i=c$ for $c\in\{0,\ldots,b-1\}$ with probability $\frac{1}{b}$. Let $B_i^0=1_{[0,a)}(X_i)$ for $1\leq i\leq n$. Then S_γ and $Y^0=\sum c_iB_i^0$ are identically distributed. Think of $c_iB_i^0$ as an assignment of weight c_i to a random element in $\{0,\ldots,b-1\}=\mathbb{Z}/b\mathbb{Z}$. Let $\ell(j)$ be the sum of the coefficients – the load – that is assigned to $j\in\mathbb{Z}/b\mathbb{Z}$. Then $Y^0=\ell(0)+\cdots+\ell(a-1)$, i.e., Y^0 is the load of $[0,a)$. Instead of $[0,a)$ we might as well select any interval $[j,j+a)\subset\mathbb{Z}/b\mathbb{Z}$. If Y^j is the load of $[j,j+a)$, then $S_\gamma\sim Y^j$, and $\mathbb{P}(S_\gamma\geq t)=\frac{1}{b}\sum\mathbb{P}(Y^j\geq t)$. We need to prove that $\sum\mathbb{P}(Y^j\geq t)\leq a$.

Let Ω be the sample space of the X_i. For $\omega\in\Omega$, let $J(\omega)$ be the cardinality of $\mathcal{J}(\omega)=\{j\colon Y^j(\omega)\geq t\}\subset\mathbb{Z}/b\mathbb{Z}$. In particular, $\mathbb{P}(S_\gamma\geq t)=\frac{1}{b}\mathbb{E}[J]$. It suffices to prove that $J\leq a$. Assume that $J(\omega)\geq a$ for some $\omega\in\Omega$. Apply Lemma 4.3 to the counting measure to find
$$\left|\bigcap_{l=0}^{k}(\mathcal{J}(\omega)-la)\right|\geq(k+1)a-kb.$$
Note that $i\in\mathcal{J}(\omega)-j$ if and only if $[i+j,i+j+a)$ has load $\geq t$. Therefore, there are at least $(k+1)a-kb$ elements i such that the intervals $[i,i+a),[i+a,i+2a),\ldots,[i+ka,i+(k+1)a)$ all have load $\geq t$. The intersection of these $k+1$ intervals is equal to
$$I_i=[i,i+(k+1)a-kb).$$
It has load $\geq(k+1)t-k$ by Lemma 4.3. Its complement I_i^c has load $\leq k+1-(k+1)t<t$. If $j\in\mathcal{J}(\omega)$ then $[j,j+a)$ has load $\geq t$ and therefore it intersects I_i. The family $\mathcal{F}=\{I_i\colon i\in\bigcap_{l=0}^{k}(\mathcal{J}(\omega)-la)\}$ and the family $\mathcal{G}=\{[j,j+a)\colon j\in\mathcal{J}(\omega)\}$ are cross-intersecting. Lemma 4.2 applies since the length of I_i is $(k+1)a-kb$ and since $a\leq b-((k+1)a-kb)$. We conclude that $J(\omega)\leq a$. □

The proof of Theorem 4.4 depends on showing that J is bounded by $k+1$, for which we need the assumption that $p<\frac{k+1}{k+2}$. If $p=\frac{k+1}{k+2}$ then J can no longer be bounded by $k+1$. Yet, by a careful analysis of $J=k+2$ we can push the result to the hypothenuse $p=t$, as follows.

Proposition 4.5. *If $p = t = \frac{k+1}{k+2}$ for some positive integer k, then bold play is optimal if $k > 1$, and $c_1 = c_2 = c_3 = \frac{1}{3}$ is optimal if $k = 1$.*

Proof: By Proposition 3.1 we may restrict our attention to finite γ. We adopt the notation of the proof of the previous theorem. Let n be the number of non-zero coefficients in γ. As before, we assign weights c_i uniformly randomly to elements of $\{0, \ldots, k+1\}$. Let $\ell(j)$ be the load of j and let $Y^j = 1 - \ell(j)$ be the load of the complement. Note that Y^j is identically distributed to S_γ and that it reaches the threshold if and only if $\ell(j) \leq \frac{1}{k+2}$. For $\omega \in \Omega$ let $J(\omega)$ be the number of $Y^j(\omega)$ that reach the threshold. We have $\frac{1}{k+2}\mathbb{E}[J] = \mathbb{P}(S_\gamma \geq t)$. In the proof above, we showed that $J \leq k+1$ if $t > p = \frac{k+1}{k+2}$. This is no longer true now that we have $t = p$. It may happen that $J(\omega) = k+2$ in which case all $Y^j(\omega)$ are equal to $\frac{k+1}{k+2}$ and all loads $\ell(j)$ are equal to $\frac{1}{k+2}$. Note that this can only happen if all c_i are bounded by $\frac{1}{k+2}$, so $n \geq k+2$.

We think of the weights as being assigned one by one $c_1, c_2, \ldots, c_n$ in increasing order. In particular, c_{n-1} and c_n are assigned last. If $J = k+2$, then either k or $k+1$ of the loads are equal to $\frac{1}{k+2}$ before c_{n-1} and c_n are assigned. In the first case, there are two remaining loads $< \frac{1}{k+2}$ and the probability that c_{n-1} are c_n are assigned here is $\frac{2}{(k+2)^2}$. In the second case, there is only one remaining load $< \frac{1}{k+2}$ and the probability that c_{n-1} and c_n are assigned here is $\frac{1}{(k+2)^2}$. We conclude that $\mathbb{P}(J = k+2) \leq \frac{2}{(k+2)^2}$ and therefore

$$\mathbb{E}[J] \leq (k+2)\mathbb{P}(J = k+2) + (k+1)\mathbb{P}(J < k+2) = k + 1 + \mathbb{P}(J = k+2)$$

is bounded by $k + 1 + \frac{2}{(k+2)^2}$. Thus we obtain $\mathbb{P}(S_\gamma \geq t) \leq \frac{k+1}{k+2} + \frac{2}{(k+2)^3}$. This bound is reached if $k = 1$ and $c_1 = c_2 = c_3 = \frac{1}{3}$.

We restrict our attention to $k > 1$ and we prove that bold play is optimal. In other words, we need to prove $\mathbb{P}(S_\gamma \geq t) \leq \frac{k+1}{k+2}$. We first consider the case that $c_{n-1} \neq c_n$. We may assume that $J(\omega) = k+2$ for some ω, because the proof of Theorem 4.4 holds if there is no such ω. If there are two remaining loads $< \frac{1}{k+2}$ before c_{n-1} and c_{n-2} are assigned, then there is a unique assignment of c_{n-1} and c_n to complete all loads to $\frac{1}{k+2}$. Since $k > 1$, there are at least two loads $\ell(i_1) = \ell(i_2) = \frac{1}{k+2}$ before the final two weights are assigned. Let $\bar{\omega}$ assign c_j for $j < n-1$ in the same way as ω, but it reassigns c_{n-1} to i_1 and c_n to i_2. Then $J(\bar{\omega}) = k$, because the loads at i_1 and i_2 exceed the threshold. We can reconstruct ω from $\bar{\omega}$ because the loads at i_1 and i_2 are the only ones that exceed the threshold for $\bar{\omega}$, and their values are different because $c_{n-1} \neq c_n$. We have a $1-1$ correspondence between $\omega \in \{J = k+2\}$ and $\bar{\omega} \in \{J = k\}$. Let $\mathcal{E} = \{J = k+2\}$ and let $\mathcal{F} = \{\bar{\omega} \colon \omega \in \mathcal{E}\}$. Then $\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{E})$ and $\mathcal{E} \cap \mathcal{F} = \emptyset$. This implies that

$$\mathbb{E}[J] \leq (k+2)\mathbb{P}(\mathcal{E}) + k\mathbb{P}(\mathcal{F}) + (k+1)\mathbb{P}(\mathcal{E}^c \cap \mathcal{F}^c) \leq k+1.$$

In particular $\mathbb{P}(S_\gamma \geq t) \leq \frac{k+1}{k+2}$ and bold play is optimal.

Finally, we consider the remaining case that $c_{n-1} = c_n$ and assume that $J(\omega) = k+2$. In this case, we may switch the order of assigning c_{n-1} and c_n. Let $\omega' \in \Omega$ represent this switch (it may be equal to ω if the assignments are the same). Again, let i_1 and i_2 be two locations for which the loads have already been completed before c_{n-1} and c_n are placed. Let $\{\bar{\omega}, \bar{\omega}'\}$ be the elements which assign the first $n-2$ coefficients in the same way, but assigns the final two elements to i_1 and i_2. In particular, $J(\bar{\omega}) = J(\bar{\omega}') = k$. We can reconstruct $\{\omega, \omega'\}$ from $\{\bar{\omega}, \bar{\omega}'\}$, the correspondence is injective, so again $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$ and we conclude in the same way that bold play is optimal. □

If bold play is stable, as discussed in the introduction of this paper, then Proposition 4.5 would imply that bold play is optimal if $p \leq \frac{k+1}{k+2} \leq t$ for $k > 1$. We verify this in one very specific case in the range $p < \frac{k+1}{k+2} = t$.

Proposition 4.6. *If $p = \frac{2k+1}{2k+3}$ and $t = \frac{k+1}{k+2}$, then bold play is optimal if $k > 1$.*

Proof: We again randomly distribute the coefficients of a finite γ over $2k+3$ locations. Let $Y^j = \ell(j)+\ldots+\ell(2k+j)$ be the load of the discrete interval $[j, 2k+j+1)$, where as before we compute modulo $2k+3$. Then $S_\gamma \sim Y^j$ and the sum of all Y^j is equal to $2k+1$. Let J be the number of Y^j that reach the threshold. Not all Y^j can reach the threshold and therefore $J \leq 2k+2$. Then

$$\mathbb{P}(S_\gamma \geq t) = \frac{\mathbb{E}[J]}{2k+3} \leq \frac{2k+1}{2k+3}\mathbb{P}(J \leq 2k+1) + \frac{2k+2}{2k+3}\mathbb{P}(J = 2k+2).$$

We need to prove that $\mathbb{P}(S_\gamma \geq t) \leq \frac{2k+1}{2k+3}$. If $\mathbb{P}(J = 2k+2) = 0$ then we are done. Therefore, we may assume that $\mathbb{P}(J = 2k+2) > 0$. If $J \geq 2k+2$ then only one of the Y^j does not meet the threshold and without loss of generality we may assume it is Y^2, which has load $1-\ell(0)-\ell(1)$. The other Y^j reach the threshold, and since the sum of all Y^j is equal to $2k+1$, we find that

$$2k+1 \geq (2k+2)t + 1 - \ell(0) - \ell(1). \tag{4.1}$$

In other words, $\ell(0)+\ell(1) \geq \frac{2}{k+2}$. If $\ell(0) > \frac{1}{k+2}$ then Y^j only reaches the threshold if it includes $\ell(0)$. Only $2k+1$ of the Y^j include $\ell(0)$, contradicting our assumption that $J = 2k+2$. Therefore $\ell(0) \leq \frac{1}{k+2}$ and since the same applies to $\ell(1)$ we have that $\ell(0) = \ell(1) = \frac{1}{k+2}$ and we have equality in 4.1. In particular, $Y^j = t$ for all j other than 2. It follows that $\ell(j)+\ell(j-1) = \frac{1}{k+2}$ for all loads other than $\ell(1)+\ell(0)$. In particular, $\ell(2) = \ell(2k+2) = 0$ which implies that the loads alternate between zero and $\frac{1}{k+2}$: $\ell(i) = \frac{1}{k+2}$ if i is odd and $\ell(i) = 0$ if $i > 0$ is even. All non-zero loads are equal and only two non-zero loads are consecutive. Obviously, the probability of this arrangement is low. There are exactly $2k+3$ such arrangements. There are also $2k+3$ arrangements in which the non-zero loads are consecutive. In this case $J = k+2 \leq 2k$. It follows that $\mathbb{P}(J \leq 2k) \geq P(J = 2k+2)$, which implies that $\mathbb{E}[J] \leq 2k+1$. Bold play is optimal. $\square$

These results conclude our analysis of the upper right-hand block of Figure 1.1. An array of triangles along the hypothenuse remains to be filled in that part. Numerical results of Csóka (2015) suggest that bold play is optimal for all of these triangles, except for the one touching on $\{(p,p)\colon \frac{1}{2} \leq p \leq \frac{2}{3}\}$. In the next section we will confirm that bold play is not optimal for this particular triangle.

5. High threshold

We now consider the case $p \leq \frac{1}{2} < t$, when bold play again comes down to a single stake $c_1 = 1$. We introduce some further notation. If γ is finite and has $\leq n$ non-zero coefficients, then $\mathcal{F}_{t,\gamma}$ denotes the family of $V \subset \{1, 2, \ldots, n\}$, such that $\sum_{i\in V} c_i \geq t$. A family of sets is *intersecting* if no two elements are disjoint. Two standard examples of intersecting families are $\mathcal{F}_1$, the family of all V such that $1 \in V$, and $\mathcal{F}_{>n/2}$, the family of all subsets such that $|V| > n/2$. Since $t > \frac{1}{2}$ each $\mathcal{F}_{t,\gamma}$ is intersecting. Note that $\mathcal{F}_1$ corresponds to $\mathcal{F}_{t,\gamma}$ with $c_1 = 1$. In other words, $\mathcal{F}_1$ is the intersecting family that corresponds to bold play. We write $p(V) = p^{|V|}(1-p)^{n-|V|}$ so that

$$\mathbb{P}(S_\gamma \geq t) = \sum_{V \in \mathcal{F}_{t,\gamma}} p(V). \tag{5.1}$$

The problem of maximizing the tail probability then turns into determining the intersecting family $\mathcal{F}_{t,\gamma}$ that has a maximal weighted sum over its elements V. Such problems are well studied in extremal combinatorics, see e.g. Filmus (2017) for recent results. Fishburn et al. (1986) settled the problem of maximizing

$$p(\mathcal{F}) = \sum_{V \in \mathcal{F}} p(V)$$

over all intersecting families $\mathcal{F}$:

Theorem 5.1 (Fishburn et al)**.** *For a fixed n, let $\mathcal{F}$ be any intersecting family of subsets from $\{1,\ldots,n\}$. If $p \le \frac{1}{2}$ then $p(\mathcal{F})$ is maximized by $\mathcal{F}_1$. If $p \ge \frac{1}{2}$ and n is odd, then $p(\mathcal{F})$ is maximized by $\mathcal{F}_{>n/2}$.*

Proof: Following Fishburn et al. (1986). First suppose n is odd. At most one of V and V^c can be in $\mathcal{F}$. If $p \ge \frac{1}{2}$, then $p(V) \ge p(V^c)$ if $|V| \ge |V^c|$. Therefore $p(\mathcal{F})$ is maximal if out of each V and V^c the largest set is in $\mathcal{F}$. It follows that $\mathcal{F}_{>n/2}$ maximizes $p(\mathcal{F})$ if n is odd and $p \ge \frac{1}{2}$.

Now consider an arbitrary n and $p \le \frac{1}{2}$. Let $c_a = |\mathcal{F}^a|$ be the cardinality of the subfamily $\mathcal{F}^a = \{V \in \mathcal{F} \colon |V| = a\}$. Since at most one of V and V^c can be in $\mathcal{F}$, we have that $c_a + c_{n-a} \le \binom{n}{a}$. Since $p \le \frac{1}{2}$ we now have $p(V) \ge p(V^c)$ if $|V| \le |V^c|$. If $a = \frac{n}{2}$ then V and V^c are equally large, we just need to be sure that one of them is in $\mathcal{F}$. For $a < \frac{n}{2}$ we want to maximize c_a under the constraint $c_a + c_{n-a} \le \binom{n}{a}$. By the Erdős-Ko-Rado theorem, if $a < \frac{n}{2}$ then c_a is maximized by a family of subsets that contain one common element. Such a family satisfies $c_a + c_{n-a} = \binom{n}{a}$, maximizes each c_a, and contains one of each V and V^c. It follows that $p(\mathcal{F}_1)$ is maximal if $p \le \frac{1}{2}$. □

As an immediate corollary we can take care of the upper left-hand block in Figure 1.1.

Corollary 5.2. *If $p \le \frac{1}{2} < t$ then bold play is optimal.*

Proof: If $t > \frac{1}{2}$ then $\mathcal{F}_{t,\gamma}$ is intersecting and $\mathcal{F}_1$ corresponds to bold play. □

As another corollary, we can say something more about the upper right-hand block.

Corollary 5.3. *If $\frac{1}{2} < p \le t \le \frac{2}{3}$ then bold play is not optimal.*

Proof: Choose k maximal such that $t \le \frac{k+1}{2k+1}$ and let $n = 2k+1$. The family $\mathcal{F}_{>n/2}$ corresponds to $\mathcal{F}_{t,\gamma}$ for $c_1 = \cdots = c_{2k+1} = \frac{1}{2k+1}$, which is not bold play. It is the unique maximizer of $p(\mathcal{F})$ among $\mathcal{F}_{t,\gamma}$ for γ such that $c_{2k+2} = 0$. □

The remaining part of the upper left-hand block has $t = \frac{1}{2}$. In this case $\mathcal{F}_{t,\gamma}$ may no longer be intersecting, but we can also use Theorem 5.1 to settle that part.

Corollary 5.4. *If $p \le t = \frac{1}{2}$ then bold play is optimal.*

Proof: Note that we may restrict our attention to $\gamma = (c_1, c_2, \ldots)$ such that $c_1 \le \frac{1}{2}$ and that bold play corresponds to $(\frac{1}{2}, \frac{1}{2}, 0, \ldots)$.

$$\begin{aligned}\mathbb{P}(S_\gamma \ge \tfrac{1}{2}) &= p\mathbb{P}(S_\gamma \ge \tfrac{1}{2} \mid \beta_1 = 1) + (1-p)\mathbb{P}(S_\gamma \ge \tfrac{1}{2} \mid \beta_1 = 0)\\ &\le p + (1-p)\mathbb{P}(S_\gamma \ge \tfrac{1}{2} \mid \beta_1 = 0)\end{aligned}$$

If $\bar{\gamma} = \frac{1}{1-c_1}(c_2, c_3, \ldots)$ then $\mathbb{P}(S_\gamma \ge \frac{1}{2} \mid \beta_1 = 0) = \mathbb{P}(S_{\bar{\gamma}} \ge \frac{1}{2(1-c_1)}) \le p$ by Theorem 5.1. We find that $\mathbb{P}(S_\gamma \ge \frac{1}{2}) \le p + (1-p)p$ with equality for bold play. □

6. Unfavorable odds

Finally, we consider the lower left-hand block $p \le t < \frac{1}{2}$, which appears to be the most challenging choice of parameters for Csóka's conjecture. We introduce some more notions and results from extremal combinatorics. A family $\mathcal{F} \subset 2^{\{1,\ldots,n\}}$ has *matching number* k, denoted by $\nu(\mathcal{F}) = k$, if the maximum number of pairwise disjoint $V \in \mathcal{F}$ is equal to k. In particular, $\mathcal{F}$ is intersecting if and only if $\nu(\mathcal{F}) = 1$. Note that $\nu(\mathcal{F}_{t,\gamma}) \le \lfloor \frac{1}{t} \rfloor$. A family $\mathcal{F}^u$ is *u-uniform* if all its elements have cardinality u. According to the Erdős matching conjecture Alon et al. (2012); Frankl (2013); Frankl and Kupavskii (2019), if $n \ge (k+1)u$ then the maximum cardinality of a u-uniform family of matching number $\le k$ is either attained by $\mathcal{F}^u_k$, the family of all u-subsets containing at least one element from $\{1,\ldots,k\}$, or by $\mathcal{F}^u_{[(k+1)u-1]}$, the family containing all u-subsets from $\{1,\ldots,(k+1)u-1\}$. Frankl (2013) settled the matching conjecture for $n \ge (2k+1)u - k$, in which case $\mathcal{F}^u_k$ has maximum cardinality. For recent progress on the conjecture, see Frankl and Kupavskii (2019) and the references therein.

Theorem 6.1. *If $p < \frac{1}{2k+1}$ and $\frac{1}{k+1} < t$ for some positive integer k then bold play is optimal.*

Proof: We need to prove that $\mathbb{P}(S_\gamma \geq t) \leq 1 - (1-p)^k$ for finite $\gamma = (c_1, c_2, \ldots)$. For a fixed γ let n be such that $c_j = 0$ if $j > n$ and let $\mathcal{F}_{t,\gamma}^j$ denote the subfamily of sets of cardinality j. Then we may write

$$\mathbb{P}(S_\gamma \geq t) = \sum_{\mathcal{F}_{t,\gamma}} p(V) = \sum_j |\mathcal{F}_{t,\gamma}^j| p^j (1-p)^{n-j}.$$

By Frankl's result, we can bound $|\mathcal{F}_{t,\gamma}^j|$ by $\binom{n}{j} - \binom{n-k}{j}$ if $(2k+1)j - k \leq n$. For larger j we simply bound by $\binom{n}{j}$. In this way we get that $\mathbb{P}(S_\gamma \geq t)$ is bounded by

$$\sum_{j \leq \frac{n+k}{2k+1}} \left(\binom{n}{j} - \binom{n-k}{j} \right) p^j (1-p)^{n-j} + \sum_{j > \frac{n+k}{2k+1}} \binom{n}{j} p^j (1-p)^{n-j}$$

which is equal to

$$1 - \sum_{j \leq \frac{n+k}{2k+1}} \binom{n-k}{j} p^j (1-p)^{n-j} = 1 - (1-p)^k \mathbb{P}\left(X \leq \frac{n+k}{2k+1} \right)$$

for $X \sim \mathrm{Bin}(n-k, p)$. By our assumptions, there exists a $c < 1$ such that $p < \frac{c}{2k+1}$. If $n \to \infty$ then $\mathbb{P}\left(X \leq \frac{n+k}{2k+1} \right) \to 1$ since $\mathbb{E}[X] = (n-k)p < \frac{(n-k)c}{2k+1}$. □

There is a gap between the shaded region and the hypothenuse in the lower left-hand block in Figure 1.1. We are able to settle a few isolated cases by the same arguments as in the proofs of Propositions 4.5 and 4.6.

Proposition 6.2. *Bold play is optimal if $t = \frac{1}{3}$ and $p = \frac{1}{b}$ for an integer $b \geq 3$.*

Proof: We assume that γ is finite and randomly assign its coefficients to $\{0, 1, \ldots, b-1\}$. The random variable Y is equal to the load at zero $\ell(0)$. We need to prove that

$$\mathbb{P}\left(Y \geq \frac{1}{3} \right) \leq p + (1-p)p + (1-p)^2 p = r$$

which is the success probability of bold play. Let K be the number of loads reaching the threshold of $\frac{1}{3}$ before the last two coefficients c_{n-1} and c_n are assigned. Obviously, K is either equal to 0 or 1 or 2. We will show that $\mathbb{P}(Y \geq \frac{1}{3} | K = j) \leq r$ for $j \in \{0, 1, 2\}$.

If $K = 0$ then Y can only reach the threshold if at least one of the last two coefficients is placed in 0. This happens with probability $p + (1-p)p < r$.

If $K = 1$ then one load has reached the threshold before the last two coefficients are placed. This load is in 0 with probability p. If the load is not in 0, then at least one of the remaining two coefficients has to be placed there. This happens with probability $p + (1-p)p$. We conclude that

$$\mathbb{P}\left(Y \geq \frac{1}{3} \,\middle|\, K = 1 \right) \leq p + (1-p)(p + (1-p)p) = r.$$

If $K = 2$ then two loads have already reached the threshold. The probability that one of these two loads is in 0 is $2p$. If none of the two loads is in 0, then $\ell(0)$ can only reach the threshold if the two last coefficients are assigned to 0. The probability that this happens is p^2.

$$\mathbb{P}\left(Y \geq \frac{1}{3} \,\middle|\, K = 2 \right) \leq 2p + (1-2p)p^2 \leq r$$

if $p \leq \frac{1}{3}$. □

Csóka's numerical results indicate that bold play is optimal if $p \leq \frac{1}{3} \leq t$. This problem has a recreational flavor, see Berrevoets et al. (2020).

7. Binomial tails

If Conjecture 1.1 holds, then the tail probability is maximized by a Bernoulli average $\bar{X}_k$ and we need to determine the optimal k. It is more convenient to state this in terms of binomials. For a fixed p and t, maximize

$$\mathbb{P}\left(\mathrm{Bin}(k,p) \geq kt\right)$$

for a positive integer k. Since the probability increases with k as long as kt does not pass an integer, we may restrict our attention to k such that $kt \leq n < (k+1)t$ for some integer n. In other words, we need to only consider $k = \lfloor \frac{n}{t} \rfloor$ for $n \in \mathbb{N}$. If $t = \frac{1}{a}$ is the reciprocal of an integer a, then the k are multiples of a. Maximizing the tail probability for this case is a classical problem. In 1693 John Smith asked which k is optimal if $a = 6$ and $p = \frac{1}{6}$. Or in his original words, which of the following events is most likely: fling at least one six with 6 dice, or at least two sixes with 12 dice, or at least three sixes with 18 dice. The problem was communicated by Samuel Pepys to Isaac Newton, who computed the probabilities. Chaundy and Bullard (1960) gave a very nice historical description (more history can be found in Koornwinder and Schlosser (2008, 2013)) and solved the problem. It is a special case of a more general result by Jogdeo and Samuels (1968).

Theorem 7.1 (Chaundy and Bullard)**.** *For an integer $a > 1$, $\mathbb{P}(\mathrm{Bin}(ka, \frac{1}{a}) \geq k)$ is maximal for $k = 1$. Even more so, the tail probabilities strictly decrease with k.*

In other words, if $p = t = \frac{1}{a}$ and if Csóka's conjecture holds, then bold play is optimal. By stability, one would expect that bold play remains optimal for $p \leq \frac{1}{a} \leq t$. In other words, one would expect that the following generalization of Chaundy and Bullard's inequality holds: $\mathbb{P}(\mathrm{Bin}(\lfloor k/t \rfloor, p) \geq k)$ is maximal for $k = 1$ if $p \leq \frac{1}{a} \leq t$. This appears to be unknown.

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