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Transportation of diffuse random measures on $\mathbb{R}^d$

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Abstract. We consider two jointly stationary and ergodic random measures ξ and η on $\mathbb{R}^d$ with equal finite intensities, assuming ξ to be diffuse (non-atomic). An allocation is a random mapping taking $\mathbb{R}^d$ to $\mathbb{R}^d \cup \{\infty\}$ in a translation invariant way. We construct allocations transporting the diffuse ξ to arbitrary η, under the mild condition of existence of an 'auxiliary' point process which is needed only in the case when η is diffuse. When that condition does not hold we show by a counterexample that an allocation transporting ξ to η need not exist.

1. Introduction

Mass transportation is an important and lively research area. We refer to Villani (2009) for an extensive monograph on optimal transports. A more recent addition to the literature is transports between random measures (and in particular balancing allocations), which connects to the classical topic in several ways. For instance it was shown in Hoffman et al. (2006); Last and Thorisson (2009) that balancing transports between stationary random measures correspond to certain couplings (shift-couplings) of the associated Palm measures, while Huesmann and Sturm (2013) studied quantitative optimality of a balancing allocation.

Before going further we need to establish some notation and terminology. An *allocation* is a random mapping $\tau : x \mapsto \tau(x)$ taking $\mathbb{R}^d$ to $\mathbb{R}^d \cup \{\infty\}$ in a translation-invariant (equivariant, covariant) way; note that in this paper the invariance is included in the definition of allocation. Let ξ and η be jointly stationary and ergodic random measures on $\mathbb{R}^d$ ($d \geq 1$) with finite identical intensities $\lambda_\xi = \lambda_\eta$. Joint ergodicity of ξ and η means that the distribution of (ξ, η) takes only the values 0 and 1 on translation invariant sets, and joint stationarity means that (with $\overset{D}{=}$ denoting

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identity in distribution)

$$(\xi(x+\cdot), \eta(x+\cdot)) \stackrel{D}{=} (\xi, \eta), \quad x \in \mathbb{R}^d.$$

Say that an allocation *balances* the *source* ξ and the *destination* η if it transports ξ to η, that is, if (a.s.) the image of the measure ξ under τ is η,

$$\xi(\tau \in \cdot) = \eta.$$

See Section 2 for exact framework and definitions. See Remark 9.5 for historical notes beyond those in this introduction.

In the present paper we construct allocations balancing a *diffuse* (a *non-atomic*) source ξ and an *arbitrary* destination η. In order to explain the point of the paper, let us outline two remarkable examples.

Extra head: The search for balancing allocations goes back to Liggett's surprising idea of 'how to choose a head at random' – an *extra head* – in a two-sided sequence of i.i.d. coin tosses. If there is a head at the origin, it is an extra head (the other coins are i.i.d.). If there is a tail at the origin, move the origin to the right counting heads and tails until you have more heads than tails. Then you are at a head and it is an extra head; see Liggett (2002).

This can be restated as follows. If η is the simple point process (on $\mathbb{Z}$) formed by the heads, and T is the location of the extra head, then $\eta(T+\cdot)$ has the same distribution as the Palm version η° of η,

$$\eta(T+\cdot) \stackrel{D}{=} \eta^\circ. \tag{1.1}$$

This means that we have constructed a *shift-coupling* of η and η°.

Allocations provide a proof of this result as follows. For each $n \in \mathbb{Z}$ let $\tau(n)$ be the location of the 'extra head' found when starting from n rather than from the origin 0. Then the map $\tau : \mathbb{Z} \to \mathbb{Z} \cup \infty$ is an allocation that leaves the heads where they are, while τ turns out to be (a.s.) a bijection from the tails to the heads. Thus if we let μ be the counting measure on $\mathbb{Z}$ then τ balances μ and 2η (or, equivalently, τ balances $\mu/2$ and η). According to Hoffman et al. (2006); Last and Thorisson (2009), this implies that (1.1) holds.

Stable marriage of Poisson and Lebesgue: Liggett's idea led Hoffman, Holroyd and Peres to solving an open problem from the mid nineties: how to find an *extra point* in a Poisson process. Let η be a stationary Poisson process on $\mathbb{R}^d$ with intensity 1. Associate disjoint cells of volume 1 to the points of η as follows. Expand balls simultaneously from all the points of η. If the ball of a particular point has accumulated volume 1 before it hits another ball, then this ball is the cell of that point. If not, then continue expanding to accumulate volume when reaching space that has not already been reached by another ball; stop when volume 1 has been accumulated. It turns out (a.s.) that in this way each point obtains a cell of volume 1 and that the cells are disjoint and cover $\mathbb{R}^d$. Now let T be the point of the cell containing the origin. If the origin is shifted to T and that point is removed then the remaining points of $\eta(T+\cdot)$ form a stationary Poisson process with intensity 1. Thus T is an extra point; see Hoffman et al. (2006).

This again means that we have constructed a shift-coupling of η and its Palm version η°, that is, (1.1) holds. And again, allocations provide a proof of that result as follows. For each $x \in \mathbb{R}^d$ let $\tau(x)$ be the point of the cell containing x and let μ be Lebesgue measure on $\mathbb{R}^d$. Then τ is (clearly) an allocation balancing μ and η. According to Hoffman et al. (2006); Last and Thorisson (2009), this implies that (1.1) holds.

What makes allocations particularly interesting are the shift-coupling results in the above examples. In both examples the source μ is translation-invariant and non-random, but the shift-coupling result extends to general ξ and η as follows. If a stochastic process (or a random measure) X is stationary and ergodic jointly with ξ and η, and τ transports ξ to η, then by shifting the origin to $\tau(0)$ the *Palm version* of X w.r.t. ξ turns into the Palm version of X w.r.t. η. See (2.10) in the next section, and Last and Thorisson (2009), for this result. See Aldous and Thorisson (1993); Thorisson (1996) for the origin of shift-coupling.

Unbiased Skorokhod embedding: In Last et al. (2014, 2018) it is shown in the one-dimensional case, $d = 1$, that if ξ is diffuse (non-atomic), η is arbitrary, and ξ and η are mutually singular then they are balanced by the allocation τ defined by

$$\tau(x) = \inf\{t > x : \xi([x,t]) \le \eta([x,t]\}, \quad x \in \mathbb{R}. \tag{1.2}$$

Local times of Brownian motion are diffuse and (see Geman and Horowitz (1973)) the two-sided standard Brownian motion $(B_x)_{x\in\mathbb{R}}$ is a Palm version of the σ-finite stationary Brownian measure. Thus (see Last et al. (2014)) the shift-coupling result for general ξ and η can be applied with $\xi = \ell^0$ the local time at 0 and with $\eta = \int \ell^y \nu(dy)$ where ℓ^y is the local time at y and where ν is a probability measure without atom at 0 (to ensure mutual singularity). This yields the following *unbiased* Skorokhod embedding: $(B_{\tau(0)+x})_{x\in\mathbb{R}}$ is a two-sided standard Brownian motion with distribution ν at $x = 0$. It is said to be *unbiased* because not only the one-sided $(B_{\tau(0)+x})_{x\ge 0}$, but also the two-sided $(B_{\tau(0)+x})_{x\in\mathbb{R}}$, is Brownian. The same approach results in various embeddings when applied to local times associated with Brownian motion, e.g. extra excursion (see Last et al. (2018)) and (see Pitman and Tang (2015)) extra Brownian bridge.

In the present paper we consider the d dimensional case when ξ is diffuse, η is arbitrary, and ξ and η need *not* be mutually singular. It turns out that there are special cases where balancing allocations do *not* exist (Section 8). In order to guarantee the existence of a balancing allocation we impose the mild condition of the existence of a non-zero simple point process χ on $\mathbb{R}^d$ with finite intensity λ_χ and such that ξ, η and χ are jointly stationary and ergodic. We call this simple point process χ *auxiliary.*

The following theorem is the main result of the paper. Note that the auxiliary χ is only needed when η is purely diffuse.

Theorem 1.1. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$. Let ξ be diffuse (non-atomic) and*

$$0 < \lambda_\xi = \lambda_\eta < \infty.$$

Then there exists an allocation balancing ξ and η if one of the following conditions holds:

(a) *η has a non-zero discrete component;*

(b) *η is diffuse and there exists an auxiliary χ.*

Condition (a) covers discrete η and, in particular, point processes. Note that under condition (a) an auxiliary χ always exists and can be chosen as a *factor* of η, that is, as a measurable and equivariant (w.r.t. translation) function of η. When the discrete component of η has isolated atoms then χ can be taken to be the support of η. And although in general the support of the discrete component need not consist of isolated points (it can even be dense), there exists a constant $c > 0$ such that the following simple point process (here δ_x is the measure with mass 1 at x)

$$\chi = \sum_x \mathbf{1}\{\eta(\{x\}) > c\}\delta_x \tag{1.3}$$

is non-zero.

Under condition (b), there are also cases where an auxiliary χ exists as a factor of (ξ, η). But the counterexample in Section 8 shows that the condition of the existence of an auxiliary χ cannot simply be removed from (b). There are diffuse ξ and η such that an allocation transporting ξ to η

does not exist. However, if extension of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is allowed, then that obstacle can be overcome.

Corollary 1.2. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$. Let ξ be diffuse and*

$$0 < \lambda_\xi = \lambda_\eta < \infty.$$

Extend $(\Omega, \mathcal{F}, \mathbb{P})$ to support a stationary Poisson process χ on $\mathbb{R}^d$ which is independent of ξ and η. Then there exists an allocation balancing ξ and η.

As stressed above, an important property of a balancing allocation τ is the shift-coupling at (2.10). When specialised to the case $\xi = \lambda_\eta$ times Lebesgue measure, then (2.10) turns into (2.11), that is, into (1.1) with $T = \tau(0)$. This is called *extra head scheme* in Holroyd and Peres (2005). Note however that removing a 'head' (or some pattern like an excursion) from η°, does not result in a copy of η except in very special cases such as coin tossing, the Poisson process and Brownian motion.

According to Theorem 1.1 applied with $\xi = \lambda_\eta$ times Lebesgue measure, the result (1.1) always holds with T a function of η alone, unless η is diffuse. In that case, (1.1) still holds with T a function of η, provided there exists an auxiliary χ that is a factor of η. Finally, according to Corollary 1.2, if external randomisation is allowed then (1.1) always holds for some T, also when η is diffuse. (Actually, due to ergodicity and the definition of Palm probabilities at (2.4), the distributions of η and η° have the same (zero-one) values on invariant sets. Thus, according to an abstract existence result in Thorisson (1996) for shift-coupling on groups, (1.1) always holds for some T defined on an extended probability space. But the constructions of T in the present paper are explicit.)

The plan of the paper is as follows. Section 2 collects some preliminaries on stationary random measures, balancing allocations, and Palm theory. Section 3 prepares for the proof of Theorem 1.1. The theorem is then proved in Sections 4-7, where allocations are constructed in four exhaustive cases: discrete η with only isolated atoms in Section 4, discrete η with some accumulating atoms in Section 5, diffuse η (η with no atoms) in Section 6, and η with discrete and diffuse parts in Section 7. We give algorithmically explicit constructions in the discrete cases, and less explicit in the diffuse. Section 8 proves by counterexample that the auxiliary χ cannot simply be removed from part (b) of Theorem 1.1. Section 9 concludes with remarks.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with expectation operator $\mathbb{E}$. A *random measure* (resp. *point process*) ξ on $\mathbb{R}^d$ (equipped with its Borel σ-field $\mathcal{B}(\mathbb{R}^d)$) is a kernel from Ω to $\mathbb{R}$ such that $\xi(\omega, C) < \infty$ (resp. $\xi(\omega, C) \in \mathbb{Z}_+$) for $\mathbb{P}$-a.e. ω and all compact $C \subset \mathbb{R}^d$; see e.g. Kallenberg (2002); Last and Penrose (2018). A point process ξ is called *simple* if $\xi(\{x\}) \in \{0, 1\}$, $x \in \mathbb{R}^d$, except on a set with probability zero. Further, let $(\Omega, \mathcal{F})$ be equipped with a *measurable flow* $\theta_x \colon \Omega \to \Omega$, $x \in \mathbb{R}^d$. This is a family of mappings such that $(\omega, x) \mapsto \theta_x\omega$ is measurable, θ_0 is the identity on Ω and

$$\theta_x \circ \theta_y = \theta_{x+y}, \quad x, y \in \mathbb{R}^d, \tag{2.1}$$

where $\circ$ denotes composition. An *allocation* Holroyd and Peres (2005); Last and Thorisson (2009) is a measurable mapping $\tau \colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d \cup \{\infty\}$ that is *equivariant* in the sense that

$$\tau(\theta_y\omega, x - y) = \tau(\omega, x) - y, \quad x, y \in \mathbb{R}^d,\ \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{2.2}$$

We illustrate these concepts with a simple but illustrative example.

Example 2.1. Take Ω as the space of all locally finite sets $\omega \subset \mathbb{R}^d$, equipped with the usual σ-field and define $\theta_x\omega := \omega - x$, $x \in \mathbb{R}^d$. The counting measure $\xi(\omega)$ supported by ω defines a (discrete) random measure ξ. An example of an allocation τ is to take $\tau(\omega, x)$ as the point of $\omega \in \Omega$ closest to $x \in \mathbb{R}^d$, using lexicographic order to break ties. (For $\omega = \emptyset$ we set $\tau(\omega, \cdot) \equiv \infty$.)

We assume that the measure $\mathbb{P}$ is *stationary*; that is

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d,$$

where θ_x is interpreted as a mapping from $\mathcal{F}$ to $\mathcal{F}$ in the usual way:

$$\theta_x A := \{\theta_x \omega : \omega \in A\}, \quad A \in \mathcal{F},\, x \in \mathbb{R}^d.$$

A random measure ξ on $\mathbb{R}^d$ is said to be *stationary* if

$$\xi(\theta_x \omega, C - x) = \xi(\omega, C), \quad C \in \mathcal{B}(\mathbb{R}^d),\, x \in \mathbb{R}^d, \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{2.3}$$

Abusing our notation by defining the shifts θ_x, $x \in \mathbb{R}^d$, also for measures on $\mathbb{R}^d$ in the obvious way, we obtain from (2.3) and stationarity of $\mathbb{P}$ that

$$\theta_x \xi \stackrel{D}{=} \xi, \quad x \in \mathbb{R}^d.$$

The *invariant* σ-field $\mathcal{I} \subset \mathcal{F}$ is the class of all sets $A \in \mathcal{F}$ satisfying $\theta_x A = A$ for all $x \in \mathbb{R}^d$. We also assume that $\mathbb{P}$ is *ergodic*; that is for any $A \in \mathcal{I}$, we have $\mathbb{P}(A) \in \{0, 1\}$ (see however Remark 9.1).

Let ξ be a stationary random measure on $\mathbb{R}^d$ with positive and finite *intensity*

$$\lambda_\xi := \mathbb{E}\xi[0, 1]^d.$$

The *Palm probability measure* $\mathbb{P}_\xi$ of ξ (with respect to $\mathbb{P}$) is defined by

$$\mathbb{P}_\xi(A) := \lambda_\xi^{-1} \lambda_d(B)^{-1}\, \mathbb{E} \int \mathbf{1}_B(x) \mathbf{1}_A(\theta_x)\, \xi(dx), \quad A \in \mathcal{F}, \tag{2.4}$$

where $B \subset \mathbb{R}^d$ is a Borel set with positive and finite Lebesgue measure $\lambda_d(B)$. This definition does not depend on B. The expectation operator associated with $\mathbb{P}_\xi$ is denoted by $\mathbb{E}_\xi$. Any multiple $c\lambda_d$ of Lebesgue measure is a (rather trivial) stationary random measure. In this case we obtain from stationarity of $\mathbb{P}$ that

$$\mathbb{P}_{c\lambda_d} = \mathbb{P}. \tag{2.5}$$

An allocation τ *balances* two random measures ξ and η if

$$\mathbb{P}(\xi(\{s \in \mathbb{R} : \tau(s) = \infty\}) > 0) = 0 \tag{2.6}$$

and the image measure of ξ under τ is η, that is,

$$\int \mathbf{1}\{\tau(s) \in C\}\, \xi(ds) = \eta(C), \quad C \in \mathcal{B}(\mathbb{R}^d),\ \mathbb{P}\text{-a.e.} \tag{2.7}$$

The balancing properties (2.6) and (2.7) imply easily that

$$\lambda_\xi = \lambda_\eta. \tag{2.8}$$

By Last and Thorisson (2009, Theorem 4.1) we then have the *shift-coupling*

$$\mathbb{P}_\xi(\theta_{\tau(0)} \in \cdot) = \mathbb{P}_\eta. \tag{2.9}$$

Remark 2.2. In this remark we consider the shift-coupling result (2.9) in terms of random elements. Let X be a random element that can be translated by $t \in \mathbb{R}^d$, for instance a random measure, or a random field, or the identity on Ω. Then X, ξ and η, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ above, are jointly stationary and ergodic.

A *Palm version of* X *w.r.t.* ξ is any random element with the distribution $\mathbb{P}_\xi(X \in \cdot)$. In particular, according to (2.5), a Palm version of X w.r.t. Lebesgue measure is X itself. What is generally called a *Palm version* of a random measure ξ is a random measure ξ° with the distribution $\mathbb{P}_\xi(\xi \in \cdot)$. That is, ξ° is a Palm version of ξ with respect to itself.

The informal interpretation of ξ° in the case when ξ is a simple point process is that ξ° behaves like ξ conditioned on having a point at the origin. When ξ is a Poisson process then (and only then) ξ° can be obtained simply by placing an extra point at the origin, $\xi^\circ = \xi + \delta_0$. In the ergodic case, which is assumed here, another informal interpretation of ξ° is that ξ° behaves like ξ with origin shifted to a uniformly chosen point of ξ, – or when ξ is not a simple point process, to a uniformly chosen location in the mass of ξ. (These interpretations are motivated by limit theorems.)

From (2.9) we obtain the following shift-coupling result. If τ is an allocation balancing ξ and η, then a shift of the origin to $\tau(0)$ turns a Palm version of X w.r.t. ξ into a Palm version of X w.r.t. η,

$$\mathbb{P}_\xi(\theta_{\tau(0)}X \in \cdot) = \mathbb{P}_\eta(X \in \cdot). \tag{2.10}$$

In particular, from (2.5) and (2.10) with $X = \eta$, we obtain the following result. If τ balances the Lebesgue-measure multiple $\lambda_\eta \lambda_d$ and η, then a shift of the origin to $\tau(0)$ turns the stationary η into a Palm version η° of η,

$$\theta_{\tau(0)}\eta \overset{D}{=} \eta^\circ. \tag{2.11}$$

On the other hand, from (2.5) and (2.10) with $X = \xi$, we obtain the reverse result. If τ balances ξ and the Lebesgue-measure multiple $\lambda_\xi \lambda_d$, then a shift of the origin to $\tau(0)$ turns the a Palm version ξ° of ξ into the stationary ξ,

$$\theta_{\tau(0)}\xi^\circ \overset{D}{=} \xi. \tag{2.12}$$

If ξ is the only source of randomness, then, as a rule, allocations balancing ξ and a multiple of Lebesgue measure do not exist, see Holroyd and Peres (2005) and also Section 8.

3. Allocations to Discrete Random Measures

Let ξ and η be two jointly stationary and ergodic random measures on $\mathbb{R}^d$; see Section 2. For the remainder of this paper we assume $\lambda_\xi > 0$ and $\lambda_\eta > 0$. Assume that η is a discrete random measure with locally finite support. Let η^* be the simple point process with the same support as η. Assume also that ξ, η and η^* have positive and finite intensities λ_ξ, λ_η and λ_{η^*} respectively. (The assumptions $\lambda_{\eta^*} < \infty$ has been made by convenience and could be removed.) We consider an allocation τ with the property

$$\xi(\{z \in \mathbb{R}^d : \tau(z) \notin \eta^* \cup \{\infty\}\}) = 0, \quad \mathbb{P}\text{-a.s.} \tag{3.1}$$

Define

$$C^\tau(z) := \{y \in \mathbb{R}^d \colon \tau(y) = z\}, \quad z \in \mathbb{R}^d.$$

Note that $C^\tau(z)$ is random.

Let $\alpha \in (0,\infty)$. The allocation τ is said to have *appetite* α (w.r.t. (ξ,η)) if (3.1) and the following two properties hold. First we have almost surely that

$$\eta^*(\{x \in \mathbb{R}^d : \xi(C^\tau(x)) > \alpha\eta\{x\}\}) = 0. \tag{3.2}$$

Second the probability that

$$\xi(\{z \in \mathbb{R}^d : \tau(z) = \infty\}) > 0 \quad \text{and} \quad \eta^*(\{x \in \mathbb{R}^d : \xi(C^\tau(x)) < \alpha\eta\{x\}\}) > 0 \tag{3.3}$$

is zero.

Proposition 3.1. *Assume that the allocation τ has appetite α for some $\alpha \in (0, \lambda_\xi \lambda_\eta^{-1}]$. Then τ is α-balanced, that is we have a.s. that $\eta(\{x \in \mathbb{R}^d : \xi(C^\tau(x)) \neq \alpha\eta\{x\}\}) = 0$. Moreover, we have that $\lambda_\xi \mathbb{P}^0_\xi(\tau(0) \neq \infty) = \alpha\lambda_\eta$.*

Proof: We generalise the proof of Last and Penrose (2018, Theorem 10.9). We start with a general result, that might be of independent interest. Let $g\colon \Omega \times \Omega \to [0,\infty)$. Then

$$\lambda_\xi \mathbb{E}_\xi \mathbf{1}\{\tau(0) \neq \infty\} g(\theta_0, \theta_{\tau(0)}) = \lambda_{\eta^*} \mathbb{E}_{\eta^*} \int_{C(0)} g(\theta_x, \theta_0)\, \xi(dx), \tag{3.4}$$

where we abbreviate $C(z) := C^\tau(z)$, $z \in \mathbb{R}^d$. This follows from Neveu's exchange formula (see e.g. Last and Penrose (2018, Remark 3.7)) applied to the function $h(\omega, x) := g(\omega, \theta_x \omega)\mathbf{1}\{\tau(\omega, 0) = x\}$ (and replacing (η, ξ) by (ξ, η^*)).

Let

$$A := \{\text{there exists } x \in \eta \text{ such that } \xi(C(x)) < \alpha\eta\{x\}\}.$$

This event is invariant. It follows from (3.3) that

$$\mathbb{P}_\xi(A) = \mathbb{P}_\xi(\{\tau(0) \neq \infty\} \cap A).$$

Therefore we obtain from (3.4) that

$$\lambda_\xi \mathbb{P}_\xi(A) = \lambda_{\eta^*}\, \mathbb{E}_{\eta^*} \mathbf{1}_A \xi(C(0)).$$

By definition (2.4) of the Palm probability measure of η^* we hence obtain for each Borel set $B \subset \mathbb{R}^d$ with $0 < \lambda_d(B) < \infty$ that

$$\begin{aligned}\lambda_\xi \mathbb{P}_\xi(A) &= \lambda_d(B)^{-1} \mathbb{E} \int_B \mathbf{1}\{\theta_x \in A\} \xi \circ \theta_x(C(0, \theta_x))\, \eta^*(dx)\\ &= \lambda_d(B)^{-1} \mathbb{E} \mathbf{1}_A \int_B \xi(C(x))\, \eta^*(dx),\end{aligned} \tag{3.5}$$

where we have used the invariance of A and

$$\begin{aligned}\xi \circ \theta_x(C(\theta_x, 0)) &= \int \mathbf{1}\{\tau(\theta_x, y) = 0\}\, \xi(\theta_x\omega, dy) = \int \mathbf{1}\{\tau(y+x) = x\}\, \xi(\theta_x\omega, dy)\\ &= \int \mathbf{1}\{\tau(y) = x\}\, \xi(dy) = \xi(C(x)).\end{aligned}$$

Using (3.2) and denoting the invariant σ-field by $\mathcal{I}$, this yields

$$\begin{aligned}\lambda_\xi \mathbb{P}_\xi(A) &\leq \lambda_d(B)^{-1} \alpha\, \mathbb{E}[\mathbf{1}_A \eta(B)]\\ &= \lambda_d(B)^{-1} \alpha\, \mathbb{E}[\mathbf{1}_A \mathbb{E}[\eta(B) \mid \mathcal{I}]]\\ &= \alpha\lambda_\eta\, \mathbb{P}(A) \leq \lambda_\xi \mathbb{P}(A) = \lambda_\xi \mathbb{P}_\xi(A),\end{aligned} \tag{3.6}$$

where we have used ergodicity to get the second equality (almost surely) and the assumption $\alpha \leq \lambda_\xi \lambda_\eta^{-1}$ to get the second inequality. Therefore the above inequalities are in fact equalities. Hence (3.5) and the right-hand side of (3.6) coincide, yielding that

$$\mathbb{E}\mathbf{1}_A \int_B (\alpha\eta\{x\} - \xi(C(x))\, \eta^*(dx) = 0.$$

Taking $B \uparrow \mathbb{R}^d$ and using montone convergence (justified by (3.2)), this yields

$$\mathbf{1}_A \int (\alpha\eta\{x\} - \xi(C(x)))\, \eta^*(dx) = 0, \quad \mathbb{P}\text{-a.s.}$$

Hence we have $\mathbb{P}$-a.s. on A that $\xi(C(x)) = \alpha\eta\{x\}$ for all $x \in \eta^*$. By definition of A this is possible only if $\mathbb{P}(A) = 0$. This implies the first assertion.

To prove the second assertion we use (3.4) with $g \equiv 1$ to obtain that $\lambda_\xi \mathbb{P}_\xi(\tau(0) \neq \infty) = \lambda_{\eta^*} \alpha\, \mathbb{E}_{\eta^*} \eta\{0\}$. Since it follows straight from the definitions that $\lambda_{\eta^*} \mathbb{E}_{\eta^*} \eta\{0\} = \lambda_\eta$ we can conclude the assertion. □

4. Destination Isolated Atoms

The spatial version of the Gale–Shapley allocation introduced in Hoffman et al. (2006) balances Lebesque measure to a simple point process. The simplified description of it in the introductory Poisson-Lebesgue example is not an effective way of proving that it actually works, the efficient way is algorithmic. We now extend this allocation to balance a diffuse ξ to an η consisting of isolated atoms. The extension is needed because unlike the Lebesgue measure a diffuse measure can have positive mass on lower dimensional sets like the boundaries of balls. Motivated by the point-optimal stable allocation introduced in Hoffman et al. (2006), we formulate an algorithm providing an allocation of appetite α.

The idea behind the algorithm (in the case $\alpha = 1$) can be sketched as follows. In the first round of the algorithm assign a *preference set* to each η-atom, that is, a set of sites in $\mathbb{R}^d$ that the atom *proposes* to. We do this by a finite recursion (note that the following four items can be reduced to one item when ξ is Lebesgue measure):

- From each η-atom blow up a "first" closed ball until you have gathered ξ-mass at least equal to the mass of that atom, m_1 say.
- Put $m_2 = m_1$ minus the ξ-mass in the **interior** of the "first" ball. This remaining mass m_2 is thus part of a ξ-mass sitting on the boundary of a d dimensional ball.
- Then blow up a "second" closed (d dimensional) ball from (e.g.) the lexicographically lowest location on the boundary of the "first" ball (think of it as a pole) until you have gathered ξ-mass on the **boundary** of the "first" ball that is at least m_2. This mass m_2 is sitting on a closed cap of the boundary of the "first" ball. Put $m_3 = m_2$ minus the ξ-mass of the interior of that cap (the relative interior of the cap w.r.t. the boundary, the sphere). This remaining mass, m_3, is a part of a mass sitting on the boundary of a $d-1$ dimensional ball.
- Repeat this down the dimensions until the η-atom has a *preference set* (the union of these interior sets) of ξ-mass exactly m_1 (because the final cap will be a circle segment and its boundary will have at most two points and their ξ-mass is zero since ξ is diffuse). Note that the preference sets of different η-atoms may overlap. Note also that a preference set of an η-atom can contain other η-atoms.

Now let each site that lies in at least one preference set of an η-atom put the closest of those atoms on a shortlist, using lexicographic order to break ties. The atoms associated with the other preference sets are rejected. Each atom has now a rejection set, a subset of its preference set containing sites that rejected the proposal.

Repeat the above procedure recursively by blowing up a ball around each η-atom restricted to the complement of its associated rejection set (thus extending its preference set) and, after each round, add the new rejections to its rejection set. One of two things can happen for a site $z \in \mathbb{R}^d$. Either it never appears in one of the preference sets. Then z has no partner and is allocated to ∞. Or it eventually shortlists a single point x. Then z is allocated to x.

In the algorithm we will use the notation $D(t-) := \bigcup_{s<t} D(s)$ for an increasing family of sets $D(s) \subset \mathbb{R}^d$, $s > 0$. If μ is a measure on $\mathbb{R}^d$ and $\mu\{x\} > 0$ we write $x \in \mu$.

Algorithm 4.1. Let $\alpha > 0$, $\mu \neq 0$ be discrete with isolated atoms and ν be diffuse with infinite mass. For $n \in \mathbb{N}$, $x \in \mu$ and $z \in \mathbb{R}^d$, define the sets

$$C_n(x) \subset \mathbb{R}^d \qquad \text{(the set of sites \textit{claimed}, or \textit{preferred}, by } x \text{ at stage } n),$$

$$R_n(x) \subset \mathbb{R}^d \qquad \text{(the set of sites \textit{rejecting} } x \text{ during the first } n \text{ stages)},$$

$$A_n(z) \subset \mu \qquad \text{(the set of points of } \mu \text{ claiming site } z \text{ in the first } n \text{ stages)},$$

via the following recursion in n. Start with setting $R_0(x) := \emptyset$.

(1) This first step is a recursion within the recursion. Fix $x \in \mu$ and mostly suppress it in the notation until step (1) is over. Define

$$\begin{aligned} S_{n,1}(s) &:= B(x,s) = \text{ the ball with center } x \text{ and radius } s > 0, \\ \beta_{n,1} &:= \alpha\mu\{x\}, \\ s_{n,1} &:= \inf\{s \geq 0 : \nu\big(S_{n,1}(s) \setminus R_{n-1}\big) \geq \beta_{n,1}\}, \\ \Delta_{n,1} &:= S_{n,1}(s_{n,1}) \setminus S_{n,1}(s_{n,1}-) \qquad \text{note that } \Delta_{n,1} := \partial B(x, s_{n,1}). \end{aligned}$$

For $k = 1, \dots, d-1$, proceed recursively as follows. Let $y_{n,k} \in \mathbb{R}^d$ be the lexicographically lowest element of $\Delta_{n,k}$ and set

$$\begin{aligned} S_{n,k+1}(s) &:= B(y_{n,k}, s) \cap \Delta_{n,k} \\ \beta_{n,k+1} &:= \beta_{n,k} - \nu\big(S_{n,k}(s_{n,k}-) \setminus R_{n-1}\big) \\ s_{n,k+1} &:= \inf\{s \geq 0 : \nu\big(S_{n,k+1}(s) \setminus R_{n-1}\big) \geq \beta_{n,k+1}\} \\ \Delta_{n,k+1} &:= S_{n,k+1}(s_{n,k+1}) \setminus S_{n,k+1}(s_{n,k+1}-). \end{aligned}$$

Then $\Delta_{n,d}$ contains at most two elements. Since ν is diffuse this implies that

$$\nu\big(S_{n,d}(s_{n,d}) \setminus R_{n-1}\big) = \nu\big(S_{n,d}(s_{n,d}-) \setminus R_{n-1}\big) = \beta_{n,d}.$$

Now set

$$C_n := S_{n,1}(s_{n,1}-) \cup \cdots \cup S_{n,d}(s_{n,d}-).$$

Since C_n is a disjoint union we have

$$\nu(C_n \setminus R_{n-1}) = \nu(S_{n,1}(s_{n,1}-) \setminus R_{n-1}) + \cdots + \nu(S_{n,d}(s_{n,d}-) \setminus R_{n-1}).$$

Since

$$\beta_{n,k+1} := \beta_{n,k} - \nu\big(S_{n,k}(s_{n,k}-) \setminus R_{n-1}\big) \qquad \text{and} \qquad \nu\big(S_{n,d}(s_{n,d}-) \setminus R_{n-1})\big) = \beta_{n,d}$$

this yields

$$\nu(C_n \setminus R_{n-1}) = (\beta_{n,1} - \beta_{n,2}) + \cdots + (\beta_{n,d-1} - \beta_{n,d}) + \beta_{n,d} = \beta_{n,1}.$$

Thus, due to $\beta_{n,1} = \alpha\mu\{x\}$, we obtain

$$\nu(C_n \setminus R_{n-1}) = \alpha\mu\{x\}.$$

(2) Recall that x was suppressed in the above step. We now make it explicit and write $C_n(x)$ instead of only C_n. For $z \in \mathbb{R}^d$, define

$$A_n(z) := \{x \in \mu : z \in C_n(x)\}.$$

If $A_n(z) \neq \emptyset$ then define

$$\tau_n(z) := l(\{x \in A_n(z) : \|z - x\| = d(z, A_n(z))\})$$

as the point *shortlisted* by site z at stage n, where $l(B)$ denotes the lexicographic minimum of a finite non-empty set $B \subset \mathbb{R}^d$ and where $d(z, A_n(z))$ is the distance of z from the set $A_n(z)$. If $A_n(z) = \emptyset$ then define $\tau_n(z) := \infty$.

(3) For $x \in \mu$, define

$$R_n(x) := \{z \in C_n(x) : \tau_n(z) \neq x\}.$$

Now define a mapping $\tau^\alpha(\nu,\mu,\cdot)\colon \mathbb{R}^d \to \mathbb{R}^d \cup \{\infty\}$ as follows. If $\tau_n(z) = \infty$ for all $n \in \mathbb{N}$ put $\tau^\alpha(\nu,\mu,z) := \infty$. Otherwise, set $\tau^\alpha(\nu,\mu,z) := \lim_{n\to\infty} \tau_n(z)$. We argue as follows that this limit exists. Defining $C_0(x) := \{x\}$ for all $x \in \mu$, we assert that for all $n \in \mathbb{N}$ the following holds:

$$C_n(x) \supset C_{n-1}(x), \quad x \in \mu, \tag{4.1}$$

$$A_n(z) \supset A_{n-1}(z), \quad z \in \mathbb{R}^d, \tag{4.2}$$

$$R_n(x) \supset R_{n-1}(x), \quad x \in \mu. \tag{4.3}$$

This is proved by induction; clearly (4.1) implies (4.2) and (4.2) implies (4.3), while (4.3) implies that (4.1) holds for the next value of n. By (4.2), $\|\tau_n(z) - z\|$ is decreasing in n, and hence, since μ is locally finite, there exist $x \in \mu$ and $n_0 \in \mathbb{N}$ such that $\tau_n(z) = x$ for all $n \geq n_0$. In this case we define $\tau^\alpha(\nu,\mu,z) := x$. If $\nu(\mathbb{R}^d) < \infty$ or $\mu(\mathbb{R}^d) = 0$ we set $\tau^\alpha(\nu,\mu,z) := \infty$. We shall now prove that τ^α (applied with ξ and η instead of ν and μ) has the following property defined in Section 3.

Lemma 4.2. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$ such that ξ is diffuse, η is discrete with locally finite support and $\lambda_\xi\lambda_\eta > 0$. Let $\alpha > 0$. Then τ defined on $\Omega \times \mathbb{R}^d$ by $\tau(\omega,x) := \tau^\alpha(\xi(\omega),\eta(\omega),x)$ is an allocation with appetite α.*

Proof: It follows by induction over the stages of Algorithm 4.1 that the mappings τ_n are measurable as functions of ν, μ and z, where measurability in ν and μ refers to the standard σ-field on the space of locally finite measures; see e.g. Last and Penrose (2018). (The proof of this fact is left to the reader.) Hence τ^α is measurable. Moreover it is clear that τ^α and hence also τ has the required covariance property. Next we note that $\mathbb{P}(\xi(\mathbb{R}^d) = \eta(\mathbb{R}^d) = \infty) = 1$, a consequence of ergodicity and $\lambda_\xi\lambda_\eta > 0$.

In the remainder of the proof we fix two locally finite measures ν and $\mu \neq 0$. We assume that ν is diffuse and satisfies $\nu(\mathbb{R}^d) = \infty$, while μ is assumed to be discrete with purely isolated atoms. Upon defining $\tau^\alpha(\nu,\mu,\cdot)$ we noted that for each $z \in \mathbb{R}^d$, either $\tau^\alpha(\nu,\mu,z) = \infty$ or $\tau_n(z) = x$ for some $x \in \mu$ and all sufficiently large $n \in \mathbb{N}$. Therefore

$$\mathbf{1}\{\tau^\alpha(\nu,\mu,z) = x\} = \lim_{n\to\infty} \mathbf{1}\{z \in C_n(x) \setminus R_{n-1}(x)\}, \quad z \in \mathbb{R}^d. \tag{4.4}$$

On the other hand, by Algorithm 4.1(1) we have $\nu(C_n(x) \setminus R_{n-1}(x)) \leq \alpha\mu\{x\}$, so that (3.2) follows from Fatou's lemma.

As in Section 3 we set

$$C^{\tau^\alpha}(x) := \{z \in \mathbb{R}^d : \tau^\alpha(\nu,\mu,z) = x\}.$$

We now show that $\{z \in \mathbb{R}^d : \tau^\alpha(\nu,\mu,z) = \infty\} \neq \emptyset$ and $\{x \in \mu : \nu(C^{\tau^\alpha}(x)) < \alpha\mu\{x\}\} \neq \emptyset$ cannot hold simultaneously, implying the event at (3.3) to have probability zero. For that purpose we assume the strict inequality $\nu(C^{\tau^\alpha}(x)) < \alpha\mu\{x\}$ for some $x \in \mu$. By (4.4) this implies that there exist $n_0 \in \mathbb{N}$ and $\alpha_1 < \alpha$ such that $\nu(C_n(x) \setminus R_{n-1}(x)) \leq \alpha_1\mu\{x\}$ for $n \geq n_0$. Let $C_\infty(x) := \cup_{n=1}^\infty C_n(x)$. We assert that $C_\infty(x) = \mathbb{R}^d$. Assume on the contrary this is not the case. By construction, there exist $r_n(x) > 0$, $n \in \mathbb{N}$, such that $B^0(x, r_n(x)) \subset C_n(x) \subset B(x, r_n(x))$, where $B^0(x, r_n(x))$ is the interior of $B(x, r_n(x))$. Since $C_\infty(x) \neq \mathbb{R}^d$ and the sets C_n are increasing, we have $r_\infty(x) := \lim_{n\to\infty} r_n(x) < \infty$. Then $C_\infty(x) \subset B(x, r_\infty(x))$ is bounded and there exists $n \geq n_0$ such that $\nu(C_\infty(x) \setminus C_n(x)) \leq \mu\{x\}(\alpha - \alpha_1)/2$. Hence we obtain (since $R_{n-1}(x) \subset C_n(x)$)

$$\nu(C_\infty(x) \setminus R_{n-1}(x)) = \nu(C_\infty(x) \setminus C_n(x)) + \nu(C_n(x) \setminus R_{n-1}(x)) \leq \alpha_2\mu\{x\},$$

where $\alpha_2 := (\alpha + \alpha_1)/2 < \alpha$. By definition of the algorithm this implies that $C_\infty(x)$ is a strict subset of $C_n(x)$. This contradiction shows that $C_\infty(x) = \mathbb{R}^d$. Now taking $z \in \mathbb{R}^d$, we hence have $z \in C_n(x)$ for some $n \geq 1$, so that z shortlists either x or some closer point of μ. In either case, $\tau^\alpha(\nu,\mu,z) \neq \infty$. □

Proposition 4.3. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$ such that ξ is diffuse, η is discrete with locally finite support (isolated atoms) and $0 < \lambda_\xi = \lambda_\eta < \infty$. Then τ defined on $\Omega \times \mathbb{R}^d$ by $\tau(\omega, x) := \tau^1(\xi(\omega), \eta(\omega), x)$ is an allocation balancing ξ and η.*

Proof: By Lemma 4.2 τ is an allocation of appetite 1. Since $\lambda_\xi = \lambda_\eta$ we can apply Proposition 3.1 to see that $\eta(\{x \in \mathbb{R}^d : \xi(C^\tau(x)) \neq \eta\{x\}\}) = 0$ holds almost surely and moreover that $\mathbb{P}_\xi(\tau(0) \neq \infty) = 1$. These two facts imply the desired balancing property of τ. □

The allocation in Proposition 4.3 is stable. In Section 5 we shall consider a general discrete η. But the balancing allocation will not be stable anymore.

Remark 4.4. Proposition 4.3 can already be found as Proposition 4.37 in Haji-Mirsadeghi and Khezeli (2016). There the authors used a site-optimal version of a stable allocation while ours is point-optimal.

5. Destination a discrete random measure

In this section we deal with a random measure η which is discrete but not necessarily with a locally finite support (isolated atoms). We need to introduce some notation. If $A \subset \Omega \times \mathbb{R}^d$ is measurable then we identify A with the mapping $\omega \mapsto A(\omega) := \{x \in \mathbb{R}^d : (\omega, x) \in A\}$. If ξ is a random measure on $\mathbb{R}^d$, then we define for each $\omega \in \Omega$ the restriction of $\xi(\omega)$ to $A(\omega)$ by $\xi_A(\omega) := \int \mathbf{1}\{x \in \cdot, (\omega, x) \in A\}\xi(\omega, dx)$. Clearly ξ_A is again a random measure.

Proposition 5.1. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$ such that ξ is diffuse, η is discrete and $\infty > \lambda_\xi \geq \lambda_\eta > 0$. Then there exists a measurable $A \subset \Omega \times \mathbb{R}^d$ and an allocation τ balancing ξ_A and η. If $\lambda_\xi = \lambda_\eta$, then τ balances ξ and η.*

Proof: Write $\eta = \sum_{n=1}^\infty \eta_n$, where $\eta_1, \eta_2, \dots$ are the mutually singular discrete random measures

$$\eta_n = \sum_x \mathbf{1}\{1/n \leq \eta\{x\} < 1/(n-1)\}\eta(\{x\})\delta_x$$

which all have non-accumulating (isolated) points. (In the definition of η_1 we use the convention $1/0 := \infty$.)

Throughout this proof we consider the stable allocation τ^1 with appetite 1; see the definition preceding Lemma 4.2. Starting with $\xi^1 := \xi$ we define sequences of measurable sets $A_n \subset \Omega \times \mathbb{R}^d$, $n \in \mathbb{N}$, and of stationary random measures $(\xi_n)_{n\geq 1}$ and $(\xi^n)_{n\geq 1}$ recursively, by setting for each $n \in \mathbb{N}$,

$$\begin{aligned} A_n &:= \{x \in \mathbb{R}^d : \tau^1(\xi^n, \eta_n, x) \neq \infty\}, \\ \xi_n &:= \xi^n_{A_n}, \\ \xi^{n+1} &:= \xi^n_{\mathbb{R}^d \setminus A_n}. \end{aligned}$$

Set $B_n := A_1 \cup \dots \cup A_n$. Using Proposition 3.1 one can prove by induction that $\xi(A_{n+1} \cap B_n) = 0$, and $\lambda_{\xi_n} = \lambda_{\eta_n}$. Note that $\sum_{n=1}^\infty \xi_n = \xi_A$, where $A := \cup_{n=1}^\infty A_n$.

The calculation below shows that it is no restriction of generality to assume that the sets A_n are disjoint. Therefore we can define an allocation τ by

$$\tau(x) := \begin{cases} \tau^1(\xi^n, \eta_n, x), & \text{if } x \in A_n \text{ for some } n \in \mathbb{N}, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.1}$$

Then we obtain for each Borel set $C \subset \mathbb{R}^d$ that

$$\begin{aligned}\int \mathbf{1}\{\tau(x) \in C\}\xi_A(dx) &= \sum_{n=1}^{\infty} \int \mathbf{1}\{\tau(x) \in C\}\xi_n(dx) \\ &= \sum_{n=1}^{\infty} \int \mathbf{1}\{\tau^1(\xi^n, \eta_n, x) \in C\}\xi_n(dx) \\ &= \sum_{n=1}^{\infty} \eta_n(C) = \eta(C).\end{aligned}$$

Therefore τ is balancing ξ_A and η.

Assume now that $\lambda_\xi = \lambda_\eta$. Then we obtain for each Borel set $C \subset \mathbb{R}^d$ that

$$\mathbb{E}\xi_A(C) = \sum_{n=1}^{\infty} \mathbb{E}\xi_n(C) = \sum_{n=1}^{\infty} \lambda_{\xi_n}\lambda_d(C) = \sum_{n=1}^{\infty} \lambda_{\eta_n}\lambda_d(C) = \lambda_\eta\lambda_d(C) = \lambda_\xi\lambda_d(C).$$

Therefore, $\mathbb{E}\xi_A(C) = \mathbb{E}\xi(C)$. Since $\xi_A \leq \xi$, this implies that $\xi_A = \xi$ $\mathbb{P}$-a.s. □

6. Destination a Diffuse Random Measure

In this section we deal with a diffuse destination η in the case when there exists an auxiliary simple point process χ. The key idea is to use the allocation from the isolated-atoms case (Proposition 4.3) to map both ξ and η to χ creating pairs of ξ-cells and η-cells of mass one associated with each point of χ, and then to transport the ξ-mass of the ξ-cells into the η-mass of the η-cells by passing through Lebesgue measure on $[0, 1]$.

Proposition 6.1. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$ such that $0 < \lambda_\xi = \lambda_\eta < \infty$. Assume further that ξ and η are both diffuse and that there exists an auxiliary simple point process χ with finite intensity λ_χ. Then there exists an allocation τ balancing ξ and η.*

Proof: Note that an allocation balances the diffuse ξ and η if and only if it balances $a\xi$ and $a\eta$ for any positive constant a, in particular for $a = \lambda_\chi/\lambda_\xi = \lambda_\chi/\lambda_\eta$. So it is no restriction to assume that the common intensity of ξ and η is the same as that of χ, that is, $\lambda_\xi = \lambda_\eta = \lambda_\chi$. We can then apply Proposition 4.3 to the pair ξ and χ and to the pair η and χ.

For each point s of χ let A_s and B_s, respectively, be the resulting allocation cells of ξ and η that are mapped to s. Fix $t \in \mathbb{R}^d$ and let S_t be the point of χ such that $t \in A_{S_t}$. Let $\xi_t = \xi(\cdot \cap A_{S_t})$ be the restriction of ξ to A_{S_t} and let $\eta_t = \eta(\cdot \cap B_{S_t})$ be the restriction of η to B_{S_t}. Since χ is a simple point process (with mass one at each of its points) both ξ_t and η_t are (random) probability measures.

Let ϕ be a measurable bijection from $\mathbb{R}^d$ to $\mathbb{R}$ such that ϕ^{-1} is also measurable. Shift the origin 0 to S_t to obtain (random) probability measures $\theta_{S_t}\xi_t$ and $\theta_{S_t}\eta_t$ that are concentrated on the shifted cells $A_{S_t} - S_t$ and $B_{S_t} - S_t$. Let F_t and G_t be the (random) distribution functions of ϕ under these probability measures, that is, for $x \in \mathbb{R}$

$$\begin{aligned}F_t(x) &= \theta_{S_t}\xi_t(\phi \leq x), \\ G_t(x) &= \theta_{S_t}\eta_t(\phi \leq x).\end{aligned}$$

Note that F_t is a continuous function since ξ does not have any atoms and since ϕ is a bijection. Thus

$$\text{the distribution of } F_t(\phi) \text{ under } \theta_{S_t}\xi_t \text{ is uniform on } [0,1].$$

With G_t^{-1} the generalized inverse (quantile function) of G_t, this in turn implies that $G_t^{-1}(F_t(\phi))$ under $\theta_{S_t}\xi_t$ has the distribution function G_t. Finally, since ϕ is a bijection this implies that

$$\text{the distribution of } \phi^{-1}(G_t^{-1}(F_t(\phi))) \text{ under } \theta_{S_t}\xi_t \text{ is } \theta_{S_t}\eta_t.$$

In other words, the mapping $x \mapsto \phi^{-1}(G_t^{-1}(F_t(\phi(x))))$ transports the measure $\theta_{S_t}\xi_t$ on $A_{S_t}-S_t$ into the measure $\theta_{S_t}\eta_t$ on $B_{S_t}-S_t$. Shifting back to the original origin, this means that the mapping $x \mapsto S_t + \phi^{-1}(G_t^{-1}(F_t(\phi(x-S_t)))$ transports the measure ξ_t on A_{S_t} into the measure η_t on B_{S_t}. Thus, the allocation rule τ defined by

$$\tau(t) = S_t + \phi^{-1}(G_t^{-1}(F_t(\phi(t-S_t)))), \quad t \in \mathbb{R}^d,$$

balances ξ and η. □

Remark 6.2. The above construction, using a point process to split space into pairs of cells and then allocate mass through Lebesgue measure, dates back to an informal note from 2012. It was a part of a brief attempt of the authors to extend the Brownian motion results of Last et al. (2014) to higher dimensional random fields. The obvious question is when this 'auxiliary' process does exist. We discussed this with several colleagues and the existence problem did become part of the PhD topic of Ali Khezeli, see Remark 8.1. In his thesis he used a result on optimal transport to balance the finite masses of the pairs of allocation cells, under certain restrictions on the diffuse source.

7. Destination having Discrete and Diffuse Parts

In this section we finish the proof of Theorem 1.1 by dealing with the case when the destination η contains both discrete and diffuse parts.

Proposition 7.1. *Assume that ξ and η are jointly stationary and ergodic random measures on $\mathbb{R}^d$ such that $\lambda_\xi = \lambda_\eta < \infty$. Assume further that ξ is diffuse and that η is mixed, that is,*

$$\eta = \eta^{\text{disc}} + \eta^{\text{diff}}$$

where η^{disc} and η^{diff} are nonzero random measures that are discrete and diffuse respectively. Then there exists an allocation τ balancing ξ and η.

Proof: Note that the measures ξ, η, η^{disc}, η^{diff} are jointly stationary and ergodic since η^{disc} and η^{diff} can be obtained from η in an translation invariant way. Now apply Proposition 5.1 to ξ and η^{disc} using $\lambda_\xi \geq \lambda_{\eta^{\text{disc}}}$ to obtain a measurable $A \subset \Omega \times \mathbb{R}^d$ and an allocation τ^{disc} balancing ξ_A and η^{disc}. Then, with χ the auxiliary simple point process defined at (1.3), apply Proposition 6.1 to ξ_{A^c} and η^{diff} using $\lambda_{\xi_{A^c}} = \lambda_{\eta^{\text{diff}}}$ to obtain an allocation τ^{diff} balancing ξ_{A^c} and η^{diff}. Finally, define an allocation τ by

$$\tau(\omega, x) = \begin{cases} \tau^{\text{disc}}(\omega, x), & \text{if } (\omega, x) \in A, \\ \tau^{\text{diff}}(\omega, x), & \text{if } (\omega, x) \in A^c. \end{cases}$$

It is easy to see that τ balances ξ and η. □

8. Counterexample

In this section we show that the auxiliary χ cannot simply be removed from Theorem 1.1. There are diffuse ξ and η such that an allocation transporting ξ to η does not exist (without external randomisation).

Here is a specific example in two dimensions, $\mathbb{R}^2$. Let N be a canonical stationary Poisson process on the y-axis with intensity 1; canonical means that N is the only source of randomness. Let ξ be formed by the one-dimensional Lebesgue measure on the lines parallel to the x-axis going through the points of N. Let $\eta = \lambda_2 =$ the Lebesgue measure on $\mathbb{R}^2$. The measures ξ and η are diffuse, jointly stationary and ergodic, and have the same intensity 1.

Suppose there exists a balancing allocation τ_N which maps $\mathbb{R}^2$ to $\mathbb{R}^2$ for each fixed value of N in such a way that the image measure of ξ under τ_N is η. The Palm version of N is $N^\circ = N + \delta_0$ and the Palm version of ξ is $\xi^\circ = \xi + \lambda_1$, where λ_1 denotes one-dimensional Lebesgue measure on the x-axis. According to (2.12), the existence of the allocation τ_N balancing ξ and the two-dimensional Lebesgue measure $\eta = \lambda_2$ would yield the following shift-coupling of ξ° and ξ,

$$\theta_{\tau_N(0)}\xi^\circ \overset{D}{=} \xi.$$

With T_N the y-axis coordinate of $\tau_N(0)$, this implies that

$$\theta_{T_N}N^\circ \overset{D}{=} N.$$

But, according to Holroyd and Peres (2005), such a T_N does not exist when the only sorce of randomness is N.

Remark 8.1. When reading a preliminary version of this paper, Ali Khezeli pointed out to the authors the following interesting problem, formulated in his PhD-thesis from 2016 (in Persian). Suppose that ξ is a diffuse random measure with no invariant directions. This means that there is (almost surely) no vector $x \neq 0$ such $\xi = \theta_{tx}\xi$ for all $t \in \mathbb{R}$. Does ξ have a stationary point process factor? A positive answer would bring us much closer to a complete characterisation of the existence of balancing factor allocations (for a diffuse source). Note that our counterexample has an invariant direction.

9. Remarks

Remark 9.1. The assumption of ergodicity has been made for simplicity and can be relaxed. The assumption $\lambda_\xi = \lambda_\eta$ has then to be replaced by

$$\mathbb{E}[\xi[0,1] \mid \mathcal{I}] = \mathbb{E}[\eta[0,1] \mid \mathcal{I}], \quad \mathbb{P}\text{-a.e.}$$

We refer to Last and Thorisson (2009); Thorisson (1996) for more detail on this point.

Remark 9.2. A natural question (asked in Haji-Mirsadeghi and Khezeli (2016) for instance) is whether there exists a balancing allocation factor τ if the source ξ is Lebesgue measure or, more generally, absolutely continuous with respect to Lebesgue measure. We have proved the answer to be positive whenever the destination η is *not* diffuse (not purely non-atomic), and also when η is diffuse if we assume that there exists an auxiliary point process factor χ.

But this assumption is not necessary, a balancing allocation factor τ *can* exist *even* when η is diffuse and *no* auxiliary point process factor χ exists. An example of this is obtained by swapping source and destination in Section 8: take $\xi = \lambda_2 =$ the Lebesgue measure on $\mathbb{R}^2$ and let η be formed by the one-dimensional Lebesgue measure on the lines parallel to the x-axis going through the points of N where N is a one-dimensional Poisson process on the y-axis. If N is the only source of randomness then, according to Section 8, there exists *no* auxiliary χ. However, there exists a balancing allocation factor τ: for example, take $\tau(x, y) = (x, \tau_1(y))$ where τ_1 is an allocation balancing Lebesgue measure on the line (the y-axis) and the Poisson process N.

Thus, the existence of an auxiliary point process is not a complete characterisation of the existence of a balancing allocation factor when the source is diffuse.

Remark 9.3. If the source ξ is not diffuse, then the question asked in this paper is in most cases not very meaningful. If, for instance, the destination η is diffuse, then a balancing allocation cannot exist. But even otherwise such allocations can only exist in special cases, for instance, if both ξ and η are simple point processes.

Remark 9.4. The following slight modification of the construction in Section 6 (where the destination is diffuse) can be used to obtain allocations in the cases treated in Section 5 and Section 7. Let χ be the simple point process defined at (1.3). In the proof of Theorem 6.1 remove the first paragraph, replace the allocation cells B_s of η by the Voronoi cells V_s of χ, and then modify χ by letting it have mass $\eta(V_s)$ at each point s. After this modification we have $\lambda_\xi = \lambda_\eta = \lambda_\chi$ and can apply Proposition 4.3 to the pair ξ and χ. Let A_s be the allocation cell of ξ that is mapped to s. This yields F_t and G_t that are cumulative mass functions of measures that need not have mass 1 but only have the same finite mass, $F_t(\infty) = \xi(A_{S_t}) = \eta(V_{S_t}) = G_t(\infty)$. Replace first the word *distribution function* by *cumulative mass function* and then the word *distribution* by *image measure*. The rest of the proof now goes through as it stands.

This method thus yields allocations in all the three cases where η does not consist of isolated atoms. We have chosen here to treat each of those cases separately because treating discrete measures as in Section 5 is more explicit then this alternative method.

Remark 9.5. Here are some further historical notes. Allocations finding extra heads in coin tosses on the d dimensional grid and extra points in the d dimensional Poisson process were constructed in Holroyd and Liggett (2001); Holroyd and Peres (2005); Hoffman et al. (2006); Chatterjee et al. (2010) by Liggett, Holroyd, Hoffman, Peres et.al. More generally, the allocations transporting Lebesgue measure to the points of stationary ergodic finite-intensity point processes produce the Palm versions of the processes. The construction in Hoffman et al. (2006) involved a Gale-Shapley algorithm resulting in a 'stable' allocation while the construction in Chatterjee et al. (2010) used a gravitational force field to obtain an 'economical' allocation. In Huesmann (2016), unique optimal allocations between jointly stationary random measures on geodesic manifolds were constructed, assuming the (Palm) average cost to be finite and the source to be absolutely continuous. Stable transports between general (jointly stationary) random measures ξ and η on $\mathbb{R}^d$ were constructed and studied in Haji-Mirsadeghi and Khezeli (2016). If ξ is diffuse and η is a point process, then these transports boil down to allocations, see Remarks 4.4, 6.2 and 8.1.

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