ALEA, Lat. Am. J. Probab. Math. Stat. **20**, 463–496 (2023)
DOI: 10.30757/ALEA.v20-16

Stochastic wave equation with Lévy white noise

Raluca Balan

University of Ottawa, Department of Mathematics and Statistics, 150 Louis Pasteur Private, Ottawa, Ontario, K1G 0P8, Canada.
E-mail address: rbalan@uottawa.ca
URL: https://mysite.science.uottawa.ca/rbalan

Abstract. In this article, we study the stochastic wave equation on the entire space $\mathbb{R}^d$, driven by a space-time Lévy white noise with possibly infinite variance (such as the α-stable Lévy noise). In this equation, the noise is multiplied by a Lipschitz function $\sigma(u)$ of the solution. We assume that the spatial dimension is $d = 1$ or $d = 2$. Under general conditions on the Lévy measure of the noise, we prove the existence of the solution, and we show that, as a function-valued process, the solution has a càdlàg modification in the local fractional Sobolev space of order $r < 1/4$ if $d = 1$, respectively $r < -1$ if $d = 2$.

1. Introduction

The study of stochastic partial differential equation (SPDEs) using the random field approach was initiated in the lecture notes Walsh (1986), and has become a very broad area in stochastic analysis since then. Major efforts have been dedicated to understanding the behaviour of solutions of SPDEs driven by space-time Gaussian white noise, as a natural replacement for the Brownian motion that is used in the classical theory of stochastic differential equations (SDEs). In the seminal article Dalang (1999), the author introduced the spatially-homogeneous Gaussian noise and developed the major tools for the study of SPDEs using random fields, laying the foundation of a general theory. These tools were embraced very quickly by a large scientific community and yielded spectacular results, especially when combined with techniques from Malliavin calculus. We include here just a small sample from a very large set of important contributions to this area: Conus and Dalang (2008); Hu and Nualart (2009); Foondun and Khoshnevisan (2009); Conus et al. (2013); Chen and Dalang (2015); Huang et al. (2017); Chen (2017).

We should also mention that there exists an alternative way for studying SPDEs, using cylindrical processes as noise, and there is a vast literature dedicated to stochastic evolution equations with this type of noise. We refer the reader to the excellent monograph Peszat and Zabczyk (2007) and the references therein. The recent papers Jakubowski and Riedle (2017); Kosmala and Riedle (2021) contain significant advances on stochastic integration with respect to cylindrical Lévy noise,

Received by the editors March 10th, 2022; accepted January 27th, 2023.
2010 *Mathematics Subject Classification.* 60H15, 60G60, 60G51.
Key words and phrases. Stochastic partial differential equations, random fields, space-time Lévy white noise.
The author has been supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

while Griffiths and Riedle (2021) gives a comparison between the cylindrical approach and the random field approach, which complements the similar comparison that was done by Dalang and Quer-Sardanyons (2011) in the Gaussian case.

SDEs driven by Lévy processes have been in the literature as long as their Brownian motion counterparts, and their origin can be traced back to Itô's memoir Ito (1951). There exist many breakthrough contributions to the area of SDEs driven by Lévy processes (or more generally, by discontinuous semi-martingales), and several excellent monographs were published on this subject (for instance Applebaum and Wu, 2000; Protter, 2004; Bichteler, 2002; Jacod and Shiryaev, 2003). However, the study of SPDEs driven by Lévy noise using the random field approach is not so well-developed as the Gaussian case. The majority of the existing works have focused so far only on the stochastic heat equation. We recall below some known results for the heat equation, which are relevant for the present article. But first, we need to introduce some notation.

Throughout this article, we let $L=\{L(A);A\in\mathcal{B}_0(\mathbb{R}_+\times\mathbb{R}^d)\}$ be a *pure-jump Lévy space-time white noise*, defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$:

$$L(A)=b|A|+\int_{A\times\{|z|\leq 1\}} z\widetilde{J}(dt,dx,dz)+\int_{A\times\{|z|>1\}} zJ(dt,dx,dz), \tag{1.1}$$

where $\mathcal{B}_0(\mathbb{R}_+\times\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}_+\times\mathbb{R}^d$ with finite Lebesgue measure, $|A|$ is the Lebesgue measure of A, $b\in\mathbb{R}$, J is a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}$ of intensity $dtdx\nu(dz)$, $\widetilde{J}$ is the compensated version of J, and ν is a Lévy measure on $\mathbb{R}$, i.e. a measure satisfying the following conditions:

$$\int_{\mathbb{R}}(|z|^2\wedge 1)\nu(dz)<\infty \quad \text{and} \quad \nu(\{0\})=0. \tag{1.2}$$

An important particular case is when L is an *α-stable Lévy noise*, i.e. an α-stable random measure with control measure given by the Lebesgue measure multiplied by a constant, as defined in Samorodnitsky and Taqqu (1994). In this case,

$$\nu(dz)=\Big(c_+z^{-\alpha-1}1_{(0,\infty)}(z)+c_-(-z)^{-\alpha-1}1_{(-\infty,0)}(z)\Big)dz$$

for some $\alpha\in(0,2)$, $c_+>0$, $c_->0$. For the symmetric α-stable Lévy noise, $c_+=c_-$.

The noise L is a natural space-time extension of a classical Lévy process (with no Gaussian component), which we recall can be written as

$$X(t)=at+\int_{[0,t]\times\{|z|\leq 1\}} z\widetilde{N}(dt,dz)+\int_{[0,t]\times\{|z|>1\}} zN(dt,dz), \quad t\geq 0,$$

where $a\in\mathbb{R}$, N is a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ of intensity $dt\nu(dz)$, and ν is a Lévy measure on $\mathbb{R}$. The process $\{X(t)\}_{t\geq 0}$ has a càdlàg modification, and the measure ν gives information about the size of the jumps of the sample paths of this modification.

In the space-time framework, we still speak about the third component of J as the "jump" component, although we do not identify a time-indexed process whose paths have these "jumps". In particular, the "large jumps" (corresponding to $|z|>1$) control the moments of L, in the sense that $\mathbb{E}|L(A)|^p<\infty$ if and only if $\int_{|z|>1}|z|^p\nu(dz)<\infty$, for any $p>0$. In the present paper, we are interested in the case when $\int_{|z|>1}|z|^2\nu(dz)$ may be infinite, and $L(A)$ may have infinite variance, as it happens in the α-stable case. (The case of finite variance Lévy noise is interesting too, since one can develop Malliavin calculus techniques similar to the Gaussian case; see Nualart and Nualart (2018); Di Nunno et al. (2009) for a very readable introduction to this subject.

Consider now the following stochastic heat equation with noise L:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{L}(t,x), & t>0, x \in \mathbb{R}^d \quad (d \geq 1), \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

The solution of this equation is a predictable process which satisfies the integral equation:

$$u(t,x) = (g_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} g_{t-s}(x-y)\sigma(u(s,y))L(ds,dy),$$

where $g_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$ is the heat kernel, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. We say that a random field $\{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is *predictable* if it is measurable with respect to the σ-field $\mathcal{P} = \mathcal{P}_0 \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}_0$ is the predictable σ-field on $\Omega \times \mathbb{R}_+$.

In Saint Loubert Bié (1998), it was proved that if the measure ν satisfies

$$\int_{\mathbb{R}} |z|^p \nu(dz) < \infty \quad \text{for some} \quad p \in [1,2], \tag{1.4}$$

and

$$p < 1 + \frac{2}{d} \tag{1.5}$$

then equation (1.3) has a unique solution which satisfies:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|u(t,x)|^p < \infty.$$

Condition (1.5) comes from the requirement $\int_0^t \int_{\mathbb{R}^d} g_{t-s}^p(x-y)dyds < \infty$ and forces $p<2$ if $d \geq 2$. A more severe restriction is (1.4), since it excludes the α-stable Lévy noise. This has been an open problem for some time, which was partially solved in Balan (2014); Chong (2017a) by replacing $\mathbb{R}^d$ in (1.3) by a bounded domain $D \subset \mathbb{R}^d$. Previous investigations related to this problem can be found for instance in Albeverio et al. (1998); Applebaum and Wu (2000); Mueller (1998); Mytnik (2002), the last two references dealing with α-stable Lévy noise and a non-Lipschitz function σ.

A major breakthrough was made in Chong (2017b), where it was showed that equation (1.3) has a solution, if there exist some exponents $p>0, q>0$ such that p satisfies (1.5), $p/(2+2/d-p) < q \leq p$, and

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < \infty \quad \text{and} \quad \int_{\{|z|>1\}} |z|^q \nu(dz) < \infty.$$

This condition holds for the α-stable Lévy noise with $\alpha < 1 + 2/d$. If $p<1$, it is assumed in addition that $b = \int_{\{|z|\leq 1\}} z\nu(dz)$, a condition which holds for the symmetric α-stable Lévy noise. The novel ideas of Chong (2017b) are to use different exponents p and q for $|z| \leq 1$ and $|z|>1$, and a spatially-dependent truncation function $h(x)$. The constant truncation function $h(x)=1$ used in Balan (2014) for the α-stable Lévy noise is problematic for equations on the entire domain $\mathbb{R}^d$. Unlike the Gaussian case, the solution constructed in Chong (2017b) is not obtained as the limit of the sequence of Picard's iterations, being defined as $u(t,x) = u_N(t,x)$ if $t \leq \tau_N$, where u_N is the solution of equation (1.3) in which L is replaced by the truncated noise:

$$\begin{aligned} L_N(A) &= b|A| + \int_{A\times\{|z|\leq 1\}} z\widetilde{J}(dt,dx,dz) + \int_{A\times\{1<|z|\leq Nh(x)\}} zJ(dt,dx,dz) \\ &=: b|A| + L^M(A) + L_N^P(A) \end{aligned} \tag{1.6}$$

where the indices M and P come from "martingale", respectively "compound-Poisson". The stopping times τ_N are given by:

$$\tau_N = \inf\left\{T>0; \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>Nh(x)\}} J(dt,dx,dz) > 0\right\},$$

where $h(x) = 1 + |x|^{\eta}$ for some $\frac{d}{q} < \eta < \frac{2-d(p-1)}{p-q}$. Since $u_N(t,x) = u_{N+1}(t,x)$ a.s. on the event $\{t \leq \tau_N\}$, $u(t,x)$ is well-defined. In addition, it was shown in Chong (2017b) that $\mathbb{E}|u_N(t,x)|^p$ is finite and uniformly bounded for (t,x) in a compact set. But the solution may not be unique. The asymptotic behaviour of the moments of $u(t,x)$ has been studied in the subsequent paper Chong and Kevei (2019). The path properties of the solution have been studied in Chong et al. (2019). The recent preprint Berger et al. (2021) focuses on the case $\sigma(u) = \beta u$, $\beta > 0$ and $\nu(-\infty, 0) = 0$, and establishes uniqueness of the solution, together with deep intermittency-type properties.

In this article, we analyze the stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{L}(t,x), & t > 0, x \in \mathbb{R}^d \quad (d \leq 2), \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.7}$$

where σ is a Lipschitz function on $\mathbb{R}$, u_0 and v_0 are deterministic functions on $\mathbb{R}^d$, and L is a Lévy space-time white noise given by (1.1). The reason we restrict ourselves to the case $d \leq 2$ is that in dimensions $d \geq 3$, the fundamental solution G_t of the wave equation is not a function (it is a measure if $d = 3$ and a distribution if $d \geq 4$).

A predictable random field $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a **solution** of (1.7) if it satisfies the integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))L(ds,dy).$$

The stochastic integral on the right-hand side of this equation is defined as in Chong (2017b), using the concept of *Daniell mean*. We refer the reader to the appendix for the definition of this integral.

Here G is the fundamental solution of the wave equation on $\mathbb{R}_+ \times \mathbb{R}^d$: for any $t > 0$

$$G_t(x) = \begin{cases} \dfrac{1}{2} 1_{\{|x|<t\}} & \text{if } d = 1 \\ \dfrac{1}{2\pi} \dfrac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}} & \text{if } d = 2 \end{cases} \tag{1.8}$$

and w is the solution of the wave equation $\frac{\partial u}{\partial t} - \Delta u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with the same initial conditions as (1.7), given by:

$$w(t,x) = (G_t * v_0)(x) + \frac{\partial}{\partial t}(G_t * u_0)(x).$$

We assume that the functions u_0 and v_0 satisfy the following conditions:

- $(d = 1)$ u_0 is bounded and continuous, and v_0 is bounded and measurable
- $(d = 2)$ $u_0 \in C^1(\mathbb{R}^2)$ and there exists $q_0 \in (2, \infty]$ such that $u_0, \nabla u_0, v_0 \in L^{q_0}(\mathbb{R}^2)$.

Under these conditions, w is jointly continuous and $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |w(t,x)| < \infty$ (see for instance, Lemma 4.2 of Dalang and Quer-Sardanyons, 2011).

Our first goal is to prove that the solution to equation (1.7) exists. As far as we know, this problem has not been studied in the literature before. In dimension $d = 2$, one difficulty is the fact that the fundamental solution $G_t(x)$ of the wave operator has singularites on the boundary of the set $|x| < t$, which lead to lengthy calculations (see for instance Dalang and Frangos, 1998; Millet and Sanz-Solé, 1999 for the case of Gaussian noise). Another problem is the fact that $(G_t)_{t>0}$ does not have the semigroup property. Luckily, some extremely useful (and highly non-trivial) properties of G have been recently obtained by Bolaños Guerrero et al. (2021), which lead to very impressive results about the asymptotic behaviour of the spatial average of the solution of the wave equation with spatially-homogeneous Gaussian noise. These properties will play an important role in this article, for the proof of the existence of the solution. More precisely, the proof of Theorem 2.6 below involves working with convolutions of the form $G_t^p * G_s^p$; see relation (2.18) below. In the case of the heat equation, these are dealt with in Chong (2017b) using properties of the normal distribution, which reduce essentially to the semigroup property of the heat kernel $g_t(x)$, since $g_t^p(x) = (2\pi t)^{d(1-p)/2} g_t(x)$. In the case of the wave equation, it is not obvious how to work with such convolutions, especially in the case $d = 2$. If $d = 1$, the problem is not so difficult since $(G_t * G_s)/(ts)$ is the law of the sum of two independent random variables with uniform distributions on $(-t, t)$, respectively $(-s, s)$.

Our second goal is to show that the solution of equation (1.7) has a càdlàg modification, when viewed as a process with values in a suitable fractional Sobolev space. A similar phenomenon has been studied in Chong et al. (2019) for the heat equation. The starting point of this analysis is a quick look at the behavior of G_t when $t = 0$. In the case of the heat equation, $g_0(x) := \lim_{t\to 0} g_t(x)$ is 0 if $x \neq 0$ and ∞ if $x = 0$, in any dimension d; hence, we can say that $g_0 = \delta_0$ (the Dirac delta distribution at 0). In the case of the wave equation, the situation is different in the case $d = 2$ compared with $d = 1$: $G_0 = \delta_0$ if $d = 2$, and $G_0 = 2^{-1} 1_{\{0\}}$ if $d = 1$ ($1_{\{0\}}$ is indicator function of $\{0\}$). Since the atoms (T_i, X_i, Z_i) of J contribute a value $G_{t-T_i}(x - X_i)Z_i$ to the solution $u(t, x)$, the function $u(t, \cdot)$ will likely live in the same function space as $G_{t-T_i}(\cdot - X_i)$. Moreover, the regularity of the path $t \mapsto u(t, \cdot)$ is related to regularity of $t \mapsto G_{t-T_i}(\cdot - X_i)$. In Section 3 below, we will show that the solution to equation (1.7) has a càdlàg modification with values in the fractional Sobolev space $H^r_{\text{loc}}(\mathbb{R}^d)$, for any $r < -1$ if $d = 2$, respectively for any $r < 1/4$ if $d = 1$.

The article is organized in two parts: in Section 2 we show the existence of the solution to equation (1.7), and in Section 3 we study the path properties of this solution. We recall that the $L^p(\Omega)$-norm is defined by:

$$\|X\|_p = (\mathbb{E}|X|^p)^{1/p} \quad \text{if } p \geq 1 \quad \text{and} \quad \|X\|_p = \mathbb{E}|X|^p \quad \text{if } p \in (0, 1).$$

2. Existence of solution

In this section, we prove the existence of solution to equation (1.7). We proceed as in Chong (2017b) in the case of the heat equation. Let L_N be the truncated noise given by (1.6), with $h(x) = 1 + |x|^\eta$ for some $\eta > 0$. We assume that ν satisfies the following assumption:

Assumption A. There exist $0 < q \leq p \leq 2$ such that

$$\gamma_1 = \int_{\{|z|\leq 1\}} |z|^p \nu(dz) < \infty \quad \text{and} \quad \gamma_2 = \int_{\{|z|>1\}} |z|^q \nu(dz) < \infty. \tag{2.1}$$

If $p < 1$, assume that $b = \int_{\{|z|\leq 1\}} z\nu(dz)$.

Remark 2.1. *(i)* In addition to (2.1), we will need that $\int_0^t \int_{\mathbb{R}^d} G^p_{t-s}(x - y)dyds < \infty$. If $d = 1$, this imposes no restrictions on p, so we can take $p = 2$. But if $d = 2$, we encounter the restriction $p < 2$, which is the same as condition (1.5) that is needed for the heat equation in dimension $d = 2$.

(ii) We will see below that the value p from Assumption A plays an important role in the analysis of the solution u_N of the wave equation with truncated noise L_N: $u_N(t, x)$ has a finite p-th moment! Since in dimension $d = 1$, we can take $p = 2$, this means that $u_N(t, x)$ has finite second moment.

(iii) If L is an α-stable Lévy noise, condition (2.1) holds for any $q < \alpha < p$. So in dimension $d = 1$, the solution of the equation with truncated noise has finite second moment, although the noise itself does not have this property. This shows the drastic impact of the truncation of the noise on the behaviour of the solution.

(iv) Unlike the case of the heat equation, we do not require a lower bound on the exponent q in (2.1). In Chong (2017b), the fact that the upper bound given by (3.13) has to be summable in n, combined with the restriction $\eta > d/q$ (from Lemma 3.2 ibid.) yields the condition $q > \frac{dp}{d+2-d(p-1)}$. For the wave equation, we obtain a different upper bound, which is summable regardless of the value of q (see the proof of Theorem 2.6 below).

(v) In the recent preprint Berger et al. (2021), the authors have obtained the existence and uniqueness of solution to the heat equation (1.3) with $\sigma(u) = u$ (known as the parabolic Anderson model) under different integrability conditions on the small jumps and the large jumps than (2.1). The methods of Berger et al. (2021) rely on the special form of the heat kernel. Finding similar methods that would show the existence and uniqueness of solution to the wave equation (1.7) with $\sigma(u) = u$ (known as the hyperbolic Anderson model) remains an open problem.

When Assumption A holds with $p < 1$, the truncated noise L^N has no drift:

$$\begin{aligned} L_N(A) &= \int_{A\times\{|z|\leq 1\}} zJ(dt,dx,dz) + \int_{A\times\{1<|z|\leq Nh(x)\}} zJ(dt,dx,dz) \\ &=: L^Q(A) + L_N^P(A). \end{aligned} \tag{2.2}$$

The following lemma is an important tool for controlling the p-th moments of stochastic convolutions of G with the truncated noise L_N, and is a reformulation of Lemma 3.3 of Chong (2017b). We believe that the condition $p \leq 2$ is needed for this result, since its proof relies essentially on the maximal inequality given by Theorem B.3. This condition is missing in Chong (2017b). Note that this result was stated in Chong (2017b) for the fundamental solution of the heat equation, but it remains valid for general functions.

Lemma 2.2 (Lemma 3.3 of Chong, 2017b)**.** *Suppose that Assumption A holds. Let G be a non-negative function such that $\int_0^T \int_{\mathbb{R}^d} \big(G_t^p(x) + G_t(x)\big)dxdt < \infty$. For any predictable processes X, X_1 and X_2 and for any $t \in [0,T]$ and $x \in \mathbb{R}^d$,*

$$\mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)X(s,y)L_N(ds,dy)\right|^p \leq$$
$$C_T \int_0^t \int_{\mathbb{R}^d} \big(G_{t-s}^p(x-y) + G_{t-s}(x-y)1_{\{p\geq 1\}}\big)\big(1 + \mathbb{E}|X(s,y)|^p\big)h(y)^{p-q}dyds$$

and

$$\mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)X_1(s,y)L_N(ds,dy) - \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)X_2(s,y)L_N(ds,dy)\right|^p \leq$$
$$C_T \int_0^t \int_{\mathbb{R}^d} \big(G_{t-s}^p(x-y) + G_{t-s}(x-y)1_{\{p\geq 1\}}\big)\mathbb{E}|X_1(s,y) - X_2(s,y)|^p h(y)^{p-q}dyds,$$

where C_T is a constant that depends on T (and also on $K, p, q, \gamma_1, \gamma_2$, but not on h), and p, q are the constants from Assumption A.

We will need several properties of G. In both cases $d = 1$ and $d = 2$, $\int_{\mathbb{R}^d} G_t(x)dx = t$, and the Fourier transform of G_t is:

$$\mathcal{F}G_t(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}G_t(x)dx = \frac{\sin(t|\xi|)}{|\xi|}, \quad \text{for all } \xi \in \mathbb{R}^d, t > 0. \tag{2.3}$$

Note that

$$\int_{\mathbb{R}^d} G_t^p(x)dx = \begin{cases} 2^{1-p}t & \text{for any } p > 0, \text{ if } d = 1 \\ \dfrac{(2\pi)^{1-p}}{2-p}t^{2-p} & \text{for any } p \in (0,2), \text{ if } d = 2 \end{cases} \tag{2.4}$$

If $d = 1$, $\int_{\mathbb{R}} G^p(t,x)|x|^\gamma dx = \frac{2^{1-p}}{\gamma+1}t^{\gamma+1}$ for any $p > 0$ and $\gamma > -1$. If $d = 2$,

$$\int_{\mathbb{R}^2} G^p(t,x)|x|^\gamma dx \leq \frac{(2\pi)^{1-p}}{2-p}t^{2-p+\gamma} \quad \text{for any } p \in (0,2) \text{ and } \gamma > 0, \tag{2.5}$$

and

$$G_t^p(x) \leq (2\pi t)^{q-p} G_t^q(x) \quad \text{for any } p<q. \tag{2.6}$$

In the case of the wave equation, $(G_t)_{t\geq 0}$ does not have the semigroup property. Fortunately, the recent article Bolaños Guerrero et al. (2021) contains some very useful properties of G, from which one can deduce a sub-semigroup-type property of $(G_t)_{t\geq 0}$. When $d=1$, it is not difficult to see that for any $r<s<t$ and $x\in\mathbb{R}$,

$$(G_{t-s} * G_{s-r})(x) \leq \frac{1}{2}(t-r)G_{t-r}(x). \tag{2.7}$$

This is relation (2.6) of Nualart and Zheng (2022). For $d=2$, we have the following highly non-trivial result.

Lemma 2.3 (Lemma 4.3 of Bolaños Guerrero et al., 2021)**.** *Assume that $d=2$. Let $q\in(\frac{1}{2},1)$ and $\delta\in[1,1/q]$ be arbitrary. Then, for any $0<r<t$ and $x\in\mathbb{R}^2$,*

$$\int_r^t (G_{t-s}^{2q} * G_{s-r}^{2q})^{\delta}(x)ds \leq A_q (t-r)^{1-\delta(2q-1)} G_{t-r}^{\delta(2q-1)}(x), \tag{2.8}$$

where $A_q>0$ is a constant depending on q.

Using (2.8) and Hölder's inequality, we obtain that for any $q\in(\frac{1}{2},1)$ and $p<2q$,

$$\int_r^t (t-s)^{2q-p}(s-r)^{2q-p}(G_{t-s}^{2q} * G_{s-r}^{2q})(x)ds \leq C_{p,q}(t-r)^{2(q-p+1)} G_{t-r}^{2q-1}(x), \tag{2.9}$$

where $C_{p,q}>0$ is a constant depending on p and q.

We will need the following result, which existed in the first version of Bolaños Guerrero et al. (2021) posted on arXiv (Lemma 4.3 in preprint arXiv:2003.10346v1), but was removed from the final version of Bolaños Guerrero et al. (2021). The authors have confirmed in personal communication that this result is correct. We list it here with a reference to Nualart and Zheng (2022), where it was used too.

Lemma 2.4 (Lemma 2.5.(b) of Nualart and Zheng, 2022)**.** *Assume that $d=2$. Let $q\in(\frac{1}{2},1)$ and $p\in(0,1)$ be such that $p+2q\leq 3$. For any $0<r<t$ and $x\in\mathbb{R}^2$,*

$$\int_r^t (G_{t-s}^{2q} * G_{s-r}^{p})(x)ds \leq B_{p,q}(t-r)^{3-p-2q} 1_{\{|x|<t-r\}},$$

where $B_{p,q}>0$ is a constant depending on p and q.

We will need also the following elementary result about a multiple beta-type integral.

Lemma 2.5. *For any $\beta_1>-1,\ldots,\beta_n>-1$,*

$$\int_{T_n(t)} \prod_{j=1}^n (t_{j+1}-t_j)^{\beta_j} dt_1\ldots dt_n = \frac{\prod_{j=1}^n \Gamma(\beta_j+1)}{\Gamma(\sum_{j=1}^n \beta_j+n+1)} t^{\sum_{j=1}^n \beta_j+n},$$

where $T_n(t)=\{(t_1,\ldots,t_n)\in(0,t)^n; t_1<\ldots<t_n\}$ and $t_{n+1}=t$.

We consider the equation with truncated noise L_N:

$$\begin{cases} \dfrac{\partial^2 u_N}{\partial t^2}(t,x) = \Delta u_N(t,x) + \sigma(u_N(t,x))\dot{L}_N(t,x) & t>0, x\in\mathbb{R}^d \quad (d\leq 2) \\ u_N(0,x)=u_0(x), \quad \dfrac{\partial u_N}{\partial t}(0,x)=v_0(x) & x\in\mathbb{R}^d \end{cases} \tag{2.10}$$

A predictable process u_N is a **solution** of (2.10) if it satisfies the integral equation:

$$u_N(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_N(s,y))L_N(ds,dy). \tag{2.11}$$

We say that two random fields $\{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ and $\{Y(t,x); t \geq 0, x \in \mathbb{R}^d\}$ are *modifications* of each other if $\mathbb{P}(X(t,x) = Y(t,x)) = 1$ for almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

We are now ready to prove the existence of solution for the equation with truncated noise. Note that unlike the case of the heat equation considered in Chong (2017b), we do not need to impose any additional conditions on p, q and η, for a *fixed* truncation level N. The additional condition $\eta > d/q$ which is imposed in the proof of Theorem 2.7 below guarantees that we can paste together the solutions u_N for different truncation levels N, to produce a solution for the equation with non-truncated noise.

Theorem 2.6. *(i) If $d = 1$, suppose that there exists $0 < q \leq 2$ such that $\int_{|z|>1} |z|^q \nu(dz) < \infty$. Then equation (2.10) has a solution $u_N = \{u_N(t,x); t \geq 0, x \in \mathbb{R}\}$. Moreover,*

$$\sup_{t \in [0,T]} \sup_{|x| \leq R} \mathbb{E}|u_N(t,x)|^2 < \infty, \tag{2.12}$$

for any $T > 0$ and $R > 0$.

(ii) If $d = 2$, suppose that Assumption A holds with $p < 2$. Then equation (2.10) has a unique solution $u_N = \{u_N(t,x); t \geq 0, x \in \mathbb{R}^d\}$. Moreover,

$$\sup_{t \in [0,T]} \sup_{|x| \leq R} \mathbb{E}|u_N(t,x)|^p < \infty, \tag{2.13}$$

for any $T > 0$ and $R > 0$, where p is the constant from Assumption A.

Proof: To provide a unified argument for both cases, we let $p = 2$ when $d = 1$. We consider the sequence $\{u_N^{(n)}(t,x)\}_{n \geq 0}$ of Picard's iterations (specific to the truncation level N), defined by $u_N^{(0)}(t,x) = w(t,x)$,

$$u_N^{(n+1)}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_N^{(n)}(s,y)) L_N(ds,dy) \quad n \geq 0. \tag{2.14}$$

By Lemma 6.2 of Chong (2017a), $u_N^{(n)}$ has a predictable modification. We work with this modification when defining $u_N^{(n+1)}$. By induction on n, it can be proved that for any $t > 0$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}|u_N^{(n)}(t,x)|^p \leq C_{n,t}(1 + |x|^{n\eta(p-q)}),$$

where $C_{n,t}$ is a positive constant that depends on n, t and is increasing in t. For this, we use Lemma 2.2 and the fact that for any $\gamma > 0$,

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y)(1+|y|^\gamma) dy ds \leq C_{\gamma,p,t}(1+|x|^\gamma)$$

where $C_{\gamma,p,t}$ is a positive constant that depends on γ, p, t and is increasing in t. Hence, for any $t > 0$ and $x \in \mathbb{R}^d$, $u_N^{(n)}(t,x)$ is finite a.s.

It suffices to prove that:

$$\sum_{n \geq 1} \sup_{t \in [0,T]} \sup_{|x| \leq R} \|u_N^{(n)}(t,x) - u_N^{(n-1)}(t,x)\|_p < \infty. \tag{2.15}$$

This will imply that $\{u_N^{(n)}(t,x)\}_{n \geq 0}$ is a Cauchy sequence in $L^p(\Omega)$, uniformly in $(t,x) \in [0,T] \times \{x \in \mathbb{R}^d; |x| \leq R\}$. We denote its limit by $u_N(t,x)$, that is:

$$\sup_{(t,x) \in [0,T]} \sup_{|x| \leq R} \mathbb{E}|u_N^{(n)}(t,x) - u_N(t,x)|^p \to 0 \quad \text{as} \quad n \to \infty. \tag{2.16}$$

Once (2.15) is shown, the rest of the proof will be the same as for the heat equation (Theorem 3.1 of Chong, 2017b). But there is a delicate part regarding the existence of a predictable modification of u_N. We include this argument here since it is not given in Chong (2017b).

We fix (t,x) and let $V_N^{(n)}(s,y)=G_{t-s}(x-y)u_N^{(n)}(s,y)h(y)^{\frac{p-q}{p}}$ for any $(s,y)\in(0,t)\times\mathbb{R}^d$. Then $V_N^{(n)}$ is predictable, since $u_N^{(n)}$ is so. The proof of (2.15) given below shows that

$$\sum_{n\geq 1}\|V_N^{(n)}-V_N^{(n-1)}\|_{L^p(\Omega\times(0,t)\times\mathbb{R}^d)}<\infty.$$

Hence, $\{V_N^{(n)}\}_{n\geq 0}$ is a Cauchy sequence in $L^p(\Omega\times(0,t)\times\mathbb{R}^d)$. We denote by V_N its limit in this space. So, there exists a subsequence $N'\subset\mathbb{N}$ such that $V_N^{(n)}(\omega,s,y)\to V_N(\omega,s,y)$ as $n\to\infty$, for almost all (ω,s,y), i.e. for $(\omega,s,y)\in B^c$, for some set $B\subset\Omega\times(0,t)\times\mathbb{R}^d$ with $(\mathbb{P}\times\text{Leb}\times\text{Leb})(B)=0$. Therefore, V_N is predictable. Due to the form of $V_N^{(n)}(s,y)$, for any $(\omega,s,y)\in B^c$ such that $|x-y|<t-s$, the sequence $\{u_N^{(n)}(\omega,s,y)\}_{n\geq 0}$ converges to some $u_N'(\omega,s,y)$ as $n\to\infty$, and $V_N(\omega,s,y)=G_{t-s}(x-y)u_N'(\omega,s,y)h(y)^{\frac{p-q}{p}}$. Hence, u_N' is predictable. An argument based on Fubini's theorem, (2.16) and the uniqueness of the limit shows that u_N' is a modification of u_N.

Finally, to prove that u_N' verifies the integral equation (2.11), we let $n\to\infty$ in (2.14). We let $v_N^{(n)}(s,y)=G_{t-s}(x-y)\sigma(u_N^{(n)}(s,y))$ and $v_N(s,y)=G_{t-s}(x-y)\sigma(u_N'(s,y))$. Then $\{v_N^{(n)}\}_{n\geq 0}$ converges to v_N as $n\to\infty$ in the Daniell mean $\|\cdot\|_{L_N,p}$, which implies the convergence of the stochastic integrals of these processes with respect to L_N.

We proceed now with the proof of (2.15). Here C is a constant which depends on N,p,q,η, and may be different from line to line. We consider separately the cases $p<1$ and $p\geq 1$.

Case 1. $p<1$. By Lemma 2.2 and the Lipschitz property of σ,

$$\mathbb{E}|u_N^{(n+1)}(t,x)-u_N^{(n)}(t,x)|^p\leq C\int_0^t\int_{\mathbb{R}^d}G_{t-s}^p(x-y)\mathbb{E}|u_N^{(n)}(s,y)-u_N^{(n-1)}(s,y)|^p h(y)^{p-q}dyds.$$

Iterating this inequality, we obtain:

$$\mathbb{E}|u_N^{(n)}(t,x)-u_N^{(n-1)}(t,x)|^p\leq C^n\int_{T_n(t)}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n G_{t_{i+1}-t_i}^p(x_{i+1}-x_i)\prod_{i=1}^n h(x_i)^{p-q}d\mathbf{x}d\mathbf{t},$$

where $\mathbf{t}=(t_1,\ldots,t_n)$, $\mathbf{x}=(x_1,\ldots,x_n)$ and we use the convention $t_{n+1}=t$ and $x_{n+1}=x$.

We use following (generalized) Hölder's inequality: for any non-negative integrable function f on $(\mathbb{R}^d)^n$, and non-negative functions $g_1,\ldots,g_n$ on $\mathbb{R}^d$,

$$\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n g_i(x_i)f(x_1,\ldots,x_n)d\mathbf{x}\leq\prod_{i=1}^n\left(\int_{(\mathbb{R}^d)^n}g_i^n(x_i)f(x_1,\ldots,x_n)d\mathbf{x}\right)^{1/n}.$$

We apply this to $f(x_1,\ldots,x_n)=\prod_{i=1}^n G_{t_{i+1}-t_i}^p(x_{i+1}-x_i)$ and $g_i=h^{p-q}$ for all $i\leq n$:

$$\mathbb{E}|u_N^{(n)}(t,x)-u_N^{(n-1)}(t,x)|^p\leq C^n\int_{T_n(t)}\prod_{i=1}^n\left(\int_{(\mathbb{R}^d)^n}\prod_{j=1}^n G_{t_{j+1}-t_j}^p(x_{j+1}-x_j)h(x_i)^{n(p-q)}d\mathbf{x}\right)^{1/n}d\mathbf{t}$$

$$=C^n\int_{T_n(t)}\prod_{i=1}^n\left(\int_{(\mathbb{R}^d)^n}\prod_{j=1}^n G_{t_{j+1}-t_j}^p(x_j)h(x-\sum_{j=i}^n x_j)^{n(p-q)}d\mathbf{x}\right)^{1/n}d\mathbf{t}$$

$$\leq C^n\frac{1}{n}\sum_{i=1}^n\int_{T_n(t)}\int_{(\mathbb{R}^d)^n}\prod_{j=1}^n G_{t_{j+1}-t_j}^p(x_j)h(x-\sum_{j=i}^n x_j)^{n(p-q)}d\mathbf{x}d\mathbf{t},$$

where for the last line we used the fact that the geometric mean is smaller than the arithmetic mean. We now use the special form of h:

$$h(x)^{n(p-q)}=(1+|x|^{\eta})^{n(p-q)}\leq 2^{n(p-q)}(1+|x|^{n\gamma})\quad\text{for all }x\in\mathbb{R}^d,$$

where $\gamma=\eta(p-q)$. Hence, for any $x\in\mathbb{R}^d$ with $|x|\leq R$, we have:

$$h(x-\sum_{j=i}^{n}x_j)^{n(p-q)}\leq 2^{n(p-q+\gamma)}(1+R^{n\gamma}+|\sum_{j=i}^{n}x_j|^{n\gamma}).$$

It follows that

$$\mathbb{E}|u_N^{(n)}(t,x)-u_N^{(n-1)}(t,x)|^p\leq C^n\Big\{\big(1+R^{n\gamma}\big)A^{(n)}(t)+B^{(n)}(t)\Big\}, \tag{2.17}$$

where

$$A^{(n)}(t)=\int_{T_n(t)}\int_{(\mathbb{R}^d)^n}\prod_{j=1}^{n}G^p_{t_{j+1}-t_j}(x_j)d\mathbf{x}d\mathbf{t}$$

and $B^{(n)}(t)=\frac{1}{n}\sum_{i=1}^{n}B_i^{(n)}(t)$, with

$$B_i^{(n)}(t)=\int_{T_n(t)}\int_{(\mathbb{R}^d)^n}\prod_{j=1}^{n}G^p_{t_{j+1}-t_j}(x_j)|\sum_{j=i}^{n}x_j|^{n\gamma}d\mathbf{x}d\mathbf{t}. \tag{2.18}$$

Using (2.4) and Lemma 2.5, we see that:

$$A^{(n)}(t)=C^n\int_{T_n(t)}\prod_{j=1}^{n}(t_{j+1}-t_j)^a d\mathbf{t}=C^n\frac{t^{(a+1)n}}{\Gamma((a+1)n+1)}\quad \text{with } a=\begin{cases}1 & \text{if } d=1\\ 2-p & \text{if } d=2\end{cases}$$

Hence

$$\sum_{n\geq 1}C^n\sup_{t\leq T}A^{(n)}(t)<\infty. \tag{2.19}$$

It remains to show that a similar relation holds for $B^{(n)}(t)$. This involves a study of convolutions of the form $G_t^p * G_s^p$. We treat separately the cases $d=1$ and $d=2$.

In the estimates below, we will use the convention $\prod_{\emptyset}=1$.

a) Case $d=1$. Using the fact that $G_t^p(x)=2^{1-p}G_t(x)$ and $\int_{\mathbb{R}}G_t(x)dx=t$, we have

$$\begin{aligned}B_i^{(n)}(t)&=C^n\int_{T_n(t)}\int_{\mathbb{R}^n}\prod_{j=1}^{n}G_{t_{j+1}-t_j}(x_j)|\sum_{j=i}^{n}x_j|^{n\gamma}d\mathbf{x}d\mathbf{t}\\ &=C^n\int_{T_n(t)}\prod_{j=1}^{i-1}(t_{j+1}-t_j)\int_{\mathbb{R}^{n-i+1}}\prod_{j=i}^{n}G_{t_{j+1}-t_j}(x_j)|\sum_{j=i}^{n}x_j|^{n\gamma}dx_i\dots dx_n d\mathbf{t}.\end{aligned}$$

If $i=n$, there is no convolution involved, since

$$\begin{aligned}B_n^{(n)}(t)&=C^n\int_{T_n(t)}\prod_{j=1}^{n-1}(t_{j+1}-t_j)\int_{\mathbb{R}}G_{t-t_n}(x_n)|x_n|^{n\gamma}dx_n d\mathbf{t}\\ &=C^n\int_{T_n(t)}\prod_{j=1}^{n-1}(t_{j+1}-t_j)(t-t_n)^{n\gamma+1}d\mathbf{t}\leq\frac{C^n t^{n(\gamma+2)}}{n!}.\end{aligned}$$

If $i \leq n-1$, we use the change of variable $x_i \to z = \sum_{j=i}^n x_j$, followed by the semigroup-type property (2.7). We obtain:

$$B_i^{(n)}(t) = C^n \int_{T_n(t)} \prod_{j=1}^{i-1} (t_{j+1}-t_j) \int_{\mathbb{R}^{n-i+1}} \prod_{j=i+1}^{n} G_{t_{j+1}-t_j}(x_j) G_{t_{i+1}-t_i}(z - \sum_{j=i+1}^n x_j)|z|^{n\gamma} dz dx_{i+1}\dots dx_n d\mathbf{t}$$

$$= C^n \int_{\mathbb{R}} |z|^{n\gamma} \int_{T_n(t)} \prod_{j=1}^{i-1}(t_{j+1}-t_j) \int_{\mathbb{R}^{n-i-1}} \prod_{j=i+2}^{n} G_{t_{j+1}-t_j}(x_j) (G_{t_{i+2}-t_{i+1}} * G_{t_{i+1}-t_i})(z - \sum_{j=i+2}^n x_j) dx_{i+2}\dots dx_n d\mathbf{t} dz$$

$$\leq C^n \int_{\mathbb{R}} |z|^{n\gamma} \int_{T_n(t)} \prod_{j=1}^{i-1}(t_{j+1}-t_j)(t_{i+2}-t_i) \int_{\mathbb{R}^{n-i-2}} \prod_{j=i+3}^{n} G_{t_{j+1}-t_j}(x_j) (G_{t_{i+3}-t_{i+2}} * G_{t_{i+2}-t_i})(z - \sum_{j=i+3}^n x_j) dx_{i+3}\dots dx_n d\mathbf{t} dz.$$

We use again inequality (2.7). We continue in this manner. At the last step, we obtain:

$$B_i^{(n)}(t) \leq C^n \int_{\mathbb{R}} |z|^{n\gamma} \int_{T_n(t)} \prod_{j=1}^{i-1}(t_{j+1}-t_j)(t_{i+2}-t_i)\dots(t_n-t_i)(G_{t-t_n} * G_{t_n-t_i})(z) d\mathbf{t} dz$$

$$\leq C^n \int_{T_n(t)} \prod_{j=1}^{i-1}(t_{j+1}-t_j)(t_{i+2}-t_i)\dots(t_n-t_i)(t-t_i)\left(\int_{\mathbb{R}} |z|^{n\gamma} G_{t-t_i}(z)dz\right) d\mathbf{t}$$

$$= C^n \int_{T_n(t)} \prod_{j=1}^{i-1}(t_{j+1}-t_j)(t_{i+2}-t_i)\dots(t_n-t_i)(t-t_i)^{n\gamma+2} d\mathbf{t}$$

$$\leq C^n t^{n-i+n\gamma+1} \int_{T_n(t)} \prod_{j=1}^{i-1}(t_{j+1}-t_j) d\mathbf{t} = \frac{C^n t^{n(\gamma+2)}}{\Gamma(i+n)} \leq \frac{C^n t^{n(\gamma+2)}}{n!},$$

using Lemma 2.5 and the monotonicity of the Γ function. Hence

$$\sum_{n\geq 1} C^n \sup_{t\leq T} B^{(n)}(t) < \infty. \tag{2.20}$$

b) Case $d=2$. If $i=n$, using relations (2.4) and (2.5) and Lemma 2.5, we obtain:

$$B_n^{(n)}(t) = C^{n-1} \int_{T_n(t)} \prod_{j=1}^{n-1}(t_{j+1}-t_j)^{2-p} \int_{\mathbb{R}^2} G_{t-t_n}^p(x_n)|x_n|^{n\gamma} dx_n d\mathbf{t}$$

$$\leq C^n \int_{T_n(t)} \prod_{j=1}^{n-1}(t_{j+1}-t_j)^{2-p}(t-t_n)^{2-p+n\gamma} d\mathbf{t}$$

$$= C^n \frac{\Gamma(3-p)^{n-1}\Gamma(n\gamma+3-p)}{\Gamma(n(3-p+\gamma)+1)} t^{n(3-p+\gamma)} \leq C^n \frac{1}{(n!)^{3-p}} t^{n(3-p+\gamma)}.$$

(For the last inequality, we used Stirling's formula.) If $i=n-1$, then by (2.4), we have

$$B_{n-1}^{(n)}(t)=C^{n-2}\int_{T_n(t)}\prod_{j=1}^{n-2}(t_{j+1}-t_j)^{2-p}\int_{(\mathbb{R}^2)^2}G_{t-t_n}^p(x_n)G_{t_n-t_{n-1}}^p(x_{n-1})|x_{n-1}+x_n|^{n\gamma}dx_{n-1}dx_n d\mathbf{t}$$

$$=C^{n-2}\int_{\mathbb{R}^2}|z|^{n\gamma}\int_{T_n(t)}\prod_{j=1}^{n-2}(t_{j+1}-t_j)^{2-p}(G_{t-t_n}^p*G_{t_n-t_{n-1}}^p)(z)d\mathbf{t}dz.$$

Fix an arbitrary $r\in(\frac{1}{2},1)$. Since $p<1<2r$, we can pass from G^p to G^{2r}, using (2.6). So,

$$B_{n-1}^{(n)}(t)\leq C^n\int_{\mathbb{R}^2}|z|^{n\gamma}\int_{T_n(t)}\prod_{j=1}^{n-2}(t_{j+1}-t_j)^{2-p}(t-t_n)^{2r-p}(t_n-t_{n-1})^{2r-p}(G_{t-t_n}^{2r}*G_{t_n-t_{n-1}}^{2r})(z)d\mathbf{t}dz.$$

We now use estimate (2.9) for the dt_n integral on (t_{n-1},t), followed by (2.5). We obtain:

$$B_{n-1}^{(n)}(t)\leq C^{n+1}\int_{T_{n-1}(t)}\prod_{j=1}^{n-2}(t_{j+1}-t_j)^{2-p}(t-t_{n-1})^{2(r-p+1)}\left(\int_{\mathbb{R}^2}|z|^{n\gamma}G_{t-t_{n-1}}^{2r-1}(z)dz\right)dt_1\dots dt_{n-1}$$

$$\leq C^{n+2}\int_{T_{n-1}(t)}\prod_{j=1}^{n-2}(t_{j+1}-t_j)^{2-p}(t-t_{n-1})^{5-2p+n\gamma}dt_1\dots dt_{n-1}$$

$$=C^{n+2}\frac{\Gamma(3-p)^{n-2}\Gamma(n\gamma+6-2p)}{\Gamma(n(3-p+\gamma)+1)}t^{n(3-p+\gamma)}\leq C^n\frac{1}{(n!)^{3-p}}t^{n(3-p+\gamma)},$$

using Lemma 2.5 and Stirling's formula.

If $i\leq n-2$ and $n\geq 3$, we use (2.4) and the change of variable $x_i\mapsto z=\sum_{j=i}^n x_j$:

$$B_i^{(n)}(t)=C^{i-1}\int_{T_n(t)}\prod_{j=1}^{i-1}(t_{j+1}-t_j)^{2-p}\int_{(\mathbb{R}^2)^{n-i+1}}\prod_{j=i}^{n}G_{t_{j+1}-t_j}^p(x_j)|\sum_{j=i}^n x_j|^{n\gamma}dx_i\dots dx_n d\mathbf{t}$$

$$=C^{i-1}\int_{\mathbb{R}^2}|z|^{n\gamma}\int_{T_n(t)}\prod_{j=1}^{i-1}(t_{j+1}-t_j)^{2-p}\int_{(\mathbb{R}^2)^{n-i-1}}\prod_{j=i+2}^{n}G_{t_{j+1}-t_j}^p(x_j)$$

$$(G_{t_{i+2}-t_{i+1}}^p*G_{t_i-t_{i+1}}^p)(z-\sum_{j=i+2}^n x_j)dx_{i+2}\dots dx_n d\mathbf{t}dz.$$

In the convolution above, we pass from G^p to G^{2r}, using (2.6). We get:

$$B_i^{(n)}(t)\leq C^{i+1}\int_{\mathbb{R}^2}|z|^{n\gamma}\int_{\{0<t_1<\ldots<t_i<t_{i+2}<\ldots<t_n<t\}}\prod_{j=1}^{i-1}(t_{j+1}-t_j)^{2-p}\int_{(\mathbb{R}^2)^{n-i-1}}\prod_{j=i+2}^{n}G_{t_{j+1}-t_j}^p(x_j)$$

$$\left(\int_{t_i}^{t_{i+2}}(t_{i+2}-t_{i+1})^{2r-p}(t_{i+1}-t_i)^{2r-p}(G_{t_{i+2}-t_{i+1}}^{2r}*G_{t_i-t_{i+1}}^{2r})(z-\sum_{j=i+2}^n x_j)dt_{i+1}\right)$$

$$dx_{i+2}\dots dx_n dt_1\dots dt_i dt_{i+2}\dots dt_n dz.$$

We use estimate (2.9) for the dt_{i+1} integral on (t_i,t_{i+2}). We obtain:

$$B_i^{(n)}(t)\leq C^{i+2}\int_{\mathbb{R}^2}|z|^{n\gamma}\int_{\{0<t_1<\ldots<t_i<t_{i+2}<\ldots<t_n<t\}}\prod_{j=1}^{i-1}(t_{j+1}-t_j)^{2-p}(t_{i+2}-t_i)^{2(r-p+1)}$$

$$\int_{(\mathbb{R}^2)^{n-i-1}}\prod_{j=i+2}^{n}G_{t_{j+1}-t_j}^p(x_j)G_{t_{i+2}-t_i}^{2r-1}(z-\sum_{j=i+2}^n x_j)dx_{i+2}\dots dx_n dt_1\dots dt_i dt_{i+2}\dots dt_n dz.$$

We now pass from $G^p_{t_{i+3}-t_{i+2}}(x_{i+2})$ to $G^{2r}_{t_{i+3}-t_{i+2}}(x_{i+2})$, again using (2.6). Hence

$$B_i^{(n)}(t) \leq C^{i+3} \int_{\mathbb{R}^2} |z|^{n\gamma} \int_{\{0<t_1<\ldots<t_i<t_{i+2}<\ldots<t_n<t\}} \prod_{j=1}^{i-1} (t_{j+1}-t_j)^{2-p}(t_{i+2}-t_i)^{2(r-p+1)}$$

$$(t_{i+3}-t_{i+2})^{2r-p} \int_{(\mathbb{R}^2)^{n-i-2}} \prod_{j=i+3}^{n} G^p_{t_{j+1}-t_j}(x_j)(G^{2r}_{t_{i+3}-t_{i+2}} * G^{2r-1}_{t_{i+2}-t_i})(z-\sum_{j=i+3}^{n} x_j)$$
$$dx_{i+3}\ldots dx_n dt_1 \ldots dt_i dt_{i+2} \ldots dt_n dz$$

$$\leq C^{i+3} \int_{\mathbb{R}^2} |z|^{n\gamma} \int_{\{0<t_1<\ldots<t_i<t_{i+3}<\ldots<t_n<t\}} \prod_{j=1}^{i-1} (t_{j+1}-t_j)^{2-p}(t_{i+3}-t_i)^{4r-3p+2}$$

$$\int_{(\mathbb{R}^2)^{n-i-2}} \prod_{j=i+3}^{n} G^p_{t_{j+1}-t_j}(x_j) \left(\int_{t_i}^{t_{i+3}} (G^{2r}_{t_{i+3}-t_{i+2}} * G^{2r-1}_{t_{i+2}-t_i})(z-\sum_{j=i+3}^{n} x_j) dt_{i+2} \right)$$
$$dx_{i+3}\ldots dx_n dt_1 \ldots dt_i dt_{i+3} \ldots dt_n dz.$$

By Lemma 2.4, the dt_{i+2} integral is bounded by $C(t_{i+3}-t_i)^{4(1-r)} 1_{\{|z-\sum_{j=i+3}^n x_j|<t_{i+3}-t_i\}}$. Note that each term $G^p_{t_{j+1}-t_j}(x_j)$ contains the indicator $1_{\{|x_j|<t_{j+1}-t_j\}}$, and

$$\prod_{j=i+3}^{n} 1_{\{|x_j|<t_{j+1}-t_j\}} 1_{\{|z-\sum_{j=i+3}^n x_j|<t_{i+3}-t_i\}} \leq 1_{\{|z|<t-t_i\}}.$$

Using the fact that $\int_{\{|z|<t-t_i\}} |z|^{n\gamma} dz \leq C(t-t_i)^{n\gamma+2} \leq C t^{n\gamma+2}$, followed by (2.4) and Lemma 2.5, we obtain:

$$B_i^{(n)}(t) \leq C^{i+5} t^{n\gamma+2} \int_{\{0<t_1<\ldots<t_i<t_{i+3}<\ldots<t_n<t\}} \prod_{j=1}^{i-1} (t_{j+1}-t_j)^{2-p}(t_{i+3}-t_i)^{6-3p}$$

$$\int_{(\mathbb{R}^2)^{n-i-2}} \prod_{j=i+3}^{n} G^p_{t_{j+1}-t_j}(x_j) dx_{i+3}\ldots dx_n dt_1 \ldots dt_i dt_{i+3} \ldots dt_n$$

$$= C^{n+3} t^{n\gamma+2} \int_{\{0<t_1<\ldots<t_i<t_{i+3}<\ldots<t_n<t\}} \prod_{j=1}^{i-1} (t_{j+1}-t_j)^{2-p}(t_{i+3}-t_i)^{6-3p}$$

$$\prod_{j=i+3}^{n} (t_{j+1}-t_j)^{2-p} dt_1 \ldots dt_i dt_{i+3} \ldots dt_n$$

$$= C^{n+3} t^{n\gamma+2} \frac{\Gamma(3-p)^{n-3}\Gamma(7-3p)}{\Gamma(n(3-p)-2+1)} t^{n(3-p)-2} \leq C^n \frac{1}{(n!)^{3-p}} t^{n(3-p+\gamma)}.$$

For the last inequality, we used the fact that for any $a>0$ and $b \in \mathbb{R}$, $\Gamma(an+b+1) \geq C^n_{a,b}(n!)^a$, where $C_{a,b}>0$ is a constant depending on a and b. This is a consequence of Stilrling's formula and the fact that $\Gamma(x+b) \sim \Gamma(x)x^b$ as $x \to \infty$.

To summarize, we have proved that:

$$B_i^{(n)} \leq C^n \frac{1}{(n!)^{3-p}} t^{n(3-p+\gamma)} \quad \text{for all } i=1,\ldots,n.$$

This shows that relation (2.20) also holds in the case $d=2$.

Case 2. Assume that $p \geq 1$. In this case, we work with the function $G_{t,p}(x) = G^p_t(x) + G_t(x)$ instead of $G^p_t(x)$. Relation (2.17) holds using the same argument as in case $p<1$, but with constants

$A^{(n)}(t)$ and $B_i^{(n)}(t)$ given by:

$$A^{(n)}(t) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^n G_{t_{j+1}-t_j,p}(x_j) d\mathbf{x} d\mathbf{t}$$

$$B_i^{(n)}(t) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^n G_{t_{j+1}-t_j,p}(x_j) |\sum_{j=i}^n x_j|^{n\gamma} d\mathbf{x} d\mathbf{t}.$$

We treat separately the cases $d=1$ and $d=2$.

a) *Case* $d=1$. Since $G_t^p(x) = 2^{1-p} G_t(x)$, we can use the same argument as for $p<1$.

b) *Case* $d=2$. This case is treated similarly to the case $p<1$. We use the following properties. For any $t>0$, $\int_{\mathbb{R}^2} G_{t,p}(x)dx = c_p t^{2-p} + t$, with $c_p = \frac{(2\pi)^{1-p}}{2-p}$. For any $\gamma>0$,

$$\int_{\mathbb{R}^2} G_{t,p}(x)|x|^\gamma dx \leq c_{p,T}\, t^{2-p+\gamma} \quad \text{for all } t \leq T,$$

with $c_{p,T} = c_p + T^{p-1}$. For any $r \in (\frac{1}{2}, 1)$ with $p < 2r$, we have

$$G_{t,p}(x) \leq c_{r,p,T} t^{2r-p} G_t^{2r}(x) \quad \text{for any } t \leq T,$$

with $c_{r,p,T} = (2\pi)^{2r-1}(1+T^{p-1})$. □

The following result gives the existence of solution to equation (2.10).

Theorem 2.7. *Under the hypotheses of Theorem 2.6, equation (1.7) has a solution u. Moreover, there exists a sequence* $(\tau_N)_{N\geq 1}$ *of stopping times with* $\tau_N \uparrow \infty$ *a.s. such that*

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\big[|u(t,x)|^p 1_{\{t\leq \tau_N\}}\big] < \infty, \tag{2.21}$$

for any $T>0$, $R>0$ *and* $N\geq 1$, *where* $p=2$ *if* $d=1$ *and* p *is the constant from Assumption A if* $d=2$.

Proof: Let $h(x) = 1+|x|^\eta$, where $\eta > d/q$ and q is the constant from Assumption A. Let

$$\tau_N = \inf\{t>0; J([0,t]\times\{(x,z); |z|>Nh(x)\})>0\}, \tag{2.22}$$

Since the set $[0,t]\times\{(x,z);|z|>Nh(x)\}$ is unbounded, $J([0,t]\times\{(x,z);|z|>Nh(x)\})$ may be infinite, which would mean that τ_N is not well-defined. This delicate issue is addressed in the proof of Lemma 3.2 of Chong (2017b), where it is proved that for any $T>0$ and $N\geq 1$, with probability 1,

$$J \text{ has finitely many points in } [0,T]\times\{(x,z);|z|>Nh(x)\}. \tag{2.23}$$

Since the details of this argument are missing in Chong (2017b), we include them below. Let $V_0=\emptyset$ and $V_n = \{x\in\mathbb{R}^d; |x|\leq b_n\}$ be such that $\text{Leb}(V_n)=n$. Let $U_n = V_n \backslash V_{n-1}$ for any $n\geq 1$. Then $(U_n)_{n\geq 1}$ is a partition of $\mathbb{R}^d$ with $\text{Leb}(U_n)=1$. For fixed $T>0$ and $N\geq 1$, let

$$F_n = [0,T]\times\{(x,z)\in U_n\times\mathbb{R}; |z|>Nh(x)\}.$$

Under the condition $\eta > d/q$, it can be proved that $\sum_{n\geq 1} P(J(F_n)>0)<\infty$. By the Borel-Cantelli lemma, it follows that with probability 1, there exists $n_0\geq 1$ such that $J(F_n)=0$ for all $n\geq n_0$. This means that, with probability 1, J has no points in the set

$$\bigcup_{n\geq n_0} F_n = [0,T]\times\{(x,z); |x|>b_{n_0-1}, |z|>Nh(x)\}.$$

Since J has finitely many points in the set $[0,T]\times\{(x,z);|x|\leq b_{n_0-1}, |z|>Nh(x)\}$, relation (2.23) follows.

The argument above shows that with probability 1, for any $T \in \mathbb{Q}_+$, there exists N_0 large enough such that for all $N \geq N_0$, $J([0,T] \times \{(x,z); |z| > Nh(x)\}) = 0$ (and therefore $\tau_N > T$). This proves that $\tau_N \uparrow \infty$ a.s. when $N \to \infty$.

Let u_N be the process given by Theorem 2.6. As in the last part of the proof of Theorem 3.1 of Chong (2017b), it can be proved that for any $N \geq 1$, $u_N(t,x) = u_{N+1}(t,x)$ a.s. on $\{t \leq \tau_N\}$. This argument uses the fact that on the event $\{\tau_N > T\}$

$$L([0,t] \times A) = L_N([0,t] \times A) = L_{N'}([0,t] \times A)$$

for any $N' > N, t \in [0,T]$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$. Moreover, the process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ given by $u(t,x) = u_N(t,x)$ if $t \leq \tau_N$ is a solution to equation (1.7). □

It was suggested by the referee that the second condition in (2.1) (about the existence of q) and special form $h(x) = 1 + |x|^\eta$ may not be needed, and one can consider simply $h(x) = 1$. This would be right if the goal would be to prove the existence of solution for the equation with truncated noise. Indeed, this truncation procedure was used in Balan (2014) for the α-stable Lévy noise. The problem is to paste together these solutions to produce a solution to equation (1.7). We explain this below. Consider the truncated noise

$$\bar{L}_N(A) = b|A| + \int_{A \times \{|z| \leq 1\}} z\widetilde{J}(dt,dx,dz) + \int_{A \times \{1 < |z| \leq N\}} zJ(dt,dx,dz).$$

Lemma 2.2 remains valid with $h(x) = 1$, if $p \in (0,2]$ is a value such that $\int_{|z| \leq 1} |z|^p \nu(dz) < \infty$. We have the following result.

Theorem 2.8. *a) If $d = 1$, equation (2.10) with noise $\bar{L}_N$ instead of L_N has a unique solution $\bar{u}_N$, and this solution satisfies:*

$$\sup_{(t,x) \in [0,T]} \mathbb{E}|\bar{u}_N(t,x)|^2 < \infty.$$

b) If $d = 2$ and there exists $p \in (0,2)$ such that $\int_{|z| \leq 1} |z|^p \nu(dz) < \infty$, then equation (2.10) with noise $\bar{L}_N$ instead of L_N has a unique solution $\bar{u}_N$, and this solution satisfies:

$$\sup_{(t,x) \in [0,T]} \mathbb{E}|\bar{u}_N(t,x)|^p < \infty.$$

Proof: If $d = 1$, we let $p = 2$. In both cases, $\int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y)dyds < \infty$. Define the Picard iterations: $\bar{u}_N^{(0)}(t,x) = w(t,x)$ and

$$\bar{u}_N^{(n+1)}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(\bar{u}_N^{(n)}(s,y))\bar{L}_N(ds,dy) \quad n \geq 0.$$

Let $H_n(t) = \sup_{t \in [0,T]} \mathbb{E}|\bar{u}_N^{(n)}(t,x) - \bar{u}_N^{(n-1)}(t,x)|^p$. By Lemma 2.2 with $h(x) = 1$, we obtain:

$$H_{n+1}(t) \leq C \int_0^t H_n(s)J(t-s)ds$$

where $J(t-s) = \int_0^t \int_{\mathbb{R}^d} \Big(G_{t-s}^p(x-y) + G_{t-s}(x-y)1_{\{p \geq 1\}}\Big)dyds$. By Lemma 15 of Dalang (1999), $\sum_{n \geq 1} \sup_{t \leq T} H_n(t)^{1/p} < \infty$. (In fact, this lemma shows that $H_n(t) \leq Ca_n$, where $(a_n)_n$ satisfies $\sum_{n \geq 1} a_n^{1/b} < \infty$ for any $b > 0$.) It follows $\{\bar{u}_N^{(n)}(t,x)\}_{n \geq 1}$ is a Cauchy sequence in $L^p(\Omega)$, uniformly in $[0,T] \times \mathbb{R}^d$. We denote by $\bar{u}_N(t,x)$ its limit.

To show that $\bar{u}_N$ has a predictable modification, let $\bar{V}_N^{(n)}(s,y) = G_{t-s}(x-y)\bar{u}_N(s,y)$. If $p \in (0,1)$, then

$$\|\bar{V}_N^{(n)} - \bar{V}_N^{(n-1)}\|_{L^p(\Omega \times (0,t) \times \mathbb{R}^d)} = \int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y)\mathbb{E}|\bar{u}_N^{(n)}(s,y) - \bar{u}_N^{(n-1)}(s,y)|^p dyds \leq Ca_n,$$

and therefore

$$\sum_{n\geq 1} \|\bar{V}_N^{(n)} - \bar{V}_N^{(n-1)}\|_{L^p(\Omega\times(0,t)\times\mathbb{R}^d)} \leq C \sum_{n\geq 1} a_n < \infty.$$

(This relation also holds if $p \in [1,2]$, using the fact that $\sum_{n\geq 1} a_n^{1/p} < \infty$.) The existence of the predictable modification follows as in the proof of Theorem 2.6. This modification is the unique solution of equation (2.10) with L_N replaced by $\bar{L}_N$. □

Remark 2.9. The natural stopping time associated with the truncation $h(x) = 1$ is

$$\bar{\tau}_N = \inf\{t > 0; J([0,t] \times \mathbb{R}^d \times \{|z| > N\}) > 0\}$$

Unfortunately, unlike (2.23), we could not find an argument to show that J has finitely many points in the (unbounded) set $[0,t] \times \mathbb{R}^d \times \{|z| > N\}$. Hence, $\bar{\tau}_N$ may not be well-defined, as mentioned in Remark 27 of Balan (2014). Because of this, we could not proceed as in the last part of the proof of Theorem 2.7 to obtain the existence of a solution of equation (1.7), based on the solutions $\bar{u}_N$. This explains why we cannot drop the condition about q in (2.1): the existence of q allows us to choose a suitable η such that (30) holds, and hence τ_N is well-defined.

3. Path properties of the solution

In this section, we fix $T > 0$ and we study the path properties of the solution u to equation (1.7) on the interval $[0,T]$. More precisely, we will show that the process $\{u(t,\cdot)\}_{t\in[0,T]}$ has a modification which is càdlàg (i.e. right-continuous with left limits) in a suitable space.

We say that two processes $\{X(t)\}_{t\in[0,T]}$ and $\{Y(t)\}_{t\in[0,T]}$ (defined on the same probability space) are *modifications* of each other if $\mathbb{P}(X(t) = Y(t)) = 1$ for almost all $t \in [0,T]$.

Let $h(x) = 1 + |x|^\eta$ for some $\eta > d/q$, and

$$\tau_N = \inf\{t \in [0,T]; J([0,t] \times \{(x,z); |z| > Nh(x)\}) > 0\}. \tag{3.1}$$

Compared with (2.22), the infimum is now taken over $[0,T]$. It follows that with probability 1, for N large enough, $\tau_N = \infty$ and $u(t,x) = u_N(t,x)$ for all $t \in [0,T]$. So, it suffices to study the path properties of u_N for fixed $N \geq 1$.

We consider the fractional Sobolev space of order $r \in \mathbb{R}$:

$$H^r(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|^2_{H^r(\mathbb{R}^2)} := \int_{\mathbb{R}^2} |\mathcal{F}f(\xi)|^2(1+|\xi|^2)^r d\xi < \infty\}$$

and the local fractional Sobolev space of order r:

$$H^r_{\text{loc}}(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2); \varphi f \in H^r(\mathbb{R}^2) \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^2)\}$$

We say that $f_n \to f$ in $H^r_{\text{loc}}(\mathbb{R}^2)$ if $f_n\varphi \to f\varphi$ in $H^r(\mathbb{R}^2)$, for any $\varphi \in C_c^\infty(\mathbb{R}^2)$. The embedding map of $H^r(\mathbb{R}^2)$ into $H^r_{\text{loc}}(\mathbb{R}^2)$ is continuous.

Before investigating the path properties of the solution u, we need to examine the regularity of the function G. A key estimate is the following: for any $t > 0$ and $\xi \in \mathbb{R}$,

$$\frac{\sin^2(t|\xi|)}{|\xi|^2} \leq 2(t^2 \vee 1)\frac{1}{1+|\xi|^2}. \tag{3.2}$$

Using this inequality, it follows that for any $t > 0$,

$$\int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2(1+|\xi|^2)^r d\xi \leq 2(t^2 \vee 1)\int_{\mathbb{R}} \left(\frac{1}{1+|\xi|^2}\right)^{1-r} d\xi. \tag{3.3}$$

Therefore, $G_t \in H^r(\mathbb{R}^d)$ for any $r < 1 - d/2$ and $t > 0$.

By convention, we let $G_t(x) = 0$ for any $t < 0$ and $x \in \mathbb{R}^d$. To define the function G at time 0, we consider the limit as $t \to 0+$. We treat separately the cases $d = 1$ and $d = 2$. We start with the case $d = 2$ since it is more involved.

3.1. *Case* $d = 2$. In this case, $G_t \in H^r(\mathbb{R}^2)$ for any $r < 0$ and $t > 0$. On the other hand, $G_0 = \delta_0$ (the Dirac delta distribution at 0) since

$$G_0(x) := \lim_{t \to 0+} G_t(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

The Dirac delta distribution δ_x is in $H^r(\mathbb{R}^2)$ if and only if $r < -1$, since $\mathcal{F}\delta_x(\xi) = e^{-i\xi \cdot x}$.

The following result will allow us to analyze the compound-Poisson component of u^N.

Lemma 3.1. *If $d = 2$, for any $t_0 \in [0, T]$, $x_0 \in \mathbb{R}^2$, the map $[0, T] \ni t \mapsto G_{t-t_0}(\cdot - x_0)$ is càdlàg in $H^r(\mathbb{R}^2)$ for any $r < -1$.*

Proof: We have to prove that the function $F : [0, T] \to H^r(\mathbb{R}^2)$ given by

$$F(t) = \begin{cases} G_{t-t_0}(\cdot - x_0) & \text{if } t > t_0 \\ \delta_{x_0} & \text{if } t = t_0 \\ 0 & \text{if } t < t_0 \end{cases}$$

is càdlàg. Note that F is continuous at any point $t > t_0$, since

$$\begin{aligned} \|F(t+h) - F(t)\|^2_{H^r(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \left| \frac{\sin((t+h)|\xi|)}{|\xi|} - \frac{\sin(t|\xi|)}{|\xi|} \right|^2 (1 + |\xi|^2)^r d\xi \\ &= 4 \int_{\mathbb{R}^2} \frac{(1 + |\xi|^2)^r}{|\xi|^2} \sin^2\left(\frac{h|\xi|}{2}\right) \cos^2 \frac{(2t+h)|\xi|}{2} d\xi \to 0 \end{aligned}$$

as $h \to 0$, by the dominated convergence theorem. To justify the application of this theorem we use the fact that $\sin^2(\frac{h|\xi|}{2}) \leq C \frac{|\xi|^2}{1+|\xi|^2}$ and $\int_{\mathbb{R}^2} (1 + |\xi|^2)^{r-1} d\xi < \infty$.

Clearly, F is continuous at any point $t < t_0$. F is right-continuous at t_0 since

$$\begin{aligned} \|F(t_0 + h) - F(t_0)\|_{H^2_r(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\mathcal{F}G_h(\cdot - x_0)(\xi) - \mathcal{F}\delta_{x_0}(\xi)|^2 (1 + |\xi|^2)^r d\xi \\ &= \int_{\mathbb{R}^2} \left| e^{-i\xi \cdot x_0} \frac{\sin(h|\xi|)}{|\xi|} - e^{-i\xi \cdot x_0} \right|^2 (1 + |\xi|^2)^r d\xi \to 0 \end{aligned}$$

as $h \to 0+$, by the dominated convergence theorem. Clearly, F has left limit 0 at t_0. □

We consider first the case of bounded function σ. The compact support property of G turns out to be very useful, and compensates for its lack of smoothness.

Theorem 3.2. *Suppose that $d = 2$ and Assumption A holds with $p < 2$. Assume that σ is bounded. Let $\{u(t, x); t \in [0, T], x \in \mathbb{R}^2\}$ be the solution to equation (1.7) on the interval $[0, T]$, constructed in Theorem 2.7 but with stopping times $(\tau_N)_{N \geq 1}$ defined by (3.1). Then the process $\{u(t, \cdot)\}_{t \in [0,T]}$ has a càdlàg modification with values in $H^r_{\text{loc}}(\mathbb{R}^2)$, for any $r < -1$.*

Proof: We follow the lines of the proof of Proposition 2.14 of Chong et al. (2019) for the heat equation. As mentioned above, it suffices to prove the result for u_N, for arbitrary $N \geq 1$. We consider separately the cases $p \geq 1$ and $p < 1$, p being the constant from Assumption A.

Case 1. $p \geq 1$. Recall that u_N satisfies the integral equation (2.11). Using decomposition (1.6) of L_N, we write:

$$u_N(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy) + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma(u_N(s,y))L_N^P(ds,dy) + b\int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma(u_N(s,y))dyds =: w(t,x) + u_N^1(t,x) + u_N^2(t,x) + u_N^3(t,x). \tag{3.4}$$

We treat $w(t,x)$. Since w is jointly continuous and bounded, the map $t \mapsto w(t,\cdot)$ is continuous in $H^r_{\rm loc}(\mathbb{R}^2)$ (see the proof of Lemma 2.13 of Chong et al., 2019).

We treat $u_N^1(t,x)$. Let $A \geq T$ be arbitrary and $K = \{y \in \mathbb{R}^2; |y| \leq 2A\}$. We use the decomposition $u_N^1(t,x) = u_N^{1,1}(t,x) + u_N^{1,2}(t,x)$, where

$$u_N^{1,1}(t,x) = \int_0^t \int_K G_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy) \tag{3.5}$$

$$u_N^{1,2}(t,x) = \int_0^t \int_{K^c} G_{t-s}(x-y)\sigma(u_N(s,y))L^M(ds,dy). \tag{3.6}$$

Suppose that $|x| \leq A$. Note that $G_{t-s}(x-y)$ contains the indicator of the set $\{y; |x-y| < t-s\}$ and any element in this set satisfies:

$$|y| \leq |y-x| + |x| \leq t - s + A \leq T + A \leq 2A.$$

This shows that $u_N^{1,2}(t,\cdot)1_{\{|x|\leq A\}} = 0$ for any $A \geq T$.

It remains to examine $u_N^{1,1}(t,x)$. Similarly to (2.33) of Chong et al. (2019), for any $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\langle \mathcal{F}u^{1,1}(t,\cdot), \varphi\rangle = \int_{\mathbb{R}^2} \left(\int_0^t \int_K e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right)\varphi(\xi)d\xi.$$

The proof of this relation is based on applying twice the stochastic Fubini theorem given by Theorem A.3.1 of Chong et al. (2019). For the first application of this theorem, we need to check that

$$\int_{\mathbb{R}^2} |\mathcal{F}\varphi(x)| \left(\int_0^t \int_K \int_{\{|z|\leq 1\}} G_{t-s}^p(x-y)\mathbb{E}|\sigma(u_N(s,y))|^p|z|^p\nu(dz)dyds\right)^{1/p} dx < \infty.$$

This follows using Assumption A, (2.13), and the fact that $\int_{\mathbb{R}^2} G_{t-s}^p(x-y)dy = c_p(t-s)^{2-p}$. For the second application, we need to check that

$$\int_{\mathbb{R}^2} |\varphi(\xi)| \left(\int_0^t \int_K \int_{\{|z|\leq 1\}} \left|e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\right|^p \mathbb{E}|\sigma(u_N(s,y))|^p|z|^p\nu(dz)dyds\right)^{1/p} d\xi < \infty,$$

which follows using Assumption A, (2.13), and the fact that $\int_0^t \left|\frac{\sin((t-s)|\xi|)}{|\xi|}\right|^p ds \leq \int_0^t (t-s)^p ds$.

It follows that the Fourier transform of $u_N^{1,1}(t,\cdot)$ is the following function:

$$a_\xi(t) := \mathcal{F}u_N^{1,1}(t,\cdot)(\xi) = \int_0^t \int_K e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy). \tag{3.7}$$

Note that for any $t \in [0,T]$, $u_N^{1,1}(t,\cdot) \in H^r(\mathbb{R}^2)$ with probability 1, since

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^2} |a_\xi(t)|^2 (1+|\xi|^2)^r d\xi\right] \\
&= \int_{\mathbb{R}^2} \mathbb{E}\left|\int_0^t \int_K \int_{|z|\le 1} e^{-i\xi\cdot y} \frac{\sin((t-s)|\xi|)}{|\xi|} \sigma(u_N(s,y)) z \widetilde{J}(ds,dy,dz)\right|^2 (1+|\xi|^2)^r d\xi \\
&= \int_{\mathbb{R}^2} \left(\int_0^t \int_K \int_{|z|\le 1} \frac{\sin^2((t-s)|\xi|)}{|\xi|^2} \mathbb{E}|\sigma(u_N(s,y))|^2 |z|^2 \nu(dz) dy ds\right) (1+|\xi|^2)^r d\xi \\
&\le C \int_{\mathbb{R}^2} \left(\int_0^t \frac{\sin^2((t-s)|\xi|)}{|\xi|^2} ds\right) (1+|\xi|^2)^r d\xi < \infty,
\end{aligned}$$

where for the last inequality above, we used the fact that σ is bounded (since relation (2.13) may not hold for $p=2$). The last integral is finite since $r<0$ (using (3.2)).

We prove that $\{u_N^{1,1}(t,\cdot)\}_{t\in[0,T]}$ is stochastically continuous as $H^r(\mathbb{R}^2)$-valued process, i.e.

$$\mathbb{E}\Big[\|u_N^{1,1}(t+h,\cdot) - u_N^{1,1}(t,\cdot)\|^2_{H^r(\mathbb{R}^2)}\Big] = 0 \quad \text{as} \quad h \to 0. \tag{3.8}$$

To see this, note that

$$\begin{aligned}
&|a_\xi(t+h) - a_\xi(t)|^2 \le \\
&2\left\{\left|\int_0^t \int_K e^{-i\xi\cdot y} \frac{\sin((t+h-s)|\xi|) - \sin((t-s)|\xi|)}{|\xi|} \sigma(u_N(s,y)) L^M(ds,dy)\right|^2 + \right. \\
&\qquad \left. \left|\int_t^{t+h} \int_K e^{-i\xi\cdot y} \frac{\sin((t+h-s)|\xi|)}{|\xi|} \sigma(u_N(s,y)) L^M(ds,dy)\right|^2\right\}.
\end{aligned} \tag{3.9}$$

Hence, using again the fact that σ is bounded, we have

$$\begin{aligned}
&\mathbb{E}\Big[|a_\xi(t+h) - a_\xi(t)|^2\Big] \le \\
&2\left\{\int_0^t \int_K \frac{\sin^2((t+h-s)|\xi|) - \sin((t-s)|\xi|)}{|\xi|^2} \mathbb{E}|\sigma(u_N(s,y))|^2 |z|^2 \nu(dz) dy ds + \right. \\
&\qquad \left. \int_t^{t+h} \int_K \frac{\sin^2((t+h-s)|\xi|)}{|\xi|^2} \mathbb{E}|\sigma(u_N(s,y))|^2 |z|^2 \nu(dz) dy ds\right\} \\
&\le C\left\{\int_0^t \frac{\sin^2((t+h-s)|\xi|) - \sin((t-s)|\xi|)}{|\xi|^2} ds + \int_t^{t+h} \frac{\sin^2((t+h-s)|\xi|)}{|\xi|^2} ds\right\} \\
&= C\left\{\frac{4}{|\xi|^2} \sin^2\left(\frac{h|\xi|}{2}\right) \int_0^t \cos^2 \frac{2(t-s)+h}{2} ds + \int_0^h \frac{\sin^2(s|\xi|)}{|\xi|^2} ds\right\} = C|h|^2,
\end{aligned}$$

and therefore, by the dominated convergence theorem,

$$\mathbb{E}\Big[\|u_N^{1,1}(t+h,\cdot) - u_N^{1,1}(t,\cdot)\|^2_{H^r(\mathbb{R}^2)}\Big] = \int_{\mathbb{R}^2} \mathbb{E}\Big[|a_\xi(t+h) - a_\xi(t)|^2\Big] (1+|\xi|^2)^r d\xi \to 0 \quad \text{as} \quad h \to 0.$$

To justify the application of this theorem, we use inequality (3.2) and the fact that $r<0$.

To conclude that $\{u_N^{1,1}(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification with values in $H^r(\mathbb{R}^2)$, we apply Theorems 1 and 5 of Gīhman and Skorohod (1974). For this, we need to show that there exists $\delta>0$ such that for any $t\in[0,T]$ and for any $h>0$ such that $t+h, t-h \in [0,T]$,

$$\mathbb{E}\Big[\|u_N^{1,1}(t+h,\cdot) - u_N^{1,1}(t,\cdot)\|^2_{H^r(\mathbb{R}^2)} \|u_N^{1,1}(t-h,\cdot) - u_N^{1,1}(t,\cdot)\|^2_{H^r(\mathbb{R}^2)}\Big] \le Ch^{1+\delta}. \tag{3.10}$$

To prove this, note that

$$\mathbb{E}\Big[\|u_N^{1,1}(t+h,\cdot)-u_N^{1,1}(t,\cdot)\|_{H^r(\mathbb{R}^2)}^2\|u_N^{1,1}(t-h,\cdot)-u_N^{1,1}(t,\cdot)\|_{H^r(\mathbb{R}^2)}^2\Big]= \int_{(\mathbb{R}^2)^2}\mathbb{E}\Big[|a_\xi(t+h)-a_\xi(t)|^2|a_\eta(t-h)-a_\eta(t)|^2\Big](1+|\xi|^2)^r(1+|\eta|^2)^r d\xi d\eta. \tag{3.11}$$

We combine (3.9) with the similar bound for the increment over $[t-h,t]$, namely:

$$|a_\eta(t-h)-a_\eta(t)|^2\leq 2\left\{\left|\int_0^t\int_K e^{-i\eta\cdot y}\frac{\sin((t-h-s)|\eta|)-\sin((t-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2+\left|\int_{t-h}^t\int_K e^{-i\eta\cdot y}\frac{\sin((t-h-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2\right\}.$$

It follows that

$$E_{t,h}(\xi,\eta):=\mathbb{E}\Big[|a_\xi(t+h)-a_\xi(t)|^2|a_\eta(t-h)-a_\eta(t)|^2\Big]\leq\sum_{i=1}^4 E_{t,h}^i(\xi,\eta), \tag{3.12}$$

where

$$E_{t,h}^1(\xi,\eta)=\mathbb{E}\left[\left|\int_0^t\int_K e^{-i\xi\cdot y}\frac{\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2 \left|\int_0^t\int_K e^{-i\eta\cdot y}\frac{\sin((t-h-s)|\eta|)-\sin((t-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2\right]$$

$$E_{t,h}^2(\xi,\eta)=\mathbb{E}\left[\left|\int_0^t\int_K e^{-i\xi\cdot y}\frac{\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2 \left|\int_{t-h}^t\int_K e^{-i\eta\cdot y}\frac{\sin((t-h-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2\right]$$

$$E_{t,h}^3(\xi,\eta)=\mathbb{E}\left[\left|\int_t^{t+h}\int_K e^{-i\xi\cdot y}\frac{\sin((t+h-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2 \left|\int_0^t\int_K e^{-i\eta\cdot y}\frac{\sin((t-h-s)|\eta|)-\sin((t-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2\right]$$

$$E_{t,h}^4(\xi,\eta)=\mathbb{E}\left[\left|\int_t^{t+h}\int_K e^{-i\xi\cdot y}\frac{\sin((t+h-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2 \left|\int_{t-h}^t\int_K e^{-i\eta\cdot y}\frac{\sin((t-h-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^2\right].$$

To estimate these terms, we use Cauchy-Schwarz inequality. Using Theorem B.1 and the fact that σ is bounded,

$$\mathbb{E}\left[\left|\int_0^t \int_K e^{-i\xi\cdot y}\frac{\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^4\right] \leq$$

$$C\left\{\left(\int_0^t \frac{|\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)|^2}{|\xi|^2}ds\right)^2 + \int_0^t \frac{|\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)|^4}{|\xi|^4}ds\right\} \leq Ch^4$$

and

$$\mathbb{E}\left[\left|\int_t^{t+h} \int_K e^{-i\xi\cdot y}\frac{\sin((t+h-s)|\xi|))}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^4\right] \leq$$

$$C\left\{\left(\int_t^{t+h} \frac{\sin^2((t+h-s)|\xi|)}{|\xi|^2}ds\right)^2 + \int_t^{t+h} \frac{\sin^4((t+h-s)|\xi|)}{|\xi|^4}ds\right\} \leq Ch^5.$$

Similar bounds are valid for the other two terms which involve $t-h$. This proves that (3.10) holds with $\delta = 3$. So, $\{u_N^{1,1}(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification with values in $H^r(\mathbb{R}^2)$.

We treat $u_N^2(t,x)$. If J has points (T_i, X_i, Z_i) in $[0,T]\times\mathbb{R}^2\times\mathbb{R}$, then

$$u_N^2(t,x) = \sum_{i\geq 1} G_{t-T_i}(x-X_i)\sigma(u_N(T_i,X_i))Z_i 1_{\{T_i\leq t\}}1_{\{1<|Z_i|\leq Nh(X_i)\}}.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be arbitrary. Pick $A>0$ such that supp $\varphi \subset \{x\in\mathbb{R}^2; |x|\leq A\}$, Since $G_{t-T_i}(x-X_i)$ contains the indicator of the set $|x-X_i|<t-T_i$, we have $|X_i|<|x|+t\leq A+T$, for any $|x|\leq A$. Hence, for any $x\in\mathbb{R}^2$,

$$u_N^2(t,x)\varphi(x) = \sum_{i\geq 1} G_{t-T_i}(x-X_i)\varphi(x)\sigma(u_N(T_i,X_i))Z_i 1_{\{T_i\leq t\}}1_{\{|X_i|<A+T,1<|Z_i|\leq Nh(X_i)\}}.$$

This sum contains finitely many terms, since $S=\{(x,z); |x|<A+T, 1<|z|<Nh(x)\}$ is a bounded set in $\mathbb{R}^2\times\mathbb{R}$. We look at one of these terms: for any $i\geq 1$ fixed, the function $[0,T]\ni t\mapsto G_{t-T_i}(\cdot - X_i)\varphi$ is càdlàg in $H^r(\mathbb{R}^2)$, by Lemma 3.1.

We claim that $[0,T]\ni t\mapsto u_N^2(t,\cdot)$ is càdlàg in $H^r_{\text{loc}}(\mathbb{R}^2)$. Right-continuity is clear, since for any $t\in[0,T]$ and for any $\varphi\in C_c^\infty(\mathbb{R}^2)$,

$$\lim_{h\to 0+} u_N^2(t+h,\cdot)\varphi = u_N^2(t,\cdot)\varphi \quad \text{in} \quad H^r(\mathbb{R}^2).$$

For the existence of left limit, we have to prove that for any $t\in[0,T]$, there exists $\phi(t)\in H^r_{\text{loc}}(\mathbb{R}^2)$ such that

$$\lim_{h\to 0+} u_N^2(t-h,\cdot)\varphi = \phi(t)\varphi \quad \text{in} \quad H^r(\mathbb{R}^2).$$

To prove this, let $\varphi\in C_c^\infty(\mathbb{R}^2)$ be arbitrary. Pick A such that supp $\varphi\subset\{x; |x|\leq A\}$. By Lemma 3.1, for any $i\geq 1$, there exists $\phi_i(t)\in H^r(\mathbb{R}^2)$ such that $\lim_{h\to 0+} G_{t-h-T_i}(\cdot - X_i) = \phi_i(t)$ in $H^r(\mathbb{R}^2)$, and hence in $H^r_{\text{loc}}(\mathbb{R}^2)$ (since the embedding of $H^r(\mathbb{R}^2)$ into $H^r_{\text{loc}}(\mathbb{R}^2)$ is continuous). It follows that $\lim_{h\to 0+} G_{t-h-T_i}(\cdot - X_i)\varphi = \phi_i(t)\varphi$ in $H^r(\mathbb{R}^2)$ for any $i\geq 1$, and

$$u_N^2(t-h,\cdot)\varphi = \sum_{i\geq 1} G_{t-h-T_i}(\cdot - X_i)\varphi\sigma(u_N(T_i,X_i))Z_i 1_{\{T_i\leq t\}}1_{\{|X_i|<A+T,1<|Z_i|\leq Nh(X_i)\}}$$

converges in $H^r(\mathbb{R}^2)$ as $h \to 0+$ to

$$\sum_{i\geq 1} \phi_i(t)\varphi\sigma(u_N(T_i,X_i))Z_i 1_{\{T_i\leq t\}}1_{\{|X_i|<A+T, 1<|Z_i|\leq Nh(X_i)\}} =: \phi(t)\varphi.$$

We treat $u_N^3(t,x)$. By Lemma 2.13 of Chong et al. (2019). the function $t \mapsto u^3(t,\cdot)$ is continuous in $H^r_{\rm loc}(\mathbb{R}^2)$, since $(t,x) \mapsto G_t(x)$ is integrable on $[0,T]\times \mathbb{R}^2$ and σ is bounded.

Case 2. $p \in (0,1)$. Using the non-drift decomposition (2.2) of L_N, we write

$$\begin{aligned} u_N(t,x) &= w(t,x) + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma(u_N(s,y))L^Q(ds,dy) + \\ &\qquad \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma(u_N(s,y))L_N^P(ds,dy) \\ &=: w(t,x) + u_N^1(t,x) + u_N^2(t,x). \end{aligned} \tag{3.13}$$

The term u_N^2 is the same as in Case 1.

To treat u_N^1, we use the same argument as in Case 1, with L^M replaced by L^Q. We mention only the changes, and we omit the details. Let $A \geq T$ be arbitrary. Using the compact set $K = \{y \in \mathbb{R}^2; |y| \leq 2A\}$, we write $u_N^1(t,x) = u_N^{1,1}(t,x) + u_N^{1,2}(t,x)$ with

$$u_N^{1,1}(t,x) = \int_0^t \int_K G_{t-s}(x-y)\sigma(u_N(s,y))L^Q(ds,dy) \tag{3.14}$$

$$u_N^{1,2}(t,x) = \int_0^t \int_{K^c} G_{t-s}(x-y)\sigma(u_N(s,y))L^Q(ds,dy), \tag{3.15}$$

The same argument as above shows that $u_N^{1,2}(t,\cdot)1_{\{|x|\leq A\}} = 0$.

It remains to study $u_N^{1,1}$. In this case, the Fourier transform of $u_N^{1,1}(t,\cdot)$ is given by

$$a_\xi(t) := \mathcal{F}u_N^{1,1}(t,\cdot)(\xi) = \int_0^t \int_K e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))L^Q(ds,dy), \tag{3.16}$$

This follows by applying twice the stochastic Fubini theorem given by Theorem A.3.2 of Chong et al. (2019). For the first application of this theorem, we need to check that

$$\int_0^t \int_K \int_{\{|z|\leq 1\}} \mathbb{E}\left[\left(\int_{\mathbb{R}^2} G_{t-s}(x-y)|\sigma(u_N(s,y))\mathcal{F}\varphi(x)|dx\right)^p\right] |z|^p \nu(dz)dyds < \infty.$$

This follows using Assumption A, (2.13), $|\mathcal{F}\varphi(x)| \leq \|\varphi\|_{L^1(\mathbb{R}^2)}$ and $\int_{\mathbb{R}^2} G_{t-s}(x-y)dx = t-s$. For the second application of Fubini's theorem, we need to check that

$$\int_0^t \int_K \int_{\{|z|\leq 1\}} \mathbb{E}\left[\left(\int_{\mathbb{R}^2} \left|e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))\varphi(\xi)\right| d\xi\right)^p\right] |z|^p \nu(dz)dyds < \infty.$$

This follows using Assumption A, (2.13), and the fact that

$$\int_{\mathbb{R}^2} \frac{|\sin((t-s)|\xi|)|}{|\xi|}|\varphi(\xi)|d\xi \leq \|\varphi\|_{L^2(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} \frac{\sin^2((t-s)|\xi|)}{|\xi|^2}d\xi\right)^{1/2} \leq C,$$

since $\sin^2((t-s)|\xi|)/|\xi|^2$ can be bounded by $(t-s)^2$ if $|\xi| \leq 1$ and by $1/|\xi|^2$ if $|\xi| > 1$.

For any $t \in [0,T]$, $u_N^{1,1}(t,\cdot) \in H^r(\mathbb{R}^2)$ with probability 1, because

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^2} |a_\xi(t)|^2 (1+|\xi|^2)^r d\xi\right] \\
&= \int_{\mathbb{R}^2} \mathbb{E}\left|\int_0^t \int_K \int_{|z|\le 1} e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))zJ(ds,dy,dz)\right|^2 (1+|\xi|^2)^r d\xi \\
&\le C\int_{\mathbb{R}^2}\left\{\int_0^t\int_K\int_{|z|\le 1}\frac{\sin^2((t-s)|\xi|)}{|\xi|^2}\mathbb{E}|\sigma(u_N(s,y))|^2|z|^2\nu(dz)dyds + \right.\\
&\qquad \left.\mathbb{E}\left[\left(\int_0^t\int_K\int_{|z|\le 1}\frac{|\sin((t-s)|\xi|)|}{|\xi|}|\sigma(u_N(s,y))||z|\nu(dz)dyds\right)^2\right]\right\}(1+|\xi|^2)^r d\xi \\
&\le C\int_{\mathbb{R}^2}\left\{\int_0^t \frac{\sin^2((t-s)|\xi|)}{|\xi|^2}ds + \left(\int_0^t\frac{|\sin((t-s)|\xi|)|}{|\xi|}ds\right)^2\right\}(1+|\xi|^2)^r d\xi < \infty,
\end{aligned}$$

using Lemma B.2, the fact that σ is bounded, $\int_{|z|\le 1}|z|\nu(dz)<\infty$ and $r<-1$.

The fact that $\{u_N^{1,1}(t,\cdot)\}_{t\in[0,T]}$ is stochastically continuous in $H^r(\mathbb{R}^2)$ follows from

$$\mathbb{E}|a_\xi(t+h)-a_\xi(t)|^2 \le Ch^2,$$

which is proved using an inequality similar to (3.9) with L^M replaced by L^Q, followed by an application of Lemma B.2. Finally, relation (3.10) also holds with $\delta=3$. To prove this, we proceed as in Case 1, replacing L^M by L^Q, and applying Lemma B.2 with $p=4$. □

We consider now the case of an unbounded function σ.

Theorem 3.3. *Suppose that $d=2$ and Assumption A holds with $p<2$. Let $\{u(t,x); t\in[0,T], x\in \mathbb{R}^2\}$ be the solution to equation (1.7) on the interval $[0,T]$, constructed in Theorem 2.7 but with stopping times $(\tau_N)_{N\ge 1}$ defined by (3.1). Then the process $\{u(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification with values in $H^r_{\rm loc}(\mathbb{R}^2)$, for any $r<-1$.*

Proof: We use the same argument as in the proof of Theorem 2.15 of Chong et al. (2019) for the heat equation. It suffices to prove the result for u_N, for arbitrary $N\ge 1$. We consider separately the cases $p\ge 1$ and $p\in(0,1)$.

Case 1. $p\ge 1$. We use decomposition (3.4).

We treat $u_N^1(t,x)$. We write $u_N^1(t,x)=u_N^{1,1}(t,x)+u_N^{1,2}(t,x)$ with $u_N^{1,1}(t,x)$ and $u_N^{1,2}(t,x)$ given by (3.5), respectively (3.6). Then $u_N^{1,2}(t,\cdot)1_{\{|x|\le A\}}=0$ for any $A\ge T$.

It remains to treat $u_N^{1,1}(t,x)$. The Fourier transform of $u_N^{1,1}(t,\cdot)$ is still given by (3.7) (the above justification of this formula does not use the boundedness of σ). But if σ is unbounded, it is not immediately clear why $\int_{\mathbb{R}^2}|a_\xi(t)|^2(1+|\xi|^2)^r d\xi<\infty$ a.s.

Consider the truncated function $\sigma_n(u)=\sigma(u)1_{\{|u|\le n\}}$, and define

$$u_{N,n}^{1,1}(t,x)=\int_0^t\int_K G_{t-s}(x-y)\sigma_n(u_N(s,y))L^M(ds,dy).$$

Since σ_n is bounded, $\{u_{N,n}^{1,1}(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification in $H^r(\mathbb{R}^2)$, for any $n\ge 1$. This follows from the proof of Theorem 3.2. Since the uniform limit of a sequence of càdlàg functions is càdlàg, it suffices to show that with probability 1, the sequence $\{u_{N,n}^{1,1}(t,\cdot)\}_{n\ge 1}$ converges to $u_N^{1,1}(t,\cdot)$

as $n \to \infty$, uniformly in $t \in [0,T]$. To achieve this, we will use a change of measure, as in Chong et al. (2019). Let $\sigma_{(n),N}(s,y) = \sigma(u_N(s,y))1_{\{|u_N(s,y)|>n\}}$. Then

$$u_N^{1,1}(t,x) - u_{N,n}^{1,1}(t,x) = \int_0^t \int_K G_{t-s}(x-y)\sigma_{(n),N}(s,y)L^M(ds,dy)$$

$$a_\xi^{(n)}(t) := \mathcal{F}\big(u_N^{1,1}(t,\cdot) - u_{N,n}^{1,1}(t,\cdot)\big)(\xi) = \int_0^t \int_K e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma_{(n),N}(s,y)L^M(ds,dy)$$

$$\sup_{t\in[0,T]} \|u_N^{1,1}(t,\cdot) - u_{N,n}^{1,1}(t,\cdot)\|_{H^r(\mathbb{R}^2)}^2 \leq \int_{\mathbb{R}^2}(1+|\xi|^2)^r \sup_{t\in[0,T]}|a_\xi^{(n)}(t)|^2 d\xi. \tag{3.17}$$

To evaluate $a_\xi^{(n)}(t)$, we use the fact that $\frac{\sin((t-s)|\xi|)}{|\xi|} = \int_s^t \cos((t-r)|\xi|)dr$. By the stochastic Fubini theorem given by Theorem A.3.1 of Chong et al. (2019),

$$a_\xi^{(n)}(t) = \int_0^t \cos((t-r)|\xi|)\left(\int_0^r \int_K e^{-i\xi\cdot y}\sigma_{(n),N}(s,y)L^M(ds,dy)\right)dr.$$

To justify the application of this theorem, we need to check that:

$$\int_0^t \left(\int_0^r \int_K \int_{|z|\leq 1} \mathbb{E}|e^{-i\xi\cdot y}\cos((t-r)|\xi|)\sigma_{(n),N}(s,y)z|^p \nu(dz)dyds\right)^{1/p} dr < \infty.$$

This follows by Assumption A, (2.13) and the bound $|\cos((t-r)|\xi|)| \leq 1$. For any $t \in [0,T]$,

$$|a_\xi^{(n)}(t)| \leq \sup_{r\in[0,T]}\left|\int_0^r \int_K e^{-i\xi\cdot y}\sigma_{(n),N}(s,y)L^M(ds,dy)\right| \cdot \int_0^t |\cos((t-r)|\xi|)|dr,$$

and hence

$$\sup_{t\in[0,T]}|a_\xi^{(n)}(t)| \leq T \sup_{r\in[0,T]}\left|\int_0^r \int_K e^{-i\xi\cdot y}\sigma_{(n),N}(s,y)L^M(ds,dy)\right|. \tag{3.18}$$

We intend to apply Theorem A.4 of Chong et al. (2019), to the random measure L^M. First, we check that $\sigma(u_N)1_K \in L^{1,p}(L^M)$ (we refer to Appendix A of Chong et al., 2019 for the notation). To see this, we use relation (A.3) of Chong et al. (2019), Assumption A and (2.13):

$$\|\sigma(u_N)1_K\|_{L^M,p}^p \leq C\int_0^T \int_K \int_{\{|z|\leq 1\}} \mathbb{E}|\sigma(u_N(s,y))|^p |z|^p \nu(dz)dyds < \infty.$$

By Theorem A.4 of Chong et al. (2019), there exists a probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ which is equivalent to $\mathbb{P}$, such that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is bounded, $\frac{d\mathbb{P}}{d\mathbb{Q}} \in L^{\frac{2}{2-p}}(\Omega,\mathcal{F},\mathbb{P})$, L^M is an L^2-random measure under $\mathbb{Q}$ and $\sigma(u_N)1_K \in L^{1,2}(L^M,\mathbb{Q})$. Hence,

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left|\int_0^t \int_K e^{-i\xi\cdot y}\sigma_{(n),N}(s,y)L^M(ds,dy)\right|^2\right] \leq \|\sigma_{(n),N}1_K\|_{L^M,2,\mathbb{Q}}^2,$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation with respect to $\mathbb{Q}$.

We return to (3.17). We use inequality (3.18), then we take expectation with respect to $\mathbb{Q}$. We obtain that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\|u_N^{1,1}(t,\cdot) - u_{N,n}^{1,1}(t,\cdot)\|_{H^r(\mathbb{R}^2)}^2\right] \\
&\leq T^2 \int_{\mathbb{R}^2}(1+|\xi|^2)^r \mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left|\int_0^t \int_K e^{-i\xi\cdot y}\sigma_{(n),N}(s,y)L^M(ds,dy)\right|^2\right] d\xi \\
&\leq T^2 \|\sigma_{(n),N}1_K\|_{L^M,2,\mathbb{Q}}^2 \int_{\mathbb{R}^2}(1+|\xi|^2)^r d\xi \to 0 \quad \text{as} \quad n\to\infty,
\end{aligned}$$

using the (stochastic) dominated convergence theorem given by Theorem A.1 of Chong et al. (2019), applied to the random measure L^M (under the probability measure $\mathbb{Q}$). In our case, the integrand $\sigma_{(n),N}1_K \to 0$ converges pointwise to 0 as $n \to \infty$, and is bounded by the random function $|\sigma(u_N)1_K|$ which belongs to $L^{1,2}(L^M, \mathbb{Q})$.

Therefore, there exists a subsequence $N' \subset \mathbb{N}$ such that with probability 1,

$$\sup_{t\in[0,T]} \|u_N^{1,1}(t,\cdot) - u_{N,n}^{1,1}(t,\cdot)\|_{H^r(\mathbb{R}^2)} \to 0 \quad \text{as} \quad n \to \infty, n \in N'.$$

Since $\{u_{N,n}^{1,1}(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification in $H^r(\mathbb{R}^2)$ for any $n \geq 1$, the process $\{u_N^{1,1}(t,\cdot)\}_{t\in[0,T]}$ will inherit this property too.

We treat $u_N^2(t,x)$. The same argument as in the proof of Theorem 3.2 shows that $\{u_N^2(t,\cdot)\}_{t\in[0,T]}$ is càdlàg in $H^r_{\text{loc}}(\mathbb{R}^2)$.

We treat $u_N^3(t,x)$. Let $A \geq T$ be arbitrary and $K = \{y \in \mathbb{R}^2; |y| \leq 2A\}$. We write $u^3(t,x) = u^{3,1}(t,x) + u^{3,2}(t,x)$, where

$$u^{3,1}(t,x) = b\int_0^t \int_K G_{t-s}(x-y)\sigma(u_N(s,y))dyds \tag{3.19}$$

$$u^{3,2}(t,x) = b\int_0^t \int_{K^c} G_{t-s}(x-y)\sigma(u_N(s,y))dyds. \tag{3.20}$$

As for $u^{1,2}$, we see that $u^{3,2}(t,\cdot)1_{\{|x|\leq A\}} = 0$, using the compact support property of G.

It remains to treat $u^{3,1}$. For this, we consider again the truncated function σ_n as above and we let

$$u_{N,n}^{3,1}(t,x) = b\int_0^t \int_K G_{t-s}(x-y)\sigma_n(u_N(s,y))dyds.$$

Since σ_n is bounded, $\{u_{N,n}^{3,1}(t,\cdot)\}_{t\in[0,T]}$ is continuous in $H^r_{\text{loc}}(\mathbb{R}^2)$, by Lemma 2.13 of Chong et al. (2019).

Similarly to (3.18) (but with $dsdy$ instead of $L^M(ds,dy)$), we have

$$\begin{aligned}\sup_{t\in[0,T]} |\mathcal{F}\big(u_N^{3,1}(t,\cdot) - u_{N,n}^{3,1}(t,\cdot)\big)(\xi)| &\leq bT \sup_{r\in[0,T]} \left|\int_0^r \int_K e^{-i\xi\cdot y}\sigma_{(n),N}(s,y)dyds\right| \\ &\leq bT\int_0^T \int_K |\sigma_{(n),N}(s,y)|dyds,\end{aligned}$$

and hence

$$\sup_{t\in[0,T]} \|u_N^{3,1}(t,\cdot) - u_{N,n}^{3,1}(t,\cdot)\|^2_{H^r(\mathbb{R}^2)} \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that $\{u_N^{3,1}(t,\cdot)\}_{t\in[0,T]}$ is continuous in $H^r_{\text{loc}}(\mathbb{R}^2)$.

Case 2. $p \in (0,1)$. We use (3.13). The term u_N^2 is the same as in Case 1. To treat u_N^1, let $A \geq T$ be arbitrary and $K = \{y \in \mathbb{R}^2; |y| \leq 2A\}$. We write $u_N^1 = u_N^{1,1} + u_N^{1,2}$, where $u_N^{1,1}$ and $u_N^{1,2}$ are given by (3.14), respectively (3.15). Then $u_N^{1,2}(t,\cdot)1_{\{|x|\leq A\}} = 0$.

It remains to study $u_N^{1,1}$. The Fourier transform of $u_N^{1,1}(t,\cdot)$ is given by (3.16). The justification of this formula does not use the fact that σ is bounded. Let $\sigma_n(u) = \sigma(u)1_{\{|u|\leq n\}}$ and define

$$u_{N,n}^{1,1}(t,x) = \int_0^t \int_K G_{t-s}(x-y)\sigma_n(u_N(s,y))L^Q(ds,dy).$$

Since σ_n is bounded, $\{u^{1,1}_{N,n}(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification in $H^r(\mathbb{R}^2)$. This follows from the proof of Theorem 3.2 in the case $p<1$. As in Case 1, replacing L^M by L^Q, we have:

$$a^{(n)}_{\xi}(t) := \mathcal{F}\big(u^{1,1}_N(t,\cdot)-u^{1,1}_{N,n}(t,\cdot)\big)(\xi) = \int_0^t\int_K e^{-i\xi\cdot y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma_{(n),N}(s,y)L^Q(ds,dy)$$
$$= \int_0^t \left(\int_0^r\int_K e^{-i\xi\cdot y}\cos((t-r)|\xi|)\sigma_{(n),N}(s,y)L^Q(ds,dy)\right)dr.$$

The last equality above is due to the stochastic Fubini theorem given by Theorem A.3.2 of Chong et al. (2019). To justify the application of this theorem, we need to check that

$$\int_0^t\int_K\int_{\{|z|\leq 1\}}\mathbb{E}\left[\left(\int_s^t |e^{-i\xi\cdot y}\cos((t-r)|\xi|)\sigma_{(n),N}(s,y)|dr\right)^p\right]|z|^p\nu(dz)dyds<\infty.$$

This follows using Assumption A, (2.13) and the bound $|\cos((t-r)|\xi|)|\leq 1$.

We now apply the change of measure of Theorem A.4 of Chong et al. (2019) to the random measure L^Q. To check that $\sigma(u_N)1_K\in L^{1,p}(L^Q)$, we use Lemma A.2.3 of Chong et al. (2019), Assumption A and (2.13):

$$\|\sigma(u_N)1_K\|_{L^Q,p}\leq \int_0^T\int_K\int_{\{|z|\leq 1\}}\mathbb{E}|\sigma(u_N(s,y))|^p|z|^p\nu(dz)dyds<\infty.$$

The rest of the proof is the same as in Case 1. □

3.2. *Case* $d=1$. In this case, $G_t\in H^r(\mathbb{R})$ for any $r<1/2$ and $t>0$. On the other hand, $G_0=\frac{1}{2}1_{\{0\}}$ since

$$G_0(x):=\lim_{t\to 0+}G_t(x)=\begin{cases}0 & \text{if } x\neq 0\\ 1/2 & \text{if } x=0\end{cases}$$

Therefore, $G_0\in H^r(\mathbb{R})$ for any $r\in\mathbb{R}$. For $t>0$, $G_t\in H^r(\mathbb{R})$ if $r<1/2$, due to (3.3).

The following result is the counterpart of Lemma 3.1 for $d=1$.

Lemma 3.4. *If $d=1$, for any $t_0\in[0,T]$, $x_0\in\mathbb{R}^2$, the map $[0,T]\ni t\mapsto G_{t-t_0}(\cdot-x_0)$ is càdlàg in $H^r(\mathbb{R}^2)$ for any $r<1/2$.*

Proof: We have to prove that the function $F:[0,T]\to H^r(\mathbb{R}^2)$ given by

$$F(t)=\begin{cases}G_{t-t_0}(\cdot-x_0) & \text{if } t>t_0\\ 2^{-1}1_{\{x_0\}} & \text{if } t=t_0\\ 0 & \text{if } t<t_0\end{cases}$$

is càdlàg. The fact that F is continuous at any point $t>t_0$ follows as in the proof of Lemma 3.4. F is right-continuous at t_0 since

$$\|F(t_0+h)-F(t_0)\|_{H^2_r(\mathbb{R})}=\int_{\mathbb{R}}|\mathcal{F}G_h(\cdot-x_0)(\xi)-\frac{1}{2}\mathcal{F}1_{\{x_0\}}(\xi)|^2(1+|\xi|^2)^r d\xi$$
$$=\int_{\mathbb{R}^2}\left|e^{-i\xi\cdot x_0}\frac{\sin(h|\xi|)}{|\xi|}\right|^2(1+|\xi|^2)^r d\xi\to 0$$

as $h\to 0+$, by the dominated convergence theorem. □

Unlike the case $d=2$, the solution u_N of the equation with truncated noise has finite moments of order p, for any $p\geq 2$. Therefore, we do not need to consider separately the case of a bounded function σ. Note that we impose a new restriction $r<1/4$ (compared with Lemma 3.4) which comes from the analysis of increments; this introduces additional requirements on the initial conditions u_0 and v_0.

Theorem 3.5. *Assume that $d=1$ and there exists $q>0$ such that*

$$\int_{\{|z|>1\}} |z|^q \nu(dz) < \infty.$$

In addition to the initial conditions mentioned at the beginning, we suppose that $u_0, v_0 \in L^1(\mathbb{R})$ and $|\mathcal{F}u_0(\xi)| \leq c\frac{1}{1+|\xi|^2}$ for any $\xi \in \mathbb{R}$ for some constant $c>0$.

Let $\{u(t,x); t\in[0,T], x\in\mathbb{R}\}$ be the solution to equation (1.7) on the interval $[0,T]$, constructed in Theorem 2.7 but with stopping times $(\tau_N)_{N\geq 1}$ defined by (3.1). Then the process $\{u(t,\cdot)\}_{t\in[0,T]}$ has a càdlàg modification with values in $H^r_{\text{loc}}(\mathbb{R}^2)$, for any $r<1/4$.

Proof: Since $\int_{|z|>1} |z|^q \nu(dz)$ is nondecreasing in q, we can assume that $q \leq 2$. Therefore, Assumption A holds with $p \geq 2$ arbitrary and $q \in (0,2]$. By Theorem 2.6,

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}|u_N(t,x)|^p < \infty \quad \text{for any} \quad p \geq 2, \tag{3.21}$$

for any $T>0$ and $R>0$. As mentioned above, it is enough to prove the result for u_N. Since $p>1$ in Assumption A, we use decomposition (3.4).

We treat $w(t,x)$. We have

$$w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}[u_0(x+t) - u_0(x-t)] =: w^1(t,x) + w^2(t,x).$$

We treat separately the two terms. Note that $\mathcal{F}w^1(t,\cdot)(\xi) = \mathcal{F}G_t(\xi)\mathcal{F}v_0(\xi)$ and

$$\begin{aligned}\|w(t+h,\cdot) - w(t,\cdot)\|^2_{H^r(\mathbb{R})} &= \int_{\mathbb{R}} \frac{|\sin((t+h)|\xi|) - \sin(t|\xi|)|^2}{|\xi|^2} |\mathcal{F}v_0(\xi)|^2 (1+|\xi|^2)^r d\xi \\ &= \int_{\mathbb{R}} \frac{4}{|\xi|^2} \sin^2\left(\frac{h|\xi|}{2}\right) \cos^2\left(\frac{2t+h}{2}|\xi|\right) |\mathcal{F}v_0(\xi)|^2 (1+|\xi|^2)^r d\xi \to 0 \quad \text{as} \quad h \to 0,\end{aligned}$$

by the dominated convergence theorem, using the fact that $|\mathcal{F}v_0(\xi)| \leq \|v_0\|_{L^1(\mathbb{R})}$ and $\frac{1}{|\xi|^2}\sin^2\left(\frac{h|\xi|}{2}\right) \leq C\frac{1}{1+|\xi|^2}$ for any $\xi \in \mathbb{R}$. To apply this theorem, we only need $r<1/2$.

By direct calculation, $\mathcal{F}w^2(t,\cdot)(\xi) = i\sin(\xi t)\mathcal{F}u_0(\xi)$. Hence,

$$\begin{aligned}\|w^2(t+h,\cdot) - w^2(t,\cdot)\|^2_{H^r(\mathbb{R})} &= \int_{\mathbb{R}} |\sin(\xi(t+h)) - \sin(\xi t)|^2 |\mathcal{F}u_0(\xi)|^2 (1+|\xi|^2)^r d\xi \\ & \int_{\mathbb{R}} 4\sin^2\left(\frac{h|\xi|}{2}\right) \cos^2\left(\frac{(2t+h)|\xi|}{2}\right) |\mathcal{F}u_0(\xi)|^2 (1+|\xi|^2)^r d\xi \to 0 \quad \text{as} \quad h \to 0,\end{aligned}$$

by the dominated convergence theorem, using the fact that $|\mathcal{F}u_0(\xi)|^2 \leq c\frac{1}{1+|\xi|^2}$ for all $\xi \in \mathbb{R}$, and $r<1/2$.

We treat $u^1_N(t,x)$. We use the decomposition $u_N = u^1_N + u^2_N$ where u^1_N, u^2_N are given by (3.5), respectively (3.6), and $K=[-2A,2A]$. As in the case $d=2$, $u^{1,2}_N(t,\cdot)1_{[-A,A]} = 0$ for any $A \geq T$. So it suffices to consider $u^{1,1}_N$.

The same argument as in the proof of Theorem 3.2 shows that $\{u^{1,1}_N(t,\cdot)\}_{t\in[0,T]}$ is stochastically continuous as $H^r(\mathbb{R}^2)$-valued process, for any $r<1/2$. Instead of using the boundedness of σ, we now use (3.21) and the fact that σ is Lipschitz.

Next, we prove (3.10). For this, we use (3.11) and (3.12). By the Cauchy-Schwarz inequality,

$$E^1_{t,h}(\xi,\eta) \leq I_1^{1/2} I_2^{1/2}, \quad E^2_{t,h}(\xi,\eta) \leq I_1^{1/2} I_3^{1/2}, \quad E^3_{t,h}(\xi,\eta) \leq I_4^{1/2} I_2^{1/2}, \quad E^4_{t,h}(\xi,\eta) \leq I_4^{1/2} I_3^{1/2},$$

where

$$I_1 = \mathbb{E}\left[\left|\int_0^t \int_K e^{-i\xi y}\frac{\sin((t+h-s)|\xi| - \sin((t-s)|\xi|))}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^4\right]$$

$$I_2 = \mathbb{E}\left[\left|\int_0^t \int_K e^{-i\xi y}\frac{\sin((t-h-s)|\eta| - \sin((t-s)|\eta|))}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^4\right]$$

$$I_3 = \mathbb{E}\left[\left|\int_{t-h}^t \int_K e^{-i\xi y}\frac{\sin((t-h-s)|\eta|)}{|\eta|}\sigma(u_N(s,y))L^M(ds,dy)\right|^4\right]$$

$$I_4 = \mathbb{E}\left[\left|\int_t^{t+h} \int_K e^{-i\xi y}\frac{\sin((t+h-s)|\xi|}{|\xi|}\sigma(u_N(s,y))L^M(ds,dy)\right|^4\right].$$

We apply Theorem B.1, followed by Cauchy-Schwarz inequality and (3.21):

$$\begin{aligned}
I_1 &\leq C\mathbb{E}\left[\left(\int_0^t \int_K \frac{|\sin((t+h-s)|\xi|) - \sin((t-s)|\xi|))|^2}{|\xi|^2}|\sigma(u_N(s,y))|^2 dyds\right)^2\right] + \\
&\quad C\mathbb{E}\left[\int_0^t \int_K \frac{|\sin((t+h-s)|\xi|) - \sin((t-s)|\xi|))|^4}{|\xi|^4}|\sigma(u_N(s,y))|^4 dyds\right] \\
&\leq C\int_0^t \int_K \frac{|\sin((t+h-s)|\xi|) - \sin((t-s)|\xi|))|^4}{|\xi|^4}\mathbb{E}|\sigma(u_N(s,y))|^4 dyds \\
&\leq C\int_0^t \frac{|\sin((t+h-s)|\xi|) - \sin((t-s)|\xi|))|^4}{|\xi|^4}ds \\
&\leq C\frac{1}{|\xi|^4}\left|\sin\left(\frac{h|\xi|}{2}\right)\right|^4 = C\frac{1}{|\xi|^\varepsilon}\left|\sin\left(\frac{h|\xi|}{2}\right)\right|^\varepsilon \cdot \frac{1}{|\xi|^{4-\varepsilon}}\left|\sin\left(\frac{h|\xi|}{2}\right)\right|^{4-\varepsilon} \\
&\leq Ch^\varepsilon\left(\frac{1}{1+|\xi|^2}\right)^{\frac{4-\varepsilon}{2}} \quad \text{for any} \quad \varepsilon \in [0,4]
\end{aligned}$$

and

$$\begin{aligned}
I_4 &\leq C\mathbb{E}\left[\left(\int_t^{t+h} \int_K \frac{|\sin((t+h-s)|\xi|)|^2}{|\xi|^2}|\sigma(u_N(s,y))|^2 dyds\right)^2\right] + \\
&\quad C\mathbb{E}\left[\int_t^{t+h} \int_K \frac{|\sin((t+h-s)|\xi|)|^4}{|\xi|^4}|\sigma(u_N(s,y))|^4 dyds\right] \\
&\leq C\int_t^{t+h} \int_K \frac{|\sin((t+h-s)|\xi|)|^4}{|\xi|^4}\mathbb{E}|\sigma(u_N(s,y))|^4 dyds \\
&\leq C\int_t^{t+h} \frac{|\sin((t+h-s)|\xi|)|^4}{|\xi|^4}ds = C\int_0^h \frac{|\sin(s|\xi|)|^4}{|\xi|^4}ds \\
&= C\int_0^h \frac{|\sin(s|\xi|)|^\varepsilon}{|\xi|^\varepsilon}\cdot\frac{|\sin(s|\xi|)|^{4-\varepsilon}}{|\xi|^{4-\varepsilon}}ds \leq C\left(\frac{1}{1+|\xi|^2}\right)^{\frac{4-\varepsilon}{2}}\int_0^h s^\varepsilon ds \\
&= Ch^{1+\varepsilon}\left(\frac{1}{1+|\xi|^2}\right)^{\frac{4-\varepsilon}{2}} \quad \text{for any} \quad \varepsilon \in [0,4].
\end{aligned}$$

Similarly, for any $\varepsilon \in [0,4]$,

$$I_2 \leq Ch^\varepsilon\left(\frac{1}{1+|\eta|^2}\right)^{\frac{4-\varepsilon}{2}} \quad \text{and} \quad I_3 \leq Ch^{1+\varepsilon}\left(\frac{1}{1+|\eta|^2}\right)^{\frac{4-\varepsilon}{2}}.$$

It follows that

$$E_{t,h}(\xi,\eta) \leq Ch^{\varepsilon}\left(\frac{1}{1+|\xi|^2}\right)^{\frac{4-\varepsilon}{4}}\left(\frac{1}{1+|\eta|^2}\right)^{\frac{4-\varepsilon}{4}}$$

and

$$\mathbb{E}[\|u(t+h,\cdot)-u(t,\cdot)\|^2_{H^r(\mathbb{R})}\cdot\|u(t-h,\cdot)-u(t,\cdot)\|^2_{H^r(\mathbb{R})}] \leq Ch^{\varepsilon}\left[\int_{\mathbb{R}}\left(\frac{1}{1+|\xi|^2}\right)^{\frac{4-\varepsilon}{4}-r}d\xi\right]^2.$$

The last integral converges if and only if $\varepsilon < 2-4r$. On the other hand, we need to choose $\varepsilon > 1$. This introduces the restriction $r < 1/4$. This proves (3.10). By Theorems 1 and 5 of Gīhman and Skorohod (1974), the process $\{u^1_N(t,\cdot)\}_{t\in[0,T]}$ has a cádlág modification in $H^r(\mathbb{R})$ for any $r < 1/4$.

We treat $u^2_N(t,x)$. Using the same argument as in the proof of Theorem 3.2 and Lemma 3.4, we see that the process $\{u^2_N(t,\cdot)\}_{t\in[0,T]}$ is càdlàg in $H^r(\mathbb{R})$ for any $r < 1/2$.

We treat $u^3_N(t,x)$. Let $A > 0$ be arbitrary and $K = [-2A, 2A]$. We write $u^3_N(t,x) = u^{3,1}_N(t,x) + u^{3,2}_N(t,x)$, where $u^{3,1}_N$ and $u^{3,2}_N$ are given by (3.19) and (3.20). Using the compact support property of G, we see that $u^{3,2}_N(t,\cdot)1_{[-A,A]} = 0$ for any $A > T$.

So, it suffices to consider $u^{3,1}_N$. Note that the Fourier transform of $u^{3,1}_N(t,\cdot)$ is given by

$$\mathcal{F}u^{3,1}_N(t,\cdot)(\xi) = b\int_0^t\int_K e^{-i\xi y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))dyds.$$

We consider separately the cases when σ is bounded and unbounded.

Case 1. (σ is bounded) By Minkowski's inequality,

$$\begin{aligned}\|u^{3,1}_N(t,\cdot)\|_{H^r(\mathbb{R})} &= \|\mathcal{F}u^{3,1}_N(t,\cdot)\|_{L^2(\mathbb{R},(1+|\xi|^2)^r d\xi)} \\ &\leq b\int_0^t\int_K\left(\int_{\mathbb{R}}\frac{\sin^2((t-s)|\xi|)}{|\xi|^2}(1+|\xi|^2)^r d\xi\right)^{1/2}|\sigma(u_N(s,y))|dyds \\ &\leq C\left[\int_{\mathbb{R}}\left(\frac{1}{1+|\xi|^2}\right)^{1-r}d\xi\right]^{1/2} < \infty \quad \text{since} \quad r < 1/2.\end{aligned}$$

We claim that $\{u^{3,1}_N(t,\cdot)\}_{t\in[0,T]}$ is continuous in $H^r(\mathbb{R})$, for any $r < 1/2$. To see this, we write

$$\begin{aligned}\mathcal{F}\big(u^{3,1}_N(t+h,\cdot)-u^{3,1}_N(t,\cdot)\big)(\xi) = b\int_t^{t+h}\int_K\frac{\sin((t+h-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))dyds+ \\ b\int_0^t\int_K e^{-i\xi y}\frac{\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)}{|\xi|}\sigma(u_N(s,y))dyds.\end{aligned}$$

Then applying Minkowski's inequality as above, we obtain:

$$\begin{aligned}&\|u^{3,1}_N(t+h,\cdot)-u^{3,1}_N(t,\cdot)\|_{H^r(\mathbb{R})} \leq \\ &b\int_t^{t+h}\int_K\left(\int_{\mathbb{R}}\frac{\sin^2((t+h-s)|\xi|)}{|\xi|^2}(1+|\xi|^2)^r d\xi\right)^{1/2}|\sigma(u_N(s,y))|dyds+ \\ &b\int_0^t\int_K\left(\int_{\mathbb{R}}\frac{|\sin((t+h-s)|\xi|)-\sin((t-s)|\xi|)|^2}{|\xi|^2}(1+|\xi|^2)^r d\xi\right)^{1/2}|\sigma(u_N(s,y))|dyds\end{aligned}$$

and the last two integrals converge to 0 as $h \to 0$, by the dominated convergence theorem.

Case 2. (σ is general) For any $n \geq 1$, let $\sigma_n(x) = \sigma(x)1_{\{|x|\leq n\}}$ and define

$$u^{3,1}_{N,n}(t,x) = b\int_0^t\int_K G_{t-s}(x-y)\sigma_n(u_N(s,y))dyds.$$

By Case 1 above, $\{u_{N,n}^{3,1}(t,\cdot)\}_{t\in[0,T]}$ is continuous in $H^r(\mathbb{R})$, for any $n\geq 1$ and $r<1/2$. We fix $r<1/2$. We will prove that along a subsequence, with probability 1, $\{u_{N,n}^{3,1}\}_{n\geq 1}$ converges to $u_N^{3,1}$ in $H^r(\mathbb{R})$ as $n\to\infty$, uniformly in $t\in[0,T]$. Since the uniform limit of continuous functions is continuous, $\{u_N^{3,1}(t,\cdot)\}_{t\in[0,T]}$ will be continuous in $H^r(\mathbb{R})$.

We denote $\sigma_{(n),N}(s,y)=\sigma(u_N(s,y))-\sigma_n(u_N(s,y))$. By Fubini theorem,

$$\begin{aligned}\mathcal{F}\big(u_N^{3,1}(t,\cdot)-u_{N,n}^{3,1}(t,\cdot)\big)(\xi)&=b\int_0^t\int_K e^{-i\xi y}\frac{\sin((t-s)|\xi|)}{|\xi|}\sigma_{(n),N}(s,y)dyds\\&=b\int_0^t\cos((t-r)|\xi|)\left(\int_0^r\int_K e^{-i\xi y}\sigma_{(n),N}(s,y)dyds\right)dr.\end{aligned}$$

Hence,

$$\sup_{t\in[0,T]}|\mathcal{F}\big(u_N^{3,1}(t,\cdot)-u_{N,n}^{3,1}(t,\cdot)\big)(\xi)|\leq b\left(\int_0^T\int_K|\sigma_{(n),N}(s,y)|dyds\right)\left(\int_0^T|\cos(r|\xi|)|dr\right).$$

We claim that

$$\int_0^T|\cos(r|\xi|)|dr\leq C\left(\frac{1}{1+|\xi|^2}\right)^{1/2}.$$

To see this, assume that $T\in\big((\frac{2k-1}{2}\vee 0)\pi,\frac{2k+1}{2}\pi\big)$ for some integer $k\geq 0$. Say k is even, $k=2m$ for some integer $m\geq 0$. (The case when k is odd is similar.) Then

$$\int_0^T|\cos(r|\xi|)|dr=\frac{\sin(\frac{\pi}{2}|\xi|)}{|\xi|}+\sum_{\ell=1}^{m-1}\frac{\sin(\frac{4\ell+1}{2}\pi)-\sin(\frac{4\ell-1}{2}\pi)}{|\xi|}-\sum_{\ell=1}^{m}\frac{\sin(\frac{4\ell-1}{2}\pi)-\sin(\frac{4\ell-3}{2}\pi)}{|\xi|},$$

and we use the fact that for any $a<b$,

$$\frac{|\sin(a|\xi|)-\sin(b|\xi|)|}{|\xi|}\leq 2\frac{|\sin\left(\frac{a-b}{2}|\xi|\right)|}{|\xi|}\leq 2\left\{2\left[\left(\frac{a-b}{2}\right)^2\vee 1\right]\frac{1}{1+|\xi|^2}\right\}^{1/2}.$$

It follows that

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|\mathcal{F}\big(u_N^{3,1}(t,\cdot)-u_{N,n}^{3,1}(t,\cdot)\big)(\xi)|^2\Big]&\leq C\mathbb{E}\left[\frac{1}{1+|\xi|^2}\left(\int_0^T\int_K|\sigma_{(n),N}(s,y)|dyds\right)^2\right]\\&\leq C\frac{1}{1+|\xi|^2}\int_0^T\int_K\mathbb{E}|\sigma_{(n),N}(s,y)|^2dyds\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u_N^{3,1}(t,\cdot)-u_{N,n}^{3,1}(t,\cdot)\|_{H^r(\mathbb{R})}^2\Big]\\&\qquad\leq C\left(\int_{\mathbb{R}}\frac{1}{1+|\xi|^2}(1+|\xi|^2)^r d\xi\right)\left(\int_0^T\int_K\mathbb{E}|\sigma_{(n),N}(s,y)|^2dyds\right).\end{aligned}$$

The last integral converges to 0 as $h\to 0$, by the dominated convergence theorem. Hence, there exists a subsequence $N'\subset\mathbb{N}$ such that $\sup_{t\in[0,T]}\|u_N^{3,1}(t,\cdot)-u_{N,n}^{3,1}(t,\cdot)\|_{H^r(\mathbb{R})}^2\to 0$ a.s. when $n\to\infty$, $n\in N'$.

□

Appendix A. Stochastic integral

For the reader's convenience, we here include the definition of the stochastic integral with respect to an L^p-random measure. We refer the reader to Chong (2017a,b); Chong et al. (2019) for more details.

Let $\widetilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ and $\mathcal{P} = \mathcal{P}_0 \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}_0$ is the predictable σ-field on $\Omega \times \mathbb{R}_+$. Let $p \in [0, \infty)$ be arbitrary. Given a sequence $(\widetilde{\Omega}_k)_{k\geq 1} \subset \mathcal{P}$ satisfying $\widetilde{\Omega}_k \uparrow \widetilde{\Omega}$, a map $M : \mathcal{P}_M = \bigcup_{k\geq 1} \mathcal{P}|_{\widetilde{\Omega}_k} \to L^p(\Omega)$ is called an *L^p-random measure* if for any disjoint sets $(A_i)_{i\geq 1}$ in $\mathcal{P}_M$ such that $\bigcup_{i\geq 1} A_i \in \mathcal{P}_M$, we have $M(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} M(A_i)$ in $L^p(\Omega)$, and some additional adaptedness conditions hold.

If S is a *simple integrand* of the form $S = \sum_{i=1}^k a_i 1_{A_i}$ for some $a_i \in \mathbb{R}$ and $A_i \in \mathcal{P}_M$, then the stochastic integral of S with respect to M is given by:

$$I^M(S) = \int_0^\infty \int_{\mathbb{R}^d} S(t,x) M(dt, dx) = \sum_{i=1}^k a_i M(A_i).$$

Let $\mathcal{S}_M$ be the set of all simple integrands. The *Daniell mean* of a process $H = \{H(t,x); t \geq 0, x \in \mathbb{R}^d\}$ with respect to M is defined by

$$\|H\|_{M,p} = \sup_{S \in \mathcal{S}_M; |S| \leq |H|} \|I^M(S)\|_p.$$

A predictable process H is said to be *p-integrable with respect to M* if there exists a sequence $(S_n)_{n\geq 1} \subset \mathcal{S}_M$ such that $\|S_n - H\|_{M,p} \to 0$ as $n \to \infty$. In this case, the *stochastic integral* of H with respect to M is defined by:

$$I^M(S) = \lim_{n\to\infty} I^M(S_n) \quad \text{in} \quad L^p(\Omega).$$

We denote by $L^{1,p}(M)$ the set of p-integrable processes with respect to M. Then the map $I^M : L^{1,p}(M) \to L^p(\Omega)$ is a contraction. In these definitions, we omit writing p, if $p = 0$.

Appendix B. Moment Inequalities

In this section, we include the moment inequalities which were used in the sequel. The first result is a Rosenthal-type maximal inequality (see also Theorem 1 of Marinelli and Röckner, 2014).

Theorem B.1 (Theorem 2.3 of Balan and Ndongo, 2016)**.** *For any predictable process H and for any $p \geq 2$,*

$$\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} H(s,x,z) \widetilde{J}(ds,dx,dz)\right|^p\right] \leq$$
$$C_p \left\{ \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} H^2(t,x,z)\nu(dz)dxdt\right)^{p/2}\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(t,x,z)|^p \nu(dz)dxdt\right]\right\},$$

where $C_p > 0$ is a constant depending on p.

To control the p-th moments of integrals with respect to J, we use the following result.

Lemma B.2. *For any predictable process H such that $\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(t,x,z)|\nu(dz)dxdt < \infty$ a.s. and for any $p \geq 2$,*

$$\mathbb{E}\left|\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} H(t,x,z)J(dt,dx,dz)\right|^p \leq$$
$$C_p \left\{ \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} H^2(t,x,z)\nu(dz)dxdt\right)^{p/2}\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(t,x,z)|^p \nu(dz)dxdt\right] \right.$$
$$\left. + \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(t,x,z)|\nu(dz)dxdt\right)^p\right] \right\},$$

where $C_p > 0$ is a constant depending on p.

Proof: This follows from Theorem B.1, writing $\int HdJ = \int Hd\widetilde{J} + \int Hdtdx\nu(dz)$. □

The next result considers the case $p \leq 2$.

Theorem B.3. *a) For any predictable process H and for any $p \in [1,2]$,*

$$\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} H(s,x,z)\widetilde{J}(ds,dx,dz)\right|^p\right] \leq C_p \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(t,x,z)|^p \nu(dz)dxdt\right],$$

where $C_p > 0$ is a constant depending on p.

b) For any predictable process H and for any $p \in (0,1)$,

$$\mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} H(s,x,z)J(ds,dx,dz)\right|^p \leq \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(t,x,z)|^p \nu(dz)dxdt\right].$$

Proof: a) The process $M_t = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} H(s,x,z)\widetilde{J}(ds,dx,dz)$ is a martingale with quadratic variation:

$$[M]_t = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} H^2(s,x,z)J(ds,dx,dz).$$

By the Burkholder-Davis-Gundy inequality, since $p \geq 1$, $\mathbb{E}(\sup_{t\leq T}|M_t|^p) \leq C_p \mathbb{E}([M]_t^{p/2})$. Since $p/2 \leq 1$, it can be proved that:

$$[M]_t^{p/2} \leq \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(s,x,z)|^p J(ds,dx,dz)$$

(see e.g. the proof of Lemma 8.22 of Peszat and Zabczyk, 2007). The conclusion follows taking expectation.

b) This follows using the inequality $|x+y|^p \leq |x|^p + |y|^p$. We refer to the proof of Lemma A.2.(3) of Chong et al. (2019). □

Acknowledgements

The author is grateful to Carsten Chong and Gennady Samorodnitsky for useful discussions. She would also like to thank an anonymous referee for insightful comments which lead to the discussion about the case $h(x) = 1$ at the end of Section 2.

References

Albeverio, S., Wu, J.-L., and Zhang, T.-S. Parabolic SPDEs driven by Poisson white noise. *Stochastic Process. Appl.*, **74** (1), 21–36 (1998). MR1624076.

Applebaum, D. and Wu, J.-L. Stochastic partial differential equations driven by Lévy space-time white noise. *Random Oper. Stochastic Equations*, **8** (3), 245–259 (2000). MR1796675.

Balan, R. M. SPDEs with α-stable Lévy noise: a random field approach. *Int. J. Stoch. Anal.*, pp. Art. ID 793275, 22 (2014). MR3166748.

Balan, R. M. and Ndongo, C. B. Intermittency for the wave equation with Lévy white noise. *Statist. Probab. Lett.*, **109**, 214–223 (2016). MR3434981.

Berger, Q., Chong, C., and Lacoin, H. The stochastic heat equation with multiplicative Lévy noise: existence, moments and intermittency. *ArXiv Mathematics e-prints* (2021). arXiv: 2111.07988.

Bichteler, K. *Stochastic integration with jumps*, volume 89 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (2002). ISBN 0-521-81129-5. MR1906715.

Bolaños Guerrero, R., Nualart, D., and Zheng, G. Averaging 2d stochastic wave equation. *Electron. J. Probab.*, **26**, Paper No. 102, 32 (2021). MR4290504.

Chen, L. and Dalang, R. C. Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab.*, **43** (6), 3006–3051 (2015). MR3433576.

Chen, X. Moment asymptotics for parabolic Anderson equation with fractional time-space noise: in Skorokhod regime. *Ann. Inst. Henri Poincaré Probab. Stat.*, **53** (2), 819–841 (2017). MR3634276.

Chong, C. Lévy-driven Volterra equations in space and time. *J. Theoret. Probab.*, **30** (3), 1014–1058 (2017a). MR3687248.

Chong, C. Stochastic PDEs with heavy-tailed noise. *Stochastic Process. Appl.*, **127** (7), 2262–2280 (2017b). MR3652413.

Chong, C., Dalang, R. C., and Humeau, T. Path properties of the solution to the stochastic heat equation with Lévy noise. *Stoch. Partial Differ. Equ. Anal. Comput.*, **7** (1), 123–168 (2019). MR3916265.

Chong, C. and Kevei, P. Intermittency for the stochastic heat equation with Lévy noise. *Ann. Probab.*, **47** (4), 1911–1948 (2019). MR3980911.

Conus, D. and Dalang, R. C. The non-linear stochastic wave equation in high dimensions. *Electron. J. Probab.*, **13**, no. 22, 629–670 (2008). MR2399293.

Conus, D., Joseph, M., and Khoshnevisan, D. On the chaotic character of the stochastic heat equation, before the onset of intermitttency. *Ann. Probab.*, **41** (3B), 2225–2260 (2013). MR3098071.

Dalang, R. C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.*, **4**, no. 6, 29 (1999). MR1684157.

Dalang, R. C. and Frangos, N. E. The stochastic wave equation in two spatial dimensions. *Ann. Probab.*, **26** (1), 187–212 (1998). MR1617046.

Dalang, R. C. and Quer-Sardanyons, L. Stochastic integrals for spde's: a comparison. *Expo. Math.*, **29** (1), 67–109 (2011). MR2785545.

Di Nunno, G., Øksendal, B., and Proske, F. *Malliavin calculus for Lévy processes with applications to finance*. Universitext. Springer-Verlag, Berlin (2009). ISBN 978-3-540-78571-2. MR2460554.

Foondun, M. and Khoshnevisan, D. Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.*, **14**, no. 21, 548–568 (2009). MR2480553.

Gīhman, u. I. and Skorohod, A. V. *The theory of stochastic processes. I.* Die Grundlehren der mathematischen Wissenschaften, Band 210. Springer-Verlag, New York-Heidelberg (1974). MR0346882.

Griffiths, M. and Riedle, M. Modelling Lévy space-time white noises. *J. Lond. Math. Soc. (2)*, **104** (3), 1452–1474 (2021). MR4332483.

Hu, Y. and Nualart, D. Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields*, **143** (1-2), 285–328 (2009). MR2449130.

Huang, J., Lê, K., and Nualart, D. Large time asymptotics for the parabolic Anderson model driven by spatially correlated noise. *Ann. Inst. Henri Poincaré Probab. Stat.*, **53** (3), 1305–1340 (2017). MR3689969.

Ito, K. On stochastic differential equations. *Mem. Amer. Math. Soc.*, **4**, 51 (1951). MR40618.

Jacod, J. and Shiryaev, A. N. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition (2003). ISBN 3-540-43932-3. MR1943877.

Jakubowski, A. and Riedle, M. Stochastic integration with respect to cylindrical Lévy processes. *Ann. Probab.*, **45** (6B), 4273–4306 (2017). MR3737911.

Kosmala, T. and Riedle, M. Stochastic integration with respect to cylindrical Lévy processes by p-summing operators. *J. Theoret. Probab.*, **34** (1), 477–497 (2021). MR4207499.

Marinelli, C. and Röckner, M. On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI*, volume 2123 of *Lecture Notes in Math.*, pp. 293–315. Springer, Cham (2014). MR3330821.

Millet, A. and Sanz-Solé, M. A stochastic wave equation in two space dimension: smoothness of the law. *Ann. Probab.*, **27** (2), 803–844 (1999). MR1698971.

Mueller, C. The heat equation with Lévy noise. *Stochastic Process. Appl.*, **74** (1), 67–82 (1998). MR1624088.

Mytnik, L. Stochastic partial differential equation driven by stable noise. *Probab. Theory Related Fields*, **123** (2), 157–201 (2002). MR1900321.

Nualart, D. and Nualart, E. *Introduction to Malliavin calculus*, volume 9 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge (2018). ISBN 978-1-107-61198-6; 978-1-107-03912-4. MR3838464.

Nualart, D. and Zheng, G. Central limit theorems for stochastic wave equations in dimensions one and two. *Stoch. Partial Differ. Equ. Anal. Comput.*, **10** (2), 392–418 (2022). MR4439987.

Peszat, S. and Zabczyk, J. *Stochastic partial differential equations with Lévy noise: An evolution equation approach*, volume 113 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (2007). ISBN 978-0-521-87989-7. MR2356959.

Protter, P. E. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, second edition (2004). ISBN 3-540-00313-4. MR2020294.

Saint Loubert Bié, E. Étude d'une EDPS conduite par un bruit poissonnien. *Probab. Theory Related Fields*, **111** (2), 287–321 (1998). MR1633586.

Samorodnitsky, G. and Taqqu, M. S. *Stable non-Gaussian random processes: Stochastic models with infinite variance*. Stochastic Modeling. Chapman & Hall, New York (1994). ISBN 0-412-05171-0. MR1280932.

Walsh, J. B. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pp. 265–439. Springer, Berlin (1986). MR876085.