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A note on $3d$-monochromatic random waves and cancellation

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Abstract. In this note we prove that the asymptotic variance of the nodal length of complex-valued monochromatic random waves restricted to an increasing domain in $\mathbb{R}^3$ is linear in the volume of the domain. Put together with previous results this shows that a Central Limit Theorem holds true for 3-dimensional monochromatic random waves. We compare with the variance of the nodal length of the real-valued 2-dimensional monochromatic random waves where a faster divergence rate is observed, this fact is connected with Berry's cancellation phenomenon. Moreover, we show that a concentration phenomenon takes place.

1. Introduction

During the last decade an impressive effort has been dedicated to understanding the behavior of geometric functionals of random waves in the high energy limit. Motivation often comes from Quantum Chaos as it is concerned with the behavior of Laplace eigenfunctions as the frequency or the energy level (equivalently, the eigenvalue) tends to infinity. For a general picture of the field, see the seminal papers Berry (2002); Berry and Dennis (2000), the recent reviews Wigman (2022); Marinucci (2021); Vidotto (2022) and the background section in Krishnapur et al. (2013, S.1.6) for an example of an early stage discussion on this topic.

Some instances of geometric functionals are the length of nodal (i.e: zero) curves, the number of critical points and the number of connected components of the zero sets, see e.g. Berry and Dennis (2000). Within this general framework, we focus our attention on the nodal length of complex-valued random waves defined on $\mathbb{R}^3$ and compare it to the well studied case of the nodal length of real-valued random waves defined on $\mathbb{R}^2$. The analogy between these two cases is apparent when looking at the covariance functions, the analytic expressions and the expansions of the nodal lengths. Indeed, since we use real analytic methods we can think the former random waves as $\mathbb{R}^2$-valued. See sections 4, 5 and 6.

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A central family of random fields (either real or complex-valued, defined on $\mathbb{R}^d$, $d \geq 3$), representing monochromatic waves, is the so-called Berry's Random Wave Model (RWM for short). Berry (1977) conjectured that the RWM can serve as a model for deterministic 'generic' Laplace eigenfunctions on manifolds with negative curvature in the high energy regime. Furthermore, the RWM can be thought of as a universal model since it can be proved that, in some local sense, it is the limit of other important Laplace random eigenfunctions models as Random Spherical Harmonics (RSH) (Wigman, 2010, Eq.(1.8)-(1.9)) or Arithmetic Random Waves (ARW) defined in the flat torus (Krishnapur et al., 2013, S1.6.1). See also Canzani and Hanin (2020) and Gass (2023) for the case of Riemannian manifolds and Romaniega and Sartori (2022) for some deterministic related constructions. For a discussion on these approximations, see section 1.6. in Krishnapur et al. (2013).

The complex (resp. real)-valued RWM can be defined as the complex (resp. real)-valued centered stationary isotropic Gaussian random field ψ defined on $\mathbb{R}^d$ having covariance function

$$r(x,y) = \mathbb{E}[\psi(x)\overline{\psi(y)}] = c_d \frac{J_\lambda(|x-y|)}{|x-y|^\lambda}, \tag{1.1}$$

where c_d is a normalizing constant, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$ and J_λ is the Bessel function of the first kind with index $\lambda = \frac{d-2}{2}$. Alternatively, the RWM can be defined by stating that its spectral measure is uniform on the unit sphere. In the real case, the conjugation in (1.1) shall be omitted.

More generally, set $\psi_k(x) = \psi(kx) : k \geq 1$. Note that $\psi = \psi_1$. Then, it is well known that for $k \geq 1$, ψ_k is a solution of the Helmholtz equation

$$\Delta\psi_k(x) + k^2\psi_k(x) = 0, \ x \in \mathbb{R}^d,$$

which explains the terminologies frequency, energy or eigenvalue used to refer to k^2. See section 1.1. in Nourdin et al. (2019).

It is worth mentioning that the study of the nodal sets of ψ on growing domains is equivalent to the study of the nodal sets of ψ_k restricted to a fixed domain in the high energy limit, i.e: as $k \to \infty$, see remark 3.6. in Dalmao et al. (2021) and remark 1.3 in Nourdin et al. (2019). In this note we choose to work on the former framework because our natural domain is the non-compact manifold $\mathbb{R}^3$. See the discussion in section 3 below.

The nodal sets of these models were deeply studied in the last years, though the major part of the literature focuses on the real-valued 2-dimensional case. The papers Nourdin et al. (2019) (concerning real and complex-valued 2 dimensional RWM) and Dalmao et al. (2021) (concerning complex-valued 3-dimensional RWM) are key for our work. See also the recent works Peccati and Vidotto (2020), Notarnicola et al. (2023) dealing with functional convergence on the 2-dimensional case and the reviews Marinucci (2021); Vidotto (2022); Wigman (2022).

In the real-valued 2-dimensional case the RWM soon showed a surprising behavior concerning the high energy limit of the variance of their nodal length (i.e: the length of their zero curves) : it diverges far more slowly than anticipated (similar results hold true for some related models as RSH and ARW). More precisely, while the expectation and the variance of the nodal length were expected to be of the same order as $k \to \infty$, rephrasing equations (1.7)-(1.8) in Nourdin et al. (2019), it happens that

$$\mathrm{Var}(\mathcal{L}_k(B)) \sim \ \log(\mathbb{E}[\mathcal{L}_k(B)]), \text{ as } k \to \infty,$$

where B is a fixed domain in $\mathbb{R}^2$ and $\mathcal{L}_k(B)$ is the nodal length associated to ψ_k restricted to B. This fact was first predicted by Berry (2002) where he expressed that this behavior is due to 'an obscure cancellation phenomenon'.

The first rigorous confirmations of the so called Berry's cancellation arrived in Wigman (2010) in the case of RSH, in Marinucci et al. (2016) for ARW in the flat torus and in Nourdin et al. (2019)

for the planar RWM. One important feature of the last reference is that it was the first one to deal with non-compact manifolds, which is also the case we are interested in.

The early results, obtained using Rice formulas, did not allow a clear understanding of the cancellation phenomenon, see e.g. Berry (2002); Wigman (2010); Krishnapur et al. (2013). The use of Wiener Chaos techniques, see e.g. Marinucci et al. (2016); Nourdin et al. (2019); Dalmao et al. (2019), provided a precise explanation of the cancellation phenomenon in terms of the Wiener (a.k.a. chaotic or Hermite) expansion of the nodal length which combines in a subtle manner geometric and probabilistic aspects. Roughly speaking, the nodal length restricted to a Borel set B, $\mathcal{L}(B)$ say, can be expanded in the L^2-sense in the form

$$\mathcal{L}(B) = \sum_{q=0}^{\infty} I_{2q}(B),$$

where the components $I_{2q}(B)$ are orthogonal random variables. In particular, $I_0(B)$ equals the expectation of $\mathcal{L}(B)$ and the rest of the terms are combinations of integrals involving polynomials on the underlying process and its derivative. It was observed that the second component $I_2(B)$ vanishes due to geometric reasons reflected in the coefficients of the expansion, see Notarnicola (2021) for a general treatment of this issue. The cancellation of the coefficients in the second term in the expansion avoids the integrals appearing in the computation of the variance of $I_2(B)$ to produce higher rates of divergence, as those which do take place in the case of non-zero levels, see e.g. Marinucci et al. (2016). See section 3 below.

In Dalmao et al. (2021) complex-valued 3-dimensional Berry's general model in growing domains was considered. In particular, for monochromatic random waves (verifying (1.1) with $d = 3$ or, equivalently, verifying (2.2) below) it was shown that the variance of the second component in the Wiener expansion vanishes asymptotically and that the variance of $\mathcal{L}(B)$ grows at most linearly in the volume $\mathrm{vol}(B)$ as the underlying domain $B \uparrow \mathbb{R}^3$. Nevertheless, no lower bound was pursued. Consequently, the possibility of having a strictly lower order variance remained open. In view of the behavior of the variance in the real-valued 2-dimensional case, the question about the true order of the variance arises.

In the present note, we establish that the variance of the nodal length of the complex-valued 3-dimensional RWM is linear (with a strictly positive coefficient) in the volume of the growing domain, which coincides with the order of the expectation. This behavior differs from the observed one in the real valued 2-dimensional case Nourdin et al. (2019) where the variance diverges faster than the expectation,

$$\mathbb{E}[\mathcal{L}(B)] \approx \mathrm{area}(B); \quad \mathrm{Var}(\mathcal{L}(B)) \approx \ \mathrm{area}(B)\log(\mathrm{area}(B)), \text{ as } B \uparrow \mathbb{R}^2.$$

A rough explanation may be as follows. In both cases the second component $\mathrm{Var}(I_2(B))$ vanishes due to geometric reasons as said above. The key difference between these cases lies in the integrability of the covariance functions which is a fact of probabilistic nature (short vs. long memory, say). In the 2-dimensional case $r(x, y) = c_2 J_0(|x - y|)$ belongs to $L^6(\mathbb{R}^2)$ but not to $L^4(\mathbb{R}^2)$ while in the 3-dimensional case $r(x, y) = J_{1/2}(|x - y|)/|x - y|^{1/2}$ belongs to $L^4(\mathbb{R}^3)$. Grosso modo, see lemma 9 in Dalmao et al. (2021), once normalized by the area or volume, according to the dimension, the variance of $I_{2q}(B)$ is written as an integral of the $2q$-th power of r (actually, the product of $2q$ factors chosen from r and its derivatives). Thus, the normalized variance of $I_4(B)$ diverges in dimension 2 while the rest of the variances (i.e: those of $I_{2q}(B) : q \geq 3$ in 2d, those of $I_{2q}(B) : q \geq 2$ in 3d), once normalized, converge. Observe that it follows that the variance is of lower order than the square of the expectation as B increases, implying the concentration of the probability. See section 3 for some details.

To prove our result, we carry on a careful analysis of the variance of the fourth component of the Wiener expansion of $\mathcal{L}(B)$ which involves a few dozen terms. We need to compute explicitly each coefficient in the expansion and to find the asymptotic of the involved integrals. Since we look for a lower bound for the variance, we can omit a convenient number of nonnegative terms.

Let us fix some notation: cs denotes an unimportant constant whose value may change from one line to another, $a_R \approx b_R$ means that $\lim_{R\to\infty} \frac{a_R}{b_R} = c$ with $c > 0$ and $a_R \sim b_R$ when $\lim_{R\to\infty} \frac{a_R}{B_R} = 1$, we write $a_R \gtrsim b_R$ when there exists $B'_R \sim b_R$ such that $a_R \geq B'_R$.

2. Problem setting and main result

Define the 3-dimensional monochromatic random wave model (RWM) as the complex-valued centered stationary isotropic Gaussian random field

$$\psi(x) := \xi(x) + i\eta(x), \quad x \in \mathbb{R}^3, \tag{2.1}$$

such that its real and imaginary parts ξ, η are centered independent Gaussian random fields with covariance function

$$r(x) := \text{sinc}(|x|), \tag{2.2}$$

where sinc stands for the cardinal sine function $\text{sinc}(\cdot) = \frac{\sin(\cdot)}{\cdot}$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$. Observe that this coincides with (2.2) for $d = 3$.

For any bounded domain B in $\mathbb{R}^3$, we introduce the nodal curve $\mathcal{Z}(B)$ and the nodal length $\mathcal{L}(B)$ by

$$\mathcal{Z}(B) := \left\{x \in B : |\psi(x)| = 0\right\},$$
$$\mathcal{L}(B) := \text{length}(\mathcal{Z}(B)).$$

In Dalmao et al. (2021) it was proved that $\mathbb{E}[\mathcal{L}(B)] = \frac{1}{3\pi}\text{vol}(B)$ (Corollary 2 with $\lambda = \frac{1}{3}$), that $\text{Var}(\mathcal{L}(B)) = O(\text{vol}(B))$ as $B \uparrow \mathbb{R}^3$ and that a Central Limit Theorem takes place (proposition 10). Since the limit variance was not known to be positive, the limit law could be degenerated.

The main result of the present note is contained in the next theorem.

Theorem 2.1. *Let ψ be defined as in (2.1)-(2.2). Consider B_R the centered ball of radius R in $\mathbb{R}^3$, then there exists a constant $c > 0$ such that*

$$\lim_{R\to\infty} \frac{\text{Var}(\mathcal{L}(B_R))}{\text{vol}(B_R)} \geq c.$$

Observe that in Dalmao et al. (2021) the underlying domain was $Q_n = [-n, n]^3$ as $n \to \infty$ while here we use B_R as $R \to \infty$. We choose the latter for simplicity of exposition, but one can easily adapt the computation of the expectation and the variance from B_R to Q_n and conversely. Observe that Nourdin et al. (2019) considers general convex domains. It may be instructive to read section 2.4 in Wigman (2022). When dealing with the asymptotic distribution of the nodal length, the case of Q_n is simpler.

3. Comparison with the real-valued 2-dimensional case

We begin with a common feature of both cases. As said above, in both cases, the second component in the Wiener expansion vanishes. This cancellation avoids having a higher order of the variances since the covariance functions are not in L^2.

Regarding the growing domain regime, that is, the setting of this paper. In view of the results in Dalmao et al. (2021), theorem 2.1 implies that the variance, and the expectation, of the nodal length are exactly of the same order as the volume of B_R. Recall that in the real-valued 2-dimensional case the ratio variance/expectation was of the order of $\log(\text{area}(B))$ as the domain $B \uparrow \mathbb{R}^2$.

This fact is connected to another important difference with the real-valued 2-dimensional case. Here, each component of the Wiener expansion of $\mathcal{L}(B)$ plays a role in the asymptotic variance, that is, there is no dominating term as in the real-valued 2-dimensional case. This is due to the fact that the covariance function belongs to $L^4(\mathbb{R}^3)$ implying that all the integrals involved in the computation of the variance are convergent. To show this point, consider one of the terms involved in the variance of the sixth component of the expansion.

$$\int_{B_R\times B_R} r(|x-y|)^6 dxdy \sim \int_{B_R}\int_{B_R} r(|x-y|)^6 dxdy \sim \frac{2\pi^3 R^3}{3},$$

which is of the order of the volume of B_R. See section 6 for the computation of this sort of integrals. In the real-valued 2-dimensional case the analogous integral was still convergent but negligible w.r.t. to the variance of $I_4(B)$ since the covariance was in $L^6(R^2)\setminus L^4(\mathbb{R}^2)$. The existence of a single dominant chaotic component makes it possible to get quantitative central limit theorems using well-known bounds for the (e.g.: Kolmogorov, Wasserstein) distance between the distributions of r.v.s lying in a fixed chaotic component. We are not pursuing this kind of results in the present case where an infinite number of components contribute to the limit.

Now, let us consider the high energy framework on a fixed domain. We follow remark 1.3. in Nourdin et al. (2019). Recall that for $k \geq 1$, we defined $\psi_k(x) = \psi(kx)$. Besides, we can relate the nodal lengths $\mathcal{L}_k(B)$ of ψ_k and $\mathcal{L}(B)$ of ψ by using their integral representation (see (2.23) in Nourdin et al. (2019))

$$\mathcal{L}_k(B) = \int_B \delta_0(\psi_k(x))|\nabla\psi_k(x)|dx = \frac{1}{k}\int_{k\cdot B}\delta_0(\psi(y))|\nabla\psi(y)|dy = \frac{\mathcal{L}(k\cdot B)}{k}.$$

Here, as usual, we use δ_0 as a shorthand notation for an approximation of the unity. To translate the former integral into the latter, we used the change of variables $y = kx$ and denoted $k\cdot B = \{kb : b\in B\}$. Hence, for fixed B, as $k\to\infty$, we have

$$\mathbb{E}[\mathcal{L}_k(B)] = \frac{\mathbb{E}[\mathcal{L}(k\cdot B)]}{k} \sim \text{cs } k^2;$$
$$\text{Var}[\mathcal{L}_k(B)] = \frac{\text{Var}[\mathcal{L}(k\cdot B)]}{k^2} \sim \text{cs } k.$$

We see, as above, that the ratio variance / square of the expectation tends to zero in the high energy limit and, thus, there is concentration of the probability in the sense that $\frac{\mathcal{L}_k}{\mathbb{E}(\mathcal{L}_k)}$ converges in probability to 1 as $k\to\infty$. This ratio is $\approx 1/k^3$ in the complex-valued 3-dimensional case and $\approx \log(k)/k^2$ in the real-valued 2-dimensional one.

4. Preliminaries

The departure point of the proof of theorem (2.1) is the Wiener expansion of $\mathcal{L}(B_R)$ which we borrow from proposition 2 in Dalmao et al. (2021). With this expansion at hand, we bound the variance from below by that of one specific term (the fourth one) and compute it explicitly until we are sure of its positivity.

Hermite polynomials : The building blocks of the Wiener expansion are Hermite polynomials which we define recursively by $H_0(x) = 1$, $H_1(x) = x$ for $x\in\mathbb{R}$ and for $n\geq 2$ by

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x), \quad x\in\mathbb{R}.$$

We need also the multi-dimensional Hermite polynomials :

$$\tilde{H}_{\boldsymbol{\alpha}}(\boldsymbol{y}) = \prod_{i=1}^m H_{\alpha_i}(y_i), \quad \boldsymbol{\alpha} = (\alpha_i)_i \in \mathbb{N}^m \text{ and } \boldsymbol{y} = (\mathrm{y_i})_\mathrm{i} \in \mathbb{R}^\mathrm{m}.$$

It is well known, see e.g. section 8.1. in Peccati and Taqqu (2011), that Hermite polynomials form a complete orthogonal system of $L^2(\phi_m(d\boldsymbol{y}))$ being ϕ_m the standard normal density function in $\mathbb{R}^m$. Hence, for $f \in L^2(\phi_m(d\boldsymbol{y}))$, we can write in the L^2-sense

$$f(\boldsymbol{y}) = \sum_{q=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathbb{N}^m,\, |\boldsymbol{\alpha}|=q} f_{\boldsymbol{\alpha}} \tilde{H}_{\boldsymbol{\alpha}}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{R}^m,$$

with $|\boldsymbol{\alpha}| = \sum_{i=1}^m \alpha_i$ and

$$f_{\boldsymbol{\alpha}} = \frac{1}{\boldsymbol{\alpha}!} \int_{\mathbb{R}^m} f(\boldsymbol{y}) \tilde{H}_{\boldsymbol{\alpha}}(\boldsymbol{y}) \phi_m(d\boldsymbol{y}),$$

with $\boldsymbol{\alpha}! = \prod_{i=1}^m \alpha_i!$. We refer to $f_{\boldsymbol{\alpha}}$ as the $\boldsymbol{\alpha}$-th Hermite coefficient of f.

Wiener expansion of $\mathcal{L}(B_R)$: Let us introduce the Wiener expansion of the nodal length. Denote $\xi'(x) = (\xi_1(x), \xi_2(x), \xi_3(x))$ for the gradient of $\xi(x)$ and $\eta'(x) = (\eta_1(x), \eta_2(x), \eta_3(x))$ for that of $\eta(x)$. Set

$$Y(x) := \big(\xi(x), \eta(x), \sqrt{3}\xi'(x), \sqrt{3}\eta'(x)\big),$$

which is a standard normal random vector in $\mathbb{R}^8$ for each $x \in \mathbb{R}^3$. Finally, we need the formal Hermite coefficients of the Delta function Kratz and León (1997, Eq.5)

$$b_\alpha = \frac{1}{\alpha!\sqrt{2\pi}} H_\alpha(0), \quad \alpha \in \mathbb{N}. \tag{4.1}$$

Therefore, proposition 7 in Dalmao et al. (2021) states that

$$\tilde{\mathcal{L}}(B_R) := \frac{\mathcal{L}(B_R) - \mathbb{E}[\mathcal{L}(B_R)]}{\mathrm{vol}(B_R)} = \frac{1}{3} \sum_{q\geq 1} I_{2q}(B_R), \tag{4.2}$$

$$I_{2q}(B_R) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^8,\, |\boldsymbol{\alpha}|=2q} c_{\boldsymbol{\alpha}} \int_{B_R} \tilde{H}_{\boldsymbol{\alpha}}(Y(x))dx,$$

in the L^2-sense. The coefficients $c_{\boldsymbol{\alpha}}$ are defined by

$$c_{\boldsymbol{\alpha}} = b_{\alpha_1} b_{\alpha_2} a_{(\alpha_3,\ldots,\alpha_8)},$$

with $a_{(\alpha_3,\ldots,\alpha_8)}$ the Hermite coefficient of $f(\boldsymbol{y}) = \det^{\perp}(\boldsymbol{y}) := \det(\boldsymbol{y}\boldsymbol{y}^\top)^{1/2}$, $\boldsymbol{y} \in \mathbb{R}^6$.

Expectations of products of Hermite polynomials : It is well known that for jointly Gaussian X, Y

$$\mathbb{E}[H_p(X)H_q(XY)] = \delta_{pq} p! \mathbb{E}[XY]^p, \tag{4.3}$$

where δ_{pq} is Kronecker's delta.

For future use we recall a few more useful formulas, see e.g. Nourdin et al. (2019, Eq.6.76). Let X_1, X_2, X_3, X_4 be standard normal r.v. with $\mathbb{E}(X_1X_2) = \mathbb{E}(X_3X_4) = 0$, then

$$\begin{aligned} \mathbb{E}[H_2(X_1)H_2(X_2)H_2(X_3)H_2(X_4)] &= 4\mathbb{E}[X_1X_3]^2\mathbb{E}[X_2X_4]^2 + 4\mathbb{E}[X_1X_4]^2\mathbb{E}[X_2X_3]^2 \\ &\quad + 16\mathbb{E}[X_1X_3]\mathbb{E}[X_1X_4]\mathbb{E}[X_2X_3]\mathbb{E}[X_2X_4]; \\ \mathbb{E}[H_2(X_1)H_2(X_2)H_4(X_3)] &= 24\mathbb{E}[X_1X_3]^2\mathbb{E}[X_2X_3]^2; \\ \mathbb{E}[X_1X_2X_3X_4] &= \mathbb{E}[X_1X_3]\mathbb{E}[X_2X_4] + \mathbb{E}[X_1X_4]\mathbb{E}[X_2X_3]. \end{aligned} \tag{4.4}$$

5. Proof of theorem 2.1

Now, we can proceed to the proof of the main result.

Lower bound for the variance : Since the components $I_{2q}(B_R) : q \geq 1$ are orthogonal, we get

$$\mathrm{Var}(\mathcal{L}(B_R)) = \frac{1}{3}\sum_{q=1}^{\infty}\mathrm{Var}(I_{2q}(B_R)) \geq \frac{1}{3}\mathrm{Var}(I_4(B_R)).$$

Thus, we reduce our attention to the fourth component $I_4(B_R)$.

Fourth component : We can give the explicit expression of $I_4(B_R)$. From (4.2) we get

$$\begin{aligned}
I_4(B_R) = {} & b_0 b_4 a_0 \int_{B_R} H_4(\xi(x))dx + b_0 b_4 a_0 \int_{B_R} H_4(\eta(x))dx \\
& + b_2^2 a_0 \int_{B_R} H_2(\xi(x))H_2(\eta(x))dx + \sum_{k=1}^{3} b_0^2 a_{4e_k} \int_{B_R} H_4(\bar{\xi}_k(x))dx \\
& + \sum_{k=1}^{3} b_0^2 a_{4e_k} \int_{B_R} H_4(\bar{\eta}_k(x))dx + \sum_{k=1}^{3} b_0 b_2 a_{2e_k} \int_{B_R} H_2(\xi(x))H_2(\bar{\xi}_k(x))dx \\
& + \sum_{k=1}^{3} b_0 b_2 a_{2e_k} \int_{B_R} H_2(\eta(x))H_2(\bar{\eta}_k(x))dx \\
& + \sum_{k=1}^{3} b_0 b_2 a_{2e_k} \int_{B_R} H_2(\xi(x))H_2(\bar{\eta}_k(x))dx \\
& + \sum_{k=1}^{3} b_0 b_2 a_{2e_k} \int_{B_R} H_2(\eta(x))H_2(\bar{\xi}_k(x))dx \\
& + \sum_{i\neq j} b_0^2 a_{2e_i+2e_j} \int_{B_R} H_2(\bar{\xi}_i(x))H_2(\bar{\xi}_j(x))dx \\
& + \sum_{i\neq j} b_0^2 a_{2e_i+2e_j} \int_{B_R} H_2(\bar{\eta}_i(x))H_2(\bar{\eta}_j(x))dx \\
& + \sum_{i\neq j} b_0^2 a_{2e_i+2e_j} \int_{B_R} H_2(\bar{\xi}_i(x))H_2(\bar{\eta}_j(x))dx.
\end{aligned}$$

Here $\bar{\xi}_k = \sqrt{3}\xi_k$ and $\bar{\eta}_k = \sqrt{3}\eta_k$ and $\{e_1, e_2, \ldots, e_6\}$ is the canonical basis in $\mathbb{R}^6$. Equation (4.1) gives

$$b_0 = \frac{1}{\sqrt{2\pi}}; b_2 = -\frac{1}{2\sqrt{2\pi}}; b_4 = \frac{1}{8\sqrt{2\pi}}.$$

The next lemma, whose proof is presented in section 6, provides the explicit values of the relevant Hermite coefficients of $\det^{\perp}(\cdot)$.

Lemma 5.1. *The first Hermite coefficients of* $\det^{\perp}(\cdot)$ *are :*

$$a_0 = 1;\ a_{2e_k} = \frac{1}{3};\ a_{2e_i+2e_j} = \frac{1}{9}\ (i \neq j);\ a_{4e_k} = -\frac{5}{9}.$$

Using the explicit values of the coefficients we have

$$2\pi \cdot I_4(B_R) = A_1 + A_2 + A_3,$$

with

$$
\begin{aligned}
A_1 &:= \frac{1}{8}\int_{B_R} H_4(\xi) + \frac{1}{8}\int_{B_R} H_4(\eta) + \frac{1}{4}\int_{B_R} H_2(\xi)H_2(\eta) - \frac{5}{9}\sum_{k=1}^{3}\int_{B_R} H_4(\bar{\xi}_k) \\
&\quad - \frac{5}{9}\sum_{k=1}^{3}\int_{B_R} H_4(\bar{\eta}_k); \\
A_2 &:= -\frac{1}{6}\sum_{k=1}^{3}\int_{B_R} H_2(\xi)H_2(\bar{\xi}_k) - \frac{1}{6}\sum_{k=1}^{3}\int_{B_R} H_2(\eta)H_2(\bar{\eta}_k) \\
&\quad - \frac{1}{6}\sum_{k=1}^{3}\int_{B_R} H_2(\xi)H_2(\bar{\eta}_k) - \frac{1}{6}\sum_{k=1}^{3}\int_{B_R} H_2(\eta)H_2(\bar{\xi}_k); \\
A_3 &:= \sum_{i\neq j}\frac{1}{9}\int_{B_R} H_2(\bar{\xi}_i)H_2(\bar{\xi}_j) + \sum_{i\neq j}\frac{1}{9}\int_{B_R} H_2(\bar{\eta}_i)H_2(\bar{\eta}_j) \\
&\quad + \sum_{i\neq j}\frac{1}{9}\int_{B_R} H_2(\bar{\xi}_i)H_2(\bar{\eta}_j).
\end{aligned}
$$

It follows that

$$
\mathrm{Var}\big(2\pi\cdot I_4(B_R)\big) = \sum_{i,j=1}^{3}\mathrm{Cov}(A_i, A_j).
$$

This is a long computation. This partition, which may seem strange at first sight, will allow us to avoid considering quite a lot of terms.

Now, we state the equivalences for the variances and covariances of these terms, except for the variance of A_2 and A_3 which are certainly non negative and turn out to be innecssary.

Denote $A_1 = \sum_{i=1}^{5} A_{1i}$. Then, as ξ and η are independent processes we have:

$$
\begin{aligned}
\mathrm{Var}(A_1) &= 2\mathrm{Var}(A_{11}) + \mathrm{Var}(A_{13}) + 2\mathrm{Var}(A_{14}) + 4\mathrm{Cov}(A_{11}, A_{14}) \\
&\sim \frac{4\pi^3R^3}{3}\frac{4362}{35}.
\end{aligned}
$$

Similarly, denote $A_2 = \sum_{i=1}^{4} A_{2i}$ and $A_3 = \sum_{i=1}^{3} A_{3i}$. Then,

$$
\begin{aligned}
\mathrm{Cov}(A_1, A_2) &= 2\mathrm{Cov}(A_{11}, A_{21}) + 2\mathrm{Cov}(A_{13}, A_{23}) + 2\mathrm{Cov}(A_{14}, A_{21}) \\
&\gtrsim -\frac{4\pi^3R^3}{3}\frac{4}{3}. \\
\mathrm{Cov}(A_1, A_3) &= 2\mathrm{Cov}(A_{11}, A_{31}) + \mathrm{Cov}(A_{13}, A_{33}) + 2\mathrm{Cov}(A_{14}, A_{31}) \\
&\gtrsim -\frac{4\pi^3R^3}{3}\frac{1184}{105}.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Cov}(A_2, A_3) &= 2\mathrm{Cov}(A_{21}, A_{31}) + 2\mathrm{Cov}(A_{23}, A_{33}) \\
&\gtrsim -\frac{4\pi^3R^3}{3}\frac{824}{525}.
\end{aligned}
$$

Therefore, since $\mathrm{Var}(A_2)\geq 0$ and $\mathrm{Var}(A_3)\geq 0$, we have

$$
\mathrm{Var}\big(I_4(B_R)\big) \gtrsim \frac{4\pi R^3}{3}\frac{7691}{350}.
$$

The result follows. The proofs of these relations are presented in section 6.

6. Ancillary computations

In this section we include part of the long and sometimes tedious computations involved in the asymptotic variance of $I_4(B_R)$. The omitted computations are analogous to those which are presented here.

6.1. *The coefficients.* We prove Lemma 5.1. Note that $\det^\perp(\boldsymbol{y}) = |\boldsymbol{y} \wedge \boldsymbol{y}|$. Here, $\wedge$ is the standard wedge product in $\mathbb{R}^3$. Assume that $Z_1 = (Z_{11}, Z_{12}, Z_{13})$ and $Z_2 = (Z_{21}, Z_{22}, Z_{23})$ are independent standard Gaussian r.v. in $\mathbb{R}^3$.

$$a_0 = \mathbb{E}(|Z_1 \wedge Z_2|) = \mathbb{E}(\mathrm{vol}\{aZ_1 + bZ_2 : 0 \le a, b \le 1\}) = \sqrt{2}\frac{\Gamma(1)}{\Gamma(1/2)} \cdot \sqrt{2}\frac{\Gamma(3/2)}{\Gamma(1)} = 1.$$

Here we used the notation and results of Azaïs and Wschebor (2009, p.305) for the expected volume of a random paralellepiped.

Since the distribution of (Z_1, Z_2) is invariant under permutations of its coordinates, we deduce that $a_{2e_k} : k = 1, \dots, 6$ coincide. Besides, as $H_2(x) = x^2 - 1$,

$$a_{2e_1} = \mathbb{E}(|Z_1 \wedge Z_2| H_2(Z_{11})) = \frac{1}{3}\mathbb{E}(|Z_1 \wedge Z_2||Z_1|^2) - a_0.$$

To connect the latter expectation with a_0 we use spherical coordinates, writing $z_1 = \rho u$ with $\rho > 0$ and $u \in S^2$.

$$\begin{aligned}\mathbb{E}(|Z_1 \wedge Z_2||Z_1|^2) &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z_1 \wedge z_2||z_1|^2 \phi_3(dz_1)\phi_3(dz_2) \\ &= \int_{\mathbb{R}^3} \phi_3(dz_2) \int_{S^2} |u \wedge z_2| du \int_0^\infty \rho^2\rho^2\rho\phi(d\rho).\end{aligned}$$

Here we have a ρ^2 from the Jacobian, another ρ^2 from the squared norm and the final ρ from the wedge product. In the same way, we get

$$a_0 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z_1 \wedge z_2| \phi_3(dz_1)\phi_3(dz_2) = \int_{\mathbb{R}^3} \phi_3(dz_2) \int_{S^2} |u \wedge z_2| du \int_0^\infty \rho^2\rho\phi(d\rho).$$

Hence, denoting $m_k := \int_0^\infty \rho^k \phi(d\rho)$ and using the recurrence $m_{k+1} = k m_{k-1}$, we have

$$\mathbb{E}(|Z_1 \wedge Z_2||Z_1|^2) = \frac{m_5}{m_3} a_0 = 4. \tag{6.1}$$

Thus,

$$a_{2e_1} = \frac{1}{3}\frac{m_5}{m_3}a_0 - a_0 = \frac{1}{3}4 - 1 = \frac{1}{3}.$$

Again, for $i \neq j$, all the $a_{2e_i+2e_j}$ coincide. Besides,

$$a_{2e_1+2e_4} = \mathbb{E}(|Z_1 \wedge Z_2| H_2(Z_{11})H_2(Z_{21})) = \mathbb{E}(|Z_1 \wedge Z_2| Z_{11}^2 Z_{21}^2) - a_{2e_1} - a_{2e_4} - a_0.$$

Here we used that $H_2(x)H_2(y) = x^2y^2 - H_2(x) - H_2(y) - 1$. Thus

$$\begin{aligned}\mathbb{E}(|Z_1 \wedge Z_2| Z_{11}^2 Z_{21}^2) &= \frac{1}{9}\mathbb{E}(|Z_1 \wedge Z_2||Z_1|^2|Z_2|^2) \\ &= \frac{1}{9}\int_{S^2 \times S^2} |u_1 \wedge u_2| du_1 du_2 \int_0^\infty \rho_1^2\rho_1^2\rho_1\phi(d\rho_1) \int_0^\infty \rho_2^2\rho_2^2\rho_2\phi(d\rho_2) = \frac{1}{9}\frac{m_5^2}{m_3^2}a_0 = \frac{16}{9},\end{aligned} \tag{6.2}$$

where we used spherical coordinates in both z_1 and z_2 and the fact that permuting the indexes does not changes the expectation. Thus

$$a_{2e_1+2e_4} = \frac{1}{9}\frac{m_5^2}{m_3^2}a_0 - \frac{2}{3} - 1 = \frac{1}{9}.$$

Finally, $a_{4e_k} : k = 1, \dots, 6$ coincide. Besides,

$$\begin{aligned} a_{4e_k} &= \mathbb{E}(|Z_1 \wedge Z_2| H_4(Z_1)) = \mathbb{E}(|Z_1 \wedge Z_2|(Z_1^4 - 6Z_1^2 + 3)) \\ &= \frac{1}{3}\mathbb{E}(|Z_1 \wedge Z_2||Z_1|^4) - 2\mathbb{E}(|Z_1 \wedge Z_2| Z_{11}^2 Z_{21}^2) - 2\mathbb{E}(|Z_1 \wedge Z_2||Z_1|^2) + 3a_0 \end{aligned}$$

Here we used that $|(a_1, a_2, a_3)|^4 = a_1^4 + a_2^4 + a_3^4 + 2(a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2)$.

The second and third terms have been computed in (6.1) and (6.2). We look at the first one.

$$\begin{aligned} \mathbb{E}(|Z_1 \wedge Z_2||Z_1|^4) &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z_1 \wedge z_2||z_1|^4 \phi_3(dz_1)\phi_3(dz_2) \\ &= \int_{\mathbb{R}^3} \phi_3(dz_2) \int_{S^2} |u \wedge z_2| du \int_0^\infty \rho^2 \rho^4 \rho \phi(d\rho) = \frac{m_7}{m_3} a_0 = 24. \end{aligned}$$

Therefore,

$$a_{4e_k} = \frac{1}{3}\frac{m_7}{m_3}a_0 - 2\frac{1}{9}\frac{m_5^2}{m_3^2}a_0 - 2\frac{m_5}{m_3}a_0 + 3a_0 = -\frac{5}{9}.$$

6.2. *The variances and covariances of A_{ij}.* We use Gradshteyn and Ryzhik (2015) and Wolfram|Alpha to compute the radial integrals (i.e: those w.r.t. ρ).

Recall that B_R is the ball of radius R centered at 0 and set $B_R(z) = B_R + z$. Also recall that

$$\begin{aligned} r(u) = \operatorname{sinc}(u) &= \frac{\sin(u)}{u}, \\ \operatorname{sinc}'(u) &= \frac{\cos(u)}{u} - \frac{\operatorname{sinc}(u)}{u} = \frac{u\cos(u) - \sin(u)}{u^2}; \\ \operatorname{sinc}''(u) &= -\operatorname{sinc}(u) + \frac{2\operatorname{sinc}(u)}{u^2} - \frac{2\cos(u)}{u^2}. \end{aligned}$$

Besides, denoting $r_{k0}(x,y) = \frac{\partial r(x,y)}{\partial x_k}$ (and using similar notations for y), we have

$$r_{k0}(x,y) = \operatorname{sinc}'(|x-y|)\Delta_k, \qquad r_{0k}(x,y) = -\operatorname{sinc}'(|x-y|)\Delta_k, \tag{6.3}$$

with

$$\Delta_k = \frac{x_k - y_k}{|x-y|}.$$

Besides,

$$\begin{aligned} r_{kk}(x,y) &= \Big[\frac{\operatorname{sinc}'(|x-y|)}{|x-y|} - \operatorname{sinc}''(|x-y|)\Big]\frac{(x_k - y_k)^2}{|x-y|^2} - \frac{\operatorname{sinc}'(|x-y|)}{|x-y|} \\ &=: A(|x-y|)\Delta_k^2 - B(|x-y|), \end{aligned} \tag{6.4}$$

with

$$\begin{aligned} A(u) &= \frac{\operatorname{sinc}'(u)}{u} - \operatorname{sinc}''(u) = \frac{u^2\sin(u) - 3\sin(u) + 3u\cos(u)}{u^3} \\ B(\rho) &= \frac{\operatorname{sinc}'(u)}{u} = \frac{u\cos(u) - \sin(u)}{u^3}. \end{aligned} \tag{6.5}$$

For $k \neq k'$:

$$r_{kk'}(x,y) = \Big[\mathrm{sinc}''(|x-y|) - \frac{\mathrm{sinc}'(|x-y|)}{|x-y|}\Big]\frac{(y_{k'}-x_{k'})(x_k-y_k)}{|x-y|^2}$$
$$= A(|x-y|)\Delta_k\Delta_{k'} \tag{6.6}$$

To deal with the integrals of the angular parts Δ_k we consider spherical coordinates in $\partial B_\rho(x)$:

$$\begin{cases} x_1 - y_1 &= \rho\sin(\theta)\cos(\varphi), \\ x_2 - y_2 &= \rho\sin(\theta)\sin(\varphi), \\ x_3 - y_3 &= \rho\cos(\theta). \end{cases}$$

In the sequel we use sistematically (4.3) and (4.4).

(1) We consider $\mathrm{Var}(A_1)$ in detail. We have.

- $\mathrm{Var}(A_{11}) = \mathrm{Var}(A_{12}) \sim \frac{4\pi^3R^3}{3}\frac{3}{4}$, $\mathrm{Var}(A_{13}) \sim \frac{4\pi^3R^3}{3}\frac{1}{2}$. Indeed, using (4.4) we get

$$\mathrm{Var}(A_{11}) = \frac{1}{64}\int_{B_R\times B_R}\mathbb{E}(H_4(\xi(x))H_4(\xi(y)))dxdy = \frac{4!}{64}\int_{B_R\times B_R} r(|x-y|)^4dxdy.$$
$$\mathrm{Var}(A_{13}) = \frac{1}{16}\int_{B_R\times B_R}\mathbb{E}(H_2(\xi(x))H_2(\xi(y)))\mathbb{E}(H_2(\eta(x))H_2(\eta(y)))dxdy$$
$$= \frac{4}{16}\int_{B_R\times B_R} r(|x-y|)^4dxdy.$$

To compute this kind of integral we begin using the area formula as in the first display in the proof of proposition 5.1 in Nourdin et al. (2019). We have

$$\int_{B_R\times B_R} r(|x-y|)^4dxdy = \int_0^{2R}d\rho\int_{B_R}dx\int_{\partial B_\rho(x)\cap B_R} r(|x-y|)^4dy,$$

where $\partial B_\rho(x)$ is the sphere of radius ρ centered at x. Following Nourdin et al. (2019, eq.(5.64)) we split this integral.

$$\int_{B_R\times B_R} r(|x-y|)^4dxdy = \int_0^R d\rho\int_{B_{R-\rho}}dx\int_{\partial B_\rho(x)} r(|x-y|)^4dy$$
$$+ \int_0^R d\rho\int_{B_R - B_{R-\rho}}dx\int_{\partial B_\rho(x)\cap B_R} r(|x-y|)^4dy$$
$$+ \int_R^{2R} d\rho\int_{B_R}dx\int_{\partial B_R(x)\cap B_R} r(|x-y|)^4dy.$$

Consider the first integral. Note that for $y\in\partial B_\rho(x)$ we have $|x-y| = \rho$. Hence,

$$\int_0^R d\rho\int_{B_{R-\rho}}dx\int_{\partial B_\rho(x)} r(|x-y|)^4dy = \int_0^R \mathrm{vol}(B_{R-\rho})\mathrm{area}(\partial B_\rho(x))r(\rho)^4d\rho$$
$$= \frac{16\pi^2}{3}\int_0^R (R-\rho)^3\rho^2\frac{\sin(\rho)^4}{\rho^4}d\rho = \frac{16\pi^2}{3}R^3\int_0^R \big(1-\frac{\rho}{R}\big)^3\frac{\sin(\rho)^4}{\rho^2}d\rho$$
$$\sim_{R\to\infty} \frac{16\pi^2}{3}R^3\int_0^\infty\frac{\sin(\rho)^4}{\rho^2}d\rho = 2\frac{4\pi^3R^3}{3}.$$

Consider the second integral.

$$\int_0^R d\rho \int_{B_R\setminus B_{R-\rho}} dx \int_{\partial B_\rho(x)\cap B_R} r(|x-y|)^4 dy \le \frac{16\pi^2}{3}\int_0^R \big(R^3-(R-\rho)^3\big)\rho^2 r(\rho)^4 d\rho$$
$$\le \frac{16\pi^2}{3}\int_0^R \big(3R - \frac{3R^2}{\rho}+\rho\big)\sin(\rho)^4 d\rho = O(R^2\log(R)) = o(R^3).$$

Similarly, for the third integral we have

$$\int_R^{2R} d\rho \int_{B_R} dx \int_{\partial B_R(x)\cap B_R} r(|x-y|)^4 dy \le \int_R^{2R} \mathrm{vol}(B_R)\mathrm{area}(\partial B_\rho(x)) r(\rho)^4 d\rho$$
$$\le \frac{16\pi^2R^3}{3}\int_R^{2R}\frac{\sin(\rho)^4}{\rho^2}d\rho \le \frac{16\pi^2R^3}{3}\int_R^{2R}\frac{1}{\rho^2}d\rho = \frac{8\pi^2R^2}{3} = o(R^3).$$

In conclusion

$$\int_{B_R\times B_R} r(|x-y|)^4 dxdy = \frac{8\pi^3R^3}{3} + O(R^2\log(R)) \sim 2\frac{4\pi^3R^3}{3},$$

as claimed.

- $\mathrm{Cov}(A_{11},A_{14}) = \mathrm{Cov}(A_{12},A_{15}) \sim -\frac{4\pi^3R^3}{3}\frac{21}{5}$. Indeed,

$$\mathrm{Cov}(A_{11},A_{14}) = -\frac{5}{72}\sum_{k=1}^3\int_{B_R\times B_R}\mathbb{E}(H_4(\xi(x))H_4(\bar{\xi}_k(y)))dxdy.$$

Recall that $\bar{\xi}_k(x) = \sqrt{3}\xi_k(x)$.

$$\int_{B_R\times B_R}\mathbb{E}(H_4(\xi(x))H_4(\bar{\xi}_k(y)))dxdy = 9\cdot 4!\int_{B_R\times B_R} r_{0k}(x,y)^4 dxdy$$
$$= 9\cdot 4!\int_0^{2R}d\rho\int_{B_R}dx\int_{\partial B_\rho(x)\cap B_R} r_{0k}(x,y)^4 dy.$$

Recall that $r_{0k}(x,y)$ was computed in (6.3). We split the integral as before, the dominant term is

$$\sum_{k=1}^3\int_0^R d\rho\int_{B_{R-\rho}}dx\int_{\partial B_\rho(x)} r_{0k}(x,y)^4 dy$$
$$= \sum_{k=1}^3\int_0^R \mathrm{vol}(B_{R-\rho})\mathrm{sinc}'(\rho)^4\rho^2 d\rho\int_{[0,\pi]\times[0,2\pi]}\Delta_k^4\sin(\theta)d\theta d\varphi$$

The radial part is similar to the previous one.

$$\int_0^R \mathrm{vol}(B_{R-\rho})\mathrm{sinc}'(\rho)^4\rho^2 d\rho = \frac{4\pi R^3}{3}\int_0^R\big[1-\frac{\rho}{R}\big]^3\mathrm{sinc}'(\rho)^4\rho^2 d\rho$$
$$\sim \frac{4\pi R^3}{3}\int_0^\infty \mathrm{sinc}'(\rho)^4\rho^2 d\rho = \frac{4\pi R^3}{3}\frac{7\pi}{60}.$$

The angular part equals $\frac{12\pi}{5}$. Indeed,

$$\begin{aligned}\sum_{k=1}^{3}\int_{[0,\pi]\times[0,2\pi]}&\Delta_k^4\sin(\theta)d\theta d\varphi\\ &=\int_{[0,\pi]\times[0,2\pi]}\Big[\sin^5(\theta)\cos^4(\varphi)+\sin^5(\theta)\sin^4(\varphi)+\cos^4(\theta)\sin(\theta)\Big]d\theta d\varphi\\ &=\frac{16}{15}\cdot\frac{3\pi}{4}+\frac{16}{15}\cdot\frac{3\pi}{4}+\frac{2}{5}2\pi=\frac{12\pi}{5}.\end{aligned}$$

In the sequel we omit the explicit computations of the angular parts since they are analogous. (The remainder term behaves as in the previous case). In conclusion

$$\sum_{k=1}^{3}\int_{B_R\times B_R}\mathbb{E}(H_4(\xi(x))H_4(\bar{\xi}_k(y)))dxdy\sim 9\cdot 4!\frac{4\pi R^3}{3}\frac{7\pi}{60}\frac{12\pi}{5}=\frac{4\pi R^3}{3}\frac{1512}{25}.$$

Multiplying by $-\frac{5}{72}$ the result follows.

- $\mathrm{Var}(A_{14})=\mathrm{Var}(A_{15})=\frac{488}{7}$. Indeed,

$$\mathrm{Var}(A_{14})=\frac{25}{81}\sum_{k,k'=1}^{3}\int_{B_R\times B_R}\mathbb{E}(H_4(\bar{\xi}_k(x))H_4(\bar{\xi}_k(y)))dxdy.$$

We begin with the diagonal terms.

$$\int_{B_R\times B_R}\mathbb{E}(H_4(\bar{\xi}_k(x))H_4(\bar{\xi}_k(y)))dxdy=81\cdot 4!\int_{B_R\times B_R}r_{kk}(x,y)^4dxdy.$$

Recall that r_{kk} is given by (6.4) and (6.5). Thus,

$$\begin{aligned}\int_{B_R\times B_R}r_{kk}(x,y)^4dxdy&\sim\frac{4\pi R^3}{3}\int_0^R[1-\frac{\rho}{R}]^3\int_{\partial B_\rho(x)}r_{kk}(x,y)^4dyd\rho\\ &\sim\frac{4\pi R^3}{3}\int_0^\infty\int_{\partial B_\rho(x)}r_{kk}(x,y)^4dyd\rho.\end{aligned}$$

The fourth power gives rise to several terms that we treat separately. For the first one, using spherical coordinates as before, we have

$$\begin{aligned}\sum_{k=1}^{3}\int_0^\infty\int_{\partial B_\rho(x)}A(|x-y|)^4\Delta_k^8dxdy&=\int_0^\infty A(\rho)^4\rho^2d\rho\cdot\int_{[0,\pi]\times[0,2\pi]}\Delta_k^8\sin(\theta)d\theta d\varphi\\ &=\frac{11\pi}{140}\frac{12\pi}{9}=\frac{11\pi^2}{105}.\end{aligned}$$

Moving to the next terms, with similar arguments we have

$$\sum_{k=1}^{3}\int_0^\infty\int_{\partial B_\rho(x)} A(\rho)^3\Delta_k^6 B(\rho)dyd\rho = \frac{\pi}{70}\frac{12\pi}{7} = \frac{6\pi^2}{245}$$

$$\sum_{k=1}^{3}\int_0^\infty\int_{\partial B_\rho(x)} A(\|x-y\|)^2\Delta_k^4 B(\|x-y\|)^2 dxdy = \frac{2\pi}{315}\frac{12\pi}{5} = \frac{8\pi^2}{525},$$

$$\sum_{k=1}^{3}\int_0^\infty\int_{\partial B_\rho(x)} A(\rho)\Delta_k^2 B(\rho)^3 dyd\rho = \frac{17\pi}{3780}\frac{12\pi}{3} = \frac{17\pi^2}{945},$$

$$\int_0^\infty\int_{\partial B_\rho(x)} B(\rho)^4 dyd\rho = 4\pi\frac{17\pi}{2835} = \frac{68\pi^2}{2835}$$

Gathering all together.

$$\begin{aligned}\sum_{k=1}^{3}&\int_{B_R\times B_R}\mathbb{E}(H_4(\bar\xi_k(x))H_4(\bar\xi_k(y)))dxdy\\ &\sim 81\cdot 4!\frac{4\pi R^3}{3}\Big[\frac{11\pi^2}{105} - 4\frac{6\pi^2}{245} + 6\frac{8\pi^2}{525} - 4\frac{17\pi^2}{945} + 3\frac{68\pi^2}{2835}\Big]\\ &= 81\cdot 4!\frac{4\pi^3 R^3}{3}\frac{361}{3675} = \frac{4\pi^3R^3}{3}\frac{233928}{1225}.\end{aligned}$$

We move to the case $k\neq k'$.

$$\int_{B_R\times B_R}\mathbb{E}(H_4(\bar\xi_k(x))H_4(\bar\xi_{k'}(y)))dxdy = 81\cdot 4!\int_{B_R\times B_R} r_{kk'}(x,y)^4 dxdy.$$

Recall that $r_{kk'}$, $k\neq k'$, is given by (6.6).

$$\begin{aligned}\sum_{k\neq k'}\int_{B_R\times B_R} r_{kk'}(x,y)^4dxdy &\sim \frac{4\pi R^3}{3}\sum_{k\neq k'}\int_0^\infty\int_{\partial B_\rho(x)} r_{kk'}(x,y)^4 dyd\rho\\ &= \frac{4\pi R^3}{3}\int_0^\infty A(\rho)^4\rho^2 d\rho\cdot\sum_{k\neq k'}\int_{[0,\pi]\times[0,2\pi]}\Delta_k^4\Delta_{k'}^4\sin(\theta)d\theta d\varphi\\ &= \frac{4\pi R^3}{3}\frac{11\pi}{140}\frac{24\pi}{105} = \frac{4\pi^3R^3}{3}\frac{22}{1225}.\end{aligned}$$

Thus, multiplying this last result by $81\cdot 4!$ and summing it to the diagonal terms:

$$\begin{aligned}\sum_{k,k'=1}^{3}\int_{B_R\times B_R}\mathbb{E}(H_4(\bar\xi_k(x))H_4(\bar\xi_{k'}(y)))dxdy &= \frac{4\pi^3R^3}{3}\Big[\frac{233928}{1225} + \frac{42768}{1225}\Big]\\ &= \frac{4\pi^3R^3}{3}\frac{39528}{175}.\end{aligned}$$

Multiplying by $25/81$ we get the result.

(2) Let us consider $\mathrm{Cov}(A_1, A_2)$. There are many terms which vanish since ξ and η are independent random fields.

$$\begin{aligned}\mathrm{Cov}(A_{11}, A_{23}) &= \mathrm{Cov}(A_{11}, A_{24}) = \mathrm{Cov}(A_{12}, A_{23}) = \mathrm{Cov}(A_{12}, A_{24}) = \mathrm{Cov}(A_{13}, A_{21})\\ &= \mathrm{Cov}(A_{13}, A_{24}) = \mathrm{Cov}(A_{14}, A_{22}) = \mathrm{Cov}(A_{14}, A_{23}) = \mathrm{Cov}(A_{14}, A_{24})\\ &= \mathrm{Cov}(A_{15}, A_{21}) = \mathrm{Cov}(A_{15}, A_{23}) = \mathrm{Cov}(A_{15}, A_{24}) = 0.\end{aligned}$$

We treat the rest of them.

- $\mathrm{Cov}(A_{11}, A_{21}) = \mathrm{Cov}(A_{12}, A_{22}) \sim -\frac{4\pi^3 R^3}{3}\frac{1}{2}$. Indeed,

$$\sum_{k=1}^{3} \int_{B_R \times B_R} \mathbb{E}\big[H_4(\xi(x))H_2(\xi(y))H_2(\bar{\xi}_k(y))\big]dxdy$$
$$= 3 \cdot 4! \sum_{k=1}^{3} \int_{B_R \times B_R} r(x,y)^2 r_{0k}(x,y)^2 dxdy$$
$$= 3 \cdot 4! \sum_{k=1}^{3} \int_{B_R \times B_R} \mathrm{sinc}(\rho)^2 \mathrm{sinc}'(\rho)^2 \Delta_k^2 dxdy \sim 3 \cdot 4! \frac{4\pi R^3}{3}\frac{\pi}{12}4\pi = \frac{4\pi^3 R^3}{3}24.$$

Multiplying by $-1/48$ we get the result.

- $\mathrm{Cov}(A_{13}, A_{23}) = \mathrm{Cov}(A_{13}, A_{24}) \sim -\frac{4\pi^3 R^3}{3}\frac{1}{6}$. Indeed,

$$\sum_{k=1}^{3} \int_{B_R \times B_R} \mathbb{E}\big[H_2(\xi(x))H_2(\xi(y))\big]\mathbb{E}\big[H_2(\eta(y))H_2(\bar{\eta}_k(y))\big]dxdy$$
$$= 3 \cdot 4 \sum_{k=1}^{3} \int_{B_R \times B_R} r(x,y)^2 r_{0k}(x,y)^2 dxdy \sim 3 \cdot 4 \frac{4\pi R^3}{3}\frac{\pi^2}{3} = \frac{4\pi^3 R^3}{3}4.$$

We used the computations in $\mathrm{Cov}(A_{11}, A_{21})$. Multiplying by $-1/24$ we get the result.

- $\mathrm{Cov}(A_{14}, A_{21}) = \mathrm{Cov}(A_{15}, A_{22}) \geq 0$. We have

$$\mathrm{Cov}(A_{14}, A_{21}) = \frac{5}{54} \sum_{k,k'=1}^{3} \int_{B_R \times B_R} \mathbb{E}\big[H_4(\bar{\xi}_k(x))H_2(\xi(y))H_2(\bar{\xi}_{k'}(y))\big]dxdy$$
$$= \frac{5}{54} 27 \cdot 4! \sum_{k,k'=1} \int_{B_R \times B_R} r_{k0}(x,y)^2 r_{kk'}(x,y)^2 dxdy \geq 0.$$

Thus, we can omit it.

(3) We move to $\mathrm{Cov}(A_1, A_3)$. Again there are several vanishing terms.

$$\begin{aligned}\mathrm{Cov}(A_{11}, A_{32}) &= \mathrm{Cov}(A_{11}, A_{33}) = \mathrm{Cov}(A_{12}, A_{31}) = \mathrm{Cov}(A_{12}, A_{33}) \\ &= \mathrm{Cov}(A_{13}, A_{31}) = \mathrm{Cov}(A_{13}, A_{32}) = \mathrm{Cov}(A_{14}, A_{32}) \\ &= \mathrm{Cov}(A_{14}, A_{33}) = \mathrm{Cov}(A_{15}, A_{31}) = \mathrm{Cov}(A_{15}, A_{33}) = 0.\end{aligned}$$

We look at the rest of the terms.

- $\mathrm{Cov}(A_{11}, A_{31}) = \mathrm{Cov}(A_{12}, A_{32}) \geq 0$. Indeed,

$$\mathrm{Cov}(A_{11}, A_{31}) = \frac{1}{72} \sum_{i \neq j} \int_{B_R \times B_R} \mathbb{E}\big[H_4(\xi(x))H_2(\bar{\xi}_i(y))H_2(\bar{\xi}_j(y))\big]dxdy$$
$$= \frac{1}{72} 9 \cdot 4! \sum_{i \neq j} \int_{B_R \times B_R} r_{0i}(x,y)^2 r_{0j}(x,y)^2 dxdy \geq 0.$$

- $\mathrm{Cov}(A_{13}, A_{33}) \geq 0$. we have,

$$\mathrm{Cov}(A_{13}, A_{33}) = \frac{1}{36}\sum_{i\neq j}\int_{B_R\times B_R}\mathbb{E}\big[H_2(\xi(x))H_2(\bar{\xi}_i(y))\big]\mathbb{E}\big[H_2(\eta(x))H_2(\bar{\eta}_j(y))\big]dxdy$$

$$= \frac{1}{36}9\cdot 4\sum_{i\neq j}\int_{B_R\times B_R} r_{0i}(x,y)^2 r_{0j}(x,y)^2 dxdy \geq 0.$$

- $\mathrm{Cov}(A_{14}, A_{31}) = \mathrm{Cov}(A_{15}, A_{32}) \sim -\frac{4\pi^3R^3}{3}\frac{592}{105}$. We have,

$$\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R}\mathbb{E}\big[H_4(\bar{\xi}_k(x))H_2(\bar{\xi}_i(y))H_2(\bar{\xi}_j(y))\big]dxdy$$

$$= 81\cdot 4!\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R} r_{ki}(x,y)^2 r_{kj}(x,y)^2 dxdy$$

We begin with the case $k\neq i, k\neq j$, $i\neq j$, we write $k\neq i\neq j$ for short.

$$\sum_{k\neq i\neq j}\int_{B_R\times B_R} r_{ki}(x,y)^2 r_{kj}(x,y)^2 dxdy = \sum_{k\neq i\neq j}\int_{B_R\times B_R} A(|x-y|)^4\Delta_k^4\Delta_i^2\Delta_j^2 dxdy$$

$$= \frac{4\pi R^3}{3}\frac{11\pi}{140}\frac{24\pi}{315} = \frac{4\pi^3R^3}{3}\frac{22}{3675}.$$

Now, we consider the case $k = i \neq j$, ($k = j\neq i$ is equal).

$$\sum_{k\neq j}\int_{B_R\times B_R} r_{kk}(x,y)^2 r_{kj}(x,y)^2 dxdy$$

$$= \sum_{k\neq j}\int_{B_R\times B_R}(A(|x-y|)^2\Delta_k^2 - B(|x-y|))^2 A(|x-y|)^2\Delta_k^2\Delta_j^2 dxdy.$$

The first integral

$$\sum_{k\neq j}\int_{B_R\times B_R} A(|x-y|)^4\Delta_k^6\Delta_j^2 dxdy \sim \frac{4\pi R^3}{3}\frac{11\pi}{140}\frac{8\pi}{21} = \frac{4\pi^3R^3}{3}\frac{22}{735}.$$

The radial part is as in the previous case. The second integral.

$$\sum_{k\neq j}\int_{B_R\times B_R} A(|x-y|)^3 B(|x-y|)\Delta_k^4\Delta_j^2 dxdy \sim \frac{4\pi R^3}{3}\frac{\pi}{70}\frac{24\pi}{35} = \frac{4\pi^3R^3}{3}\frac{12}{1225}.$$

The third integral is

$$\sum_{k\neq j}\int_{B_R\times B_R} A(|x-y|)^2 B(|x-y|)^2\Delta_k^2\Delta_j^2 dxdy \sim \frac{4\pi R^3}{3}\frac{2\pi}{315}\frac{8\pi}{5} = \frac{4\pi^3R^3}{3}\frac{16}{1575}.$$

Thus,

$$\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R}\mathbb{E}\big[H_4(\bar{\xi}_k(x))H_2(\bar{\xi}_i(y))H_2(\bar{\xi}_j(y))\big]dxdy$$

$$\sim 81\cdot 4!\frac{4\pi^3R^3}{3}\Big[\frac{22}{3675} + 2\big(\frac{22}{735} - 2\frac{12}{1225} + \frac{16}{1575}\big)\Big] = \frac{4\pi R^3}{3}\frac{15984}{175}.$$

The result follows multiplying by $-5/81$.

(4) Finally, we consider $\mathrm{Cov}(A_2, A_3)$. We have

$$\begin{aligned}\mathrm{Cov}(A_{21}, A_{32}) &= \mathrm{Cov}(A_{21}, A_{33}) = \mathrm{Cov}(A_{22}, A_{31}) = \mathrm{Cov}(A_{22}, A_{33}) \\ &= \mathrm{Cov}(A_{23}, A_{31}) = \mathrm{Cov}(A_{23}, A_{32}) = \mathrm{Cov}(A_{24}, A_{31}) = \mathrm{Cov}(A_{24}, A_{32}) = 0.\end{aligned}$$

Besides, we have.

- $\mathrm{Cov}(A_{21}, A_{31}) = \mathrm{Cov}(A_{22}, A_{32}) = -\frac{4\pi^3 R^3}{3}\frac{1304}{3675}$.

$$\begin{aligned}&\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R}\mathbb{E}\big[H_2(\xi(x))H_2(\bar{\xi}_k(x))H_2(\bar{\xi}_i(y))H_2(\bar{\xi}_j(y))\big]dxdy \\ &\quad= 27\cdot 4\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R} r_{0i}(x,y)^2 r_{kj}(x,y)^2 dxdy \\ &\quad+ 27\cdot 4\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R} r_{0j}(x,y)^2 r_{ki}(x,y)^2 dxdy \\ &\quad+ 27\cdot 16\sum_{k=1}^{3}\sum_{i\neq j}\int_{B_R\times B_R} r_{0i}(x,y) r_{0j}(x,y) r_{ki}(x,y) r_{kj}(x,y) dxdy\end{aligned}$$

We begin with the first two integrals (their sum over $k, i\neq j$ coincide). Consider $k\neq i$, $k\neq j$ (and $i\neq j$), we write $i\neq k\neq j$ for short.

$$\begin{aligned}&\sum_{i\neq k\neq j}\int_{B_R\times B_R} r_{0j}(x,y)^2 r_{ki}(x,y)^2 dxdy \\ &\qquad= \sum_{i\neq k\neq j}\int_{B_R\times B_R}\mathrm{sinc}'(|x-y|)^2 A(|x-y|)^2\Delta_i^2\Delta_k^2\Delta_j^2 dxdy \\ &\qquad\qquad\sim \frac{4\pi R^3}{3}\frac{23\pi}{420}\frac{8\pi}{35} = \frac{4\pi^3 R^3}{3}\frac{46}{3675}.\end{aligned}$$

Now, take $k = j\neq i$.

$$\begin{aligned}&\sum_{i\neq k}\int_{B_R\times B_R} r_{0k}(x,y)^2 r_{ki}(x,y)^2 dxdy \\ &\quad= \sum_{i\neq k}\int_{B_R\times B_R}\mathrm{sinc}'(|x-y|)^2 A(|x-y|)^2\Delta_i^2\Delta_k^4 dxdy \sim \frac{4\pi R^3}{3}\frac{23\pi}{420}\frac{24\pi}{35} = \frac{4\pi^3 R^3}{3}\frac{46}{1225}.\end{aligned}$$

Finally, take $k = i\neq j$.

$$\begin{aligned}&\sum_{j\neq k}\int_{B_R\times B_R} r_{0j}(x,y)^2 r_{kk}(x,y)^2 dxdy \\ &\quad= \sum_{j\neq k}\int_{B_R\times B_R}\mathrm{sinc}'(|x-y|)^2 (A(|x-y|)\Delta_k^2 - B(|x-y|))^2\Delta_j^2 dxdy \\ &\quad= \frac{4\pi R^3}{3}\Big[\frac{23\pi}{420}\frac{24\pi}{35} - 2\frac{\pi}{42}\frac{8\pi}{5} + \frac{2\pi}{105}12\pi\Big] = \frac{4\pi^3 R^3}{3}\frac{698}{3675}.\end{aligned}$$

We move to the third integral. Take $k \neq i$, $k \neq j$ and $i \neq j$. Write $i \neq k \neq j$ for short.

$$\sum_{i\neq k\neq j} \int_{B_R\times B_R} r_{0i}(x,y) r_{0j}(x,y) r_{ki}(x,y) r_{kj}(x,y) dxdy$$
$$= \sum_{i\neq k\neq j} \int_{B_R\times B_R} \operatorname{sinc}'(|x-y|)^4 \Delta_i^2 \Delta_j^2 \Delta_k^2 \, dxdy \sim \frac{4\pi R^3}{3} \frac{7\pi}{60} \frac{8\pi}{35} = \frac{4\pi^3 R^3}{3} \frac{2}{75}.$$

Finally, take $k = j \neq i$ (the case $k = i \neq j$ yields the same value).

$$\sum_{i\neq k} \int_{B_R\times B_R} r_{0i}(x,y) r_{0k}(x,y) r_{ki}(x,y) r_{kk}(x,y) dxdy$$
$$= \sum_{i\neq k} \int_{B_R\times B_R} \operatorname{sinc}'(|x-y|)^2 A(|x-y|)(A(|x-y|)\Delta_k^2 - B(|x-y|))\Delta_i^2 \Delta_k^2 \, dxdy$$
$$\sim \frac{4\pi R^3}{3}\Big[\frac{23\pi}{420}\frac{24\pi}{35} - \frac{\pi}{42}\frac{8\pi}{5}\Big] = -\frac{4\pi^3 R^3}{3}\frac{2}{3675}.$$

Gathering all together

$$\sum_{k=1}^{3} \sum_{i\neq j} \int_{B_R\times B_R} \mathbb{E}\big[H_2(\xi(x)) H_2(\bar{\xi}_k(x)) H_2(\bar{\xi}_i(y)) H_2(\bar{\xi}_j(y))\big] dxdy$$
$$= 27\cdot 4 \frac{4\pi^3 R^3}{3}\Big[2\big(\frac{46}{3675} + \frac{46}{1225} + \frac{698}{3675}\big) + 4\big(\frac{2}{75} - 2\frac{2}{3675}\big)\Big] = 27\cdot 4\frac{4\pi^3 R^3}{3}\frac{652}{3675}.$$

Multiplying by $-1/54$ we get the result.

- $\operatorname{Cov}(A_{23}, A_{33}) = \operatorname{Cov}(A_{24}, A_{33}) \sim -\frac{4\pi^3 R^3}{3}\frac{316}{735}$.

$$\sum_{k=1}^{3} \sum_{i\neq j} \int_{B_R\times B_R} \mathbb{E}\big[H_2(\xi(x)) H_2(\bar{\xi}_i(y))\big]\mathbb{E}\big[H_2(\bar{\eta}_k(x)) H_2(\bar{\eta}_j(y))\big] dxdy$$
$$= 27\cdot 4 \sum_{k=1}^{3} \sum_{i\neq j} \int_{B_R\times B_R} r_{0i}(x,y)^2 r_{kj}(x,y)^2 dxdy = 27\cdot 4\frac{4\pi^3 R^3}{3}\frac{158}{735}.$$

Multiplying by $-1/54$ we get the result. This integral coincides with the first two terms in $\operatorname{Cov}(A_{21}, A_{31})$.

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