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On SDEs for Bessel Processes in low dimension and path-dependent extensions

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Abstract. The Bessel process in low dimension ($0 \leq \delta \leq 1$) is not an Itô process and it is a semimartingale only in the cases $\delta = 1$ and $\delta = 0$. In this paper we first characterize it as the unique solution of an SDE with distributional drift or more precisely its related martingale problem. In a second part, we introduce a suitable notion of *path-dependent Bessel processes* and we characterize them as solutions of path-dependent SDEs with distributional drift.

1. Introduction

The class of Bessel processes is one of the most important classes of diffusion processes with values in $\mathbb{R}_+$. It is a family of strong Markov processes parameterized by $\delta \in \mathbb{R}_+$ (called the *dimension*), which has deep connections with the radial behavior of the Brownian motion, square-root diffusions, conformally invariant processes, etc. Bessel processes have been largely investigated in the literature, we refer the reader to e.g. Mansuy and Yor (2008); Zambotti (2017); Revuz and Yor (1999) (Section 2.3, Chapter 3 and Chapter XI, respectively) for an overview on Bessel processes.

Let $x_0 \geq 0$. We recall that a *Bessel process* X (with initial condition x_0, dimension $\delta \geq 0$ and denoted by $\mathrm{BES}^\delta(x_0)$) is defined as the square root of the so-called *squared Bessel process* (with initial condition $s_0 = x_0^2$, dimension $\delta \geq 0$ and denoted by $\mathrm{BESQ}^\delta(x_0^2)$), which is characterized as

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the pathwise unique solution of the SDE

$$dS_t = 2\sqrt{|S_t|}dW_t + \delta dt, S_0 = x_0^2, \tag{1.1}$$

where W is a standard Brownian motion.

When $\delta > 1$ it is possible to characterize X as (pathwise unique non-negative) solution of

$$dX_t = \frac{\delta - 1}{2} X_t^{-1} dt + dW_t, \tag{1.2}$$

where W is again a standard Brownian motion, see for instance Exercise (1.26) of Chapter IX in Revuz and Yor (1999). From now on the letter W will always denominate such a process. In particular X is an Itô process. For $0 \le \delta \le 1$, the integral $\int_0^t X_s^{-1} ds$ does not converge and $\mathrm{BES}^\delta(x_0)$ is a non-semimartingale process, except for $\delta = 1$ and $\delta = 0$, see Revuz and Yor (1999); Jeanblanc et al. (2009), Chapter XI Section 1 and Section 6.1, respectively. If $0 < \delta < 1$, see for instance Bertoin (1991) it is known that

$$X_t = x_0 + \frac{\delta - 1}{2} \text{ p.v. } \int_0^t \frac{ds}{X_s} ds + W_t, t \ge 0, \tag{1.3}$$

where p.v. stands for principal value defined by

$$\text{p.v. } \int_0^t \frac{ds}{X_s} ds := \int_{\mathbb{R}_+} (L_t^X(a) - L_t^X(0)) a^{\delta - 2} da,$$

where L^X is the local time of X, defined as a density occupation measure. For details, see e.g. Mansuy and Yor (2008).

The drift in decomposition (1.3) is a zero energy additive functional in the language of Markov processes and $\mathrm{BES}^\delta(x_0)$ is a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process. As a consequence, in the low dimensional regime, (1.2) does not correctly represent the paths of $\mathrm{BES}^\delta(x_0)$. Representation (1.3) can be interpreted as the Dirichlet process decomposition of $\mathrm{BES}^\delta(x_0)$. For further details, we refer the reader to the works Zambotti (2017); Engelbert and Wolf (1998); Mansuy and Yor (2008) and other references therein.

Typical examples of low-dimensional Bessel processes appear in the theory of Schramm-Loewner evolution, see e.g. Lawler (2005). Two-parameter family of Schramm-Loewner evolution $SLE(\kappa, \kappa - 4)$ defined in Lawler et al. (2003) provides a source of examples of BES^δ flows with very singular behavior when $\delta = 1 - \frac{4}{\kappa}, \kappa > 4$. In fact, the final right-boundary of $\mathrm{SLE}(\kappa, \kappa - 4)$ processes is described by the excursions of $\mathrm{BES}^\delta(x_0)$. We refer the reader to Dubédat (2006) for more details. We also drive attention to Beliaev et al. (2020) for more recent applications of low-dimensional Bessel processes starting at the origin.

In this work, we characterize $\mathrm{BES}^\delta(x_0)$, for $0 \le \delta \le 1$, as the unique solution of an SDE with distributional drift. The main result of this paper states that one natural way to investigate the SDE dynamics of low-dimensional Bessel processes is the interpretation of the singular drift $x \mapsto \frac{1}{x}$ as the derivative in the sense of Schwartz distributions of the function $x \mapsto \log|x|$ rather than principal values via local times. For this purpose, we interpret (1.2) as a strong-martingale problem previously introduced by Russo and Trutnau (2007). In this case, for $0 \le \delta \le 1$, we prove $\mathrm{BES}^\delta(x_0)$ is the unique non-negative solution of a suitable strong-martingale problem starting at $x_0 \ge 0$. A non-Markovian extension is also considered for SDEs with singular drifts of the form

$$\frac{\delta - 1}{2} \frac{1}{X_t} + \Gamma(t, X^t),$$

where Γ is a path-dependent non-anticipative functional satisfying some technical conditions and X^t will be given in (2.2). Our analysis is inspired by the series of works Flandoli et al. (2003, 2004);

Russo and Trutnau (2007) which treat Markovian SDEs of the form

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \tag{1.4}$$

where σ and b are continuous functions on $\mathbb{R}$. Moreover σ is strictly positive and one supposes the existence of the function

$$\Sigma(x) := 2\int_0^x \frac{b'}{\sigma^2}(y)dy, x \in \mathbb{R}, \tag{1.5}$$

as a suitable limit via regularization. We stress that b' is the derivative of some function b in the sense of distributions. Assuming (1.5), the Markovian operator L is defined by the authors as

$$Lf = (e^{\Sigma}f')'\frac{e^{-\Sigma}\sigma^2}{2}, \tag{1.6}$$

where f belongs to the domain

$$D_L = \{f \in C^1(\mathbb{R}) | f'e^{\Sigma} \in C^1(\mathbb{R})\},$$

see e.g. Flandoli et al. (2003), Section 2 and also Ohashi et al. (2022) Proposition 4.1.

When σ and b' are functions then previous expression equals

$$Lf = \frac{\sigma^2}{2}f'' + b'f'. \tag{1.7}$$

In Ohashi et al. (2022), we have studied the class of SDEs

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \tag{1.8}$$

for some classes of functionals Γ.

In this paper, we will investigate existence and uniqueness of an SDE of the type (1.8), where $\sigma = 1$, but b is no more a continuous function. More precisely, we focus on the SDE

$$dX_t = dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \ X_0 \stackrel{d}{=} \delta_{x_0}, \tag{1.9}$$

where b is given by

$$b(x) = \begin{cases} \frac{\delta-1}{2}\log|x|, x \in \mathbb{R}^* & | \quad \delta \neq 1 \\ H(x), x \in \mathbb{R} & | \quad \delta = 1, \end{cases} \tag{1.10}$$

and H is the Heaviside function and $\mathbb{R}^* = \mathbb{R} - \{0\}$. Then, (1.3) is considered as a particular case of the SDE (1.9) with distributional drift b' and $\Gamma = 0$. Even though b is no longer a continuous function, (1.5) can still be defined in such a way that $\Sigma \equiv 2b$ and (1.6) holds. We distinguish the two cases: $0 \leq \delta < 1$ and $\delta = 1$.

- $0 \leq \delta < 1$. If b is given by (1.10), then (1.5) implies

  $$\exp(-\Sigma(x)) = |x|^{1-\delta}. \tag{1.11}$$

  At this point, representation (1.6) for $L^\delta = L$ yields

  $$L^\delta f(x) = \frac{f''(x)}{2} + \frac{(\delta-1)f'(x)}{2x}, \ x \neq 0. \tag{1.12}$$

- $\delta = 1$. In this case, $b(x) = H(x)$. So (1.6) yields

  $$L^1 f(x) = \frac{f''(x)}{2} + \delta_0 f'(x), \ x \neq 0, \tag{1.13}$$

  where δ_0 is the Dirac measure at zero. Those expressions are perfectly well-defined for $f \in \mathcal{D}_{L^\delta}$ defined in Section 3.2 below. The product $\delta_0 f'$ for $f \in D_L$ is necessarily zero.

We then study the (possibly non-Markovian) martingale problem associated with the operator

$$\mathcal{L}^\delta f = L^\delta + \Gamma f',$$

in a suitable domain. The notion of martingale problem related to $\mathcal{L}^\delta$ is given by Definition 2.2. The notion of strong martingale problem related to the domain of L^δ and an underlying Brownian motion W is given by Definition 2.3, which borrows the one in Russo and Trutnau (2007). It has to be compared with the notion of strong existence and pathwise uniqueness of an SDE. In particular, it represents the corresponding notion of strong solution of SDEs in the framework of martingale problems.

Sections 3.2, 3.3, 3.4 present a series of results concerning existence/uniqueness for the SDE (1.9) in Markovian case, for $0 \le \delta < 1$. In particular, Propositions 3.6 and 3.15 show the low-dimensional Bessel process $\mathrm{BES}^\delta(x_0)$ as the unique non-negative solution of the strong martingale problem associated with L^δ for $0 < \delta < 1$ and $x_0 \ge 0$. A similar discussion concerns the case $\delta = 1$, see Section 3.7, Propositions 3.23 and 3.24. We remark that in the case $\delta = 1$, results for pathwise uniqueness, see Harrison and Shepp (1981) were already available in the literature.

In Section 3.5 we connect the martingale problem related to Bessel processes to one related to an extended domain which includes the harmonic function

$$h(x) = \mathrm{sign}(x)\frac{|x|^{2-\delta}}{2-\delta}, x \in \mathbb{R}. \tag{1.14}$$

We also give general conditions on the marginal law of a generic process which is solution of the basic martingale problem to solve the one with extended domain. This is fulfilled for instance by the Bessel process with dimension $\delta > 0$. Related considerations are discussed when $\delta = 0$.

In Section 4, we establish existence and uniqueness of the martingale problem associated with the non-Markovian SDE (1.9) under the condition that Γ is bounded; see Propositions 4.2 and 4.11. Proposition 4.8 proves existence when Γ is unbounded with some technical conditions. Theorem 4.16 illustrate sufficient conditions on Γ to have well-posedness of the strong martingale problem.

We highlight that Aryasova and Pilipenko (2011) have established uniqueness for (1.3) of non-negative solutions X, when $0 \le \delta \le 1$, under the condition that the solution X spends zero time at the point zero, i.e.,

$$\mathbb{E}\Big[\int_0^\infty 1_{\{0\}}(X_s)ds\Big] = 0. \tag{1.15}$$

In contrast to Aryasova and Pilipenko (2011) we do not suppose that assumption and we provide uniqueness among all non-negative solutions.

One important objective of the paper is the definition of *path-dependent Bessel process*. Let $\delta \ge 2$ be an integer. Similarly to the classical Markovian case with integer dimension, a path-dependent Bessel type process (as solution of (1.9)) appears considering the dynamics of a δ-dimensional Brownian motion β with drift having a radial intensity proportional to a non-anticipative functional Γ.

More precisely, let Y be a solution to

$$dY_t = d\beta_t + \Gamma(t, \|Y_s\|_{\mathbb{R}^\delta}, s \le t)\frac{Y_t}{\|Y_t\|_{\mathbb{R}^\delta}}dt, \tag{1.16}$$

Then $X_t := \|Y_t\|_{\mathbb{R}^\delta}$, i.e. the Euclidean norm in $\mathbb{R}^\delta$, solves (1.9). Indeed, if Y is a solution of (1.16), then a formal application of Itô's formula to $\rho_t := \|Y_t\|^2_{\mathbb{R}^\delta}$ and Lévy's characterization theorem for local martingales show that

$$d\rho_t = 2\sqrt{\rho_t}dW_t + 2\sqrt{\rho_t}\Gamma(t, \sqrt{\rho_s}, s \le t)dt + \delta dt. \tag{1.17}$$

A subsequent formal application of Itô's formula shows that $X_t = \sqrt{\rho_t}$ solves (1.9). Our result concerns the extension of that model to the singular case represented by $\delta \in [0, 1]$.

The paper is organized as follows. After this Introduction we recall the notations and some important results from Ohashi et al. (2022). Then we introduce specific preliminary considerations. Section 3 is devoted to the case of Bessel processes in low dimension, under the perspective of strong martingale problems. Section 4 discusses the case of non-Markovian perturbations of Bessel processes.

2. About path-dependent martingale problems

2.1. *Preliminary notations, definitions and results.*

In this section we recall the general notation and some necessary results from Ohashi et al. (2022).

Let I be an interval of $\mathbb{R}$. For $k \in \mathbb{N}$, $C^k(I)$ will denote the space of real functions defined on I having continuous derivatives till order k. Such space is endowed with the uniform convergence topology on compact sets for the functions and all derivatives. Generally $I = \mathbb{R}$, $\mathbb{R}_+ := [0, +\infty[$, $\mathbb{R}_- :=]-\infty, 0]$, $[0, T]$, for some fixed positive real T. If there is no ambiguity $C^k(\mathbb{R})$ will be simply indicated by C^k. The space of continuous functions on I will be denoted by $C(I)$. Given an a.e. bounded real function f, $|f|_\infty$ will denote the essential supremum.

We recall some notions from Flandoli et al. (2003). For us all filtrations $\mathfrak{F}$ fulfill the usual conditions. When no filtration is specified, we mean the canonical filtration of an underlying process. Otherwise, the canonical filtration associated with a process X is denoted by $\mathfrak{F}^X$.

A sequence (X^n) of continuous processes indexed by $[0, T]$ is said to converge u.c.p. to some process X whenever $\sup_{t \in [0,T]} |X^n_t - X_t|$ converges to zero in probability.

We consider a locally bounded functional

$$\Gamma : \Lambda \to \mathbb{R}, \tag{2.1}$$

where

$$\Lambda := \{(s, \eta^s), s \in [0, T], \eta \in C([0, T])\}$$

and

$$\eta^t_s = \begin{cases} \eta_s, & \text{if} \quad s \le t \\ \eta_t, & \text{if} \quad s > t. \end{cases} \tag{2.2}$$

By convention, we extend Γ from Λ to $[0, T] \times C([0, T])$ by setting (in a non-anticipating way)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]).$$

All along the paper E will denote $\mathbb{R}$ or $\mathbb{R}_+$.

Let us consider some locally bounded Borel functions $\sigma, b' : E \to \mathbb{R}$. In this case the path-dependent SDE

$$\begin{cases} dX_t &= \sigma(X_t) dW_t + b'(X_t) dt + \Gamma(t, X^t) dt \\ X_0 &= \xi, \end{cases} \tag{2.3}$$

for some deterministic initial condition ξ taking values in E, makes perfectly sense, see Section 5 of Ohashi et al. (2022), in particular one can speak about strong existence, pathwise uniqueness, existence and uniqueness in law. (2.3) is denominated by $E(\sigma, b', \Gamma)$. Proposition 3.2 in Ohashi et al. (2022) implies the following.

Proposition 2.1. *Let $b' : E \to \mathbb{R}$ be a locally bounded function. We set $Lf = \frac{\sigma^2}{2} f'' + b' f'$, $f \in C^2(E)$. A couple $(X, \mathbb{P})$ is a solution of $E(\sigma, b', \Gamma)$, if and only if, under $\mathbb{P}$,*

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds \tag{2.4}$$

is a local martingale, where $Lf = \frac{\sigma^2}{2} f'' + b' f'$, for every $f \in C^2(E)$.

In this paper, we will be interested in a formal $E(\sigma, b', \Gamma)$ where $\sigma = 1$ but b' is the derivative of some specific Borel discontinuous function. The formulation is inspired by Proposition 2.1 which states that the SDE is equivalent to a specific martingale problem. We will consider formal PDE operators of the type $L : \mathcal{D}_L(E) \subset C^1(E) \to C(E)$, where Lf gives formally $\frac{\sigma^2}{2} f'' + b'f'$. When b', σ are locally bounded functions then $\mathcal{D}_L(E) = C^2(E)$. In that case, the notion of martingale problem is (since the works of Stroock and Varadhan Stroock and Varadhan (2006)) is a concept related to solutions of SDEs in law.

Definition 2.2.

(1) We say that a continuous stochastic process X solves (with respect to a probability $\mathbb{P}$ on some measurable space $(\Omega, \mathcal{F})$) the martingale problem related to

$$\mathcal{L}f := Lf + \Gamma f', \tag{2.5}$$

with initial condition $\nu = \delta_{x_0}, x_0 \in E$, with respect to a domain $\mathcal{D}_L(E)$ if

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s)\Gamma(s, X^s) ds, \tag{2.6}$$

is a $\mathbb{P}$-local martingale for all $f \in \mathcal{D}_L(E)$.

We will also say that the couple $(X, \mathbb{P})$ is a solution of (or $(X, \mathbb{P})$ solves) the martingale problem with respect to $\mathcal{D}_L(E)$.

(2) If a solution exists we say that the martingale problem above *admits existence*.

(3) We say that the martingale problem above *admits uniqueness* if any two solutions $(X^i, \mathbb{P}^i)$, $i = 1, 2$ (on some measurable space $(\Omega, \mathcal{F})$) have the same law.

In the sequel, when the measurable space $(\Omega, \mathcal{F})$ is self-explanatory it will be often omitted.

Below we introduce the analogous notion of strong existence and pathwise uniqueness for our martingale problem, see also Ohashi et al. (2022) for the case when b' is the derivative of a continuous function and Russo and Trutnau (2007) for the case $\Gamma = 0$. In both cases we had $E = \mathbb{R}$.

Definition 2.3.

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathfrak{F} = (\mathcal{F}_t)$ be the canonical filtration associated with a fixed Brownian motion W. Let $x_0 \in E$. We say that a continuous $\mathfrak{F}$-adapted E-valued process X such that $X_0 = x_0$ is a **solution to the strong martingale problem** (related to (2.5), σ) with respect to $\mathcal{D}_L(E)$ and W (with related filtered probability space) (with $X_0 = x_0$) if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s)\Gamma(s, X^s) ds = \int_0^t f'(X_s)\sigma(X_s) dW_s, \tag{2.7}$$

for all $f \in \mathcal{D}_L(E)$.

(2) We say that the martingale problem related to (2.5) and σ with respect to $\mathcal{D}_L(E)$ admits **strong existence** if for every $x_0 \in E$, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{F})$, where $\mathfrak{F} = (\mathcal{F}_t)$ is the canonical filtration associated with a Brownian motion W, there is a process X solving the strong martingale problem (related to (2.5) and σ) with respect to $\mathcal{D}_L(E)$ and W with $X_0 = x_0$.

(3) We say that the martingale problem (related to (2.5)) with respect to $\mathcal{D}_L(E)$ admits **pathwise uniqueness** if given $(\Omega, \mathcal{F}, \mathbb{P})$ and a Brownian motion W and $X^i, i = 1, 2$ are solutions to the strong martingale problem with respect to $\mathcal{D}_L(E)$ and W with $\mathbb{P}[X_0^1 = X_0^2] = 1$ then X^1 and X^2 are indistinguishable.

The mention E will be often omitted when $E = \mathbb{R}$. For instance $C^1(E), C^2(E), \mathcal{D}_L(E)$, will be simply denoted by $C^1, C^2, \mathcal{D}_L$.

3. Martingale problem for Bessel processes

3.1. *Preliminary considerations.*

In this section, we are going to introduce and investigate well-posedness for a martingale problem related to a Bessel process. In this section again W will denote a standard Brownian motion. We recall that the rigorous definition of the Bessel process is the following. A non-negative process X is said to be a Bessel process starting at x_0 with dimension $\delta \geq 0$ (notation $BES^\delta(x_0)$) if $S = X^2$ is a squared Bessel process starting at $s_0 = x_0^2$ of dimension δ. S is denoted by $BESQ^\delta(s_0)$, we recall in particular that it is the pathwise unique solution of (1.1).

As is shown in Proposition 2.13 in Chapter 5 of Karatzas and Shreve (1991) (see also Zambotti (2017, Chapter 3)) (1.1) admits pathwise uniqueness. Since $x \mapsto \sqrt{|x|}$ has linear growth it has weak existence and so by Yamada-Watanabe theorem it also admits strong existence.

Remark 3.1. For $\delta > 1$, we know that the Bessel process X fulfills

$$X_t = x_0 + \frac{\delta - 1}{2}\int_0^t X_s^{-1} ds + W_t. \tag{3.1}$$

We recall that for $\delta > 2$, X is even transient and it never touches zero, see Revuz and Yor (1999, Chapter XI). As anticipated, when $\delta = 1$ or $\delta = 0$ X is still a semimartingale. Unfortunately if $0 < \delta < 1$ that is no more the case, see Chapter 10 of Mansuy and Yor (2008), and X is just a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process.

Our point of view consists in rewriting (3.1) under the form

$$X_t = x_0 + \int_0^t b'(X_s) ds + W_t, \tag{3.2}$$

where W is a Brownian motion and b' is the derivative of the function $b(x) = \frac{\delta-1}{2}\log|x|$, at least when $\delta < 1$. In other words we make use of the "analytical" p.v. of $x \mapsto \frac{1}{x}$ which is the derivative of $\log$. That object is, on $\mathbb{R}$, a Schwartz distribution and not a function, which nevertheless coincides with $x \mapsto \frac{1}{x}$ on $\mathbb{R}^*$. This indeed explains (3.1) and takes into account the "relevant" time spent by the Bessel process at zero.

In the case $\delta = 1$, for similar reasons, and taking into account the fact that the Bessel process is a reflected Brownian motion, we naturally choose b to be a Heaviside function so that b' is the δ-Dirac measure at zero.

We are going to construct two settings: one for $0 \leq \delta < 1$ and another one for $\delta = 1$. In what follows, we should recall $\mathbb{R}^* = \mathbb{R} - \{0\}$.

3.2. *The framework for* $0 \leq \delta < 1$.

According to the considerations in Section 3.1, the natural form of the operator $L^\delta := L$ (outside zero) is expected to be of the form

$$L^\delta f(x) = \frac{f''(x)}{2} + \frac{(\delta - 1)f'(x)}{2x}, \tag{3.3}$$

for $f \in C^2(\mathbb{R}^*)$.

As anticipated, we fix $b : \mathbb{R} \to \mathbb{R}$, $b(x) = \frac{\delta-1}{2}\log|x|$, $x \neq 0$ and $\sigma \equiv 1$. $x \mapsto \frac{\delta-1}{2x}$, appearing in (3.3) coincides with b' restricted to $\mathbb{R}^*$. Formally speaking, Σ as in (1.5) gives $\Sigma(x) = 2b(x)$, so

$$\exp(-\Sigma(x)) = |x|^{1-\delta}, x \in \mathbb{R}. \tag{3.4}$$

The expression (3.3) can also be expressed as

$$L^\delta f(x) = \frac{|x|^{1-\delta}}{2}(|x|^{\delta-1}f')', x \neq 0. \tag{3.5}$$

The problem is to provide a natural extension for $x = 0$, which constitutes the critical point.

We have now to specify the natural domain of L^δ, in such a way that it is compatible with (3.5).

Definition 3.2. We will denote by $\mathcal{D}_{L^\delta}$ the set of $f \in C(\mathbb{R}) \cap C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-)$ such that the following holds.

(a) There is a continuous function $g : \mathbb{R} \to \mathbb{R}$ extending $x \mapsto f'(x)|x|^{\delta-1}, x \neq 0$.
(b) There is a continuous function $G : \mathbb{R} \to \mathbb{R}$, extending $x \mapsto g'(x)|x|^{1-\delta}, x \neq 0$, (i.e. $2L^\delta f(x)$, according to (3.5)) to $\mathbb{R}$.

We define then

$$L^\delta f := \frac{G}{2}. \tag{3.6}$$

Proposition 3.3.

(1) Suppose $\delta > 0$. Then $\mathcal{D}_{L^\delta} = \mathcal{D}_\delta := \{f \in C^2(\mathbb{R}) | f'(0) = 0\}$ and

$$L^\delta f(x) = \begin{cases} \frac{f''(x)}{2} + \frac{(\delta-1)f'(x)}{2x} & : \quad x \neq 0 \\ \delta \frac{f''(0)}{2} & : \quad x = 0. \end{cases} \tag{3.7}$$

(2) Suppose $\delta = 0$. Then $\mathcal{D}_{L^0} = \mathcal{D}_0$, where

$$\mathcal{D}_0 := \{f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-) | f'(0) = 0\} \tag{3.8}$$

and

$$L^0 f(x) = \begin{cases} \frac{f''(x)}{2} - \frac{f'(x)}{2x} & : \quad x \neq 0 \\ 0 & : \quad x = 0. \end{cases} \tag{3.9}$$

Proof: We first show the inclusion $\mathcal{D}_{L^\delta} \subset \mathcal{D}_\delta$. Suppose $f \in \mathcal{D}_{L^\delta}$. We have

$$\lim_{x\to 0} f'(x) = \lim_{x \to 0} |x|^{1-\delta} g(x) = 0. \tag{3.10}$$

This obviously implies that $f \in C^1(\mathbb{R})$ and $f'(0) = 0$. Taking into account (3.3), we have

$$\begin{aligned} f''(0+) &:= \lim_{x\to 0+} f''(x) = \lim_{x \to 0+} \left(G(x) - (\delta - 1)\frac{f'(x)}{x} \right) \\ &= G(0) - (\delta - 1) \lim_{x\to 0+} \frac{f'(x)}{x} = G(0) - (\delta-1) f''(0+), \\ f''(0-) &:= \lim_{x\to 0-} f''(x) = \lim_{x \to 0-} \left(G(x) - (\delta - 1)\frac{f'(x)}{x} \right) \\ &= G(0) - (\delta - 1) \lim_{x\to 0-} \frac{f'(x)}{x} = G(0) - (\delta-1) f''(0-), \end{aligned}$$

by L'Hospital rule. This implies that

$$\delta f''(0+) = G(0) = \delta f''(0-). \tag{3.11}$$

To show that $f \in \mathcal{D}_\delta$, it remains to show that $f''(0+) = f''(0-)$ when $\delta \neq 0$. This obviously follows from (3.11), which shows the inclusion $\mathcal{D}_{L^\delta} \subset \mathcal{D}_\delta$ for all $\delta \in [0, 1[$. Now, (3.11), (3.3) and (3.6) show in particular (3.7) and (3.9).

We prove now the opposite inclusion $\mathcal{D}_\delta \subset \mathcal{D}_{L^\delta}$. Let $f \in \mathcal{D}_\delta$, in particular such that $f'(0) = 0$. We need to prove that it fulfills the properties (a) and (b) characterizing $\mathcal{D}_{L^\delta}$. We set $g(x) := f'(x)|x|^{\delta-1}, x \neq 0$ and $g(0) := 0$. By L'Hospital rule we can show that $\lim_{x\to 0} g(x) = 0$, so that g is continuous at zero. This proves property (a) characterizing $\mathcal{D}_{L^\delta}$. Taking the derivative of g on $\mathbb{R}^*$ we get

$$g'(x) = f''(x)|x|^{\delta-1} + (\delta - 1) f'(x) \operatorname{sign}(x) |x|^{\delta-2}. \tag{3.12}$$

Concerning property (b), as $x \mapsto G(x) := g'(x)|x|^{1-\delta}$ is continuous on $\mathbb{R}^*$ it is enough to show that $\lim_{x\to 0} G(x)$ exists. By (3.12) we obtain

$$G(x) = f''(x) + (\delta - 1)\,\mathrm{sign}(x)\frac{1}{|x|}f'(x) = f''(x) + (\delta - 1)\frac{f'(x)}{x}, x \neq 0.$$

We recall that $f''(0+)$ and $f''(0-)$ exist. Taking the limit when x goes to zero from the right and from the left, by L'Hospital rule, we get

$$\begin{aligned} G(0+) &= f''(0+) + (\delta - 1)f''(0+) = \delta f''(0+), \\ G(0-) &= f''(0-) + (\delta - 1)f''(0-) = \delta f''(0-). \end{aligned}$$

Distinguishing the cases $\delta > 0$ (in this case $f''(0+) = f''(0-)$) and $\delta = 0$, we show that $G(0+) = G(0-)$ and finally G extends continuously to 0. This concludes the proof of the two properties (a) and (b) and so the inclusion $\mathcal{D}_\delta \subset \mathcal{D}_{L^\delta}$. □

Remark 3.4. In fact one could consider a larger domain $\hat{\mathcal{D}}_{L^\delta}$ constituted by the functions $f \in C(\mathbb{R}) \cap C^2(\mathbb{R}^*)$ fulfilling the conditions (a) and (b) before (3.6). Consider for instance the L^δ-harmonic function h defined in (3.37). That function does not belong to $\mathcal{D}_{L^\delta}$ because it has no second left and right-derivative in 0, but it is an element of $\hat{\mathcal{D}}_{L^\delta}$. In fact that domain is too large for our purposes of investigation of well-posedness. Formulating the martingale problem replacing $\mathcal{D}_{L^\delta}$ with $\hat{\mathcal{D}}_{L^\delta}$, it will be easier to show uniqueness, but more difficult to formulate existence. Suppose that $(X, \mathbb{P})$ is a solution to previous martingale problem, making use of the domain $\mathcal{D}_{L^\delta}$. The natural question is to know if $(X, \mathbb{P})$ is still a solution to the martingale problem formulated making use of $\hat{\mathcal{D}}_{L^\delta}$ instead of $\mathcal{D}_{L^\delta}$. This will be possible under a restricting condition on the law of X, see Section 3.5; this condition will be fulfilled by the Bessel process starting from a point $x_0 \neq 0$ for instance.

In the sequel we will denote by $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ the set of functions $f : \mathbb{R}_+ \to \mathbb{R}$ which are restrictions of functions $\hat{f}$ belonging to $\mathcal{D}_{L^\delta}$. We recall that, sometimes, we will also denote $\mathcal{D}_{L^\delta}(\mathbb{R}) := \mathcal{D}_{L^\delta}$. We will also denote $L^\delta f$ as the restriction to $\mathbb{R}_+$ of $L^\delta \hat{f}$. (3.7) shows that this notation is coherent. This convention will be made also for $\delta = 1$ in Section 3.7.

Starting from Section 3.3, we will make use of convergence properties for functions and processes according to the remark below.

Remark 3.5.

(1) If $g : \mathbb{R} \to \mathbb{R}$ is continuous (therefore uniformly continuous on compacts) then $g_n(x) = f\left(x + \frac{1}{n}\right)$ converges to g uniformly on compacts.
(2) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a continuous stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. If $g_n : \mathbb{R} \to \mathbb{R}$ is a sequence of functions that converges uniformly on compacts of $\mathbb{R}$ to a function g then $g_n(X)$ converges to $g(X)$ u.c.p.

3.3. *The martingale problem in the full line case when* $0 \le \delta < 1$.

Proposition 3.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a Brownian motion W. Let $x_0 \ge 0, 0 \le \delta < 1$. Let S be the solution of* (1.1) *(necessarily non-negative by comparison theorem) with $s_0 = x_0^2$, so that $X = \sqrt{S}$ is a $BES^\delta(x_0)$ process.*

Then X solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and W. In particular, for every $f \in \mathcal{D}_{L^\delta}$

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s)ds = \int_0^t f'(X_s)dW_s. \tag{3.13}$$

Remark 3.7.

(1) Suppose that S is a non-negative solution of an SDE of the type (1.1), where the Brownian motion W is replaced by a continuous semimartingale whose martingale component is a Brownian motion. Then (3.13) still holds for every $f \in \mathcal{D}_{L^\delta}$.
(2) For $\delta = 0$ and $x_0 = 0$, $BESQ^0(0)$ is the null process. By Proposition 3.3 $L^0 f(0) = 0$ for all $f \in \mathcal{D}_{L^0}$, obviously $f(0) - f(0) - \int_0^t L^\delta f(0) ds \equiv 0$ and (3.13) holds.

Proof (of Proposition 3.6):
We consider immediately the case of Remark 3.7 (1) and suppose W to be a semimartingale such that $[W]_t \equiv t$. Let $X = \sqrt{S}$, where S is a $BESQ^\delta(s_0)$, let $f \in \mathcal{D}_{L^\delta}$ and define $f_n : \mathbb{R}_+ \to \mathbb{R}$ as $f_n(y) = f\left(\sqrt{y + \frac{1}{n}}\right)$. Clearly $f_n \in C^2(\mathbb{R}_+)$. Applying Itô's formula we have

$$f_n(S_t) = f_n(S_0) + \int_0^t \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{\sqrt{S_s + \frac{1}{n}}} \sqrt{S_s} dW_s + \int_0^t \delta \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{2\sqrt{S_s + \frac{1}{n}}} ds$$
$$+ \int_0^t \left[\frac{1}{2} f''\left(\sqrt{S_s + \frac{1}{n}}\right) - \frac{1}{2} \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{\sqrt{S_s + \frac{1}{n}}}\right] \left[\frac{S_s}{S_s + \frac{1}{n}}\right] ds, \tag{3.14}$$

which can be rewritten as

$$f_n(S_t) = f_n(S_0) + \int_0^t \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{\sqrt{S_s + \frac{1}{n}}} \sqrt{S_s} dW_s + \int_0^t \frac{1}{2} f''\left(\sqrt{S_s + \frac{1}{n}}\right) \left[\frac{S_s}{S_s + \frac{1}{n}}\right] ds$$
$$+ \frac{1}{2} \int_0^t \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{\sqrt{S_s + \frac{1}{n}}} \left[\delta - \frac{S_s}{S_s + \frac{1}{n}}\right] ds. \tag{3.15}$$

The first integral converges to

$$\int_0^t f'\left(\sqrt{S_s}\right) dW_s, \tag{3.16}$$

u.c.p. by Remark 3.5, with $g = f' \in C(\mathbb{R}_+)$.

Secondly, applying Remark 3.5 with $g = f''$ and, taking into account the fact that $\frac{S_s}{S_s + \frac{1}{n}} \leq 1$, together with Lebesgue's dominated convergence theorem, the second integral in (3.15) converges u.c.p. to

$$\frac{1}{2} \int_0^\cdot f''\left(\sqrt{S_s}\right) ds. \tag{3.17}$$

We set now $\ell : \mathbb{R}_+ \longrightarrow \mathbb{R}$, the continuous function defined by

$$\ell(x) = \begin{cases} \frac{f'(x)}{x} & : \quad x \neq 0. \\ f''(0+) & : \quad x = 0. \end{cases}$$

The third integral can be rewritten as

$$\frac{1}{2} \int_0^t \ell\left(\sqrt{S_s + \frac{1}{n}}\right) \left[\delta - \frac{S_s}{S_s + \frac{1}{n}}\right] ds.$$

By Remark 3.5 with $g = \ell$ and, similarly as above, again Lebesgue's dominated convergence the previous expression converges u.c.p. to

$$\int_0^t \ell\left(\sqrt{S_s}\right) \left(\frac{\delta - 1}{2}\right) ds. \tag{3.18}$$

Finally (3.16), (3.17) and (3.18) allow to conclude the proof of (3.13). □

Below, if $x \ge 0$, we denote by X^x the $BES^\delta(x)$ process, being the square root of a solution of (1.1) with $s_0 = x^2$.

Corollary 3.8. *Let $x_0 \in \mathbb{R}, 0 \le \delta < 1$. The martingale problem with respect to $\mathcal{D}_{L^\delta}$, with initial condition $X_0 = x_0$ admits strong existence.*

(1) If $x_0 \ge 0$, X^{x_0} solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and W.
(2) If $x_0 \le 0$, $-X^{-x_0}$ solves the same strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and $-W$.

Proof: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a Brownian motion W. We set $s_0 = x_0^2$. We know that (1.1) admits a strong solution S. Then, by Proposition 3.6 $X = \sqrt{S}$ is a solution for the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and W with initial condition $|x_0|$.

So, if $x_0 \ge 0$ then strong existence is established. If $x_0 < 0$ then we show below that $-X$ also solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and $-W$.

Let $f \in \mathcal{D}_{L^\delta}$. Then obviously $f_-(x) := f(-x) \in \mathcal{D}_{L^\delta}$ and

$$L^\delta f_-(x) = L^\delta f(-x).$$

Therefore, since X solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and W, for all $f \in \mathcal{D}_{L^\delta}$ we have

$$f_-(X_t) - f_-(x_0) - \int_0^t Lf_-(X_s)ds = \int_0^t f_-'(X_s)dW_s,$$

which implies

$$f(-X_t) - f(-x_0) - \int_0^t L^\delta f(-X_s)ds = \int_0^t f'(-X_s)d(-W)_s.$$

Thus $-X$ also solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and $-W$.

□

Proposition 3.9. *Let us suppose $0 < \delta < 1$. The martingale problem with respect to $\mathcal{D}_{L^\delta}$ does not admit (in general) uniqueness in law.*

Proof: Let S be the $BESQ^\delta(0)$. By Corollary 3.8, we know that $X^+ = \sqrt{S}$ and $X^- = -\sqrt{S}$ solve the martingale problem with respect to an underlying probability $\mathbb{P}$.

Obviously X does not have the same law as $-X$ since X is positive and $-X$ is negative. □

Remark 3.10. If the initial condition x_0 is different from zero, for instance positive, then uniqueness also fails since we can exhibit two solutions. The first one is still the classical Bessel process, the second one behaving as the first one until it reaches zero and then it behaves like minus a Bessel. We recall that, when $\delta \le 1$, the corresponding Bessel process reaches $\{0\}$ a.s., see (ii) in the considerations after Corollary (1.4), Chapter XI in Revuz and Yor (1999).

For proving indeed results for uniqueness, we will need the following.

Proposition 3.11. *Let $0 \le \delta < 1$. Let $(X, \mathbb{P})$ be a solution (not necessarily positive) of the martingale problem with respect to $\mathcal{D}_{L^\delta}$. Then $S = X^2$ is a squared Bessel process.*

Proof: We first show that

$$M_t^1 := X_t^2 - x_0^2 - \delta t \tag{3.19}$$

is a local martingale and

$$X_t^4 = x_0^4 + 2(2+\delta)\int_0^t X_s^2 ds + M_t^2, \tag{3.20}$$

where M^2 is a local martingale. Clearly, $f_1(x) := x^2 \in \mathcal{D}_{L^\delta}$ because $f \in C^2(\mathbb{R})$ and $f_1'(0) = 0$. By Proposition 3.3 $L^\delta f_1(x) \equiv \delta$, which shows (3.19). On the other hand, obviously $f_2(x) := x^4 \in \mathcal{D}_{L^\delta}$

and then, by Proposition 3.3, $L^\delta f_2(x) = 2(2+\delta)x^2$, so (3.20) follows. Now, setting $S := X^2$, by integration by parts and using (3.19) we have

$$[M^1]_t = [S]_t = S_t^2 - s_0^2 - 2\int_0^t S_s dS_s = X_t^4 - x_0^4 - 2\delta \int_0^t X_s^2 ds + M_t, \tag{3.21}$$

where M is a local martingale. This implies

$$X_t^4 = x_0^4 + 2\delta \int_0^t X_s^2 ds + [M^1]_t - M_t. \tag{3.22}$$

We remark that (3.22) and (3.20) provide two decompositions of the semimartingale X^4. By uniqueness of the semimartingale decomposition we can identify the bounded variation component, which implies

$$[M^1]_t = 4\int_0^t X_s^2 ds, t \in [0,T]. \tag{3.23}$$

Possibly enlarging the probability space, we consider an independent Brownian motion $\mathcal{W}$. We define now the process

$$W_t := \int_0^t 1_{\{X_s \neq 0\}} \frac{dM_s^1}{2|X_s|} + \int_0^t 1_{\{X_s = 0\}} d\mathcal{W}_s, \ t \geq 0. \tag{3.24}$$

W is a Brownian motion taking into account the fact that $[W]_t \equiv t$ together with Lévy's characterization of Brownian motion. At this point we define $\tilde{M}^1 = 2\int_0^\cdot |X_s| dW_s$. We get

$$\begin{aligned} \tilde{M}_t^1 &= 2\int_0^t 1_{\{X_s \neq 0\}} |X_s| dW_s = \int_0^t 1_{\{X_s \neq 0\}} |X_s| \frac{1}{|X_s|} dM_s^1 \\ &= \int_0^t 1_{\{X_s \neq 0\}} dM_s^1. \end{aligned}$$

This yields

$$[\tilde{M}^1 - M^1]_t = \int_0^t 1_{\{X_s = 0\}} d[M^1]_s = 4\int_0^t X_s^2 1_{\{X_s = 0\}} ds = 0, \ t \geq 0,$$

Consequently $M^1 = \tilde{M}^1$, hence, (3.19) yields that the process S is a (weak) solution of the SDE

$$dS_s = \delta ds + 2\sqrt{|S_s|} dW_s, \tag{3.25}$$

which shows that S is a $BESQ^\delta(s_0)$, $s_0 = x_0^2$.

□

Proposition 3.9 shows that no uniqueness on the real line holds when $\delta > 0$. Surprisingly, if $\delta = 0$ then uniqueness holds.

Remark 3.12. Suppose $\delta = 0$.

(1) Assume $x_0 = 0$. By Proposition 3.11 if $(X, \mathbb{P})$ is a solution of the martingale problem, then X^2 is (under $\mathbb{P}$) a $BESQ^0(0)$ which is the null process; this fact shows uniqueness.
(2) Suppose x_0 different from zero (for instance strictly positive). If $(X, \mathbb{P})$ is a solution to the strong martingale problem, then, by Proposition 3.11, under $\mathbb{P}$, $S := X^2$ is a $BESQ^0(x_0^2)$.
    In particular S is a solution of (1.1) with respect to some suitable Brownian motion W. Then, the strong Markov property shows that, whenever S reaches zero it is forced to remain there.

At the level of strong martingale problem we have the following.

Proposition 3.13. *Let $0 \leq \delta < 1$. Let X be a non-negative solution to the strong martingale problem with respect to $\mathcal{D}_{L^\delta}, \sigma$ and a Brownian motion W. Then $S = X^2$ is a solution to* (1.1).

Proof: Let us suppose that X is a solution of the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and a Brownian motion W. Setting $S := X^2$ and applying (2.7) with $f_1(x) = x^2$ we get

$$S_t = s_0 + 2\int_0^t \sqrt{|S_s|}dW_s + \delta t, t \in [0,T],$$

with $s_0 = x_0^2$. □

3.4. *The martingale problem in the* $\mathbb{R}_+$*-case.*

We remain still with the case $0 \le \delta < 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a Brownian motion W. We will be interested in non-negative solutions X for the strong martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W, which means that

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s)ds = \int_0^t f'(X_s)dW_s, \tag{3.26}$$

for all $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$. Proposition 3.14 below states the existence result. It follows directly from the $\mathbb{R}$-case, see Proposition 3.6.

Proposition 3.14. *Let $0 \le \delta < 1$. The process $BES^\delta(x_0)$ as stated in Proposition 3.6 solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W. In particular, the martingale problem related to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ admits strong existence.*

Proposition 3.15. *The martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W admits pathwise uniqueness.*

Proof: Let us suppose that $(X, \mathbb{P})$ is a solution of the martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W. This implies the same with respect to $\mathcal{D}_{L^\delta}$. By Proposition 3.13 $S = X^2$ is a solution of (1.1) for some Brownian motion W. The result follows by the pathwise uniqueness of the SDE (1.1) and the positivity of X.

□

3.5. *The martingale problem related to an extended domain.*

In this section we answer to the question raised in Remark 3.4. Indeed, for some aspects, one could be interested in a formulation of the martingale problem with respect to the extended domain $\hat{D}_{L^\delta}$ defined in Remark 3.4 in order to include the harmonic function (1.14).

Proposition 3.16. *Let $(X_t)_{t\ge 0}$ be a solution to the martingale problem with respect to $\mathcal{D}_{L^\delta}$. Suppose the following.*

i) For almost all $t \in\,]0,T]$ the law of X_t admits density p_t.

ii) $\lim_{|x|\to 0} \int_0^T |x|^{1-\delta} p_t(x)dt = 0.$

Then (X_t) is also a solution to the martingale problem with respect to $\hat{\mathcal{D}}_{L^\delta}$.

Remark 3.17. An analogous statement is valid for the strong martingale problem.

Proof (of Proposition 3.16): Let $f \in \hat{D}_{L^\delta}$ and consider a smooth bounded function $\chi : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that

$$\chi(x) = \begin{cases} 1, & x \le -1 \\ 0, & x \ge 0 \\ S(x), & x \in [-1,0], \end{cases} \tag{3.27}$$

for some bounded function $S : [-1,0] \longrightarrow [0,1]$ with $S(0) = 0, S(-1) = 1$. For every $n \ge 1$ we define $\chi_n : \mathbb{R} \longrightarrow \mathbb{R}_+$ as

$$\chi_n(x) := \chi\left(\frac{1}{2} - n|x|\right).$$

Notice that

$$\chi_n(x) = \begin{cases} 0, & |x| \le \frac{1}{2n} \\ 1, & |x| \ge \frac{3}{2n} \\ \in [0,1], & \text{otherwise}. \end{cases}$$

We have $\chi_n'(x) = \chi'\left(\frac{1}{2} - n|x|\right)(-n\text{sign}(x))$, so that

$$|\chi_n'(x)| \le n||\chi'||_\infty I_{\{\frac{1}{2n} \le |x| \le \frac{3}{2n}\}}(x), \ x \in \mathbb{R}. \tag{3.28}$$

For every $n \ge 1$ we define $f_n : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that

$$\begin{cases} f_n(0) = f(0) \\ f_n' = f'\chi_n. \end{cases} \tag{3.29}$$

Clearly $f_n \in \mathcal{D}_{L^\delta}$, so

$$f_n(X_t) - f_n(X_0) - \int_0^t L^\delta f_n(X_s)ds \tag{3.30}$$

is a local martingale. Obviously $f_n \to f$ and $f_n' \to f'$ uniformly on each compact. We show below that

$$\int_0^\cdot L^\delta f_n(X_s)ds \xrightarrow{u.c.p.} \int_0^\cdot L^\delta f(X_s)ds. \tag{3.31}$$

By (3.5) and (3.29) we get

$$L^\delta f_n(x) = \frac{|x|^{1-\delta}}{2}(|x|^{\delta-1}f_n')'(x) = \chi_n(x)L^\delta f(x) + \frac{1}{2}\chi_n'(x)f'(x).$$

Since χ_n converges to 1 uniformly on each compact, then $\int_0^\cdot \chi_n(X_s)L^\delta f(X_s)ds$ converges u.c.p. to $\int_0^\cdot L^\delta f(X_s)ds$. To prove (3.31) it remains to prove that

$$\int_0^t \chi_n'(X_s)f'(X_s)ds \xrightarrow{u.c.p.} 0. \tag{3.32}$$

For this, by (3.28) we have

$$\mathbb{E}\left(\sup_{t\le T}\left|\int_0^t \chi_n'(X_s)f'(X_s)ds\right|\right) \le \mathbb{E}\left(\int_0^T \left|\chi_n'(X_s)f'(X_s)\right| ds\right) \le \tag{3.33}$$

$$\le n||\chi'||_\infty \mathbb{E}\left(\int_0^T |f'(X_t)| I_{\{\frac{1}{2n}|X_t|<\frac{3}{2n}\}}(X_t)dt\right) = n||\chi'||_\infty \int_0^T \int_{\frac{1}{2n}}^{\frac{3}{2n}} |f'(x)|p_t(x)dxdt.$$

Let g be the continuous functions such that for $x \ne 0$ we have $g(x) = f'(x)|x|^{\delta-1}$. (3.33) gives

$$I(n) := n||\chi'||_\infty \int_0^T \int_{\frac{1}{2n}}^{\frac{3}{2n}} |f'(x)|p_t(x)dxdt = n||\chi'||_\infty \int_0^T \int_{\frac{1}{2n}}^{\frac{3}{2n}} |g(x)||x|^{1-\delta}p_t(x)dxdt.$$

Let $\varepsilon > 0$. Taking into account hypothesis ii) in the statement, there exists $A > 0$ such that for $|x| \le A$, we have $\int_0^T |x|^{1-\delta}p_t(x)dt < \varepsilon$. Consequently, for $|x| \le A$

$$I(n) \le ||\chi'||_\infty \sup_{|x|\le A} |g(x)|\varepsilon.$$

Taking the lim sup when n goes to infinity and since ε is arbitrary we get $\limsup_{n\to+\infty} I(n) = 0$ and consequently (3.32).

Since the space of local martingales is closed under the u.c.p. convergence then, taking the limit on (3.30) when $n \to \infty$, we conclude that $f(X_s) - f(X_0) - \int_0^t L^\delta f(X_s)ds$ is a local martingale. □

Proposition 3.18. *Let (X_t) be the Bessel process of dimension $\delta \in [0,1]$ starting from $x_0 > 0$. Then the following holds.*

i) For every $t > 0$ *the law of* X_t *admits a density* p_t.

ii) $\lim_{|x|\to 0^+} \int_0^T |x|^{1-\delta} p_t(x) = 0.$

Before doing the proof we recall that I_ν is the modified Bessel function of first kind (see Abramowitz and Stegun (1964), section 10) with $\nu = \frac{\delta}{2} - 1$. To prove Proposition 3.18 we will make use of the estimate stated in the following lemma.

Lemma 3.19. *$I_\nu(z) \le C \exp(z)$, for some constant C and $z \in \mathbb{R}$ large enough.*

Proof: In Abramowitz and Stegun (1964) equation 9.6.20 (p.376) we have

$$I_\nu(z) = \frac{1}{2\pi}\int_0^\pi \exp(z\cos(\theta))\cos(\nu\theta z)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty \exp(-z\cosh(t) - \nu t)dt =: I_1(z) - I_2(z).$$

For $z > 0$ we get

$$|I_1(z)| \le \frac{1}{2}\exp(z).$$

Concerning $I_2(z)$ we first observe that $-z\cosh(t) - \nu t \le (-z-\nu)t$ for $t \ge 0$. Let $R > -\nu$. For $z > R$ we get

$$|I_2(z)| \le \frac{1}{\pi}\int_0^\infty \exp(-t(R+\nu))dt = \frac{1}{\pi(R+\nu)}.$$

Consequently the result follows. □

Proof (of Proposition 3.18):

i) We recall (see Jeanblanc et al. (2009), chapter 6 equation 6.2.2 and Appendix A) that for $X_0 = x_0$

$$p_s(y) = \frac{y}{s}\left(\frac{y}{x_0}\right)^\nu \exp\left(-\frac{x_0^2 + y^2}{2s}\right) I_\nu\left(\frac{x_0 y}{s}\right), \tag{3.34}$$

ii) Since X is non-negative, we can remove the absolute value from $|x|$. By (3.34) and Lemma 3.19 we have

$$x^{1-\delta}\int_0^T p_t(x)dt \le C\frac{x^{1-\delta+1+\frac{\delta}{2}-1}}{x_0^{\frac{\delta}{2}-1}}\int_0^T \exp\left(-\frac{x_0^2+x^2}{2t} + \frac{x_0 x}{t}\right)\frac{1}{t}dt$$
$$\le C x^{1-\frac{\delta}{2}} x_0^{1-\frac{\delta}{2}} \int_0^T \exp\left(-\frac{(x_0-x)^2}{2t}\right)\frac{1}{t}dt.$$

For $t > 0$ and $x < x_0$ we set $\tilde{t} := \frac{(x_0-x)^2}{t}$, so $dt = -\frac{(x_0-x)^2}{\tilde{t}^2}$. That gives us

$$C(xx_0)^{1-\frac{\delta}{2}}\int_{\frac{(x_0-x)^2}{T}}^\infty \exp\left(\frac{-\tilde{t}}{2}\right)\frac{1}{\tilde{t}}d\tilde{t}. \tag{3.35}$$

Since previous integral converges to

$$\int_{\frac{x_0^2}{T}}^\infty \exp\left(\frac{-\tilde{t}}{2}\right)d\tilde{t},$$

when $x \to 0$, then (3.35) converges to zero. So the proof is concluded.

□

Remark 3.20. We remark that item (ii) of Proposition 3.18 is not fulfilled for a Bessel process starting from $x_0 = 0$, see Proposition 3.21. In this case, if one replaces the initial domain $\mathcal{D}_{L^\delta}$ with its extended domain the Bessel process fulfills a martingale problem where one has to add a supplementary term in the operator L^δ. This research is developed in an ongoing draft, which goes beyond the scope of the present paper.

Proposition 3.21. *Let (X_t) be the Bessel process with dimension $\delta \in [0,1]$ starting at $x_0 = 0$. Then, the following holds.*

i) For every $t > 0$ the law of X_t admits a density p_t

ii) For every $t > 0$ $\lim_{x \to 0^+} \int_0^t x^{1-\delta} p_s(x) ds = \frac{2^{2-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} t^{1-\frac{\delta}{2}} \frac{1}{2-\delta}.$

where Γ is the Gamma function given by $\Gamma(a) = \int_0^\infty x^{a-1} \exp(-x) dx, a > 0$.

Proof: According to equation 6.2.2 in Jeanblanc et al. (2009) we have

$$p_t(x) = \frac{2^\nu t^{-(\nu+1)}}{\Gamma(\nu+1)} x^{2\nu+1} \exp\left(-\frac{x^2}{2t}\right).$$

Consequently, since $\nu = \frac{\delta}{2} - 1$, we get

$$x^{\delta-1} \int_0^t p_s(x) ds = \frac{2^{1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} \int_0^t \exp\left(-\frac{x^2}{2s}\right) s^{-\frac{\delta}{2}} ds$$

For $s > 0$ and $x > 0$ we set $\tilde{s} = \frac{x^2}{s}$, $ds = -\frac{x^2}{\tilde{s}^2} d\tilde{s}$. We obtain

$$\frac{2^{1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} \int_{\frac{x^2}{t}}^\infty \frac{x^2}{\tilde{s}^2} \left(\frac{\tilde{s}}{x^2}\right)^{\frac{\delta}{2}} \exp\left(-\frac{\tilde{s}}{2}\right) d\tilde{s} = \frac{x^{2-\delta} 2^{1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} \int_{\frac{x^2}{t}}^\infty \tilde{s}^{-(\frac{\delta}{2})} \exp\left(\frac{\delta}{2}\right) d\tilde{s} =$$
$$= \frac{2^{1-\frac{\delta}{2}}}{\Gamma(\frac{-\tilde{s}}{2})} \frac{1}{x^{\delta-2}} \int_{\frac{x^2}{t}}^\infty \tilde{s}^{\frac{\delta}{2}-2} \exp\left(-\frac{\tilde{s}}{2}\right) d\tilde{s}$$

Since the integral and $\frac{1}{x^{\delta-2}}$ go to ∞ when $x \to 0^+$ then, by L'Hospital rule,

$$\lim_{x \to 0^+} \int_0^t x^{1-\delta} p_s(x) ds = -\frac{2^{1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} \lim_{x \to 0^+} \frac{\frac{2x}{t} (\frac{x^2}{t})^{\frac{\delta}{2}-2} \exp(-\frac{x^2}{t})}{(\delta-2) x^{\delta-3}} =$$
$$= -\frac{2^{2-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} t^{1-\frac{\delta}{2}} \frac{1}{\delta-2} \lim_{x \to 0^+} x^0 = \frac{2^{2-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} t^{1-\frac{\delta}{2}} \frac{1}{2-\delta}.$$

□

3.6. *On an alternative approach to treat the martingale problem on the full line.*

A priori we could have approached the martingale problem related to Bessel processes by the technique of Flandoli et al. (2003).

(1) Thereby, the authors handled martingale problems related to operators $L : \mathcal{D}_L \subset C^1(\mathbb{R}) \to \mathbb{R}$ of the form $Lf = \frac{\sigma^2}{2} f'' + b' f'$, where b is the derivative of a continuous function, σ is strictly positive continuous and Σ is defined as (1.5). The idea was to consider an L-harmonic function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(0) = 0$ and $h' = e^{-\Sigma}$. In Flandoli et al. (2003), L was also expressed in the form (1.6). The proof of well-posedness of the martingale problem thereby was based on a non-explosion condition (3.16) in Proposition 3.13 in Flandoli et al. (2003) and the fact that $\sigma_0 := (\sigma e^{-\Sigma}) \circ h^{-1}$ is strictly positive and so the SDE (for every fixed initial condition)

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s) dW_s, \tag{3.36}$$

is well-posed.

(2) Consider $\delta \in [0,1[$. As far as the martingale problem (for the Bessel process) on the full line is concerned, we could have tried to adapt similar methods. We observe that $L := L^\delta$ is also expressed in the form (1.6), which in our case gives (3.5). Taking into account (3.4), we have

$$h(x) = \mathrm{sign}(x)\frac{|x|^{2-\delta}}{2-\delta}, x \in \mathbb{R}. \tag{3.37}$$

Since h is bijective, one can show that (3.16) in Proposition 3.13 in Flandoli et al. (2003) is automatically satisfied. Moreover

$$\sigma_0(y) = \mathrm{sign}(y)(2-\delta)^{\frac{1-\delta}{2-\delta}}|y|^{\frac{1-\delta}{2-\delta}}. \tag{3.38}$$

Following the same idea as in in Proposition 3.2 of Flandoli et al. (2003), one can show that the well-posedness of the Bessel martingale problem (with respect to $\hat{\mathcal{D}}_{L^\delta}$) is equivalent to the well-posedness (in law) of (3.36). Here $\sigma_0(0) = 0$, but (3.36) is still well-posed even if

$$\int_0^\varepsilon \frac{1}{\sigma_0^2}(y)dy = +\infty,\ \forall \varepsilon > 0. \tag{3.39}$$

In fact in that case (3.39) corresponds to the Engelbert-Schmidt criterion (see Theorem 5.7 in Karatzas and Shreve (1991, Chapter 5).

(3) The criterion (3.39) can be reformulated here saying that the quantity

$$\frac{1}{(2-\delta)^{\frac{2-2\delta}{2-\delta}}}\int_0^\epsilon y^{\frac{2\delta-2}{2-\delta}}dy,\ \forall \varepsilon > 0, \tag{3.40}$$

is infinite. Now, (3.40) is always finite for any $\delta > 0$. This confirms that (3.36) has no uniqueness in law on $\mathbb{R}$, with σ_0 defined in (3.38), when $\delta \in]0,1[$. So, the non-uniqueness observed in Proposition 3.9 is not astonishing.

(4) On the other hand, when $\delta = 0$, then (3.40) is infinite, which implies uniqueness in law.

(5) We drive the attention on the fact that the considerations of this section concern the martingale problem with respect to the extended domain $\hat{\mathcal{D}}_{L^\delta}$ and for the case $x_0 \neq 0$.

3.7. *The framework for $\delta = 1$.*

Let W be a standard Brownian motion on some underlying probability space. By definition, a Bessel process of dimension $\delta = 1$ starting at $x_0 \geq 0$ is a non-negative process X such that $S := X^2$ is a $BESQ^1(x_0^2)$. On the other hand, in the literature such a Bessel process X is also characterized as a non-negative strong solution of

$$X_t = x_0 + W_t + L_t^X(0), t \in [0,T], \tag{3.41}$$

where $L^X(0)$ is a non-decreasing process only increasing when $X = 0$, i.e.

$$\int_{[0,T]} \varphi(s)dL_s^X(0) = \int_{[0,T]} \varphi(s)1_{\{X_s=0\}}dL_s^X(0),$$

for every generic Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. In particular, X is a semimartingale. Indeed, let X be a non-negative solution of (3.41), then by an easy application of Itô's formula for semimartingales, setting $S := X^2$, we have

$$\begin{aligned} S_t &= x_0^2 + 2\int_0^t X_s dW_s + \int_0^t X_s dL_s^X(0) + \frac{1}{2}2t \\ &= x_0^2 + 2\int_0^t \sqrt{S_s}dW_s + \int_0^t X_s 1_{\{X_s=0\}}dL_s^X(0) + t \\ &= x_0^2 + 2\int_0^t \sqrt{S_s}dW_s + t, \end{aligned}$$

which implies that S is a $BESQ^1(x_0^2)$ and so X is a $BES^1(x_0)$. This shows in particular that (3.41) admits pathwise uniqueness. Existence and uniqueness of (3.41) can be seen via the Skorokhod problem, see Harrison and Shepp (1981).

In this section, we represent alternatively X as a non-negative solution of a (strong) martingale problem. As we mentioned at the beginning of Section 3, we have fixed

$$b(x) = H(x) = \begin{cases} 1 & : \quad x \geq 0 \\ 0 & : \quad x < 0. \end{cases}$$

Formally speaking we get

$$\Sigma(x) = 2\int_0^x \delta_0(y)dy = 2H(x),$$

where H is the Heaviside function. Coming back to the expression (1.6), it is natural to set

$$L^1 f = (\exp(2H)f')'\frac{\exp(-2H)}{2}, \; f \in C^2(\mathbb{R}^*). \tag{3.42}$$

This gives of course

$$L^1 f = \frac{f''}{2}, \; f \in C^2(\mathbb{R}^*). \tag{3.43}$$

Analogously to the case $\delta \in]0,1[$ and applying the same principle as for the domain characterization in the case $\delta \in [0,1[$, we naturally arrive to

$$\mathcal{D}_{L^1} = \{f \in C^2 | f'(0) = 0\}.$$

Since $L^1 f$ has to be continuous, (3.43) gives

$$L^1 f = \frac{f''}{2}, \; \forall f \in \mathcal{D}_{L^1}. \tag{3.44}$$

The PDE operator L^1 appearing at (3.44) coincides with the generator of Brownian motion. However, the domain of that generator is larger since it is $C^2(\mathbb{R})$.

Remark 3.22. The same preliminary analysis of Section 3.3 about the martingale problem related to $0 \leq \delta < 1$ in the $\mathbb{R}$-case extends to the case $\delta = 1$. More precisely, Proposition 3.6, Corollary 3.8, Proposition 3.9 and Remark 3.10 hold. This is stated below.

Proposition 3.23.

(1) There is a process $BES^1(x_0)$ solving the strong martingale problem with respect to $\mathcal{D}_{L^1}$ and W.
(2) The martingale problem related to L^1 with respect to $\mathcal{D}_{L^1}$ admits (in general) no uniqueness.

Similarly to Corollary 3.8, the processes $BES^1(x_0)$ (resp. $-BES^1(-x_0)$) is a solution to the strong martingale problem with respect to $\mathcal{D}_{L^1}$ and an underlying Brownian motion W (resp. $-W$)). Other solutions on the real line are the so-called *skew Brownian motions* which will be possibly investigated more in detail in a future work. For this last one, we can mention the works of Harrison and Shepp (Harrison and Shepp (1981)) and Le Gall (Le Gall (1984)).

Concerning the $\mathbb{R}_+$-case, let again $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with the canonical filtration $\mathfrak{F}^W$ of a Brownian motion W.

By using the same arguments as for Propositions 3.14 and 3.15, we get the following result.

Proposition 3.24. *There is a process $BES^1(x_0)$ solving the strong martingale problem with respect to $\mathcal{D}_{L^1}(\mathbb{R}_+)$ and W. Moreover, the martingale problem admits pathwise uniqueness with respect to $\mathcal{D}_{L^1}(\mathbb{R}_+)$.*

4. Martingale problem related to the path-dependent Bessel process

4.1. *Generalities.*

Now we are going to treat a non-Markovian martingale problem which is a perturbation of the Bessel process $BES^\delta(x_0)$, $0 \le \delta \le 1, x_0 \ge 0$. More precisely, we want to analyze existence and uniqueness of solutions to the martingale problem related to the SDE

$$X_t = x_0 + W_t + \int_0^t b'(X_s)ds + \int_0^t \Gamma(s, X^s)ds, \tag{4.1}$$

where Γ is the same path-dependent functional as in (2.1), and b is as in (1.10).

Proposition 4.1. *Suppose $\delta = 0, x_0 = 0$. Let W be a standard Brownian motion. The null process is a solution to the strong martingale problem (in the sense of Definition 2.3) with respect to $\mathcal{D}_{L^\delta}$ and W.*

In presence of a path-dependent drift Γ, under suitable conditions, Corollary 4.17 allows to show that the null process is still the unique solution of the corresponding strong martingale problem.

4.2. *The martingale problem in the path-dependent case: existence in law.*

We recall that a pair $(X, \mathbb{P})$ is a solution for the martingale problem related to $\mathcal{L}$ in the sense of Definition 2.2 with $L = L^\delta$ with respect to $\mathcal{D}_{L^\delta}$ (resp. $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$), $0 \le \delta \le 1$, if for all $f \in \mathcal{D}_{L^\delta}$ (resp. $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$),

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \tag{4.2}$$

is a $\mathbb{P}$-local martingale.

A first criterion of existence can be stated if Γ is measurable and bounded.

Proposition 4.2. *Suppose that Γ is bounded. Then the martingale problem related to $\mathcal{L}$ (defined in (2.5)) admits existence with respect to $\mathcal{D}_{L^\delta}$. Moreover we have the following.*

(1) If the initial condition is $x_0 \ge 0$, then the solution can be constructed to be non-negative.
(2) If the initial condition is $x_0 \le 0$, then the solution can be constructed to be non-positive.

Proof: Let $x_0 \ge 0$. Given a Brownian motion W, by Propositions 3.14 and 3.23, there exists a solution X to the (even strong) martingale problem related to (2.5) (with $\Gamma = 0$) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W. That solution is in fact a $BES^\delta(x_0)$. In particular, for all $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$,

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s)ds = \int_0^t f'(X_s)dW_s. \tag{4.3}$$

Since the Bessel process is non-negative, (4.3) also holds for $f \in \mathcal{D}_{L^\delta}$. As Γ is bounded then, by Novikov's condition

$$N_t = \exp\left(\int_0^t \Gamma(s, X^s)dW_s - \frac{1}{2}\int_0^t \Gamma^2(s, X^s)ds\right),$$

is a martingale. By Girsanov's Theorem

$$B_t := W_t - \int_0^t \Gamma(s, X^s)ds,$$

is a Brownian motion under the probability measure $\mathbb{Q}$ such that $d\mathbb{Q} = N_T d\mathbb{P}$. Then, we can rewrite (4.3) as

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s)ds - \int_0^t f'(X_s)\, dB_s - \int_0^t f'(X_s)\,\Gamma(s, X^s)ds = 0.$$

Since $\int_0^t f'(X_s)\,dB_s$ is a $\mathbb{Q}-$local martingale, $(X,\mathbb{Q})$ happens to be a solution to the martingale problem in the sense of Definition 2.2 with respect to $\mathcal{D}_{L^\delta}$.

Suppose now that $x_0 \le 0$. The process X defined as $-BES^\delta(-x_0)$ is a solution of (4.3), with W replaced with $-W$. Then a similar procedure as for the case $x_0 \ge 0$ works. This shows existence for the martingale problem on $\mathcal{D}_{L^\delta}$.

Let us discuss the sign of the solution. Suppose that $x_0 \ge 0$ (resp. $x_0 \le 0$). Then, our construction starts with $BES^\delta(x_0)$ (resp. $-BES^\delta(-x_0)$) which is clearly non-negative (resp. non-positive). The constructed solution is again non-negative (resp. non-positive) since it is supported by an equivalent probability measure. □

Remark 4.3. As we have mentioned in Proposition 3.9 and its extension to $\delta = 1$, the martingale problem in the sense of Definition 2.2 admits no uniqueness in general, at least with respect to $\mathcal{D}_{L^\delta}$, i.e. on the whole line.

4.3. *Some preliminary results on a path-dependent SDE.*

Before studying a new class of path-dependent martingale problems we recall some results stated in Section 4.5 of Ohashi et al. (2022).

Let $\sigma_0 : \mathbb{R} \to \mathbb{R}$. Let $\bar\Gamma : \Lambda \to \mathbb{R}$ be a generic Borel functional. Related to it we formulate the following, which was Assumption 4.25 in Ohashi et al. (2022).

Assumption 4.4.

(1) There exists a function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\epsilon l^{-2}(u)du = \infty$ for all $\epsilon > 0$ and

$$|\sigma_0(x) - \sigma_0(y)| \le l(|x-y|).$$

(2) σ_0 has at most linear growth.

(3) There exists $K > 0$ such that

$$|\bar\Gamma(s,\eta^1) - \bar\Gamma(s,\eta^2)| \le K\left(|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)|dr\right),$$

for all $s \in [0,T], \eta^1, \eta^2 \in C([0,T])$.

(4) $\bar\Gamma_\infty := \sup_{s\in[0,T]} |\bar\Gamma(s,0)| < \infty$.

The proposition below was the object of Ohashi et al. (2022, Proposition 4.27).

Proposition 4.5. *Let $y_0 \in \mathbb{R}$. Suppose the validity of Assumption 4.4. Then $E(\sigma_0, 0, \bar\Gamma)$, i.e.*

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s)dW_s + \int_0^t \bar\Gamma(s, Y^s)ds, \tag{4.4}$$

admits pathwise uniqueness.

The lemma below was the object of Ohashi et al. (2022, Lemma 4.28).

Lemma 4.6. *Suppose the validity of the assumptions of Proposition 4.5. Let Y be a solution of (4.4) and $m \ge 2$ an integer. Then there exists a constant $C > 0$, depending on the linear growth constant of σ_0, y_0, K, T, m and the quantity (4) in Assumption 4.4 such that*

$$\mathbb{E}\left(\sup_{t\le T} |Y_s|^m\right) \le C.$$

4.4. *A new class of solutions to the martingale problem.*

Besides Proposition 4.2, Proposition 4.8 below and Proposition 4.9 provide a new class of solutions to the martingale problem related to $\mathcal{L}$ with respect to $\mathcal{D}_{L^\delta}$. We consider now a particular case of $\bar{\Gamma}$, which is associated with Γ:

$$\bar{\Gamma}(s,\eta) := 2\sqrt{|\eta(s)|}\Gamma(s,\sqrt{|\eta^s|}) + \delta,\ \ s \in [0,T]\ \eta \in C([0,T]). \tag{4.5}$$

Next, we introduce a growth assumption on Γ.

Assumption 4.7. Γ is continuous and there exists a constant K such that, for every $(s,\eta) \in \Lambda$ we have

$$|\Gamma(s,\eta)| \le K\left(1 + \sup_{r\in[0,T]} \sqrt{|\eta(r)|}\right).$$

Proposition 4.8. . *Let $\delta \in [0,1]$. Suppose that Γ fulfills Assumption 4.7. Then, we have the following.*

(1) Let $s_0 \ge 0$. The path-dependent SDE

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{|S_s|}dW_s + \int_0^t 2\sqrt{|S_s|}\Gamma\left(s,\sqrt{|S^s|}\right)ds, \delta \ge 0, \tag{4.6}$$

admits existence in law, see Definition A.4 of Appendix in Ohashi et al. (2022).

(2) The constructed solution of (4.6) in item (1) is non-negative.

(3) Let $x_0 \ge 0$. The martingale problem related to $\mathcal{L}f = L^\delta f + \Gamma f'$ (see Definition 2.2, (2.5)) admits existence with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$.

Proof:

We remark that the hypothesis on Γ implies that $\bar{\Gamma}$ has linear growth, i.e. there is a constant K such that

$$\bar{\Gamma}(t,\eta^t) \le K(1 + \sup_{s\in[0,t]} |\eta(s)|), \forall (t,\eta) \in [0,T] \times C([0,T]). \tag{4.7}$$

For item (1), we start truncating Γ. Let $N > 0$. Let us define, for $s \in [0,T], \eta \in C([0,T])$,

$$\begin{aligned}\Gamma^N(s,\eta) &:= (\Gamma(s,\eta^s) \vee (-N)) \wedge N,\\ \bar{\Gamma}^N(s,\eta) &:= 2\sqrt{|\eta(s)|}\Gamma^N(s,\sqrt{|\eta|}) + \delta.\end{aligned}$$

We consider the SDE

$$\begin{cases} dS_t = & 2\sqrt{|S_t|}dW_t + \bar{\Gamma}^N(t,S)\,dt, \\ S_0 = & s_0. \end{cases} \tag{4.8}$$

We set $x_0 := \sqrt{s_0}$. Since Γ^N is bounded, by Proposition 4.2, the martingale problem related to $\mathcal{L}$ with respect to $\mathcal{D}_{L^\delta}$, admits a solution $(X,\mathbb{P})$ which is non-negative. By Proposition 4.9 the SDE (4.8) admits existence in law and in particular there exists a solution S^N (which is necessarily non-negative) on some probability space $(\Omega,\mathcal{F},\bar{\mathbb{P}}^N)$. By Itô's formula, this implies that (on the mentioned space),

$$M_t^N := f(S_t^N) - f(S_0^N) - \int_0^t f'(S_s^N)\bar{\Gamma}^N\left(s,S^N\right)ds - 2\int_0^t f''(S_s^N)|S_s^N|ds, \tag{4.9}$$

is a martingale for all $f \in C^2$ with compact support. This will be used later.

We want first to show that the family of laws $(\bar{\mathbb{Q}}^N)$ of (S^N) is tight. For this we are going to use the Kolmogorov-Centsov Theorem. We denote by $\bar{\mathbb{E}}^N$ the expectation related to $\bar{\mathbb{P}}^N$. According to Problem 4.11 in Section 2.4 of Karatzas and Shreve (1991), it is enough to find constants $\alpha,\beta > 0$ realizing

$$\sup_N \bar{\mathbb{E}}^N(|S_t^N - S_s^N|^\alpha) \le c|t-s|^{1+\beta}; s,t \in [0,T], \tag{4.10}$$

for some constant $c > 0$. Indeed, we will show (4.10) for $\alpha = 6$ and $\beta = 1$. By (4.8) and Burkholder-Davis-Gundy inequality there exists a constant c_6 such that, for $0 \le s \le t \le T$,

$$\bar{\mathbb{E}}^N(|S_t^N - S_s^N|^6) \le c_6 \left(\bar{\mathbb{E}}^N \left(\int_s^t (|S_r^N|) dr \right)^3 + \bar{\mathbb{E}}^N \left(\int_s^t \bar{\Gamma}^N \left(r, |S^N|\right) dr \right)^6 \right). \tag{4.11}$$

By (4.7), there exists a constant $\mathcal{C}_1$ where

$$|\bar{\Gamma}^N(s,\eta)| \le 2\sqrt{|\eta(s)|}|\Gamma(s, \sqrt{|\eta|})| + \delta = |\bar{\Gamma}(s,\eta)| \le \mathcal{C}_1 \left(1 + \sup_{r \le s} |\eta(s)|\right), \tag{4.12}$$

for every $(s,\eta) \in \Lambda$, uniformly in N. By Jensen's inequality and (4.12), there exists a constant $\mathcal{C}_2 > 0$, only depending on T and on $\bar{\Gamma}$, but not on N, such that

$$\bar{\mathbb{E}}^N(|S_t^N - S_s^N|^6) \le \mathcal{C}_2 \left((t-s)^2 \bar{\mathbb{E}}^N \left(\sup_{s \le t} |S_s^N|^3 \right) + (t-s)^5 \bar{\mathbb{E}}^N \left(\sup_{s \le t} |S_s^N|^6 \right) \right).$$

By Lemma 4.6, the quantity

$$\bar{\mathbb{E}}^N \left(\sup_{s \le T} |S_s^N|^3 + \sup_{s \le T} |S_s^N|^6 \right),$$

is bounded uniformly in N and therefore (4.10) holds. Consequently, the family of laws $(\bar{\mathbb{Q}}^N)$ of (S^N) under $(\bar{\mathbb{P}}^N)$ is tight. We can therefore extract a subsequence which, for simplicity, we will still call $\bar{\mathbb{Q}}^N$ that converges weakly to a probability measure $\bar{\mathbb{Q}}$ on $(C[0,T], \mathfrak{B}(C[0,T]))$.

We denote by $\mathbb{E}^N$ the expectation with respect to $\bar{\mathbb{Q}}^N$. Let $0 \le s \le t \le T$ and let $F : C([0,s]) \to \mathbb{R}$ be a bounded and continuous function. By (4.9), if S is the canonical process we have

$$\mathbb{E}^N((\tilde{M}_t^N - \tilde{M}_s^N) F(S_r, 0 \le r \le s)) = 0, \tag{4.13}$$

where

$$\tilde{M}_t^N := f(S_t) - f(S_0) - \int_0^t f'(S_s) \bar{\Gamma}^N (s, S)\, ds - 2 \int_0^t f''(S_s)|S_s| ds. \tag{4.14}$$

By Skorokhod's convergence theorem, there exists a sequence of processes (Y^N) and a process Y both on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, converging u.c.p. to Y as $N \to +\infty$. Indeed (Y^N) and Y can be seen as random elements taking values in the state space $(C[0,T], \mathfrak{B}(C[0,T]))$.

Moreover, the law of Y^N is $\bar{\mathbb{Q}}^N$, so that

$$\mathbb{E}^{\mathbb{Q}}((\overline{M}_t^N - \overline{M}_s^N) F(Y_r^N, 0 \le r \le s)) = 0, \tag{4.15}$$

where

$$\overline{M}_t^N := f(Y_t^N) - f(S_0) - \int_0^t f'(Y_r^N) \bar{\Gamma}^N \left(s, Y^N\right) ds - 2 \int_0^t f''(Y_s^N)|Y_r^N| dr. \tag{4.16}$$

We wish to pass to the limit when $N \to \infty$ using Lebesgue dominated convergence theorem and obtain

$$\mathbb{E}^{\mathbb{Q}}((\overline{M}_t - \overline{M}_s) F(Y_r, 0 \le r \le s)) = 0, \tag{4.17}$$

with

$$\overline{M}_t := f(Y_t) - f(S_0) - \int_0^t f'(Y_s) \bar{\Gamma}\,(s, Y)\, ds - 2 \int_0^t f''(Y_r)|Y_r| dr. \tag{4.18}$$

For this it remains to prove that, when $N \to \infty$

$$\mathbb{E}^{\mathbb{Q}} \left(\int_s^t f'(Y_r^N) \bar{\Gamma}^N(r, Y^N) dr \right) \to \mathbb{E}^{\mathbb{Q}} \left(\int_s^t f'(Y) \bar{\Gamma}(r, Y) dr \right) \tag{4.19}$$

and

$$\mathbb{E}^{\mathbb{Q}} \left(\int_s^t f''(Y_r^N)|Y_r^N| dr \right) \to \mathbb{E}^{\mathbb{Q}} \left(\int_s^t f''(Y_r)|Y_r| dr \right), \tag{4.20}$$

as $N \to \infty$. Below, we only prove (4.19) since (4.20) follows similarly.

Note that (4.19) is true, if and only if,

$$\lim_{N\to\infty} I_1(N) = 0, \quad \lim_{N\to\infty} I_2(N) = 0,$$

where

$$I_1(N) := \mathbb{E}^{\mathbb{Q}}\left[\int_s^t f'(Y_r^N)(\bar{\Gamma}^N(r,Y^N) - \bar{\Gamma}(r,Y^N))dr\right],$$
$$I_2(N) := \mathbb{E}^{\mathbb{Q}}\left[\int_s^t f'(Y_r^N)\bar{\Gamma}(r,Y^N) - f'(Y_r)\bar{\Gamma}(r,Y)dr\right].$$

By (4.7) and (4.12), we have

$$I_1(N) \le ||f'||_\infty \mathbb{E}^{\mathbb{Q}}\left[1_{\{\sup_{r\in[0,T]}|\Gamma(r,Y^{N,r})|>N\}}\int_s^t |\bar{\Gamma}^N(r,Y^N) - \bar{\Gamma}(r,Y^N)|dr\right] \le$$
$$\le 2KT||f'||_\infty \mathbb{E}^{\mathbb{Q}}\left[1_{\{\sup_{r\in[0,T]}|\Gamma(r,Y^N)|>N\}}(1+\sup_{r\in[0,T]}|Y_r^N|)\right].$$

By Cauchy-Schwarz's inequality, there exists a non-negative constant $C(f,T,K)$ such that

$$I_1(N)^2 \le C(f,T,K) I_{11}(N) I_{12}(N), \tag{4.21}$$

where

$$I_{11}(N) := \mathbb{Q}\left(\sup_{r\in[0,T]}|\Gamma(r,Y^N)| > N\right),$$
$$I_{12}(N) := \mathbb{E}^{\mathbb{Q}}\left[1+\sup_{r\in[0,T]}|Y_r^N|^2\right].$$

By Chebyshev's inequality we have

$$I_{11}(N) \le \frac{1}{N^2}\mathbb{E}^{\mathbb{Q}}\left[\sup_{r\in[0,T]}|\Gamma(r,Y^N)|^2\right] \le \frac{2K}{N^2}\mathbb{E}^{\mathbb{Q}}\left[1+\sup_{r\in[0,T]}|Y_r^N|^2\right].$$

Consequently, $\lim_{N\to\infty} I_{11}(N) = 0$ because of Lemma 4.6. On the other hand, again by Lemma 4.6, $I_{12}(N)$ is bounded in N and so by (4.21), we get $\lim_{N\to\infty} I_1(N) = 0$.

Concerning $I_2(N)$, we have

$$I_2(N)^2 \le T\int_s^t \mathbb{E}^{\mathbb{Q}}\left[|f'(Y_r^N)\bar{\Gamma}(r,Y^N) - f'(Y_r)\bar{\Gamma}(r,Y)|^2\right]dr. \tag{4.22}$$

By Lemma 4.6, there exists a constant C not depending on N such that

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{r\in[0,T]}|Y_r^N|^4\right] \le C,$$

and, consequently, by Fatou's Lemma

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{r\in[0,T]}|Y_r|^4\right] \le C.$$

Let $r \in [0,T]$. We have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|f'(Y_r^N)\bar{\Gamma}(r,Y^N) &- f'(Y_r)\bar{\Gamma}(r,Y)|^4] \\ &\leq 8||f'||_\infty^4 K^4\left(2+\mathbb{E}^{\mathbb{Q}}\left[\sup_{r\in[0,T]}|Y_r^N|^4+\sup_{r\in[0,T]}|Y_r|^4\right]\right) \\ &\leq 16||f'||_\infty^4 K^4(1+C).\end{aligned} \tag{4.23}$$

So the sequence

$$|f'(Y_r^N)\bar{\Gamma}(r,Y^N) - f'(Y_r)\bar{\Gamma}(r,Y)|^2$$

is uniformly integrable. We fix again $r \in [0,T]$. Since f' and $\bar{\Gamma}$ are continuous it follows that

$$\mathbb{E}^{\mathbb{Q}}\left[|f'(Y_r^N)\bar{\Gamma}(r,Y^N) - f'(Y_r)\bar{\Gamma}(r,Y)|^2\right] \longrightarrow 0, \tag{4.24}$$

as $N \to \infty$. Now (4.23) and Cauchy-Schwarz implies that

$$\mathbb{E}^{\mathbb{Q}}[|f'(Y_r^N)\bar{\Gamma}(r,Y^N) - f'(Y_r)\bar{\Gamma}(r,Y)|^2] \leq 4||f'||_\infty^2 K^2\sqrt{1+C}. \tag{4.25}$$

This time (4.24), (4.25) and Lebesgue's dominated theorem show that the entire Lebesgue integral of (4.24) on $[s,t]$ converges to 0. Finally, $\lim_{N\to\infty} I_2(N) = 0$ so that we conclude to (4.19) and, consequently, (4.17). Therefore, $(Y,\mathbb{Q})$ solve the martingale problem of the type (2.4) as in Proposition 2.1 with

$$Lf(x) = 2|x|f''(x) + \delta f'(x)$$

and $\bar{\Gamma}$ replacing Γ. By Proposition 2.1, this concludes the proof of item (1).

Concerning item (2), the previously constructed Y is a (weak) solution to (4.6) under the probability $\mathbb{Q}$. Since it is a limit of non-negative solutions, it will also be non-negative.

Item (3) follows from Proposition 4.9 below.

□

4.5. *Equivalence between martingale problem and SDE in the path-dependent case.*

We state here an important result establishing the equivalence between the martingale problem and a path-dependent SDE of squared Bessel type. Let $0 \leq \delta \leq 1$.

Proposition 4.9. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let X be a stochastic process and we denote $S = X^2$.*

(1) $(|X|,\mathbb{P})$ is a solution to the martingale problem related to (2.5) with respect to $\mathcal{D}_{L^\delta}$, if and only if, the process S is a solution of (4.6) for some Brownian motion W.

(2) Let W be a standard Brownian motion (with respect to $\mathbb{P}$). Then $|X|$ is a solution to the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and W, if and only if, S is a solution of (4.6).

Remark 4.10. In the statement of Proposition 4.9, $\mathcal{D}_{L^\delta}$ can be replaced with $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$, provided that $|X|$ is replaced by X.

Proof (of Proposition 4.9): We discuss item (1).

Concerning the direct implication, by choosing $f_1(x) = x^2, f_2(x) = x^4$ we have $L^\delta f_1(x) = \delta, L^\delta f_2(x) = 2(2+\delta)x^2$. By definition of the martingale problem, the two processes ($t \in [0,T]$)

$$M_t := X_t^2 - X_0^2 - \delta t - \int_0^t 2|X_s|\Gamma(s,|X^s|)ds \tag{4.26}$$

and

$$N_t := X_t^4 - X_0^4 - 2(2+\delta)\int_0^t X_s^2 ds - 4\int_0^t |X_s|^3\Gamma(s,|X^s|)ds, \tag{4.27}$$

are $\mathfrak{F}^X$-local martingales.

Since $S = X^2$, by (4.26) we have $[S] = [M]$. By integration by parts and (4.26), we have

$$[M]_t = [X^2]_t = X_t^4 - X_0^4 - 2\int_0^t X_s^2 dX_s^2 = X_t^4 - X_0^4 - 2\delta\int_0^t X_s^2 ds - 4\int_0^t |X_s|^3\Gamma(s,|X^s|)ds + M^1,$$

where M^1 is a local martingale. Therefore

$$X_t^4 - X_0^4 = M^1 + 2\delta\int_0^t X_s^2 ds + 4\int_0^t |X_s|^3\Gamma(s,|X^s|)ds + [M]_t, \quad t\in[0,T]. \tag{4.28}$$

(4.28) and (4.27) give us two decompositions of the semimartingale X^4; by the uniqueness of the semimartingale decomposition, $[M]_t = 4\int_0^t X_s^2 ds$. Defining W similarly as in (3.24) with M replacing M^1, (possibly enlarging the probability space) one can show that W is a standard Brownian motion and

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{S_s}dW_s + \int_0^t 2\sqrt{S_s}\Gamma(s,\sqrt{S^s})ds, t\in[0,T].$$

Concerning the converse implication, suppose that $S = X^2$ solves (4.6) for some Brownian motion W. Then S solves

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{|S_s|}d\widetilde{W}_s, t\in[0,T], \tag{4.29}$$

where

$$\widetilde{W}_t := W_t + \int_0^t \Gamma(s,\sqrt{|S^s|})ds, t\in[0,T].$$

Let $f\in\mathcal{D}_{L^\delta}$; by Proposition 3.6 and Remark 3.7 we have

$$f(|X_t|) - f(|X_0|) - \int_0^t L^\delta f(|X_s|)ds = \int_0^t f'(|X_s|)d\widetilde{W}_s. \tag{4.30}$$

Consequently

$$\begin{aligned} M_t^f &:= f(|X_t|) - f(|x_0|) - \int_0^t L^\delta f(|X_s|)ds - \int_0^t f'(|X_s|)\Gamma(s,|X^s|)ds \\ &= \int_0^t f'(|X_s|)dW_s, \end{aligned}$$

is an $\mathfrak{F}^X$-local martingale. Then, $(|X|,\mathbb{P})$ solve the martingale problem related to (2.5) with respect to $\mathcal{D}_{L^\delta}$ in the sense of Definition 2.2. On the other hand, $|X|$ also solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}$ and W. This concludes the proof of item (1).

As far as item (2) is concerned, the converse implication argument can be easily adapted to the argument for the proof of the converse implication in (1). Concerning the direct implication, we define f_1 as in the proof of item (1). By (2.7), (4.26) and the fact that

$$M = 2\int_0^\cdot f_1'(|X_s|)dW_s = 2\int_0^\cdot \sqrt{S_s}dW_s,$$

we obtain (4.6). This concludes the proof.

□

4.6. *The martingale problem in the path-dependent case: uniqueness in law.*

A consequence of Girsanov's theorem gives us the following.

Proposition 4.11. *Let $0\le\delta\le 1$. Suppose that Γ is bounded. The martingale problem related to (2.5) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ admits uniqueness.*

Remark 4.12. Let $x_0\ge 0$ (resp. $x_0\le 0$). By Proposition 4.2, every solution of the aforementioned martingale problem is non-negative (resp. non-positive).

Proof (of Proposition 4.11):

Let $(X^i, \mathbb{P}^i), i = 1, 2$ be two solutions to the martingale problem related to $\mathcal{L}f = Lf + \Gamma f'$ with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$. By Proposition 4.9, $S^i = (X^i)^2$ is a solution of (4.6), for some Brownian motion W^i and $\mathbb{P}^i$. We define the random variable (which is also a Borel functional of X^i)

$$V_t^i = \exp\left(-\int_0^t \Gamma(s, X^i) dW_s^i - \frac{1}{2}\int_0^t \left(\Gamma(s, X^i)\right)^2 ds\right).$$

By the Novikov's condition, it is a $\mathbb{P}^i$-martingale. This allows us to define the probability $d\mathbb{Q}^i = V_T^i d\mathbb{P}^i$. By Girsanov's theorem, for $i = 1, 2$, under $\mathbb{Q}^i$, $B_t^i := W_t^i + \int_0^t \Gamma(s, X^{i,s})ds$ is a Brownian motion. Therefore, S^i is a solution of (4.6) with $\Gamma = 0$, under $\mathbb{Q}^i$. Now (4.6) (with $\Gamma = 0$) admits pathwise uniqueness and therefore uniqueness in law, by Yamada-Watanabe theorem. Consequently S^i (under $\mathbb{Q}^i$), $i = 1, 2$ have the same law and the same holds of course for $X^i, i = 1, 2$. Hence, for every Borel set $B \in \mathfrak{B}(C[0, T])$ we have

$$\mathbb{P}^1\{X^1 \in B\} = \int_\Omega \frac{1}{V_T^1(X^1)} 1_{\{X^1 \in B\}} d\mathbb{Q}^1 = \int_\Omega \frac{1}{V_T^2(X^2)} 1_{\{X^2 \in B\}} d\mathbb{Q}^2 = \mathbb{P}^2\{X^2 \in B\}.$$

So, X^1 under $\mathbb{P}^1$ has the same law as X^2 under $\mathbb{P}^2$. Finally the martingale problem related to (2.5) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ admits uniqueness. □

4.7. *Path-dependent Bessel process: results on pathwise uniqueness.*

In this section, $\bar{\Gamma}$ is the same as the one defined in (4.5), i.e.

$$\bar{\Gamma}(s, \eta) := 2\sqrt{|\eta(s)|}\Gamma(s, \sqrt{|\eta^s|}) + \delta, s \in [0, T]\ \eta \in C([0, T]).$$

At this point, we can state a pathwise uniqueness theorem. For this purpose, we state the following assumption.

Assumption 4.13.

(1) There exists a constant $K > 0$ such that, for every $s \in [0, T], \eta^1, \eta^2 \in C([0, T])$, we have $|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K\left(|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)|dr\right)$.
(2) $\sup_{t \in [0,T]} |\bar{\Gamma}(t, 0)| < \infty$.

Remark 4.14.

(1) $\sigma_0(y) = 2\sqrt{|y|}$ has linear growth.
(2) Defining $l(x) = 2\sqrt{x}, x \geq 0$, we have $\int_0^\epsilon l^{-2}(u)du = \infty$ for every $\epsilon > 0$ and $|l(x) - l(y)| \leq l(|x - y|), x, y \in \mathbb{R}_+$.

Remark 4.15. Note that, by Remark 4.14, Assumption 4.13 implies Assumption 4.4.

We start the analysis by considering equation (4.6). For the definitions of strong existence and pathwise uniqueness for path-dependent SDEs, see Definitions A.2 and A.3 of Ohashi et al. (2022).

Theorem 4.16. *Suppose Assumptions 4.13 and 4.7.*

(1) (4.6) *admits pathwise uniqueness.*
(2) (4.6) *admits strong existence.*
(3) Suppose $x_0 \geq 0$*. Every solution of* (4.6) *with* $s_0 = x_0^2$ *is non-negative.*

Proof:

(1) We remark that (4.6) is of the form (4.4). The result follows from Proposition 4.5 and Remark 4.14.
(2) By Proposition 4.8, we have existence in law. By an extension of Yamada-Watanabe theorem to the path-dependent case, strong existence holds for (4.6).

(3) Suppose $x_0 \geq 0$. By Proposition 4.8 (2), (4.6) admits even existence in law of a non-negative solution. By Yamada-Watanabe theorem extended to the path-dependent case, pathwise uniqueness implies uniqueness in law, so that the above-mentioned solution has to be non-negative.

□

We are now able to state the following.

Corollary 4.17. *Suppose that $\bar{\Gamma}$ (defined in (4.5)) fulfills Assumptions 4.13 and 4.7. Then the strong martingale problem related to (2.5) (see Definition 2.3) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W admits strong existence and pathwise uniqueness.*

Proof: By Theorem 4.16, the equation (4.6) admits a unique strong solution which is non-negative. Proposition 4.9 and Remark 4.10 allow us to conclude the proof.

□

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