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Asymptotic formula for the conjunction probability of smooth stationary Gaussian fields

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Abstract. Let $\{X_i(t) : t \in S \subset \mathbb{R}^d\}_{i=1,2,\dots,n}$ be independent copies of a stationary centered Gaussian field with almost surely smooth sample paths. In this paper, we are interested in the conjunction probability defined as $\mathrm{P}\left(\exists t \in S : X_i(t) \geq u, \forall i = 1, 2, \dots, n\right)$ for a given threshold level u. As $u \to \infty$, we will provide an asymptotic formula for the conjunction probability. This asymptotic formula is derived from the behaviour of the volume of the set of local maximum points. The proof relies on a result of Azaïs and Wschebor (2014) describing the shape of the excursion set of a stationary centered Gaussian field. Our result partially confirms the validity of the Euler characteristic method.

1. Introduction

Let X be a real-valued stationary centered Gaussian field with unit variance and almost surely smooth sample paths. Assume more that it is defined on a compact set $S \subset \mathbb{R}^d$. Consider n independent copies $\{X_i(t); i = 1, 2, \dots, n\}$ of X. In this paper, we are interested in the behaviour of *conjunction probability*,

$$\mathrm{P}\left(\exists t \in S : X_i(t) \geq u, \forall i \in \overline{1,n}\right), \tag{1.1}$$

where u is a threshold level. The probability above can also be expressed in different manners: equivalently, it can be rewritten as the probability that the conjunction set (excursion set)

$$C_u = \{t \in S : X_i(t) \geq u, \forall i \in \overline{1,n}\}$$

is non-empty or that the maximum of the smallest value among the fields exceeds u

$$\mathrm{P}\left(\sup_{t\in S} \min_{1\leq i\leq n} X_i(t) \geq u\right). \tag{1.2}$$

When $n = 1$, the expression (1.2) is simply the tail distribution of the maximum of a stationary Gaussian field. Even in this simple case, finding the exact value of the tail distribution is very challenging, see Azaïs and Wschebor (2009). Hence, an interesting approach is to provide the

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asymptotic estimates for the tail distribution when $u \to +\infty$. This problem has been studied extensively in the literature. One could mention three main techniques to deal with it: Double-sum method (see (Pickands, 1969; Piterbarg, 1996)), Euler characteristic method (see (Adler and Taylor, 2007; Taylor, 2006; Taylor et al., 2005)) and Rice method (see (Azaïs and Delmas, 2002; Azaïs and Wschebor, 2008, 2009)). See also (Sun, 1993; Takemura and Kuriki, 2002).

The first method was introduced by Pickands (1969) for stationary Gaussian "α processes" and later was extended to non-stationary processes and to non-Gaussian ones by Piterbarg (1996). Here the authors provide the estimates containing some non-explicit constants: *the Pickands constant* that depends on the local self-similarity exponent α of the process.

The second method was provided by Adler and Taylor (2007) and concerns differentiable processes. It is an important tool for studying the geometry of random surfaces. Given S a compact convex (for instance) subset of $\mathbb{R}^d$ and a positive constant ϵ, the ϵ- neighborhood of S, denoted by $S^{+\epsilon}$, is defined as

$$S^{+\epsilon} = \{t \in \mathbb{R}^d : \ \mathrm{dist}(t, S) \leq \epsilon\}.$$

Then Steiner formula, see Gray (2004), states that the volume of $S^{+\epsilon}$ can be expressed as a polynomial of the variable ϵ,

$$\lambda_d \left(S^{+\epsilon}\right) = \sum_{j=0}^{S} \omega_{d-j}\mu_j(S)\epsilon^{d-j}, \tag{1.3}$$

where ω_{d-j} is the volume of a $(d-j)$-dimensional unit ball, and $\mu_j(S)$'s are the geometric *Minkowski functionals* (or the *Killing-Lipschitz curvatures*) of S. In particular, $\mu_d(S)$ is equal to the volume of S; and $\mu_0(S)$ is the *Euler characteristic* of S, for example, it is equal to the number of connected components minus the number of holes inside when $d = 2$ (see Adler and Taylor (2007)). In general, one can define the Killing-Lipschitz curvatures of a manifold with positive reach through Weyl tube formula (see (4.8)).

The main idea of the Euler characteristic method is that the excursion probability could be approximated by the expectation of the Euler characteristic of the random excursion set C_u. The heuristic argument is as follows: when the level u is large, if the excursion set is non-empty, then it is usually a simply-connected domain corresponding to a unique local maximum point. Under the condition that the random field X is an isotropic centered Gaussian field with unit variance and unit covariance matrix of the derivatives, Adler and Taylor (2007) proposed the *Gaussian kinematic formula* to calculate the expectation of the Euler characteristic of the excursion set as

$$\mathrm{E}(\mu_0(C_u)) = \sum_{i=0}^{d} \rho_i\mu_i(S), \tag{1.4}$$

where ρ_i's are the Euler characteristic densities defined as

$$\rho_0 = \overline{\Phi}(u) = \int_u^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx,$$

$$\rho_i = (2\pi)^{-(i+1)/2}H_{i-1}(u)e^{-u^2/2} = (2\pi)^{-i/2}H_{i-1}(u)\varphi(u), \forall i > 0,$$

with $\varphi(u) = e^{-u^2/2}/\sqrt{2\pi}$, and $H_j(x) = (-1)^n e^{\frac{x^2}{2}}\dfrac{d^j}{dx^j}e^{\frac{-x^2}{2}}$ is the Hermite polynomial of degree j; and $\mu_j(S)$'s are the geometric functionals of S, defined as above.

The third method, the Rice method, is based on local maxima and leads to the same approximation as in RHS of (1.4). It gives also an upper bound. The first proof of validity is due to Piterbarg (1981). The expectation given in (1.4) is proved to be a very accurate approximation when the domain S is "nice" in the sense that it is a tamed and locally convex subset of $\mathbb{R}^d$ (see Adler and Taylor (2007, Theorem 14.3.3)). Note that in the case both methods apply, the Euler characteristic method gives extra terms with respect to the Double-sum method, and thus it is more accurate, see Azaïs and Mourareau (2019).

In this paper, we are interested in the case $n \geq 2$. The motivation of this problem comes from the statistical applications in neurology, for example, to determine whether the functional organization of the brain for language differs according to sex (see Worsley and Friston (2000)). In this application, $X_i(t)$ is the value of image i at the location $t \in \mathbb{R}^d$ representing the intensity with respect to some actions. Here both the Double-sum method and Euler characteristic method are still useful.

By the Double-sum method, (Dębicki et al., 2014, 2015) considered the one-dimensional processes and proved that

$$\mathrm{P}\left(\sup_{t\in[0,T]}\min_{1\leq i\leq n} X_i(t) > u\right) = H_{n,2}Tu\overline{\Phi}^n(u)(1+o(1)),$$

where $H_{n,2}$ is so-called the *generalized Pickands constant* defined as

$$H_{n,2} = \lim_{a\downarrow 0}\frac{1}{a}\mathrm{P}\left(\max_{k\geq 1} Z(ak) \leq 0\right),$$

with

$$Z(t) = \min_{1\leq i\leq n}\left(\sqrt{2}Y_i(t) - t^2 + E_i\right),$$

here Y_i's are independent copies of a centered Gaussian process $Y(t)$ with covariance function $\mathrm{Cov}(Y(t), Y(s)) = |ts|$, $\forall t, s \geq 0$, and E_i's are mutually independent unit mean exponential random variables being further independent of Y_i's. The expansion above must be understood, as in the rest of the paper, as $u \to +\infty$.

Debicki et al also considered non-stationary processes and mentioned that their result could be extended to Gaussian fields but at the cost of heavy notations. Note that their work deals with a wider class of random processes than those considered here that are smooth stationary ones.

By the Euler characteristic method, Worsley and Friston (2000) considered the upper-triangular Toeplitz matrix R defined as

$$R = \begin{pmatrix} \rho_0/b_0 & \rho_1/b_1 & \dots & \rho_d/b_d \\ 0 & \rho_0/b_0 & \dots & \rho_{d-1}/b_{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_0/b_0 \end{pmatrix}, \tag{1.5}$$

where $b_i = \Gamma((i+1)/2)/\Gamma(1/2)$ with $\Gamma(.)$ the Euler gamma function, and the ρ_i's are the Euler characteristic densities as defined above. They gave the heuristic argument that

$$\mathrm{P}(C_u \neq \emptyset) = \mathrm{P}\left(\sup_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) \approx \mathrm{E}(\mu_0(C_u)) = (1, 0, \dots, 0)R^n\mu(S), \tag{1.6}$$

where $\mu(S) = (\mu_0(S)b_0, \mu_1(S)b_1, \dots, \mu_d(S)b_d)$ is the column vector of the scaled Minkowski functionals of S. However, to prove that $\mathrm{E}(\mu_0(C_u))$ is a good approximation is still an open question. For further discussion, see also (Alodat, 2011; Alodat et al., 2010; Worsley, 1994).

Let us consider the particular case of random processes (i.e. $d = 1$). Then the matrix R defined in (1.5) becomes

$$R = \begin{pmatrix} \rho_0 & \rho_1/b_1 \\ 0 & \rho_0 \end{pmatrix}, \quad R^n = \begin{pmatrix} \rho_0^n & n\rho_0^{n-1}\rho_1/b_1 \\ 0 & \rho_0^n \end{pmatrix}.$$

Also note that in this case the domain S is the interval $[0, T]$ with $\mu_0([0, T]) = 1$ and $\mu_1([0, T]) = T$. Therefore, *the validity of the Euler characteristic*, is equivalent to

$$\mathrm{P}\left(\sup_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) \approx \rho_0^n + n\rho_0^{n-1}\rho_1 T = \overline{\Phi}^n(u) + n\overline{\Phi}^{n-1}(u)\varphi(u).T/\sqrt{2\pi}.$$

That would give an extra term in comparing with Double-sum method. And since

$$\overline{\Phi}(u) = \varphi(u)\left(\frac{1}{u} + o\left(\frac{1}{u}\right)\right),$$

it could imply that

$$H_{n,2} = \frac{n}{\sqrt{2\pi}}. \tag{1.7}$$

In a recent paper Pham (2020), the equality in (1.7) has been proved to be true. The proof exploits the one-dimensional structure of the processes, and uses Rice formula to calculate the expected number of *"up-crossing"* of the level u while the other processes are all greater than u. However, this idea seems hard to extend to higher dimensions.

In this paper, we consider the conjunction problem from another point of view. Our approach relies on a result of Azaïs and Wschebor (2014) describing the geometry of the excursion set. There they established a relation between the tail distribution of the maximum and the volume and the perimeter of the index set. In Azaïs and Pham (2016), this idea has been used to provide the asymptotic formula of the tail of the maximum corresponding to the coefficients of the volume of the ϵ-neighborhood of a non-locally convex index set. With the same spirit, we will give a one-term asymptotic formula for the conjunction probability where the coefficient comes from the local geometry (or local volume) of the conjunction set (see Proposition 2.2).

Before stating the main result of this paper, we present here the technical assumptions on the considered fields.

Assumption A**:** Assume X a random field defined on a ball $B \subset \mathbb{R}^d$ containing the domain S such that X satisfies:

i. S is a compact subset of $\mathbb{R}^d$ satisfying that it is the closure of its interior, and its boundary is the union of a finite number of $\mathcal{C}^2$ and $d-1$ dimensional compact domains.
ii. X is a stationary centered Gaussian field with unit variance and $\mathrm{Var}(X'(t))$ is the identity matrix.
iii. Almost surely the paths of $X(t)$ are of class $\mathcal{C}^3$.
iv. For all $s \neq t \in B$, the distribution of $(X(s), X(t), X'(s), X'(t))$ does not degenerate.
v. For all $t \in B$ and γ in the unit sphere $\mathcal{S}^{d-1}$, the distribution of $(X(t), X'(t), X''(t)\gamma)$ does not degenerate.

Our main result is the following.

Theorem 1.1. *Let $X_i(t), 1 \leq i \leq n$, be independent copies of a Gaussian field X satisfying Assumption (A). Then as u tends to infinity,*

$$\mathrm{P}\left(\max_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) = u^{d-n}\varphi^n(u)\left[\frac{\lambda_d(S)}{(2\pi)^{d/2}}\sum_{k_n=0}^{d}\sum_{k_{n-1}=d-k_n}^{d}\cdots\sum_{k_2=(n-2)d-(k_n+k_{n-1}+\ldots+k_3)}^{d}\right.$$
$$\left.\frac{\omega_d}{\omega_{\sum_{i=2}^n k_i-(n-2)d}\prod_{i=2}^n\omega_{d-k_i}} \times \frac{d!}{[\sum_{i=2}^n k_i-(n-2)d]!.\prod_{i=2}^n(d-k_i)!} + o(1)\right], \tag{1.8}$$

where ω_k stands for the volume of a k-dimensional unit ball.

The proof of the main theorem consists of two propositions presented in Section 2. We prove these propositions and consider some special examples in Sections 3 and 4. In Section 5, we compare our result with the corresponding term in the prediction given by the Euler characteristic method. Although the two formulas seem to be different, their values coincide. Therefore, in some sense, our result partially confirms the validity of the Euler characteristic method.

Throughout this paper, we will use the following notation.

- $\lambda_k(.)$ stands for the usual k-dimensional Lebesgue measure.

- $B(t, r)$ stands for the ball of radius r at center t.
- For an n-dimensional vector $\mathbf{r} = (r_1, \dots, r_n)$ and an n-tuple of non-negative integers $\mathbf{m} = (m_1, \dots, m_n)$, the notations $\mathbf{r}^{\mathbf{m}}$ stands for

$$\mathbf{r}^{\mathbf{m}} = r_1^{m_1} r_2^{m_2} \dots r_n^{m_n},$$

  and $\|\mathbf{m}\|$ is the l_1-norm of the vector.
- For a given set $S \subset \mathbb{R}^d$ and a positive constant ϵ, the ϵ- neighborhood of S, denoted by $S^{+\epsilon}$, is defined as

$$S^{+\epsilon} = \{t \in \mathbb{R}^d : \ \mathrm{dist}(t, S) \le \epsilon\}.$$

- For a given set $S \subset \mathbb{R}^d$ and a small enough positive constant ϵ, the set $S^{-\epsilon}$ is defined as

$$S^{-\epsilon} = \{t \in \mathbb{R}^d : \ B(t, \epsilon) \subset S\}.$$

- ω_k is the volume of a k-dimensional unit ball.
- $\varphi(.)$ and $\overline{\Phi}(.)$ are the density and tail distribution functions of a standard normal random variable.

2. Proof of the main theorem

The main result can be easily deduced from following two propositions.

Proposition 2.1. *Let $X_i(t), 1 \le i \le n$, be n independent copies of a Gaussian field X satisfying Assumption (A). Assume that for a fixed point t_1 and small enough $r_1, r_2, \dots, r_n$, there exist constants $k > 0$ and $C_{\boldsymbol{m}}$ such that*

$$\lambda_{(n-1)d}\left((t_2, \dots, t_n) : \bigcap_{1 \le i \le n} B(t_i, r_i) \neq \emptyset\right) = \sum_{\|\boldsymbol{m}\| = k} C_{\boldsymbol{m}} \boldsymbol{r}^{\boldsymbol{m}}. \tag{2.1}$$

Then, as u tends to infinity,

$$\mathrm{P}\left(\max_{t \in S} \min_{1 \le i \le n} X_i(t) \ge u\right) = u^{nd-n-k} \varphi^n(u) \left(\frac{2^{k/2} \lambda_d(S)}{(2\pi)^{nd/2}} \sum_{\|\boldsymbol{m}\| = k} C_{\boldsymbol{m}} \prod_{i=1}^{n} \Gamma(1 + m_i/2) + o(1)\right).$$

Proposition 2.2. *For a fixed point t_1 in the parameter space and for $r_1, r_2, \dots, r_n > 0$ small enough, we have:*

$$\begin{aligned}
&\lambda_{(n-1)d}\left((t_2, \dots, t_n) \in \mathbb{R}^{d(n-1)} : \bigcap_{1 \le i \le n} B(t_i, r_i) \neq \emptyset\right) \\
&= \sum_{k_n=0}^{d} \sum_{k_{n-1}=d-k_n}^{d} \dots \sum_{k_2=(n-2)d-(k_n+k_{n-1}+\dots+k_3)}^{d} r_1^{(n-1)d - \sum_{i=2}^n k_i} \times \\
&\prod_{i=2}^{n} \left(r_i^{k_i} \omega_{k_i}\right) \frac{\omega_d \omega_{(n-1)d - \sum_{i=2}^n k_i}}{\omega_{\sum_{i=2}^n k_i - (n-2)d} \prod_{i=2}^n \omega_{d-k_i}} \times \frac{d!}{[\sum_{i=2}^n k_i - (n-2)d]!. \prod_{i=2}^n (d-k_i)!}.
\end{aligned} \tag{2.2}$$

Indeed, the explicit values of $k = (n-1)d$ and $C_{\mathbf{m}}$'s are provided in Proposition 2.2. Then substituting them in Proposition 2.1, we get the asymptotic formula for the conjunction probability as

$$\begin{aligned}
&\mathrm{P}\left(\max_{t\in S}\min_{1\le i\le n} X_i(t)\ge u\right)\\
=&u^{d-n}\varphi^n(u)\left[\frac{2^{(n-1)d/2}\lambda_d(S)}{(2\pi)^{nd/2}}\sum_{k_n=0}^{d}\sum_{k_{n-1}=d-k_n}^{d}\dots\sum_{k_2=(n-2)d-(k_n+k_{n-1}+\ldots+k_3)}^{d}\right.\\
&\Gamma\left(\frac{(n-1)d-\sum_{i=2}^n k_i}{2}+1\right)\omega_{(n-1)d-\sum_{i=2}^n k_i}\prod_{i=2}^n\Gamma\left(\frac{k_i}{2}+1\right)\omega_{k_i}\times\\
&\left.\frac{\omega_d}{\omega_{\sum_{i=2}^n k_i-(n-2)d}\prod_{i=2}^n\omega_{d-k_i}}\times\frac{d!}{[\sum_{i=2}^n k_i-(n-2)d]!.\prod_{i=2}^n(d-k_i)!}+o(1)\right].
\end{aligned}$$

Using the fact that $\omega_k=\dfrac{\pi^{k/2}}{\Gamma(1+k/2)}$, the proof is completed.

3. Proof of Proposition 2.1

In the proof of Proposition 2.1, we need the following two lemmas. The first lemma describes the role of (2.1).

Lemma 3.1. *Assume that there exist constants k and $C_{\boldsymbol{m}}$'s such that the equality in (2.1) holds. Then for small enough $r_1, r_2, \dots, r_n$, we have*

$$\lambda_{nd}\left((t_1,\dots,t_n):\ t_1\in S^{+r_1},\ \bigcap_{1\le i\le n}B(t_i,r_i)\neq\emptyset\right)\le\lambda_d(S)\sum_{\|\boldsymbol{m}\|=k}C_{\boldsymbol{m}}\boldsymbol{r}^{\boldsymbol{m}}+O\Big(\sum_{\|\boldsymbol{m}\|=k+1}\boldsymbol{r}^{\boldsymbol{m}}\Big),\quad(3.1)$$

and

$$\lambda_{nd}\left((t_1,\dots,t_n):\ t_1\in S^{-r_1},\forall i;\ \bigcap_{1\le i\le n}B(t_i,r_i)\neq\emptyset\right)\ge\lambda_d(S)\sum_{\|\boldsymbol{m}\|=k}C_{\boldsymbol{m}}\boldsymbol{r}^{\boldsymbol{m}}+O\Big(\sum_{\|\boldsymbol{m}\|=k+1}\boldsymbol{r}^{\boldsymbol{m}}\Big).\quad(3.2)$$

Proof: It is clear that

$$\begin{aligned}
&\lambda_{nd}\left((t_1,\dots,t_n):\ t_1\in S^{+r_1},\ \bigcap_{1\le i\le n}B(t_i,r_i)\neq\emptyset\right)\\
=&\int\mathbb{I}_{\{t_1\in S^{+r_1}\}}dt_1\int\mathbb{I}_{\{\bigcap_{1\le i\le n}B(t_i,r_i)\neq\emptyset\}}dt_2\dots dt_n=\lambda_d(S^{+r_1})\sum_{\|\mathbf{m}\|=k}C_{\mathbf{m}}\mathbf{r}^{\mathbf{m}}.
\end{aligned}$$

Then (3.1) follows from the facts that

$$\lambda_d(S^{+r_1})\le\lambda_d(S)+\lambda_d(\partial S^{+r_1}),$$

where ∂S stands for the boundary of S that is the union of a finite number of $\mathcal{C}^2$ and $d-1$ dimensional compact domains T_j's; and for each domain T_j, one has the tube formula that (see Gray (2004))

$$\lambda_d(T_j^{+r_1})=O(r_1).$$

Similarly, to prove (3.2), we have

$$\begin{aligned}
&\lambda_{nd}\left((t_1,\dots,t_n):\ t_1\in S^{-r_1},\ \bigcap_{1\le i\le n}B(t_i,r_i)\neq\emptyset\right)\\
=&\lambda_d(S^{-r_1})\sum_{\|\mathbf{m}\|=k}C_{\mathbf{m}}r^m\ge\big(\lambda_d(S)-\lambda_d(\partial S^{+r_1})\big)\sum_{\|\mathbf{m}\|=k}C_{\mathbf{m}}\mathbf{r}^{\mathbf{m}},
\end{aligned}$$

and the proof follows easily. □

The second lemma is due to Azaïs and Wschebor (2014).

Lemma 3.2. *Let X be a random field satisfying Assumption (A) and α be a real number, $0 < \alpha < 1$. Then the following event occurs with high probability, in the sense that there exist two constants $C, c > 1$ such that its probability is at least equal to $1 - Ce^{-cu^2/2}$. The event is described by :*

- *The field has only one local maximum point $t_0 \in \overset{\circ}{B}$ with value $X(t_0) \in [u, u+1]$,*
- *and the excursion set*

$$K_u := \{s \in B : \; X(s) \geq u\}$$

consists of only one connected component, and moreover,

$$B(t_0, \underline{r}) \subset K_u \subset B(t_0, \overline{r}),$$

where $\underline{r} = \sqrt{2\dfrac{X(t_0) - u}{X(t_0) + u^\alpha}}$ *and* $\overline{r} = \sqrt{2\dfrac{X(t_0) - u}{X(t_0) - u^\alpha}}$.

From Lemma 3.2, for each $i = 1, \dots, n$, with high probability, the following event H_i occurs: there exists only one local maximum of $X_i(t)$ at the location $t_i \in \overset{\circ}{B_i} = \overset{\circ}{B}$ with value in $[u, u+1]$, and the corresponding excursion set $K_{u,i} := \{s \in B_i : \; X_i(s) \geq u\}$ satisfies that

$$B(t, \underline{r_i}) \subset K_{u,i} \subset B(t_i, \overline{r_i}),$$

where $\underline{r_i} = \sqrt{2\dfrac{X_i(t_i) - u}{X_i(t_i) + u^\alpha}}$ and $\overline{r_i} = \sqrt{2\dfrac{X_i(t_i) - u}{X_i(t_i) - u^\alpha}}$.

Moreover, if for some $i \in \{1, \dots, n\}$, the complement of the above event H_i occurs, then

$$\begin{aligned}
&\mathrm{P}\left(\bar{H}_i \cap \{\sup_{t\in S}\min_{1\leq k\leq n} X_k(t) \geq u\}\right)\\
\leq &\mathrm{P}\left(\bar{H}_i\right)\mathrm{P}\left(\sup_{t\in S}\min_{1\leq k\leq n, k\neq i} X_k(t) \geq u\right)\\
\leq &(const)e^{-cu^2/2}\left(\lambda_d(S)u^{d-1}\varphi(u)\right)^{n-1} = o(u^{nd-n-k}\varphi^n(u)).
\end{aligned}$$

Here we use the following inequality (see Piterbarg (1996))

$$\mathrm{P}(\max_{t\in S} X_i(t) \geq u) \leq (const)\lambda_d(S)u^{d-1}\varphi(u).$$

Therefore, from the fact that

$$\begin{aligned}
\mathrm{P}\left(\sup_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) &= \mathrm{P}(\exists t \in S : \; t \in K_{u,i} \forall i = 1, \dots, n)\\
&= \mathrm{P}(S \cap K_{u,1} \cap \ldots \cap K_{u,n} \neq \emptyset),
\end{aligned}$$

we obtain the upper bound

$$\mathrm{P}\left(\sup_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) \leq \mathrm{P}(S \cap B(t_1, \overline{r_1}) \cap \ldots \cap B(t_n, \overline{r_n}) \neq \emptyset) + o(u^{nd-n-k}\varphi^n(u)),$$

and the lower bound

$$\mathrm{P}\left(\sup_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) \geq \mathrm{P}(S \cap B(t_1, \underline{r_1}) \cap \ldots \cap B(t_n, \underline{r_n}) \neq \emptyset) + o(u^{nd-n-k}\varphi^n(u)).$$

- At first, we deal with the upper bound. By the Markov inequality, it is at most equal to

$$
\begin{aligned}
&\mathrm{P}(\exists t=(t_1,\dots,t_n)\in B^{\otimes n}:\ \forall i=1,\dots,n,\ X_i(t)\,\text{has a local maximum at}\,t_i,\\
&\qquad X_i(t_i)\in[u,u+1], t_1\in S^{+\overline{r_1}}\ \text{and}\ \underset{1\le i\le n}{\cap} B(t_i,\overline{r_i})\neq\emptyset)+o(u^{nd-n-k}\varphi^n(u))\\
\le&\mathrm{E}(\mathrm{card}\{t=(t_1,\dots,t_n)\in B^{\otimes n}:\ \forall i=1,\dots,n,\ X_i(t)\,\text{has a local maximum at}\,t_i,\\
&\qquad X_i(t_i)\in[u,u+1], t_1\in S^{+\overline{r_1}}\ \text{and}\ \underset{1\le i\le n}{\cap} B(t_i,\overline{r_i})\neq\emptyset\})+o(u^{nd-n-k}\varphi^n(u)),
\end{aligned}
$$

where $B^{\otimes n}$ stands for the Cartesian product set $B\times\ldots\times B$.

By applying of Rice formula to the vector-valued Gaussian field $Z(t)=(X_1'(t_1),\dots,X_n'(t_n))$ with $t=(t_1,\dots,t_n)\in B^{\otimes n}$, the above expectation is equal to

$$
\begin{aligned}
E=&\int_{[u,u+1]^{\otimes n}} du_1\dots du_n\int_{B^{\otimes n}} dt\ p_{X_1(t_1),\dots,X_n(t_n),X_1'(t_1),\dots,X_n'(t_n)}(u_1,\dots,u_n,0,\dots,0)\\
&\times\mathrm{E}\left(\left|\prod_{i=1}^n \det\left(X_i''(t_i)\right)\mathbb{I}_{\{X_i''(t_i)\preceq 0\}}\right|\mathbb{I}_{\{t_1\in S^{+\overline{r_1}}\}}\mathbb{I}_{\{\underset{1\le i\le n}{\cap}B(t_i,\overline{r_i})\neq\emptyset\}}\mid X_i(t_i)=u_i, X_i'(t_i)=0\,\forall i\right),
\end{aligned}
$$

where $p_{X_1(t_1),\dots,X_n(t_n),X_1'(t_1),\dots,X_n'(t_n)}(.)$ is the joint density function of the random vector

$$(X_1(t_1),\dots,X_n(t_n),X_1'(t_1),...,X_n'(t_n)).$$

Using (3.1) and the fact that the fields X_i's are independent and $X_i'(t_i)$ is independent to $X_i(t_i)$ and $X_i''(t_i)$, we have

$$
\begin{aligned}
E=\frac{\lambda_d(S)}{(2\pi)^{nd/2}}&\int_{[u,u+1]^{\otimes n}}\prod_{i=1}^n\mathrm{E}\left(\left|\det\left(X_i''(t_i)\right)\mathbb{I}_{\{X_i''(t_i)\preceq 0\}}\right|\mid X_i(t_i)=u_i\right)\varphi(u_i)\\
&\times\left(\sum_{\|\mathbf{m}\|=k}C_{\mathbf{m}}\overline{\mathbf{r}}^{\mathbf{m}}+O(\sum_{\|\mathbf{m}\|=k+1}\overline{\mathbf{r}}^{\mathbf{m}})\right)du_1\dots du_n.
\end{aligned}
\tag{3.3}
$$

Note that under the condition $X_i(t_i)=u_i$ then $\overline{r_i}$ is no more random and is equal to

$$\overline{r_i}=\sqrt{2\frac{u_i-u}{u_i-u^\alpha}}.$$

Using the fact that (see Azaïs and Delmas (2002))

$$\mathrm{E}\left(|\det(X_i'(t))|\mathbb{I}_{\{X_i''(t)\preceq 0\}}\mid X_i(t)=u_i, X_i'(t)=0\right)=u_i^d+O\left(u_i^{d-2}\right)\ \text{as}\ u_i\to\infty,$$

then

$$
\begin{aligned}
&\int_u^{u+1}\overline{r_i}^{m_i}\mathrm{E}\left(\left|\det\left(X_i''(t_i)\right)\mathbb{I}_{\{X_i''(t_i)\preceq 0\}}\right|\mid X_i(t_i)=u_i, X_i'(t_i)=0\right)\varphi(u_i)du_i\\
\simeq&\int_u^{u+1}u_i^d\left(2\frac{u_i-u}{u_i-u^\alpha}\right)^{m_i/2}\varphi(u_i)du_i.
\end{aligned}
$$

By the change of variable $u_i=u+x/u$, the above integral is equal to

$$
\begin{aligned}
&\int_u^{u+1}(u+x/u)^d\left(\frac{2x/u}{u+x/u-u^\alpha}\right)^{m_i/2}\varphi(u+x/u)dx/u\\
\simeq&2^{m_i/2}u^{d-(m_i+1)}\varphi(u)\int_0^u x^{m_i/2}e^{-x}dx\simeq 2^{m_i/2}u^{d-(m_i+1)}\varphi(u)\Gamma(1+m_i/2).
\end{aligned}
$$

Therefore, for each vector $\mathbf{m} = (m_1, \dots, m_n)$ with l_1-norm k,

$$\begin{aligned}
&\int_{[u,u+1]^{\otimes n}} \prod_{i=1}^{n} \mathrm{E}\left(\left| \det\left(X_i''(t_i)\right) \mathbb{I}_{\{X_i''(t_i) \preceq 0\}} \right| \mid X_i(t_i) = u_i \right) \varphi(u_i) \overline{\mathbf{r}}^{\mathbf{m}} du_1 \dots du_n \\
=&\prod_{i=1}^{n} \int_u^{u+1} \overline{r_i}^{m_i} \mathrm{E}\left(\left| \det\left(X_i''(t_i)\right) \mathbb{I}_{\{X_i''(t_i) \preceq 0\}} \right| \mid X_i(t_i) = u_i \right) \varphi(u_i) du_i \\
\simeq&\prod_{i=1}^{n} 2^{m_i/2} u^{d-(m_i+1)} \varphi(u) \Gamma(1 + m_i/2) \\
=&2^{k/2} u^{nd-n-k} \varphi^n(u) \prod_{i=1}^{n} \Gamma(1 + j_i/2).
\end{aligned}$$

Hence, by substituting this into (3.3),

$$E = u^{nd-n-k} \varphi^n(u) \left(\frac{2^{k/2} \lambda_d(S)}{(2\pi)^{nd/2}} \sum_{\|\mathbf{m}\|=k} C_{\mathbf{m}} \prod_{i=1}^{n} \Gamma(1 + m_i/2) + o(1) \right).$$

- For the lower bound, recall that

$$S^{-\underline{r_1}} = \{t \in S: \ B(t, \underline{r_1}) \subset S\}.$$

Then the lower bound is at least equal to

$$\begin{aligned}
&\mathrm{P}\left(\exists t \in B^{\otimes n}: \ t_1 \in S^{-\underline{r_1}}, \forall i: \ \begin{array}{c} X_i(t) \text{ has a local maximum at } t_i, \\ X_i(t_i) \in [u, u+1], \text{and } \bigcap\limits_{1 \le i \le n} B(t_i, \underline{r_i}) \neq \emptyset \end{array} \right) + o(u^{nd-n-k} \varphi^n(u)) \\
&= \mathrm{P}(M_{\underline{r}} \ge 1) + o(u^{nd-n-k} \varphi^n(u)) \\
&\ge \mathrm{E}(M_{\underline{r}}) - \mathrm{E}(M_{\underline{r}}(M_{\underline{r}} - 1))/2 + o(u^{nd-n-k} \varphi^n(u))
\end{aligned}$$

where

$$M_{\underline{r}} = \mathrm{card} \left\{ \begin{array}{c} t \in B^{\otimes n}: \ t_1 \in S^{-\underline{r_1}}, \ \bigcap\limits_{1 \le i \le n} B(t_i, \underline{r_i}) \neq \emptyset, \text{ and } \forall i: \\ X_i(t) \text{ has a local maximum at } t_i, X_i(t_i) \in [u, u+1] \end{array} \right\}.$$

It is clear that

$$M_{\underline{r}} \le M = \mathrm{card}\{t = (t_1, \dots, t_n) \in B^{\otimes n}: \ X_i(t) \text{ has a local maximum at } t_i, X_i(t_i) \ge u, \forall i\}.$$

Then applying the Rice formula and using the independent property of the given fields, we have

$$
\begin{aligned}
&\mathrm{E}(M_{\underline{r}}(M_{\underline{r}}-1)) \le \mathrm{E}(M.(M-1))\\
=&\int_{[u,\infty)^{\otimes n}\times[u,\infty)^{\otimes n}} dydz \int_{B^{\otimes n}\times B^{\otimes n}} dtds\\
&\times \mathrm{E}\Big(\Big|\prod_{i=1}^{n} \det\left(X_i''(t_i)\right)\mathbb{I}_{\{X_i''(t_i)\preceq 0\}} \det\left(X_i''(s_i)\right)\mathbb{I}_{\{X_i''(s_i)\preceq 0\}}\Big|\\
&\qquad\qquad \mid X_i(t_i)=y_i, X_i'(t_i)=0, X_i(s_i)=z_i, X_i'(s_i)=0,\ \forall i\Big)\\
&\times p_{X_1(t_1),\dots,X_n(t_n),X_1'(t_1),\dots,X_n'(t_n),X_1(s_1),\dots,X_n(s_n),X_1'(s_1),\dots,X_n'(s_n)}(y,0,z,0)\\
=&\prod_{i=1}^{n}\int_{[u,\infty)\times[u,\infty)} dy_idz_i \int_{B\times B} dt_ids_i \times p_{X_i(t_i),X_i'(t_i),X_i(s_i),X_i'(s_i)}(y_i,0,z_i,0)\\
&\times \mathrm{E}\left(\left|\det\left(X_i''(t_i)\right)\mathbb{I}_{\{X_i''(t_i)\preceq 0\}} \det\left(X_i''(s_i)\right)\mathbb{I}_{\{X_i''(s_i)\preceq 0\}}\right|\right.\\
&\qquad\qquad \left.\mid X_i(t_i)=y_i, X_i'(t_i)=0, X_i(s_i)=z_i, X_i'(s_i)=0\right)\\
=&\prod_{i=1}^{n}\mathrm{E}(M_i.(M_i-1)),
\end{aligned}
$$

where

$$M_i = \mathrm{card}\left\{t_i \in \overset{\circ}{B}_i:\ X_i(.)\ \text{ has a local maximum at } t_i,\ X(t)\ge u\right\}.$$

In Azaïs and Delmas (2002), it is proved that there exist two constants $C, c > 1$ such that

$$\mathrm{E}(M_i.(M_i-1)) \le Ce^{-cu^2/2}.$$

Hence we have

$$\mathrm{E}(M.(M-1)) = o(u^{nd-n-k}\varphi^n(u)).$$

The calculation of the expectation $\mathrm{E}(\mathcal{M}_{\underline{r}})$ can be done similarly as in the upper bound part and we obtain the same asymptotic formula. Then the result follows.

4. Proof of Proposition 2.2 and some examples

Before giving the proof of Proposition 2.2, we would like to consider three interesting examples where we can give a much simpler formula for the volume in (2.1), and therefore a simpler asymptotic formula for the conjunction probability thanks to Proposition 2.1. We hope that through these examples, the readers can get the intuition about the basic ideas of the detailed proof.

4.1. *First example:* $n = 2$. This example corresponds to the practical application mentioned in the introduction. It is clear that

$$\{t_2:\ B(t_1,r_1)\cap B(t_2,r_2)\neq\emptyset\} = B(t_1, r_1+r_2).$$

Therefore

$$
\begin{aligned}
&\lambda_d\left(t_2:\ B(t_1,r_1)\cap B(t_2,r_2)\neq\emptyset\right) = \lambda_d\left(B(t_1,r_1+r_2)\right)\\
=&\frac{\pi^{d/2}}{\Gamma(1+d/2)}(r_1+r_2)^d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}\sum_{j=0}^{d}\binom{d}{j} r_1^j r_2^{d-j},
\end{aligned}
$$

here we use again the fact that the volume of a d-dimensional unit ball is $\pi^{d/2}/\Gamma(1+d/2)$.

Then we have an immediate consequence of Proposition 2.1 as follows.

Corollary 4.1. *Consider $X_i(t), 1 \le i \le 2$, two independent copies of a Gaussian field X satisfying Assumption (A). Then as u tends to infinity,*

$$\begin{aligned}&\mathrm{P}\left(\max_{t\in S}\{\min(X_1(t),X_2(t))\}\ge u\right)\\ =&\frac{u^{d-2}\varphi^2(u)\lambda_d(S)}{(2\pi)^{d/2}\Gamma(1+d/2)}\left(\sum_{j=0}^{d}\binom{d}{j}\Gamma(1+j/2)\Gamma(1+(d-j)/2)+o(1)\right).\end{aligned} \tag{4.1}$$

Proof: We substitute the following parameters in the statement of the main theorem

$$k=d,\quad \mathbf{m}=(j,d-j),\ \text{and}\ C_{\mathbf{m}}=\frac{\pi^{d/2}\binom{d}{j}}{\Gamma(1+d/2)}.$$

□

Remark. Let us now consider the estimation given by the Euler characteristic method. It is clear that (1.6) becomes

$$(1,0,\dots,0)R^2\mu(S).$$

Here the term corresponding to $\mu_d(S)$ (or $\lambda_d(S)$) is

$$\lambda_d(S)b_d\sum_{i=0}^{d}\frac{\rho_i}{b_i}\frac{\rho_{d-i}}{b_{d-i}}.$$

From the definition of the Euler characteristic densities ρ_i's, this term is equivalent to

$$\frac{\lambda_d(S)u^{d-2}\varphi^2(u)}{(2\pi)^{d/2}}\Gamma((d+1)/2)\Gamma(1/2)\sum_{i=0}^{d}\frac{1}{\Gamma((i+1)/2)\Gamma((d-i+1)/2)}.$$

Comparing with the asymptotic formula given in (4.1), it is surprising to see that

$$\begin{aligned}&\Gamma((d+1)/2)\Gamma(1/2)\sum_{i=0}^{d}\frac{1}{\Gamma((i+1)/2)\Gamma((d-i+1)/2)}\\ =&\frac{1}{\Gamma(1+d/2)}\sum_{i=0}^{d}\binom{d}{i}\Gamma(1+i/2)\Gamma(1+(d-i)/2).\end{aligned}$$

Indeed, we will prove that for every $i=0,\dots,d$,

$$\frac{\Gamma((d+1)/2)\Gamma(1/2)}{\Gamma((i+1)/2)\Gamma((d-i+1)/2)}=\frac{1}{\Gamma(1+d/2)}\binom{d}{i}\Gamma(1+i/2)\Gamma(1+(d-i)/2). \tag{4.2}$$

The equality (4.2) is equivalent to

$$\begin{aligned}&\Gamma(d/2+1/2)\Gamma(d/2+1)\Gamma(1/2)\\ =&\binom{d}{i}\Gamma(i/2+1/2)\Gamma(i/2+1)\Gamma((d-i)/2+1/2)\Gamma((d-i)/2+1),\end{aligned}$$

that is true from the Legendre duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\Gamma(2z), \tag{4.3}$$

and from $\Gamma(n+1)=n!,\quad \Gamma(1/2)=\sqrt{\pi}$.

4.2. *Second example:* $d = 1$. In this subsection, we would like to revisit the conjunction probability of stationary centered Gaussian processes. Although that the corresponding result given in Pham (2020) is more powerful and more informative than the asymptotic formula given in Theorem 1.1, it would be nice to reprove that

$$H_{n,2} = \frac{n}{\sqrt{2\pi}}.$$

The affirmative answer is deduced by the following lemma.

Lemma 4.2. *For a fixed point t_1 on the real axis and small enough fixed radii $r_1, r_2, \ldots, r_n$, we have*

$$\lambda_{n-1}\left((t_2, \ldots, t_n) \in \mathbb{R}^{n-1} : \bigcap_{1 \le i \le n} B(t_i, r_i) \neq \emptyset\right) = 2^{n-1} \sum_{i=1}^{n} \left(\prod_{j \neq i} r_j\right). \tag{4.4}$$

Proof: We will prove by induction on n.
- For $n = 2$, it is obvious as in the above subsection.
- Assume that the statement is true from 2 to $n - 1$.
- For n-tuple $(t_1, t_2, \ldots, t_n)$, we would like to calculate the volume as the following integral

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \mathbb{I}_{\{\bigcap_{1 \le i \le n} B(t_i, r_i) \neq \emptyset\}} dt_2 \ldots dt_n \\ &= \int_{\mathbb{R}^{n-2}} \mathbb{I}_{\{\bigcap_{1 \le i \le n-1} B(t_i, r_i) \neq \emptyset\}} dt_2 \ldots dt_{n-1} \int_{\mathbb{R}} \mathbb{I}_{\left\{B(t_n, r_n) \cap \left(\bigcap_{1 \le i \le n-1} B(t_i, r_i)\right) \neq \emptyset\right\}} dt_n. \end{aligned} \tag{4.5}$$

Again by induction, it is clear that if the intersection $\bigcap_{1 \le i \le n-1} B(t_i, r_i)$ is non-empty, it is an interval. Therefore

$$\int_{\mathbb{R}} \mathbb{I}_{\left\{B(t_n, r_n) \cap \left(\bigcap_{1 \le i \le n-1} B(t_i, r_i)\right) \neq \emptyset\right\}} dt_n = \lambda_1\left(\bigcap_{1 \le i \le n-1} B(t_i, r_i)\right) + 2r_n,$$

and the considering volume is equal to

$$\int_{\mathbb{R}^{n-2}} \mathbb{I}_{\{\bigcap_{1 \le i \le n-1} B(t_i, r_i) \neq \emptyset\}} \left(\lambda_1\left(\bigcap_{1 \le i \le n-1} B(t_i, r_i)\right) + 2r_n\right) dt_2 \ldots dt_{n-1}. \tag{4.6}$$

By inductive hyphothesis,

$$2r_n \int_{\mathbb{R}^{n-2}} \mathbb{I}_{\{\bigcap_{1 \le i \le n-1} B(t_i, r_i) \neq \emptyset\}} \ldots dt_{n-1} = 2r_n . 2^{n-2} \sum_{i=1}^{n-1} \left(\prod_{1 \le j \le n-1,\, j \neq i} r_j\right).$$

For the rest term in the integral (4.6), let us introduce a new variable y corresponding to the point in the intersection, and we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n-2}} \mathbb{I}_{\{\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\neq\emptyset\}}\lambda_1\left(\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\right) dt_2\dots dt_{n-1}\\
=&\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}} \mathbb{I}_{\{\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\neq\emptyset\}}\mathbb{I}_{\{y\in\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\}} dt_2\dots dt_{n-1}dy\\
=&\int_{B(t_1,r_1)} dy\left(\int_{\mathbb{R}^{n-2}} \mathbb{I}_{\{y\in\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\}} dt_2\dots dt_{n-1}\right)\\
=&\int_{B(t_1,r_1)} dy\left(\int_{B(y,r_2)} dy_2\dots\int_{B(y,r_{n-1})} dt_{n-1}\right)\\
=&\int_{B(t_1,r_1)}\left(\prod_{i=2}^{n-1}(2r_i)\right)dy = 2^{n-1}\prod_{i=1}^{n-1} r_i,
\end{aligned}$$

where the equality in the third line follows from Fubini theorem. The result follows easily. □

Applying Proposition 2.1 in this case with respect to $k=n-1$, $\mathbf{m}$ is an n-dimensional vector with $n-1$ unit entries and only one zero entry and $C_{\mathbf{m}}=2^{n-1}$, we obtain the following corollary.

Corollary 4.3. *Let $X_i(t), 1\le i\le n$, be the independent copies of a Gaussian process X satisfying Assumption (A). Then as u tends to infinity,*

$$\mathrm{P}\left(\max_{t\in[0,T]}\min_{1\le i\le n} X_i(t)\ge u\right) = u^{-(n-1)}\varphi^n(u)\left(\frac{nT}{\sqrt{2\pi}}+o(1)\right).$$

4.3. *Third example: $d=2$.*

Lemma 4.4. *For a fixed point t_1 in the plane and small enough fixed radii $r_1,r_2,\dots,r_n$, we have*

$$\begin{aligned}
&\lambda_{2(n-1)}\left((t_2,\dots,t_n)\in\mathbb{R}^{2(n-1)}:\ \underset{1\le i\le n}{\cap} B(t_i,r_i)\neq\emptyset\right)\\
=&\pi^{n-1}\sum_{i=1}^{n}\left(\prod_{j\neq i} r_j^2\right)+2\pi^{n-1}\sum_{1\le i<j\le n}\left(r_ir_j\prod_{k\neq i,j} r_k^2\right).
\end{aligned}$$

Proof: We will prove by induction on n.

- Case $n=2$ has been considered in Subsection 4.1.
- For general $n\ge 3$, we have

$$\begin{aligned}
&\lambda_{2(n-1)}\left((t_2,t_3,\dots,t_n)\in\mathbb{R}^{2(n-1)}:\ \underset{1\le i\le n}{\cap} B(t_i,r_i)\neq\emptyset\right)\\
=&\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\left[\mathbb{I}_{\{\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\neq\emptyset\}}\int_{\mathbb{R}^2}\mathbb{I}_{\{B(t_n,r_n)\cap\left(\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\right)\neq\emptyset\}} dt_n\right]\\
=&\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\left[\mathbb{I}_{\{\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\neq\emptyset\}}\lambda_2\left(\left(\underset{1\le i\le n-1}{\cap} B(t_i,r_i)\right)^{+r_n}\right)\right].
\end{aligned}$$

Since the intersection $\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)$ is a convex set then by Steiner formula,

$$\lambda_2\left(\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)^{+r_n}\right) = \pi r_n^2 + r_n.\mathrm{peri}\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right) + \lambda_2\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right),$$

where *peri(.)* stands for the perimeter of the set.

Therefore, the considering volume is equal to

$$\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\mathbb{I}_{\{\bigcap_{1\le i\le n-1} B(t_i,r_i)\neq\emptyset\}} \left[\pi r_n^2 + r_n.\mathrm{peri}\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right) + \lambda_2\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)\right]. \tag{4.7}$$

– For the first term in (4.7), by the inductive hypothesis,

$$\begin{aligned}&\pi r_n^2\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\mathbb{I}_{\{\bigcap_{1\le i\le n-1} B(t_i,r_i)\neq\emptyset\}}\\ =&\pi r_n^2\left[\pi^{n-2}\sum_{i=1}^{n-1}\left(\prod_{j\neq i} r_j^2\right) + 2\pi^{n-2}\sum_{1\le i<j\le n-1}\left(r_ir_j\prod_{k\neq i,j} r_k^2\right)\right].\end{aligned}$$

– For the third term in (4.7), we introduce a new variable y corresponding to the point in the intersection, and we use Fubini theorem to obtain that

$$\begin{aligned}&\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\lambda_2\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right) = \int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\left[\int_{\bigcap_{1\le i\le n-1} B(t_i,r_i)} dy\right]\\ =&\int_{B(t_1,r_1)} dy\left[\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\prod_{i=2}^{n-1}\mathbb{I}_{\{t_i\in B(y,r_i)\}}\right] = \pi^{n-1}\prod_{i=1}^{n-1} r_i^2.\end{aligned}$$

– For the second term in (4.7), let us denote $S(t,r)$ the circle with radius r at center point t, i.e. the boundary of the disk $B(t,r)$. It is clear that the perimeter of the intersection $\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)$ is the sum of the lengths of the arcs on each circle $S(t_i,r_i), i=\overline{1,n-1}$. For the first kind with respect to the arc on $S(t_1,r_1)$, again by Fubini theorem, we have

$$\begin{aligned}&r_n\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\left[\int_{S(t_1,r_1)}\mathbb{I}_{\{y\in\bigcap_{2\le i\le n-1} B(t_i,r_i)\}}dy\right]\\ =&r_n\int_{S(t_1,r_1)} dy\left[\int_{\mathbb{R}^{2(n-2)}} dt_2\dots dt_{n-1}\prod_{i=2}^{n-1}\mathbb{I}_{\{t_i\in B(y,r_i)\}}\right] = 2\pi^{n-1}r_1r_n\prod_{i=2}^{n-1} r_i^2.\end{aligned}$$

For the second kind with respect to the arc on $S(t_i, r_i)$ with $i = 2, \dots, n-1$. Without loss of generality, we consider the arc on $S(t_2, r_2)$. We have

$$
\begin{aligned}
&r_n \int_{\mathbb{R}^{2(n-2)}} dt_2 \dots dt_{n-1} \left[\int_{S(t_2,r_2)} \mathbb{I}_{\{y \in \bigcap_{i \neq 2} B(t_i,r_i)\}} dy \right] \\
=&r_n \int_{B(t_1,r_1+r_2)} dt_2 \int_{S(t_2,r_2)} dy \mathbb{I}_{\{y\in B(t_1,r_1)\}} \left[\int_{\mathbb{R}^{2(n-3)}} dt_3 \dots dt_{n-1} \prod_{i=3}^{n-1} \mathbb{I}_{\{t_i \in B(y,r_i)\}} \right] \\
=&r_n \pi^{n-3} \prod_{i=3}^{n-1} r_i^2 \int_{B(t_1,r_1+r_2)} dt_2 \int_{S(t_2,r_2)} \mathbb{I}_{\{y\in B(t_1,r_1)\}} dy = r_n \pi^{n-3} \prod_{i=3}^{n-1} r_i^2 \int_{B(t_1,r_1)} dy \int_{S(y,r_2)} dt_2 \\
=&2\pi^{n-1} r_n r_1^2 r_2 \prod_{i=3}^{n-1} r_i^2.
\end{aligned}
$$

The result follows by summing up three terms in (4.7). □

From the above lemma, we can apply Proposition 2.1 to deduce the following corollary.

Corollary 4.5. *Consider $X_i(t), 1 \le i \le n$, being independent copies of a two-dimensional Gaussian field X satisfying Assumption (A). Then as u tends to infinity,*

$$
\mathrm{P}\left(\max_{t\in S} \min_{1\le i\le n} X_i(t) \ge u \right) = u^{2-n} \varphi^n(u) \left[\frac{\lambda_2(S)}{2\pi} \left(n + \frac{n(n-1)\pi}{4} \right) + o(1) \right].
$$

4.4. *Proof of Proposition 2.2.* As the readers can see in Subsections 4.2 and 4.3, the basic ideas of the proofs are Fubini theorem to change the order of the variables in the integrals, and Steiner formula to calculate the area (length) of the ϵ- neighborhood of some sets. In general, these ideas are still useful. Let us introduce Weyl tube formula, that is a generalization of Steiner formula, and also Crofton formula that will be used later.

4.4.1. *Preliminaries on Weyl tube formula and Crofton formula.* - **Weyl tube formula**: Let M be an m-dimensional manifold with positive reach or critical radius (see Adler and Taylor (2007)) embedded in R^d which is endowed with the canonical Riemannian structure on R^d. Then for any positive ϵ less than the critical radius of M, the Lebesgue volume of the ϵ- neighborhood of M in R^d is given by

$$
\lambda_d \left(M^{+\epsilon} \right) = \sum_{j=0}^{m} \epsilon^{d-j} \omega_{d-j} \mu_j(M), \tag{4.8}
$$

where $\mu_0(M), \mu_1(M), \dots, \mu_d(M)$ are the intrinsic Killing-Lipschitz curvatures of M, that do not depend on the ambient space $\mathbb{R}^d$. Note that when M is convex, then its critical radius equals to infinity, therefore the above Weyl tube fomula holds true for any positive ϵ and becomes Steiner formula.

- **Crofton formula**: Borrowing the notations from Klain and Rota (1997) and Kratz and Vadlamani (2018), let $\mathrm{Gr}(d,k)$ be the Grassmanian of all $k-$ dimensional linear subspaces of $\mathbb{R}^d$ with the invariant Haar measure ν_k^d. Let $\mathrm{Graff}(d,k)$ be the affine Grassmannian of all k-dimensional affine subspaces of $\mathbb{R}^d$. We define the measure λ_k^d on $\mathrm{Graff}(d,k)$ that is invariant under the group of Euclidean motions as follows.

Given a k-dimensional affine subspaces $V^* \in \mathrm{Graff}(d,k)$. Let $V^{*,\perp}$ be the maximal linear subspace of $\mathbb{R}^d$ orthogonal to V^* and containing the origin. There is a unique maximal linear subspace $V \in \mathrm{Gr}(d,k)$ orthogonal to $V^{*,\perp}$. It means that V is the k-dimensional linear subspace parallel to V^*. Denote p by the intersection point between V^* and $V^{*,\perp} = V^\perp$. Thus, any $V^* \in \mathrm{Graff}(d,k)$ corresponds one-to-one to a pair $(V,p) \in \mathrm{Gr}(d,k) \times V^\perp$, in the sense that $V^* = V + p$, where the plus symbol stands for the translation.Then, the measure λ_k^d on Graff(d,k) is defined as, for any real-valued measurable function f on Graff(d,k),

$$\int_{\mathrm{Graff}(d,k)} f(V^*)d\lambda_k^d(V^*) = \int_{\mathrm{Gr}(d,k)} \int_{V^\perp} f(V+p)d\nu_k^d(V)dp,$$

where dp denotes the ordinary Lebesgue measure on $V^\perp \cong \mathbb{R}^{d-k}$.

In a special case, let M be a suitable compact subset of $\mathbb{R}^d$ and function f be the Killing-Lipschitz curvatures of the intersection $M \cap V^*$, we have the Crofton formula as

$$\int_{\mathrm{Graff}(d,k)} \mu_j(M\cap V^*)d\lambda_k^d(V^*) = \begin{bmatrix} d-k+j \\ j \end{bmatrix} \mu_{d-k+j}(M), \tag{4.9}$$

where the flag symbol stands for

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{\omega_m}{\omega_n\omega_{m-n}}\binom{m}{n}.$$

4.4.2. *Detailed proof.* It is clear that

$$\begin{aligned}
I &= \lambda_{(n-1)d}\left((t_2,\dots,t_n)\in\mathbb{R}^{d(n-1)} : \bigcap_{1\le i\le n} B(t_i,r_i)\neq\emptyset\right)\\
&= \int_{\mathbb{R}^{(n-1)d}} \mathbb{I}_{\{\bigcap_{1\le i\le n} B(t_i,r_i)\neq\emptyset\}} dt_2\dots dt_n\\
&= \int_{\mathbb{R}^{(n-2)d}} \mathbb{I}_{\{\bigcap_{1\le i\le n-1} B(t_i,r_i)\neq\emptyset\}} dt_2\dots dt_{n-1}\int_{\mathbb{R}^d} \mathbb{I}_{\{B(t_n,r_n)\cap\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)\neq\emptyset\}} dt_n\\
&= \int_{\mathbb{R}^{(n-2)d}} \mathbb{I}_{\{\bigcap_{1\le i\le n-1} B(t_i,r_i)\neq\emptyset\}} \lambda_d\left(\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)^{+r_n}\right) dt_2\dots dt_{n-1}
\end{aligned}$$

By Weyl tube formula in (4.8),

$$\lambda_d\left(\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right)^{+r_n}\right) = \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\mu_{d-k_n}\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right).$$

Substituting this expansion in the integral, we have

$$\begin{aligned}
I &= \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\int_{\mathbb{R}^{(n-2)d}} \mu_{d-k_n}\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\right) dt_2\dots dt_{n-1}\\
&= \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\int_{\mathbb{R}^{(n-2)d}} dt_2\dots dt_{n-1}\int_{\mathrm{Graff}(d,k_n)} \mu_0\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\cap V^*\right) d\lambda_{k_n}^d(V^*)\\
&= \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\int_{\mathbb{R}^{(n-2)d}} dt_2\dots dt_{n-1}\int_{\mathrm{Gr}(d,k_n)} d\nu_{k_n}^d(V)\int_{V^\perp} \mu_0\left(\bigcap_{1\le i\le n-1} B(t_i,r_i)\cap (V+p)\right) dp\\
&= \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\int_{\mathbb{R}^{(n-2)d}} dt_2\dots dt_{n-1}\int_{\mathrm{Gr}(d,k_n)} d\nu_{k_n}^d(V)\int_{V^\perp} \mathbb{I}_{\{p\in \bigcap_{1\le i\le n-1} B(t_i,r_i)|_{V^\perp}\}} dp,
\end{aligned}$$

where the second line follows from Crofton formula (4.9) applied to the case $j=0$; the third line follows from the definition of the measure $\lambda_{k_n}^d$ and the last line follows from the fact that the intersection $\bigcap_{1\le i\le n-1} B(t_i,r_i)\cap(V+p)$ is empty or a non-empty convex, with Euler characteristic 0 or 1. The value depends on whether the point p is on the orthogonal projection of $\bigcap_{1\le i\le n-1} B(t_i,r_i)$ on the subspace $V^\perp$.

By the Fubini theorem, we continue as

$$I=\sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\int_{\mathbb{R}^{(n-3)d}} dt_2\dots dt_{n-2}\int_{\mathrm{Gr}(d,k_n)} d\nu_{k_n}^d(V)\times$$
$$\int_{V^\perp}\mathbb{I}_{\{p\in \bigcap_{1\le i\le n-2} B(t_i,r_i)|_{V^\perp}\}}dp\int_{\mathbb{R}^d}\mathbb{I}_{\left\{B(t_{n-1},r_{n-1})\cap\left(\bigcap_{1\le i\le n-2}B(t_i,r_i)\cap(V+p)\right)\neq\emptyset\right\}}dt_{n-1}.$$

Here using again Weyl tube formula (4.8), we have

$$\begin{aligned}
I=&\sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\int_{\mathbb{R}^{(n-3)d}} dt_2\dots dt_{n-2}\int_{\mathrm{Gr}(d,k_n)} d\nu_{k_n}^d(V)\times\\
&\int_{V^\perp}\mathbb{I}_{\{p\in \bigcap_{1\le i\le n-2} B(t_i,r_i)|_{V^\perp}\}}\sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}}\mu_{d-k_{n-1}}\left(\bigcap_{1\le i\le n-2}B(t_i,r_i)\cap(V+p)\right)dp\\
=&\sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}}\int_{\mathbb{R}^{(n-3)d}} dt_2\dots dt_{n-2}\times\\
&\int_{\mathrm{Gr}(d,k_n)} d\nu_{k_n}^d(V)\int_{V^\perp}\mu_{d-k_{n-1}}\left(\bigcap_{1\le i\le n-2}B(t_i,r_i)\cap(V+p)\right)dp\\
=&\sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}}\int_{\mathbb{R}^{(n-3)d}} dt_2\dots dt_{n-2}\times\\
&\int_{\mathrm{Graff}(d,k)}\mu_{d-k_{n-1}}\left(\bigcap_{1\le i\le n-2}B(t_i,r_i)\cap V^*\right)d\lambda_k^d(V^*)\\
=&\sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n}\sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}}\begin{bmatrix} d-k_n+(d-k_{n-1})\\ d-k_{n-1}\end{bmatrix}\times\\
&\int_{\mathbb{R}^{(n-3)d}}\mu_{d-k_n+(d-k_{n-1})}\left(\bigcap_{1\le i\le n-2}B(t_i,r_i)\right)dt_2\dots dt_{n-2},
\end{aligned}$$

where the last line follows from the Crofton formula (4.9).

In summary, we have proved that

$$\int_{\mathbb{R}^{(n-2)d}}\mu_{d-k_n}\left(\bigcap_{1\le i\le n-1}B(t_i,r_i)\right)dt_2\dots dt_{n-1}=\sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}}\begin{bmatrix} d-k_n+d-k_{n-1}\\ d-k_{n-1}\end{bmatrix}$$
$$\times\int_{\mathbb{R}^{(n-3)d}}\mu_{d-(k_n+k_{n-1}-d)}\left(\bigcap_{1\le i\le n-2}B(t_i,r_i)\right)dt_2\dots dt_{n-2},$$

then using this argument repeatedly, we obtain that

$$I = \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n} \sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}} \begin{bmatrix} d-k_n+d-k_{n-1} \\ d-k_{n-1} \end{bmatrix} \times$$

$$\sum_{k_{n-2}=d-k_n+d-k_{n-1}}^{d} r_{n-2}^{k_{n-2}}\omega_{k_{n-2}} \begin{bmatrix} d-k_n+d-k_{n-1}+d-k_{n-2} \\ d-k_{n-2} \end{bmatrix} \times \ldots \times$$

$$\sum_{k_2=(n-2)d-(k_n+k_{n-1}+\ldots+k_3)}^{d} r_2^{k_2}\omega_{k_2} \begin{bmatrix} (d-k_n)+\ldots+(d-k_2) \\ d-k_2 \end{bmatrix} \mu_{(n-1)d-\sum_{i=2}^n k_i}\left(B(t_1,r_1)\right).$$

By comparing two formulas

$$\lambda_d\left(B(t_1,r_1)^{+\epsilon}\right) = \sum_{j=0}^{d} \epsilon^{d-j}\omega_{d-j}\mu_j(M)$$

and

$$\lambda_d\left(B(t_1,r_1)^{+\epsilon}\right) = \lambda_d\left(B(t_1,r_1+\epsilon)\right) = \omega_d(r_1+\epsilon)^d = \omega_d\sum_{j=0}^{d}\binom{d}{j}\epsilon^{d-j}r_1^j,$$

it is clear that

$$\mu_j\left(B(t_1,r_1)\right) = \frac{\omega_d}{\omega_{d-j}}\binom{d}{j}r_1^j.$$

Thus we have

$$I = \sum_{k_n=0}^{d}\sum_{k_{n-1}=d-k_n}^{d}\cdots\sum_{k_2=(n-2)d-(k_n+k_{n-1}+\ldots+k_3)}^{d} r_1^{(n-1)d-\sum_{i=2}^n k_i}\times$$

$$\prod_{i=2}^{n}\left(r_i^{k_i}\omega_{k_i}\begin{bmatrix}(d-k_n)+\ldots+(d-k_i)\\ d-k_i\end{bmatrix}\right)\frac{\omega_d}{\omega_{\sum_{i=2}^n k_i-(n-2)d}}\binom{d}{(n-1)d-\sum_{i=2}^n k_i}.$$

Observe that

$$\frac{\omega_d}{\omega_{\sum_{i=2}^n k_i-(n-2)d}}\binom{d}{(n-1)d-\sum_{i=2}^n k_i}\prod_{i=2}^{n}\begin{bmatrix}(d-k_n)+\ldots+(d-k_i)\\ d-k_i\end{bmatrix}$$

$$=\frac{\omega_d}{\omega_{\sum_{i=2}^n k_i-(n-2)d}}\binom{d}{(n-1)d-\sum_{i=2}^n k_i}\prod_{i=2}^{n-1}\frac{\omega_{(d-k_n)+\ldots+(d-k_i)}}{\omega_{(d-k_n)+\ldots+(d-k_{i+1})}\times\omega_{d-k_i}}\binom{(d-k_n)+\ldots+(d-k_i)}{d-k_i}$$

$$=\frac{\omega_d\omega_{(n-1)d-\sum_{i=2}^n k_i}}{\omega_{\sum_{i=2}^n k_i-(n-2)d}\prod_{i=2}^n\omega_{d-k_i}}\times\frac{d!}{[\sum_{i=2}^n k_i-(n-2)d]!.\prod_{i=2}^n(d-k_i)!},$$

this completes the proof.

5. Comparing with Euler characteristic method

In this section, we would like to compare our result given in the main theorem with the prediction given by the Euler characteristic method. This prediction is defined in (1.6) as

$$(1,0,\ldots,0)R^2\mu(S).$$

Note that the indexes of the rows and columns of matrix R varies from 0 to d, the term corresponding to $\mu_d(S)$ (or $\lambda_d(S)$) is

$$\lambda_d(S)b_d\sum_{0\le h_1\le\ldots\le h_{n-1}\le d}\frac{\rho_{h_1}}{b_{h_1}}\frac{\rho_{h_2-h_1}}{b_{h_2-h_1}}\cdots\frac{\rho_{d-h_{n-1}}}{b_{d-h_{n-1}}},$$

and it is equivalent to

$$\sum_{0\le h_1\le\ldots\le h_{n-1}\le d}\frac{(2\pi)^{-d/2}\varphi^n(u)u^{d-n}\lambda_d(S)\Gamma(1/2)^{n-1}\Gamma((d+1)/2)}{\Gamma((h_1+1)/2)\Gamma((h_2-h_1+1)/2)\ldots\Gamma((d-h_{n-1}+1)/2)}.$$

To prove that the above sum coincides with the asymptotic formula (1.8), we need to show that

$$\sum_{0\le h_1\le\ldots\le h_{n-1}\le d}\frac{\Gamma(1/2)^{n-1}\Gamma((d+1)/2)}{\Gamma((h_1+1)/2)\Gamma((h_2-h_1+1)/2)\ldots\Gamma((d-h_{n-1}+1)/2)}$$
$$=\sum_{k_n=0}^{d}\sum_{k_{n-1}=d-k_n}^{d}\cdots\sum_{k_2=(n-2)d-\sum_{i=3}^n k_i}^{d}\frac{\omega_d}{\omega_{\sum_{i=2}^n k_i-(n-2)d}\prod_{i=2}^n\omega_{d-k_i}}\times\frac{d!}{[\sum_{i=2}^n k_i-(n-2)d]!\prod_{i=2}^n(d-k_i)!}.$$

Indeed, we rewrite the indices $(h_1,h_2,\ldots,h_{n-1})$ as

$$h_1=d-k_n,\ h_2=(d-k_n)+(d-k_{n-1}),\ldots,h_{n-1}=\sum_{i=2}^n(d-k_i),$$

and it can be checked one-by-one that

$$\frac{\Gamma(1/2)^{n-1}\Gamma((d+1)/2)}{\Gamma((d-k_n+1)/2)\Gamma((d-k_{n-1}+1)/2)\ldots\Gamma((d-k_2+1)/2)\Gamma\left(\left(d+1-\sum_{i=2}^n(d-k_i)\right)/2\right)}$$
$$=\frac{\omega_d}{\omega_{d-\sum_{i=2}^n(d-k_i)}\prod_{i=2}^n\omega_{d-k_i}}\times\frac{d!}{[d-\sum_{i=2}^n(d-k_i)]!\prod_{i=2}^n(d-k_i)!}.$$

The equality follows easily from Legendre duplication formula (4.3) as in Subsection 4.1.

In conclusion, we give a one-term expansion for the conjunction probability of smooth Gaussian fields. It is interesting to see that this expansion coincides with the first term of the heuristic prediction given by the Euler characteristic method although they look different at first sight. Since the heuristic prediction consists of $d+1$ terms, it is natural to ask that

"Could we prove the full validity of the Euler characteristic method for the conjunction probability as for the tail distribution of a smooth Gaussian field?"

We would like to leave this interesting question for future research.

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