Two-sided immigration, emigration and symmetry properties of self-similar interval partition evolutions

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Abstract. Forman et al. (2020+) constructed (α,θ)-interval partition evolutions for $\alpha \in (0,1)$ and $\theta \geq 0$, in which the total sums of interval lengths ("total mass") evolve as squared Bessel processes of dimension 2θ, where $\theta \geq 0$ acts as an immigration parameter. These evolutions have pseudo-stationary distributions related to regenerative Poisson–Dirichlet interval partitions. In this paper we study symmetry properties of (α,θ)-interval partition evolutions. Furthermore, we introduce a three-parameter family $\mathrm{SSIP}^{(\alpha)}(\theta_1,\theta_2)$ of self-similar interval partition evolutions that have separate left and right immigration parameters $\theta_1 \geq 0$ and $\theta_2 \geq 0$. They also have squared Bessel total mass processes of dimension 2θ, where $\theta = \theta_1 + \theta_2 - \alpha \geq -\alpha$ includes the usual parameter range of the two-parameter Poisson–Dirichlet distribution – negative θ can be interpreted as an overall emigration. Under the constraint $\max\{\theta_1,\theta_2\} \geq \alpha$, we prove that an $\mathrm{SSIP}^{(\alpha)}(\theta_1,\theta_2)$-evolution is pseudo-stationary for a new distribution on interval partitions, whose ranked sequence of lengths has Poisson–Dirichlet distribution with parameters α and θ, but we are unable to cover all parameters without developing a limit theory for composition-valued Markov chains, which we do in a sequel paper.

1. Introduction.

In this paper, we construct a three-parameter family of interval partition evolutions that generalises a two-parameter family recently introduced by Forman et al. (2020c). When projected onto ranked interval lengths (Forman et al., 2023), their evolutions yield Poisson–Dirichlet(α,θ) diffusions in the cases when $\theta \geq 0$ (Petrov, 2009). Members of the two-parameter family were used in Forman et al. (2018) to construct the Aldous diffusion that has the Brownian continuum random tree as its stationary distribution, solving a problem posed by Aldous (1999). The three-parameter family is relevant since it captures for each $\alpha \in (0,1)$ the full Poisson–Dirichlet parameter range $\theta > -\alpha$. This extended range is crucial for potential generalisations of the Aldous diffusion to
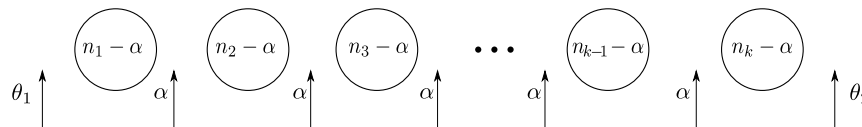
FIGURE 1.1. The rates at which new customers arrive in the ordered Chinese restaurant. In the setting of Rogers and Winkel (2022), $\theta_2 = \alpha$.

continuum random trees that include multifurcating ones such as stable trees (Chen et al., 2009; Curien and Kortchemski, 2014; Duplantier et al., 2021; Duquesne and Le Gall, 2002; Haas and Miermont, 2004; Haas et al., 2009; Le Gall and Miermont, 2011). In the present paper, we focus on the interval partition diffusions and their symmetry and stationarity properties. Before introducing these processes, we provide some motivation in the related setting of composition-valued Markov chains.

Rogers and Winkel (2022) introduced a family of continuous-time Markov chains with two parameters $\alpha \in (0,1)$ and $\theta_1 \geq 0$, taking values on the space $\mathcal{C}$ of vectors of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$:

$$\mathcal{C} = \{(n_1, n_2, \ldots, n_k) \colon k \geq 0, \ n_1, n_2, \ldots, n_k \in \mathbb{N}\}.$$

Each element in $\mathcal{C}$ is also known as an *integer composition*. Their model is in the framework of Pitman's two-parameter Chinese restaurant processes (Pitman (2006)) and their ordered variants (James, 2006; Pitman and Winkel, 2009). In their language, we interpret the process $(C(t), t \geq 0)$ as describing the fluctuating numbers of customers sitting at tables arranged in a line. At any time $t \geq 0$ with $C(t) = (n_1, n_2, \ldots, n_k)$, we say there are $k \in \mathbb{N}$ occupied tables and for $1 \leq i \leq k$ the i-th table enumerated from left to right has $n_i \in \mathbb{N}$ customers. The transition rates of $(C(t), t \geq 0)$ are, as follows – see also the illustration in Figure 1.1:

1. for each occupied table with n_i customers, a new customer joins this table at rate $n_i - \alpha$;
2. at rate θ_1, a new customer starts a new table to the left of the leftmost occupied table;
3. between each pair of two neighbouring occupied tables, a new customer starts a new table there at rate α;
4. at rate α, a new customer starts a new table to the right of the rightmost occupied table;
5. each customer leaves at rate 1.

Without the departure rates in 5. and ignoring the order of tables, this model corresponds to Pitman's two-parameter extension of the Dubins–Pitman Chinese restaurant process, with parameters $\alpha \in (0,1)$ and $\theta_1 \geq 0$. It is worth noting that the original parameter constraint of Pitman's model is $\theta_1 > -\alpha$, while ordered variants have been restricted to $\theta_1 \geq 0$ in James (2006); Pitman and Winkel (2009); Rogers and Winkel (2022).

It is natural to extend the model to a three-parameter version with $\alpha \in (0,1), \theta_1, \theta_2 \geq 0$. Specifically, we replace the transition rate in 4. by the following 4′.:

4′. at rate θ_2, a new customer starts a new table to the right of the rightmost table.

The total departure rate of n for $n = n_1 + n_2 + \cdots + n_k$ customers is set against a total arrival rate of $n + \theta$, commonly interpreted as down and up-rates of a birth-and-death process with immigration for $\theta > 0$, but also including cases $\theta = \theta_1 + \theta_2 - \alpha \in (-\alpha, 0)$, which we interpret as emigration.

Rogers and Winkel (2022) conjectured that, for the ordered Chinese restaurant Markov chain with parameter (α, θ_1), if $n^{-1}C(0)$ converges to a limit, then the process $(n^{-1}C(2nt), t \geq 0)$ has a scaling limit, which is a so-called *self-similar interval partition evolution with parameter (α, θ_1)* introduced in Forman et al. (2021, 2020c). Naturally, we expect such convergence to also hold in the generalised three-parameter $(\alpha, \theta_1, \theta_2)$ setting. Indeed, we establish this in a sequel paper (Shi and Winkel, 2020), building in part on the constructions and study of the expected limiting diffusion in the current work.

1.1. *Main results.* For $M \geq 0$, an *interval partition* $\beta = \{U_i, i \in I\}$ of $[0, M]$ is a (countable) collection of disjoint open intervals $U_i = (a_i, b_i) \subseteq (0, M)$, such that the (compact) set of partition points $G(\beta) := [0, M] \setminus \bigcup_{i \in I} U_i$ has zero Lebesgue measure. We refer to the intervals $U \in \beta$ as *blocks* and to their lengths $|U|$ as their *masses*. We similarly refer to $\|\beta\| := M = \sum_{U \in \beta} |U|$ as the *total mass* of β. We denote by $\mathcal{I}_H$ the set of all interval partitions of $[0, M]$ for all $M \geq 0$. This space is equipped with the metric d_H that is obtained by applying the Hausdorff metric to the sets of partition points: for every $\gamma, \gamma' \in \mathcal{I}_H$,

$$d_H(\gamma, \gamma') := \inf \left\{ r \geq 0 \colon G(\gamma) \subseteq \bigcup_{x \in G(\gamma')} [x - r, x + r], \ G(\gamma') \subseteq \bigcup_{x \in G(\gamma)} [x - r, x + r] \right\}. \quad (1.1)$$

The metric space $(\mathcal{I}_H, d_H)$ is not complete, but it generates a Polish topology (Forman et al., 2020b). For $c > 0$, define a *scaling map* by

$$c\beta := \{(ca, cb) \colon (a, b) \in \beta\}, \qquad \beta \in \mathcal{I}_H.$$

For $\beta \in \mathcal{I}_H$, we write $\mathrm{rev}(\beta) := \{(\|\beta\| - b, \|\beta\| - a) \colon (a, b) \in \beta\} \in \mathcal{I}_H$ for the left-right reversal of β. In Gnedin and Pitman (2005); Pitman and Winkel (2009), a two-parameter family of interval partitions was introduced that places blocks of Poisson–Dirichlet masses $\mathtt{PD}^{(\alpha)}(\theta)$ into a (right-)regenerative random order, where a random interval partition $\overline{\beta}$ of $[0, 1]$ is called "right-regenerative" if for all $s \in (0, 1)$, the remaining interval partition to the right of the first partition point $R_s := \inf(G(\overline{\beta}) \cap [s, 1])$ to the right of s, is a scaled copy of $\overline{\beta}$, in the sense that given $R_s < 1$, we have $(1 - R_s)^{-1}\{(a - R_s, b - R_s) \colon (a, b) \in \overline{\beta} \cap [R_s, 1]\} \overset{d}{=} \overline{\beta}$. We denote the left-right reversals of these right-regenerative Poisson–Dirichlet interval partitions by $\mathtt{PDIP}^{(\alpha)}(\theta)$, $\alpha \in (0, 1)$, $\theta \geq 0$. They inherit an analogous left-regenerative property at $L_s = \sup(G(\beta) \cap [0, s])$, $s \in (0, 1)$. For $\theta = \alpha$, this is the distribution of the excursion intervals of a (squared) Bessel bridge (Pitman and Yor, 1997) with dimension parameter 2α.

Let $(\beta_a)_{a \in \mathcal{A}}$ be a family of interval partitions indexed by a totally ordered countable set $(\mathcal{A}, \preceq)$. Let $S_\beta(a-) := \sum_{b \prec a} \|\beta_b\|$. We define the natural *concatenation* of their blocks by

$$\underset{a \in \mathcal{A}}{\bigstar} \beta_a := \left\{ \left(x + S_\beta(a-), y + S_\beta(a-)\right) \colon (x, y) \in \beta_a, \ a \in \mathcal{A} \right\}.$$

When $\mathcal{A} = \{1, 2\}$, we denote this by $\beta_1 \star \beta_2$.

Let us recall from Forman et al. (2020c) the transition kernels of the two-parameter family of interval partition evolutions. The kernels have the branching property (with immigration) under which each initial block of mass $b > 0$ contributes independently to time y with probability $1 - e^{-b/2y}$. Specifically, for $r = 1/2y > 0$ and $b > 0$, we consider independent $G \sim \mathtt{Gamma}(\alpha, r)$, $\overline{\beta} \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$, and a $(0, \infty)$-valued random variable $L_{b,r}^{(\alpha)}$ with Laplace transform

$$\mathbb{E}\left[e^{-\lambda L_{b,r}^{(\alpha)}}\right] = \left(\frac{r + \lambda}{r}\right)^\alpha \frac{e^{br^2/(r+\lambda)} - 1}{e^{br} - 1}. \quad (1.2)$$

As can be read from Forman et al. (2021, Lemma 3.5) or verified directly, this is also equal to

$$\mathbb{E}\left[\exp\left(-\lambda \sum_{1 \leq i \leq N} Z_i\right) \,\Big|\, N > 0\right]$$

for independent $N \sim \mathtt{Poisson}(br)$, $Z_1 \sim \mathtt{Gamma}(1 - \alpha, r)$ and $Z_n \sim \mathtt{Exponential}(r)$, $n \geq 2$. Then we define the distribution $\mu_{b,r}^{(\alpha)}$ of a random interval partition as

$$\mu_{b,r}^{(\alpha)} = e^{-br}\delta_\emptyset + (1 - e^{-br})\mathbb{P}\left(\{(0, L_{b,r}^{(\alpha)})\} \star G\overline{\beta} \in \cdot\right). \quad (1.3)$$

Definition 1.1 (SSIP$^{(\alpha)}(\theta)$-evolution). Fix $\alpha \in (0,1)$, $\theta \geq 0$. An SSIP$^{(\alpha)}(\theta)$-evolution is an $\mathcal{I}_H$-valued diffusion, whose transition semigroup $(\kappa_y^{\alpha,\theta}, y \geq 0)$ has the following form. Let $\beta \in \mathcal{I}_H$ and $y > 0$. Then $\kappa_y^{\alpha,\theta}(\beta, \cdot)$ is defined to be the distribution of

$$G^y \overline{\beta}_0 \star \bigstar_{U \in \beta} \beta_U^y \tag{1.4}$$

for independent $G^y \sim \mathtt{Gamma}(\theta, 1/2y)$, $\overline{\beta}_0 \sim \mathtt{PDIP}^{(\alpha)}(\theta)$, and $\beta_U^y \sim \mu_{|U|,1/2y}^{(\alpha)}$, $U \in \beta$, where $|U|$ denotes the length of the interval U.

Informally speaking, the description of the semigroup can be interpreted by analogy with the Chinese restaurant Markov chains introduced above. Each β_U^y is the contribution of the initial table U together with new tables added between U and any contributions of initial tables to the right of U. And $G^y \overline{\beta}_0$ represents "immigrating" tables to the far left.

While the semigroup property is not obvious from this definition, it was shown in Forman et al. (2020c) that SSIP$^{(\alpha)}(\theta)$-evolutions exist as self-similar path-continuous Hunt process in $(\mathcal{I}_H, d_H)$. In Section A.1, we formally recall a construction from spectrally positive stable Lévy processes with jumps marked by continuous paths derived from squared Bessel processes. We recall here that for any $m \geq 0$, $\delta \in \mathbb{R}$, there is a unique strong solution of the equation

$$Z_t = m + \delta t + 2 \int_0^t \sqrt{|Z_s|} dB_s,$$

where $(B_t, t \geq 0)$ is a standard Brownian motion. The first hitting time of zero $\tau_0(Z) := \inf\{t \geq 0 \colon Z_t = 0\}$ is almost surely finite if and only if $\delta < 2$. When $\delta < 2$, the law of $\tau_0(Z)$ is described in (Göing-Jaeschke and Yor, 2003, Equation (13)) as the distribution of $m/2G$ with $G \sim \mathtt{Gamma}(1 - \delta/2, 1)$. Furthermore, we define the *lifetime of Z* by

$$\zeta(Z) := \infty, \text{ if } \delta > 0, \quad \text{and} \quad \zeta(Z) := \tau_0(Z), \text{ if } \delta \leq 0. \tag{1.5}$$

We write $\mathtt{BESQ}_m(\delta)$ for the law of $Z := (Z_{t \wedge \zeta(Z)}, t \geq 0)$, which we will refer to as a *squared Bessel process starting from m* with dimension parameter δ. When $\delta \leq 0$, this process is absorbed at 0 at the end of its a.s. finite lifetime. We denote by $\mathtt{BESQ}_m^\dagger(\delta)$ the law of $(Z_{t \wedge \tau_0(Z)}, t \geq 0)$, which differs from $\mathtt{BESQ}_m(\delta)$ only for $\delta \in (0,2)$. While $\mathtt{BESQ}(0)$, respectively $\mathtt{BESQ}(\delta)$, is a continuous-state branching process, with immigration at rate $\delta > 0$, the case $\delta < 0$ is naturally interpreted as emigration at rate $|\delta|$. We refer to Göing-Jaeschke and Yor (2003) for general properties of such squared Bessel processes.

Specifically, in the construction of SSIP$^{(\alpha)}(\theta)$ from marked Lévy processes, jumps of height z are marked by the squared Bessel bridges of dimension $4 + 2\alpha$ from 0 to 0 and of length z, which are also $\mathtt{BESQ}(-2\alpha)$ excursions Pitman and Yor (1982). This construction reveals how block masses evolve as independent $\mathtt{BESQ}(-2\alpha)$-processes, with further blocks created between existing blocks at a dense set of times, each evolving as a $\mathtt{BESQ}(-2\alpha)$-excursion.

The contribution $G^y \overline{\beta}_0$ in (1.4) can be interpreted as "immigration" at rate $\theta \geq 0$, on the left-hand side. We note that the semigroup is left-right-reversible for $\theta = \alpha$. Specifically, we have the following consequence of the symmetry properties of the semigroup of SSIP$^{(\alpha)}(\alpha)$-evolutions.

Proposition 1.2. *Let $(\beta^y, y \geq 0)$ be an SSIP$^{(\alpha)}(\alpha)$-evolution starting from $\beta \in \mathcal{I}_H$. Then the left-right reversal $(\mathrm{rev}(\beta^y), y \geq 0)$ is an SSIP$^{(\alpha)}(\alpha)$-evolution starting from $\mathrm{rev}(\beta)$.*

Recall that the object of the paper is to generalise the SSIP$^{(\alpha)}(\theta_1)$-evolutions to the three-parameter setting with $\alpha \in (0,1), \theta_1, \theta_2 \geq 0$, obtaining the continuum analogues of the three-parameter Chinese restaurant Markov chains. This symmetry property above suggests that the rightmost copy of $G\overline{\beta}$ in (1.3) may similarly be interpreted as immigration at rate α. But with positive probability, no initial block contributes to time $y > 0$, and modifying the two-parameter
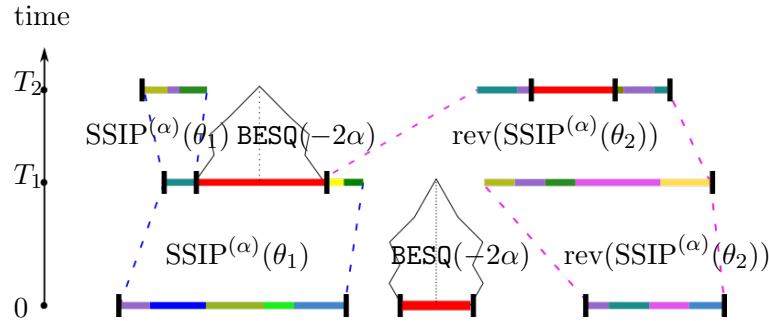
FIGURE 1.2. We illustrate an $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1,\theta_2)$-evolution. At time zero, the initial interval partition is decomposed into three parts: the "middle" block coloured in red and the interval partitions formed by the blocks to the left and the right of the red block respectively. As time increases (the vertical direction), the size of the middle block evolves according to $\mathtt{BESQ}(-2\alpha)$, the left part is an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution, and the right part is the left-right reversal of an $\mathrm{SSIP}^{(\alpha)}(\theta_2)$-evolution, until the time T_1 when the middle block is absorbed at zero. Then we decompose the value at T_1 around its longest block, which is coloured in red and regarded as the new middle block. We next continue in a similar way.

model at the level of semigroups to include left-hand immigration at rate $\theta_1 \geq 0$ and right-hand immigration at rate $\theta_2 \geq 0$ is a challenge. We propose two approaches.

Our first approach to the desired three-parameter process is motivated from properties of the discrete model. Observe from Figure 1.1 that we can split at any occupied table in the middle, then before the time when the last customer leaves this table, the part to its left evolves according to a Chinese restaurant chain with parameter (α,θ_1) and the part to its right is a left-right-reversed Chinese restaurant chain with parameter (α,θ_2). By analogy, for the continuum model, we construct a three-parameter family of $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1,\theta_2)$-evolutions from $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolutions, $\mathtt{BESQ}(-2\alpha)$ processes and left-right reversals of $\mathrm{SSIP}^{(\alpha)}(\theta_2)$-evolutions by repeatedly decomposing around a "middle" block, as made precise in the following definition and illustrated in Figure 1.2.

Definition 1.3. Fix $\alpha \in (0,1)$ and $\theta_1,\theta_2 \geq 0$. Let $\beta \in \mathcal{I}_H$ and set $T_0 := 0$, $\beta^0 := \beta$. Inductively, for any $n \geq 0$, conditionally given $(\beta^y, 0 \leq y \leq T_n)$, proceed as follows.

- If $\beta^{T_n} = \emptyset$, set $T_i := T_n$, $i \geq n+1$, and $\beta^y := \emptyset$, $y \geq T_n$.
- If $\beta^{T_n} \neq \emptyset$, denote by $U^{(n)}$ the longest interval in β^{T_n}, taking the leftmost of these if it is not unique. Let $\beta_1^{(n)} := \beta^{T_n} \cap [0, \inf U^{(n)}] \in \mathcal{I}_H$ be the partition to the left of $U^{(n)}$ and record the remainder to the right of $U^{(n)}$ in $\beta_2^{(n)} \in \mathcal{I}_H$ such that

$$\beta^{T_n} = \beta_1^{(n)} \star \{(0, |U^{(n)}|)\} \star \beta_2^{(n)}.$$

Build three independent processes: $\mathrm{SSIP}^{(\alpha)}(\theta_j)$-evolutions $\boldsymbol{\gamma}_j^{(n)} = (\gamma_j^{(n)}(r), r \geq 0)$ started from $\beta_1^{(n)}$ and $\mathrm{rev}(\beta_2^{(n)})$ respectively, $\mathbf{f}^{(n)} \sim \mathtt{BESQ}(-2\alpha)$ started from $|U^{(n)}|$ and absorbed at $\zeta(\mathbf{f}^{(n)}) := \inf\{z \geq 0 : \mathbf{f}^{(n)}(z) = 0\}$. Set $T_{n+1} := T_n + \zeta(\mathbf{f}^{(n)})$ and

$$\beta^{T_n+s} := \gamma_1^{(n)}(s) \star \{(0, \mathbf{f}^{(n)}(s))\} \star \mathrm{rev}(\gamma_2^{(n)}(s)), \quad 0 \leq s \leq \zeta(\mathbf{f}^{(n)}).$$

If $T_n \uparrow T_\infty < \infty$, set $\beta^y := \emptyset$, $y \geq T_\infty$. We refer to $\boldsymbol{\beta} = (\beta^y, y \geq 0)$ as an $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1,\theta_2)$-evolution, a *self-similar interval partition evolution*, to θ_1 and θ_2 as *left and right immigration parameters*, and to $(\|\beta^y\|, y \geq 0)$ as the *total mass process*.

The choice of the longest interval is not essential, but is one way to achieve, as we will show, that this process either continues forever without hitting $\emptyset$ or reaches $\emptyset$ continuously in finite time. Another natural choice would be a size-biased block.

The subscript $\dagger$ in $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$ acknowledges that these processes are absorbed ("killed") when they reach $\emptyset$. When $\theta > 0$, this is not the case for $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions. While almost surely neither process visits $\emptyset$ when $\theta \geq 1$, the state $\emptyset$ is an instantaneously reflecting boundary state of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions when $\theta \in (0,1)$. This relates to the well-known boundary behaviour of squared Bessel processes, which are the total mass evolutions of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions (Forman et al., 2020c, Proposition 1.3(iii)).

We will check carefully that $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolutions are well-defined as path-continuous processes in $(\mathcal{I}_H, d_H)$ and establish the following properties.

Theorem 1.4. *For each $\alpha \in (0,1)$, $\theta_1 \geq 0$ and $\theta_2 \geq 0$, an $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolution $(\beta^y, y \geq 0)$ has the following properties:*

  (i) *it is a path-continuous Hunt process in $(\mathcal{I}_H, d_H)$;*
  (ii) *the total mass process $(\|\beta^y\|, y \geq 0)$ is $\mathrm{BESQ}_{\|\beta^0\|}^{\dagger}(2\theta)$, where $\theta = \theta_1 + \theta_2 - \alpha \geq -\alpha$;*
  (iii) *it is self-similar with index 1 in the sense that, for each $c > 0$, $(c\beta^{y/c}, y \geq 0)$ is an $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from $c\beta^0$;*
  (iv) *when $\theta_2 = \alpha$, an $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \alpha)$-evolution is an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution killed at its first hitting time of $\emptyset$.*

The items (i)–(iii) of Theorem 1.4 are analogues of Forman et al. (2020c, Proposition 1.3(i)–(iii)) and (iv) is suggested by the discrete Chinese restaurant model illustrated in Figure 1.1, as we have explained above Definition 1.3. A key part of the proof is to show that $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolutions reach $\emptyset$ continuously. We will obtain the Markov property by applying Dynkin's criterion to the triple-valued process whose components are the mass of the "middle" block and the interval partitions on either side. This process inherits the Markov property from $\mathrm{SSIP}^{(\alpha)}(\theta_j)$-evolutions, $j = 1, 2$, and $\mathrm{BESQ}(-2\alpha)$ via standard results (Meyer, 1975; Bect, 2007) about suitably restarting Markov processes at stopping times.

Our second approach aims to address a number of questions that arise in the first approach. One is to construct $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions that have $\emptyset$ as a reflecting boundary when $\theta \in (0,1)$. A second question is whether, like $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolutions, these processes have pseudo-stationarity properties in the sense that for an initial distribution to be determined, the marginal distribution at all times is not the same, but the same as for an independent random multiple of the initial interval partition. Specifically, we restrict our attention to parameters $\theta_1 \geq \alpha$ and $\theta_2 \geq 0$ consider the left-right reversal of an $\mathrm{SSIP}^{(\alpha)}(\theta_2)$-evolution. Morally (e.g. in the ordered Chinese restaurant setting of Figure 1.1), this process has left-hand immigration parameter α instead of θ_1. Therefore, we have to find a way to add further immigration at rate $\theta_1 - \alpha$ in between the existing immigration at rate α. To make this precise, we use $\mathrm{SSIP}^{(\alpha)}(0)$-excursions away from $\emptyset$ that we recall from Forman et al. (2020c) in Section 4.1 and study Poissonian constructions. See Definition 4.3 for a formal definition of $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions.

Theorem 1.5. *Let $\theta_1 \geq \alpha$ and $\theta_2 \geq 0$. Set $\theta = \theta_1 + \theta_2 - \alpha$. Then any $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolution with $\theta \in (0,1)$ starting from any initial distribution has a recurrent extension that we call an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution, which has total mass evolution $\mathrm{BESQ}(2\theta)$. In the case when $\theta \in (0,1)$, as well as for $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2) = \mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$ when $\theta \geq 1$ or $\theta = 0$, the following holds. For independent $B \sim \mathrm{Beta}(\theta_1 - \alpha, \theta_2)$, $B' \sim \mathrm{Beta}(1 - \alpha, \theta_1)$ and $\overline{\beta}_j \sim \mathrm{PDIP}^{(\alpha)}(\theta_j)$, $j = 1, 2$, the*

distribution of the random interval partition of $[0, 1]$

$$\bar{\gamma} := B(1 - B')\bar{\beta}_1 \star \{(0, BB')\} \star (1 - B)\mathrm{rev}(\bar{\beta}_2) \tag{1.6}$$

is a pseudo-stationary distribution of an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolution* $(\beta^y, y \geq 0)$ *in the following sense:* *if* $\beta^0 = \|\beta^0\|\bar{\gamma}$ *and* $\|\beta^0\|$ *is independent of* $\bar{\gamma}$, *then, for every fixed* $y \geq 0$, $\beta^y \stackrel{d}{=} \|\beta^y\|\bar{\gamma}$. *Here we use the convention* $B = 1$ *when* $\theta_2 = 0$, *and* $B = 0$ *when* $\theta_2 > 0$ *and* $\theta_1 = \alpha$.

The way immigration on the left produces a multiple of $\bar{\beta}_1 \sim \mathrm{PDIP}^{(\alpha)}(\theta_1)$ relates to the representation (Pitman, 2006, (5.26)) of $\mathrm{PD}^{(\alpha)}(\theta_1)$ as an $(\alpha, 0)$-fragmentation of $\mathrm{PD}^{(0)}(\theta_1)$, in which every part of a $\mathrm{PD}^{(0)}(\theta_1)$-sequence is fragmented independently into $\mathrm{PD}^{(\alpha)}(0)$-proportions. See also Pitman and Yor (1997, Proposition 21). We noted in Forman et al. (2020c, Proposition 3.6 and its proof) that this has a refinement to interval partitions where each block of $\mathrm{PDIP}^{(0)}(\theta_1)$ is fragmented according to $\mathrm{PDIP}^{(\alpha)}(0)$.

The two approaches and associated three-parameter models allow us to study further related processes and properties that generalise straightforwardly from corresponding results for the two-parameter family of Forman et al. (2020c), but they also leave several questions open, which merit further exploration.

- The self-similarity of the construction allows us to de-Poissonize in the sense that we can time-change an $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$- or equivalently an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution $(\beta^y, y \geq 0)$ by the time-change $\tau(u) := \inf\left\{y \geq 0 \colon \int_0^y \|\beta^z\|^{-1}dz > u\right\}$, $u \geq 0$, and normalise to unit total mass $\beta^{\tau(u)}/\|\beta^{\tau(u)}\|$, $u \geq 0$. This yields a Hunt process, which, in the context of Theorem 1.5, will have the distribution of (1.6) as a stationary distribution.

- $Q = (Q_k, k \geq 1) \sim \mathrm{PD}^{(\alpha)}(\theta)$ is well-known to have an α-diversity

$$\mathscr{D}_Q := \Gamma(1 - \alpha)\lim_{h\downarrow 0} h^\alpha \#\{k \geq 1 \colon Q_k > h\} \in (0, \infty) \quad \text{a.s..}$$

  For $\mathrm{PDIP}^{(\alpha)}(\theta)$, this total diversity property was strengthened in Pitman and Winkel (2009, Proposition 6(iv)) to the existence of diversities of $\beta \cap [0, t]$, $t \geq 0$. With a bit of work to control diversities when times T_n, $n \geq 1$, accumulate in Definition 1.3, it can be deduced from corresponding properties of $\mathrm{SSIP}^{(\alpha)}(\theta_i)$, $i = 1, 2$, see Forman et al. (2020c, Theorem 1.4(i)), that $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-, as well as $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions have continuously evolving diversity processes $y \mapsto \mathscr{D}_{\beta^y}$, where $\mathscr{D}_\beta(t) := \Gamma(1 - \alpha)\lim_{h\downarrow 0} h^\alpha \#\{(a, b) \in \beta \colon |b - a| > h, b \leq t\}$, $t \geq 0$.

- The second approach is subject to the restriction $\theta_1 \geq \alpha$, or $\max\{\theta_1, \theta_2\} \geq \alpha$ by left-right reversal arguments. Recall $\theta := \theta_1 + \theta_2 - \alpha$. If we distinguish according to the absorbing, reflecting and transient boundary behaviours of the $\mathrm{BESQ}(2\theta)$ total mass process, when $\theta \leq 0$, $\theta \in (0, 1)$ and $\theta \geq 1$, respectively, this restriction excludes all absorbing cases and some cases in the reflecting regime, but the entire transient regime is already covered. We address the remaining cases in a sequel paper (Shi and Winkel, 2020), where we take a third approach to the three-parameter family. This involves establishing $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$ as scaling limits of the Chinese restaurant Markov chain introduced at the beginning of this introduction. In the two-parameter special case this scaling limit result was stated as a conjecture in Rogers and Winkel (2022). See also Petrov (2009); Rivera-Lopez and Rizzolo (2023) for a different approach to convergence results in the corresponding de-Poissonized setting.

- We show in Forman et al. (2023) that de-Poissonized $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-, $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$- and $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolutions, are such that the associated process of projections onto ranked block sizes are Petrov's $\mathrm{PD}^{(\alpha)}(\theta)$-diffusions, where we note that $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolutions cover only $\theta \geq 0$, while $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolutions here cover Petrov's full parameter range $\theta > -\alpha$,

as $\theta := \theta_1 + \theta_2 - \alpha \geq -\alpha$ when $\alpha \in (0,1)$, as well as the boundary case $\theta = -\alpha$ of processes that degenerate by absorption in $\{(0,1)\} \in \mathcal{I}_H$ or $(1,0,0,\ldots)$, respectively.

1.2. *Organisation of the paper.* The structure of this paper is as follows. We first recall in Section 2 the topology (Forman et al., 2020b) and main examples (Gnedin and Pitman, 2005; Pitman and Winkel, 2009) of interval partitions, and we slightly develop results from Forman et al. (2020a, 2021), here discussing symmetry properties of $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolutions including a proof of Proposition 1.2. In Section 3, we introduce triple-valued $\mathrm{SSIP}^{(\alpha)}_{\dagger}(\theta_1,\theta_2)$-evolutions, study their properties, and prove Theorem 1.4. In Section 4, we make precise the construction of $\mathrm{SSIP}^{(\alpha)}(\theta_1,\theta_2)$-evolutions when $\theta_1 \geq \alpha$ and prove Theorem 1.5. In the Appendix A we recall from Forman et al. (2020a, 2021, 2020c) the construction of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions from marked Lévy processes and Poisson random measures, and we use it to prove two key technical lemmas needed in Section 3.

2. Preliminaries on the transition description of the two–parameter family $\mathrm{SSIP}^{(\alpha)}(\theta)$, $\alpha \in (0,1)$, $\theta \geq 0$.

Throughout this paper, we fix a parameter $\alpha \in (0,1)$. In this section, we recall and develop some properties of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions for $\theta \geq 0$. Specifically, Section 2.1 briefly revisits the topology on $\mathcal{I}_H$ of Forman et al. (2020b). In Section 2.2, we discuss the two-parameter family $\mathrm{PDIP}^{(\alpha)}(\theta)$ of interval partitions (Gnedin and Pitman, 2005; Pitman and Winkel, 2009) that arise as stationary distributions and in transition kernels. In Section 2.3, we discuss symmetry properties and include a short proof of Proposition 1.2.

2.1. *The topology generated by the metric space $(\mathcal{I}_H, d_H)$.* Recall from the introduction that $\mathcal{I}_H$ is the space of interval partitions endowed with the metric d_H of (1.1). We next endow $\mathcal{I}_H$ with another metric d'_H introduced in Forman et al. (2020b). Let $[n] := \{1,2,\ldots,n\}$. For $\beta, \gamma \in \mathcal{I}_H$, a *correspondence* from β to γ is a finite sequence of ordered pairs of intervals $(U_1, V_1),\ldots,(U_n, V_n) \in \beta \times \gamma$, $n \geq 0$, where the sequences $(U_j)_{j \in [n]}$ and $(V_j)_{j \in [n]}$ are each strictly increasing in the left-to-right ordering of the interval partitions. The Hausdorff *distortion* of a correspondence $(U_j, V_j)_{j \in [n]}$ from β to γ, denoted by $\mathrm{dis}_H(\beta, \gamma, (U_j, V_j)_{j \in [n]})$, is defined to be the maximum of the following two quantities:

1. $\sum_{j \in [n]} \left| |U_j| - |V_j| \right| + \|\beta\| - \sum_{j \in [n]} |U_j|$,

2. $\sum_{j \in [n]} \left| |U_j| - |V_j| \right| + \|\gamma\| - \sum_{j \in [n]} |V_j|$,

For $\beta, \gamma \in \mathcal{I}_H$ we define

$$d'_H(\beta, \gamma) := \inf_{n \geq 0,\, (U_j, V_j)_{j \in [n]}} \mathrm{dis}_H\big(\beta, \gamma, (U_j, V_j)_{j \in [n]}\big), \tag{2.1}$$

where the infimum is over all correspondences from β to γ.

Lemma 2.1 (Theorems 2.3–2.4 of Forman et al. (2020b)). *The metric spaces $(\mathcal{I}_H, d_H)$ and $(\mathcal{I}_H, d'_H)$ generate the same separable topology. The space $(\mathcal{I}_H, d'_H)$ is complete, while $(\mathcal{I}_H, d_H)$ is not complete. In particular, the topology is Polish.*

2.2. *Poisson–Dirichlet interval partitions* $\mathrm{PDIP}^{(\alpha)}(\theta)$.

Definition 2.2 ($\mathrm{PDIP}^{(\alpha)}(\theta)$). Fix $\alpha \in (0,1)$ and $\theta \geq 0$. Let $(Z_{\alpha,\theta}(t),\, t \geq 0)$ denote a subordinator with Laplace exponent

$$\Phi_{\alpha,\theta}(q) := \frac{q\Gamma(q+\theta)\Gamma(1-\alpha)}{\Gamma(q+\theta+1-\alpha)} \quad \text{if } \theta > 0, \quad \text{or} \quad \Phi_{\alpha,0}(q) := \frac{\Gamma(q+1)\Gamma(1-\alpha)}{\Gamma(q+1-\alpha)}, \qquad q \geq 0.$$

Let $(Z_{\alpha,\theta}(t-),\ t \geq 0)$ denote the left-continuous version of this subordinator. We write $\mathtt{PDIP}^{(\alpha)}(\theta)$ to denote the law of the random interval partition

$$\left\{ \left(e^{-Z_{\alpha,\theta}(t)}, e^{-Z_{\alpha,\theta}(t-)} \right) : t \geq 0,\ Z_{\alpha,\theta}(t-) \neq Z_{\alpha,\theta}(t) \right\}.$$

This is referred to as a *Poisson-Dirichlet (α, θ) interval partition.*

We refer to Gnedin and Pitman (2005, Section 8) and Kyprianou and Pardo (2022, Chapter 4) for applications of these "β-subordinators" to study respectively a two-parameter family of regenerative composition structures and hypergeometric Lévy processes. They also contain further references. A Poisson–Dirichlet (α, θ) interval partition is the reversal of a *regenerative (α, θ) interval partition* studied in Gnedin and Pitman (2005) and Pitman and Winkel (2009). It also describes the limiting proportions of customers at tables in the ordered Chinese restaurant process of Pitman and Winkel (2009). The special case $\mathtt{PDIP}^{(\alpha)}(0)$ can be obtained from an α-stable subordinator $(\sigma(t), t \geq 0)$ as the interval partition of $[0,1]$ obtained from the complement of the range of $1 - \sigma(t)$, $t \geq 0$, restricted to $[0,1]$, or equivalently as the left-right reversal of the interval partition formed by the excursion intervals in $[0,1]$ of a (squared) Bessel process of dimension $2 - 2\alpha$, including the incomplete excursion stopped at time 1. In Forman et al. (2020c) we noted the following alternative representation, which builds the general $\mathtt{PDIP}^{(\alpha)}(\theta)$ from $\mathtt{PDIP}^{(\alpha)}(0)$, refining the $\mathtt{PD}^{(\alpha)}(\theta)$ analogue of Pitman (2006, (5.26)) and Pitman and Yor (1997, Proposition 21).

Lemma 2.3 (Proposition 3.6 and its proof in Forman et al. (2020c)). *Let $B_i \sim \mathtt{Beta}(\theta, 1)$, $\overline{\beta}_i \sim \mathtt{PDIP}^{(\alpha)}(0)$, $i \geq 1$, be independent. Then*

$$\underset{k=\infty}{\overset{1}{\bigstar}} (1 - B_k) \left(\prod_{i=1}^{k-1} B_i \right) \overline{\beta}_k \sim \mathtt{PDIP}^{(\alpha)}(\theta),$$

where the indexation of the concatenation operator means that the $(k+1)$st term is placed to the left of the kth, $k \geq 1$.

We also record here from Forman et al. (2021, Proposition 2.2(iv)) a decomposition for easier reference: with independent $B \sim \mathtt{Beta}(\alpha, 1 - \alpha)$ and $\overline{\beta} \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$, we have

$$\{(0, 1 - B)\} \star B\overline{\beta} \sim \mathtt{PDIP}^{(\alpha)}(0). \tag{2.2}$$

2.3. $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions and their left-right reversals. Recall from the introduction the definition of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions via their transition kernels and the terminology "total mass process" for the process $(\|\beta^y\|, y \geq 0)$ associated with any interval partition evolution $(\beta^y, y \geq 0)$. In this section we define left-right-reversed $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions and also provide an (elementary) proof of Proposition 1.2. Let $\beta \in \mathcal{I}_H$ and recall that we denote its left-right reversal by

$$\mathrm{rev}(\beta) := \{(\|\beta\| - b, \|\beta\| - a) : (a, b) \in \beta\}.$$

Definition 2.4. Let $\alpha \in (0, 1)$, $\theta \geq 0$, and $\beta_0 \in \mathcal{I}_H$. Let $(\beta^y, y \geq 0)$ be an $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolution starting from $\mathrm{rev}(\beta_0)$. Then we call the process $(\mathrm{rev}(\beta^y), y \geq 0)$ a *(left-right-)reversed self-similar interval partition evolution with parameters α and θ*, abbreviated as $\mathrm{RSSIP}^{(\alpha)}(\theta)$-evolution, starting from β_0.

Proposition 2.5. $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions and $\mathrm{RSSIP}^{(\alpha)}(\theta)$-evolutions are path-continuous Hunt processes. Their total mass processes are $\mathtt{BESQ}(2\theta)$-processes.

Proof: The claims for $\mathrm{RSSIP}^{(\alpha)}(\theta)$-evolutions follow from the corresponding result for $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions, see Forman et al. (2020c, Theorem 1.4), since left-right reversal $\mathrm{rev}: (\mathcal{I}_H, d_H) \to (\mathcal{I}_H, d_H)$ is a total-mass-preserving homeomorphism. $\square$

Lemma 2.6 (Corollary 10.2 of Gnedin and Pitman (2005))**.** *Let* $\beta \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$. *Then* $\mathrm{rev}(\beta) \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$.

Let $(\beta^y, y \geq 0)$ be an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution starting from any $\beta_0 \in \mathcal{I}_H$. Recall that Proposition 1.2 claims that left-right-reversing β^y, $y \geq 0$, yields another $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution.

Proof of Proposition 1.2: Recall that $(\beta^y, y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution starting from $\beta \in \mathcal{I}_H$. Let $(\widehat{\beta}^z, z \geq 0)$ be an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution starting from $\mathrm{rev}(\beta_0)$. Fix any $y > 0$. We first show that $\widehat{\beta}^y$ has the same law as $\mathrm{rev}(\beta^y)$.

Using Definition 1.1 and notation therein, we can write $\beta^y = G_0^y \overline{\beta}_0^y \star \bigstar_{U \in \beta} \beta_U^y$ for independent $G_0^y \sim \mathtt{Gamma}(\alpha, 1/2y)$, $\overline{\beta}_0^y \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$, and $\beta_U^y \sim \mu_{|U|, 1/2y}^{(\alpha)}$, $U \in \beta_0$. By (1.3), we further have the decomposition $\beta_U^y = (0, L_U^y) \star G_U^y \overline{\beta}_U^y$ for all $U \in \beta_0$, where the independent random variables L_U^y have distribution given by (1.2) (with parameter $b = |U|$ and $r = 1/2y$), $G_U^y \sim \mathtt{Gamma}(\alpha, 1/2y)$, and $\overline{\beta}_U^y \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$.

For each $U = (a, b) \in \beta_0$, write $U' = (\|\beta_0\| - b, \|\beta_0\| - a) \in \mathrm{rev}(\beta_0)$ for the corresponding interval in $\mathrm{rev}(\beta_0)$. In particular, $|U'| = |U|$. Using Definition 1.1 and noticing that the law of β_U^y depends only on the length $|U|$, we deduce that

$$\widehat{\beta}^y \stackrel{d}{=} G_0^y \overline{\beta}_0^y \star \bigstar_{U' \in \mathrm{rev}(\beta_0)} \beta_U^y, \tag{2.3}$$

where the right-hand side is the concatenation of the same interval partitions β_U^y but the order is according to the corresponding U' in $\mathrm{rev}(\beta_0)$. It remains to prove that the right-hand side of (2.3) and $\mathrm{rev}(\beta^y)$ have the same law.

We easily deduce from (1.3) that a.s. only a finite number of those $(\beta_U^y, U \in \beta_0)$ are non-empty. Let us now break things down according to indices of the non-empty ones. Take any $\{U_1, \ldots, U_k\} \subseteq \beta_0$. We henceforth constrain our discussion to the event that exactly those interval partitions $\beta_{U_1}^y, \ldots, \beta_{U_k}^y$ are non-empty. Conditionally on this event, we have

$$\beta^y = G_0^y \overline{\beta}_0^y \star (0, L_1^y) \star G_{U_1}^y \overline{\beta}_{U_1}^y \star \cdots \star (0, L_k^y) \star G_{U_k}^y \overline{\beta}_{U_k}^y.$$

Then we have the identity

$$\mathrm{rev}(\beta^y) = G_{U_k}^y \mathrm{rev}(\overline{\beta}_{U_k}^y) \star (0, L_k^y) \star \cdots \star G_{U_1}^y \mathrm{rev}(\overline{\beta}_{U_1}^y) \star (0, L_1^y) \star G_0^y \mathrm{rev}(\overline{\beta}_0^y).$$

On the other hand, on this event we have

$$G_0^y \overline{\beta}_0^y \star \bigstar_{U' \in \mathrm{rev}(\beta_0)} \beta_U^y = G_0^y \overline{\beta}_0^y \star (0, L_k^y) \star G_{U_k}^y \overline{\beta}_{U_k}^y \star \cdots \star (0, L_1^y) \star G_{U_1}^y \overline{\beta}_{U_1}^y.$$

Since $\overline{\beta}_{U_i}^y \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$ are i.i.d., by Lemma 2.6 we have i.i.d. $\mathrm{rev}(\overline{\beta}_{U_i}^y) \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$. Combining with the two representations above, we deduce that $\mathrm{rev}(\beta^y)$ and $G_0^y \overline{\beta}_0^y \star \bigstar_{U' \in \mathrm{rev}(\beta_0)} \beta_U^y$ have the same conditional distribution. This leads to the conclusion that $\widehat{\beta}^y$ and $\mathrm{rev}(\beta^y)$ have the same (unconditional) law.

As the initial state β_0 was arbitrary and because of the Markov property and the identity of the marginal distributions, we identify the finite-dimensional distributions. Finally, we note that the left-right-reversed $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution $(\mathrm{rev}(\beta^y), y \geq 0)$ is also path-continuous, and this completes the identification as an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution. $\qquad\square$

Corollary 2.7. *An* $\mathrm{SSIP}^{(\alpha)}(\alpha)$- *and an* $\mathrm{RSSIP}^{(\alpha)}(\alpha)$-*evolution starting from the same initial state have the same distribution.*

Proof: For any $\beta_0 \in \mathcal{I}_H$, let $\beta := (\beta^y, y \geq 0)$ be an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution starting from $\mathrm{rev}(\beta_0)$. By definition, $(\mathrm{rev}(\beta^y), y \geq 0)$ is a $\mathrm{RSSIP}^{(\alpha)}(\alpha)$-evolution starting from β_0. At the same time, it follows from Proposition 1.2 that $(\mathrm{rev}(\beta^y), y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution starting from β_0. $\qquad\square$

We also recall the following relationship between $\mathrm{SSIP}^{(\alpha)}(0)$- and $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolutions, which we will apply as it stands and in combination with Definition 2.4 as a relationship between $\mathrm{RSSIP}^{(\alpha)}(0)$- and $\mathrm{RSSIP}^{(\alpha)}(\alpha)$-evolutions. Let $L\colon \mathcal{I}_H \to [0,\infty)$ be the map from an interval partition to the mass of its leftmost interval, or zero if none exists (i.e. if either the interval partition is empty or has an accumulation of blocks at its left end).

Lemma 2.8 (Proposition 3.15 of Forman et al. (2021)). *For $m > 0$ and $\gamma \in \mathcal{I}_H$, consider an $\mathrm{SSIP}^{(\alpha)}(0)$-evolution $(\widetilde{\beta}^y, y \geq 0)$ starting from $\{(0,m)\} \star \gamma$, consider $\mathbf{f} \sim \mathtt{BESQ}_m(-2\alpha)$ and an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution $(\beta^y, y \geq 0)$ starting from γ, independent of each other. Write $Y = \inf\{y > 0 \colon L(\widetilde{\beta}^{y-}) = 0\}$ for the lifetime of the original leftmost interval of $(\widetilde{\beta}^y, y \geq 0)$. Then*

$$\left(\widetilde{\beta}^y,\, y \in [0,Y)\right) \overset{d}{=} \left(\{(0,\mathbf{f}(y))\} \star \beta^y,\, y \in [0,\zeta(\mathbf{f}))\right).$$

Note that the total masses exhibit the extended additivity of $\mathtt{BESQ}(-2\alpha)$ and $\mathtt{BESQ}(2\alpha)$ to give $\mathtt{BESQ}(0)$ up to time Y, as studied in Pitman and Winkel (2018). Finally, we recall the following consequence of the form of the semigroups of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions.

Proposition 2.9. *Consider an independent pair consisting of an $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolution $(\beta_1^y,\, y \geq 0)$ starting from $\beta_1^0 \in \mathcal{I}_H$ and an $\mathrm{SSIP}^{(\alpha)}(0)$-evolution $(\beta_2^y,\, y \geq 0)$ starting from $\beta_2^0 \in \mathcal{I}_H$. Then $(\beta_1^y \star \beta_2^y,\, y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolution starting from $\beta_1^0 \star \beta_2^0$.*

3. $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolutions for $\theta_1, \theta_2 \geq 0$, and the proof of Theorem 1.4.

3.1. *Triple-valued* $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-*evolutions.* Fix $\alpha \in (0,1)$. Let $\mathcal{J} := \left(\mathcal{I}_H \times (0,\infty) \times \mathcal{I}_H\right) \cup \{(\emptyset, 0, \emptyset)\}$ and equip $\mathcal{J}$ with the metric

$$d_{\mathcal{J}}((\beta_1, m, \beta_2), (\beta_1', m', \beta_2')) := d_H(\beta_1, \beta_1') + |m - m'| + d_H(\beta_2, \beta_2').$$

The space $(\mathcal{J}, d_{\mathcal{J}})$ is a Borel subset of a Polish space, since $(\mathcal{I}_H, d_H)$ is a Polish space by Lemma 2.1.

We define a function $\phi \colon \mathcal{I}_H \to \mathcal{J}$, as follows. Let $\beta \in \mathcal{I}_H$. For the purpose of defining $\phi(\beta)$, let U be the longest interval in β; if the longest interval is not unique, then we take U to be the leftmost longest interval. Then we set

$$\phi(\beta) := (\beta \cap (0, \inf U), |U|, \beta \cap (\sup U, \|\beta\|) - \sup U). \tag{3.1}$$

By convention, $\phi(\emptyset) := (\emptyset, 0, \emptyset)$.

Definition 3.1 (Triple-valued $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolution). Let $\theta_1, \theta_2 \geq 0$ and $(\beta_1^0, m^0, \beta_2^0) \in \mathcal{J}$. We define a $\mathcal{J}$-valued $\mathrm{SSIP}_{\dagger}^{(\alpha)}(\theta_1, \theta_2)$-evolution $((\beta_1^y, m^y, \beta_2^y), y \geq 0)$ starting from $(\beta_1^0, m^0, \beta_2^0)$ by the following construction.

Set $T_0 := 0$. For $n \geq 0$, suppose by induction that we have constructed the process for the time interval $[0, T_n]$.

- If $(\beta_1^{T_n}, m^{T_n}, \beta_2^{T_n}) = (\emptyset, 0, \emptyset)$, then we set $T_i := T_n$ for every $i \geq n+1$, and $(\beta_1^y, m^y, \beta_2^y) := (\emptyset, 0, \emptyset)$, $y \geq T_n$.
- If $(\beta_1^{T_n}, m^{T_n}, \beta_2^{T_n}) \neq (\emptyset, 0, \emptyset)$, then conditionally given $(\boldsymbol{\gamma}_1^{(k)}, \mathbf{f}^{(k)}, \boldsymbol{\gamma}_2^{(k)})$, $0 \leq k \leq n-1$, consider, independently, an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution $\boldsymbol{\gamma}_1^{(n)}$ starting from $\beta_1^{T_n}$, an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution $\boldsymbol{\gamma}_2^{(n)}$ starting from $\beta_2^{T_n}$, and $\mathbf{f}^{(n)} \sim \mathtt{BESQ}_{m^{T_n}}(-2\alpha)$. Set $T_{n+1} := T_n + \zeta(\mathbf{f}^{(n)})$ and

$$\left(\beta_1^{T_n+y}, m^{T_n+y}, \beta_2^{T_n+y}\right) := \left(\gamma_1^{(n)}(y), \mathbf{f}^{(n)}(y), \gamma_2^{(n)}(y)\right), \qquad 0 \leq y < \zeta(\mathbf{f}^{(n)}).$$

Furthermore, with ϕ the function defined in (3.1), we set

$$\left(\beta_1^{T_{n+1}}, m^{T_{n+1}}, \beta_2^{T_{n+1}}\right) := \phi\left(\beta_1^{T_{n+1}-} \star \beta_2^{T_{n+1}-}\right).$$

We refer to T_n, $n \geq 1$, as the *renaissance times* and $T_\infty := \sup_{n \geq 1} T_n \in [0, \infty]$ as the *degeneration time*. If $T_\infty < \infty$, then by convention we set $(\beta_1^y, m^y, \beta_2^y) := (\emptyset, 0, \emptyset)$ for all $y \geq T_\infty$.

By construction, the process $(\beta^y := \beta_1^y \star \{(0, m^y)\} \star \beta_2^y, y \geq 0)$ satisfies the Definition 1.3 of an $\mathcal{I}_H$-valued $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-*evolution* starting from $\beta^0 := \beta_1^0 \star \{(0, m^0)\} \star \beta_2^0$.

The following observation is a direct consequence of the construction.

Proposition 3.2 (Left-right reversal). *For $\theta_1, \theta_2 \geq 0$, let $((\beta_1^y, m^y, \beta_2^y), y \geq 0)$ be a $\mathcal{J}$-valued $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution and $(\beta^y, y \geq 0)$ its associated $\mathcal{I}_H$-valued process. Then*

$$(\mathrm{rev}(\beta_2^y), m^y, \mathrm{rev}(\beta_1^y)), \qquad y \geq 0$$

is a $\mathcal{J}$-valued $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_2, \theta_1)$-evolution, and its associated $\mathcal{I}_H$-valued process is $(\mathrm{rev}(\beta^y), y \geq 0)$.

Proof: By the construction in Definition 3.1, we have the identity, for every $n \geq 0$ and $y \in [T_n, T_{n+1})$,

$$(\mathrm{rev}(\beta_2^y), m^y, \mathrm{rev}(\beta_1^y)) = \left(\mathrm{rev}\big(\gamma_2^{(n)}(y - T_n)\big), \mathbf{f}^{(n)}(y - T_n), \mathrm{rev}\big(\gamma_1^{(n)}(y - T_n)\big) \right).$$

Since the distributions that determine block sizes in the transition kernels of Definition 1.1 are diffuse and by the independence properties of $(\zeta(\mathbf{f}^{(n)}), \boldsymbol{\gamma}_1^{(n)}, \boldsymbol{\gamma}_2^{(n)})$,

$$\text{the longest interval in } \beta^{T_{n+1}-} = \beta_1^{T_{n+1}-} \star \beta_2^{T_{n+1}-} \text{ is a.s. unique.}$$

Therefore,

$$\left(\mathrm{rev}(\beta_2^{T_{n+1}}), m^{T_{n+1}}, \mathrm{rev}(\beta_1^{T_{n+1}}) \right) = \phi\left(\mathrm{rev}(\beta^{T_{n+1}-}) \right) = \phi\left(\mathrm{rev}(\beta_2^{T_{n+1}-}) \star \mathrm{rev}(\beta_1^{T_{n+1}-}) \right) \quad \text{a.s..}$$

These observations show that $\left((\mathrm{rev}(\beta_2^y), m^y, \mathrm{rev}(\beta_1^y)), y \geq 0 \right)$ satisfies the definition of a $\mathcal{J}$-valued $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_2, \theta_1)$-evolution. $\square$

3.2. *The $\mathcal{I}_H$-valued process.* The aim of this section is to prove Theorem 1.4. Let us begin by identifying the total mass process of $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$.

Theorem 3.3 (Total mass of an $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution). *For $\alpha \in (0, 1)$ and $\theta_1, \theta_2 \geq 0$, let $(\beta^y, y \geq 0)$ be an $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution. Then $(\|\beta^y\|, y \geq 0) \sim \mathrm{BESQ}_{\|\beta^0\|}^\dagger(2\theta)$ with $\theta := \theta_1 + \theta_2 - \alpha$, i.e. a $\mathrm{BESQ}_{\|\beta^0\|}(2\theta)$ killed at its first hitting time of zero.*

To prove Theorem 3.3, we need two lemmas. The first addresses the problem that it is possible that the renaissance times T_n accumulate, i.e. $T_\infty < \infty$; we would like to understand the behaviour near the degeneration time T_∞.

Lemma 3.4. *Let $\theta_1, \theta_2 \geq 0$. Let $(\beta^y, y \geq 0)$ be an $\mathcal{I}_H$-valued $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution with renaissance times T_n, $n \geq 1$, and degeneration time T_∞. If $\mathbb{P}(T_\infty < \infty) > 0$, then conditionally on $T_\infty < \infty$, the total mass $\|\beta^{T_n}\|$ converges almost surely to zero as $n \to \infty$.*

Indeed, we will see in Corollary 3.6 that $\mathbb{P}(T_\infty < \infty)$ is either 0 or 1, depending on the value of θ only. We prove this lemma in Appendix A.4. Second, recall from Pitman and Winkel (2018) a generalised additivity property of squared Bessel processes, with the dimension parameters possibly being negative.

Lemma 3.5 (Pitman and Winkel (2018, Proposition 1)). *For any $\delta_1, \delta_2 \in \mathbb{R}$, and $b_1, b_2 \geq 0$, let $Z_1 \sim \mathrm{BESQ}_{b_1}(\delta_1)$ and $Z_2 \sim \mathrm{BESQ}_{b_2}(\delta_2)$ be independent. Let T be a stopping time relative to the filtration $(\mathcal{F}_t, t \geq 0)$ generated by the pair of processes (Z_1, Z_2), with $T \leq \zeta(Z_1) \wedge \zeta(Z_2)$, where the lifetime ζ is defined as in (1.5). Given $\mathcal{F}_T$, let $Z_3 \sim \mathrm{BESQ}_{Z_1(T) + Z_2(T)}(\delta_1 + \delta_2)$. Then the process Z defined as follows is a $\mathrm{BESQ}_{b_1 + b_2}(\delta_1 + \delta_2)$:*

$$Z(t) = \begin{cases} Z_1(t) + Z_2(t), & \text{if } 0 \le t \le T, \\ Z_3(t-T), & \text{if } t > T. \end{cases}$$

Proof of Theorem 3.3: Let $W \sim \mathtt{BESQ}_1^\dagger(2\theta)$ be independent of everything else. Recall the construction described in Definition 3.1. With notation therein, define for every $i \ge 1$ a process Z_i by

$$Z_i(x) = \begin{cases} \|\beta^x\|, & \text{if } 0 \le x \le T_i, \\ \mathbf{1}\{\|\beta^{T_i}\| \ne 0\}\|\beta^{T_i}\| W\big((x-T_i)/\|\beta^{T_i}\|\big), & \text{if } x > T_i. \end{cases} \tag{3.2}$$

These processes are constructed on the same (large enough) probability space. We first prove by induction that each Z_i is a $\mathtt{BESQ}_{\|\beta^0\|}^\dagger(2\theta)$. Conditionally on $(\|\beta^x\|, x \le T_i)$,

$$\big(\mathbf{1}\{\|\beta^{T_i}\| \ne 0\}\|\beta^{T_i}\| W\big((x-T_i)/\|\beta^{T_i}\|\big), x \ge 0\big) \quad \text{has distribution } \mathtt{BESQ}_{\|\beta^{T_i}\|}^\dagger(2\theta),$$

by the scaling property of $\mathtt{BESQ}^\dagger(2\theta)$. Note that by Definition 1.1 and independence, $\|\beta^{T_i}\| = 0$ only happens with positive probability when $\theta_1 = \theta_2 = 0$, and then there is $T_j = T_i$ and $Z_j \equiv Z_i$ for all $j \ge i$; in this case $2\theta = -2\alpha < 0$ and $\mathtt{BESQ}_0^\dagger(-2\alpha)$ and $\mathtt{BESQ}_0(-2\alpha)$ are the distribution of the constant zero process.

Let $\boldsymbol{\gamma}_1^{(0)}$ and $\boldsymbol{\gamma}_2^{(0)}$ be as in Definition 3.1. By Proposition 2.5, the total mass evolutions of $\boldsymbol{\gamma}_1^{(0)}$ and $\boldsymbol{\gamma}_2^{(0)}$ are independent $\mathtt{BESQ}_{\|\beta_1^0\|}(2\theta_1)$ and $\mathtt{BESQ}_{\|\beta_2^0\|}(2\theta_2)$ respectively, also independent of $\mathbf{f}^{(0)} \sim \mathtt{BESQ}_{\|m^0\|}(-2\alpha)$. Noticing that $\|\beta^x\| = \|\gamma_1^{(0)}(x)\| + \mathbf{f}^{(0)}(x) + \|\gamma_2^{(0)}(x)\|$ for all $x \le T_1$, we deduce from Lemma 3.5 that $Z_1 \sim \mathtt{BESQ}_{\|\beta^0\|}^\dagger(2\theta)$, since we have $2\theta_1 - 2\alpha + 2\theta_2 = 2\theta$.

Suppose by induction that for some $i \ge 1$, for each $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from any state in $\mathcal{J}$, its corresponding process Z_j' as in (3.2) is a $\mathtt{BESQ}^\dagger(2\theta)$ for each $j \le i$. By the construction in Definition 3.1, conditionally on $(\boldsymbol{\gamma}_1^{(0)}, \mathbf{f}^{(0)}, \boldsymbol{\gamma}_2^{(0)})$, the process

$$(\widetilde{\beta}_1^y, \widetilde{m}^y, \widetilde{\beta}_2^y) := (\beta_1^{T_1+y}, m^{T_1+y}, \beta_2^{T_1+y}), \quad y \ge 0,$$

is an $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from $(\beta_1^{T_1}, m^{T_1}, \beta_2^{T_1})$. Define

$$\widetilde{Z}_i(x) = \begin{cases} \|\beta^{T_i+x}\|, & \text{if } 0 \le x \le T_{i+1} - T_1, \\ \mathbf{1}\left\{\|\beta^{T_{i+1}}\| \ne 0\right\}\|\beta^{T_{i+1}}\| W\left((x-T_{i+1})/\|\beta^{T_{i+1}}\|\right), & \text{if } x > T_{i+1} - T_1. \end{cases}$$

Then there is the identity

$$Z_{i+1}(x) = \begin{cases} \|\beta^x\|, & \text{if } 0 \le x \le T_i, \\ \widetilde{Z}_i(x-T_i), & \text{if } x > T_i. \end{cases}$$

Given the triple $(\boldsymbol{\gamma}_1^{(0)}, \mathbf{f}^{(0)}, \boldsymbol{\gamma}_2^{(0)})$, by the inductive hypothesis $\widetilde{Z}_i$ has conditional distribution $\mathtt{BESQ}_{\|\beta^{T_1}\|}^\dagger(2\theta)$. Consequently, by the $i = 1$ case, we have $Z_{i+1} \sim \mathtt{BESQ}_{\|\beta^0\|}^\dagger(2\theta)$. This completes the induction step. We conclude that $(Z_n)_{n \ge 1}$ is a sequence of processes, in which each Z_n is a $\mathtt{BESQ}_{\|\beta^0\|}^\dagger(2\theta)$, and $(Z_n(y), y \le T_n) = (\|\beta^y\|, y \le T_n)$.

We now prove the theorem for the case $\theta = \theta_1 + \theta_2 - \alpha < 1$. For each $n \ge 1$, let $\tau_0(Z_n)$ be the first hitting time of zero by Z_n. Then $T_n \le \tau_0(Z_n)$ and $\tau_0(Z_n) - T_n = \|\beta^{T_n}\|\tau_0(W)$, where $\tau_0(W)$ is the first hitting time of zero by W and has the law of $1/2G$ with $G \sim \mathtt{Gamma}(1-\theta, 1)$. We deduce that the distribution of the degeneration time T_∞ is stochastically dominated by $\|\beta^0\|/2G$ and is thus a.s. finite. Then it follows from Lemma 3.4 that a.s. $\lim_{n \to \infty} \|\beta^{T_n}\| = 0$. Therefore, $\tau_0(Z_n)$ converges to T_∞ a.s., as $n \to \infty$. We conclude that $(Z_n(y \wedge \tau_0(Z_n)), y \ge 0)$ converges a.s. uniformly to $(\|\beta^{y \wedge T_\infty}\|, y \ge 0)$, and the limiting process is $\mathtt{BESQ}_{\|\beta^0\|}^\dagger(2\theta)$.

We finally study the case $\theta = \theta_1 + \theta_2 - \alpha \geq 1$. By the connection between $\|\beta\|$ and the processes $(Z_n)_{n \geq 1}$, it suffices to prove that, for every $a > 0$, $\mathbb{P}(T_\infty < a) = 0$. On the event $\{T_\infty < a\}$, Lemma 3.4 leads to $\lim_{n \to \infty} Z_n(T_n) = \lim_{n \to \infty} \|\beta^{T_n}\| = 0$. Therefore, for any $\delta > 0$, we have

$$\{T_\infty < a\} \subseteq \bigcup_{n \in \mathbb{N}} \{Z_n(T_n) < \delta, T_\infty < a\}.$$

Note that $\{Z_n(T_n) < \delta, T_\infty < a\} = \{\|\beta^{T_n}\| < \delta, T_\infty < a\} \subseteq \{\inf_{y \in [0, T_n]} \|\beta^y\| < \delta, T_\infty < a\} =: A_n$. The sequence of events $(A_n, n \in \mathbb{N})$ is increasing. By monotone convergence, we have

$$\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} \{Z_n(T_n) < \delta, T_\infty < a\} \right) \leq \lim_{n \to \infty} \mathbb{P}(A_n) \leq \limsup_{n \to \infty} \mathbb{P}\left(\inf_{t \in [0, a]} Z_n(t) < \delta \right) = \mathbb{P}\left(\inf_{t \in [0, a]} Z_1(t) < \delta \right),$$

using the fact that $Z_n \sim \mathtt{BESQ}^\dagger_{\|\beta^0\|}(2\theta) = \mathtt{BESQ}_{\|\beta^0\|}(2\theta)$. As $2\theta \geq 2$, $\lim_{\delta \downarrow 0} \mathbb{P}\left(\inf_{t \in [0,a]} Z_1(t) < \delta \right) = 0$. As a result, we have $\mathbb{P}(T_\infty < a) = 0$. This completes the proof. $\qquad\square$

We have obtained the following dichotomy in the preceding proof.

Corollary 3.6. *For $\theta_1, \theta_2 \geq 0$, let $(\beta^y, y \geq 0)$ be an $\mathcal{I}_H$-valued $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolution starting from $\beta^0 \neq \emptyset$, with renaissance times T_n, $n \geq 0$, and degeneration time T_∞. Set $\theta = \theta_1 + \theta_2 - \alpha$.*

    (i) *If $\theta \geq 1$, then a.s. $T_\infty = \infty$ and $\beta^y \neq \emptyset$ for every $y \geq 0$.*

    (ii) *If $\theta < 1$, then a.s. $T_\infty < \infty$ and $\lim_{y \uparrow T_\infty} \|\beta^y\| = 0$.*

Proposition 3.7. *For $\theta_1, \theta_2 \geq 0$, a $\mathcal{J}$-valued $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolution is Borel right Markov on $(\mathcal{J}, d_{\mathcal{J}})$, but not Hunt.*

Here, we use Sharpe's definition of Borel right Markov and Hunt processes, see e.g. Li (2011, Definition A.18): a Borel right Markov process is a Markov process on a Radon space (such as a Borel subset of a Polish space) with a transition semigroup that is Borel measurable in the initial state, with right-continuous sample paths and with the strong Markov property under a right-continuous filtration. A Hunt process is furthermore quasi-left-continuous, i.e. left-continuous along strictly increasing sequences of stopping times.

Proof: The $\mathcal{J}$-valued process takes values in $\mathcal{J}$, which is a Borel subset of a Polish space. It has càdlàg paths, as a consequence of the path-continuity of the $\mathrm{SSIP}^{(\alpha)}(\theta_1)$- and $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolutions used in the construction, and from Corollary 3.6.

We know that squared Bessel processes, $\mathrm{SSIP}^{(\alpha)}(\theta_1)$- and $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolutions are Borel right Markov processes. In the construction of a $\mathcal{J}$-valued process, we kill a Borel right Markov process with finite lifetime, and at the end of the lifetime, give birth to a new one according to a (degenerate) probability kernel. This entails that the $\mathcal{J}$-valued process is itself a Borel right Markov process; see e.g. Meyer (1975, Théorème 1) or Bect (2007, Théorème II 3.18).

Finally, the $\mathcal{J}$-valued process is not Hunt. To see this, we consider the increasing sequence of stopping times $\tau_n := \inf\{y \geq 0 \colon m^y < 1/n\}$, $n \geq 1$. Then τ_n convergences to the first renaissance time T_1 as $n \to \infty$. But the $\mathcal{J}$-valued process has a jump at T_1 with strictly positive probability; so it is not quasi-left continuous. $\qquad\square$

The $\mathcal{J}$-valued $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolution is only of secondary importance to us, as we are more interested in its associated $\mathcal{I}_H$-valued process. However, the definition of the $\mathcal{I}_H$-valued process a priori depends on the initial choice of the "middle" block. Even with the natural choice of the longest block, the size of this block is typically exceeded by other blocks during the evolution. To establish the Markov property of the $\mathcal{I}_H$-valued processes, we will view them as projections of $\mathcal{J}$-valued processes. Indeed, we will show that two $\mathcal{I}_H$-valued $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolutions started from

different choices of middle block indeed have the same law. The following lemma, whose proof is postponed to Appendix A.4, deals with this problem.

Lemma 3.8. *Let* $\theta_1, \theta_2 \geq 0$. *Let* $((\beta_1^y, m^y, \beta_2^y), y \geq 0)$ *and* $((\widetilde{\beta}_1^y, \widetilde{m}^y, \widetilde{\beta}_2^y), y \geq 0)$ *be two* $\mathcal{J}$-*valued* $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-*evolutions, with* $(\beta^y, y \geq 0)$ *and* $(\widetilde{\beta}^y, y \geq 0)$ *being their associated* $\mathcal{I}_H$-*valued evolutions. Suppose that*

$$\beta_1^0 \star (0, m^0) \star \beta_2^0 = \widetilde{\beta}_1^0 \star (0, \widetilde{m}^0) \star \widetilde{\beta}_2^0,$$

then we can couple these two processes such that their associated $\mathcal{I}_H$-*valued evolutions are almost surely equal.*

Proof of Theorem 1.4: (i) The space $(\mathcal{I}_H, d_H)$ is Polish by Lemma 2.1. The path-continuity for the $\mathcal{I}_H$-valued process between renaissance times follows from the path-continuity of any $\mathrm{SSIP}^{(\alpha)}(\theta_i)$-evolutions, $i = 1, 2$. The path-continuity at times T_n, $n \geq 1$, holds by construction since the parts of $\phi(\beta)$ in (3.1) have concatenation β. By Corollary 3.6, the path-continuity at time T_∞ holds almost surely whenever $T_\infty < \infty$.

We have proved in Proposition 3.7 that the corresponding $\mathcal{J}$-valued process is a Borel right Markov process. The $\mathcal{I}_H$-valued process is a projection of the $\mathcal{J}$-valued process. Since functions of Markov processes are not always Markov processes, we apply Dynkin's criterion (Dynkin, 1965, Theorem 10.13) as a sufficient condition for when they are. Specifically, we have to check that for any two initial states in $\mathcal{J}$ that map to the same state in $\mathcal{I}_H$, projecting the time-y states of the two $\mathcal{J}$-valued evolutions gives identically distributed random variables in $\mathcal{I}_H$. By Lemma 3.8, the two random variables in question can be coupled to be equal, so indeed they have the same distribution. By Dynkin (1965, Theorem 10.13), the projected process, i.e. $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$ is also a Borel right Markov process. The Hunt property then follows from the path-continuity.

(ii) The claimed total mass process was established in Theorem 3.3.

(iii) The same scaling property holds for $\mathtt{BESQ}(-2\alpha)$, by Göing-Jaeschke and Yor (2003, A.3), and for $\mathrm{SSIP}^{(\alpha)}(\theta_i)$ evolutions, $i = 1, 2$, by Forman et al. (2020c, Theorem 1.4) and hence also for $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$. Keeping track of Definition 3.1, we easily check the self-similarity for a $\mathcal{J}$-valued $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolution. Using Lemma 3.8, we deduce the self-similarity of $\mathcal{I}_H$-valued processes.

(iv) For the final claim, we may assume that the $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \alpha)$-evolution $(\beta^y, y \geq 0)$ is associated with a $\mathcal{J}$-valued process $((\beta_1^y, m^y, \beta_2^y), y \geq 0)$. In the notation of Definition 3.1, we have

$$\beta^y = \gamma_1^{(n)}(y) \star \left\{ \left(0, \mathbf{f}^{(n)}(y)\right) \right\} \star \gamma_2^{(n)}(y), \qquad 0 \leq y \leq T_{n+1} - T_n, \ n \geq 0.$$

Conditionally on $(\beta_1^{T_n}, m^{T_n}, \beta_2^{T_n})$, the RHS is an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution starting from β^{T_n}, by Lemma 2.8 and Proposition 2.9. Using this fact and Lemma 3.4, the remainder of this proof is analogous to the proof of Theorem 3.3; details are left to the reader. □

4. Construction of $\mathrm{SSIP}(\theta_1, \theta_2)$-evolutions for $\theta_1 \geq \alpha$, $\theta_2 \geq 0$, and the proof of Theorem 1.5.

In this section we make precise the construction of $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions in the case $\theta_1 \geq \alpha$. The approach does not depend on the construction of $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolutions nor indeed on any developments in Section 3, except when we establish the connections between the two processes in Section 4.5.

4.1. *Distributional properties of* $\mathrm{SSIP}^{(\alpha)}(\theta)$-*evolutions.* Let us now recall a Poissonian construction of $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolutions from Forman et al. (2020c). Let $\mathcal{E}_\mathcal{I}$ be the space of continuous excursions $(\beta^y, y \geq 0)$ on $(\mathcal{I}_H, d_H)$ away from $\emptyset$, with lifetime $\zeta := \inf\{y > 0 \colon \beta^y = \emptyset\}$ and $\beta^y = \emptyset$ for $y \geq \zeta$. The following two statements are straightforward consequences of results in Forman et al. (2020c).

For the reader's convenience, in the Appendix A we will recall relevant materials from Forman et al. (2020c) and prove Propositions 4.1 and 4.2.

Proposition 4.1. *There exists a sigma-finite measure $\Lambda^{(\alpha)}$ on $\mathcal{E}_{\mathcal{I}}$, that is specified by the following:*
  (i) $\Lambda^{(\alpha)}(\zeta > y) = \alpha/y$, $y > 0$;
  (ii) *Under $\Lambda^{(\alpha)}(\cdot \mid \zeta > y)$, $\beta^y \sim \text{Exponential}(1/2y) \cdot \text{PDIP}(\alpha, 0)$ and, given $(\beta^z, z \leq y)$, the conditional law of the process $(\beta^{y+z}, z \geq 0)$ is an $\text{SSIP}^{(\alpha)}(0)$-evolution starting from β^y.*

Moreover, $\Lambda^{(\alpha)}$ has the self-similarity: for $c > 0$, define a mapping $\Phi_c \colon \mathcal{E}_{\mathcal{I}} \to \mathcal{E}_{\mathcal{I}}$ by $(\beta^y, y \geq 0) \mapsto (c\beta^{y/c}, y \geq 0)$. Then the image of $\Lambda^{(\alpha)}$ via Φ_c is $c\,\Lambda^{(\alpha)}$.

In the sense of Pitman and Yor (1982), $\Lambda^{(\alpha)}$ is an excursion measure of the diffusion process $\text{SSIP}^{(\alpha)}(0)$-evolution, though $\emptyset$ is not an entrance boundary.

Proposition 4.2. *Let $\overleftarrow{\mathbf{Z}}$ be a Poisson random measure with intensity $(\theta/\alpha)\text{Leb} \otimes \Lambda^{(\alpha)}$ on $(-\infty, 0) \times \mathcal{E}_{\mathcal{I}}$.*

  (i) *Set*
$$\overleftarrow{\beta}^y = \bigstar_{\text{points } (s, \boldsymbol{\beta}_s) \text{ of } \overleftarrow{\mathbf{Z}} \colon s \in [0,y]\downarrow} \beta_s^{y-s}, \qquad y \geq 0, \tag{4.1}$$

  where $\downarrow$ indicates that the concatenation is by decreasing order of s. Then $(\overleftarrow{\beta}^y, y \geq 0)$ is an $\text{SSIP}^{(\alpha)}(\theta)$-evolution starting from $\emptyset$.
  (ii) *Fix $b > 0$. Set*
$$\overleftarrow{\beta}^y_{(b)} = \bigstar_{\text{points } (s, \boldsymbol{\beta}_s) \text{ of } \overleftarrow{\mathbf{Z}} \colon s \in [0, b \wedge y]\downarrow} \beta_s^{y-s}, \qquad y \geq 0. \tag{4.2}$$

  Then $(\overleftarrow{\beta}^y_{(b)}, y \in [0, b])$ is an $\text{SSIP}^{(\alpha)}(\theta)$-evolution starting from $\emptyset$. Moreover, conditionally on $(\overleftarrow{\beta}^y_{(b)}, y \in [0, b])$, the process $(\overleftarrow{\beta}^{b+z}_{(b)}, z \geq 0)$ is an $\text{SSIP}^{(\alpha)}(0)$-evolution starting from $\overleftarrow{\beta}^b_{(b)}$.

Intuitively, each atom $(s, (\beta_s^z, z \geq 0))$ of $\overleftarrow{\mathbf{Z}}$ represents immigration at time s to the left of the current population. Then the state of an $\text{SSIP}^{(\alpha)}(\theta)$-evolution at time y is the concatenation of all immigrants and their descendants alive at time y from left to right.

Fix $\theta > 0$. Consider an $\text{SSIP}^{(\alpha)}(\theta)$-evolution $(\overleftarrow{\beta}^y, y \geq 0)$ starting from $\emptyset$, constructed as in (4.1). Fix $y > 0$. Let $G^y \sim \text{Gamma}(\theta, (2y)^{-1})$ and $\overline{\beta}_0 \sim \text{PDIP}^{(\alpha)}(\theta)$ be independent and denote the law of $G^y\overline{\beta}_0$ by $\text{Gamma}(\theta, (2y)^{-1}) \cdot \text{PDIP}^{(\alpha)}(\theta)$. Then we know from the semigroup description in Definition 1.1 that
$$\overleftarrow{\beta}^y \sim \text{Gamma}(\theta, (2y)^{-1}) \cdot \text{PDIP}^{(\alpha)}(\theta). \tag{4.3}$$
For future usage, we explore in more detail the components of $\overleftarrow{\beta}^y$. By Proposition 4.1, the atoms of $\overleftarrow{\mathbf{Z}}$ that survive to time y, can be listed as $\big(s_k(y), (\beta_k^z(y), z \geq 0)\big)_{k \geq 1}$, with immigration times $s_k(y)$ listed in increasing order and

$$\sum_{k \geq 1} \delta(s_k(y)) \quad \text{is a Poisson random measure on } [0, y) \text{ with intensity } \theta(y - s)^{-1}ds. \tag{4.4}$$

Note that $\lim_{k \to \infty} \uparrow s_k(y) = y$. In the sequel, we fix y and omit the parameter that indicates the dependence of $s_k = s_k(y)$ on y for simplicity. Then, as in (4.1), $\overleftarrow{\beta}^y$ is the concatenation of $\beta_{s_k}^{y-s_k}$, $k \geq 1$, in deceasing order of k. Moreover, using the Poisson property and Beta-Gamma algebra, see Forman et al. (2020c, proof of Proposition 3.6) for details, we can express these components in terms of families of independent identically distributed (i.i.d.) random variables, as follows:

$$\left(\beta_{s_k}^{y-s_k}, \quad k \geq 1\right) \overset{d}{=} \left(E_k \prod_{i=1}^{k} B_i \bar{\beta}_k, \quad k \geq 1\right) \overset{d}{=} \left(G(1 - B_k) \prod_{i=1}^{k-1} B_i \bar{\beta}_k, \quad k \geq 1\right), \tag{4.5}$$

where $(\bar{\beta}_i)_{i \geq 1}$ are i.i.d. $\mathrm{PDIP}^{(\alpha)}(0)$, $(B_i)_{i \geq 1}$ are i.i.d. $\mathrm{Beta}(\theta - \alpha, 1)$, $(E_i)_{i \geq 1}$ are i.i.d. $\mathrm{Exp}((2y)^{-1})$ and $G \sim \mathrm{Gamma}(\theta, (2y)^{-1})$, independent of each other. Fix any $j \geq 1$. Let $G' \sim \mathrm{Gamma}(\theta, (2y)^{-1})$ and $\bar{\gamma} \sim \mathrm{PDIP}^{(\alpha)}(\theta)$ be independent, further independent of $(E_k, B_k, \bar{\beta}_k)_{k \leq j}$. It follows that

$$\left(\underset{k=\infty}{\overset{j+1}{\bigstar}} \beta_{s_k}^{y-s_k}, \quad \left(\beta_{s_k}^{y-s_k} \right)_{k \leq j} \right)$$

$$\overset{d}{=} \left(\prod_{i=1}^{j} B_i \underset{\ell=\infty}{\overset{1}{\bigstar}} E_{j+\ell} \prod_{i=1}^{\ell} B_{j+i} \bar{\beta}_{j+\ell}, \quad \left(E_k \prod_{i=1}^{k} B_i \bar{\beta}_k \right)_{k \leq j} \right)$$

$$\overset{d}{=} \left(G' \prod_{i=1}^{j} B_i \bar{\gamma}, \quad \left(E_k \prod_{i=1}^{k} B_i \bar{\beta}_k \right)_{k \leq j} \right), \tag{4.6}$$

where the second equality is from the observation that $\bigstar_{k=\infty}^{1} E_{j+k} \prod_{i=1}^{k} B_{j+i} \bar{\beta}_{j+k}$ is independent of $(E_k, B_k, \bar{\beta}_k)_{k \leq j}$ and has distribution $\mathrm{Gamma}(\theta, (2y)^{-1}) \cdot \mathrm{PDIP}^{(\alpha)}(\theta)$ by (4.5) and (4.3).

It will be useful to also describe the distribution of (4.5) conditionally given $(s_k, k \geq 1)$. Specifically, we know from (2.2) and Forman et al. (2020c, Lemma 3.5) that, given $(s_k, k \geq 1)$, the interval partitions $\beta_k^{y-s_k}$, $k \geq 1$, are conditionally independent, with

$$\beta_k^{y-s_k} \overset{d}{=} \{(0, H_k^y)\} \star \gamma_k^y, \qquad k \geq 1, \tag{4.7}$$

where $H_k^y \sim \mathrm{Gamma}(1-\alpha, (2(y-s_k))^{-1})$ and $\gamma_k^y \sim \mathrm{Gamma}(\alpha, (2y-s_k)^{-1}) \cdot \mathrm{PDIP}^{(\alpha)}(\alpha)$ are independent, $k \geq 1$. In particular, we note that the dependence on $\theta > 0$ is only via the sequence of immigration times $(s_k, k \geq 1)$, of clades surviving to time y.

4.2. *Definition, Markov property and path-continuity of* $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolutions with* $\theta_1 \geq \alpha$. Fix $\theta_1 \geq \alpha$ and let $\overleftarrow{\mathbf{Z}}$ be a Poisson random measure on $[0, \infty) \times \mathcal{E}_\mathcal{I}$ with intensity $(\theta_1/\alpha - 1)\mathrm{Leb} \otimes \Lambda^{(\alpha)}$. Rather than using (4.1), we modify this construction and define $(\overleftarrow{\beta}^y, y \geq 0)$ by left-right-reversing each immigrating interval partition excursion $\boldsymbol{\beta}_s = (\beta_s^z, z \geq 0)$:

$$\overleftarrow{\beta}^y := \underset{\text{points } (s, \boldsymbol{\beta}_s) \text{ of } \overleftarrow{\mathbf{Z}} : s \in [0, y] \downarrow}{\bigstar} \mathrm{rev}\left(\beta_s^{y-s} \right), \qquad y \geq 0. \tag{4.8}$$

Definition 4.3 ($\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution with $\min(\theta_1, \theta_2) \geq \alpha$)**.**

(i) For $\theta_1 \geq \alpha$, $\theta_2 \geq 0$, and $\gamma \in \mathcal{I}_H$. Let $(\overrightarrow{\beta}^y, y \geq 0)$ be an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution starting from γ, and define $(\overleftarrow{\beta}^y, y \geq 0)$ as in (4.8), based on a Poisson random measure $\overleftarrow{\mathbf{Z}}$ on $[0, \infty) \times \mathcal{E}_\mathcal{I}$ with intensity $(\theta_1/\alpha - 1)\mathrm{Leb} \otimes \Lambda^{(\alpha)}$. Set

$$\beta^y := \overleftarrow{\beta}^y \star \overrightarrow{\beta}^y, \qquad y \geq 0.$$

Then the $\mathcal{I}_H$-valued process $(\beta^y, y \geq 0)$ is called an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolution* starting from γ.

(ii) For $\theta_1 \geq 0$ and $\theta_2 \geq \alpha$, an $\mathcal{I}_H$-valued process $(\beta^y, y \geq 0)$ is called an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolution*, if its left-right reversal $(\mathrm{rev}(\beta^y), y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\theta_2, \theta_1)$-evolution.

When $\theta_1, \theta_2 \geq \alpha$, an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution can be defined by both parts in Definition 4.3. We still have to justify that there is no ambiguity in this definition, as there is no obvious left-right-symmetry in the construction.

Proposition 4.4. *For* $\theta_1, \theta_2 \geq \alpha$, *the left-right reversal of an* $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolution is an* $\mathrm{SSIP}^{(\alpha)}(\theta_2, \theta_1)$-*evolution starting from the left-right-reversed initial state.*

The proof of Proposition 4.4 is postponed to Section 4.3.

In what follows, we will always implicitly restrict ourselves to $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions with $\theta_1 \geq \alpha$. The corresponding results for the other case, with $\theta_2 \geq \alpha$, follow straightforwardly.

Proposition 4.5 (Total mass). *The total mass of an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution is a $\mathtt{BESQ}(2\theta)$ with $\theta := \theta_1 + \theta_2 - \alpha$.*

Proof: $(\overleftarrow{\underline{\beta}}^y, y \geq 0)$ has the same total mass as an $\mathrm{SSIP}^{(\alpha)}(\theta_1 - \alpha)$-evolution, which is a $\mathtt{BESQ}(2(\theta_1 - \alpha))$ starting from zero. Then we conclude by the additivity of squared Bessel processes. $\square$

Proposition 4.6 (Self-similarity). *Let $(\beta^y, y \geq 0)$ be an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from γ. Then for every $c > 0$, the process $(c\beta^{y/c}, y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from $c\gamma$.*

Proof: Since an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution possesses the self-similarity property (Forman et al., 2020c, Theorem 1.4(ii)), it suffices to prove that

$$(\overleftarrow{\underline{\beta}}^y, y \geq 0) \stackrel{d}{=} (c\overleftarrow{\underline{\beta}}^{y/c}, y \geq 0) \tag{4.9}$$

with $(\overleftarrow{\underline{\beta}}^y, y \geq 0)$ defined as in (4.8) associated with the Poisson random measure $\overleftarrow{\underline{\mathbf{Z}}}$.

Map each atom $(s, (\beta_s^z, z \geq 0))$ of $\overleftarrow{\underline{\mathbf{Z}}}$ to $(cs, (c\beta_s^{z/c}, z \geq 0))$. Then the image $\overleftarrow{\underline{\mathbf{Z}}}'$ is a Poisson random measure with intensity $c^{-1}(\theta_1/\alpha - 1)\mathrm{Leb} \otimes (\Phi_c)_* \Lambda^{(\alpha)} = (\theta_1/\alpha - 1)\mathrm{Leb} \otimes \Lambda^{(\alpha)}$, where $(\Phi_c)_* \Lambda^{(\alpha)}$ is the pushforward measure, due to the self-similarity of $\Lambda^{(\alpha)}$ given in Proposition 4.1. That is $\overleftarrow{\underline{\mathbf{Z}}}' \stackrel{d}{=} \overleftarrow{\underline{\mathbf{Z}}}$. Note that we have the identity

$$\bigstar_{\text{points } (r, \boldsymbol{\gamma}_r) \text{ of } \overleftarrow{\underline{\mathbf{Z}}}' : r \in [0,y] \downarrow} \mathrm{rev}\left(\gamma_r^{y-r}\right) = \bigstar_{\text{points } (s, \boldsymbol{\beta}_s) \text{ of } \overleftarrow{\underline{\mathbf{Z}}} : s \in [0, y/c] \downarrow} \mathrm{rev}\left(c\,\beta_s^{y/c-s}\right) = c\overleftarrow{\underline{\beta}}^{y/c}, \qquad y \geq 0.$$

This leads to (4.9). $\square$

Proposition 4.7 (Markov property). *Let $(\beta^y, y \geq 0)$ be an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from γ. For any $y \geq 0$, conditionally on $(\beta^x, x \leq y)$ the process $(\beta^{y+z}, z \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from β^y.*

Proof: Recall that $\beta^r := \overleftarrow{\underline{\beta}}^r \star \overrightarrow{\beta}^r, r \geq 0$. Fix $y \geq 0$, we have the decomposition $\beta^{y+z} = \overleftarrow{\underline{\beta}}_{-y}^z \star \underline{\beta}_{-y}^z \star \overrightarrow{\beta}^{y+z}$, where

$$\overleftarrow{\underline{\beta}}_{-y}^z := \bigstar_{\text{points } (s, \boldsymbol{\beta}_s) \text{ of } \overleftarrow{\underline{\mathbf{Z}}} : s \in (y, y+z] \downarrow} \mathrm{rev}\left(\beta_s^{y+z-s}\right), \qquad z \geq 0,$$

and

$$\underline{\beta}_{-y}^z := \bigstar_{\text{points } (s, \boldsymbol{\beta}_s) \text{ of } \overleftarrow{\underline{\mathbf{Z}}} : s \in [0, y] \downarrow} \mathrm{rev}\left(\beta_s^{y+z-s}\right), \qquad z \geq 0.$$

Conditionally on $((\overleftarrow{\beta}^x, \overrightarrow{\beta}^x), x \leq y)$, it follows from Proposition 4.2(ii) that $(\underline{\beta}_{-y}^z, z \geq 0)$ is an $\mathrm{RSSIP}^{(\alpha)}(0)$-evolution starting from $\overleftarrow{\beta}^y$ and from the Markov property of $\mathrm{SSIP}^{(\alpha)}(\theta_2)$-evolutions that $\overrightarrow{\beta}^{y+z}, z \geq 0$ is an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution starting from $\overrightarrow{\beta}^y$. Then, by Proposition 2.9, conditionally on $(\beta^x, x \leq y)$, the concatenation $(\underline{\beta}_{-y}^z \star \overrightarrow{\beta}^{y+z}, z \geq 0)$ is an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution starting from $\beta^y = \overleftarrow{\beta}^y \star \overrightarrow{\beta}^y$. Furthermore, the Poisson property shows that $(\overleftarrow{\underline{\beta}}_{-y}^z, z \geq 0)$ is independent of $(\underline{\beta}_{-y}^z \star \overrightarrow{\beta}^{y+z}, z \geq 0)$ and has the same law as $(\overleftarrow{\underline{\beta}}^z, z \geq 0)$. By Definition 4.3 we deduce that $(\beta^{y+z}, z \geq 0)$ is an $\mathrm{RSSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from β^y. $\square$

Proposition 4.8 (Path-continuity). *An $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution a.s. has continuous paths.*

Proof: It follows from the same arguments as in Forman et al. (2020c, proof of Proposition 3.2) that $(\overleftarrow{\beta}^y, y \geq 0)$ defined as in (4.8) a.s. has continuous paths in $(\mathcal{I}_H, d_H)$. Let us only give a sketch here. Fix $y_0 \geq 0$ and $\epsilon > 0$. For each $z < y_0$, recall that $\overleftarrow{\beta}^z$ is the concatenation of $\mathrm{rev}(\beta_s^{z-s})$, where $(s, (\beta_s^r, r \geq 0))$ is an atom of $\overleftarrow{\mathbf{Z}}$. For a suitably chosen $\delta > 0$, we will separate the atoms into two parts: those with $s \in (z - \delta, z)$ and those with $s \leq z - \delta$. For the first part, by Forman et al. (2020c, Lemma 3.7), a.s. we can choose $\delta > 0$ small enough such that

$$\sup_{z \in [0, y_0)} \sum_{s \in ((z-\delta) \wedge 0, z)} \|\beta_s^{z-s}\| < \epsilon.$$

For the second part, only a finite number $k(\delta) \geq 0$ of atoms $(s_i, (\beta_{s_i}^z, z \geq 0))_{i \leq k(\delta)}$ with $s_i < y_0$ have lifetime longer than δ and it is known that each process $(\beta_{s_i}^z, z \geq 0)$ is continuous. So there exists $\delta > \delta' > 0$ such that for each $i \leq k(\delta)$ we have

$$\sup_{y,z \in [0, y_0), |y-z| < \delta'} d_H(\beta_{s_i}^z, \beta_{s_i}^y) < \epsilon / k(\delta).$$

Therefore,

$$\sup_{y,z \in [0, y_0), |y-z| < \delta'} d_H(\overleftarrow{\beta}^z, \overleftarrow{\beta}^y) < 3\epsilon.$$

We conclude that $(\overleftarrow{\beta}^y, y \geq 0)$ is continuous. Combining this with the path-continuity of an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution, we deduce the claim. □

4.3. *Identification of the two-parameter family of* $\mathrm{SSIP}^{(\alpha)}(\theta)$-*evolutions,* $\alpha \in (0,1)$, $\theta \geq 0$.

Proposition 4.9. *An* $\mathrm{SSIP}^{(\alpha)}(\theta_1, \alpha)$-*evolution starting from* γ *is an* $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-*evolution starting from* γ.

As a consequence of Proposition 4.9, we can address the apparent lack of left-right-symmetry in Definition 4.3. Proposition 4.4 follows immediately from the following statement.

Proposition 4.10. *For* $\theta_1, \theta_2 \geq \alpha$ *and* $\gamma \in \mathcal{I}_H$, *consider three independent processes, an* $\mathrm{SSIP}^{(\alpha)}(\alpha)$-*evolution* $(\widehat{\beta}^y, y \geq 0)$ *starting from* γ, *and* $\mathrm{SSIP}^{(\alpha)}(\theta_j, 0)$-*evolutions* $(\beta_j^y, y \geq 0)$, $j = 1, 2$, *starting from* $\emptyset$. *Then* $\beta^y := \beta_1^y \star \widehat{\beta}^y \star \mathrm{rev}(\beta_2^y)$, $y \geq 0$ *is an* $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolution starting from* γ.

Proof of Proposition 4.10: First note that $(\beta_1^y, y \geq 0)$ has the same distribution as $(\overrightarrow{\beta}^y, y \geq 0)$ in (4.8), by Definition 4.3. Hence, we need to show that $(\widehat{\beta}^y \star \mathrm{rev}(\beta_2^y) \, y \geq 0)$ is an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution starting from γ, which we defined as the left-right reversal of $\mathrm{SSIP}^{(\alpha)}(\theta_2)$ starting from $\mathrm{rev}(\gamma)$. By Proposition 4.9, it suffices to show that $(\beta_2^y \star \mathrm{rev}(\widehat{\beta}^y), y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\theta_2, \alpha)$-evolution starting from $\mathrm{rev}(\gamma)$. This follows straight from Definition 4.3 and Proposition 1.2. The final claim follows from the representation in the first part and Proposition 1.2 since $\mathrm{rev}(\beta^y) = \beta_2^y \star \mathrm{rev}(\widehat{\beta}^y) \star \mathrm{rev}(\beta_1^y)$. □

In preparation of proving Proposition 4.9, we consider a Poisson random measure $\overleftarrow{\mathbf{Z}}^{(\mathrm{r})}$ on $[0, \infty) \times \mathcal{E}_\mathcal{I}$ with intensity $(\theta_1/\alpha - 1)\mathrm{Leb} \otimes \Lambda^{(\alpha)}$, whose atoms consisting of immigration times and excursions we label as *red*, and an independent Poisson random measure $\overleftarrow{\mathbf{Z}}^{(\mathrm{b})}$ on $[0, \infty) \times \mathcal{E}_\mathcal{I}$ with intensity $\mathrm{Leb} \otimes \Lambda^{(\alpha)}$, whose atoms we label as *blue*. Let

$$\overleftarrow{\beta}^z = \bigstar_{(s, \boldsymbol{\beta}_s) \text{ points of } \overleftarrow{\mathbf{Z}}^{(\mathrm{r})} + \overleftarrow{\mathbf{Z}}^{(\mathrm{b})} : s \in [0,z] \downarrow} \overleftarrow{\beta}_s^{z-s}, \qquad z \geq 0.$$

Then $(\overleftarrow{\beta}^z, z \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution, by Proposition A.5. We seek to compare this process with an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \alpha)$-evolution; to this end, let us explore this two-colour model in more detail.
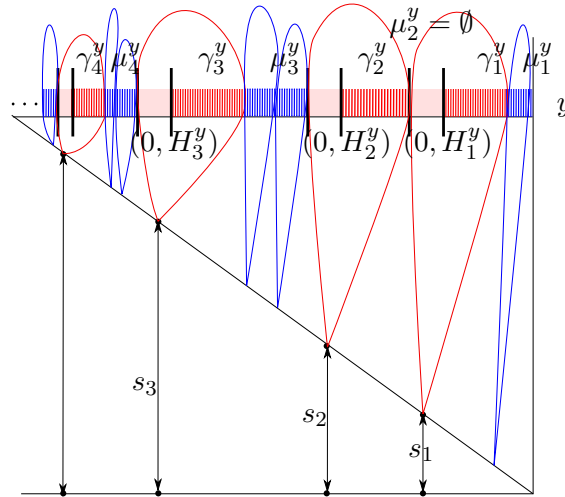
FIGURE 4.3. We illustrate the state of an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$ at time $y > 0$. The contribution of each surviving red excursion contains a leftmost block of mass H_i^y and the remaining (red-shaded) part γ_i^y. To the right of each surviving red excursion, there is a finite number (possibly zero) of blue excursions that form μ_i^y. Excursions not surviving to time y are omitted.

Fix any $y \geq 0$. Denote by $(s_k, \boldsymbol{\beta}_{s_k})_{k \geq 1}$ the *red* atoms of $\overleftarrow{\mathbf{Z}}^{(\mathrm{r})}$ whose excursions surviving to time y i.e. $\beta_{s_k}^{y-s_k} \neq \emptyset$, with s_k in increasing order. The distribution of $(s_k, k \geq 1)$ is given in (4.4), with $\theta = \theta_1 - \alpha$. We also read from (4.7) the conditional distribution given $(s_k, k \geq 1)$ of the interval partition $\beta_{s_k}^{y-s_k} \overset{d}{=} \{(0, H_k^y)\} \star \gamma_k^y$.

Given the immigration times $(s_k, k \geq 1)$ of the red excursions and $s_0 := 0$, let μ_k^y be the contribution at time y of the *blue* excursions that are immigrating at times in the interval $[s_{k-1}, s_k)$, i.e.

$$\mu_k^y := \underset{(s, \boldsymbol{\beta}_s) \text{ points of } \overleftarrow{\mathbf{Z}}^{(\mathrm{b})} : s \in [s_{k-1}, s_k) \downarrow}{\bigstar} \beta_s^{y-s}, \quad k \geq 1.$$

Note that μ_k^y may be empty. Then we have a decomposition of $\overleftarrow{\beta}^y$, as illustrated in Figure 4.3:

$$\overleftarrow{\beta}^y = \overset{1}{\underset{k=\infty}{\bigstar}} \left(\{(0, H_k^y)\} \star \gamma_k^y \star \mu_k^y \right). \tag{4.10}$$

Given $(s_k, k \geq 1)$, these H_k^y, γ_k^y, μ_k^y, $k \geq 1$, are conditionally independent. To identify the conditional distribution of $\gamma_k^y \star \mu_k^y$ given $(s_k^y, k \geq 1)$, note that display (4.3) yields that

$$\underset{(s, \boldsymbol{\beta}_s) \text{ points of } \overleftarrow{\mathbf{Z}}^{(\mathrm{b})} : s \in [s_k, y] \downarrow}{\bigstar} \beta_s^{y-s} \sim \mathtt{Gamma}\big(\alpha, (2(y - s_k))^{-1}\big) \cdot \mathtt{PDIP}^{(\alpha)}(\alpha),$$

which coincides with the conditional distribution of γ_k^y given in (4.7), and this interval partition is conditionally independent of μ_k^y. As a result, given $(s_k, k \geq 1)$ and writing $s_0 = 0$, the interval partitions $\gamma_k^y \star \mu_k^y$, $k \geq 1$, are conditionally independent, and using (4.3) again, we obtain

$$\gamma_k^y \star \mu_k^y \overset{d}{=} \underset{(s, \boldsymbol{\beta}_s) \text{ points of } \overleftarrow{\mathbf{Z}}^{(\mathrm{b})} : s \in [s_{k-1}, y] \downarrow}{\bigstar} \beta_s^{y-s} \sim \mathtt{Gamma}\big(\alpha, (2(y - s_{k-1}))^{-1}\big) \cdot \mathtt{PDIP}^{(\alpha)}(\alpha).$$

Next, define an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \alpha)$-evolution by $\widetilde{\beta}^y = \overleftarrow{\beta}^y \star \overrightarrow{\beta}^y$, $y \geq 0$, where $(\overleftarrow{\beta}, y \geq 0)$ is given by (4.8) and $(\overrightarrow{\beta}^y, y \geq 0)$ is an $\mathrm{RSSIP}^{(\alpha)}(\alpha)$-evolution starting from $\emptyset$. Since (4.8) is only left-right-reversing within surviving clades, the distribution of the increasing sequence of immigration times is again
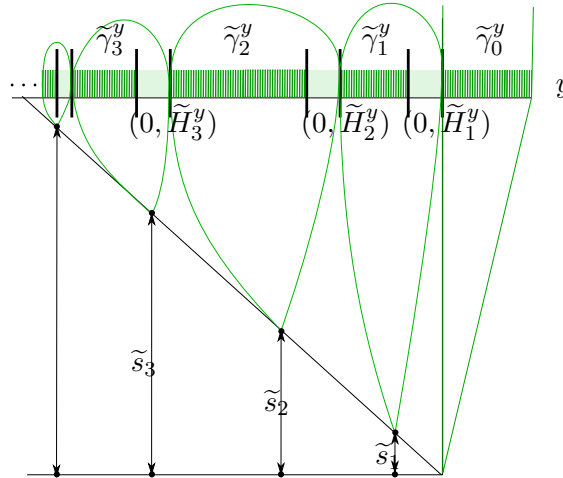
FIGURE 4.4. The value of an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \alpha)$ at time $y > 0$. The surviving immigration of $\overleftarrow{\beta}_{\theta_1-\alpha}$ to time y are coloured in green, with each one composed by a rightmost interval and the remaining part (green-shaded).

given by (4.4), with $\theta = \theta_1 - \alpha$, now based on $\overleftarrow{\underline{\mathbf{Z}}}$. We denote these by $(\widetilde{s}_k, \ k \geq 1)$ and also write $\widetilde{s}_0 := 0$. We further read from (4.7) and (4.8) that

$$(\underline{\overleftarrow{\beta}}^y, \vec{\beta}^y) \stackrel{d}{=} \left(\overset{1}{\underset{k=\infty}{\bigstar}} \widetilde{\gamma}_k^y \star \{(0, \widetilde{H}_k^y)\}, \ \widetilde{\gamma}_0^y \right), \tag{4.11}$$

where, given $(\widetilde{s}_k, \ k \geq 1)$, we have conditionally independent $\widetilde{H}_k^y \sim \mathtt{Gamma}(1 - \alpha, (2(y - \widetilde{s}_k^y))^{-1})$, $k \geq 1$, and $\widetilde{\gamma}_k^y \sim \mathtt{Gamma}(\alpha, (2(y - \widetilde{s}_k))^{-1}) \cdot \mathtt{PDIP}^{(\alpha)}(\alpha)$, $k \geq 0$. See Figure 4.4.

Summarizing, we have the following statement.

Lemma 4.11. *Fix any $y \geq 0$. With notation as above, we have $(s_k, \ k \geq 1) \stackrel{d}{=} (\widetilde{s}_k, \ k \geq 1)$. Moreover, the conditional distribution of $\big((H_k^y, \gamma_k^y \star \mu_k^y), \ k \geq 1 \big)$ given $(s_k, \ k \geq 1)$ is the same as the conditional distribution of $\big((\widetilde{H}_k^y, \widetilde{\gamma}_{k-1}^y), \ k \geq 1 \big)$ given $(\widetilde{s}_k, \ k \geq 1)$.*

Proof of Proposition 4.9: For fixed $y \geq 0$, using Lemma 4.11 and its notation, we have the identity in law

$$\left(\overset{1}{\underset{i=k}{\bigstar}} \big(\{(0, H_i^y)\} \star \gamma_i^y \star \mu_i^y \big) \right) \star \hat{\beta}^y \stackrel{d}{=} \{(0, \widetilde{H}_k^y)\} \star \left(\overset{1}{\underset{i=k-1}{\bigstar}} \big(\widetilde{\gamma}_i^y \star \{(0, \widetilde{H}_i^y)\} \big) \right) \star \widetilde{\gamma}_0^y \star \hat{\beta}^y, \qquad k \geq 1,$$

where $(\hat{\beta}^z, z \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(0)$-evolution starting from γ, independent of everything else. In the limit $k \to \infty$, the LHS has the law at time y of an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution starting from γ. For the RHS, since it follows from Corollary 2.7 that $\widetilde{\gamma}_0^y \star \hat{\beta}^y$ has the law of a $\mathrm{RSSIP}^{(\alpha)}(\alpha)$-evolution at time y starting from γ, the RHS has, in the limit $k \to \infty$, the law of an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \alpha)$-evolution at time y, starting from γ. So we have identified the one-dimensional marginals. It follows from this observation above and the Markov properties of both processes that they have the same finite-dimensional distributions. The claim follows from the path-continuity. $\square$

We end this section by deriving two decompositions of Poisson–Dirichlet interval partitions from the correspondence in Lemma 4.11. They have a similar flavour to Pitman and Winkel (2009, Corollary 8), but are different.

Corollary 4.12. *For $\theta > \alpha$, $\rho > 0$, let $(\bar{\beta}_n)_{n\geq 1}$ be i.i.d. $\mathtt{PDIP}^{(\alpha)}(0)$, $(B_n)_{n\geq 1}$ be i.i.d. $\mathtt{Beta}(\theta-\alpha,1)$, $(E_n)_{n\geq 1}$ be i.i.d. $\mathtt{Exp}(\rho)$, and $\gamma \sim \mathtt{Gamma}(\alpha,\rho) \cdot \mathtt{PDIP}^{(\alpha)}(\alpha)$. Then we have the identity*

$$\left(\bigstar_{n=\infty}^{1} \left(E_n \prod_{i=1}^{n} B_i \right) \mathrm{rev}(\bar{\beta}_n) \right) \star \gamma \stackrel{d}{=} \mathtt{Gamma}(\theta,\rho) \cdot \mathtt{PDIP}^{(\alpha)}(\theta). \tag{4.12}$$

Proof: Consider $\widetilde{\beta}^y \stackrel{d}{=} \bigstar_{k=\infty}^{1} \left(\widetilde{\gamma}_k^y \star \{(0,\widetilde{H}_k^y)\} \right) \star \widetilde{\gamma}_0^y$ given in (4.11), i.e. a decomposition of an $\mathtt{SSIP}^{(\alpha)}(\theta,\alpha)$-evolution at time y. With $y = 1/2\rho$, by (4.5) and (2.2) we have

$$\left(\widetilde{\gamma}_n^{1/2\rho} \star \left\{ (0,\widetilde{H}_n^{1/2\rho}) \right\}, \;\; n \geq 1 \right) \stackrel{d}{=} \left(\left(E_n \prod_{i=1}^{n} B_i \right) \mathrm{rev}(\bar{\beta}_n), \;\; n \geq 1 \right).$$

Then $\widetilde{\beta}^{1/2\rho}$ can be written as the LHS of (4.12). On the other hand, $\widetilde{\beta}^{1/2\rho} \sim \mathtt{Gamma}(\theta,\rho) \cdot \mathtt{PDIP}^{(\alpha)}(\theta)$ by Proposition 4.9 and (4.3). $\qquad\square$

Corollary 4.13. *For $\theta > \alpha$ and $\rho > 0$, let $(\bar{\beta}_n)_{n\geq 1}$ be i.i.d. $\mathtt{PDIP}^{(\alpha)}(0)$, $(B_n)_{n\geq 1}$ i.i.d. $\mathtt{Beta}(\theta,1)$, $(E_n)_{n\geq 1}$ i.i.d. $\mathtt{Exp}(\rho)$, $G \sim \mathtt{Gamma}(\alpha,\rho)$, $\bar{\gamma} \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$, and K have geometric distribution on $\mathbb{N}$ with success probability $1 - \alpha/\theta$. Then we have the identity*

$$\left(G \prod_{i=1}^{K} B_i \right) \bar{\gamma} \star \left(\bigstar_{n=K-1}^{1} \left(E_n \prod_{i=1}^{n} B_i \right) \mathrm{rev}(\bar{\beta}_n) \right) \stackrel{d}{=} \mathtt{Gamma}(\alpha,\rho) \cdot \mathtt{PDIP}^{(\alpha)}(\alpha).$$

Proof: Using the decomposition of $\overleftarrow{\beta}^{1/2\rho}$ given in (4.10) and Lemma 4.11, and notation therein, we look at the interval partition to the right of the rightmost red interval, i.e. $\gamma_1^{1/2\rho} \star \mu_1^{1/2\rho}$. The Poisson property shows that the first (from the right) red excursion is the K-th one among all points of $\overleftarrow{\mathbf{Z}}^{(\mathrm{r})} + \overleftarrow{\mathbf{Z}}^{(\mathrm{b})}$ surviving to time $1/2\rho$. Using (4.5) and (2.2), we have the representation of the left-hand side. By Lemma 4.11, $\gamma_1^{1/2\rho} \star \mu_1^{1/2\rho}$ has the same law as $\widetilde{\gamma}_0^{1/2\rho} \sim \mathtt{Gamma}(\alpha,\rho) \cdot \mathtt{PDIP}^{(\alpha)}(\alpha)$. $\qquad\square$

4.4. *Pseudo-stationarity of* $\mathtt{SSIP}^{(\alpha)}(\theta_1,\theta_2)$*-evolutions, and the proof of Theorem 1.5.* Recall from the introduction that we call a distribution μ on unit-mass interval partitions in $\mathcal{I}_H$ *pseudo-stationary* for an interval partition evolution if starting the evolution from an independently scaled multiple of a μ-distributed interval partition, the marginal distributions at all positive times have the same form. In other words, the evolution only changes the distribution of the total mass, but keeps the distribution of the interval partition normalised to unit total mass invariant. Let us first study $\mathtt{SSIP}^{(\alpha)}(\theta_1,\theta_2)$-evolutions starting from $\emptyset$.

Proposition 4.14. *For $\theta_1 \geq \alpha$ and $\theta_2 \geq 0$, let $(\beta^y, y \geq 0)$ be an $\mathtt{SSIP}^{(\alpha)}(\theta_1,\theta_2)$-evolution starting from $\emptyset$. Then at any fixed time $y \geq 0$ we have*

$$\beta^y \stackrel{d}{=} V G_1^y \bar{\beta}_1 \star \{(0, V G_0^y)\} \star G_2^y \bar{\beta}_2,$$

*where $(V, \bar{\beta}_1, \bar{\beta}_2, G_1^y, G_0^y, G_2^y)$ is a family of independent random variables with $V \sim \mathtt{Beta}(\theta_1-\alpha,1)$, $\bar{\beta}_1 \sim \mathtt{PDIP}^{(\alpha)}(\theta_1)$, $\mathrm{rev}(\bar{\beta}_2) \sim \mathtt{PDIP}^{(\alpha)}(\theta_2)$, $G_1^y \sim \mathtt{Gamma}(\theta_1,1/2y)$, $G_0^y \sim \mathtt{Gamma}(1-\alpha,1/2y)$, and $G_2^y \sim \mathtt{Gamma}(\theta_2,1/2y)$. By convention $V = 0$ when $\theta_1 = \alpha$ and $G_2^y = 0$ when $\theta_2 = 0$.
    In other words,*

$$\beta^y \stackrel{d}{=} G^y \left(V_1(1-V_2)\bar{\beta}_1 \star \{(0,V_1 V_2)\} \star (1-V_1)\bar{\beta}_2 \right),$$

where $G^y \sim \mathtt{Gamma}(\theta_1+\theta_2-\alpha,1/2y)$, $V_1 \sim \mathtt{Beta}(\theta_1-\alpha,\theta_2)$ and $V_2 \sim \mathtt{Beta}(1-\alpha,\theta_1)$ are independent, further independent of $(\bar{\beta}_1, \bar{\beta}_2)$.

Proof: We write $\beta^y = \underleftarrow{\beta}^y \star \overrightarrow{\beta}^y, y \geq 0$ as in Definition 4.3. It is known from Forman et al. (2020c, Proposition 3.6) that $\mathrm{rev}(\overrightarrow{\beta}^y) \sim \mathtt{Gamma}(\theta_2, 1/2y) \cdot \mathtt{PDIP}^{(\alpha)}(\theta_2)$, i.e. $\overrightarrow{\beta}^y \overset{d}{=} G_2^y \bar{\beta}_2$.

Using the two-colour correspondence described in Lemma 4.11 and its notation, we have $\underleftarrow{\beta}^y \overset{d}{=} \gamma^y \star \{(0, A^y)\}$, where $\gamma^y := \bigstar_{i=\infty}^{2} (\{(0, H_i^y)\} \star \gamma_i^y \star \mu_i^y)$ is the concatenation of all interval partitions to the left of the rightmost red one, and $A^y := H_1^y$ is the mass of the leftmost block of the rightmost red interval partition. Let us enumerate all immigrants in $\underleftarrow{\mathbf{Z}}^{(\mathrm{r})} + \underleftarrow{\mathbf{Z}}^{(\mathrm{b})}$ surviving to time y from right to left and denote by K the index of the first red one (i.e. μ_1^y is the concatenation of contributions from $K-1$ blue ones). Then K clearly has a geometric distribution with success probability $1 - \alpha/\theta_1$.

Applying (4.5) and (4.6) to the $\mathtt{SSIP}^{(\alpha)}(\theta_1)$-evolution associated with $\underleftarrow{\mathbf{Z}}^{(\mathrm{r})} + \underleftarrow{\mathbf{Z}}^{(\mathrm{b})}$ and using (2.2), we deduce that, conditionally on $\{K = k\}$, we have $(\gamma^y, A^y) \overset{d}{=} \left(G_1^y \left(\prod_{i=1}^{k} B_i \right) \bar{\beta}_1, \ G_0^y \prod_{i=1}^{k} B_i \right)$, where $G_1^y \sim \mathtt{Gamma}(\theta_1, 1/2y)$, $G_0^y \sim \mathtt{Gamma}(1 - \alpha, 1/2y)$, $\bar{\beta}_1 \sim \mathtt{PDIP}^{(\alpha)}(\theta_1)$, and $(B_i)_{i \geq 1}$ is an i.i.d. sequence of $\mathtt{Beta}(\theta_1, 1)$; they are all independent. So we complete the proof of the first statement by checking that $V := \prod_{i=1}^{K} B_i \sim \mathtt{Beta}(\theta_1 - \alpha, 1)$, which follows from the calculation of moments: for every $r \in \mathbb{N}$, we have

$$\mathbb{E}\left[\left(\prod_{i=1}^{K} B_i \right)^r \right] = \sum_{k=1}^{\infty} \left(\frac{\theta_1}{\theta_1 + r} \right)^k \frac{\theta_1 - \alpha}{\theta_1} \left(\frac{\alpha}{\theta_1} \right)^{k-1} = \frac{\theta_1 - \alpha}{\theta_1 - \alpha + r}.$$

Since $(G_1^y V, G_0^y V) \overset{d}{=} \left((1 - V_2)(G_1^y + G_0^y)V, V_2(G_1^y + G_0^y)V \right)$ and $\left((G_1^y + G_0^y)V, G_2^y \right) \overset{d}{=} G^y(V_1, (1 - V_1))$, the second statement follows from the first one. $\square$

Corollary 4.15. *Let $\theta \geq \alpha$, $\rho > 0$. Consider independent $B \sim \mathtt{Beta}(\theta - \alpha, \alpha)$, $B' \sim \mathtt{Beta}(1 - \alpha, \theta)$, $\bar{\gamma}_1 \sim \mathtt{PDIP}^{(\alpha)}(\theta)$, and $\bar{\gamma}_2 \sim \mathtt{PDIP}^{(\alpha)}(\alpha)$. Then we have*

$$B(1 - B')\bar{\gamma}_1 \star \{(0, BB')\} \star (1 - B)\bar{\gamma}_2 \sim \mathtt{PDIP}^{(\alpha)}(\theta).$$

Proof: This follows from the marginals of Proposition 4.14 with $\theta_1 = \theta$, $\theta_2 = \alpha$ and $y = 1/2\rho$, and from the marginals of an $\mathtt{SSIP}^{(\alpha)}(\theta)$-evolution recalled in (4.3), noting that they must be equal by Proposition 4.9. $\square$

By using very similar arguments as in Forman et al. (2020c, proof of Proposition 3.15), respectively, we deduce from Proposition 4.14 the following consequence of Proposition 4.14, and we can similarly prove Theorem 1.5.

Lemma 4.16. *For $\theta_1 \geq \alpha$, $\theta_2 \geq 0$ and $\rho > 0$, let $(G, B, B', \bar{\gamma}_1, \bar{\gamma}_2)$ be an independent quintuple with $G \sim \mathtt{Gamma}(\theta_1 + \theta_2 - \alpha, \rho)$, $B \sim \mathtt{Beta}(\theta_1 - \alpha, \theta_2)$, $B' \sim \mathtt{Beta}(1 - \alpha, \theta_1)$, $\bar{\gamma}_1 \sim \mathtt{PDIP}^{(\alpha)}(\theta_1)$, and $\mathrm{rev}(\bar{\gamma}_2) \sim \mathtt{PDIP}^{(\alpha)}(\theta_2)$. Let $(\beta^y, y \geq 0)$ be an $\mathtt{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from*

$$\gamma := G \left(B(1 - B')\bar{\gamma}_1 \star \{(0, BB')\} \star (1 - B)\bar{\gamma}_2 \right).$$

Then at any $y \geq 0$, the interval partition β^y has the same distribution as $(2y\rho + 1)\gamma$.

Proof of Theorem 1.5: When $\theta > 0$, the pseudo-stationarity claims for $\mathtt{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions can be deduced from Lemma 4.16 by self-similarity and Laplace inversion, as in the corresponding proof of Forman et al. (2020c, Proposition 1.3(iv)) for $\mathtt{SSIP}^{(\alpha)}(\theta)$-evolutions. Note, in particular, that Theorem 1.4(ii) entails that $\|\beta^y\|$ is independent of $\bar{\gamma}$, since the conditional distribution of the total mass process given the initial interval partition only depends on the initial total mass $\|\beta^0\|$, not on the independent unit-mass interval partition $\bar{\gamma} = \beta^0/\|\beta^0\|$.

When $\theta = 0$, i.e. $\theta_1 = \alpha$ and $\theta_2 = 0$, then by definition the process is an $\mathtt{RSSIP}^{(\alpha)}(0)$. Then the pseudo-stationary distribution is the reversal of $\mathtt{PDIP}^{(\alpha)}(0)$ by Forman et al. (2021, Theorem 1.5), as desired.

That these processes are recurrent for $\theta \in (0,1)$ is a consequence of the recurrence of their total mass processes $\mathtt{BESQ}(2\theta)$. It remains to show that $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions are extensions of $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolutions. This is the achieved in Proposition 4.17. $\qquad\square$

4.5. *Identification of stopped* $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolutions as* $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-*evolutions.* Finally, we show that the two approaches to define a three-parameter family of interval partition evolutions with left and right immigration lead to the same processes (when stopped upon first reaching $\emptyset$).

Proposition 4.17. *An* $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-*evolution starting from* γ *and stopped when first hitting* $\emptyset$, *is an* $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-*evolution starting from* γ.

Proof: Recall the construction of an $\mathrm{SSIP}^{(\alpha)}_\dagger(\theta_1, \theta_2)$-evolution $(\beta^y, y \geq 0)$ in Definition 1.3. By Proposition 4.9, $\boldsymbol{\gamma}_1^{(0)}$ has the same distribution as the process

$$\overleftarrow{\underline{\beta}}^y_{-1} \star \overrightarrow{\beta}^y_1, \qquad y \geq 0,$$

where $(\overleftarrow{\underline{\beta}}^y_{-1}, y \geq 0)$ is as in (4.8) and $(\overrightarrow{\beta}^y_1, y \geq 0)$ is an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution. Then we can write

$$\beta^y = \overleftarrow{\underline{\beta}}^y_{-1} \star \overrightarrow{\beta}^y_+, \quad \text{where } \overrightarrow{\beta}^y_+ = \overrightarrow{\beta}^y_1 \star \{(0, \mathbf{f}^{(0)}(y))\} \star \gamma_2^{(0)}(y), \quad 0 \leq y \leq \zeta(\mathbf{f}^{(0)}),$$

is an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution, by Lemma 2.8 and Proposition 2.9. Comparing with Definition 4.3, the process $(\beta^y, 0 \leq y \leq \zeta(\mathbf{f}^{(0)}))$ can be viewed as an $\mathrm{SSIP}^{(\alpha)}(\theta_1, \theta_2)$-evolution stopped at the lifetime of the block starting from the middle interval $(\|\beta_1^0\|, \|\beta_1^0\| + m^0)$. Because of the Markov properties of both processes, Theorem 1.4 and Proposition 4.7, we can continue using these arguments to complete the proof inductively. $\qquad\square$

Appendix A. Proofs of technical lemmas

We complete the proofs of Lemmas 3.4 and 3.8 in Appendix A.4. These proofs depend on the marked Lévy process constructions of Forman et al. (2020a, 2022, 2020c) that we recall here. We also explain how Propositions 4.1 and 4.2 can be deduced from these constructions.

A.1. *Construction of* $\mathrm{SSIP}^{(\alpha)}(0)$ *from marked stable Lévy processes.* Let $\mathcal{E}$ be the space of non-negative càdlàg excursions away from zero, i.e.

$$\mathcal{E} = \{f: [0, \infty) \to [0, \infty) \text{ is càdlàg, and } \exists z \geq 0 \text{ s.t. } f(y) = 0, \forall y \geq z\}$$

For any $f \in \mathcal{E}$, let $\zeta(f) := \sup\{t \geq 0: f(t) > 0\}$. Fix $\alpha \in (0,1)$. Pitman and Yor (1982, Section 3) construct a sigma-finite excursion measure associated with $\mathtt{BESQ}(-2\alpha)$ on (the subspace of continuous excursions in) the space $\mathcal{E}$, which we scale to a measure $\nu_{\mathtt{BESQ}}^{(-2\alpha)}$ such that

$$\nu_{\mathtt{BESQ}}^{(-2\alpha)}(f: \zeta(f) \geq y) = \frac{\alpha}{2^\alpha \Gamma(1-\alpha)\Gamma(1+\alpha)} y^{-1-\alpha}, \qquad y > 0,$$

and under $\nu_{\mathtt{BESQ}}^{(-2\alpha)}$, conditional on $\zeta(f) = y$ for $0 < y < \infty$, the process f is the squared Bessel bridge with dimension parameter $4 + 2\alpha$, from 0 to 0 over $[0, y]$, see Revuz and Yor (1999, Section 11.3).

Following Forman et al. (2020a), let $\mathbf{N}$ be a Poisson random measure on $[0, \infty) \times \mathcal{E}$ with intensity measure $\mathrm{Leb} \otimes \nu_{\mathtt{BESQ}}^{(-2\alpha)}$. Each atom of $\mathbf{N}$, which is an excursion function in $\mathcal{E}$, shall be referred to as a *spindle*, in view of illustrations of $\mathbf{N}$ as in Figure A.5. Indeed, it will be useful to have a designated name other than "excursion" in order to avoid confusion with excursions of Lévy processes that we will also encounter. For $t \geq 0$ the following limit exists (see Sato (1999, Theorem 19.2)):

$$\xi_{\mathbf{N}}(t) := \lim_{z \downarrow 0} \left(\int_{[0,t] \times \{g \in \mathcal{E}\, :\, \zeta(g) > z\}} \zeta(f) \mathbf{N}(ds, df) - \frac{(1+\alpha)t}{(2z)^\alpha \Gamma(1-\alpha)\Gamma(1+\alpha)} \right). \qquad (\mathrm{A.1})$$
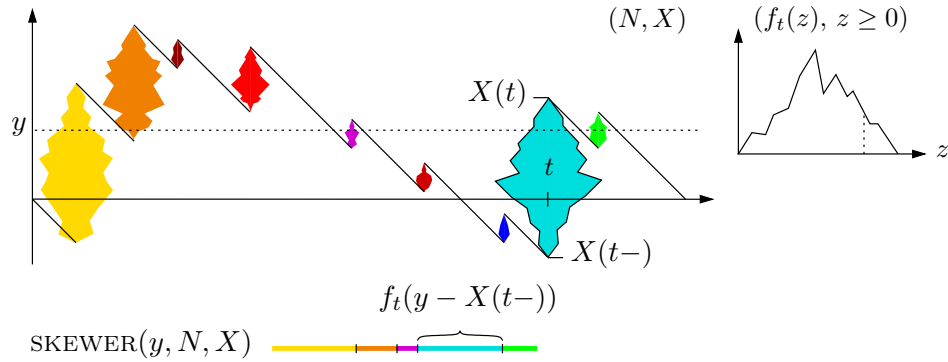
FIGURE A.5. A (discrete analogue of a) scaffolding X with spindle marks: (N, X), and the spindle f_t at scaffolding time t. The skewer SKEWER(y, N, X) extracts from each spindle that crosses level y the spindle width at that level and builds an interval partition in which block sizes are spindle widths, placed in left-to-right order without leaving gaps, as if on a skewer that is pushed through spindles from left to right.

The process $\xi_{\mathbf{N}} := (\xi_{\mathbf{N}}(t), t \geq 0)$, particularly its time parametrisation is not of direct relevance to us, and we call it the *scaffolding function* of $\mathbf{N}$ so that we view the spindles associated with its jumps as being placed onto this scaffolding. In the present setting, it is a spectrally positive stable Lévy process of index $1 + \alpha$, with Lévy measure $\nu_{\mathtt{BESQ}}^{(-2\alpha)}(\zeta(f) \in dy)$ and Laplace exponent $q^{1+\alpha}/2^\alpha \Gamma(1 + \alpha)$, $q \geq 0$. We will also use the term scaffolding function for relevant concatenations of stable processes below.

For $x > 0$, let $\mathbf{f} \sim \mathtt{BESQ}_x(-2\alpha)$, independent of $\mathbf{N}$. Then $\mathbf{f}$ is $\mathcal{E}$-valued (actually taking values in the subspace of excursions that are continuous after an initial positive jump). Write $\mathtt{Clade}_x(\alpha)$ for the law of a *clade of initial mass x*, which is a random point measure on $[0, \infty) \times \mathcal{E}$ defined by

$$\mathrm{CLADE}(\mathbf{f}, \mathbf{N}) := \delta(0, \mathbf{f}) + \mathbf{N} \mid_{(0, T_{-\zeta(\mathbf{f})}(\mathbf{N})] \times \mathcal{E}}, \quad \text{where } T_{-y}(\mathbf{N}) := \inf\{t \geq 0 \colon \xi_{\mathbf{N}}(t) = -y\}. \tag{A.2}$$

We also write $\mathrm{len}(\mathrm{CLADE}(\mathbf{f}, \mathbf{N})) := T_{-\zeta(\mathbf{f})}(\mathbf{N})$ for its length, which is a.s. finite. For $\gamma \in \mathcal{I}_H$, let $(\mathbf{N}_U, U \in \gamma)$ be a family of independent clades, with each $\mathbf{N}_U \sim \mathtt{Clade}_{|U|}(\alpha)$. Then we define $\mathbf{N}_\gamma$, a random point measure on $[0, \infty) \times \mathcal{E}$, by the concatenation of $(\mathbf{N}_U, U \in \gamma)$:

$$\mathbf{N}_\gamma := \bigstar_{U \in \gamma} \mathbf{N}_U := \sum_{U \in \gamma} \int \delta(g(U) + t, f) \mathbf{N}_U(dt, df), \quad \text{where } g(U) := \sum_{V \in \gamma \colon \sup V \leq \inf U} \mathrm{len}(\mathbf{N}_V). \tag{A.3}$$

We denote the distribution of $\mathbf{N}_\gamma$ by $\mathbf{P}_\gamma^{\alpha, 0}$.

Definition A.1 (Skewer)**.** Let $N = \sum_{i \in \mathbb{N}} \delta(t_i, f_i)$ for some $(t_i, f_i) \in [0, T] \times \mathcal{E}$ and $X \colon [0, T] \to \mathbb{R}$ càdlàg and such that

$$\sum_{t \in [0, T] \colon \Delta X(t) > 0} \delta(t, \Delta X(t)) = \sum_{i \in \mathbb{N}} \delta(t_i, \zeta(f_i)).$$

For $y \geq 0$ and $t \in [0, T]$, set

$$M^y(t) = \int_{[0, t] \times \mathcal{E}} f\big(y - X(s-)\big) \mathbf{1}\{y \geq X(s-)\} N(ds, df) = \sum_{i \in \mathbb{N} \colon t_i \in [0, t]} f_i\big(y - X(t_i-) \mathbf{1}\{y \geq X(t_i-)\}\big),$$

with the convention that $M^y(0-) = 0$. The *skewer* of the pair (N, X) at level y is the interval partition

$$\mathrm{SKEWER}(y, N, X) := \{(M^y(t-), M^y(t)) \colon M^y(t-) < M^y(t), t \in [0, T]\}, \tag{A.4}$$

If ξ_N as in (A.1) exists, then we denote the *skewer process of N* by

$$\overline{\mathrm{SKEWER}}(N) := (\mathrm{SKEWER}(y, N, \xi_N), y \geq 0).$$

See Figure A.5 for an illustration of $\textsc{skewer}(y, N, X)$ in the natural extension of this definition to point measures with finitely many atoms.

Proposition A.2 (Theorem 1.8 of Forman et al. (2021)). *For $\gamma \in \mathcal{I}_H$, let $\mathbf{N}_\gamma \sim \mathbf{P}_\gamma^{\alpha,0}$ be as in (A.3). Then $\overline{\textsc{skewer}}(\mathbf{N}_\gamma)$ is an $\textrm{SSIP}^{(\alpha)}(0)$-evolution starting from γ.*

Note that the skewer process is indexed by the non-negative levels of the scaffolding function, while the time axis of the scaffolding function induces the left-to-right order within the interval partition. This means we need to be careful when we refer to "time", so it will often be clearest if we distinguish "scaffolding time t" and "level y", but we will also continue to refer to "time y" in an $\textrm{SSIP}^{(\alpha)}(0)$, which in the context of a clade construction is the same as "level y".

The Markov property of $\textrm{SSIP}^{(\alpha)}(0)$-evolutions at time $y > 0$ corresponds to a decomposition of $\mathbf{N}_\gamma$ at scaffolding level y in $\xi_{\mathbf{N}_\gamma}$. Indeed, the $\textrm{SSIP}^{(\alpha)}(0)$-evolution after time y is described by the the spindles of $\mathbf{N}_\gamma$ that the $\xi_{\mathbf{N}_\gamma}$ places above level y, and these spindles naturally split into clades according to the excursions of $\xi_{\mathbf{N}_\gamma}$ above level y, which (for all excursions above almost all levels) start by a jump that corresponds to the upper part of a spindle straddling level y.

More precisely, denote by $S_\gamma(t) = \textrm{Leb}(\{u \le t \colon \xi_{\mathbf{N}_\gamma}(u) \ge y\})$, where Leb denote the Lebesgue measure, the amount of time up to scaffolding time $t \in [0, \textrm{len}(\mathbf{N}_\gamma)]$ that the scaffolding $\xi_{\mathbf{N}_\gamma}$ has spent above level y. Then we define the associated point measure

$$N^{\ge y}(\mathbf{N}_\gamma) := \sum_{\textrm{points } (t, f_t) \textrm{ of } \mathbf{N}_\gamma} \left(\mathbf{1}\{\xi_{\mathbf{N}_\gamma}(t-) \ge y\} \delta(S_\gamma(t), f_t) + \mathbf{1}\{\xi_{\mathbf{N}_\gamma}(t-) < y < \xi_{\mathbf{N}_\gamma}(t)\} \delta(S_\gamma(t), \hat{f}_t^y) \right),$$

where $\hat{f}_t^y(s) = f(y - \xi_{\mathbf{N}_\gamma}(t-) + s)$, $s \in [\xi_{\mathbf{N}_\gamma}(t) - y]$, is the part of the spindle f_t above level y. We can similarly define $N^{\le y}(\mathbf{N}_\gamma)$ based on (part-)spindles below level y and refer to the σ-algebra generated by $N^{\le y}(\mathbf{N}_\gamma)$ as the *history below level y*. Then the point measures $\mathbf{N}_\gamma$ satisfy the following property, which we refer to as *Markov-like property*.

Proposition A.3 (Proposition 6.6 of Forman et al. (2020a)). *Let $\mathbf{N}_\gamma \sim \mathbf{P}_\gamma^{\alpha,0}$ and $y > 0$. Then conditionally on the history below level y, we have $N^{\ge y}(\mathbf{N}_\gamma) \sim \mathbf{P}_{\beta^y}^{\alpha,0}$, where $\beta^y := \textsc{skewer}(y, \mathbf{N}_\gamma, \xi_{\mathbf{N}_\gamma})$.*

A.2. *Clade construction of* $\textrm{SSIP}^{(\alpha)}(\theta)$-*evolutions with* $\theta > 0$. Let $\alpha \in (0, 1)$ and $\theta > 0$. The clade construction of $\textrm{SSIP}^{(\alpha)}(\theta)$-evolutions is a Poissonian construction, based on a different type of clade, clades without an initial spindle. To define these clades, first consider again a Poisson random measure $\mathbf{N}$ on $[0, \infty) \times \mathcal{E}$ with intensity measure $\textrm{Leb} \otimes \nu_{\textrm{BESQ}}^{(-2\alpha)}$ and $\xi_{\mathbf{N}}$ its associated scaffolding defined as in (A.1). For $y > 0$, let $T^{-y} := \inf\{t \ge 0 \colon \xi_{\mathbf{N}}(t) < -y\}$, and $T^{(-y)-} := \sup_{z \in (0,y)} T^{-z}$. We recall from Forman et al. (2020c, Section 2.3) that there is a sigma-finite measure $\nu_{\perp \textrm{cld}}^{(\alpha)}$ on a suitable space $(\mathcal{N}, \Sigma(\mathcal{N}))$ of counting measures on $[0, \infty) \times \mathcal{E}$ that can be defined as

$$\nu_{\perp \textrm{cld}}^{(\alpha)}(A) := \mathbb{E}\left[\sum_{y \in [0,1]} \mathbf{1}\left\{ \mathbf{N}|_{[T^{(-y)-}, T^{-y})}^{\leftarrow} \in A \right\} \right], \qquad A \in \Sigma(\mathcal{N}), \tag{A.5}$$

where $\mathbf{N}|_{[a,b)}^{\leftarrow} := \sum_{(s,f) \textrm{ points of } \mathbf{N} \colon s \in [a,b)} \delta(s - a, f)$ for every $a < b$. Since each interval $[T^{(-y)-}, T^{-y})$ is the interval of an excursion of $\xi_{\mathbf{N}}$ above the minimum, the sigma-finite measure $\nu_{\perp \textrm{cld}}^{(\alpha)}$ captures the distribution of the associated point measure of spindles during such an excursion under the Itô excursion measure of the spectrally positive stable Lévy process $\xi_{\mathbf{N}}$ reflected at its running minimum process. See Forman et al. (2020c, Section 2.3) for more detailed discussion on $\nu_{\perp \textrm{cld}}^{(\alpha)}$.

Lemma A.4 (Equations (2.17)–(2.18) in Forman et al. (2020c)). *Let $\mathbf{N}_x = \delta(0, \mathbf{f}) + \mathbf{N}^\circ \sim \textrm{Clade}_x(\alpha)$ be a clade of initial mass $x > 0$ and write $T^{-y} := \inf\{t \ge 0 \colon \xi_{\mathbf{N}^\circ}(t) \le -y\}$, $y \in [0, \zeta(\mathbf{f}))$. Then*

conditionally given $\zeta(\mathbf{f}) = s$,

$$\sum_{y \in [0,\zeta(\mathbf{f})): T^{(-y)-} < T^{-y}} \delta\left(\zeta(\mathbf{f}) - y, \mathbf{N}^\circ|_{[T^{(-y)-}, T^{-y})}^\leftarrow\right),$$

is a Poisson random measure on $[0, s] \times \mathcal{N}$ *with intensity measure* $\mathrm{Leb} \otimes \nu_{\perp\mathrm{cld}}^{(\alpha)}$.

Let $\overleftarrow{\mathbf{F}}$ be a Poisson random measure on $(-\infty, 0] \times \mathcal{N}$ with intensity measure $(\theta/\alpha)\mathrm{Leb} \otimes \nu_{\perp\mathrm{cld}}^{(\alpha)}$. We will abbreviate the distribution of $\overleftarrow{\mathbf{F}}$ to $\mathbf{P}_\emptyset^{\alpha,\theta}$. We will now use the points (s, N_s) of $\overleftarrow{\mathbf{F}}$ to provide immigration at level $|s|$ that contributes to levels $y \geq |s|$ via $\mathrm{SKEWER}(y - |s|, N_s, \xi_{N_s})$.

Proposition A.5 (Proposition 3.4 of Forman et al. (2020c)). *Let* $\overleftarrow{\mathbf{F}} \sim \mathbf{P}_\emptyset^{\alpha,\theta}$ *and*

$$\overline{\beta}^y = \underset{\text{points } (s,N_s) \text{ of } \overleftarrow{\mathbf{F}}: s \in [-y, 0]}{\bigstar} \mathrm{SKEWER}(y - |s|, N_s, \xi_{N_s}), \qquad y \geq 0. \tag{A.6}$$

Then $(\overline{\beta}^y, y \geq 0)$ *is an* $\mathrm{SSIP}^{(\alpha)}(\theta)$*-evolution starting from* $\emptyset$.

We stress that each atom (s, N_s) of $\overleftarrow{\mathbf{F}}$ is in the negative half plane with $s \leq 0$. So this concatenation is in decreasing order of $|s|$, setting skewers of N_s with $|s|$ higher to the left of those with $|s|$ lower.

Recall from Proposition 2.9 that an $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolution starting from $\beta^0 \in \mathcal{I}_H$ can be constructed as $(\beta_1^y \star \beta_2^y, y \geq 0)$ for an independent pair of an $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolution $(\beta_1^y, y \geq 0)$ starting from $\emptyset$ and an $\mathrm{SSIP}^{(\alpha)}(0)$-evolution $(\beta_2^y, y \geq 0)$ starting from β^0. In particular, we can combine Propositions A.2 and A.5 to obtain a clade construction of an $\mathrm{SSIP}^{(\alpha)}(\theta)$-evolution starting from β^0, based on an independent pair $(\overleftarrow{\mathbf{F}}, \mathbf{N}_{\beta^0}) \sim \mathbf{P}_\emptyset^{\alpha,\theta} \otimes \mathbf{P}_{\beta^0}^{\alpha,0}$.

Finally, let us note the Markov-like property in this context. To this end, we denote by

$$N^{\geq y}(\overleftarrow{\mathbf{F}}) = \underset{\text{points } (s,N_s) \text{ of } \overleftarrow{\mathbf{F}}: s \in [-y, 0]}{\bigstar} N_s^{\geq y-s}$$

the point measure of spindles that takes into account all immigration at levels $s \leq y$ and builds a point measure from the associated (part-)spindles above level $y = s + (y - s)$. We collect the immigration at levels $s > y$ in a point measure $\overleftarrow{\mathbf{F}}^{\geq y} := \overleftarrow{\mathbf{F}}|_{(y,\infty)}^\leftarrow$ on $[0,\infty) \times \mathcal{N}$. We include the remaining spindles below level y, captured in $N_s^{\leq y-s}$, from immigration at levels $s \leq y$, in the history below level y.

Proposition A.6 (Lemma 3.10 of Forman et al. (2020c)). *Let* $(\overleftarrow{\mathbf{F}}, \mathbf{N}_{\beta^0}) \sim \mathbf{P}_\emptyset^{\alpha,\theta} \otimes \mathbf{P}_{\beta^0}^{\alpha,0}$ *and* $y > 0$. *Denote by* $(\beta^y, y \geq 0)$ *the associated* $\mathrm{SSIP}^{(\alpha)}(\theta)$*-evolution. Then conditionally given the history below level* y, *we have* $(\overleftarrow{\mathbf{F}}^{\geq y}, N^{\geq y}(\overleftarrow{\mathbf{F}}) \star N^{\geq y}(\mathbf{N}_{\beta^0})) \sim \mathbf{P}_\emptyset^{\alpha,\theta} \otimes \mathbf{P}_{\beta^y}^{\alpha,0}$.

For more details, we refer to Forman et al. (2020c, Section 3).

A.3. *Excursions of interval partition evolutions.* Let $\Lambda^{(\alpha)}$ be the image of the measure $\nu_{\perp\mathrm{cld}}^{(\alpha)}$ via the mapping $N \mapsto \overline{\mathrm{SKEWER}}(N)$. Then $\Lambda^{(\alpha)}$ is the desired excursion measure in Section 4.1.

Proof of Proposition 4.1: Part 1 is Forman et al. (2020c, Proposition 2.12(i)). For part 2, the entrance law is given by Forman et al. (2020c, Lemma 3.5) and the Markov property is implied by Forman et al. (2020c, Corollary 3.9).

The self-similarity follows from the scaling property of $\nu_{\perp\mathrm{cld}}^{(\alpha)}$ given by Forman et al. (2020c, Lemma 2.11). Specifically, for any $N \in \mathcal{N}$, define $c \odot_{\mathrm{cld}}^{(\alpha)} N \in \mathcal{N}$ by replacing each atom (t, f) of N

with $(c^{1+\alpha}t, cf(\cdot/c))$. Then Forman et al. (2020c, Lemma 2.11) states that the image of $\nu^{(\alpha)}_{\perp\text{cld}}$ via the mapping $N \mapsto c \odot^{(\alpha)}_{\text{cld}} N$ is $c^{-1}\nu^{(\alpha)}_{\perp\text{cld}}$. Combining with the identity

$$\text{SKEWER}\big(y, c \odot^{(\alpha)}_{\text{cld}} N, \xi_{c\odot^{(\alpha)}_{\text{cld}}N}\big) = c \cdot \text{SKEWER}\big(y/c, N, \xi_N\big), \quad y > 0,$$

we have, for any measurable set A in $\mathcal{E}_{\mathcal{I}}$,

$$\begin{aligned}
\Lambda^{(\alpha)}(\Phi_c A) &= \nu^{(\alpha)}_{\perp\text{cld}}\big(\overline{\text{SKEWER}}^{-1}(\Phi_c A)\big) \\
&= \nu^{(\alpha)}_{\perp\text{cld}}\Big(c \odot^{(\alpha)}_{\text{cld}} \overline{\text{SKEWER}}^{-1}(A)\Big) = c^{-1}\nu^{(\alpha)}_{\perp\text{cld}}\big(\overline{\text{SKEWER}}^{-1}(A)\big) = c^{-1}\Lambda^{(\alpha)}(A). \qquad \square
\end{aligned}$$

Proof of Proposition 4.2: The first statement follows from Proposition A.5. The second one is a consequence of Proposition A.6. $\qquad\square$

A.4. *Proofs of Lemmas 3.4 and 3.8.*

Proof of Lemma 3.4: Let $((\beta^y_1, m^y, \beta^y_2), y \ge 0)$ be a $\mathcal{J}$-valued $\text{SSIP}^{(\alpha)}_{\dagger}(\theta_1, \theta_2)$-evolution as defined in Definition 3.1. Since total mass evolves continuously between and across any finite number of renaissance times, the total mass reaches zero continuously on any event $\{T_n = T_\infty < \infty\}$, $n \ge 0$. Hence, it suffices to show that, on the event $\{T_n \uparrow T_\infty < \infty\}$, the total mass tends to zero along the sequence $(T_n, n \ge 0)$.

Recall from (A.3) the concatenation of clades $\mathbf{N}_\gamma = \bigstar_{U\in\gamma} \mathbf{N}_U$ and from Definition A.1 notation $\overline{\text{SKEWER}}(\mathbf{N}_\gamma)$. We use the notation of Definition 3.1, consider $\mathbf{f}^{(0)} \sim \text{BESQ}_{m^0}(-2\alpha)$ and independent clade constructions

$$\gamma^{(0)}_1 = \overleftarrow{\boldsymbol{\beta}}^{(0)}_1 \star \overline{\text{SKEWER}}\Bigg(\bigstar_{U\in\beta^0_1} \mathbf{N}^{(0)}_U\Bigg) \quad \text{and} \quad \gamma^{(0)}_2 = \text{rev}\Bigg(\overleftarrow{\boldsymbol{\beta}}^{(0)}_2 \star \overline{\text{SKEWER}}\Bigg(\bigstar_{U\in\text{rev}(\beta^0_2)} \mathbf{N}^{(0)}_U\Bigg)\Bigg),$$

in the sense of (A.3) and where $\overleftarrow{\boldsymbol{\beta}}^{(0)}_i$ is built from point measures $\overleftarrow{\mathbf{F}}^{(0)}_i$ of clades as in (A.6), with intensities θ_i, $i = 1, 2$. Our strategy is to use these clades, as well as an auxiliary independent clade $\delta(0, \mathbf{f}^{(0)}) + \mathbf{N}^{(0)}_{\text{mid}} := \text{CLADE}(\mathbf{f}^{(0)}, \mathbf{N})$ associated with $\mathbf{f}^{(0)}$, to enhance Definition 3.1 and construct from these clades the entire process $((\beta^y_1, m^y, \beta^y_2), y \ge 0)$, as well as a process $(\beta^y_{\text{em}}, y \ge 0)$ that is an $\text{SSIP}^{(\alpha)}(\alpha)$-evolution during $[0, T_\infty)$ and, on $\{T_\infty < \infty\}$, proceeds continuously across T_∞, as an $\text{SSIP}^{(\alpha)}(0)$-evolution. Indeed, while the blocks of $(\beta^y_1, m^y, \beta^y_2)$ can then be thought of as a subset of the blocks of $\gamma^{(0)}_i(y)$, $i = 1, 2$, and $\mathbf{f}^{(0)}(y)$, we will make sure that β^y_{em} will contain precisely the remaining blocks ("emigration"), and the corresponding relationship of the associated total mass processes will yield the claimed asymptotics.

Specifically, suppose by induction that we have constructed the processes for the time interval $[0, T_n]$ for some $n \ge 0$ and are given families $(\mathbf{N}^{(n)}_U, U \in \beta^{T_n}_i)$, $i = 1, 2$, and point measures $\overleftarrow{\mathbf{F}}^{(n)}_i$, $i = 1, 2$, of clades, as well as another clade $\delta(0, \mathbf{f}^{(n)}) + \mathbf{N}^{(n)}_{\text{mid}}$. Furthermore, suppose that, conditionally given the history up to level T_n, in the sense of Forman et al. (2020c, (3.8) and (3.10)) and as recalled less formally in Section A.2, these clades and point measures are independent and so that

$$\gamma^{(n)}_1 = \overleftarrow{\boldsymbol{\beta}}^{(n)}_1 \star \overline{\text{SKEWER}}\Bigg(\bigstar_{U\in\beta^{T_n}_1} \mathbf{N}^{(n)}_U\Bigg) \quad \text{and} \quad \gamma^{(n)}_2 = \text{rev}\Bigg(\overleftarrow{\boldsymbol{\beta}}^{(n)}_2 \star \overline{\text{SKEWER}}\Bigg(\bigstar_{U\in\text{rev}(\beta^{T_n}_2)} \mathbf{N}^{(n)}_U\Bigg)\Bigg),$$

and $\mathbf{f}^{(n)}$ have joint conditional distributions given $((\beta^y_1, m^y, \beta^y_2), 0 \le y \le T_n)$ as in Definition 3.1. Then $T_{n+1} := T_n + \zeta(\mathbf{f}^{(n)})$,

$$(\beta^y_1, m^y, \beta^y_2) := \Big(\gamma^{(n)}_1(y - T_n), \mathbf{f}^{(n)}(y - T_n), \gamma^{(n)}_2(y - T_n)\Big), \qquad T_n \le y < T_{n+1},$$

and $(\beta_1^{T_{n+1}}, m^{T_{n+1}}, \beta_2^{T_{n+1}}) := \phi(\beta_1^{T_{n+1}-} \star \beta_2^{T_{n+1}-})$ extends the construction of the $\mathcal{J}$-valued process to $[0, T_{n+1}]$ as in Definition 3.1. To proceed with the induction, we note that $\zeta(\mathbf{f}^{(n)})$ is independent of the other clades, conditionally given the history up to level T_n, so we can apply the Markov-like properties at level $\zeta(\mathbf{f}^{(n)})$, which we recalled from Forman et al. (2020a) and Forman et al. (2020c) in Propositions A.3 and A.6. Specifically, we obtain point measures of spindles that, via (A.3), can be decomposed into clades and then grouped as $(\mathbf{N}_U^{(n+1)}, U \in \beta_i^{T_{n+1}})$, $i = 1, 2$, and we also obtain point measures $\overleftarrow{\mathbf{F}}_i^{(n+1)}$, $i = 1, 2$, of clades, as well as another clade $\delta(0, \mathbf{f}^{(n+1)}) + \mathbf{N}_{\mathrm{mid}}^{(n+1)}$ associated with the longest interval of length $m^{T_{n+1}}$, all conditionally independent given the history up to level T_{n+1}. Inductively, this completes the construction of Definition 3.1 on $[0, T_\infty)$.

Now set $\beta_{\mathrm{em}}^0 := 0$ and suppose further that we enter the induction step also with a clade $\delta(0, \mathbf{f}^{(n)}) + \mathbf{N}_{\mathrm{mid}}^{(n)}$ and an independent process

$$\beta_{\mathrm{em}}^y = \sum_{j=0}^{n-1} \mathrm{SKEWER}\left(y - T_j, \mathbf{N}_{\mathrm{mid}}^{(j)}, \zeta(\mathbf{f}^{(j)}) + \xi_{\mathbf{N}_{\mathrm{mid}}^{(j)}}\right), \qquad y \geq 0, \tag{A.7}$$

that is an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution on $[0, T_n]$ continued as an $\mathrm{SSIP}^{(\alpha)}(0)$-evolution on $[T_n, \infty)$. By Lemma 2.8, the process $\mathrm{SKEWER}\left(y, \mathbf{N}_{\mathrm{mid}}^{(n)}, \zeta(\mathbf{f}^{(n)}) + \xi_{\mathbf{N}_{\mathrm{mid}}^{(n)}}\right)$, $y \geq 0$, evolves as an $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolution on $[0, \zeta(\mathbf{f}^{(n)})]$. By the Markov-like property at level $\zeta(\mathbf{f}^{(n)})$, it continues as an $\mathrm{SSIP}^{(\alpha)}(0)$-evolution. Then the strong Markov property Forman et al. (2021, Proposition 3.14) of $\mathrm{SSIP}^{(\alpha)}(\alpha)$-evolutions yields (A.7) with n replaced by $n + 1$. Inductively, the statement holds for all $n \geq 0$, and by Poisson random measure arguments based on Lemma A.4 and Proposition A.5, this also entails the corresponding statement with $n = \infty$, and in particular, the left limit at T_∞ extends this continuously to an $\mathrm{SSIP}^{(\alpha)}(\alpha)$ on $[0, T_\infty]$.

By construction, the blocks of $(\beta_1^y, m^y, \beta_2^y)$ are all taken from the skewer at level y of clades that were used in the construction of $\boldsymbol{\gamma}_1^{(0)}$ and $\boldsymbol{\gamma}_2^{(0)}$, and from $\mathbf{f}^{(0)}$. Specifically, this holds explicitly for $0 \leq y < T_1$. For $T_n \leq y < T_{n+1}$, $n \geq 1$, we take skewers at level $y - T_n$ of clades above level T_n, which were obtained from the original clades by repeatedly applying Markov-like properties at levels $\zeta(\mathbf{f}^{(j)})$, $0 \leq j \leq n - 1$, and these levels add up to T_n. We remark that only the order, not the size of blocks, is affected by the reversals in (A.4).

This construction captures at each step all clades above the next level for use either in $(\beta_1^y, m^y, \beta_2^y)$, $0 \leq y < T_\infty$ or, via $\mathbf{N}_{\mathrm{mid}}^{(n)}$, $n \geq 0$, for use in β_{em}^y, $0 \leq y < T_\infty$. In particular, for all $0 \leq y < T_\infty$,

$$\|\beta_1^y\| + m^y + \|\beta_2^y\| = \left\|\boldsymbol{\gamma}_1^{(0)}(y)\right\| + \left\|\boldsymbol{\gamma}_2^{(0)}(y)\right\| + \left\|\mathrm{SKEWER}\left(y, \delta(0, \mathbf{f}^{(0)}) + \mathbf{N}_{\mathrm{mid}}^{(0)}\right)\right\| - \|\beta_{\mathrm{em}}^y\|. \tag{A.8}$$

On the other hand, the size of the longest interval of $((\beta_1^y \star \{(0, m^y)\} \star \beta_2^y), 0 \leq y < T_\infty)$ tends to zero along times $(T_n, n \geq 0)$, when on the event $\{T_n \uparrow T_\infty < \infty\}$: indeed, any subsequence of longest intervals of lengths exceeding $\varepsilon > 0$ would contribute lifetimes that are stochastically bounded below by the lifetimes of an independent sequence of $\mathrm{BESQ}_\varepsilon(-2\alpha)$, and such lifetimes would have an infinite sum almost surely.

In the clade construction of $\boldsymbol{\gamma}_1^{(0)}$ and $\boldsymbol{\gamma}_2^{(0)}$, the mass evolution of each block is represented by a spindle in a clade. In our construction, each spindle that starts strictly below level T_∞ is either used for $((\beta_1^y, m^y, \beta_2^y), 0 \leq y < T_\infty)$ or for $(\beta_{\mathrm{em}}^y, y \geq 0)$. But on $\{T_\infty < \infty\}$, each spindle that straddles level T_∞ must have a positive mass at level T_∞, exceeding some $\varepsilon > 0$ on an interval around T_∞. Hence it cannot be included in $((\beta_1^y, m^y, \beta_2^y), 0 \leq y < T_\infty)$. But then the RHS of (A.8) tends to 0 as $y \uparrow T_\infty$, and this completes the proof. $\square$

Proof of Lemma 3.8: Let $\mathcal{J}_2 := \mathcal{I}_H \times (0, \infty) \times \mathcal{I}_H \times (0, \infty) \times \mathcal{I}_H$ and $(\beta_0, m_0, \beta_1, m_1, \beta_2) \in \mathcal{J}_2$. We want to couple two $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$ starting respectively from $(\beta_0, m_0, \beta_1 \star \{(0, m_1)\} \star \beta_2)$ and $(\beta_0 \star \{(0, m_0)\} \star \beta_1, m_1, \beta_2)$ such that the associated $\mathcal{I}_H$-valued processes coincide.

Viewing $\mathcal{J}$-valued processes as $\mathcal{I}_H$-valued processes split around a marked block, we now construct a $\mathcal{J}_2$-valued process that captures two marked blocks. Specifically, we construct

$$((\beta_0^y, m_0^y, \beta_1^y, m_1^y, \beta_2^y),\, 0 \le y < S_N)$$

starting from $(\beta_0, m_0, \beta_1, m_1, \beta_2)$ at time $S_0 := 0$ by the following inductive steps, mimicking Definition 3.1. Suppose that we have constructed the process on $[0, S_n]$ for some $n \ge 0$ and some $(\beta_0^{S_n}, m_0^{S_n}, \beta_1^{S_n}, m_1^{S_n}, \beta_2^{S_n}) \in \mathcal{J}_2$.

- Conditionally on the history, consider, independently,
  – an $\mathrm{SSIP}^{(\alpha)}(\theta_1)$-evolution $\boldsymbol{\gamma}_0^{(n)}$ starting from $\beta_0^{S_n}$,
  – an $\mathrm{SSIP}^{(\alpha)}(\alpha) = \mathrm{RSSIP}^{(\alpha)}(\alpha)$-evolution $\boldsymbol{\gamma}_1^{(n)}$ starting from $\beta_1^{S_n}$,
  – an $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolution $\boldsymbol{\gamma}_2^{(n)}$ starting from $\beta_2^{S_n}$,
  – and $\mathbf{f}_i^{(n)} \sim \mathtt{BESQ}_{m_i^{S_n}}(-2\alpha)$, $i = 0, 1$.

  Let $\Delta_n := \min\{\zeta(\mathbf{f}_0^{(n)}), \zeta(\mathbf{f}_1^{(n)})\}$ and $S_{n+1} := S_n + \Delta_n$. Define, for $0 \le y < \Delta(S_n)$,

  $$\left(\beta_0^{S_n+y}, m_0^{S_n+y}, \beta_1^{S_n+y}, m_1^{S_n+y}, \beta_2^{S_n+y}\right) := \left(\gamma_0^{(n)}(y), \mathbf{f}_0^{(n)}(y), \gamma_1^{(n)}(y), \mathbf{f}_1^{(n)}(y), \gamma_2^{(n)}(y)\right).$$

- If $\Delta_n = \zeta(\mathbf{f}_i^{(n)})$ for some $i = 0, 1$, and $\mathbf{f}_{1-i}^{(n)}(\Delta_n)$ exceeds the length of the longest interval in $\gamma_j^{(n)}(\Delta_n)$ for all $j = 0, 1, 2$, let $N = n + 1$. The construction is complete.

- Otherwise, identify the longest interval and split the associated $\gamma_j^{(n)}(\Delta_n)$ around this interval. This results in a total of four interval partitions and two blocks. In the natural order, two of these interval partitions are adjacent. Concatenate these two and collect the now five components as $\left(\beta_0^{S_{n+1}}, m_0^{S_{n+1}}, \beta_1^{S_{n+1}}, m_1^{S_{n+1}}, \beta_2^{S_{n+1}}\right)$.

Note that (in general) we may have $N \in \mathbb{N} \cup \{\infty\}$. On the event $\{N < \infty\}$, we further continue the evolution as a $\mathcal{J}$-valued process starting from the terminal value of the $\mathcal{J}_2$-valued process, with adjacent interval partitions concatenated.

By concatenation properties of $\mathrm{SSIP}^{(\alpha)}(\theta_1)$- and $\mathrm{RSSIP}^{(\alpha)}(\theta_2)$-evolutions (Proposition 2.9 and Lemma 2.8) and by the strong Markov property of these processes applied at the stopping times S_n, $n \ge 1$, we obtain two coupled $\mathrm{SSIP}_\dagger^{(\alpha)}(\theta_1, \theta_2)$-evolutions, which induce the same $\mathcal{I}_H$-valued process, as required. Indeed, the construction of these two processes is clearly complete on $\{N < \infty\}$ and on $\{N = \infty, S_\infty = \infty\}$ with $S_\infty = \lim_{n \to \infty} S_n$. This suffices if the event $\{N = \infty, S_\infty < \infty\}$ has zero probability. Otherwise, on $\{N = \infty, S_\infty < \infty\}$ the construction of at least one process is complete and by Lemma 3.4, the total mass tends to zero along a subsequence of $(S_n, n \ge 0)$, and hence the other construction cannot remain unfinished with blocks of positive size at S_∞. □

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