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Sojourns of Stationary Gaussian Processes over a Random Interval

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Abstract. We investigate asymptotics of the tail distribution of sojourn time

$$\int_0^T \mathbb{I}(X(t) > u)dt,$$

as $u \to \infty$, where X is a centered stationary Gaussian process and T is an independent of X nonnegative random variable. The heaviness of the tail distribution of T impacts the form of the asymptotics, leading to four scenarios: the case of integrable T, the case of regularly varying T with index $\lambda = 1$ and index $\lambda \in (0, 1)$ and the case of slowly varying tail distribution of T. The derived findings are illustrated by the analysis of the class of fractional Ornstein-Uhlenbeck processes.

1. Introduction

For a given stochastic process $X(t), t \geq 0$, by

$$L_u[a, b] := \int_a^b \mathbb{I}_u (X(t)) \,\mathrm{d}t,$$

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with $\mathbb{I}_u(x) := \mathbb{I}(x > u)$, we define the sojourn time spent above a fixed level u by process X on interval $[a, b]$. The interest in analysis of distributional properties of $L_u[a, b]$ stems both from theoretical questions related to the research on the level sets of stochastic processes and from its importance in applied probability, as e.g., in finance or insurance theory, where $L_u[0, T]$, $T > 0$ may be interpreted as the total time in ruin up to time T for the risk process modeled by X; see e.g., Guérin and Renaud (2017); Landriault et al. (2020).

In the case of X being a Gaussian process, the asymptotics of the tail distribution of $L_u[0, T]$, as $u \to \infty$, was analyzed extensively in a series of papers by Berman, e.g., Berman (1985, 1987); see also the seminal monograph Berman (1992) and recent refinements Dębicki et al. (2019, 2020b).

The aim of this paper is to get the exact asymptotics of tail distribution of $L_u[0, T]$ for a class of centered stationary Gaussian processes over an independent of X random time T. The motivation to consider extremal behaviour of a stochastic process over a random time interval stems from its relevance in such problems as ruin of time-changed risk processes Fotopoulos and Luo (2011); Geman et al. (2001), resetting models den Hollander et al. (2019) or hybrid queueing models Zwart et al. (2005). We also refer to related problems on extremes of conditionally Gaussian processes and Gaussian processes with random variance Hüsler et al. (2011a,b). Using the fact that

$$\mathbb{P}\{L_u[0, T] > 0\} = \mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\},$$

the findings of this contribution also extend results obtained in Arendarczyk and Dębicki (2012); Dębicki et al. (2018); Tan and Hashorva (2013).

It appears that the form of the derived exact asymptotics strongly depends on the heaviness of the tail distribution of T, leading to four scenarios: the case of finite $\mathbb{E}T$ (scenario **D1**), the case of T having regularly varying tail distribution with index $\lambda = 1$ (scenario **D2**), $\lambda \in (0, 1)$ (scenario **D3**) and the case of slowly varying tail distribution of T (scenario **D4**); see Section 3.

Brief organisation of the rest of the paper: In Section 2 we formalize the analyzed model and introduce notation. In Section 3 we derive the tail asymptotic behavior of the sojourn time for a class of centered stationary Gaussian processes X over random interval $[0, T]$ under introduced in Section 2 scenarios **D1-D4**, respectively. Section 4 contains some examples illustrating the main findings of this contribution. All the proofs are displayed in Section 5, whereas few technical results are included in Section 6.

2. Notation and model description

Let $X(t), t \geq 0$ be a centered stationary Gaussian process with a.s. continuous trajectories, unit variance function and covariance function r satisfying

A1: $1 - r(t)$ is regularly varying at $t = 0$ with index $\alpha \in (0, 2]$;

A2: $r(t) < 1$ for all $t > 0$;

A3: $\lim_{t\to\infty} r(t)\log(t) = 0$.

Assumptions **A1-A3** cover a wide range of investigated in the literature stationary Gaussian processes, where **A3** is referred to as *Berman's condition* (see, e.g., Berman, 1992); see also Section 4.

Let function $v(\cdot)$ be such that $\lim_{u\to\infty} v(u) = \infty$ and

$$\lim_{u\to\infty} u^2(1 - r(1/v(u))) = 1. \tag{2.1}$$

By Berman (1992), $v(\cdot)$ exists and is regularly varying at infinity with index $2/\alpha$.

We are interested in the asymptotics of

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\},$$

as $u \to \infty$, where

$$L_u^*[0,T] := v(u)L_u[0,T] \tag{2.2}$$

and T is an independent of X nonnegative random variable with distribution function $F_T(\cdot)$ which belongs to one of the following distribution classes:

D1: T is integrable;

D2: T has regularly varying tail distribution with index $\lambda = 1$;

D3: T has regularly varying tail distribution with index $\lambda \in (0,1)$;

D4: T has slowly varying tail distribution.

We recall that a nonnegative random variable T has regularly varying tail distribution with index $\lambda > 0$ if for any $x > 0$

$$\lim_{t\to\infty} \frac{\mathbb{P}\left\{T > xt\right\}}{\mathbb{P}\left\{T > t\right\}} = x^{-\lambda}.$$

If $\lambda = 0$ in the above limit, then it is said that T has slowly varying tail distribution. Furthermore, T is integrable if $\lambda > 1$.

We note that, although classes **D1-D4** cover a wide collection of distribution functions, they are not exhaustive in the sense that, for example, random variable T that satisfies $\mathbb{P}\left\{T > t\right\} \sim e^{-\log t - \cos(\log t)}$ as $t \to \infty$ does not belong to any of **D1-D4**.

Define for any $x \ge 0$

$$\mathcal{B}_\alpha(x) = \lim_{S\to\infty} S^{-1}\mathcal{B}_\alpha(S,x), \tag{2.3}$$

with

$$\mathcal{B}_\alpha(S,x) = \int_{\mathbb{R}} \mathbb{P}\left\{\int_0^S \mathbb{I}_0\left(W_\alpha(s) + z\right) \mathrm{d}s > x\right\} e^{-z}\mathrm{d}z, \quad W_\alpha(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, \tag{2.4}$$

where B_α is a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0,1]$. By Theorem 2.1 in Dębicki et al. (2019), we know that $\mathcal{B}_\alpha(x)$ is positive and finite for any $x \ge 0$. Let $\mathcal{E}$ be a unit exponential random variable independent of W_α and set

$$\mathcal{G}_\alpha(x) = \mathbb{P}\left\{\int_{\mathbb{R}} \mathbb{I}_0\left(W_\alpha(s) + \mathcal{E}\right) \mathrm{d}s \le x\right\}.$$

As shown in Dębicki et al. (2020b), $\mathcal{G}_\alpha$ is continuous on $\mathbb{R}^+$, and thus by Remark 2.2 ii) in Dębicki et al. (2019)

$$\mathcal{B}_\alpha(x) = \int_x^\infty \frac{1}{y} \mathrm{d}\mathcal{G}_\alpha(y)$$

holds for all $x \in \mathbb{R}^+$. We note that $\mathcal{B}_\alpha(0)$ is equal to the classical Pickands constant; see e.g., Piterbarg (1996) or Section 10 in Berman (1992). Let

$$m(u) = \big(\mathcal{B}_\alpha(0)v(u)\Psi(u)\big)^{-1}, \tag{2.5}$$

where $\Psi(u)$ is the survival function of an $N(0,1)$ random variable. Then, by Theorem 10.5.1 in Berman (1992),

$$\mathbb{P}\left\{\sup_{t\in[0,1]} X(t) > u\right\} \sim m^{-1}(u), \quad u \to \infty. \tag{2.6}$$

In our notation $\sim$ stands for asymptotic equivalence of two functions as the argument tends to 0 or to ∞ respectively.

3. Main results

In this section we find the exact asymptotics of

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\} \tag{3.1}$$

as $u \to \infty$, under scenarios **D1-D4**, respectively. All the proofs are postponed to Section 5.

3.1. Scenario D1

3.1. **Scenario D1**. We begin with the case when T is integrable. It appears that under this scenario the main contribution to the asymptotics of (3.1) comes from Gaussian process X, whereas T contributes only by its average behavior.

Theorem 3.1. *Let $X(t), t \geq 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1-A2**. *Suppose that T is an independent of X nonnegative random variable that satisfies* **D1**. *Then for any $x \geq 0$*

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\} \sim \mathcal{B}_\alpha(x)\mathbb{E}\left\{T\right\} v(u)\Psi(u), \quad u \to \infty. \tag{3.2}$$

Theorem 2.1 in Dębicki et al. (2019) combined with Remark 2.2 therein yields

$$\mathbb{P}\left\{L_u^*[0,1] > x\right\} \sim \mathcal{B}_\alpha(x)v(u)\Psi(u), \quad u \to \infty.$$

Thus

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\} \sim \mathbb{E}\left\{T\right\}\mathbb{P}\left\{L_u^*[0,1] > x\right\}, \quad u \to \infty,$$

which implies that T contributes to the asymptotics in (3.2) only by its average behavior.

3.2. **Scenario D2**. Under this scenario the asymptotics of (3.1) is similar to the one obtained for case **D1** with the exception that T contributes to (3.1) by its integrated tail distribution rather than by its mean.

Theorem 3.2. *Let $X(t), t \geq 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1-A3**. *Suppose that T is an independent of X nonnegative random variable that satisfies* **D2**. *Then for any $x \geq 0$*

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\} \sim \mathcal{B}_\alpha(x) l(m(u)) v(u) \Psi(u), \quad u \to \infty, \tag{3.3}$$

where $l(u) = \int_0^u \mathbb{P}\left\{T > t\right\} \mathrm{d}t$.

Remark 3.3. We note that if T satisfies **D2** and is integrable, then (3.3) coincides with (3.2).

3.3. **Scenario D3**. This scenario leads to the asymptotics of (3.1) which depends only of the heaviness of the tail distribution of T.

The following continuous distribution function

$$\mathcal{F}_\alpha(x) := \mathcal{B}_\alpha^{-1}(0) \int_0^x \frac{1}{y} \mathrm{d}\mathcal{G}_\alpha(x), \quad x \geq 0 \tag{3.4}$$

plays an important role in further analysis. $\overline{\mathcal{F}_\alpha^{*k}}(x)$ denotes the tail distribution of the k-th convolution of $\mathcal{F}_\alpha$ at $x \geq 0$.

Theorem 3.4. *Let $X(t), t \geq 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1-A3**. *Suppose that T is an independent of X nonnegative random variable that satisfies* **D3**. *Then for any $x \geq 0$*

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\} \sim \lambda \sum_{k=1}^{\infty} \frac{\Gamma(k-\lambda)}{k!} \overline{\mathcal{F}_\alpha^{*k}}(x) \mathbb{P}\left\{T > m(u)\right\}, \quad u \to \infty. \tag{3.5}$$

Remark 3.5. Taking $x = 0$ in (3.5) and using

$$\lambda \sum_{k=1}^{\infty} \frac{\Gamma(k-\lambda)}{k!} = \lambda \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^\infty l^{k-\lambda-1} e^{-l} \mathrm{d}l = \lambda \int_0^\infty (1-e^{-l}) l^{-\lambda-1} \mathrm{d}l = \Gamma(1-\lambda),$$

we recover Theorem 3.2 in Arendarczyk and Dębicki (2012).

3.4. **Scenario D4**. Suppose now that T has slowly varying tail distribution. As shown in the following theorem, similarly to scenario **D3**, the asymptotics of (3.1) depends only on the asymptotic behavior of the tail distribution of T but in contrast to scenario **D3** doesn't depend on x.

Theorem 3.6. *Let $X(t), t \geq 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1-A3**. *Suppose that T is an independent of X nonnegative random variable that satisfies* **D4**. *Then for any $x \geq 0$*

$$\mathbb{P}\left\{L_u^*[0,T] > x\right\} \sim \mathbb{P}\left\{T > m(u)\right\}, \quad u \to \infty.$$

4. Examples

In this section we illustrate the results derived in Section 3 by two classes of stationary Gaussian processes: fractional Ornstein-Uhlenbeck processes and increments of fractional Brownian motions.

4.1. **Fractional Ornstein-Uhlenbeck processes**. Suppose that X is a centered stationary Gaussian process with covariance $r(t) = e^{-t^\alpha}, t \geq 0$, for $\alpha \in (0, 2]$. We call X a *fractional Ornstein-Uhlenbeck* process with index α. If $\alpha = 1$, then X is the classical Ornstein-Uhlenbeck process.

It is straightforward to check that **A1-A3** are satisfied. Thus, the following proposition holds due to Theorems 3.1-3.6.

Proposition 4.1. *Suppose that X is a fractional Ornstein-Uhlenbeck process with index $\alpha \in (0, 2]$, and T is an independent of X nonnegative random variable. Then for any $x \geq 0$, as $u \to \infty$,*
(i) If $T \in \mathbf{D1}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \mathcal{B}_\alpha(x)\mathbb{E}\{T\}(2\pi)^{-1/2}u^{2/\alpha-1}e^{-u^2/2}$.
(ii) If $T \in \mathbf{D2}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \mathcal{B}_\alpha(x)(2\pi)^{-1/2}u^{2/\alpha-1}e^{-u^2/2}\int_0^{\sqrt{2\pi}\mathcal{B}_\alpha^{-1}(0)u^{1-2/\alpha}e^{u^2/2}}\mathbb{P}\{T > t\}\,\mathrm{d}t$.
(iii) If $T \in \mathbf{D3}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \lambda\sum_{k=1}^{\infty}\frac{\Gamma(k-\lambda)}{k!}\overline{\mathcal{F}_\alpha^{*k}}(x)\mathbb{P}\left\{T > \sqrt{2\pi}\mathcal{B}_\alpha^{-1}(0)u^{1-2/\alpha}e^{u^2/2}\right\}$.
(iv) If $T \in \mathbf{D4}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \mathbb{P}\left\{T > \sqrt{2\pi}\mathcal{B}_\alpha^{-1}(0)u^{1-2/\alpha}e^{u^2/2}\right\}$.

4.2. **Increments of fractional Brownian motion**. For a standard fBm $B_\alpha(t), t \geq 0$ with Hurst index $\alpha/2 \in (0, 1)$ and $a > 0$, define

$$X_{\alpha,a}(t) := \frac{B_\alpha(t+a) - B_\alpha(t)}{a^{\alpha/2}}, \quad t \geq 0$$

One can check that $X_{\alpha,a}$ is a centered stationary Gaussian process with unit variance and covariance function

$$r(t) = \frac{(a+t)^\alpha + |a-t|^\alpha - 2t^\alpha}{2a^\alpha}, \quad t \geq 0,$$

and $1 - r(t) \sim a^{-\alpha}t^\alpha$, $t \to 0$, which verifies assumption **A1**. Similarly for $t > a$

$$|r(t)| \leq \frac{\alpha\,|1-\alpha|\,(t-a)^{\alpha-2}}{a^{\alpha-2}},$$

which confirms assumption **A3**. Thus the following proposition holds.

Proposition 4.2. *Suppose that $X_{\alpha,a}(t), t \geq 0$ with $\alpha \in (0, 2)$, $a > 0$ is independent of a nonnegative random variable T. Then for any $x \geq 0$, as $u \to \infty$,*
(i) If $T \in \mathbf{D1}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \mathcal{B}_\alpha(x)\mathbb{E}\{T\}(2\pi)^{-1/2}a^{-1}u^{2/\alpha-1}e^{-u^2/2}$.
(ii) If $T \in \mathbf{D2}$, then

$$\mathbb{P}\{L_u^*[0,T] > x\} \sim \mathcal{B}_\alpha(x)(2\pi)^{-1/2}a^{-1}u^{2/\alpha-1}e^{-u^2/2}\int_0^{\sqrt{2\pi}a\mathcal{B}_\alpha^{-1}(0)u^{1-2/\alpha}e^{u^2/2}}\mathbb{P}\{T > t\}\,\mathrm{d}t.$$

(iii) If $T \in \mathbf{D3}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \lambda\sum_{k=1}^{\infty}\frac{\Gamma(k-\lambda)}{k!}\overline{\mathcal{F}_\alpha^{*k}}(x)\mathbb{P}\left\{T > \sqrt{2\pi}a\mathcal{B}_\alpha^{-1}(0)u^{1-2/\alpha}e^{u^2/2}\right\}$;
(iv) If $T \in \mathbf{D4}$, then $\mathbb{P}\{L_u^*[0,T] > x\} \sim \mathbb{P}\left\{T > \sqrt{2\pi}a\mathcal{B}_\alpha^{-1}(0)u^{1-2/\alpha}e^{u^2/2}\right\}$.

5. Proofs

In this section we give detailed proofs of all the theorems presented in Section 3. We first give a simple extension of Theorem 7.4.1 of Berman (1992).

Lemma 5.1. *Let $X(t), t \geq 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1** *and* **A3**. *If L_u^*, $m(u)$ and $\mathcal{F}_\alpha$ are defined in (2.2), (2.5) and (3.4), respectively, then for any $s \geq 0$ and $0 < l_0 < l_1 < \infty$ we have*

$$\lim_{u\to\infty} \sup_{\tau\in[l_0,l_1]} \left| \mathbb{E}\left\{ e^{-sL_u^*[0,\tau m(u)]} \right\} - e^{-\tau \int_0^\infty (1-e^{-sx}) \mathrm{d}\mathcal{F}_\alpha(x)} \right| = 0. \tag{5.1}$$

Proof: For any $\tau > 0$, the point convergence follows from Berman's proof of Theorem 7.4.1 in Berman (1992). The uniformity of convergence on $[l_0, l_1]$ follows by monotonicity of $\mathbb{E}\left\{e^{-sL_u^*[0,\tau m(u)]}\right\}$ and by continuity of $e^{-\tau \int_0^\infty (1-e^{-sx}) \mathrm{d}\mathcal{F}_\alpha(x)}$ as functions of τ. □

Define a compound Poisson process

$$Y(t) = \sum_{i=1}^{N(t)} \xi_i, \tag{5.2}$$

where $\{N(t) : t \geq 0\}$ is a Poisson process with unit intensity, and $\{\xi_i : i \geq 1\}$ are independent and identically distributed random variables, with distribution function $\mathcal{F}_\alpha$, which are also independent of N. The following corollary of Lemma 5.1 will play an important role in the proof of Theorem 3.4.

Corollary 5.2. *If X is the Gaussian process given as in Lemma 5.1 and Y is defined in (5.2), then for any $x \geq 0$ and $0 < l_0 < l_1 < \infty$ we have*

$$\lim_{u\to\infty} \sup_{l\in[l_0,l_1]} \left| \mathbb{P}\left\{ L_u^*[0, lm(u)] > x \right\} - \mathbb{P}\left\{ Y(l) > x \right\} \right| = 0. \tag{5.3}$$

Proof: For arbitrary $l > 0$, from (5.1) we know that the Laplace transforms of $L_u^*[0, lm(u)]$ converge pointwise to the Laplace transform of $Y(l)$ as $u \to \infty$. Then by the continuity theorem for Laplace transforms (see, e.g., Exercise 15.4.3 in Klenke, 2020),

$$\mathbb{P}\left\{ L_u^*[0, lm(u)] > x \right\} \to \mathbb{P}\left\{ Y(l) > x \right\}, \quad u \to \infty$$

holds for any continuity point of the distribution function of $Y(l)$. Due to the continuity of $\mathcal{F}_\alpha$, the above limit holds for all $x > 0$. Further,

$$\left| \mathbb{P}\left\{ Y(l_1) > x \right\} - \mathbb{P}\left\{ Y(l_0) > x \right\} \right| \leq \mathbb{P}\left\{ Y(|l_1 - l_0|) > 0 \right\} = 1 - e^{-|l_1 - l_0|} \leq |l_1 - l_0|,$$

which implies that for any $x > 0$, $\mathbb{P}\{Y(l) > x\}$ is continuous in l. Finally, the uniform convergence follows by the same arguments as the proof of Lemma 5.1. For $x = 0$ in (5.3), we refer to Lemma 4.3 in Arendarczyk and Dębicki (2012). This completers the proof. □

5.1. ***Proof of Theorem* 3.1.** By (2.6), for arbitrary $\varepsilon > 0$, there exists large enough u such that

$$\mathbb{P}\left\{\sup_{t\in[0,1]} X(t) > u\right\} < (1+\varepsilon)m^{-1}(u),$$

which together with the stationarity of process X implies that for any $x \geq 0$ and $t > 0$

$$\begin{aligned}\frac{\mathbb{P}\{L_u^*[0,t] > x\}}{v(u)\Psi(u)} &\leq \frac{\mathbb{P}\left\{\sup_{s\in[0,t]} X(s) > u\right\}}{v(u)\Psi(u)} \\ &\leq (t+1)\frac{\mathbb{P}\left\{\sup_{s\in[0,1]} X(s) > u\right\}}{v(u)\Psi(u)} \\ &\leq (t+1)(1+\varepsilon)\mathcal{B}_\alpha(0).\end{aligned}$$

Consequently, for nonnegative random variable T with distribution function F_T satisfying **D1**, by the Dominated Convergence Theorem and Remark 2.2 i) in Dębicki et al. (2019) we have

$$\begin{aligned}\lim_{u\to\infty}\frac{\mathbb{P}\{L_u^*[0,T] > x\}}{v(u)\Psi(u)} &= \lim_{u\to\infty}\int_0^\infty \frac{\mathbb{P}\{L_u^*[0,t] > x\}}{v(u)\Psi(u)} \mathrm{d}F_T(t) \\ &= \mathcal{B}_\alpha(x)\int_0^\infty t\mathrm{d}F_T(t) \\ &= \mathcal{B}_\alpha(x)\mathbb{E}\{T\}.\end{aligned}$$

This completes the proof. □

5.2. ***Proof of Theorem* 3.2.** Let $A(u)$ satisfy

$$\lim_{u\to\infty} A(u)v(u) = \infty \quad \text{and} \quad \lim_{u\to\infty} A(u) = 0.$$

By Corollary 6.3, for any $x \geq 0$ and arbitrary $\varepsilon \in (0,1)$, there exist $\delta > 0$ and u_0 such that

$$\inf_{t\in[A(u),\delta m(u)]} \frac{\mathbb{P}\{L_u^*[0,t] > x\}}{t\mathcal{B}_\alpha(x)v(u)\Psi(u)} \geq 1 - 2\varepsilon, \quad u > u_0$$

and

$$\sup_{t\in[A(u),\delta m(u)]} \frac{\mathbb{P}\{L_u^*[0,t] > x\}}{t\mathcal{B}_\alpha(x)v(u)\Psi(u)} \leq 1 + 2\varepsilon, \quad u > u_0.$$

Therefore,

$$\begin{aligned}
\liminf_{u\to\infty} \frac{\mathbb{P}\left\{L_u^*[0,T]>x\right\}}{\mathcal{B}_\alpha(x)l(m(u))v(u)\Psi(u)} &\geq \liminf_{u\to\infty} \frac{\int_{A(u)}^{\delta m(u)} \mathbb{P}\left\{L_u^*[0,t]>x\right\} \mathrm{d}F_T(t)}{\mathcal{B}_\alpha(x)l(m(u))v(u)\Psi(u)} \\
&\geq (1-2\varepsilon) \liminf_{u\to\infty} \frac{\int_{A(u)}^{\delta m(u)} t\mathrm{d}F_T(t)}{l(m(u))} \\
&= (1-2\varepsilon) \liminf_{u\to\infty} \frac{\int_{0}^{\delta m(u)} t\mathrm{d}F_T(t)}{l(m(u))} \\
&= (1-2\varepsilon) \liminf_{u\to\infty} \frac{\int_{0}^{\delta m(u)} \mathbb{P}\left\{T>t\right\}\mathrm{d}t - \delta m(u)\mathbb{P}\left\{T>\delta m(u)\right\}}{l(m(u))} \\
&= (1-2\varepsilon),
\end{aligned}$$

where the last equality follows from Proposition 1.5.9a in Bingham et al. (1989) such that $l(u)$ is slowly varying at ∞ and $\lim_{u\to\infty} u\mathbb{P}\left\{T>u\right\}/l(u)=0$.

Similarly,

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{\mathbb{P}\left\{L_u^*[0,T]>x\right\}}{\mathcal{B}_\alpha(x)l(m(u))v(u)\Psi(u)} \\
&\leq \limsup_{u\to\infty} \frac{\mathbb{P}\left\{L_u^*[0,A(u)]>x\right\}\mathbb{P}\left\{T\leq A(u)\right\} + \int_{A(u)}^{\delta m(u)} \mathbb{P}\left\{L_u^*[0,t]>x\right\}\mathrm{d}F_T(t) + \mathbb{P}\left\{T>\delta m(u)\right\}}{\mathcal{B}_\alpha(x)l(m(u))v(u)\Psi(u)} \\
&\leq \limsup_{u\to\infty} \frac{A(u)\mathbb{P}\left\{T\leq A(u)\right\}}{l(m(u))} + (1+2\varepsilon)\limsup_{u\to\infty} \frac{\int_{A(u)}^{\delta m(u)} t\mathrm{d}F_T(t)}{l(m(u))} \\
&= (1+2\varepsilon),
\end{aligned}$$

where the last inequality follows from (6.19) and the same reasons as above. Since ε is arbitrary, letting $\varepsilon\to 0$, we complete the proof. □

5.3. ***Proof of Theorem* 3.4**. First, note that by Raabe's Test, the series in (3.5) converges for $\lambda\in(0,1)$. Then by integration by parts, for any $x\geq 0$ we have

$$\begin{aligned}
\int_0^\infty l^{-\lambda}\mathrm{d}\mathbb{P}\left\{Y(l)>x\right\} &= \int_0^\infty \mathbb{P}\left\{Y(l)>x\right\}\lambda l^{-\lambda-1}\mathrm{d}l - \lim_{l\to 0} l^{-\lambda}\mathbb{P}\left\{Y(l)>x\right\} \\
&= \lambda\sum_{k=1}^\infty \overline{\mathcal{F}_\alpha^{*k}}(x)\frac{1}{k!}\int_0^\infty l^{k-\lambda-1}e^{-l}\mathrm{d}l - \lim_{l\to 0} l^{-\lambda}\mathbb{P}\left\{Y(l)>x\right\} \\
&= \lambda\sum_{k=1}^\infty \frac{\Gamma(k-\lambda)}{k!}\overline{\mathcal{F}_\alpha^{*k}}(x) < \infty, \qquad (5.4)
\end{aligned}$$

where the last equality, recalling that $\lambda\in(0,1)$, follows from

$$\lim_{l\to 0} l^{-\lambda}\mathbb{P}\left\{Y(l)>x\right\} \leq \lim_{l\to 0} l^{-\lambda}(1-e^{-l}) = 0. \qquad (5.5)$$

Next, by a similar argument as that used in the proof of Theorem 3.2 in Arendarczyk and Dębicki (2012), for any $0 < l_0 < l_1 < \infty$ we have

$$\begin{aligned}\mathbb{P}\{L_u^*[0,T] > x\} &= \Big(\int_0^{l_0 m(u)} + \int_{l_0 m(u)}^{l_1 m(u)} + \int_{l_1 m(u)}^{\infty}\Big)\mathbb{P}\{L_u^*[0,l] > x\}\,\mathrm{d}F_T(l)\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\limsup_{u\to\infty}\frac{I_1}{\mathbb{P}\{T > m(u)\}} \le \limsup_{u\to\infty}\frac{\int_0^{l_0 m(u)}\mathbb{P}\left\{\sup_{s\in[0,l]} X(s) > u\right\}\mathrm{d}F_T(l)}{\mathbb{P}\{T > m(u)\}} \le \frac{\lambda}{1-\lambda} l_0^{1-\lambda}$$

and

$$\limsup_{u\to\infty}\frac{I_3}{\mathbb{P}\{T > m(u)\}} \le \limsup_{u\to\infty}\frac{\mathbb{P}\{T > l_1 m(u)\}}{\mathbb{P}\{T > m(u)\}} = l_1^{-\lambda}.$$

Further, in view of Corollary 5.2, for any given $x \ge 0$ and arbitrary $\varepsilon > 0$, we have the following upper bound

$$\begin{aligned}I_2 &= \int_{l_0}^{l_1}\mathbb{P}\{L_u^*[0,lm(u)] > x\}\,\mathrm{d}F_T(lm(u))\\ &\le (1+\varepsilon)\int_{l_0}^{l_1}\mathbb{P}\{Y(l) > x\}\,\mathrm{d}F_T(lm(u))\\ &= (1+\varepsilon)\Big(\int_{l_0}^{l_1}\mathbb{P}\{T > lm(u)\}\,\mathrm{d}\mathbb{P}\{Y(l) > x\}\\ &\qquad -\mathbb{P}\{Y(l_1) > x\}\,\mathbb{P}\{T > l_1 m(u)\} + \mathbb{P}\{Y(l_0) > x\}\,\mathbb{P}\{T > l_0 m(u)\}\Big),\end{aligned}$$

which holds for u large enough. By Theorem 1.5.2 in Bingham et al. (1989),

$$\frac{\mathbb{P}\{T > lm(u)\}}{\mathbb{P}\{T > m(u)\}} \to l^{-\lambda}$$

uniformly for $l \in [l_0, l_1]$ as $u \to \infty$. Thus by the Dominated Convergence Theorem

$$\limsup_{u\to\infty}\frac{I_2}{\mathbb{P}\{T > m(u)\}} \le (1+\varepsilon)\left(\int_{l_0}^{l_1} l^{-\lambda}\mathrm{d}\mathbb{P}\{Y(l) > x\} - \mathbb{P}\{Y(l_1) > x\}\,l_1^{-\lambda} + \mathbb{P}\{Y(l_0) > x\}\,l_0^{-\lambda}\right).$$

Similarly, we have the lower bound

$$\liminf_{u\to\infty}\frac{I_2}{\mathbb{P}\{T > m(u)\}} \ge (1-\varepsilon)\left(\int_{l_0}^{l_1} l^{-\lambda}\mathrm{d}\mathbb{P}\{Y(l) > x\} - \mathbb{P}\{Y(l_1) > x\}\,l_1^{-\lambda} + \mathbb{P}\{Y(l_0) > x\}\,l_0^{-\lambda}\right).$$

Finally, first letting $\varepsilon \to 0$, and then letting $l_0 \to 0$, $l_1 \to \infty$ in the above bounds, using (5.4) and (5.5), we complete the proof. □

5.4. ***Proof of Theorem 3.6***. According to Remark 3.3 in Arendarczyk and Dębicki (2012), we know that

$$\limsup_{u\to\infty} \frac{\mathbb{P}\{L_u^*[0,T] > x\}}{\mathbb{P}\{T > m(u)\}} \le \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{s\in[0,T]} X(s) > u\right\}}{\mathbb{P}\{T > m(u)\}} \le 1.$$

Further, by Corollary 5.2, for arbitrary $l > 0$

$$\mathbb{P}\{L_u^*[0, lm(u)] > x\} \to \mathbb{P}\{Y(l) > x\}$$

holds for any $x \ge 0$ as $u \to \infty$. Thus, for T with slowly varying tail distribution we get

$$\begin{aligned}\liminf_{u\to\infty} \frac{\mathbb{P}\{L_u^*[0,T] > x\}}{\mathbb{P}\{T > m(u)\}} &\ge \liminf_{u\to\infty} \frac{\mathbb{P}\{L_u^*[0, lm(u)] > x\}\,\mathbb{P}\{T > lm(u)\}}{\mathbb{P}\{T > m(u)\}} \\ &= \mathbb{P}\{Y(l) > x\},\end{aligned}$$

which converges to 1 as $l \to \infty$, since by the strong law of large numbers $Y(l)/l \to \mathcal{B}_\alpha^{-1}(0) > 0$. This completes the proof. □

6. Appendix

Hereafter, $C_i, i \in \mathbb{N}$ are positive constants which may be different from line to line. All vectors are column vectors unless otherwise specified. As long as it doesn't cause confusion we use $\mathbf{0}$ to denote the 2×1 column vector or the 2×2 matrix whose entries are all 0's. For a given vector (matrix) $\mathbf{Q}$, let $|\mathbf{Q}|$ denote vector (matrix) with entries equal to absolute value of respective entries of $\mathbf{Q}$.

Lemma 6.1. *Let $X(t), t \ge 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1** *and* **A3***. If $v(u)$, $\mathcal{B}_\alpha(S,x)$ and $m(u)$ are defined in (2.1), (2.4) and (2.5), respectively, then for any $A(u) > 1$ satisfying*

$$\limsup_{u\to\infty} \frac{u^2}{\log A(u)} < \infty \tag{6.1}$$

we have

$$\lim_{u\to\infty} \sup_{d\ge A(u)} \left| \frac{\mathbb{P}\left\{\sup_{s_1\in[0,S]} X(s_1/v(u)) > u, \sup_{s_2\in[0,S]} X(d + s_2/v(u)) > u\right\}}{\Psi^2(u)} - \mathcal{B}_\alpha^2(S,0)\right| = 0.$$

Proof: We borrow the argument used in the proof of Theorem 5.1 in Dębicki et al. (2019). First, for notational simplicity we define

$$\xi_{u,d}(\boldsymbol{s}) = (X(s_1/v(u)), X(d + s_2/v(u)))^T, \quad \boldsymbol{s} = (s_1, s_2) \in \boldsymbol{D} = [0, S]^2,$$

and denote by $R_{u,d}(\boldsymbol{s},\boldsymbol{t})$ the covariance matrix function of $\xi_{u,d}$, i.e.,

$$\begin{aligned} R_{u,d}(\boldsymbol{s},\boldsymbol{t}) &= \mathrm{Cov}(\xi_{u,d}(\boldsymbol{s}),\xi_{u,d}(\boldsymbol{t})) \\ &= \mathbb{E}\left\{\xi_{u,d}(\boldsymbol{s})\xi_{u,d}(\boldsymbol{t})^T\right\} \\ &= \begin{pmatrix} r(\frac{|t_1-s_1|}{v(u)}) & r(d+\frac{t_2-s_1}{v(u)}) \\ r(d+\frac{s_2-t_1}{v(u)}) & r(\frac{|t_2-s_2|}{v(u)}) \end{pmatrix} \quad \boldsymbol{s},\boldsymbol{t}\in\boldsymbol{D}. \end{aligned}$$

Then, conditioning on $\xi_{u,d}(\mathbf{0})$ we have

$$\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\xi_{u,d}(\boldsymbol{s})>\boldsymbol{u}\right\}=\iint_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\xi_{u,d}(\boldsymbol{s})>\boldsymbol{u}|\xi_{u,d}(\mathbf{0})=\boldsymbol{y}\right\}\phi(y_1,y_2;r(d))\mathrm{d}y_1\mathrm{d}y_2,$$

where $\boldsymbol{y}=(y_1,y_2)^T$ and $\phi(y_1,y_2;r(d))$ is the density function of bivariate normal random variable $\xi_{u,d}(\mathbf{0})$. By the change of variables $\boldsymbol{y}=\boldsymbol{u}+\boldsymbol{z}/u$ and using properties of conditional distribution of normal random variable (see e.g., Chapter 2.2 in Berman, 1992), we get

$$\begin{aligned} \mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\xi_{u,d}(\boldsymbol{s})>\boldsymbol{u}\right\} &= \frac{e^{-u^2}}{2\pi u^2}\iint_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\chi_{u,d}(\boldsymbol{s})-\theta_{u,d}(\boldsymbol{s},\boldsymbol{z})>\mathbf{0}\right\}f_{u,d}(\boldsymbol{z})\mathrm{d}z_1\mathrm{d}z_2 \\ &= \frac{e^{-u^2}}{2\pi u^2}\iint_{\mathbb{R}^2}\mathcal{I}_{u,d}(\boldsymbol{z})f_{u,d}(\boldsymbol{z})\mathrm{d}z_1\mathrm{d}z_2, \end{aligned}$$

where $\mathcal{I}_{u,d}(\boldsymbol{z}):=\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\chi_{u,d}(\boldsymbol{s})-\theta_{u,d}(\boldsymbol{s},\boldsymbol{z})>\mathbf{0}\right\}$ with

$$\chi_{u,d}(\boldsymbol{s})=u\left(\xi_{u,d}(\boldsymbol{s})-R_{u,d}(\boldsymbol{s},\mathbf{0})R_{u,d}^{-1}(\mathbf{0},\mathbf{0})\xi_{u,d}(\mathbf{0})\right),\quad \boldsymbol{s}\in\boldsymbol{D},$$

$$\theta_{u,d}(\boldsymbol{s},\boldsymbol{z})=u^2\left(\mathbf{1}-R_{u,d}(\boldsymbol{s},\mathbf{0})R_{u,d}^{-1}(\mathbf{0},\mathbf{0})(\mathbf{1}+\boldsymbol{z}/u^2)\right),\quad \boldsymbol{s}\in\boldsymbol{D},\boldsymbol{z}\in\mathbb{R}^2$$

and

$$f_{u,d}(\boldsymbol{z})=\frac{1}{\sqrt{1-r^2(d)}}\exp\left(\frac{1}{1+r(d)}(u^2r(d)-z_1-z_2)-\frac{(z_1-r(d)z_2)^2}{2u^2(1-r^2(d))}-\frac{z_2^2}{2u^2}\right),\quad \boldsymbol{z}\in\mathbb{R}^2.$$

Consequently, in order to show the claim it suffices to prove that for $W_\alpha(\boldsymbol{s})=(\sqrt{2}B_\alpha^{(1)}(s_1)-s_1^\alpha,\sqrt{2}B_\alpha^{(2)}(s_2)-s_2^\alpha)^T$, $\boldsymbol{s}\in\boldsymbol{D}$, where $B_\alpha^{(1)}$ and $B_\alpha^{(2)}$ are two independent fBm's with Hurst index $\alpha/2$,

$$\begin{aligned} &\lim_{u\to\infty}\sup_{d\ge A(u)}\left|\iint_{\mathbb{R}^2}\mathcal{I}_{u,d}(\boldsymbol{z})f_{u,d}(\boldsymbol{z})\mathrm{d}z_1\mathrm{d}z_2-\mathcal{B}_\alpha^2(S,0)\right| \\ &=\lim_{u\to\infty}\sup_{d\ge A(u)}\left|\iint_{\mathbb{R}^2}\left(\mathcal{I}_{u,d}(\boldsymbol{z})f_{u,d}(\boldsymbol{z})-\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}W_\alpha(\boldsymbol{s})+\boldsymbol{z}>\mathbf{0}\right\}e^{-z_1-z_2}\right)\mathrm{d}z_1\mathrm{d}z_2\right| \\ &=\lim_{u\to\infty}\sup_{d\ge A(u)}\left|\iint_{\mathbb{R}^2}\left(\mathcal{I}_{u,d}(\boldsymbol{z})f_{u,d}(\boldsymbol{z})-\mathcal{I}(\boldsymbol{z})e^{-z_1-z_2}\right)\mathrm{d}z_1\mathrm{d}z_2\right|=0, \end{aligned} \tag{6.2}$$

with

$$\begin{aligned} \mathcal{I}(\boldsymbol{z}) &:= \mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}W_\alpha(\boldsymbol{s})+\boldsymbol{z}>\mathbf{0}\right\} && (6.3)\\ &= \mathbb{P}\left\{\sup_{s\in[0,S]}\sqrt{2}B_\alpha(s)-s^\alpha>-z_1\right\}\mathbb{P}\left\{\sup_{s\in[0,S]}\sqrt{2}B_\alpha(s)-s^\alpha>-z_2\right\}. && (6.4) \end{aligned}$$

For any $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{D}$,

$$\begin{aligned}
&\mathrm{Cov}(\chi_{u,d}(\boldsymbol{s}), \chi_{u,d}(\boldsymbol{t})) \\
&= u^2 \mathrm{Cov}\big(\xi_{u,d}(\boldsymbol{s}) - R_{u,d}(\boldsymbol{s}, \boldsymbol{0}) R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}) \xi_{u,d}(\boldsymbol{0}),\ \xi_{u,d}(\boldsymbol{t}) - R_{u,d}(\boldsymbol{t},\boldsymbol{0}) R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}) \xi_{u,d}(\boldsymbol{0})\big) \\
&= u^2 \{R_{u,d}(\boldsymbol{s},\boldsymbol{t}) - R_{u,d}(\boldsymbol{s},\boldsymbol{0}) R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}) R_{u,d}(\boldsymbol{0},\boldsymbol{t})\} \\
&= u^2 \{(R_{u,d}(\boldsymbol{s},\boldsymbol{t}) - E) + (E - R_{u,d}(\boldsymbol{s},\boldsymbol{0})) + R_{u,d}(\boldsymbol{s},\boldsymbol{0})(E - R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}) R_{u,d}(\boldsymbol{0},\boldsymbol{t}))\} \qquad (6.5)
\end{aligned}$$

where E is the 2×2 identity matrix.

Since $A(u) > 1$ satisfying (6.1) tends to ∞ as $u \to \infty$, then by **A3** we have

$$\lim_{u\to\infty} \sup_{d > A(u), \boldsymbol{s} \in \boldsymbol{D}} |R_{u,d}(\boldsymbol{s},\boldsymbol{0}) - E| = \boldsymbol{0} \qquad (6.6)$$

and

$$\lim_{u\to\infty} u^2 \sup_{d \ge A(u)} |r(d)| \le \lim_{u\to\infty} \frac{u^2}{\log A(u)} \sup_{d \ge A(u)} |r(d)| \log d = 0. \qquad (6.7)$$

Therefore,

$$\lim_{u\to\infty} \sup_{d \ge A(u)} u^2 (E - R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})) = \lim_{u\to\infty} \sup_{d \ge A(u)} \frac{u^2 r(d)}{1 - r^2(d)} \begin{pmatrix} -r(d) & 1 \\ 1 & -r(d) \end{pmatrix} = \boldsymbol{0}. \qquad (6.8)$$

Note that, by **A1**, (2.1) and the Uniform Convergence Theorem (see, e.g., Theorem 1.5.2 in Bingham et al., 1989), we get

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{s \in [0,S]} \left| u^2 (1 - r(s/v(u))) - s^\alpha \right| \\
&\le \lim_{u\to\infty} \sup_{s \in [0,S]} \left| \frac{1 - r(s/v(u))}{1 - r(1/v(u))} - s^\alpha \right| + \lim_{u\to\infty} \sup_{s \in [0,S]} \left| \frac{1 - r(s/v(u))}{1 - r(1/v(u))} \right| \left| u^2 (1 - r(1/v(u))) - 1 \right| \\
&= 0.
\end{aligned}$$

Consequently,

$$\lim_{u\to\infty} \sup_{d \ge A(u), \boldsymbol{s},\boldsymbol{t} \in \boldsymbol{D}} \left| u^2 (E - R_{u,d}(\boldsymbol{s},\boldsymbol{t})) - \begin{pmatrix} |t_1 - s_1|^\alpha & 0 \\ 0 & |t_2 - s_2|^\alpha \end{pmatrix} \right| = \boldsymbol{0}. \qquad (6.9)$$

Similarly,

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{d \ge A(u), \boldsymbol{s},\boldsymbol{t} \in \boldsymbol{D}} \left| u^2 R_{u,d}(\boldsymbol{s},\boldsymbol{0}) \big(E - R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}) R_{u,d}(\boldsymbol{0},\boldsymbol{t})\big) - \begin{pmatrix} |t_1|^\alpha & 0 \\ 0 & |t_2|^\alpha \end{pmatrix} \right| \\
&\le \lim_{u\to\infty} \sup_{d \ge A(u), \boldsymbol{s},\boldsymbol{t} \in \boldsymbol{D}} \left| u^2 (E - R_{u,d}(\boldsymbol{0},\boldsymbol{t})) - \begin{pmatrix} |t_1|^\alpha & 0 \\ 0 & |t_2|^\alpha \end{pmatrix} \right| \\
&\quad + \lim_{u\to\infty} \sup_{d \ge A(u), \boldsymbol{s},\boldsymbol{t} \in \boldsymbol{D}} \left| (R_{u,d}(\boldsymbol{s},\boldsymbol{0}) - E)[u^2 (E - R_{u,d}(\boldsymbol{0},\boldsymbol{t}))] \right| \\
&\quad + \lim_{u\to\infty} \sup_{d \ge A(u), \boldsymbol{s},\boldsymbol{t} \in \boldsymbol{D}} \left| R_{u,d}(\boldsymbol{s},\boldsymbol{0}) [u^2 (E - R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}))] R_{u,d}(\boldsymbol{0},\boldsymbol{t}) \right| \\
&= \boldsymbol{0}, \qquad (6.10)
\end{aligned}$$

where in the last equality we have used (6.6), (6.8) and (6.9).

Substituting (6.9) and (6.10) into (6.5) gives

$$\lim_{u\to\infty}\sup_{\substack{d\ge A(u),\\ \boldsymbol{s},\boldsymbol{t}\in\boldsymbol{D}}}\left|\mathrm{Cov}(\chi_{u,d}(\boldsymbol{s}),\chi_{u,d}(\boldsymbol{t}))-\begin{pmatrix}|s_1|^\alpha+|t_1|^\alpha-|t_1-s_1|^\alpha & 0\\ 0 & |s_2|^\alpha+|t_2|^\alpha-|t_2-s_2|^\alpha\end{pmatrix}\right|=\boldsymbol{0}. \quad (6.11)$$

Hence, the finite-dimensional distributions of $\chi_{u,d}$ converge to that of $\{\sqrt{2}B_\alpha(\boldsymbol{s}),\boldsymbol{s}\in\boldsymbol{D}\}$ uniformly with respect to $d\ge A(u)$, where $B_\alpha(\boldsymbol{s})=(B_\alpha^{(1)}(s_1),B_\alpha^{(2)}(s_2))^T$.

Let $C(\boldsymbol{D})$ denote the Banach space of all continuous functions on $\boldsymbol{D}$ equipped with sup-norm, we now show that the measures on $C(\boldsymbol{D})$ induced by $\{\chi_{u,d}(\boldsymbol{s}),\boldsymbol{s}\in\boldsymbol{D},d\ge A(u)\}$ are uniformly tight for large u. In fact, since $\mathbb{E}\left\{\xi_{u,d}(\boldsymbol{s})|\xi_{u,d}(\boldsymbol{0})\right\}=R_{u,d}(\boldsymbol{s},\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})$ then

$$\begin{aligned}
&\mathbb{E}\left\{(\xi_{u,d}(\boldsymbol{s})-\xi_{u,d}(\boldsymbol{t}))^T\left(R_{u,d}(\boldsymbol{s},\boldsymbol{0})-R_{u,d}(\boldsymbol{t},\boldsymbol{0})\right)R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})\right\}\\
&=\mathbb{E}\left\{\mathbb{E}\left\{(\xi_{u,d}(\boldsymbol{s})-\xi_{u,d}(\boldsymbol{t}))^T|\xi_{u,d}(\boldsymbol{0})\right\}\left(R_{u,d}(\boldsymbol{s},\boldsymbol{0})-R_{u,d}(\boldsymbol{t},\boldsymbol{0})\right)R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})\right\}\\
&=\mathbb{E}\left\{\xi_{u,d}^T(\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})(R_{u,d}(\boldsymbol{0},\boldsymbol{s})-R_{u,d}(\boldsymbol{0},\boldsymbol{t}))(R_{u,d}(\boldsymbol{s},\boldsymbol{0})-R_{u,d}(\boldsymbol{t},\boldsymbol{0}))R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})\right\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\mathbb{E}\left\{\|\chi_{u,d}(\boldsymbol{s})-\chi_{u,d}(\boldsymbol{t})\|^2\right\}\\
&=u^2\Big[\mathbb{E}\left\{\|\xi_{u,d}(\boldsymbol{s})-\xi_{u,d}(\boldsymbol{t})\|^2\right\}\\
&\quad-2\mathbb{E}\left\{(\xi_{u,d}(\boldsymbol{s})-\xi_{u,d}(\boldsymbol{t}))^T(R_{u,d}(\boldsymbol{s},\boldsymbol{0})-R_{u,d}(\boldsymbol{t},\boldsymbol{0}))R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})\right\}\\
&\quad+\mathbb{E}\left\{\xi_{u,d}^T(\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})(R_{u,d}(\boldsymbol{0},\boldsymbol{s})-R_{u,d}(\boldsymbol{0},\boldsymbol{t}))(R_{u,d}(\boldsymbol{s},\boldsymbol{0})-R_{u,d}(\boldsymbol{t},\boldsymbol{0}))R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})\right\}\Big]\\
&=u^2\Big[\mathbb{E}\left\{\|\xi_{u,d}(\boldsymbol{s})-\xi_{u,d}(\boldsymbol{t})\|^2\right\}-\mathbb{E}\left\{\|(R_{u,d}(\boldsymbol{s},\boldsymbol{0})-R_{u,d}(\boldsymbol{t},\boldsymbol{0}))R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\xi_{u,d}(\boldsymbol{0})\|^2\right\}\Big]\\
&\le u^2E\|\xi_{u,d}(\boldsymbol{s})-\xi_{u,d}(\boldsymbol{t})\|^2\\
&=2u^2[1-r(|t_1-s_1|/v(u))+1-r(|t_2-s_2|/v(u))].
\end{aligned}$$

Moreover, by (2.1) and Potter's Theorem (see, e.g., Theorem 1.5.6 in Bingham et al., 1989), there exists some constant $C>1$ such that

$$u^2(1-r(s/v(u)))=u^2(1-r(1/v(u)))\frac{1-r(s/v(u))}{1-r(1/v(u))}\le C|s|^{\alpha/2}$$

holds for all $s\in[0,S]$ and all large enough u. Hence, for large enough u we get

$$\sup_{d\ge A(u)}\mathbb{E}\left\{\|\chi_{u,d}(\boldsymbol{s})-\chi_{u,d}(\boldsymbol{t})\|^2\right\}\le 2C(|t_1-s_1|^{\alpha/2}+|t_2-s_2|^{\alpha/2}) \quad (6.12)$$

for any $\boldsymbol{s},\boldsymbol{t}\in\boldsymbol{D}$, implying the uniform tightness of the measures induced by $\{\chi_{u,d}(\boldsymbol{s}),\boldsymbol{s}\in\boldsymbol{D},d\ge A(u)\}$. This together with (6.11) implies that $\chi_{u,d}$ converges weakly, as $u\to\infty$, to $\sqrt{2}B_\alpha(\boldsymbol{s}),\boldsymbol{s}\in\boldsymbol{D}$ uniformly for $d\ge A(u)$.

Further, by (6.8) and (6.9) we have

$$\lim_{u\to\infty}\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|u^2\big(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\big)-\begin{pmatrix}|s_1|^\alpha & 0\\ 0 & |s_2|^\alpha\end{pmatrix}\right| \tag{6.13}$$
$$\le \lim_{u\to\infty}\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|u^2(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0}))-\begin{pmatrix}|s_1|^\alpha & 0\\ 0 & |s_2|^\alpha\end{pmatrix}\right|$$
$$+\lim_{u\to\infty}\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|R_{u,d}(\boldsymbol{s},\boldsymbol{0})[u^2(E-R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0}))]\right|$$
$$=\boldsymbol{0}$$

and thus for any $\boldsymbol{z}\in\mathbb{R}^2$

$$\lim_{u\to\infty}\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|\theta_{u,d}(\boldsymbol{s})-(|s_1|^\alpha-z_1,|s_2|^\alpha-z_2)^T\right|$$
$$\le\lim_{u\to\infty}\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|\left[u^2\big(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\big)-\begin{pmatrix}|s_1|^\alpha & 0\\ 0 & |s_2|^\alpha\end{pmatrix}\right]\begin{bmatrix}1\\1\end{bmatrix}\right|$$
$$+\quad\lim_{u\to\infty}\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|\big(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\big)\boldsymbol{z}\right|$$
$$=\boldsymbol{0}.$$

Therefore, for each $\boldsymbol{z}\in\mathbb{R}^2$, the probability measures on $C(\boldsymbol{D})$ induced by $\{\chi_{u,d}(\boldsymbol{s})-\theta_{u,d}(\boldsymbol{s},\boldsymbol{z}),\boldsymbol{s}\in\boldsymbol{D}\}$ converge weakly, as $u\to\infty$, to that induced by $\{W_\alpha(\boldsymbol{s})+\boldsymbol{z},\boldsymbol{s}\in\boldsymbol{D}\}$ uniformly with respect to $d\ge A(u)$. Then, by the continuous mapping theorem, (6.4) and the fact that the set of discontinuity points of cumulative distribution function of $\sup_{s\in[0,S]}\sqrt{2}B_\alpha(s)-s^\alpha$ consists of at most of one point (see, e.g., Theorem 7.1 in Azaïs and Wschebor, 2009 or related Lemma 4.4 in Dębicki et al., 2020a), we get

$$\lim_{u\to\infty}\sup_{d\ge A(u)}|\mathcal{I}_{u,d}(\boldsymbol{z})-\mathcal{I}(\boldsymbol{z})|=0$$

for almost all $\boldsymbol{z}\in\mathbb{R}^2$, where $\mathcal{I}(\boldsymbol{z})$ is defined in (6.3). Further, by (6.7) we know

$$\lim_{u\to\infty}\sup_{d\ge A(u)}\left|f_{u,d}(\boldsymbol{z})-e^{-z_1-z_2}\right|=0,\quad\forall\,\boldsymbol{z}\in\mathbb{R}^2,$$

and thus for almost all $\boldsymbol{z}\in\mathbb{R}^2$

$$\lim_{u\to\infty}\sup_{d\ge A(u)}\left|\mathcal{I}_{u,d}(\boldsymbol{z})f_{u,d}(\boldsymbol{z})-\mathcal{I}(\boldsymbol{z})e^{-z_1-z_2}\right|=0. \tag{6.14}$$

Therefore, to verify (6.2), we have to put the limit into integral. In the following, we look for an integrable upper bound for $\sup_{d\ge A(u)}\mathcal{I}_{u,d}(\boldsymbol{z})f_{u,d}(\boldsymbol{z})$. We first give a lower bound for $\inf_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\theta_{u,d}(\boldsymbol{s},\boldsymbol{z})$. Let $\varepsilon(<1/2)$ be a positive constant. In view of (6.13), we know that, for sufficiently large u

$$\sup_{d\ge A(u),\boldsymbol{s}\in\boldsymbol{D}}\left|E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R_{u,d}^{-1}(\boldsymbol{0},\boldsymbol{0})\right|\le\begin{pmatrix}\varepsilon&\varepsilon\\ \varepsilon&\varepsilon\end{pmatrix},$$

and thus

$$\begin{aligned}\inf_{\substack{d\geq A(u),\\ \boldsymbol{s}\in\boldsymbol{D}}}\theta_{u,d}(\boldsymbol{s},\boldsymbol{z}) &= \inf_{d\geq A(u),\boldsymbol{s}\in\boldsymbol{D}}\{u^2(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R^{-1}_{u,d}(\boldsymbol{0},\boldsymbol{0}))\boldsymbol{1}+(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R^{-1}_{u,d}(\boldsymbol{0},\boldsymbol{0}))\boldsymbol{z}-\boldsymbol{z}\}\\ &\geq -\boldsymbol{1}-\boldsymbol{z}+\inf_{d\geq A(u),\boldsymbol{s}\in\boldsymbol{D}}\{(E-R_{u,d}(\boldsymbol{s},\boldsymbol{0})R^{-1}_{u,d}(\boldsymbol{0},\boldsymbol{0}))\boldsymbol{z}\}\\ &\geq -\boldsymbol{1}-\boldsymbol{z}-\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & \varepsilon\end{pmatrix}|\boldsymbol{z}| := h(\boldsymbol{z}), \quad \boldsymbol{z}\in\mathbb{R}^2.\end{aligned}$$

Let $\{\boldsymbol{e}_k, k=1,2,3\}$ denotes $(1,1)^T$, $(0,1)^T$ and $(1,0)^T$, respectively. By Cauchy-Schwartz inequality and (6.12), for large enough u

$$\begin{aligned}\sup_{d\geq A(u)}\mathbb{E}\left\{\left(\boldsymbol{e}_k^T(\chi_{u,d}(\boldsymbol{s})-\chi_{u,d}(\boldsymbol{t}))\right)^2\right\} &\leq \sup_{d\geq A(u)} 2\mathbb{E}\left\{\|\chi_{u,d}(\boldsymbol{s})-\chi_{u,d}(\boldsymbol{t})\|^2\right\}\\ &\leq 4C(|t_1-s_1|^{\alpha/2}+|t_2-s_2|^{\alpha/2}), \quad k=1,2,3 \quad (6.15)\end{aligned}$$

holds for any $\boldsymbol{s},\boldsymbol{t}\in\boldsymbol{D}$. Thus, by Sudakov-Fernique inequality (see, e.g., Theorem 2.9 in Adler, 1990), we have

$$\sup_{d\geq A(u)}\mathbb{E}\left\{\sup_{\boldsymbol{s}\in\boldsymbol{D}}\boldsymbol{e}_k^T\chi_{u,d}(\boldsymbol{s})\right\}\leq\mathbb{E}\left\{\sup_{\boldsymbol{s}\in\boldsymbol{D}}\sum_{i=1}^{2}2\sqrt{C}B^{(i)}_{\alpha/2}(s_i)\right\} := C_1<\infty, \quad k=1,2,3, \tag{6.16}$$

where $B^{(i)}_{\alpha/2}$'s are independent fBm's with Hurst index $\alpha/4$. Then, for all large enough u,

$$\begin{aligned}\sup_{d\geq A(u)}\mathcal{I}_{u,d}(\boldsymbol{z}) &= \sup_{d\geq A(u)}\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\chi_{u,d}(\boldsymbol{s})-\theta_{u,d}(\boldsymbol{s},\boldsymbol{z})>\boldsymbol{0}\right\}\\ &\leq \sup_{d\geq A(u)}\mathbb{P}\left\{\exists_{\boldsymbol{s}\in\boldsymbol{D}}\chi_{u,d}(\boldsymbol{s})>\inf_{d\geq A(u),\boldsymbol{s}\in\boldsymbol{D}}\theta_{u,d}(\boldsymbol{s},\boldsymbol{z})\right\}\\ &\leq \sup_{d\geq A(u)}\mathbb{P}\left\{\sup_{\boldsymbol{s}\in\boldsymbol{D}}\boldsymbol{e}_k^T\chi_{u,d}(\boldsymbol{s})>\boldsymbol{e}_k^T h(\boldsymbol{z})\right\}\\ &\leq \sup_{d\geq A(u)}\exp\left(-\frac{\left(\boldsymbol{e}_k^T h(\boldsymbol{z})-\mathbb{E}\left\{\sup_{\boldsymbol{s}\in\boldsymbol{D}}\boldsymbol{e}_k^T\chi_{u,d}(\boldsymbol{s})\right\}\right)^2}{2\mathrm{Var}_{\boldsymbol{s}\in\boldsymbol{D}}\boldsymbol{e}_k^T\chi_{u,d}(\boldsymbol{s})}\right) \quad (6.17)\\ &\leq \exp\left(-C_2\left(\boldsymbol{e}_k^T h(\boldsymbol{z})-C_1\right)^2\right), \quad \boldsymbol{z}\in\boldsymbol{Z}_k, k=1,2,3,\end{aligned}$$

where (6.17) follows from Borell-TIS inequality (see, e.g., Theorem 2.1.1 in Adler and Taylor, 2007), the last inequality follows from (6.15)-(6.16) with $C_2=(16CS^{\alpha/2})^{-1}$, and

$$\boldsymbol{Z}_1=\{(z_1,z_2)|z_1<0, z_2<0, (2\varepsilon-1)(z_1+z_2)>2+C_1\},$$

$$\boldsymbol{Z}_2=\{(z_1,z_2)|z_1>0, z_2<0, (\varepsilon-1)z_2-\varepsilon z_1>1+C_1\},$$

$$\boldsymbol{Z}_3=\{(z_1,z_2)|z_1<0, z_2>0, (\varepsilon-1)z_1-\varepsilon z_2>1+C_1\}.$$

Therefore,

$$\sup_{d\ge A(u)} \mathcal{I}_{u,d}(\boldsymbol{z}) \le g(\boldsymbol{z}) := \begin{cases} \exp\left(-C_2\left(\boldsymbol{e}_k^T h(\boldsymbol{z}) - C_1\right)^2\right), & \boldsymbol{z}\in \boldsymbol{Z}_k, k=1,2,3,\\ 1, & \boldsymbol{z}\in\mathbb{R}^2\setminus\bigcup_{k=1}^3 \boldsymbol{Z}_k,\end{cases}$$

holds for sufficiently large u. Moreover, by (6.7)

$$\begin{aligned}
&\sup_{d\ge A(u)} f_{u,d}(\boldsymbol{z})e^{z_1+z_2}\\
&= \sup_{d\ge A(u)} \frac{1}{\sqrt{1-r^2(d)}}\exp\left(\frac{u^2r(d)}{1+r(d)} + \frac{2u^2r(d)(1-r(d))(z_1+z_2)-(z_1^2-2r(d)z_1z_2+z_2^2)}{2u^2(1-r^2(d))}\right)\\
&\le \frac{3}{2}\sup_{d\ge A(u)}\exp\left(\frac{u^2r(d)}{1+r(d)} + \frac{-\frac{1+r(d)}{2}(z_2-z_1)^2-\frac{1-r(d)}{2}(z_1+z_2-2u^2r(d))^2+2u^4r^2(d)(1-r(d))}{2u^2(1-r^2(d))}\right)\\
&\le \frac{3}{2}\sup_{d\ge A(u)} e^{u^2r(d)} \le 2, \quad \boldsymbol{z}\in\mathbb{R}^2
\end{aligned}$$

holds for all large enough u, and thus

$$\sup_{d\ge A(u)} \mathcal{I}_{u,d}(\boldsymbol{z}) f_{u,d}(\boldsymbol{z}) \le 2g(\boldsymbol{z})e^{-z_1-z_2}, \quad \boldsymbol{z}\in\mathbb{R}^2.$$

We now show that $g(\boldsymbol{z})e^{-z_1-z_2}$ is integrable on $\mathbb{R}^2$. In fact,

$$\iint_{\mathbb{R}^2} g(\boldsymbol{z})e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2 = \left(\iint_{\boldsymbol{Z}_1}+\iint_{\boldsymbol{Z}_2}+\iint_{\boldsymbol{Z}_3}+\iint_{\mathbb{R}^2\setminus\bigcup_{k=1}^3\boldsymbol{Z}_k}\right) g(\boldsymbol{z})e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2,$$

where

$$\begin{aligned}
&\iint_{\boldsymbol{Z}_1} g(\boldsymbol{z})e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2\\
&\qquad\le \int_{-\infty}^0\int_{-\infty}^0 \exp\left(-C_2\left((2\varepsilon-1)z_1+(2\varepsilon-1)z_2-2-C_1\right)^2-z_1-z_2\right)\mathrm{d}z_1\mathrm{d}z_2\\
&\qquad\le \left(\int_{-\infty}^0 \exp\left(-C_2(2\varepsilon-1)^2z_1^2+\left(2C_2(2\varepsilon-1)(2+C_1)-1\right)z_1\right)\mathrm{d}z_1\right)^2\\
&\qquad< \infty,
\end{aligned}$$

$$\begin{aligned}
&\iint_{\boldsymbol{Z}_2} g(\boldsymbol{z})e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2\\
&\qquad= \iint_{\boldsymbol{Z}_3} g(\boldsymbol{z})e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2\\
&\qquad= \int_0^\infty\left(\int_{-\infty}^{\frac{\varepsilon z_1+1+C_1}{(\varepsilon-1)}} \exp\left(-C_2\left((\varepsilon-1)z_2-\varepsilon z_1-1-C_1\right)^2-z_2\right)\mathrm{d}z_2\right)e^{-z_1}\mathrm{d}z_1\\
&\qquad= \frac{e^{\frac{1+C_1}{1-\varepsilon}}}{1-\varepsilon}\int_0^\infty e^{(\frac{\varepsilon}{1-\varepsilon}-1)z_1}\mathrm{d}z_1\int_0^\infty \exp\left(-C_2z_2^2-\frac{z_2}{\varepsilon-1}\right)\mathrm{d}z_2<\infty,
\end{aligned}$$

since $\varepsilon < 1/2$, and

$$\iint_{\mathbb{R}^2\setminus\bigcup_{k=1}^3 \boldsymbol{Z}_k} g(\boldsymbol{z})e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2 \le \left(\iint_{z_1<0,z_2<0,z_1+z_2\ge\frac{2+C_1}{2\varepsilon-1}} +2\int_0^\infty\int_{\frac{\varepsilon z_1+C_1+1}{\varepsilon-1}}^\infty\right) e^{-z_1-z_2}\mathrm{d}z_1\mathrm{d}z_2$$
$$\le \left(\frac{2+C_1}{1-2\varepsilon}\right)^2 e^{\frac{2+C_1}{1-2\varepsilon}} + 2e^{\frac{1+C_1}{1-\varepsilon}}\int_0^\infty e^{(\frac{\varepsilon}{1-\varepsilon}-1)z_1}\mathrm{d}z_1 < \infty.$$

Consequently, (6.2) follows from the Dominated Convergence Theorem and (6.14). This completes the proof. □

Lemma 6.2. *Let $X(t), t \ge 0$ be a centered stationary Gaussian process with unit variance and covariance function satisfying* **A1** *and* **A3***. Let $v(u)$, $\mathcal{B}_\alpha(x)$ and $m(u)$ be defined in (2.1), (2.3) and (2.5) respectively. Then for $A(u)$ such that*

$$\lim_{u\to\infty} A(u)v(u) = \infty \quad \textit{and} \quad \lim_{u\to\infty}\frac{A(u)}{m(u)} = 0 \tag{6.18}$$

and any $x \ge 0$ we have

$$\mathbb{P}\left\{L_u^*[0, A(u)] > x\right\} \sim \mathcal{B}_\alpha(x)A(u)v(u)\Psi(u), \quad u\to\infty. \tag{6.19}$$

Proof: We follow the argument used in the proof of Theorem 2.1 in Dębicki et al. (2019). Let $A(u)$ satisfy (6.18), for any $S > 1$ define

$$\Delta_k = [kS/v(u), (k+1)S/v(u)], \quad k = 0,\dots,N_u$$

with $N_u = \lfloor A(u)v(u)/S\rfloor$, i.e., the integer part of $A(u)v(u)/S$. By stationarity of X, we have for all u positive and $x \ge 0$

$$I_1(u) \le \mathbb{P}\left\{L_u^*[0, A(u)] > x\right\} \le I_2(u),$$

where

$$I_1(u) = (N_u - 1)\mathbb{P}\left\{L_u^*\Delta_0 > x\right\} - \sum_{0\le i<k\le N_u-1} q_{i,k}(u),$$
$$I_2(u) = (N_u + 1)\mathbb{P}\left\{L_u^*\Delta_0 > x\right\} + \sum_{0\le i<k\le N_u} q_{i,k}(u),$$

with $q_{i,k}(u) = \mathbb{P}\left\{\sup_{t\in\Delta_i} X(t) > u, \sup_{t\in\Delta_k} X(t) > u\right\}$. By Theorem 5.1 in Dębicki et al. (2019) and (2.3), we have

$$\lim_{S\to\infty}\lim_{u\to\infty}\frac{N_u\mathbb{P}\left\{L_u^*\Delta_0 > x\right\}}{A(u)v(u)\Psi(u)} = \mathcal{B}_\alpha(x) \tag{6.20}$$

for any $x \ge 0$. Therefore, it suffices to show that the double sum is negligible with respect to $A(u)v(u)\Psi(u)$ as $u\to\infty$ and then as $S\to\infty$.

Let $\varepsilon^*(<2)$ be the positive root of equation $x^2-(2-\alpha)x-\frac{3}{2}\alpha = 0$ and put $\beta = \inf_{t\ge1}\{1-r(t)\}$, which by **A3** is positive. Define

$$A_0(u) = 0,\ A_1(u) = u^{\frac{\varepsilon^*-2}{\alpha}} \wedge A(u),\ A_2(u) = 1\wedge A(u),\ A_3(u) = e^{\beta u^2/8}\wedge A(u),\ A_4(u) = A(u)$$

and

$$\Lambda_l(u) = \{(i,k) : 1 \le i+1 < k \le N_u, A_{l-1}(u) < (k-i-1)S/v(u) \le A_l(u),\ l = 1,2,3,4\}.$$

Then

$$\begin{aligned}\sum_{0\le i<k\le N_u} q_{i,k}(u) &= \sum_{0\le i<N_u} q_{i,i+1}(u) + \sum_{l=1}^{4}\sum_{(i,k)\in\Lambda_l(u)} q_{i,k}(u) \\ &:= Q_0(u) + \sum_{l=1}^{4} Q_l(u).\end{aligned} \tag{6.21}$$

According to (4.7)-(4.9) in Dębicki et al. (2019) we know that

$$\limsup_{u\to\infty} \frac{Q_2(u)}{A(u)v(u)\Psi(u)} = 0, \tag{6.22}$$

$$\lim_{S\to\infty}\limsup_{u\to\infty} \frac{Q_1(u)}{A(u)v(u)\Psi(u)} = 0, \tag{6.23}$$

and

$$\lim_{S\to\infty}\limsup_{u\to\infty} \frac{Q_0(u)}{A(u)v(u)\Psi(u)} = 0. \tag{6.24}$$

Next, without loss of generality we suppose that $A(u) > 1$ (since otherwise, $Q_3(u) = 0$). Then by the stationarity of X, for sufficiently large u

$$\sup_{(i,k)\in\Lambda_3(u)} \mathbb{E}\left\{\sup_{s\in\Delta_i,t\in\Delta_k}(X(s)+X(t))\right\} \le 2\mathbb{E}\left\{\sup_{s\in[0,1]} X(s)\right\} =: C_3 < \infty,$$

$$\begin{aligned}\sup_{(i,k)\in\Lambda_3(u),s\in\Delta_i,t\in\Delta_k} \mathrm{Var}(X(s)+X(t)) &= 4 - 2\inf_{(i,k)\in\Lambda_3(u),(s,t)\in\Delta_i\times\Delta_k}\{1-r(t-s)\} \\ &\le 4-2\beta.\end{aligned}$$

The last inequality above follows from

$$\inf_{(i,k)\in\Lambda_3(u),(s,t)\in\Delta_i\times\Delta_k}\{1-r(t-s)\} \ge \inf_{(i,k)\in\Lambda_3(u),t\ge(k-i-1)S/v(u)}\{1-r(t)\} \ge \beta,$$

where we used that $(k-i-1)S/v(u) > 1$ for $(i,k)\in\Lambda_3(u)$. Then, by Borell-TIS inequality we have for large enough u

$$\begin{aligned}\sup_{(i,k)\in\Lambda_3(u)} q_{i,k}(u) &\le \sup_{(i,k)\in\Lambda_3(u)} \mathbb{P}\left\{\sup_{s\in\Delta_i,t\in\Delta_k} X(s)+X(t) > 2u\right\} \\ &\le \exp\left(-\frac{(2u-C_3)^2}{2(4-2\beta)}\right) \\ &\le \exp\left(-\frac{1+\beta/2}{2}(u-C_3/2)^2\right),\end{aligned}$$

and thus

$$\begin{aligned}\limsup_{u\to\infty}\frac{Q_3(u)}{A(u)v(u)\Psi(u)} &\le \limsup_{u\to\infty}\frac{N_u A_3(u)v(u)}{SA(u)v(u)\Psi(u)}\exp\left(-\frac{1+\beta/2}{2}(u-C_3/2)^2\right)\\ &\le \limsup_{u\to\infty}\frac{\sqrt{2\pi}uv(u)}{S^2}\exp\left(-\frac{\beta}{8}u^2+C_3u(1+\beta/2)\right)=0. \qquad (6.25)\end{aligned}$$

Further, since $e^{\beta u^2/8}$ satisfies (6.1), then by Lemma 6.1 and stationarity of X,

$$Q_4(u)\le 2N_u^2\Psi^2(u)\mathcal{B}_\alpha^2(S,0)$$

holds for u sufficiently large. Therefore,

$$\begin{aligned}\limsup_{u\to\infty}\frac{Q_4(u)}{A(u)v(u)\Psi(u)} &\le \limsup_{u\to\infty}\frac{2N_u^2\Psi^2(u)\mathcal{B}_\alpha^2(S,0)}{A(u)v(u)\Psi(u)}\\ &\le \limsup_{u\to\infty}\frac{2\mathcal{B}_\alpha^2(S,0)}{S^2\mathcal{B}_\alpha(0)}\frac{A(u)}{m(u)}=0, \qquad (6.26)\end{aligned}$$

where the last equality follows from (6.18).

Consequently, substituting (6.22)-(6.26) into (6.21) yields

$$\lim_{S\to\infty}\limsup_{u\to\infty}\frac{1}{A(u)v(u)\Psi(u)}\sum_{0\le i<k\le N_u}q_{i,k}(u)=0,$$

which together with (6.20) completes the proof. □

Corollary 6.3. *If X, $v(u)$, $\mathcal{B}_\alpha(x)$, $m(u)$ and $A(u)$ are given as in Lemma 6.2, then for any $x\ge 0$ and $\varepsilon\in(0,1)$ there exists $\delta>0$ such that*

$$\liminf_{u\to\infty}\inf_{t\in[A(u),\delta m(u)]}\frac{\mathbb{P}\left\{L_u^*[0,t]>x\right\}}{t\mathcal{B}_\alpha(x)v(u)\Psi(u)}\ge 1-\varepsilon \qquad (6.27)$$

and

$$\limsup_{u\to\infty}\sup_{t\in[A(u),\delta m(u)]}\frac{\mathbb{P}\left\{L_u^*[0,t]>x\right\}}{t\mathcal{B}_\alpha(x)v(u)\Psi(u)}\le 1+\varepsilon. \qquad (6.28)$$

Proof: Let $x\ge 0$ be fixed, recalling (5.2) we have that, for arbitrary $\varepsilon\in(0,1)$, there exists some $\delta>0$ such that

$$(1-\varepsilon/4)\le\frac{\mathbb{P}\left\{Y(t)>x\right\}}{t\overline{\mathcal{F}_\alpha}(x)}\le(1+\varepsilon/4),\quad t\in(0,\delta). \qquad (6.29)$$

For such ε and δ, suppose that (6.27) does not hold. Then, there exist two sequences $\{u_n,n\in\mathbb{N}\}$ and $\{t_n,n\in\mathbb{N}\}$ such that $u_n\to\infty$ as $n\to\infty$ and

$$\frac{\mathbb{P}\left\{L_{u_n}^*[0,t_n]>x\right\}}{t_n\mathcal{B}_\alpha(x)v(u_n)\Psi(u_n)}<1-\varepsilon,\quad t_n\in[A(u_n),\delta m(u_n)],n\in\mathbb{N}. \qquad (6.30)$$

Putting $\hat{t}_n=t_n/m(u_n)$, by (2.5) and (3.4), we get

$$\frac{\mathbb{P}\left\{L_{u_n}^*[0,\hat{t}_n m(u_n)]>x\right\}}{\hat{t}_n\overline{\mathcal{F}_\alpha}(x)}<1-\varepsilon,\quad \hat{t}_n\in[A(u_n)/m(u_n),\delta],n\in\mathbb{N}. \qquad (6.30')$$

Since sequence $\{\hat{t}_n, n \in \mathbb{N}\}$ is bounded, then there exists a convergent subsequence $\{\hat{t}_{n_k}, k \in \mathbb{N}\}$ such that $\lim_{k\to\infty} \hat{t}_{n_k} \geq 0$. If $\lim_{k\to\infty} \hat{t}_{n_k} > 0$, then by Corollary 5.2

$$\frac{\mathbb{P}\left\{L^*_{u_{n_k}}[0, \hat{t}_{n_k} m(u_{n_k})] > x\right\}}{\mathbb{P}\left\{Y(\hat{t}_{n_k}) > x\right\}} > 1 - \varepsilon/4$$

holds for sufficiently large k, which together with (6.29) implies

$$\frac{\mathbb{P}\left\{L^*_{u_{n_k}}[0, \hat{t}_{n_k} m(u_{n_k})] > x\right\}}{\hat{t}_{n_k}\overline{\mathcal{F}_\alpha}(x)} > (1 - \varepsilon/4)^2.$$

This however contradicts (6.30′). If $\lim_{k\to\infty} \hat{t}_{n_k} = 0$, then

$$\lim_{k\to\infty} t_{n_k} v(u_{n_k}) \geq \lim_{k\to\infty} A(u_{n_k}) v(u_{n_k}) = \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{t_{n_k}}{m(u_{n_k})} = \lim_{k\to\infty} \hat{t}_{n_k} = 0,$$

and thus by Lemma 6.2

$$\frac{\mathbb{P}\left\{L^*_{u_{n_k}}[0, t_{n_k}] > x\right\}}{t_{n_k}\mathcal{B}_\alpha(x) v(u_{n_k})\Psi(u_{n_k})} > 1 - \varepsilon/4$$

holds for sufficiently large k. This contradicts (6.30). An analogous argument can be used to verify (6.28). This completes the proof. □

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