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A characterization of progressively equivalent probability measures preserving the structure of a compound mixed renewal process

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Abstract. Generalizing earlier work of Delbaen & Haezendonck for given compound mixed renewal process S under a probability measure P, we characterize all those probability measures Q on the domain of P such that Q and P are progressively equivalent and S remains a compound mixed renewal process under Q with improved properties. As a consequence, we prove that any compound mixed renewal process can be converted into a compound mixed Poisson one through a change of measures.

1. Introduction

A basic method in mathematical finance is to replace the original probability measure with an equivalent martingale measure, sometimes called a risk-neutral measure. This measure is used for pricing and hedging given contingent claims (e.g. options, futures, etc.). In contrast to the situation of the classical Black-Scholes option pricing formula, where the equivalent martingale measure is unique, in actuarial mathematics that is certainly not the case.

The above fact was pointed out by Delbaen & Haezendonck in their pioneering paper Delbaen and Haezendonck (1989), as the authors "tried to create a mathematical framework to deal with finance related to risk processes" in the frame of classical Risk Theory. Thus, they were confronted with the problem of characterizing all equivalent martingale measures Q such that a compound Poisson process under an original measure P remains a compound Poisson one under Q. Delbaen & Haezendonck answered to the positive the previous problem in Delbaen and Haezendonck (1989), and applied their results to the theory of premium calculation principles (see also Embrechts (2000) for an overview). The method provided by Delbaen and Haezendonck (1989) has been successfully

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applied to many areas of insurance mathematics such as pricing (re-)insurance contracts (Holtan (2007), Haslip and Kaishev (2010)), simulation of ruin probabilities (Boogaert and De Waegenaere (1990)), risk capital allocation (Yu et al. (2012)), pricing CAT derivatives (Geman and Yor (1997), Embrechts and Meister (1997)).

However, there is one vital point about the (compound) Poisson processes which is their greatest weakness as far as practical applications are considered, and this is the fact that the variance is a linear function of time t. The latter consideration, together with some interesting real-life cases of interest in Risk Theory, which show that the interarrival times process associated with a counting process remain independent but the exponential interarrival time distribution does not fit well into the observed data (cf. e.g. Chen et al. (2013) and Wang et al. (2012)) motivated Macheras and Tzaninis (2020) to generalize the Delbaen & Haezendonck characterization for the more general renewal risk model (also known as the Sparre Andersen model).

In reality, risk portfolios are inhomogeneous and they can be seen as a mixture of smaller homogeneous portfolios which can be identified by the realization of a random variable (or a random vector) $\varTheta$. The latter interpretation leads to the notion of mixed counting processes. Based on that, Meister (1995) and later on (in a more general setup) Lyberopoulos and Macheras (2019), generalized the Delbaen & Haezendonck characterization for the more general class of compound mixed Poisson processes, and applied their results to pricing CAT futures and to the theory of premium calculation principles, respectively.

Even though (compound) mixed Poisson processes provide a better practical model than the (compound) Poisson ones, their variance includes a quadratic term t^2, which is of importance only for large values of t. The latter, together with the fact that any (compound) mixed Poisson process is a Markov one, indicates the need for a proper generalization. In an effort to overcome these deficiencies, to allow more fluctuation and to step away from a Markovian environment, we consider the mixed renewal risk model. Since the mixed renewal risk model is strictly more comprising than the mixed Poisson one, the question whether the corresponding characterizations provided in Delbaen and Haezendonck (1989), Lyberopoulos and Macheras (2019) and Macheras and Tzaninis (2020) can be extended to the more general mixed renewal risk model naturally arises, and it is precisely this problem the present paper deals with. In particular, if the process S is a compound mixed renewal process under the probability measure P, it would be interesting to characterize all probability measures Q being progressively equivalent to P and converting S into a compound mixed Poisson process under Q.

Since conditioning is involved in the definition of compound mixed renewal processes (CMRPs for short), it is natural to expect that the notion of regular conditional probabilities plays an essential role for the investigation of CMRPs. For this reason, we first give in Section 3 a characterization of CMRPs via regular conditional probabilities, see Proposition 3.4, which serves as a useful preparatory tool for the proofs of our results. Next, necessary and sufficient conditions under which P and Q are progressively equivalent are established in Proposition 3.11, which also serves as the one direction of the desired characterization and provides an explicit and tractable formula for a Radon-Nikodým derivative of Q with respect to P. The arguments used in the proof of Proposition 3.11 are essentially different from those appearing in the existing literature for compound (mixed) Poisson processes, where the property of (conditionally) stationary independent increments plays a fundamental role, but it fails, in general, in the case of CMRPs.

In Section 4, we first construct a suitable canonical probability space $(\varOmega, \varSigma, P)$ admitting a CMRP S with prescribed distributions for the structural parameter and the claim sizes and prescribed conditional distributions for the claim interarrival times, see Proposition 4.1. The latter result along with Proposition 3.11, is required for the desired characterization, in terms of regular conditional probabilities, of all those measures Q which are progressively equivalent to an original probability measure P, such that a CMRP under P remains a CMRP under Q, see Theorem 4.5. A consequence of Theorem 4.5 is that any CMRP can be converted into a compound mixed Poisson one through a change of measures technique, by choosing the “correct” Radon-Nikodým derivative, see Corollary 4.8. Another consequence of Theorem 4.5 is Proposition 4.15, which recovers and generalises some well-known measure changes/martingales associated with compound renewal processes that have appeared in the actuarial risk theory literature. Appendix A collects some long proofs. For applications of Theorem 4.5 to a characterization of equivalent martingale measures for CMRPs, and to the pricing of actuarial risks (premium calculation principles) in an insurance market possessing the property of no free lunch with vanishing risk we refer to Tzaninis and Macheras (2020), while for applications of Theorem 4.5 to the ruin problem for CMRPs we refer to Tzaninis (2022) and Tzaninis and Macheras (2020).

2. Preliminaries

$\mathbb{N}$ and $\mathbb{R}$ stand for the natural and the real numbers, respectively, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. If $d \in \mathbb{N}$ then $\mathbb{R}^d$ denotes the Euclidean space of dimension d. For a map $f : A \to E$ and for a non-empty set $B \subseteq A$ we denote by $f \restriction B$ the restriction of f to B, while $\mathbb{1}_B$ denotes the indicator function of the set B.

Throughout this paper, unless stated otherwise, $(\varOmega, \varSigma, P)$ is a fixed but arbitrary probability space. The symbol $\mathcal{L}^1(P)$ stands for the family of all $\varSigma$-measurable real-valued functions f on $\varOmega$ such that $\int |f|\, dP < \infty$. For any Hausdorff topology $\mathfrak{T}$ on a set $\varUpsilon$, by $\mathfrak{B}(\varUpsilon)$ is denoted the Borel σ-algebra on $\varUpsilon$, i.e. the σ-algebra generated by $\mathfrak{T}$, while $\mathfrak{B} := \mathfrak{B}(\mathbb{R})$ stands for the Borel σ-algebra of subsets of $\mathbb{R}$ generated by the Euclidean topology $\mathfrak{E}$ over $\mathbb{R}$. Our measure theoretic terminology is standard and generally follows Cohn (2013). For the definitions of real-valued random variables and random variables we refer to Cohn (2013) page 308. We apply notation $P_X := P_X(\theta) := \mathbf{K}(\theta)$ to mean that X is distributed according to the law $\mathbf{K}(\theta)$, where $\theta \in D \subseteq \mathbb{R}^d$ is the parameter of the distribution. We denote again by $\mathbf{K}(\theta)$ the distribution function induced by the probability distribution $\mathbf{K}(\theta)$. Notation $\mathbf{Ga}(b, a)$, where $a, b \in (0, \infty)$, stands for the law of gamma distribution (cf. e.g. Schmidt (1996) page 180). In particular, $\mathbf{Ga}(b, 1) = \mathbf{Exp}(b)$ stands for the law of exponential distribution. For the unexplained terminology of Probability and Risk Theory we refer to Schmidt (1996).

Given a real-valued random variable X on $\varOmega$ and a $D(\subseteq \mathbb{R}^d)$-random vector $\varTheta$ on $\varOmega$, a **conditional distribution** of X over $\varTheta$ is a $\sigma(\varTheta)$-$\mathfrak{B}$-Markov kernel (see Bauer (1996) Definition 36.1 for the definition) denoted by $P_{X|\varTheta} := P_{X|\sigma(\varTheta)}$ and satisfying for each $B \in \mathfrak{B}$ the equality $P_{X|\varTheta}(\bullet, B) = P\big(X^{-1}[B] \mid \sigma(\varTheta)\big)$ $P \restriction \sigma(\varTheta)$-almost surely (written a.s. for short). Clearly, for every $\mathfrak{B}(\mathbb{R}^d)$-$\mathfrak{B}$-Markov kernel k, the map $K(\varTheta)$ from $\varOmega \times \mathfrak{B}$ into $[0, 1]$ defined by means of

$$K(\varTheta)(\omega, B) := \big(k(\bullet, B) \circ \varTheta\big)(\omega) \quad \text{for any } (\omega, B) \in \varOmega \times \mathfrak{B}$$

is a $\sigma(\varTheta)$-$\mathfrak{B}$-Markov kernel. Then for $\theta = \varTheta(\omega)$ with $\omega \in \varOmega$ the probability measures $k(\theta, \bullet)$ are distributions on $\mathfrak{B}$ and so we may write $\mathbf{K}(\theta)(\bullet)$ instead of $k(\theta, \bullet)$. Consequently, in this case $K(\varTheta)$ will be denoted by $\mathbf{K}(\varTheta)$.

For any real-valued random variables X, Y on $\varOmega$ we say that $P_{X|\varTheta}$ and $P_{Y|\varTheta}$ are $P \restriction \sigma(\varTheta)$-equivalent and we write $P_{X|\varTheta} = P_{Y|\varTheta}$ $P \restriction \sigma(\varTheta)$-a.s., if there exists a P-null set $M \in \sigma(\varTheta)$ such that for any $\omega \notin M$ and $B \in \mathfrak{B}$ the equality $P_{X|\varTheta}(\omega, B) = P_{Y|\varTheta}(\omega, B)$ holds true.

A sequence $\{V_n\}_{n\in\mathbb{N}}$ of real-valued random variables on $\varOmega$ is said to be P-**conditionally (stochastically) independent over** $\sigma(\varTheta)$, if for each $n \in \mathbb{N}$ with $n \geq 2$ we have

$$P\Big(\bigcap_{k=1}^{n} \{V_k \leq v_k\} \mid \sigma(\varTheta)\Big) = \prod_{k=1}^{n} P\big(\{V_k \leq v_k\} \mid \sigma(\varTheta)\big) \quad P \restriction \sigma(\varTheta)\text{-a.s.},$$

and $\{V_n\}_{n\in\mathbb{N}}$ is said to be P-**conditionally identically distributed over** $\sigma(\varTheta)$, if

$$P\big(F \cap V_k^{-1}[B]\big) = P\big(F \cap V_m^{-1}[B]\big)$$

whenever $k, m \in \mathbb{N}$, $F \in \sigma(\Theta)$ and $B \in \mathfrak{B}$. We say that the process $\{V_n\}_{n\in\mathbb{N}}$ is **P-conditionally (stochastically) independent or identically distributed given Θ**, if it is conditionally independent or identically distributed over the σ-algebra $\sigma(\Theta)$.

For the rest of the paper, unless stated otherwise, $\Upsilon := (0, \infty)$, Θ is a d-dimensional random vector on Ω with values in $D \subseteq \mathbb{R}^d$, and we simply write "conditionally" in the place of "conditionally given Θ" whenever conditioning refers to Θ.

Remark 2.1. If the sequence $\{V_n\}_{n\in\mathbb{N}}$ is P-conditionally identically distributed, then it is P-identically distributed.

In fact, for any $n \in \mathbb{N}$ and $B \in \mathfrak{B}$ we have

$$P_{V_n}(B) = \int P\big(V_n^{-1}[B] \mid \Theta\big)\, dP = \int P\big(V_1^{-1}[B] \mid \Theta\big)\, dP = P_{V_1}(B),$$

where the second equality follows form the fact that $\{V_n\}_{n\in\mathbb{N}}$ is P-conditionally identically distributed.

Definition 2.2. Let Q_1, Q_2 be two probability measures on a σ-subalgebra $\mathcal{F}$ of Σ. We say that Q_1 and Q_2 are equivalent on $\mathcal{F}$, in symbols $Q_1 \upharpoonright \mathcal{F} \sim Q_2 \upharpoonright \mathcal{F}$, if

$$\forall\, A \in \mathcal{F} \qquad \big(Q_1(A) = 0 \Leftrightarrow Q_2(A) = 0\big).$$

For $\mathcal{F} = \Sigma$ we simply write $Q_1 \sim Q_2$, and we say that Q_1 is absolutely continuous with respect to Q_2, if every Q_2-null set in Σ is a Q_1-null set.

3. Compound Mixed Renewal Processes and Progressively Equivalent Measures

We first recall some additional background material, needed in this section.

A family $N := \{N_t\}_{t\in\mathbb{R}_+}$ of $\overline{\mathbb{R}}$-valued random variables on (Ω, Σ, P) is called a **counting process** (or a **claim number process**) if there exists a P-null set $\Omega_N \in \Sigma$ such that the process N restricted on $\Omega \setminus \Omega_N$ takes values in $\mathbb{N}_0 \cup \{\infty\}$, has right-continuous paths, presents jumps of size (at most) one, vanishes at $t = 0$ and increases to infinity. Without loss of generality we may and do assume, that $\Omega_N = \emptyset$. Denote by $T := \{T_n\}_{n\in\mathbb{N}_0}$ and $W := \{W_n\}_{n\in\mathbb{N}}$ the **arrival process** and **interarrival process**, respectively (cf. e.g. Schmidt (1996) Section 1.1 page 6, for the definitions) associated with N (see also Schmidt (1996), Theorem 2.1.1).

Furthermore, let $X := \{X_n\}_{n\in\mathbb{N}}$ be a **claim size process** with all X_n non-negative random variables, and $S = \{S_t\}_{t\in\mathbb{R}_+}$ the **aggregate claims process** induced by the pair (N, X) (cf. e.g. Schmidt (1996) page 103 for the definitions). Recall that a pair (N, X) is called a **risk process**, if N is a counting process, X is P-i.i.d. and the processes N and X are mutually P-independent (see Schmidt (1996) Section 6.1).

Let $\mathbb{T} \subseteq \mathbb{R}_+$ with $0 \in \mathbb{T}$. For a process $Y_\mathbb{T} := \{Y_t\}_{t\in\mathbb{T}}$ denote by $\mathcal{F}^Y_\mathbb{T} := \{\mathcal{F}^Y_t\}_{t\in\mathbb{T}}$ the canonical filtration of $Y_\mathbb{T}$. For $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = \mathbb{N}$ we simply write $\mathcal{F}^Y$ instead of $\mathcal{F}^Y_{\mathbb{R}_+}$ or $\mathcal{F}^Y_\mathbb{N}$, respectively. Also, we write $\mathcal{F} := \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, where $\mathcal{F}_t := \sigma(\mathcal{F}^S_t \cup \sigma(\Theta))$, for the canonical filtration generated by S and Θ, $\mathcal{F}^S_\infty := \sigma(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}^S_t)$ and $\mathcal{F}_\infty := \sigma(\mathcal{F}^S_\infty \cup \sigma(\Theta))$.

Definition 3.1. Let Q_1, Q_2 be two probability measures on Σ and $Y_\mathbb{T}$ a process on (Ω, Σ). Then Q_1 and Q_2 are said to be **progressively equivalent** with respect to $\mathcal{F}^Y_\mathbb{T}$, if Q_1 and Q_2 are equivalent on $\mathcal{F}^Y_\mathbb{T}$ (i.e. $Q_1 \upharpoonright \mathcal{F}^Y_t \sim Q_2 \upharpoonright \mathcal{F}^Y_t$) for every $t \in \mathbb{T}$.

The following definition has been introduced in Lyberopoulos and Macheras (2022) Definition 3.1, see also Macheras and Tzaninis (2018) Definition 3.2(b).

Definition 3.2. A counting process N is said to be a P**-mixed renewal process with mixing parameter** Θ **and interarrival time conditional distribution** $\mathbf{K}(\Theta)$ (written P-MRP$(\mathbf{K}(\Theta))$ for short), if the induced interarrival process W is P-conditionally independent and

$$\forall\, n \in \mathbb{N} \qquad \Big(P_{W_n|\Theta} = \mathbf{K}(\Theta) \quad \mathbf{P} \restriction \sigma(\Theta)\text{-a.s.}\Big).$$

In particular, if the distribution of Θ is degenerate at some point $\theta_0 \in D$, then the counting process N becomes a P*-renewal process with interarrival time distribution* $\mathbf{K}(\theta_0)$ (written P-RP$\big(\mathbf{K}(\theta_0)\big)$ for short).

Accordingly, an aggregate claims process S induced by a P-risk process (N, X) such that N is a P-MRP$\big(\mathbf{K}(\Theta)\big)$ is called a **compound mixed renewal process with parameters** $\mathbf{K}(\Theta)$ **and** P_{X_1} (P-CMRP$\big(\mathbf{K}(\Theta), P_{X_1}\big)$ for short). In particular, if the distribution of Θ is degenerate at $\theta_0 \in D$ then S is called a **compound renewal process with parameters** $\mathbf{K}(\theta_0)$ **and** P_{X_1} (P-CRP$\big(\mathbf{K}(\theta_0), P_{X_1}\big)$ for short).

Throughout what follows we denote again by $\mathbf{K}(\Theta)$ *and* $\mathbf{K}(\theta)$ *the conditional distribution function and the distribution function induced by the conditional probability distribution* $\mathbf{K}(\Theta)$ *and the probability distribution* $\mathbf{K}(\theta)$*, respectively.*

The following conditions will be useful for our investigations:

(a1) The pair (W, X) is P-conditionally independent.
(a2) The random vector Θ and the process X are P-(unconditionally) independent.

Next, whenever condition (a1) or (a2) holds true we shall write that the quadruplet (P, W, X, Θ) *or (if no confusion arises) the probability measure* P *satisfies (a1) or (a2), respectively.*

Since conditioning is involved in the definition of (compound) mixed renewal processes, it seems natural to investigate the relationship between such processes and regular conditional probabilities. To this purpose, we recall their definition.

Definition 3.3. Let (Z, H, R) be an arbitrary probability probability space. A family $\{P_z\}_{z\in Z}$ of probability measures on Σ is called a **regular conditional probability** (rcp for short) of P over R if

(d1) for each $E \in \Sigma$ the map $z \mapsto P_z(E)$ is H-measurable;
(d2) $\int P_z(E)\, R(dz) = P(E)$ for each $E \in \Sigma$.

We could use the term of *disintegration* instead, but it is better to reserve that term to the general case when P_z's may be defined on different domains (see Pachl (1978/79)).

If $f : \Omega \to Z$ is an inverse-measure-preserving function (i.e. $P(f^{-1}(B)) = R(B)$ for each $B \in H$), a rcp $\{P_z\}_{z\in Z}$ of P over R is called **consistent** with f if, for each $B \in H$ the equality $P_z(f^{-1}(B)) = 1$ holds for R-almost every $z \in B$.

We say that a rcp $\{P_z\}_{z\in Z}$ of P over R consistent with f is **essentially unique**, if for any other rcp $\{\widetilde{P}_z\}_{z\in Z}$ of P over R consistent with f there exists a R-null set $L \in H$ such that for any $z \notin L$ the equality $P_z = \widetilde{P}_z$ holds true.

Remark. If Σ is countably generated (cf. e.g. Cohn (2013) Section 3.4, page 102 for the definition) and P is perfect (see Faden (1985) page 291 for the definition), then there always exists a rcp $\{P_z\}_{z\in Z}$ of P over R consistent with any inverse-measure-preserving map f from Ω into Υ providing that H is countably generated (see Faden (1985) Theorem 6). So, in most cases appearing in applications (e.g. Polish spaces) rcps as above always exist.

From now on $(Z, H, R) := (D, \mathfrak{B}(D), P_\Theta)$ *and the family* $\{P_\theta\}_{\theta\in D}$ *is a rcp of* P *over* P_Θ *consistent with* Θ.

The following characterization of compound mixed renewal processes in terms of rcps is of independent interest, as it allows us to convert a CMRP into a CRP via a suitable change of measures.

It also plays a fundamental role in the characterization of progressively equivalent measures that preserve the structure of a CMRP.

Proposition 3.4. *If P satisfies conditions (a1) and (a2), the following are equivalent:*

(i) S is a P-CMRP$\big(\mathbf{K}(\Theta), P_{X_1}\big)$;

(ii) there is a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that S is a P_θ-CRP$\big(\mathbf{K}(\theta), P_{X_1}\big)$ with $P_{X_1} = (P_\theta)_{X_1}$ for every $\theta \notin L_P$.

Proof: Assertion (i) is equivalent to the fact that (N, X) is a risk process and that N is a P-MRP$\big(\mathbf{K}(\Theta)\big)$. But according to Lyberopoulos and Macheras (2019) Lemma 2.3 and Lyberopoulos and Macheras (2022) Proposition 3.1, we equivalently obtain two P_Θ-null sets $L_{P,1}, L_{P,2} \in \mathfrak{B}(D)$ such that the pair (N, X) is a P_θ-risk process for any $\theta \notin L_{P,1}$, and for any $\theta \notin L_{P,2}$ the process N is a P_θ-RP$\big(\mathbf{K}(\theta)\big)$. Let us fix an arbitrary $A \in \mathcal{F}_1^X$. Since P satisfies condition (a2), applying Lyberopoulos and Macheras (2012) Lemma 3.5, we get for any $B \in \mathfrak{B}(D)$ that

$$\int_B P_\theta(A)\, P_\Theta(d\theta) = \int_B P(A)\, P_\Theta(d\theta),$$

or equivalently that there exists a P_Θ-null set $L_{P,3,A} \in \mathfrak{B}(D)$ such that for any $\theta \notin L_{P,3,A}$ condition $P_\theta(A) = P(A)$ holds. But since $\mathcal{F}_1^X$ is countably generated, by a monotone class argument, we equivalently get a P_Θ-null set $L_{P,3} \in \mathfrak{B}(D)$ such that for any $\theta \notin L_{P,3}$ condition $(P_\theta)_{X_1} = P_{X_1}$ holds. Thus, putting $L_P := L_{P,1} \cup L_{P,2} \cup L_{P,3}$, we equivalently get that S is a P_θ-CRP$\big(\mathbf{K}(\theta), P_{X_1}\big)$ for any $\theta \notin L_P$, completing in this way the proof. □

The first statement of the next corollary is Proposition 3.1 of Lyberopoulos and Macheras (2022).

Corollary 3.5. *If P satisfies conditions (a1) and (a2) the following hold true:*

(a) the counting process N is a P-MRP$\big(\mathbf{K}(\Theta)\big)$ if and only if for P_Θ-almost all (written a.a. for short) $\theta \in D$ it is a P_θ-RP$\big(\mathbf{K}(\theta)\big)$;

(b) if N is a P-MRP$(\mathbf{K}(\Theta))$ then the event of **explosion** *$E := \{\sup_{n\in\mathbb{N}_0} T_n < \infty\}$ is a P-null set.*

Proof: Ad (a): If we assume that the distribution P_{X_n} of X_n is the Bernoulli distribution $\mathbf{B}(\alpha)$, $\alpha \in (0,1)$, for all $n \in \mathbb{N}$ (cf. e.g. Schmidt (1996) page 176 for the definition of the Bernoulli distribution), then applying Schmidt (1996) Theorem 5.1.4, we get $S = N$, and so the statement (i) (resp. (ii)) of Proposition 3.4 holds, if and only if N is a P-MRP$\big(\mathbf{K}(\Theta)\big)$ resp. a P_θ-RP$\big(\mathbf{K}(\theta)\big)$ for P_Θ-a.a. $\theta \in D$. Thus, the statement (a) follows.

Ad (b): If N is a P-MRP$\big(\mathbf{K}(\Theta)\big)$, it then follows by (a) that N is a P_θ-RP$\big(\mathbf{K}(\theta)\big)$ for P_Θ-a.a. $\theta \in D$. But since any renewal process has finite expectation (cf. e.g. Serfozo (2009) Proposition 4 on page 101) and any counting process with finite expectation has zero probability of explosion (cf. e.g. Schmidt (1996) Corollary 2.1.5), we obtain that $P_\theta(E) = 0$ for P_Θ-a.a. $\theta \in D$; hence taking into account condition (d2) we obtain $P(E) = \int_D P_\theta(E)\, P_\Theta(d\theta) = 0$. □

Remark 3.6. Let S be the aggregate claims process induced by the counting process N and the claim size process X. Fix on arbitrary $u \in \Upsilon$ and $t \in \mathbb{R}_+$, and define the function $r_t^u : \Omega \times D \to \mathbb{R}$ by means of

$$r_t^u(\omega, \theta) := u + c(\theta) \cdot t - S_t(\omega) \quad \text{for any} \quad (\omega, \theta) \in \Omega \times D, \tag{3.1}$$

where c is a positive $\mathfrak{B}(D)$-measurable function. For arbitrary but fixed $\theta \in D$, the process $r^u(\theta) := \{r_t^u(\theta)\}_{t\in\mathbb{R}_+}$ defined by $r_t^u(\theta) := r_t^u(\omega, \theta)$ for any $\omega \in \Omega$, is called the **reserve process** induced by the **initial reserve** u, the **premium intensity** or **premium rate** $c(\theta)$ and the aggregate claims process S (see Schmidt (1996) Section 7.1 pages 155-156 for the definition). The function ψ_θ defined by $\psi_\theta(u) := P_\theta(\{\inf r_t^u(\theta) < 0\})$ is called the **probability of ruin** for the reserve process $r^u(\theta)$ with respect to P_θ (see Schmidt (1996) Section 7.1 page 158 for the definition).

Define the real-valued function $R_t^u(\Theta)$ on Ω by means of $R_t^u(\Theta) := r_t^u \circ (id_\Omega \times \Theta)$. The process $R^u(\Theta) := \{R_t^u(\Theta)\}_{t\in\mathbb{R}_+}$ is called the **reserve process** induced by the initial reserve u, the **stochastic premium intensity** or **stochastic premium rate** $c(\Theta)$ and the aggregate claims process S. The function ψ defined by $\psi(u) := P(\{\inf_{t\in\mathbb{R}_+} R_t^u(\Theta) < 0\})$ is called the **probability of ruin** for the reserve process $R^u(\Theta)$ with respect to P. Applying Lyberopoulos and Macheras (2012) Proposition 3.8 and standard computations we get that the ruin probability satisfies condition

$$\psi(u) = \int_D \psi_\theta(u) P_\Theta(d\theta) \quad \text{for any } u \in \Upsilon.$$

Example 3.7. Assume that P satisfies conditions (a1) and (a2), and let S be a P-CMRP$\big(\mathbf{K}(\Theta), P_{X_1}\big)$ with $P_{X_1} = \mathbf{Exp}(\eta)$ and $\eta \in \Upsilon$. Applying Proposition 3.4 we obtain a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that S is a P_θ-CRP$\big(\mathbf{K}(\theta), P_{X_1}\big)$ for any $\theta \notin L_P$; hence we may apply Rolski et al. (1999) Corollary 6.5.2, for $\delta = \eta$ and $\gamma = R(\theta)$ in order to obtain

$$\psi_\theta(u) = \left(1 - \frac{R(\theta)}{\eta}\right) \cdot e^{-R(\theta)\cdot u} \quad \text{for any } u \geq 0,$$

where $R(\theta)$ is the unique positive solution of

$$\mathbb{E}_P\big[e^{r\cdot X_1}\big] \cdot \mathbb{E}_{P_\theta}\big[e^{-c(\theta)\cdot r\cdot W_1}\big] = 1.$$

Consequently, according to Remark 3.6, the probability of ruin for a compound mixed renewal process with exponentially distributed claim sizes is given by

$$\psi(u) = \int_D \psi_\theta(u)\, P_\Theta(d\theta) = \int_D \left(1 - \frac{R(\theta)}{\eta}\right) \cdot e^{-R(\theta)\cdot u}\, P_\Theta(d\theta) \quad \text{for any } u \geq 0.$$

In order to prove the main result of this section we need the following auxiliary lemmas.

Lemma 3.8. *Let Q be a probability measure on Σ such that $Q_\Theta \sim P_\Theta$, and let $\{Q_\theta\}_{\theta\in D}$ be a rcp of Q over Q_Θ consistent with Θ. The following hold true:*

(i) if $Q_{W_1} \sim P_{W_1}$, then there exists a P_Θ-null set $M_{P,Q} \in \mathfrak{B}(D)$ such that $(Q_\theta)_{W_1} \sim (P_\theta)_{W_1}$ for any $\theta \notin M_{P,Q}$;

(ii) if W is P- and Q-conditionally i.i.d., then there exists a P_Θ-null set $M'_{P,Q} \in \mathfrak{B}(D)$, containing the P_Θ-null set $M_{P,Q}$, and for any $\theta \notin M'_{P,Q}$ there exists a $(P_\theta)_{W_1}$-a.s. positive Radon-Nikodým derivative r_θ of $(Q_\theta)_{W_1}$ with respect to $(P_\theta)_{W_1}$, satisfying for all $n \in \mathbb{N}_0$ condition

$$Q_\theta(E) = \mathbb{E}_{P_\theta}\Big[\mathbb{1}_E \cdot \Big(\prod_{j=1}^n r_\theta(W_j)\Big)\Big] \quad \textit{for any } E \in \mathcal{F}_n^W. \tag{3.2}$$

Proof: Ad (i): Fix on arbitrary $B \in \mathfrak{B}(\Upsilon)$, and consider a Q_Θ-null set $M_{Q,B} \in \mathfrak{B}(D)$ such that $(Q_\theta)_{W_1}(B) = 0$ for any $\theta \notin M_{Q,B}$. We then get

$$Q_{W_1}(B) = \int (Q_\theta)_{W_1}(B)\, Q_\Theta(d\theta) = \int_{\Omega\setminus M_{Q,B}} (Q_\theta)_{W_1}(B)\, Q_\Theta(d\theta),$$

implying $Q_{W_1}(B) = 0$, or equivalently $P_{W_1}(B) = 0$ by $Q_{W_1} \sim P_{W_1}$. It then follows by the property (d2) of $\{P_\theta\}_{\theta\in D}$ that there exists a P_Θ-null set $M_{P,B} \in \mathfrak{B}(D)$ such that $(P_\theta)_{W_1}(B) = 0$ for any $\theta \notin M_{P,B}$. Replacing Q with P, and assuming that there exists a P_Θ-null set $N_{P,B} \in \mathfrak{B}(D)$ such that $(P_\theta)_{W_1}(B) = 0$ for any $\theta \notin N_{P,B}$ we conclude that there exists a Q_Θ-null set $N_{Q,B} \in \mathfrak{B}(D)$ such that $(Q_\theta)_{W_1}(B) = 0$ for any $\theta \notin N_{Q,B}$. Put $M_{P,Q,B} := M_{Q,B} \cup M_{P,B} \cup N_{Q,B} \cup N_{P,B} \in \mathfrak{B}(D)$. Since $P_\Theta \sim Q_\Theta$, the set $M_{P,Q,B}$ is a P_Θ-null set. But since $\mathfrak{B}(\Upsilon)$ is countably generated, by a monotone class argument we can find a P-null set $M_{P,Q} := \bigcup_{B\in\mathcal{G}_{\mathfrak{B}(\Upsilon)}} M_{P,Q,B} \in \mathfrak{B}(D)$, where $\mathcal{G}_{\mathfrak{B}(\Upsilon)}$ is a countable generator of $\mathfrak{B}(\Upsilon)$ which is closed under finite intersections, such that $(Q_\theta)_{W_1} \sim (P_\theta)_{W_1}$ for any $\theta \notin M_{P,Q}$; hence (i) follows.

Ad (ii): Since W is P- and Q-conditionally i.i.d. and $Q_\Theta \sim P_\Theta$, it follows by Lyberopoulos and Macheras (2022) Lemma 3.4, that there exists a P_Θ-null set $M_1 := M_{P,Q,1} \in \mathfrak{B}(D)$ such that W is P_θ- and Q_θ-i.i.d. for any $\theta \notin M_1$. Put $M'_{P,Q} := M_1 \cup M_{P,Q}$ and fix on arbitrary $n \in \mathbb{N}$. Assertion (i) implies that for any $\theta \notin M'_{P,Q}$ there exists a $(P_\theta)_{W_1}$-a.s. positive Radon-Nikodým derivative r_θ of $(Q_\theta)_{W_1}$ with respect to $(P_\theta)_{W_1}$ such that

$$(Q_\theta)(W_n^{-1}[B]) = \mathbb{E}_{P_\theta}\big[\mathbb{1}_{W_n^{-1}[B]} \cdot r_\theta(W_n)\big] \quad \text{for any } B \in \mathfrak{B}(\Upsilon).$$

Putting $\widetilde{\mathcal{C}}_n^W := \big\{\bigcap_{j=1}^n C_j : C_j \in \sigma(W_j)\big\}$ we have that $\widetilde{\mathcal{C}}_n^W$ is a generator of $\mathcal{F}_n^W$, closed under finite intersections, and that any $C \in \widetilde{\mathcal{C}}_n^W$ satisfies condition (3.2) by the P_θ-independence of W for any $\theta \notin M'_{P,Q}$. If by $\widetilde{\mathcal{D}}_n^W$ is denoted the family of all $E \in \mathcal{F}_n^W$ satisfying condition (3.2), it can be easily shown that it is a Dynkin class containing $\widetilde{\mathcal{C}}_n^W$; hence by Dynkin Lemma we obtain $\widetilde{\mathcal{D}}_n^W = \mathcal{F}_n^W$, i.e. condition (3.2) holds. □

Notations 3.9. **(a)** Let h be a real-valued, one to one $\mathfrak{B}(\Upsilon)$-measurable function. The class of all real-valued $\mathfrak{B}(\Upsilon)$-measurable functions γ such that $\mathbb{E}_P\big[h^{-1} \circ \gamma \circ X_1\big] = 1$ will be denoted by $\mathcal{F}_{P,h} := \mathcal{F}_{P,X_1,h}$. The class of all real-valued $\mathfrak{B}(D)$-measurable functions ξ on D such that $P_\Theta(\{\xi > 0\}) = 1$ and $\mathbb{E}_P[\xi(\Theta)] = 1$ is denoted by $\mathcal{R}_+(D) := \mathcal{R}_+(D, \mathfrak{B}(D), P_\Theta)$.
(b) Denote by $\mathfrak{M}^k(D)$ ($k \in \mathbb{N}$) the class of all $\mathfrak{B}(D)$-$\mathfrak{B}(\mathbb{R}^k)$-measurable functions on D. In the special case $k = 1$ we write $\mathfrak{M}(D) := \mathfrak{M}^1(D)$. By $\mathfrak{M}_+(D)$ will be denoted the class of all positive $\mathfrak{B}(D)$-measurable functions on D. For each $\rho \in \mathfrak{M}^k(D)$ the class of all probability measures Q on Σ which satisfy conditions (a1) and (a2), are progressively equivalent to P, and such that S is a Q-CMRP$(\mathbf{\Lambda}(\rho(\Theta)), Q_{X_1})$, is denoted by $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))} := \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta)),P,X_1}$. In the special case $d = k$ and $\rho := id_D$ we write $\mathcal{M}_{S,\mathbf{\Lambda}(\Theta)} := \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ for simplicity.
(c) For given $\rho \in \mathfrak{M}^k(D)$ and $\theta \in D$, denote by $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ the class of all probability measures Q_θ on Σ, such that $Q_\theta \restriction \mathcal{F}_t \sim P_\theta \restriction \mathcal{F}_t$ for any $t \in \mathbb{R}_+$ and S is a Q_θ-CRP$(\mathbf{\Lambda}(\rho(\theta)), (Q_\theta)_{X_1})$.

From now on, unless stated otherwise, h is as in Notations 3.9(a), $\rho \in \mathfrak{M}^k(D)$ ($k \in \mathbb{N}$) and $P \in \mathcal{M}_{S,\mathbf{K}(\Theta)}$ is the initial probability measure under which the aggregate claims process S is a P-CMRP$(\mathbf{K}(\Theta), P_{X_1})$.

Recall that a **martingale in $\mathcal{L}^1(P)$ adapted to the filtration $\mathcal{Y}_\mathbb{T}$** or a **$\mathcal{Y}_\mathbb{T}$-martingale** in $\mathcal{L}^1(P)$ is a family $Y_\mathbb{T} := \{Y_t\}_{t\in\mathbb{T}}$ of real-valued random variables in $\mathcal{L}^1(P)$ such that each Y_t is $\mathcal{Y}_t$-measurable and whenever $s \le t$ in $\mathbb{T}$ condition $\int_A Y_s\, dP = \int_A Y_t\, dP$ holds true for all $A \in \mathcal{Y}_s$. A $\mathcal{Y}_\mathbb{T}$-martingale $\{Y_t\}_{t\in\mathbb{T}}$ in $\mathcal{L}^1(P)$ is **a.s. positive** if Y_t is P-a.s. positive for each $t \in \mathbb{T}$. For $\mathcal{Y}_{\mathbb{R}_+} = \mathcal{F}$ we simply say that $Y := Y_{\mathbb{R}_+}$ is a martingale in $\mathcal{L}^1(P)$.

Lemma 3.10. *Let Q be a probability measure on Σ satisfying conditions (a1), (a2) and such that S is a Q-CMRP$(\mathbf{\Lambda}(\rho(\Theta)), Q_{X_1})$. If $\{Q_\theta\}_{\theta\in D}$ is a rcp of Q over Q_Θ consistent with Θ, and $Q_{X_1} \sim P_{X_1}$, $Q_{W_1} \sim P_{W_1}$ and $Q_\Theta \sim P_\Theta$, then there exist a P_Θ-null set $L_* \in \mathfrak{B}(D)$, containing the P_Θ-null sets L_P and L_Q appearing in Proposition 3.4, and a P_{X_1}-a.s. unique function $\gamma \in \mathcal{F}_{P,h}$ such that for any $\theta \notin L_*$*

$$Q_\theta(A) = \mathbb{E}_{P_\theta}\big[\mathbb{1}_A \cdot \widetilde{M}_t^{(\gamma)}(\theta)\big] \quad \textit{for all } 0 \le s \le t \textit{ and } A \in \mathcal{F}_s^S, \tag{3.3}$$

with

$$\widetilde{M}_t^{(\gamma)}(\theta) := \left[\prod_{j=1}^{N_t} (h^{-1} \circ \gamma \circ X_j) \cdot \frac{d\mathbf{\Lambda}(\rho(\theta))}{d\mathbf{K}(\theta)}(W_j)\right] \cdot \frac{1 - \mathbf{\Lambda}(\rho(\theta))(J_t)}{1 - \mathbf{K}(\theta)(J_t)},$$

where $J_t := t - T_{N_t}$, and the family $\widetilde{M}^{(\gamma)}(\theta) := \big\{\widetilde{M}_t^{(\gamma)}(\theta)\big\}_{t\in\mathbb{R}_+}$ is a P_θ-a.s. positive $\mathcal{F}^S$-martingale in $\mathcal{L}^1(P_\theta)$, satisfying condition $\mathbb{E}_{P_\theta}\big[\widetilde{M}_t^{(\gamma)}(\theta)\big] = 1$ for any $t \in \mathbb{R}_+$

We postpone the proof of Lemma 3.10 to Appendix A.1.

For a given aggregate claims process S on (Ω, Σ), in order to characterize the progressively equivalent measures that preserve the structure of a compound mixed renewal process (see Theorem 4.5), one has to be able to characterize the Radon-Nikodým derivatives dQ/dP. The next result provides such a characterization as well as the one direction of our main result. Its long and technical proof is postponed to Appendix A.2.

Proposition 3.11. *Let Q be a probability measure on Σ satisfying conditions (a1), (a2) and such that S is a Q-CMRP$(\mathbf{\Lambda}(\rho(\Theta)), Q_{X_1})$. If $\{Q_\theta\}_{\theta\in D}$ is a rcp of Q over Q_Θ consistent with Θ, then the following are all equivalent:*

(i) $Q \upharpoonright \mathcal{F}_t \sim P \upharpoonright \mathcal{F}_t$ *for any* $t \in \mathbb{R}_+$*;*

(ii) $Q_{X_1} \sim P_{X_1}$, $Q_{W_1} \sim P_{W_1}$ *and* $Q_\Theta \sim P_\Theta$*;*

(iii) $Q_\Theta \sim P_\Theta$ *and there exist a P_Θ-null set $L_{**} \in \mathfrak{B}(D)$, containing the P_Θ-null sets L_P and L_Q appearing in Proposition 3.4, and a P_{X_1}-a.s. unique function $\gamma \in \mathcal{F}_{P,h}$ with $\gamma = h \circ f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, such that for any $\theta \notin L_{**}$*

$$Q_\theta(A) = \mathbb{E}_{P_\theta}\big[\mathbb{1}_A \cdot \widetilde{M}_t^{(\gamma)}(\theta)\big] \quad \textit{for all } 0 \le s \le t \textit{ and } A \in \mathcal{F}_s \qquad (RRM_\theta)$$

where the family $\widetilde{M}^{(\gamma)}(\theta) := \{\widetilde{M}_t^{(\gamma)}(\theta)\}_{t\in\mathbb{R}_+}$, involved in condition (3.3), is a P_θ-a.s. positive martingale in $\mathcal{L}^1(P_\theta)$ with $\mathbb{E}_{P_\theta}[\widetilde{M}_t^{(\gamma)}(\theta)] = 1$;

(iv) *there exist an essentially unique pair $(\gamma, \xi) \in \mathcal{F}_{P,h} \times \mathcal{R}_+(D)$, where ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ and $\gamma = h \circ f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, such that*

$$Q(A) = \mathbb{E}_P\big[\mathbb{1}_A \cdot M_t^{(\gamma)}(\Theta)\big] \quad \textit{for all } 0 \le s \le t \textit{ and } A \in \mathcal{F}_s, \qquad (RRM_\xi)$$

where

$$M_t^{(\gamma)}(\Theta) := \xi(\Theta) \cdot \widetilde{M}_t^{(\gamma)}(\Theta),$$

and the family $M^{(\gamma)}(\Theta) := \big\{M_t^{(\gamma)}(\Theta)\big\}_{t\in\mathbb{R}_+}$ is a P-a.s. positive martingale in $\mathcal{L}^1(P)$ satisfying condition $\mathbb{E}_P[M_t^{(\gamma)}(\Theta)] = 1$.

Due to Proposition 3.11, under the weak conditions $Q_{X_1} \sim P_{X_1}$, $Q_{W_1} \sim P_{W_1}$ and $Q_\Theta \sim P_\Theta$ the measures P and Q are equivalent on each σ-algebra $\mathcal{F}_t$. We will show here that this result does not, in general, hold true for the σ-algebra $\mathcal{F}_\infty$.

Proposition 3.12. *Let be given $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ and $\{Q_\theta\}_{\theta\in D}$ a rcp of Q over Q_Θ consistent with Θ. The following assertions hold true:*

(i) *if there exists a P_Θ-null set $\widehat{L}_1$ in $\mathfrak{B}(D)$ such that $P_\theta = Q_\theta$ for any $\theta \notin \widehat{L}_1$, then the measures P and Q are equivalent on $\mathcal{F}_\infty$;*

(ii) *if there exists a P_Θ-null set $\widehat{L}_2$ in $\mathfrak{B}(D)$ such that $P_\theta \neq Q_\theta$ for any $\theta \notin \widehat{L}_2$, then the measures P and Q are singular on $\mathcal{F}_\infty$, i.e. there exists a set $E \in \mathcal{F}_\infty$ such that $P(E) = 0$ if and only if $Q(E) = 1$.*

Proof: First note that since $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$, Proposition 3.11 implies $Q_\Theta \sim P_\Theta$.

Ad (i): Assume that $P_\theta = Q_\theta$ for any $\theta \notin \widehat{L}_1$, and consider the family

$$\mathcal{G}_\infty := \Big\{ \bigcap_{k\in\mathbb{N}_m} A_k : A_k \in \mathcal{F}_\infty^S \cup \sigma(\Theta),\, m \in \mathbb{N} \Big\}.$$

We then have $Q \upharpoonright \mathcal{G}_\infty \sim P \upharpoonright \mathcal{G}_\infty$.

In fact, let $A \in \mathcal{G}_\infty$ such that $Q(A) = 0$. There exists a number $m \in \mathbb{N}$ and a finite sequence $\{A_k\}_{k \in \mathbb{N}_m}$ in $\mathcal{F}_\infty^S \cup \sigma(\Theta)$, where $\mathbb{N}_m := \{1, 2, \ldots, m\}$, such that $A = \bigcap_{k \in \mathbb{N}_m} A_k$. Putting

$$I_\Theta := \{k \in \mathbb{N}_m : A_k \in \sigma(\Theta)\} \quad \text{and} \quad I_H := \{k \in \mathbb{N}_m : A_k \in \mathcal{F}_\infty^S \setminus \sigma(\Theta)\}$$

we get $I_\Theta \cup I_H = \mathbb{N}_m$, $\bigcap_{k \in I_\Theta} A_k \in \sigma(\Theta)$ and $C := \bigcap_{k \in I_H} A_k \in \mathcal{F}_\infty^S$. Since $\bigcap_{k \in I_\Theta} A_k \in \sigma(\Theta)$, there exists a set $B \in \mathfrak{B}(D)$ such that $\bigcap_{k \in I_\Theta} A_k = \Theta^{-1}[B]$, implying

$$0 = Q\big(C \cap \Theta^{-1}[B]\big) = \int_B Q_\theta(C)\, Q_\Theta(d\theta).$$

If $Q_\Theta(B) = 0$ then $P_\Theta(B) = 0$ by $Q_\Theta \sim P_\Theta$; hence $P(A) = 0$.

If $Q_\Theta(B) > 0$ then there exists a Q_Θ-null set $L_{Q,C} \in \mathfrak{B}(D)$ such that $Q_\theta(C) = 0$ for Q_Θ-a.a. $\theta \in B$; hence for Q_Θ-a.a. $\theta \in B \setminus \widehat{L}_1$, implying along with the assumption $Q_\Theta \sim P_\Theta$ that $P_\theta(A) = 0$ for P_Θ-a.a. $\theta \in B \setminus \widehat{L}_1$, and so $P(A) = 0$. In the same way, replacing Q with P, we get that $P(A) = 0$ implies $Q(A) = 0$ for any $A \in \mathcal{G}_\infty$. Considering now the family $\mathcal{D}_\infty$ of all $A \in \mathcal{F}_\infty$ such that $Q(A) = 0$ if and only if $P(A) = 0$, we have that $\mathcal{D}_\infty$ is a Dynkin class containing $\mathcal{G}_\infty$. Thus, we may apply the Dynkin Lemma in order to conclude the validity of (i).

Ad (ii): By Proposition 3.4 along with Proposition 3.11(i)⇒(ii), there exists a P_Θ- and Q_Θ-null set $L_{P,Q} := L_P \cup L_Q$ such that the aggregate process S is a P_θ-CRP$(\mathbf{K}(\theta), P_{X_1})$ with $P_{X_1} = (P_\theta)_{X_1}$ and a Q_θ-CRP$\big(\mathbf{\Lambda}(\rho(\theta)), Q_{X_1}\big)$ with $Q_{X_1} = (Q_\theta)_{X_1}$ for any $\theta \notin L_{P,Q}$. Thus, applying Macheras and Tzaninis (2020) Remark 3.1, we get that for any $\theta \notin \widehat{L}_2 \cup L_{P,Q}$ the measures P_θ and Q_θ are singular on $\mathcal{F}_\infty^S$; hence on $\mathcal{F}_\infty$. Consequently, P and Q are singular on $\mathcal{F}_\infty$. □

Proposition 3.11 allows us to explicitly calculate Radon-Nikodým derivatives for various cases appearing in applications. In the next example we consider the mixed Poisson process (cf. e.g. Schmidt (1996) page 87 for its definition). A common choice for the distribution of Θ in Risk Theory is the gamma distribution. In the case of a mixed Poisson process that process is called Pólya-Lundberg process (cf. e.g. Schmidt (1996) page 100 for its definition and basic properties). In order to present our first example recall the inverted gamma distribution with parameters $a, b \in \Upsilon$ (written $\mathbf{IG}(b, a)$ for short), i.e.

$$\mathbf{IG}(b,a)(B) := \int_B \frac{b^a}{\Gamma(a)} \cdot x^{-(a-1)} \cdot e^{-b/x}\, \lambda(dx) \quad \text{for any } B \in \mathfrak{B}(\Upsilon).$$

Example 3.13. Take $D := \Upsilon$, $\rho \in \mathfrak{M}_+(\Upsilon)$ defined by means of $\rho(x) := 1/x$, $h := \ln$, $P \in \mathcal{M}_{S,\mathbf{Exp}(\Theta)}$ and $Q \in \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$, such that $P_\Theta = \mathbf{Ga}(b_1, a_1)$ and $Q_\Theta = \mathbf{IGa}(b_2, a_2)$ with $a_1, a_2, b_1, b_2 > 0$. Since the probability mesaures P and Q are progressively equivalent, we may apply Proposition 3.11 in order to conclude that there exists an essentially unique pair $(\gamma, \xi) \in \mathcal{F}_{P,h} \times \mathcal{R}_+(\Upsilon)$ such that

$$Q(A) = \int_A M_t^{(\gamma)}(\Theta)\, dP \quad \text{for all } 0 \le u \le t \text{ and } A \in \mathcal{F}_u^S,$$

where $M_t^{(\gamma)}(\Theta) := \frac{b_2^{a_2}}{b_1^{a_1}} \cdot \frac{\Gamma(a_1)}{\Gamma(a_2)} \cdot \frac{e^{b_1\Theta - b_2/\Theta}}{\Theta^{a_1+a_2}} \cdot e^{\sum_{j=1}^{N_t} \gamma(X_j)} \cdot \left(\frac{\rho(\Theta)}{\Theta}\right)^{N_t} \cdot e^{-t(\rho(\Theta)-\Theta)}$.

4. The Characterization

We first prove the following result concerning the construction of compound mixed renewal processes. Such a construction serves as a preparatory tool for the main result of this work (i.e. Theorem 4.5). To this purpose we recall the following notations concerning product probability spaces.

By $(\Omega \times \Xi, \Sigma \otimes H, P \otimes R)$ we denote the product probability space of the probability spaces (Ω, Σ, P) and (Ξ, H, R). If I is an arbitrary non-empty index set, we write P_I for the product measure on Ω^I and Σ_I for its domain.

Throughout what follows, we put $\Omega := \Upsilon^{\mathbb{N}} \times \Upsilon^{\mathbb{N}} \times D$ and $\Sigma := \mathfrak{B}(\Omega) = \mathfrak{B}(\Upsilon)_{\mathbb{N}} \otimes \mathfrak{B}(\Upsilon)_{\mathbb{N}} \otimes \mathfrak{B}(D)$ for simplicity.

Proposition 4.1 below, the proof of which is postponed to Appendix A.3, enables us to construct *canonical* probability spaces admitting compound mixed renewal processes.

Proposition 4.1. *Let μ be a probability measure on $\mathfrak{B}(D)$, and for any $n \in \mathbb{N}$ and fixed $\theta \in D$ let $P_n(\theta) := \mathbf{K}(\theta)$ and $R_n := R$ be probability measures on $\mathfrak{B}(\Upsilon)$. Assume that for any fixed $B \in \mathfrak{B}(\Upsilon)$ the function $\theta \mapsto \mathbf{K}(\theta)(B)$ is $\mathfrak{B}(D)$-measurable. Then there exist:*

- *(i) a family $\{P_\theta\}_{\theta\in D}$ of probability measures $P_\theta := \big(\mathbf{K}(\theta)\big)_{\mathbb{N}} \otimes R_{\mathbb{N}} \otimes \delta_\theta$ on Σ, where δ_θ is the Dirac measure on $\mathfrak{B}(D)$ concentrated on θ, and a probability measure P on Σ such that $\{P_\theta\}_{\theta\in D}$ is a rcp of P over μ consistent with $\Theta := \pi_D$, where π_D is the canonical projection from Ω onto D, and $P_\Theta = \mu$;*
- *(ii) a counting process N being a P-MRP$\big(\mathbf{K}(\Theta)\big)$, the interarrival process W of which satisfies condition $(P_\theta)_{W_n} = \mathbf{K}(\theta)$ for all $n \in \mathbb{N}$, a claim size process X satisfying condition $P_{X_n} = R$ for all $n \in \mathbb{N}$, such that the quadruplet (P, W, X, Θ) satisfies conditions (a1) and (a2), and an aggregate claims process S being a P-CMRP$\big(\mathbf{K}(\Theta), P_{X_1}\big)$.*

Remark 4.2. Due to Macheras and Tzaninis (2020) Lemma 3.1, we get $\mathcal{F}^S_\infty = \mathcal{F}^{(W,X)}_\infty$, implying together with Proposition 4.1 that $\Sigma = \mathcal{F}^{(W,X,\Theta)}_\infty = \mathcal{F}_\infty$.

Notation 4.3. For $\theta \in D$ let $\mathbf{K}(\theta)$ and $\mathbf{\Lambda}(\rho(\theta))$ be probability distributions on $\mathfrak{B}(\Upsilon)$. For any $n \in \mathbb{N}_0$ the class of all likelihood ratios $g_n := g_{\rho,n} : \Upsilon^{n+1} \times D \to \Upsilon$ defined by means of

$$g_n(w_1, \ldots, w_n, t, \theta) := \left[\prod_{j=1}^{n} \frac{d\mathbf{\Lambda}(\rho(\theta))}{d\mathbf{K}(\theta)}(w_j)\right] \cdot \frac{1 - \mathbf{\Lambda}(\rho(\theta))(t-w)}{1 - \mathbf{K}(\theta)(t-w)}$$

for any $(w_1, \ldots, w_n, t, \theta) \in \Upsilon^{n+1} \times D$, where $w := \sum_{j=1}^{n} w_j$, will be denoted by $\mathcal{G}_{n,\rho}$. Notation $\mathcal{G}_\rho$ stands for the set $\big\{g = \{g_n\}_{n\in\mathbb{N}_0} : g_n \in \mathcal{G}_{n,\rho}$ for any $n \in \mathbb{N}_0\big\}$ of all sequences of elements of $\mathcal{G}_{n,\rho}$.

Throughout what follows $\mathbf{K}(\theta)$, $\mathbf{\Lambda}(\rho(\theta))$ and $g \in \mathcal{G}_\rho$ are as in Notation 4.3, and P, Θ, $\{P_\theta\}_{\theta\in D}$ and S are as in Proposition 4.1.

Proposition 4.4. *Let $(\gamma, \xi) \in \mathcal{F}_{P,h} \times \mathcal{R}_+(D)$. Then for every $0 \le u \le t$ and for all $A \in \mathcal{F}_u$ condition*

$$Q(A) = \mathbb{E}_P\left[\mathbb{1}_A \cdot \xi(\Theta) \cdot \left[\prod_{j=1}^{N_t} (h^{-1} \circ \gamma \circ X_j)\right] \cdot g(W_1, \ldots, W_{N_t}, t, \Theta)\right]$$

determines a unique probability measure $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ such that ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ and $\gamma = h \circ f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}.

Proof: Let $(\gamma, \xi) \in \mathcal{F}_{P,h} \times \mathcal{R}_+(D)$ and fix on arbitrary $t \ge 0$ and $n \in \mathbb{N}$. For any $\theta \in D$ define the set-functions $\check{\mu} : \mathfrak{B}(D) \to \mathbb{R}$ and $\check{Q}_n(\theta) : \mathfrak{B}(\Upsilon) \to \mathbb{R}$, by means of

$$\check{\mu}(F) := \mathbb{E}_P\big[\mathbb{1}_{\Theta^{-1}[F]} \cdot \xi(\Theta)\big] \quad \text{for any } F \in \mathfrak{B}(D)$$

and

$$\check{Q}_n(\theta)(B_1) := \mathbb{E}_{P_\theta}\left[\mathbb{1}_{W_1^{-1}[B_1]} \cdot \left(\frac{d\mathbf{\Lambda}(\rho(\theta))}{d\mathbf{K}(\theta)} \circ W_1\right)\right] \quad \text{for any } B_1 \in \mathfrak{B}(\Upsilon),$$

respectively. Also, consider the set-function $\check{R} : \mathfrak{B}(\Upsilon) \to \mathbb{R}$, defined by means of

$$\check{R}(B_2) := \mathbb{E}_P\big[\mathbb{1}_{X_1^{-1}[B_2]} \cdot (h^{-1} \circ \gamma \circ X_1)\big] \quad \text{for any } B_2 \in \mathfrak{B}(\Upsilon).$$

Clearly $\check{\mu}$ and $\check{Q}_n(\theta)$ are probability measures on $\mathfrak{B}(D)$ and $\mathfrak{B}(\Upsilon)$, respectively, while $\check{R}$ is a probability measure by Macheras and Tzaninis (2020) Lemma 2.2(a). To show that $\check{Q}_n(\theta)$ satisfies condition

$$\check{Q}_n(\theta)(B) = \mathbf{\Lambda}(\rho(\theta))(B) \tag{4.1}$$

for any $B \in \mathfrak{B}(\Upsilon)$, put $\mathcal{C}^W := \{(0, w] : w \in \Upsilon\}$. Clearly, $\mathcal{C}^W$ is a generator of $\mathfrak{B}(\Upsilon)$, which is closed under finite intersections and satisfies condition (4.1). Denoting by $\mathcal{D}^W$ the family of all elements of $\mathfrak{B}(\Upsilon)$ satisfying condition (4.1), it can be easily shown that it is a Dynkin class containing $\mathcal{C}^W$; hence $\mathcal{D}^W = \mathfrak{B}(\Upsilon)$ by Dynkin Lemma, i.e. condition (4.1) holds for any $B \in \mathfrak{B}(\Upsilon)$.

Thus, applying Proposition 4.1 for $\check{\mu}$, $\check{Q}_n(\theta)$ and $\check{R}$ in the place of μ, $P_n(\theta)$ and R, respectively, we can construct a family $\{\check{Q}_\theta\}_{\theta \in D}$ of probability measures on Σ defined by means of $\check{Q}_\theta := \big(\mathbf{\Lambda}(\rho(\theta))\big)_{\mathbb{N}} \otimes \check{R}_{\mathbb{N}} \otimes \delta_\theta$, a probability measure $\check{Q}$ on Σ satisfying conditions (a1) and (a2) and such that $\{\check{Q}_\theta\}_{\theta \in D}$ is a rcp of $\check{Q}$ over $\check{Q}_\Theta = \check{\mu}$ consistent with Θ and S is a $\check{Q}$-CMRP$\big(\mathbf{\Lambda}(\rho(\Theta)), \check{Q}_{X_1}\big)$ with $\check{Q}_{X_1} = \check{R}$. The latter, together with the definitions of $\check{\mu}$, $\check{R}$ and $\check{Q}_n(\theta)$, implies that $\check{Q}_\Theta \sim P_\Theta$, $\check{Q}_{X_1} \sim P_{X_1}$ and $(\check{Q}_\theta)_{W_1} \sim (P_\theta)_{W_1}$ for any $\theta \in D$. But since $(\check{Q}_\theta)_{W_1} \sim (P_\theta)_{W_1}$ for any $\theta \in D$ and $\{P_\theta\}_{\theta \in D}$ and $\{\check{Q}_\theta\}_{\theta \in D}$ are rcps of P over P_Θ and of $\check{Q}$ over $\check{Q}_\Theta$, respectively, consistent with Θ, it follows easily that $\check{Q}_{W_1} \sim P_{W_1}$. Applying now Proposition 3.11, we get $\check{Q} \restriction \mathcal{F}_t \sim P \restriction \mathcal{F}_t$, implying that $\check{Q} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$, or equivalently

$$\check{Q}(A) = \int_A \xi(\Theta) \cdot \prod_{j=1}^{N_t} (h^{-1} \circ \gamma \circ X_j) \cdot g(W_1, \ldots, W_{N_t}, t, \Theta)\, dP$$

for all $0 \le u \le t$ and $A \in \mathcal{F}_u$. Thus $Q \restriction \mathcal{F}_u = \check{Q} \restriction \mathcal{F}_u$ for all $u \in \mathbb{R}_+$; hence $Q \restriction \check{\Sigma} = \check{Q} \restriction \check{\Sigma}$, where $\check{\Sigma} := \bigcup_{u \in \mathbb{R}_+} \mathcal{F}_u$, implying that Q is σ-additive on $\check{\Sigma}$ and that $\check{Q}$ is the unique extension of Q on $\Sigma = \sigma(\check{\Sigma})$, completing in this way the proof. $\square$

The following result is the desired characterization of progressively equivalent measures that preserve the structure of a compound mixed renewal process.

Theorem 4.5. *The following hold true:*

(i) for any $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ there exist an essentially unique pair $(\gamma, \xi) \in \mathcal{F}_{P,h} \times \mathcal{R}_+(D)$, where ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ and $\gamma = h \circ f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, satisfying condition (RRM_ξ);

(ii) conversely, for any pair $(\gamma, \xi) \in \mathcal{F}_{P,h} \times \mathcal{R}_+(D)$ there exists a unique probability measure $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ determined by condition (RRM_ξ), such that ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ and $\gamma = h \circ f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1};

(iii) in both cases (i) and (ii), there exists an essentially unique rcp $\{Q_\theta\}_{\theta \in D}$ of Q over Q_Θ consistent with Θ satisfying conditions $Q_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ and (RRM_θ) for any $\theta \notin L_{**}$, where L_{**} is the P_Θ-null set in $\mathfrak{B}(D)$ appearing in Proposition 3.11.

Proof: Assertions (i) and (ii) follow by Propositions 3.11 and 4.4, respectively.

Ad (iii): Since Ω is a Polish space, according to the Remark following Definition 3.3, there always exists a rcp $\{Q_\theta\}_{\theta \in D}$ of Q over Q_Θ consistent with Θ; hence assertion (iii) is an immediate consequence of Proposition 3.11. $\square$

For applications of Theorem 4.5 to a characterization of equivalent martingale measures for CMRPs, and to the pricing of actuarial risks (premium calculation principles) in an insurance market possessing the property of no free lunch with vanishing risk we refer to Tzaninis and Macheras (2020).

The following corollary shows that Theorem 3.1 of Macheras and Tzaninis (2020) is a consequence of Theorem 4.5.

Corollary 4.6. *Let $\theta \in D$ and $\widetilde{\theta} := \rho(\theta) \in \mathbb{R}^k$ $(k \in \mathbb{N})$. The following statements hold true:*

(i) for every $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\widetilde{\theta})}$ there exists a P_{X_1}-a.s. unique function $\gamma \in \mathcal{F}_{P,h}$ satisfying condition (RRM_θ);

(ii) conversely, for every function $\gamma \in \mathcal{F}_{P,h}$ there exists a unique probability measure $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\widetilde{\theta})}$ satisfying condition (RRM_θ).

Proof: Fix $\theta \in D$ and assume that μ is a probability measure on $\mathfrak{B}(D)$ such that $\mu := \delta_\theta$, where δ_θ is the Dirac measure on $\mathfrak{B}(D)$ concentrated at θ. Then according to Proposition 4.1, we get $P_\Theta(\{\theta\}) = 1$; hence without loss of generality we may and do assume that $\Theta(\omega) = \theta$ for all $\omega \in \Omega$, implying that $\sigma(\Theta) = \{\emptyset, \Omega\}$ and so $\mathcal{F}_t = \mathcal{F}_t^S$ for any $t \in \mathbb{R}_+$ and $\mathcal{F}_\infty = \mathcal{F}_\infty^S$. Thus, S is reduced to a P-CRP$\big(\mathbf{K}(\theta), P_{X_1}\big)$. If $Q_\Theta \sim P_\Theta$, then the Radon-Nikodým derivative ξ of Q_Θ with respect to P_Θ is equal to 1; hence $Q_\Theta = P_\Theta$. In this case condition (RRM_ξ) is reduced to (RRM_θ). Thus, applying Theorem 4.5 for degenerate P_Θ we obtain Theorem 3.1 of Macheras and Tzaninis (2020). $\square$

Notation 4.7. Denote by $\mathcal{F}_{P,\Theta} := \mathcal{F}_{P,\Theta,X_1}$ the class of all real-valued $\mathfrak{B}(\Upsilon \times D)$-measurable functions β on $\Upsilon \times D$ defined by means of $\beta(x,\theta) := \gamma(x) + \alpha(\theta)$ for any $x \in \Upsilon$ and $\theta \in D$, where $\gamma \in \mathcal{F}_{P,\ln}$ and $\alpha \in \mathfrak{M}(D)$. Moreover, put $S_t^{(\gamma)} := \sum_{j=1}^{N_t} \gamma(X_j)$ for any $\gamma \in \mathcal{F}_{P,\ln}$.

The next corollary shows how to convert a CMRP into a compound mixed Poisson one by choosing the "correct" Radon-Nikodým derivative.

Corollary 4.8. *If $P\big(\{\mathbb{E}_P[W_1|\Theta] \in \Upsilon\}\big) = 1$ the following hold true:*

(i) for any pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ there exists an essentially unique pair $(\beta, \xi) \in \mathcal{F}_{P,\Theta} \times \mathcal{R}_+(D)$, where ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ, such that

$$\gamma = \ln f \quad \text{and} \quad \alpha(\Theta) = \ln \rho(\Theta) + \ln \mathbb{E}_P[W_1 \mid \Theta] \quad P \restriction \sigma(\Theta)\text{-a.s.}, \tag{$*$}$$

where f is a P_{X_1}-a.s. positive Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, and

$$Q(A) = \mathbb{E}_P\Big[\mathbb{1}_A \cdot M_t^{(\beta)}(\Theta)\Big] \quad \text{for all } 0 \le s \le t \text{ and } A \in \mathcal{F}_u, \tag{RPM_ξ}$$

with

$$M_t^{(\beta)}(\Theta) := \xi(\Theta) \cdot \frac{e^{S_t^{(\gamma)} - \rho(\Theta)\cdot J_t}}{1 - \mathbf{K}(\Theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\Theta))}{d\mathbf{K}(\Theta)}(W_j),$$

where $J_t := t - T_{N_t}$;

(ii) conversely, for any pair function $(\beta, \xi) \in \mathcal{F}_{P,\Theta} \times \mathcal{R}_+(D)$ there exist a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by (RPM_ξ) and satisfying condition ($*$) and such that ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ;

(iii) in both cases (i) and (ii), there exists an essentially unique rcp $\{Q_\theta\}_{\theta \in D}$ of Q over Q_Θ consistent with Θ and a P_Θ-null set $\widetilde{L}_{**} \in \mathfrak{B}(D)$, containing the P_Θ-null set L_{**} appearing in Proposition 3.11, satisfying for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_\theta \in \mathcal{M}_{S,\mathbf{Exp}(\rho(\theta))}$,

$$\gamma = \ln f \quad \text{and} \quad \rho(\theta) = \frac{e^{\alpha(\theta)}}{\mathbb{E}_{P_\theta}[W_1]}, \tag{$\widetilde{*}$}$$

where f is a P_{X_1}-a.s. positive Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, and

$$Q_\theta(A) = \mathbb{E}_{P_\theta}\Big[\mathbb{1}_A \cdot \widetilde{M}_t^{(\beta)}(\theta)\Big] \quad \text{for all } 0 \le s \le t \text{ and } A \in \mathcal{F}_s, \tag{RPM_θ}$$

where

$$\widetilde{M}_t^{(\beta)}(\theta) := \frac{e^{S_t^{(\gamma)} - \rho(\theta)\cdot J_t}}{1 - \mathbf{K}(\theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\theta))}{d\mathbf{K}(\theta)}(W_j).$$

Proof: Fix on arbitrary $t \in \mathbb{R}_+$.

Ad (i): If (ρ, Q) is an element of $\mathfrak{M}_+(D) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$, then according to Theorem 4.5(i), there exist an essentially unique pair $(\gamma, \xi) \in \mathcal{F}_{P,\ln} \times \mathcal{R}_+(D)$, with $\gamma = \ln f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, and ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ, such that

$$Q(A) = \int_A \xi(\Theta) \cdot \frac{e^{\sum_{j=1}^{N_t} \gamma(X_j) - \rho(\Theta) \cdot J_t}}{1 - \mathbf{K}(\Theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\Theta))}{d\mathbf{K}(\Theta)}(W_j)\, dP$$

for all $0 \leq s \leq t$ and $A \in \mathcal{F}_s$. Putting $\alpha(\Theta) = \ln \rho(\Theta) + \ln \mathbb{E}_P[W_1 \mid \Theta]$ $P \upharpoonright \sigma(\Theta)$-a.s. and $\beta = \gamma + \alpha$, we get that $\beta \in \mathcal{F}_{P,\Theta}$ and condition $(*)$ is valid.

Ad (ii): Consider the pair $(\beta, \xi) \in \mathcal{F}_{P,\Theta} \times \mathcal{R}_+(D)$ and define $\rho \in \mathfrak{M}_+(D)$ by means of $\rho(\Theta) := e^{\alpha(\Theta)}/\mathbb{E}_P[W_1 \mid \Theta]$ $P \upharpoonright \sigma(\Theta)$-a.s.. Then, condition $(*)$ is satisfied, while applying Theorem 4.5(ii) for $\gamma = \beta - \alpha$ we get a unique probability measure $Q \in \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ satisfying condition (RRM_ξ) or equivalently condition (RPM_ξ).

Ad (iii): Since Ω is a Polish space, according to the Remark following Definition 3.3, there always exists a rcp $\{Q_\theta\}_{\theta \in D}$ of Q over Q_Θ consistent with Θ. By Lyberopoulos and Macheras (2012) Lemma 3.5, there exists a P_Θ-null set $U_1 \in \mathfrak{B}(D)$ such that the second equality of condition $(*)$ is equivalent to $\alpha(\theta) = \ln \rho(\theta) + \ln \mathbb{E}_{P_\theta}[W_1]$ for any $\theta \notin U_1$. The latter along with Proposition 3.11, implies that there exists a P_Θ-null set $\widetilde{L}_{**} := L_{**} \cup U_1 \in \mathfrak{B}(D)$ such that for any $\theta \notin \widetilde{L}_{**}$ conditions $(\widetilde{*})$ and (RPM_θ) hold true. Clearly, condition (RPM_θ) implies that $Q_\theta \in \mathcal{M}_{S,\mathbf{Exp}(\rho(\theta))}$ for any $\theta \notin \widetilde{L}_{**}$. $\square$

Remarks 4.9. **(a)** For the special case $P \in \mathcal{M}_{S,\mathbf{Exp}(\Theta)}$, Corollary 4.8 yields the main result of Lyberopoulos and Macheras (2019) Theorem 4.3.

(b) Note that the family of probability measure Q such that $\mathbb{E}_Q[X_1] < \infty$ and S is converted into a compound mixed Poisson process under Q is nothing else than the family of all mixed premium calculation principles appearing in Lyberopoulos and Macheras (2019) page 24 and Tzaninis and Macheras (2020) page 17.

(c) Fix on $\ell \in \{1, 2\}$. For given $\rho \in \mathfrak{M}^k(D)$, define the classes

$$\mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))} := \left\{ Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))} : \mathbb{E}_Q\left[X_1^\ell\right] < \infty \right\}$$

and

$$\mathcal{F}^\ell_{P,h} := \left\{ \gamma \in \mathcal{F}_{P,h} : \mathbb{E}_P\left[X_1^\ell \cdot (h^{-1} \circ \gamma \circ X_1)\right] < \infty \right\}.$$

It can be easily seen that Theorem 4.5 remains true, if we replace the classes $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ and $\mathcal{F}_{P,h}$ by their subclasses $\mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))}$ and $\mathcal{F}^\ell_{P,h}$, respectively. Consequently, Corollary 4.8 remains true if we replace $\mathcal{F}_{P,\Theta}$ and $\mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ by their subclasses

$$\mathcal{F}^\ell_{P,\Theta} := \left\{ \beta = \gamma + \alpha : \gamma \in \mathcal{F}^\ell_{P,\ln} \text{ and } \alpha \in \mathfrak{M}(D) \right\}$$

and $\mathcal{M}^\ell_{S,\mathbf{Exp}(\rho(\Theta))}$, respectively.

(d) Assuming in Proposition 4.1 that $\int_\Upsilon x^\ell \, R(dx) < \infty$ for $\ell \in \{1, 2\}$, we get $\mathbb{E}_P\left[X_1^\ell\right] < \infty$, implying that $P \in \mathcal{M}^\ell_{S,\mathbf{K}(\Theta)}$.

(e) Note that Lemma 3.5 of Lyberopoulos and Macheras (2012) remains true without the assumption $g \in \mathcal{L}^1(P)$ but only with the assumption that the integral $\int g \, dP$ is defined in $\mathbb{R} \cup \{-\infty, +\infty\}$.

In the next examples, applying Corollary 4.8, we show how starting from a given pair $(\beta, \xi) \in \mathcal{F}_{P,\Theta} \times \mathcal{R}_+(D)$ we can construct a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$, converting an arbitrary compound mixed renewal process S into a compound mixed Poisson one.

Throughout what follows we assume that $P\big(\{\mathbb{E}_P[W_1|\Theta] \in \Upsilon\}\big) = 1$ and that P_Θ is absolutely continuous with respect to the Lebesgue measure $\lambda \upharpoonright \mathfrak{B}([0,1]^d)$, and we denote by g the probability density function of Θ.

In our first example we show how to find a probability measure Q such that S is converted into a compound Pólya-Lundberg process under Q.

Example 4.10. Take $D := \Upsilon$ and define the function $\xi \in \mathfrak{M}_+(\Upsilon)$ by means of

$$\xi(\theta) := \frac{b^a \cdot \theta^{a-1} \cdot e^{-b\cdot\theta}}{\Gamma(a) \cdot g(\theta)} \quad \text{for any } \theta \in \Upsilon,$$

where $a, b \in \Upsilon$ are constants. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying that $\xi \in \mathcal{R}_+(\Upsilon)$. Consider the function $\beta(x,\theta) := \gamma(x) + \ln\big(\theta \cdot \mathbb{E}_{P_\theta}[W_1]\big)$ for any $(x,\theta) \in \Upsilon^2$, with $\gamma \in \mathcal{F}_{P,\ln}$; hence $\beta \in \mathcal{F}_{P,\Theta}$. According to Corollary 4.8(ii) there exists a unique pair $(\rho, Q) \in \mathfrak{M}_+(\Upsilon) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ satisfying conditions $(*)$ and (RPM_ξ), and such that ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ. In particular, it follows from condition $(*)$ that $\rho = id_\Upsilon$ P_Θ-a.s. and

$$Q_\Theta(B) = \mathbb{E}_P\big[\mathbb{1}_{\Theta^{-1}(B)} \cdot \xi(\Theta)\big] = \int_B \frac{b^a \cdot \theta^{a-1}}{\Gamma(a)} \cdot e^{-b\cdot\theta}\, \lambda(d\theta) \quad \text{for any } B \in \mathfrak{B}(\Upsilon),$$

i.e. $Q_\Theta = \mathbf{Ga}(b,a)$, implying that S is a compound Pólya-Lundberg process under Q.

The following example shows how to choose a probability measure Q under which S becomes a compound Poisson-Lognormal process. Recall the Lognormal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \in \Upsilon$ (written $\mathbf{LN}\big(\mu,\sigma^2\big)$ for short) i.e.

$$\mathbf{LN}(\mu,\sigma^2)(B) := \int_B \frac{1}{\sqrt{2\pi} \cdot \sigma \cdot x}\, e^{-\frac{(\ln x - \mu)^2}{2\cdot\sigma^2}}\, \lambda(dx) \quad \text{for any } B \in \mathfrak{B}(\Upsilon).$$

Example 4.11. Take $D := \mathbb{R}$ and define the function $\xi \in \mathfrak{M}_+(\mathbb{R})$ by means of

$$\xi(\theta) := \frac{1}{\sqrt{2\pi} \cdot \sigma \cdot g(\theta)} \cdot e^{-\frac{1}{2\cdot\sigma^2}\cdot(\theta-\mu)^2} \quad \text{for any } \theta \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma \in \Upsilon$ are constants. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying that $\xi \in \mathcal{R}_+(\mathbb{R})$. Consider the function $\beta(x,\theta) := \gamma(x) + \ln\big(e^\theta \cdot \mathbb{E}_{P_\theta}[W_1]\big)$ for any $(x,\theta) \in \Upsilon \times \mathbb{R}$, with $\gamma \in \mathcal{F}_{P,\ln}$; hence $\beta \in \mathcal{F}_{P,\Theta}$. By Corollary 4.8(ii) there exists a unique pair $(\rho, Q) \in \mathfrak{M}_+(\Upsilon) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ satisfying conditions $(*)$ and (RPM_ξ), and such that ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ. In particular, it follows from condition $(*)$ that $\rho(\theta) = e^\theta$ for P_Θ-a.a $\theta \in \mathbb{R}$, and that

$$Q_\Theta(B) = \mathbb{E}_P\big[\mathbb{1}_{\Theta^{-1}(B)} \cdot \xi(\Theta)\big] = \int_B \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2\cdot\sigma^2}\cdot(\theta-\mu)^2}\, \lambda(d\theta) \quad \text{for any } B \in \mathfrak{B},$$

i.e. $Q_\Theta = \mathbf{N}\big(\mu,\sigma^2\big)$, implying that $Q_{\rho(\Theta)} = \mathbf{LN}\big(\mu,\sigma^2\big)$ and that the aggregate claims process S is a compound Poisson-Lognormal process under Q.

In our final example we show how to convert a compound mixed renewal process into a compound Poisson-Beta one under a change of measures. Recall the Beta distribution with parameters $a, b \in \Upsilon$ (written $\mathbf{Be}(a,b)$ for short) i.e.

$$\mathbf{Be}(a,b)(B) := \int_B \frac{\Gamma(a+b)}{\Gamma(a) \cdot \Gamma(b)} \cdot x^{a-1} \cdot (1-x)^{b-1}\, \lambda(dx) \quad \text{for any } B \in \mathfrak{B}\big((0,1)\big).$$

Example 4.12. Take $D := \Upsilon^2$ and define the function $\xi \in \mathfrak{M}_+(\Upsilon^2)$ by means of

$$\xi(\theta) := \xi(\theta_1,\theta_2) := \frac{a^{b_1+b_2} \cdot \theta_1^{b_1-1} \cdot \theta_2^{b_2-1}}{\Gamma(b_1) \cdot \Gamma(b_2) \cdot g(\theta_1,\theta_2)} \cdot e^{-a(\theta_1+\theta_2)} \quad \text{for any } (\theta_1,\theta_2) \in \Upsilon^2$$

where $a, b_1, b_2 \in \Upsilon$ are constants. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying that $\xi \in \mathcal{R}_+(\Upsilon^2)$. Consider the function $\beta(x,\theta) := \gamma(x) + \ln \frac{\theta_1 \cdot \mathbb{E}_{P_\theta}[W_1]}{\theta_1+\theta_2}$ for any $(x,\theta) \in \Upsilon \times \Upsilon^2$, with $\gamma \in \mathcal{F}_{P,\ln}$; hence $\beta \in \mathcal{F}_{P,\Theta}$. Applying now Corollary 4.8(ii) we get that there exists a unique pair $(\rho, Q) \in \mathfrak{M}_+(\Upsilon) \times \mathcal{M}_{S,\mathbf{Exp}(\rho(\Theta))}$ satisfying conditions $(*)$ and (RPM_ξ), and such that ξ is a Radon-Nikodým derivative of Q_Θ with respect to P_Θ. In particular, it follows from condition $(*)$ that $\rho(\theta) = \frac{\theta_1}{\theta_1+\theta_2}$ for P_Θ-a.a. $\theta = (\theta_1, \theta_2) \in \Upsilon^2$ and

$$\begin{aligned}
Q_\Theta(B_1 \times B_2) &= \mathbb{E}_P\big[\mathbb{1}_{\Theta^{-1}(B_1\times B_2)} \cdot \xi(\Theta)\big] \\
&= \int_{B_1\times B_2} \frac{a^{b_1+b_2} \cdot \theta_1^{b_1-1} \cdot \theta_2^{b_2-1}}{\Gamma(b_1)\cdot\Gamma(b_2)} \cdot e^{-a(\theta_1+\theta_2)}\, \lambda_2(d\theta) \\
&= \int_{B_1\times B_2} \frac{a^{b_1+b_2} \cdot \theta_1^{b_1-1} \cdot \theta_2^{b_2-1}}{\Gamma(b_1)\cdot\Gamma(b_2)} \cdot e^{-a(\theta_1+\theta_2)}\, \lambda(d\theta_2)\, \lambda(d\theta_1) \\
&= \Big(\int_{B_1} \frac{a^{b_1}\theta_1^{b_1-1}}{\Gamma(b_1)} \cdot e^{-a\theta_1}\, \lambda(d\theta_1)\Big) \cdot \Big(\int_{B_2} \frac{a^{b_2}\theta_2^{b_2-1}}{\Gamma(b_2)} \cdot e^{-a\theta_2}\, \lambda(d\theta_2)\Big) \\
&= \big(Q_{\Theta_1} \otimes Q_{\Theta_2}\big)(B_1 \times B_2),
\end{aligned}$$

where $B_1, B_2 \in \mathfrak{B}(\Upsilon)$; hence $Q_{\Theta_1} = \mathbf{Ga}(b_1, a)$ and $Q_{\Theta_2} = \mathbf{Ga}(b_2, a)$ and Θ_1 and Θ_2 are Q-independent, implying that $Q_{\rho(\Theta)} = \mathbf{Be}(b_1, b_2)$ and that S is a compound Poisson-Beta process.

Another consequence of Theorem 4.5 is Proposition 4.15 which shows that the martingales $L^r := \{L^r_t\}_{t\in\mathbb{R}_+}$ and the measures Q^r appearing in Schmidli (2017) Lemma 8.4, are special instances of the martingales $\widetilde{M}^{(\gamma)}(\theta)$ and the measures Q_θ, respectively, of Theorem 4.5. Note that Schmidli (2017) Lemma 8.4, was first proven in Dassios and Embrechts (1989) Theorem 10. In order to prove the aforementioned proposition, we first need the following two auxiliary results.

Lemma 4.13. *Let u be a $[-\infty, +\infty]$-valued function on $\mathbb{R} \times D$, Z a real-valued random variable on (Ω, Σ) such that the integral $\int u\, dM$, where $M := P_{(Z,\Theta)}$ is defined in $[-\infty, +\infty]$, and let $g := u(Z, \Theta)$. The following hold true:*

(i) The integral $\int\int u(z,\theta)\,(P_\theta)_Z(dz)\, P_\Theta(d\theta)$ is defined and equals $\int u\, dM$;
(ii) $\mathbb{E}_P[g \mid \Theta] = \mathbb{E}_{P_\bullet}[u^\bullet] \circ \Theta \quad P \restriction \sigma(\Theta)$-a.s.;
(iii) $\int g\, dP = \int\int u^\bullet\, dP_\bullet\, dP_\Theta$.

Proof: Ad (i): For any $\theta \in D$ define the probability measure $\widetilde{P}_\theta$ on $\mathfrak{B}$ by means of

$$\widetilde{P}_\theta(A) := (P_\theta)_Z(A) \quad \text{for any} \quad A \in \mathfrak{B}.$$

It follows easily that $\{\widetilde{P}_\theta\}_{\theta\in D}$ is a product rcp on $\mathfrak{B}$ for M with respect to P_Θ (see Strauss et al. (2004) Definition 1.1 for the definition). By Lyberopoulos and Macheras (2012) Remark 3.4(b), we have

$$M(E) = \int \widetilde{P}_\theta(E^\theta)\, P_\Theta(d\theta) \quad \text{for any} \quad E \in \mathfrak{B}(\mathbb{R} \times D);$$

hence we can apply Lyberopoulos and Macheras (2012) Remark 3.4(c), in order to conclude that statement (i) holds.

The proof of the statements (ii) and (iii) follow in a similar way as that in Lyberopoulos and Macheras (2012) Proposition 3.8, by replacing $id_\Omega \times f$ with $Z \times \Theta$. □

Lemma 4.14. *For any $r \in \mathbb{R}_+$ such that $\mathbb{E}_P\big[e^{rX_1}\big] < \infty$ and for any $\theta \notin L_P$, where L_P is the P_Θ-null set in $\mathfrak{B}(D)$ appearing in Proposition 3.4, let $\kappa_\theta(r)$ be the unique solution to the equation*

$$M_{X_1}(r) \cdot (M_\theta)_{W_1}\big(-\kappa_\theta(r) - c(\theta)\cdot r\big) = 1, \tag{4.2}$$

where M_{X_1} *and* $(M_\theta)_{W_1}$ *are the moment generating function of* X_1 *and* W_1 *under the measures* P *and* P_θ*, respectively. (Such a solution exists by e.g. Rolski et al. (1999) Lemma 11.5.1(a)). Define the function* $\kappa : D \times \mathbb{R}_+ \to \mathbb{R}$ *by means of* $\kappa(\theta, r) := \kappa_\theta(r)$ *for any* $(\theta, r) \in D \times \mathbb{R}_+$*, and for fixed* $r \in \mathbb{R}_+$ *denote by* $\kappa_\Theta(r)$ *the random variable defined by the formula* $\kappa_\Theta(r)(\omega) := \kappa_{\Theta(\omega)}(r)$ *for any* $\omega \in \Omega$*. Then* $\kappa_\Theta(r)$ *is the* $P \upharpoonright \sigma(\Theta)$*-a.s. unique solution to the equation*

$$M_{X_1}(r) \cdot \mathbb{E}_P\Big[e^{-(\kappa_\Theta(r)+c(\Theta)\cdot r)\cdot W_1} \mid \Theta\Big] = 1 \quad P \upharpoonright \sigma(\Theta)\text{-a.s.}. \tag{4.3}$$

Proof: Note that, according to Proposition 3.4, there exists a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that the process S is a P_θ-CRP$\big(\mathbf{K}(\theta), P_{X_1}\big)$ with $P_{X_1} = (P_\theta)_{X_1}$ for any $\theta \notin L_P$. Lemma 4.13 along with Proposition 3.4, yields that equation (4.2) is equivalent to (4.3); hence (4.3) holds and $\kappa_\Theta(r)$ is the $P \upharpoonright \sigma(\Theta)$-a.s. unique solution to (4.3). $\square$

Proposition 4.15. *Let* $u, t \in \mathbb{R}_+$*. For any* $r \in \mathbb{R}_+$ *such that* $\mathbb{E}_P\big[e^{rX_1}\big] < \infty$*, and for any* $\theta \notin L_P$*, let* $\kappa_\theta(r)$ *be the unique solution to the equation (4.2), and let* $\kappa_\Theta(r)$ *be as in Lemma 4.14. Put* $\rho_r(\Theta) := \kappa_\Theta(r) + c(\Theta) \cdot r$*.*

(i) Assume that $Q := Q^r \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\Theta))}$ *with*

$$Q_{X_1}(A) := Q^r_{X_1}(A) := \frac{\mathbb{E}_P\big[\mathbb{1}_{X_1^{-1}[A]} \cdot e^{r\cdot X_1}\big]}{\mathbb{E}_P\big[e^{r\cdot X_1}\big]} \quad \text{for any } A \in \mathfrak{B}(\Upsilon)$$

and

$$Q_{W_1|\Theta}(B) := \frac{\mathbb{E}_P\big[\mathbb{1}_{W_1^{-1}[B]} \cdot e^{-\rho_r(\Theta)\cdot W_1} \mid \Theta\big]}{\mathbb{E}_P\big[e^{-\rho_r(\Theta)\cdot W_1} \mid \Theta\big]} \quad P \upharpoonright \sigma(\Theta)\text{-a.s.},$$

for any $B \in \mathfrak{B}(\Upsilon)$*, where* $Q_{W_1|\Theta}(B) := Q^r_{W_1|\Theta}(B) := \mathbf{\Lambda}(\rho_r(\Theta))(B)$ $P \upharpoonright \sigma(\Theta)$*-a.s.. There exists an essentially unique pair* $(\gamma, \xi) \in \mathcal{F}_{P,\ln} \times \mathcal{R}_+(D)$ *satisfying conditions* $\gamma(x) = r \cdot x - \ln \mathbb{E}_P[e^{r\cdot X_1}]$ *for any* $x \in \Upsilon$*, and (*RRM_ξ*) with*

$$M_t^{(\gamma,r)}(\Theta) = \xi(\Theta) \cdot \widetilde{M}_t^{(\gamma,r)}(\Theta) \;\; P \upharpoonright \sigma(\Theta)\text{-a.s.} \tag{4.4}$$

and

$$\begin{aligned}\widetilde{M}_t^{(\gamma,r)}(\Theta) &= e^{r\cdot S_t - \rho_r(\Theta)\cdot T_{N_t} + \ln \mathbb{E}_P[e^{r\cdot X_1}]} \cdot \frac{\int_{J_t}^\infty e^{-\rho_r(\Theta)\cdot w}\, P_{W_1|\Theta}(dw)}{1 - \mathbf{K}(\Theta)(J_t)} \\ &= M_{X_1}(r) \cdot e^{-r\cdot(R_t^u(\Theta)-u) + \rho_r(\Theta)\cdot J_t - \kappa_\Theta(r)\cdot t} \cdot \frac{\int_{J_t}^\infty e^{-\rho_r(\Theta)\cdot w}\, P_{W_1|\Theta}(dw)}{1 - \mathbf{K}(\Theta)(J_t)},\end{aligned}$$

where $J_t := t - T_{N_t}$*;*

(ii) conversely, for any pair (γ, ξ) *with* $\xi \in \mathcal{R}_+(D)$ *and* $\gamma(x) := r \cdot x - \ln \mathbb{E}_P\big[e^{r\cdot X_1}\big]$ *for any* $x \in \Upsilon$ *there exists a unique probability measure* $Q := Q^r \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\Theta))}$ *determined by condition (*RRM_ξ*) with* $M^{(\gamma)}(\Theta)$ *fulfilling condition (4.4);*

(iii) in both cases (i) and (ii) there exists an essentially unique rcp $\{Q_\theta\}_{\theta\in D} := \{Q^r_\theta\}_{\theta\in D}$ *of* Q *over* Q_Θ *consistent with* Θ*, satisfying for any* $\theta \notin L_{**}$*, where* L_{**} *is the* P_Θ*-null set in* $\mathfrak{B}(D)$ *appearing in Proposition 3.11, conditions* $Q_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\theta))}$ *and (*RRM_θ*) with*

$$\widetilde{M}_t^{(\gamma,r)}(\theta) = e^{r\cdot S_t - \rho_r(\theta)\cdot T_{N_t} + \ln \mathbb{E}_P[e^{r\cdot X_1}]} \cdot \frac{\int_{J_t}^\infty e^{-\rho_r(\theta)\cdot w}\,(P_\theta)_{W_1}(dw)}{1 - \mathbf{K}(\theta)(J_t)} = L_t^r(\theta),$$

where

$$L_t^r(\theta) = M_{X_1}(r) \cdot e^{-r\cdot(r_t^u(\theta)-u) + \rho_r(\theta)\cdot J_t - \kappa_\theta(r)\cdot t} \cdot \frac{\int_{J_t}^\infty e^{-\rho_r(\theta)\cdot w}\,(P_\theta)_{W_1}(dw)}{1 - \mathbf{K}(\theta)(J_t)}$$

and $L^r(\theta) := \{L_t^r(\theta)\}_{t\in\mathbb{R}_+}$ *is the martingale constructed in Rolski et al. (1999) Theorem 11.5.1.*

Proof: Fix on arbitrary $u, t, r \in \mathbb{R}_+$ so that $\mathbb{E}_P[e^{rX_1}] < \infty$.

Ad (i): Note that according to Lemma 4.14, $\kappa_\Theta(r)$ is the $P \restriction \sigma(\Theta)$-a.s. unique solution to the equation (4.3). According to Theorem 4.5(i), there exists an essentially unique pair $(\gamma, \xi) \in \mathcal{F}_{P,\ln} \times \mathcal{R}_+(D)$ satisfying conditions $\gamma(x) = r \cdot x - \ln \mathbb{E}_P[e^{rX_1}]$ for any $x \in \Upsilon$, and (RRM_ξ). Condition (4.4) now follows by conditions (RRM_ξ), (4.2) and standard computations.

Ad (ii): An easy computation justifies that $\gamma \in \mathcal{F}_{P,\ln}$; hence we may apply Theorem 4.5(ii), in order to obtain a unique probability measure $Q := Q^r \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\Theta))}$ determined by condition (RRM_ξ) with the martingale $M^{(\gamma,r)}(\Theta)$ fulfilling condition (4.4).

Ad (iii): According to Theorem 4.5(iii) there exists an essentially unique rcp $\{Q_\theta\}_{\theta \in D} := \{Q^r_\theta\}_{\theta \in D}$ of Q over Q_Θ consistent with Θ, satisfying for any $\theta \notin L_{**}$ conditions $Q_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ and (RRM_θ). Condition (RRM_θ) along with (4.2) and standard computations yields the desired martingale. $\square$

Remark 4.16. Fix on arbitrary $\theta \notin L_{**}$ and $r \in \mathbb{R}_+$ such that $\mathbb{E}_P[e^{r \cdot X_1}] < \infty$. It is worth noting that the constructions of the martingales $\widetilde{M}^{(\gamma,r)}(\theta)$ and $L^r(\theta)$ in Proposition 4.15 and Rolski et al. (1999) Theorem 11.5.1, respectively, differ essentially. While the construction of $\widetilde{M}^{(\gamma,r)}(\theta)$ is rather elementary and consists only of standard measure theoretic arguments, the construction of $L^r(\theta)$ not only requires the absolute continuity of $(P_\theta)_{W_1}$ with respect to $\lambda \restriction \mathfrak{B}([0,1])$ but also the backward Markovization of the reserve process $r^u(\theta)$ $(u \in \mathbb{R}_+)$ and the use of the theory of piecewise deterministic Markov processes (cf. e.g. Rolski et al. (1999) Chapter 11). Note also that the above proposition extends Schmidli (2017) Lemma 8.4, as well as Rolski et al. (1999) Theorem 11.5.1, to CMRPs. For applications of Proposition 4.15 to the ruin problem for CMRPs we refer to Tzaninis (2022).

Appendix A.

A.1. *Proof of Lemma 3.10*. Fix on arbitrary $t \geq 0$ and $n \in \mathbb{N}_0$. We split the proof into the following steps.

(a) There exists a P_Θ-null set $L_{P,Q} := L_P \cup L_Q \in \mathfrak{B}(D)$, such that for any $\theta \notin L_{P,Q}$ the aggregate process S is a P_θ-CRP$(\mathbf{K}(\theta), P_{X_1})$ with $P_{X_1} = (P_\theta)_{X_1}$, and a Q_θ-CRP$(\mathbf{\Lambda}(\rho(\theta)), Q_{X_1})$ with $Q_{X_1} = (Q_\theta)_{X_1}$.

In fact, due to Proposition 3.4 there exist a P_Θ-null set $L_P \in \mathfrak{B}(D)$ and a Q_Θ-null set $L_Q \in \mathfrak{B}(D)$ such that the process S is a P_θ-CRP$(\mathbf{K}(\theta), P_{X_1})$ and a Q_θ-CRP$(\mathbf{\Lambda}(\rho(\theta)), Q_{X_1})$ with $P_{X_1} = (P_\theta)_{X_1}$ and $Q_{X_1} = (Q_\theta)_{X_1}$ for any $\theta \notin L_P$ and any $\theta \notin L_Q$, respectively. Thus, taking into account that $P_\Theta \sim Q_\Theta$ by assumption, we obtain that $L_{P,Q} \in \mathfrak{B}(D)$ is a P_Θ-null set such that for any $\theta \notin L_{P,Q}$ the conclusion of (a) holds.

(b) There exists a P_Θ-null set $M'_{P,Q} \in \mathfrak{B}(D)$ such that for any $\theta \notin M'_{P,Q}$ there exists a $(P_\theta)_{W_1}$-a.s. positive Radon-Nikodým derivative r_θ of $(Q_\theta)_{W_1}$ with respect to $(P_\theta)_{W_1}$ satisfying condition

$$Q_\theta(E) = \mathbb{E}_{P_\theta}\Big[\mathbb{1}_E \cdot \Big(\prod_{j=1}^{n} r_\theta(W_j)\Big)\Big] \quad \text{for any } E \in \mathcal{F}^W_n.$$

In fact, since by assumption $P_\Theta \sim Q_\Theta$ and W is P- and Q-conditionally independent, we can apply Lemma 3.8 in order to conclude the validity of (b).

(c) There exists a P_{X_1}-a.s. unique function $\gamma \in \mathcal{F}_{P,h}$, defined by means of $\gamma := h \circ f$, where f is a P_{X_1}-a.s. positive Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1}, satisfying for every $n \in \mathbb{N}_0$ condition

$$Q(E) = \mathbb{E}_P\Big[\mathbb{1}_E \cdot \Big(\prod_{j=1}^{n} (h^{-1} \circ \gamma \circ X_j)\Big)\Big] \quad \text{for any } E \in \mathcal{F}^X_n. \tag{A.1.1}$$

In fact, since $P_{X_1} \sim Q_{X_1}$ by assumption, we can apply Macheras and Tzaninis (2020) Lemma 2.2(a), in order to get (c).

(d) There exists a P_Θ-null set $L_* \in \mathfrak{B}(D)$, containing the P_Θ-null sets L_P and L_Q, such that for any $\theta \notin L_*$ the conclusion of the lemma holds true.

In fact, put $L_* := L_{P,Q} \cup M'_{P,Q}$ and fix on an arbitrary $\theta \notin L_*$. Since $(Q_\theta)_{W_1} \sim (P_\theta)_{W_1}$ by (b), and $(Q_\theta)_{X_1} \sim (P_\theta)_{X_1}$ by (a) and (c), we can apply Macheras and Tzaninis (2020) Proposition 2.1, to complete the whole proof. □

A.2. *Proof of Proposition 3.11.* Fix on arbitrary $t \geq 0$ and $s \in [0, t]$.

Ad (i)⇒(ii): Since $P \upharpoonright \mathcal{F}_t \sim Q \upharpoonright \mathcal{F}_t$, we have $Q_\Theta \sim P_\Theta$ by $\sigma(\Theta) \subseteq \mathcal{F}_t$, while Remark 2.1 along with inclusion $\mathcal{F}_t^S \subseteq \mathcal{F}_t$ and Macheras and Tzaninis (2020) Proposition 2.1 (i)⇒ (ii), yields $Q_{X_1} \sim P_{X_1}$ an $Q_{W_1} \sim P_{W_1}$.

Ad (ii)⇒(iii): According to Lemma 3.10 there exists a P_Θ-null set $L_* \in \mathfrak{B}(D)$, containing the P_Θ-null sets L_P and L_Q appearing in Proposition 3.4, and a P_{X_1}-a.s. unique function $\gamma \in \mathcal{F}_{P,h}$ so that, for any $\theta \notin L_*$, condition (3.3) is valid with the family $\widetilde{M}^{(\gamma)}\theta)$ being a P_θ-a.s. positive $\mathcal{F}^S$-martingale in $\mathcal{L}^1(P_\theta)$ and satisfying condition $\mathbb{E}_{P_\theta}\big[\widetilde{M}_t^{(\gamma)}(\theta)\big] = 1$. We split the rest of the proof of this implication into the following steps:

(a) Consider the family of sets

$$\mathcal{G}_t := \Big\{ \bigcap_{k=1}^{m} A_k : A_k \in \mathcal{F}_t^S \cup \sigma(\Theta),\, m \in \mathbb{N} \Big\}.$$

Then there exists a P_Θ-null set $L_{**} \in \mathfrak{B}(D)$, containing the P_Θ-null sets L_P and L_Q, so that for any $\theta \notin L_{**}$ condition $Q_\theta(G) = \int_G \widetilde{M}_t^{(\gamma)}(\theta)\, dP_\theta$ for all $G \in \mathcal{G}_t$, holds true.

For the proof of (a) we follow the arguments used in Lyberopoulos and Macheras (2019) Proposition 3.4, proof of implication (ii)⇒(iii), step (e).

First note that the consistency of the rcp $\{P_\theta\}_{\theta \in D}$ with Θ yields the existence of a P_Θ-null set $V_P \in \mathfrak{B}(D)$ so that for any $\theta \notin V_P$ and $B \in \mathfrak{B}(D)$ we have $P_\theta(\Theta^{-1}[B]) = \mathbb{1}_B(\theta)$. For the same reasoning for $\{Q_\theta\}_{\theta \in D}$ in the place of $\{P_\theta\}_{\theta \in D}$, there exists a Q_Θ-null set(hence a P_Θ-null set by the assumption $Q_\Theta \sim P_\Theta$) $V_Q \in \mathfrak{B}(D)$ so that for any $\theta \notin V_Q$ and $B \in \mathfrak{B}(D)$ we have $Q_\theta(\Theta^{-1}[B]) = \mathbb{1}_B(\theta)$. Put $L_{**} := L_{**,P,Q} := L_* \cup V_P \cup V_Q$.

For any $G \in \mathcal{G}_u$ there exist an integer $m \in \mathbb{N}$ and a finite sequence $\{A_k\}_{k \in \mathbb{N}_m}$ in $\mathcal{F}_t^S \cup \sigma(\Theta)$, where $\mathbb{N}_m := \{1, 2, \ldots, m\}$, such that $G = \bigcap_{k \in \mathbb{N}_m} A_k$. Setting

$$I_\Theta := \big\{k \in \mathbb{N}_m : A_k \in \sigma(\Theta)\big\} \quad \text{and} \quad I_H := \big\{k \in \mathbb{N}_m : A_k \in \mathcal{F}_s^S \setminus \sigma(\Theta)\big\}$$

we get $I_\Theta \cup I_H = \mathbb{N}_m$, $\bigcap_{k \in I_\Theta} A_k \in \sigma(\Theta)$ and $\bigcap_{k \in I_H} A_k \in \mathcal{F}_s^S$. Since $\bigcap_{k \in I_\Theta} A_k \in \sigma(\Theta)$, there exists a set $B \in \mathfrak{B}(D)$ such that $\bigcap_{k \in I_\Theta} A_k = \Theta^{-1}[B]$. The latter, together with the consistency of $\{Q_\theta\}_{\theta \in D}$ and $\{P_\theta\}_{\theta \in D}$ with Θ, yields for any $\theta \in (D \setminus L_{**}) \cap B$ that

$$\begin{aligned} Q_\theta(G) &= Q_\theta\big(\bigcap_{k \in I_H} A_k \cap \Theta^{-1}[B]\big) = Q_\theta\big(\bigcap_{k \in I_H} A_k\big) \\ &= \mathbb{E}_{P_\theta}\big[\mathbb{1}_{\bigcap_{k \in I_H} A_k} \cdot \widetilde{M}_t^{(\gamma)}(\theta)\big] = \mathbb{E}_{P_\theta}\big[\mathbb{1}_G \cdot \widetilde{M}_t^{(\gamma)}(\theta)\big], \end{aligned}$$

where the third equality follows by Lemma 3.10. With the same reasoning for any $\theta \notin (L_{**} \cup B)$ we get

$$Q_\theta(G) = Q_\theta\big(\bigcap_{k \in I_H} A_k \cap \Theta^{-1}[B]\big) = 0 = \mathbb{E}_{P_\theta}\big[\mathbb{1}_G \cdot \widetilde{M}_t^{(\gamma)}(\theta)\big].$$

(b) For any $\theta \notin L_{**}$ the family $\widetilde{M}^{(\gamma)}(\theta)$ is a martingale in $\mathcal{L}^1(P_\theta)$ and condition (RRM_θ) holds for all $A \in \mathcal{F}_t$ true.

In fact, fix on arbitrary $\theta \notin L_{**}$ and denote by $\mathcal{D}_t$ the family of all $A \in \mathcal{F}_t$ satisfying condition (RRM_θ). By (a) we get $\mathcal{G}_t \subseteq \mathcal{D}_t$, while an easy computation justifies that $\mathcal{D}_t$ is a Dynkin class, and so by Dynkin Lemma we get that condition (RRM_θ) holds.

Clearly, we can easily obtain by condition (RRM_θ) that the family $\widetilde{M}^{(\gamma)}(\theta)$ is a martingale in $\mathcal{L}^1(P_\theta)$, since (RRM_θ) yields

$$\int_A \widetilde{M}_s^{(\gamma)}(\theta)\, dP_\theta = \int_A \widetilde{M}_s^{(\gamma)}(\theta)\, dP_\theta \quad \text{for any} \quad A \in \mathcal{F}_s.$$

Furthermore, condition (RRM_θ) implies easily that $\widetilde{M}_t^{(\gamma)}(\theta)$ is a P_θ-a.s. positive and $\mathbb{E}_{P_\theta}\big[\widetilde{M}_t^{(\gamma)}(\theta)\big] = 1$, since for $A = \Omega$ we get

$$\mathbb{E}_{P_\theta}\big[\widetilde{M}_t^{(\gamma)}(\theta)\big] = Q_\theta(\Omega) = 1$$

and by Lemma 3.10 we have $P_\theta(\{\widetilde{M}_t^{(\gamma)}(\theta) > 0\}) = 1$.

Ad (iii) $\Rightarrow$ (iv): It follows by (iii) that there exists a P_Θ-null set $L_{**} \in \mathfrak{B}(D)$ and an essentially unique function $\gamma \in \mathcal{F}_{P,h}$ with $\gamma = h \circ f$, where f is a Radon-Nikodým derivative of Q_{X_1} with respect to P_{X_1} so that for any $\theta \notin L_{**}$ condition (RRM_θ) holds.

By assumption $P_\Theta \sim Q_\Theta$ there exists a P_Θ-a.s. positive Radon-Nikodým derivative ξ of Q_Θ with respect to P_Θ. Fix on arbitrary $A \in \mathcal{F}_s$. Put $\mu := P \circ (id_\Omega \times \Theta)^{-1}$ and consider the $\mathcal{F}_s \otimes \mathfrak{B}(D)$-measurable map $v := \mathbb{1}_A \otimes M_t^{(\gamma)} : \Omega \times D \to \mathbb{R}$. We will show first that $v \in \mathcal{L}^1(\mu)$. Since $\{P_\theta\}_{\theta \in D}$ is a rcp of P over P_Θ consistent with Θ, it follows by Lyberopoulos and Macheras (2012) Proposition 3.7, that it is a product rcp on Σ for μ with respect to P_Θ (see Strauss et al. (2004) Definition 1.1 for the definition and the properties of a product rcp); hence

$$\begin{aligned}
\int v\, d\mu &= \int_D \int_\Omega v^\theta\, dP_\theta\, P_\Theta(d\theta) = \int_D \int_\Omega \mathbb{1}_A \cdot M_t^{(\gamma)}(\theta)\, dP_\theta\, P_\Theta(d\theta) \\
&= \int_D \int_\Omega \mathbb{1}_A \cdot \xi(\theta) \cdot \widetilde{M}_t^{(\gamma)}(\theta)\, dP_\theta\, P_\Theta(d\theta) = \int_D \int_\Omega \mathbb{1}_A \cdot \widetilde{M}_t^{(\gamma)}(\theta)\, dP_\theta\, Q_\Theta(d\theta) \\
&= \int_D \mathbb{E}_{P_\theta}\big[\mathbb{1}_A \cdot \widetilde{M}_t^{(\gamma)}(\theta)\big]\, Q_\Theta(d\theta) \leq \int_D \mathbb{E}_{P_\theta}\big[\widetilde{M}_t^{(\gamma)}(\theta)\big]\, Q_\Theta(d\theta) = 1,
\end{aligned}$$

where the first equality follows by Lyberopoulos and Macheras (2012) Remark 3.4(c) and the inequality follows from the statement (iii). Thus, we may apply Lyberopoulos and Macheras (2012) Proposition 3.8(ii) for $f := \Theta$ and $g := v \circ (id_\Omega \times \Theta)$ in order to conclude that

$$\mathbb{E}_P\big[\mathbb{1}_A \cdot M_t^{(\gamma)}(\Theta)\big] = \mathbb{E}_{P_\Theta}\big[\mathbb{E}_{P_\theta}[\mathbb{1}_A \cdot M_t^{(\gamma)}(\theta)]\big],$$

implying along with condition (RRM_θ)

$$Q(A) = \int Q_\theta(A) Q_\Theta(d\theta) = \mathbb{E}_{P_\Theta}\big[\mathbb{E}_{P_\theta}[\mathbb{1}_A \cdot \xi(\theta) \cdot \widetilde{M}_t^{(\gamma)}(\theta)]\big] = \mathbb{E}_P\big[\mathbb{1}_A \cdot M_t^{(\gamma)}(\Theta)\big],$$

and so condition (RRM_ξ) holds; hence the family $M^{(\gamma)}(\Theta)$ is a P-a.s. positive martingale in $\mathcal{L}^1(P)$ satisfying condition $\mathbb{E}_P\big[M_t^{(\gamma)}(\Theta)\big] = 1$.

Ad (iv)$\Rightarrow$(i): This follows by (RRM_ξ) and $P\big(\{M_t^{(\gamma)}(\Theta) > 0\}\big) = 1$. $\square$

A.3. *Proof of Proposition 4.1.* Fix on arbitrary $\theta \in D$ and $n \in \mathbb{N}$, and consider the product probability space $(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{P}_\theta)$ constructed in Macheras and Tzaninis (2020) page 51, where $\widetilde{\Omega} := \Upsilon^{\mathbb{N}} \times \Upsilon^{\mathbb{N}}$, $\widetilde{\Sigma} := \mathfrak{B}(\widetilde{\Omega}) = \mathfrak{B}(\Upsilon)_{\mathbb{N}} \otimes \mathfrak{B}(\Upsilon)_{\mathbb{N}}$ and $\widetilde{P}_\theta := \big(\otimes_{n\in\mathbb{N}} P_n(\theta)\big) \otimes R_{\mathbb{N}}$. We split the proof into the next four steps. The first two steps establish the validity of (i), while the remaining ones concern the statement (ii) of the proposition.

(a) For any fixed $F \in \widetilde{\Sigma}$ the function $\theta \mapsto \widetilde{P}_\theta(F)$ is $\mathfrak{B}(D)$-measurable.

In fact, since by assumption, for any fixed $B \in \mathfrak{B}(\Upsilon)$ each function $\theta \mapsto P_n(\theta)(B)$ is $\mathfrak{B}(D)$-measurable, it follows by a monotone class argument that the same holds true for the function $\theta \mapsto \widetilde{P}_\theta(F)$ for any fixed $F \in \widetilde{\Sigma}$.

(b) Define the set-functions $\widetilde{P} : \widetilde{\Sigma} \to \mathbb{R}_+$ and $P : \Sigma \to \mathbb{R}_+$ by means of

$$\widetilde{P}(F) := \int \widetilde{P}_\theta(F)\,\mu(d\theta) \quad \text{for all} \quad F \in \widetilde{\Sigma}.$$

and

$$P(E) := \int \widetilde{P}_\theta(E^\theta)\,\mu(d\theta) \quad \text{for each} \quad E \in \Sigma,$$

where $E^\theta := \big\{\widetilde{\omega} \in \widetilde{\Omega} : (\widetilde{\omega}, \theta) \in E\big\}$ is the θ-section of E, respectively. Then $\widetilde{P}$ and P are probability measures on $\widetilde{\Sigma}$ and Σ, respectively, such that $\{P_\theta\}_{\theta\in D}$ is a rcp of P over μ consistent with Θ and $P_\Theta = \mu$.

In fact, obviously $\widetilde{P}$ and P are probability measures on $\widetilde{\Sigma}$ and Σ, respectively. It is easy to see that $\{\widetilde{P}_\theta\}_{\theta\in D}$ is a product rcp on $\widetilde{\Sigma}$ for P with respect to μ (see Strauss et al. (2004) Definition 1.1, for the definition and its properties). Put $P_\theta := \widetilde{P}_\theta \otimes \delta_\theta$. Clearly, P_θ is a probability measure on Σ. So, we may apply Lyberopoulos and Macheras (2013) Proposition 3.5, to get that $\{P_\theta\}_{\theta\in D}$ is a rcp of P over μ consistent with the canonical projection π_D from Ω onto D. Putting $\Theta := \pi_D$ we get $P_\Theta = \mu$, completing in this way the proof of statement (i).

(c) There exists a counting process N and a claim size process X such that the quadruplet (P, W, X, Θ) satisfies conditions (a1), (a2), W is P_θ-i.i.d. and the pair (N, X) is both a P- and P_θ-risk process for any $\theta \in D$.

In fact, denote by $\pi_{\Omega\widetilde{\Omega}}$ the canonical projection from Ω onto $\widetilde{\Omega}$, and by $\widetilde{W}_n$ and $\widetilde{X}_n$ the canonical projections from Ω onto the n-coordinate of the first and the second factor of $\Omega = \Upsilon^{\mathbb{N}} \times \Upsilon^{\mathbb{N}} \times D$, respectively. Put $W_n := \widetilde{W}_n \circ \pi_{\Omega\widetilde{\Omega}}$ and $X_n := \widetilde{X}_n \circ \pi_{\Omega\widetilde{\Omega}}$ and get

$$\mathbf{K}(\theta) = (P_\theta)_{W_n} = (\widetilde{P}_\theta)_{W_n} \quad \text{and} \quad R = (P_\theta)_{X_n} = (\widetilde{P}_\theta)_{X_n}. \tag{A.3.1}$$

Since $(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{P}_\theta)$ is a product probability space and $\widetilde{W}_n$, $\widetilde{X}_n$ are the canonical projections, applying standard computations we get that the processes $W := \{W_k\}_{k\in\mathbb{N}}$ and $X := \{X_k\}_{k\in\mathbb{N}}$ are $\widetilde{P}_\theta$-independent and $\widetilde{P}_\theta$-mutually independent; hence they are P_θ-independent and P_θ-mutually independent. Putting $T_k := \sum_{m=1}^{k} W_m$ for any $k \in \mathbb{N}_0$ and $T := \{T_k\}_{k\in\mathbb{N}_0}$, we obtain that $N := \{N_t\}_{t\in\mathbb{R}_+}$ is the counting process induced by T by means of $N_t := \sum_{k=1}^{\infty} \mathbb{1}_{\{T_k \le t\}}$ for all $t \in \mathbb{R}_+$. The fact that W and X are P_θ-mutually independent along with condition (A.3.1), yields that the processes N and X are P_θ-mutually independent. Thus, the pair (N, X) is a P_θ-risk process.

Since $\{P_\theta\}_{\theta\in D}$ is a rcp of P over μ consistent with Θ by (b), applying Lyberopoulos and Macheras (2012) Lemma 4.1 along with Lyberopoulos and Macheras (2014), we obtain that W and X are P-conditionally independent, i.e., P satisfies condition (a1). Furthermore, condition (A.3.1) again together with the fact that $\{P_\theta\}_{\theta\in D}$ is a rcp of P over μ consistent with Θ by (b), implies that $P_{X_n} = R$ and condition (a2) is satisfied by P. Thus, taking into account the fact that X is P_θ-i.i.d., we may apply Lyberopoulos and Macheras (2019) Lemma 3.3(ii), in order to conclude that the process X is P-i.i.d.. Therefore the pair (N, X) is a P-risk process.

(d) The process S induced by (N, X) is a P-CMRP$(\mathbf{K}(\Theta), P_{X_1})$.

In fact, since the sequence W is P_θ-i.i.d. by (c), it follows that N is a P_θ-RP$(\mathbf{K}(\theta))$, implying together with the fact that (N, X) is a P_θ-risk process by (b), that S is a P_θ-CRP$(\mathbf{K}(\theta), P_{X_1})$ with $P_{X_1} = (P_\theta)_{X_1}$; hence taking into account the fact that P satisfies conditions (a1), (a2) by (c), we can apply Proposition 3.4 in order to get the conclusion of (d). Thus, assertion (ii) follows, completing the whole proof. □

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