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An aggregated model for Karlin stable processes

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Abstract. An aggregated model is proposed, of which the partial-sum process scales to the Karlin stable processes recently investigated in the literature. The limit extremes of the proposed model, when having regularly-varying tails, are characterized by the convergence of the corresponding point processes. The proposed model is an extension of an aggregated model proposed by Enriquez (2004) in order to approximate fractional Brownian motions with Hurst index $H \in (0, 1/2)$, and is of a different nature of the other recently investigated Karlin models which are essentially based on infinite urn schemes.

1. Introduction and main results

1.1. *Karlin stable processes.* The Karlin stable processes are a family of self-similar symmetric α-stable (SαS) stochastic processes, $\alpha \in (0, 2]$, with stationary increments that recently appeared in the literature (Durieu and Wang, 2016; Durieu et al., 2020). A Karlin SαS process has a memory parameter $\beta \in (0, 1)$. In the case $\alpha = 2$, the process becomes a fractional Brownian motion with Hurst index $H = \beta/2 \in (0, 1/2)$. Fractional Brownian motions (Kolmogorov, 1940; Mandelbrot and Van Ness, 1968) are fundamental models in stochastic processes with long-range dependence (Pipiras and Taqqu, 2017), and hence such an extension to stable processes is of its own interest.

Karlin stable processes was first discovered during the investigation of fractional Brownian motions via a limit-theorem point of view: what stochastic models may scale to a Brownian motion? This is an extensively investigated question in the literature of applied probability; see Pipiras and Taqqu (2017) for various such stochastic models. Remarkably, a few recent results focus on the following question: since a simple random walk scales to a Brownian motion, would it be possible

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to find a *correlated simple random walk* (with dependent ± 1-steps) that scales to a fractional Brownian motion? This question was answered affirmatively first by Hammond and Sheffield (2013) for $H \in (1/2, 1)$, and later on by Durieu and Wang (2016) for $H \in (0, 1/2)$. It is worth noting that for two ranges of H the correlated random walks are of completely different natures. In particular, in Durieu and Wang (2016), it was shown that the partial-sum process of the Karlin model with Rademacher randomizations scales to a fractional Brownian motion with $H \in (0, 1/2)$.

The original Karlin model is as an infinite urn scheme, of which the law on the urns has a power-law decay (Karlin, 1967; Gnedin et al., 2007), and with Rademacher randomizations it can be interpreted as a correlated random walk to be reviewed below. The Karlin stable processes first appeared in a follow-up study (Durieu et al., 2020). Therein it was shown that if the randomizations are replaced by heavy-tailed ones, that is, the random walks are now with correlated and heavy-tailed steps, then the scaling limit becomes a new SαS stable process, to which we termed the name Karlin stable process. Since then, the Karlin model and its variations have attracted attention in the literature of stochastic processes as they serve as simple models that exhibit long-range dependence. For example, the set-indexed Karlin stable random fields (Fu and Wang, 2020) include and generalize the set-indexed fractional Brownian motions (Herbin and Merzbach, 2006), and extensions to hierarchical models have also been considered (Iksanov et al., 2022; Iksanov and Kotelnikova, 2022).

We first recall the Karlin stable process $\{\zeta_{\alpha,\beta}(t)\}_{t\ge 0}$, and explain how it arises from the Karlin model with randomization as in Durieu and Wang (2016); Durieu et al. (2020). Throughout, we assume $\alpha \in (0, 2]$ and $\beta \in (0, 1)$. Then, $\zeta_{\alpha,\beta}$ is an SαS process, of which the characteristic function of finite-dimensional distributions is, for any $d \in \mathbb{N} \equiv \{1, 2, \cdots\}$, $t_1, \cdots, t_d \ge 0$, $\theta_1, \cdots, \theta_d \in \mathbb{R}$,

$$\mathbb{E}\exp\left(i\sum_{j=1}^d \theta_j \zeta_{\alpha,\beta}(t_j)\right) = \exp\left(-\frac{\beta}{\Gamma(1-\beta)\mathsf{C}_\alpha}\int_0^\infty \mathbb{E}\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\{N(t_j q)\ \mathrm{odd}\}}\right|^\alpha q^{-\beta-1}dq\right), \tag{1.1}$$

where on the right-hand side, N is a standard Poisson process (the probability spaces involved on both sides are not necessarily the same), and

$$\mathsf{C}_\alpha = \begin{cases} \left(\int_0^\infty x^{-\alpha}\sin x dx\right)^{-1}, & \text{if } \alpha \in (0,2), \\ 2, & \text{if } \alpha = 2. \end{cases} \tag{1.2}$$

See Samorodnitsky and Taqqu (1994, Eq. (1.2.9)) for other formula of C_α. It follows from the representation above that the process is self-similar with index β/α and with stationary increments (Durieu et al., 2020). Moreover, when $\alpha = 2$ it is a fractional Brownian motion with Hurst index $H = \beta/2$ up to a multiplicative constant (see (2.4) below).

It is a well-known fact on stable processes (Samorodnitsky and Taqqu, 1994) that when $\alpha \in (0, 2)$, (1.1) has a corresponding series representation. However, for our discussions later we shall need to work with another series representation of the process $\zeta_{\alpha,\beta}$ restricted to $t \in [0, 1]$ as follows. Let $\{\Gamma_\ell\}_{\ell\in\mathbb{N}}$ denote the collection of consecutive arrival times of a standard Poisson process on $\mathbb{R}_+$, $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$ i.i.d. Rademacher random variables, $\{Q_{\beta,\ell}\}_{\ell\in\mathbb{N}}$ i.i.d. copies of a β-Sibuya random variable (see (2.1) below), and $\{U_{\ell,j}\}_{\ell,j\in\mathbb{N}}$ i.i.d. uniform random variables on $(0, 1)$. Further, all four families of random variables are assumed to be independent. Then, we also have the following series representation of the Karlin stable processes, *restricted to* $t \in [0, 1]$,

$$\{\zeta_{\alpha,\beta}(t)\}_{t\in[0,1]} \stackrel{d}{=} \left\{\sum_{\ell=1}^\infty \frac{\varepsilon_\ell}{\Gamma_\ell^{1/\alpha}}\mathbf{1}_{\left\{\sum_{j=1}^{Q_{\beta,\ell}}\mathbf{1}_{\left\{U_{\ell,j}\le t\right\}}\ \mathrm{odd}\right\}}\right\}_{t\in[0,1]}, \alpha \in (0,2), \beta \in (0,1). \tag{1.3}$$

The fact that the representations (1.1) and (1.3) are equivalent is recalled in Lemma 2.1 (following a more general result in Fu and Wang (2021, Theorem 2.1)).

Now we explain the so-called *randomized Karlin model* in Durieu and Wang (2016); Durieu et al. (2020), for comparison purpose only (see Remark 1.6). Let $\{Y_n\}_{n\in\mathbb{N}}$ be i.i.d. $\mathbb{N}$-valued random variables with $\mathbb{P}(Y_1 = k) \sim k^{-1/\beta}$ as $k \to \infty$ for some $\beta \in (0,1)$ (we only present a simple version; a slowly varying function is allowed in general). Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables independent from $\{Y_n\}_{n\in\mathbb{N}}$, and assume in addition that X_1 is symmetric with $1 - \mathbb{E}\exp(i\theta X_1) \sim \sigma_X^\alpha |\theta|^\alpha$ as $\theta \to 0$. Consider the partial-sum process

$$S_n := \sum_{j=1}^n (-1)^{K_{j,Y_j}+1} X_{Y_j} = \sum_{\ell=1}^\infty X_\ell \mathbf{1}_{\{K_{n,\ell} \text{ odd}\}} \quad \text{with} \quad K_{n,\ell} := \sum_{j=1}^n \mathbf{1}_{\{Y_j=\ell\}}, \quad n, \ell \in \mathbb{N}. \tag{1.4}$$

Then, one can show that for some explicit constant $C_{\alpha,\beta}$,

$$\frac{1}{n^{\beta/\alpha}} \left\{S_{\lfloor nt \rfloor}\right\}_{t\in[0,1]} \xrightarrow{f.d.d.} C_{\alpha,\beta} \left\{\zeta_{\alpha,\beta}(t)\right\}_{t\in[0,1]}.$$

Note that when $\{X_n\}_{n\in\mathbb{N}}$ are i.i.d. Rademacher random variables, in view of the first expression in (1.4), one can write $S_n = Z_1 + \cdots + Z_n$, where $\{Z_i\}_{i\in\mathbb{N}}$ is a sequence of dependent ± 1-valued random variables, and hence the above limit theorem can be interpreted as *a correlated random walk* with ± 1 steps scaling to a fractional Brownian motion with Hurst index $H = \beta/2$. This was the motivation behind the introduction of randomization in Durieu and Wang (2016) for the original Karlin model (Karlin, 1967).

1.2. *An aggregated model.* Since the fractional Brownian motions arise from various stochastic models, and Karlin stable processes extend fractional Brownian motions to stable processes, it is natural to ask the question: whether the Karlin stable processes arise from other stochastic models? Of particular interest are the models of *aggregation* nature. It has been well-known that stochastic processes with short-range dependence, when aggregated and with appropriately chosen random parameters, may exhibit long-range dependence, and in particular fractional Brownian motions may arise in this way (Kaj and Taqqu, 2008; Mikosch and Samorodnitsky, 2007).

In this paper, we propose a one-dimensional aggregated model as follows and show that it scales to a Karlin stable process. The model actually extends a previous one by Enriquez (2004), who proposed an aggregated model that scales to a fractional Brownian motion with $H \in (0,1/2)$. (However, our formulations are slightly different; see Remark 1.4 for the original description in Enriquez (2004).) Let q be a random parameter taking values from $(0,1)$, and given q, let $\{\eta_j^{(q)}\}_{j\in\mathbb{N}}$ be a sequence of conditionally i.i.d. Bernoulli random variables with parameter q. Let $\mathcal{X}$ be a symmetric random variable, independent from q and $\{\eta_j^{(q)}\}_{j\in\mathbb{N}}$. Let $\alpha' > 0$ be another parameter. Then we introduce

$$X_j := \frac{\mathcal{X}}{q^{1/\alpha'}} \cdot (-1)^{\tau_j^{(q)}+1} \eta_j^{(q)} \quad \text{with} \quad \tau_j^{(q)} := \sum_{k=1}^j \eta_k^{(q)}, \quad j \in \mathbb{N}. \tag{1.5}$$

In words, $X_j = 0$ whenever $\eta_j^{(q)} = 0$, and for those $j \in \mathbb{N}$ such that $\eta_j^{(q)} = 1$, X_j takes the same value $\mathcal{X}/q^{1/\alpha'}$, but with alternating signs. One can check that $\{X_n\}_{n\in\mathbb{N}}$ forms a stationary sequence of random variables. The partial-sum process is then

$$S_n := \sum_{j=1}^n X_j = \frac{\mathcal{X}}{q^{1/\alpha'}} \mathbf{1}_{\left\{\tau_n^{(q)} \text{ odd}\right\}}, n \in \mathbb{N}. \tag{1.6}$$

Note that there is no summation involved in the second expression above, and $S_n \neq 0$ implies necessarily that $\tau_n^{(q)}$ is odd. The simple expression is essentially due to the alternating signs. Next, introduce

$$\left((\mathcal{X}_i, q_i, \{\eta_{i,j}^{(q_i)}\}_{j\in\mathbb{N}}, \{\tau_{i,j}^{(q_i)}\}_{j\in\mathbb{N}})\right)_{i\in\mathbb{N}} \stackrel{i.i.d.}{\sim} \left(\mathcal{X}, q, \{\eta_j^{(q)}\}_{j\in\mathbb{N}}, \{\tau_j^{(q)}\}_{j\in\mathbb{N}}\right), \tag{1.7}$$

and for each copy let $\{S_n^{(i)}\}_{n\in\mathbb{N}}$ denote the corresponding partial-sum process. We are interested in the aggregated model, for an increasing sequence of positive integers $\{m_n\}_{n\in\mathbb{N}}$,

$$\widehat{\mathcal{X}}_{n,j} := \sum_{i=1}^{m_n} \frac{\mathcal{X}_i}{q_i^{1/\alpha'}} \cdot (-1)^{\tau_{i,j}^{(q_i)}+1} \eta_{i,j}^{(q_i)}, \quad n\in\mathbb{N}, j=1,\dots,n,$$

and its corresponding partial-sum process

$$\left\{\widehat{S}_n(t)\right\}_{t\in[0,1]} := \left\{\sum_{j=1}^{\lfloor nt\rfloor} \widehat{\mathcal{X}}_{n,j}\right\}_{t\in[0,1]} \equiv \left\{\sum_{i=1}^{m_n} S_{\lfloor nt\rfloor}^{(i)}\right\}_{t\in[0,1]} \equiv \left\{\sum_{i=1}^{m_n} \frac{\mathcal{X}_i}{q_i^{1/\alpha'}} \mathbf{1}_{\left\{\tau_{i,\lfloor nt\rfloor}^{(q_i)} \text{ odd}\right\}}\right\}_{t\in[0,1]}.$$

Above, we provide three equivalent representations to better understand the process. We shall mostly use the third one in our analysis.

Now we specify the assumptions on q and $\mathcal{X}$. The random parameter q is assumed to have the probability density function

$$p(x) = x^{-\rho} L(1/x), \quad x\in(0,1), \text{ for some } \rho<1, \tag{1.8}$$

where L is a slowly varying function at infinity. The random variable $\mathcal{X}$ is assumed to be symmetric, and either to have finite second moment, or

$$\overline{F}_{|\mathcal{X}|}(x) \equiv \mathbb{P}(|\mathcal{X}|>x) \sim C_{\mathcal{X}} x^{-\alpha}, x>0, \text{ for some } \alpha>0 \text{ and } C_{\mathcal{X}}>0. \tag{1.9}$$

1.3. *Main results.* Our main results are the following two limit theorems. The first is a multivariate central limit theorem.

Theorem 1.1. *Assume (1.8) holds. Assume the symmetric random variable $\mathcal{X}$ satisfies one of the following two conditions:*

(i) $\mathbb{E}\mathcal{X}^2<\infty$, and in this case set $\alpha=2$, $C_{\mathcal{X}} := \mathbb{E}\mathcal{X}^2$.

(ii) (1.9) holds with $\alpha\in(0,2)$.

Further assume

$$\beta := \gamma - 1 + \rho \in (0,1) \quad \textit{with} \quad \gamma := \frac{\alpha}{\alpha'}. \tag{1.10}$$

Then, with m_n satisfying

$$\lim_{n\to\infty} \frac{m_n L(n)}{n^{1-\rho}} = \infty, \tag{1.11}$$

and

$$a_n = \left(C_{\mathcal{X}} \frac{\Gamma(1-\beta)}{\beta} \cdot n^\beta m_n L(n)\right)^{1/\alpha}, \tag{1.12}$$

we have

$$\left\{\frac{\widehat{S}_n(t)}{a_n}\right\}_{t\in[0,1]} \xrightarrow{f.d.d.} \{\zeta_{\alpha,\beta}(t)\}_{t\in[0,1]}.$$

Regarding scaling limits of extremes, our second result is a convergence of point processes.

Theorem 1.2. *Assume (1.8), (1.9) with $\alpha>0$ and (1.10). Assume that, in addition to (1.11), $m_n \le Cn^\kappa$ for some $\kappa\in(0,2\beta/(\alpha-2))$ if $\alpha\ge 2$ (so $\alpha=2$ means that m_n grows at a polynomial rate). We have*

$$\xi_n := \sum_{j=1}^n \delta_{\left(\sum_{i=1}^{m_n} \mathcal{X}_i \eta_{i,j}^{(q_i)}/(a_n q_i^{1/\alpha'}), j/n\right)} \Rightarrow \xi := \sum_{\ell=1}^\infty \sum_{j=1}^{Q_{\beta,\ell}} \delta_{\left(\varepsilon_\ell \Gamma_\ell^{-1/\alpha}, U_{\ell,j}\right)}, \tag{1.13}$$

in $\mathfrak{M}_p((\overline{\mathbb{R}}\setminus\{0\})\times[0,1])$, where the random variables involved in the definition of ξ are as those in (1.3).

Above and below, $\mathfrak{M}_p(E)$ is the space of Radon point measures on the metric space E, equipped with vague topology. Our reference for point processes and convergence is Resnick (1987). In the case $\alpha < 2$, Theorem 1.2 contains more information regarding the limit of the partial-sum process, and provides a second proof for Theorem 1.1, as discussed in Section 4.3. Moreover, Theorem 1.2 also implies extremal limit theorems regarding the proposed model, as explained in Section 4.4. In particular, the choice of a_n in (1.12) is such that

$$\mathbb{P}\left(\frac{|\mathcal{X}|}{q^{1/\alpha'}} > a_n x, \tau_n^{(q)} \neq 0\right) \sim \frac{x^{-\alpha}}{m_n}, \text{ for all } x > 0. \tag{1.14}$$

As the key of Theorem 1.2, a more refined conditional limit theorem given the event above is in Proposition 4.2.

We conclude the introduction with a few remarks.

Remark 1.3. For the central limit theorem, we only prove the convergence of finite-dimensional distributions to the Karlin stable process for $\alpha \in (0, 2]$, without the tightness. The tightness is a challenging issue and actually, in Durieu et al. (2020), the tightness for the randomized Karlin model was only proved for $\alpha \in (0, 1)$, and the tightness remains an open question for $\alpha \in [1, 2)$ (for the Gaussian case the tightness was proved in Durieu and Wang (2016)). It is also an open question to show that the Karlin stable process has a version in D (the Skorokhod space with J_1 topology (Billingsley, 1999)), for $\alpha \in [1, 2)$.

Remark 1.4. The main inspiration of this paper came from a paper of Enriquez (2004), and our model is in fact a generalization of a model proposed therein, and our limit theorems extend his to (non-Gaussian) stable domain of attractions.

The goal of Enriquez (2004) was to provide an approximation of fractional Brownian motion with Hurst index $H \in (0, 1)$ by aggregation of independent correlated random walks. Two models were proposed therein and the second was for $H \in (0, 1/2)$, recalled here. Consider again random variable q with probability density function

$$(1 - 2H)2^{1-2H} q^{-2H} \mathbf{1}_{\{q \in (0,1/2)\}}.$$

Then a sequence of random variables $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is constructed as follows: ε_1 is a ± 1-valued symmetric random variable and for each $n \geq 1$, the law of ε_n is determined by

$$q = \mathbb{P}(\varepsilon_{2n} = \varepsilon_{2n-1} \mid \varepsilon_1, \cdots, \varepsilon_{2n-1}, q) = 1 - \mathbb{P}(\varepsilon_{2n} = -\varepsilon_{2n-1} \mid \varepsilon_1, \cdots, \varepsilon_{2n-1}, q), n \in \mathbb{N},$$

and $\varepsilon_{2n+1} = -\varepsilon_{2n}, n \in \mathbb{N}$. Then, consider $\mathcal{X}_n := (\varepsilon_{2n-1} + \varepsilon_{2n})/(2\sqrt{q}), n \in \mathbb{N}$. In this way, their model fits into our setup with $\alpha = \alpha' = 2, \rho = 2H$ (see Enriquez (2004, p. 209) for details), whereas we consider the general case with $\alpha \in (0, 2]$ and $\alpha' > 0$. Our Theorem 1.1 includes Enriquez (2004, Corollary 3) as a special case for $\alpha = 2$ (but without tightness).

Remark 1.5. There is non-trivial dependence between the magnitude $\mathcal{X}_i/q_i^{1/\alpha'}$ and the locations $\{j = 1, \ldots, n : \eta_{i,j}^{(q_i)} = 1\}$, via q, in the aggregated model. However, the dependence disappears in the limit. It is also remarkable that while our model has three parameters ρ, α, α', the limiting Karlin stable process has only two: $\alpha \in (0, 2)$ and $\beta = \rho + \alpha/\alpha' - 1 \in (0, 1)$.

Both observations can be explained by the following representation of $\zeta_{\alpha,\beta}$ (compare also (3.3) in the proof of Theorem 1.1 later): essentially, the factor $q^{-1/\alpha'}$ in (1.5) introduces an effect of change of measures in the limit. Recall the characteristic function of $\zeta_{\alpha,\beta}$ in (1.1), and write

$$\int_0^\infty \mathbb{E}\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\{N(t_j q) \text{ odd}\}}\right|^\alpha q^{-\beta-1} dq = \int_0^\infty \mathbb{E}\left|\sum_{j=1}^d \theta_j \frac{1}{q^{\gamma/\alpha}} \mathbf{1}_{\{N(t_j q) \text{ odd}\}}\right|^\alpha q^{-\rho} dq.$$

Equivalently, for $\alpha < 2$ we have another series representation as follows

$$\{\zeta_{\alpha,\beta}(t)\}_{t\geq 0} \stackrel{d}{=} \left\{\sum_{\ell=1}^{\infty} \frac{\varepsilon_\ell}{\Gamma_\ell^{1/\alpha}} q_\ell^{-\gamma/\alpha} \mathbf{1}_{\{N_\ell(tq_\ell) \text{ odd}\}}\right\}_{t\geq 0},$$

where Γ_ℓ and q_ℓ are such that $\sum_{\ell=1}^{\infty} \delta_{(\Gamma_\ell, q_\ell)}$ is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dx(\beta/\Gamma(1-\beta))q^{-\rho}dq$, independent from the Rademacher random variables $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$. So in the limit process $\zeta_{\alpha,\beta}$, $q^{-\gamma/\alpha}$ above eventually comes from the normalization $q^{-1/\alpha'}$ in (1.5) and $q^{-\rho}$ comes from the density of q (both after $1/n$-scaling as can be read from the proof later).

Remark 1.6. In the randomized Karlin models (Durieu and Wang, 2016; Durieu et al., 2020), there are two sources of dependence. First, in (1.4), the dependence is determined by the law of certain counting numbers being odd. For $\{S_n\}_{n\in\mathbb{N}}$ in (1.4), with all $X_\ell = 1$ it is known as an *odd-occupancy* process, counting by time n how many urns have been sampled by an odd number of times. This process has been already investigated by Karlin (1967), and the motivation of such a consideration dates back to Spitzer (1964). So with i.i.d. $\{X_\ell\}_{\ell\in\mathbb{N}}$, $\{S_n\}_{n\in\mathbb{N}}$ becomes a randomized odd-occupancy process. The law of the occupancy numbers being odd, eventually, plays a crucial role in the underlying dependence structure of the limit Karlin stable process in (1.1) and (1.3). Second, the original Karlin model also has a strong combinatorial flavor, as the sampling $\{Y_n\}_{n\in\mathbb{N}}$ induces a random partition of $\mathbb{N}$, essentially related to the Pitman–Yor partition with parameters $(\beta, 0)$, to which the β-Sibuya distribution is intrinsically related (Pitman, 2006). There have been recent interests regarding various counting statistics for other combinatorial models, and often they lead to new stochastic processes of their own interest. For an example with a similar flavor, see Alsmeyer et al. (2017).

In a sense, our proposed aggregated model and the limit theorems indicate that the counting of odd-occupancy numbers is much more fundamental than the underlying random partitions for the randomized Karlin models: our proposed model has a much less combinatorial flavor than the Karlin models, and yet they lead to the same scaling limits.

Remark 1.7. We learned from Rafał Kulik the following illuminating remark regarding the factor $q^{-1/\alpha'}$ from a different aspect as in Remark 1.5. In view of (1.14), and if one ignores the restriction $\tau_n^{(q)} \neq 0$, then it is known by Breiman's lemma (Breiman, 1965) that the tail of $|\mathcal{X}/q^{1/\alpha'}|$ is determined by the heavier one of $\mathcal{X}$ and $q^{-1/\alpha'}$, which is the one of the latter here. More precisely, one readily checks that $\overline{F}_{q^{-1/\alpha'}}(x) \in RV_{-\kappa}$ with $\kappa = (1-\rho)\alpha'$, and $\beta > 0$ in our assumption (1.10) is exactly $\kappa < \alpha$, and then by Resnick (2007, Proposition 7.5) we have

$$\frac{1}{b_n}\sum_{i=1}^{m_n} \delta_{|\mathcal{X}_i|/q_i^{1/\alpha'}} \Rightarrow \sum_{\ell=1}^{\infty} \delta_{\Gamma_\ell^{-1/\kappa}},$$

for some $b_n \in RV_{1/\kappa}$. At the same time, (1.14) implies (see Proposition 4.2)

$$\frac{1}{a_n}\sum_{i=1}^{m_n} \delta_{|\mathcal{X}_i|/q_i^{1/\alpha'}} \mathbf{1}_{\left\{\tau_{i,n}^{(q_i)}>0\right\}} \Rightarrow \sum_{\ell=1}^{\infty} \delta_{\Gamma_\ell^{-1/\alpha'}}.$$

So, the restriction $\tau_{i,n}^{(q_i)} \neq 0$ could be understood as a thinning property.

The paper is organized as follows. Section 2 collects a few facts about Karlin stable processes. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 and explain its further connection to the so-called Karlin random sup-measures.

2. Representations for Karlin stable processes

We collect some facts on the Karlin stable processes that can be derived from the general Karlin stable set-indexed processes (Fu and Wang, 2021). Let Q_β denote the β-Sibuya distribution (Sibuya, 1979), so that

$$\mathbb{P}(Q_\beta = \ell) = \frac{\beta}{\Gamma(1-\beta)}\frac{\Gamma(\ell-\beta)}{\Gamma(\ell+1)}, \ell \in \mathbb{N}. \tag{2.1}$$

Let $\mathcal{C}_\ell := \bigcup_{i=1}^{\ell}\{U_i\}$ denote the union of ℓ i.i.d. random variables $\{U_i\}_{i\in\mathbb{N}}$ that are uniformly distributed over $(0,1)$. If ℓ is a random variable, then assume in addition that $\{U_i\}_{i\in\mathbb{N}}$ are independent from ℓ.

Lemma 2.1. *For $\alpha > 0$ and $\beta \in (0,1)$,*

$$\int_0^\infty \mathbb{E}_q\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\{N(t_j q)\ \mathrm{odd}\}}\right|^\alpha \frac{\beta}{\Gamma(1-\beta)} q^{-\beta-1} dq = \mathbb{E}\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\left\{|\mathcal{C}_{Q_\beta}\cap[0,t_j]|\ \mathrm{odd}\right\}}\right|^\alpha. \tag{2.2}$$

Therefore, (1.3) holds with $\alpha \in (0,2)$.

Remark 2.2. Throughout, with a little abuse of notations, when writing $\mathbb{E}_q(\cdots)$ or $\mathbb{P}_q(\cdots)$, we mean that the q appears in $(\cdots)$ is viewed as a fixed constant instead of a random variable (e.g., $\mathbb{E}_q(\cdots)$ on the left-hand side of (2.2) is viewed then as a function of q).

Proof of Lemma 2.1: First, let $\{N^{(q)}(t)\}_{t\ge 0}$ denote a Poisson process on $\mathbb{R}_+$ with constant intensity density q on $\mathbb{R}_+$. Then, for every $q > 0$ fixed,

$$\begin{aligned}
\mathbb{E}_q\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\{N(t_j q)\ \mathrm{odd}\}}\right|^\alpha &= \mathbb{E}_q\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\left\{N^{(q)}(t_j)\ \mathrm{odd}\right\}}\right|^\alpha \\
&= \sum_{\ell=1}^\infty \mathbb{E}_q\left(\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\left\{N^{(q)}(t_j)\ \mathrm{odd}\right\}}\right|^\alpha \,\middle|\, N^{(q)}(1)=\ell\right)\mathbb{P}_q(N^{(q)}(1)=\ell) \\
&= \sum_{\ell=1}^\infty \mathbb{E}_q\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\{|\mathcal{C}_\ell\cap[0,t_j]|\ \mathrm{odd}\}}\right|^\alpha \mathbb{P}_q(N^{(q)}(1)=\ell).
\end{aligned}$$

Then, the left-hand side of (2.2) becomes, by Fubini's theorem,

$$\sum_{\ell=1}^\infty \mathbb{E}\left|\sum_{j=1}^d \theta_j \mathbf{1}_{\{|\mathcal{C}_\ell\cap[0,t_j]|\ \mathrm{odd}\}}\right|^\alpha \cdot \frac{\beta}{\Gamma(1-\beta)}\int_0^\infty \mathbb{P}_q(N^{(q)}(1)=\ell) q^{-\beta-1}dq.$$

It remains to notice that the second factor above is simply

$$\frac{\beta}{\Gamma(1-\beta)}\int_0^\infty \mathbb{P}_q(N^{(q)}(1)=\ell)q^{-\beta-1}dq = \frac{\beta}{\Gamma(1-\beta)}\int_0^\infty \frac{q^\ell}{\Gamma(\ell+1)}e^{-q}q^{-\beta-1}dq = \mathbb{P}(Q_\beta=\ell).$$

The desired identity (2.2) now follows. (1.3) then follows by the well-known equivalence between stochastic-integral and series representations of SαS processes (Samorodnitsky and Taqqu, 1994, Theorem 3.10.1). □

The Karlin stable process $\zeta_{\alpha,\beta}$ has the following stochastic-integral representation (Samorodnitsky and Taqqu, 1994)

$$\{\zeta_{\alpha,\beta}(t)\}_{t\ge 0} \stackrel{d}{=} \left\{\int_{\mathbb{R}_+\times\Omega'} \mathbf{1}_{\{N'(tq)(\omega')\ \mathrm{odd}\}} M_\alpha(dq, dw')\right\}_{t\ge 0}, \tag{2.3}$$

where M_α is a SαS random measure on $\mathbb{R}_+ \times \Omega'$, for another probability space $(\Omega', \mathbb{P}')$ different from the one where the stochastic integral is defined on, with control measure $dm = (\beta/(\Gamma(1-\beta)\mathsf{C}_\alpha))q^{-\beta-1}dq\mathbb{P}'(d\omega')$. N' is a standard Poisson process on $(\Omega', \mathbb{P}')$. This is a standard so-called *doubly stochastic representation*, where the random measure M_α is defined on the by-default probability space $(\Omega, \mathbb{P})$, and $(\Omega', \mathbb{P}')$ is a different space. This representation is notationally convenient but not needed in our proofs. We refer the readers to Samorodnitsky and Taqqu (1994) for more details regarding stochastic-integral representation for stable processes.

Note that the characteristic function (1.1) and the stochastic-integral representation (2.3) both allow $\alpha = 2$, and in this case the Karlin stable process becomes (up to a multiplicative constant) a fractional Brownian motion with Hurst index $\beta/2$. A quick derivation is as follows, using (2.3) and stochastic integrals with respect to Gaussian random measures: for $0 < s < t$,

$$\begin{aligned}\operatorname{Cov}\left(\zeta_{2,\beta}(s), \zeta_{2,\beta}(t)\right) &= \frac{\beta}{\mathsf{C}_\alpha\Gamma(1-\beta)}\int_0^\infty \mathbb{P}_q\left(N(sq) \text{ odd}, N(tq) \text{ odd}\right) q^{-\beta-1}dq \\ &= \frac{\beta}{\mathsf{C}_\alpha\Gamma(1-\beta)}\int_0^\infty \frac{1}{2}(1-e^{-2qs})\frac{1}{2}(1+e^{-2q(t-s)})q^{-\beta-1}dq \\ &= \frac{1}{4}\frac{\beta}{\mathsf{C}_\alpha\Gamma(1-\beta)}\int_0^\infty (1-e^{-2qs}-e^{-2qt}+e^{-2q(t-s)})q^{-\beta-1}dq.\end{aligned}$$

(Recall that for a Poisson random variable N with $\lambda = \mathbb{E}N$, $\mathbb{P}(N \text{ is odd}) = \left(1-e^{-2\lambda}\right)/2$.) Using

$$\int_0^\infty (1-e^{-rq})q^{-\beta-1}dq = r^\beta \frac{\Gamma(1-\beta)}{\beta},$$

we have

$$\operatorname{Cov}\left(\zeta_{2,\beta}(s), \zeta_{2,\beta}(t)\right) = 2^{\beta-1}\mathsf{C}_\alpha^{-1}\cdot\frac{1}{2}\left(s^\beta + t^\beta - |t-s|^\beta\right), s,t \geq 0. \tag{2.4}$$

So $\{\zeta_{2,\beta}(t)\}_{t\geq0} \stackrel{d}{=} 2^{(\beta-1)/2}\mathsf{C}_\alpha^{-1/2}\left\{\mathbb{B}^{\beta/2}(t)\right\}_{t\geq0}$, where $\{\mathbb{B}^{\beta/2}(t)\}_{t\geq0}$ is a fractional Brownian motion with Hurst index $\beta/2$.

3. Proof of Theorem 1.1

The proof is by computing the asymptotic characteristic function. Consider the characteristic function $\phi_{\mathcal{X}}(\theta) := \mathbb{E}\exp(i\theta\mathcal{X})$. It is known that the two assumptions on $\mathcal{X}$ in Theorem 1.1 can be unified into the following condition

$$1 - \phi_{\mathcal{X}}(\theta) \sim \sigma_{\mathcal{X}}^\alpha |\theta|^\alpha \text{ as } \theta \to 0,$$

where $\sigma_{\mathcal{X}}^2 := \mathbb{E}\mathcal{X}^2/2 < \infty$ and $\sigma_{\mathcal{X}}^\alpha := C_{\mathcal{X}}/\mathsf{C}_\alpha$ when $\alpha \in (0,2)$ (see Bingham et al. (1987, Theorem 8.1.10) for the second). C_α is defined in (1.2), and $C_{\mathcal{X}}$ is the constant such that $\overline{F}_{|\mathcal{X}|}(x) \equiv \mathbb{P}(|\mathcal{X}| > x) \sim C_{\mathcal{X}}x^{-\alpha}, x > 0$ for $\alpha \in (0,2)$, as in (1.9).

Recall the characteristic function of the Karlin stable process in (1.1). We shall rewrite it in a more convenient expression for our proof. Throughout, for $d \in \mathbb{N}$, write

$$\Lambda_d := \{0,1\}^d \setminus \{(0,\cdots,0)\},$$

and for $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d) \in \Lambda_d$,

$$\langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle := \sum_{j=1}^d \theta_j\delta_j.$$

Recall that $N(t)$ is a standard Poisson process on $\mathbb{R}_+$. For $\boldsymbol{\delta} \in \Lambda_d$ and $\boldsymbol{t} = (t_1, \ldots, t_d) \in [0,1]^d$, define

$$\{\boldsymbol{N}(q\boldsymbol{t}) = \boldsymbol{\delta} \bmod 2\} := \{N(qt_j) = \delta_j \bmod 2 \text{ for all } j = 1, \cdots, d\}.$$

For $k \in \mathbb{N}_0$, we write k mod $2 = 1$ if k is odd and 0 otherwise. Observe

$$\mathbb{E}_q \left| \sum_{j=1}^d \theta_j \mathbf{1}_{\{N(t_j q) \text{ odd}\}} \right|^\alpha = \sum_{\boldsymbol{\delta} \in \Lambda_d} |\langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle|^\alpha \mathbb{P}_q(\boldsymbol{N}(q\boldsymbol{t}) = \boldsymbol{\delta} \bmod 2).$$

Therefore, with

$$\mathfrak{m}_{\alpha,\beta}(\boldsymbol{t}, \boldsymbol{\delta}) := \frac{\beta}{\Gamma(1-\beta)\mathsf{C}_\alpha} \int_0^\infty \mathbb{P}_q \left(\boldsymbol{N}(q\boldsymbol{t}) = \boldsymbol{\delta} \bmod 2 \right) q^{-\beta-1} dq,$$

we see that (1.1) becomes

$$\mathbb{E} \exp \left(i \sum_{j=1}^d \theta_j \zeta_{\alpha,\beta}(t_j) \right) = \exp \left(- \sum_{\boldsymbol{\delta} \in \Lambda_d} |\langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle|^\alpha \mathfrak{m}_{\alpha,\beta}(\boldsymbol{t}, \boldsymbol{\delta}) \right).$$

Now for the aggregated model, we start by computing the characteristic function of the finite-dimensional distributions of $S_{\lfloor nt \rfloor}/a_n$ (recall S_n in (1.6)). Write

$$A_{n,q} := \left\{ \tau_n^{(q)} \text{ is odd} \right\} \quad \text{and} \quad A^\delta := \begin{cases} A, & \text{if } \delta = 1, \\ A^c, & \text{if } \delta = 0. \end{cases}$$

For any $d \in \mathbb{N}$, $\boldsymbol{t} \in [0,1]^d$, write $n_j := \lfloor nt_j \rfloor$, $j = 1, \cdots, d$, and

$$A_{n,q,\boldsymbol{t},\boldsymbol{\delta}} := \bigcap_{j=1}^d A_{n_j,q}^{\delta_j}.$$

Then, for $\boldsymbol{\theta} \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E} \exp \left(i \sum_{j=1}^d \theta_j \frac{S_{\lfloor nt_j \rfloor}}{a_n} \right) &= \mathbb{E} \exp \left(i \sum_{j=1}^d \frac{\theta_j \mathcal{X}}{a_n q^{1/\alpha'}} \mathbf{1}_{A_{n_j,q}} \right) = \mathbb{E} \exp \left(i \sum_{\boldsymbol{\delta} \in \Lambda_d} \frac{\langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle \mathcal{X}}{a_n q^{1/\alpha'}} \mathbf{1}_{A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}} \right) \\
&= \mathbb{E} \left(\mathbb{E} \left(\exp \left(i \sum_{\boldsymbol{\delta} \in \Lambda_d} \frac{\langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle \mathcal{X}}{a_n q^{1/\alpha'}} \mathbf{1}_{A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}} \right) \middle| \mathcal{X}, q \right) \right). \qquad (3.1)
\end{aligned}$$

In the second step we used the fact that $A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}$ are disjoint for different $\boldsymbol{\delta} \in \Lambda_d$. Using the disjointness again, the inner conditional expectation of (3.1) becomes

$$\begin{aligned}
\mathbb{E} \left(\prod_{\boldsymbol{\delta} \in \Lambda_d} \exp \left(\frac{i \langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle \mathcal{X}}{a_n q^{1/\alpha'}} \mathbf{1}_{A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}} \right) \middle| \mathcal{X}, q \right) &= \mathbb{E} \left(\sum_{\boldsymbol{\delta} \in \Lambda_d} \exp \left(\frac{i \langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle \mathcal{X}}{a_n q^{1/\alpha'}} \right) \mathbf{1}_{A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}} + \mathbf{1}_{A_{n,q,\boldsymbol{t},\mathbf{0}}} \middle| \mathcal{X}, q \right) \\
&= \mathbb{E} \left(1 - \sum_{\boldsymbol{\delta} \in \Lambda_d} \left(1 - \exp \left(\frac{i \langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle \mathcal{X}}{a_n q^{1/\alpha'}} \right) \right) \mathbf{1}_{A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}} \middle| \mathcal{X}, q \right),
\end{aligned}$$

where in the second step above we used the fact that $\sum_{\boldsymbol{\delta} \in \{0,1\}^d} \mathbf{1}_{A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}} = 1$. Indeed, each $\boldsymbol{\delta}$ corresponds to one particular combination of oddness and evenness for the sequence $\{\tau_{n_j}^{(q)}\}_{j=1,\dots,d}$. Since exactly one of such combination is realized, the sum of the indicators is always 1. So we arrive at

$$\mathbb{E} \exp \left(i \sum_{j=1}^d \theta_j \frac{S_{\lfloor nt_j \rfloor}}{a_n} \right) = 1 - \sum_{\boldsymbol{\delta} \in \Lambda_d} \mathbb{E} \left(\left(1 - \phi_{\mathcal{X}} \left(\frac{\langle \boldsymbol{\theta}, \boldsymbol{\delta} \rangle}{a_n q^{1/\alpha'}} \right) \right) \mathbb{P}_q(A_{n,q,\boldsymbol{t},\boldsymbol{\delta}}) \right).$$

The key is now to establish

$$\Phi_n := \mathbb{E}\left[\left(1-\phi_{\mathcal{X}}\left(\frac{\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle}{a_n q^{1/\alpha'}}\right)\right)\mathbb{P}_q(A_{n,q,\boldsymbol{t},\boldsymbol{\delta}})\right] \sim \frac{1}{m_n}\left|\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle\right|^\alpha \mathfrak{m}_{\alpha,\beta}(\boldsymbol{t},\boldsymbol{\delta}) \text{ as } n\to\infty. \tag{3.2}$$

It then follows that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\exp\left(i\sum_{j=1}^d\theta_j\frac{\widehat{S}_n(t_j)}{a_n}\right) &= \lim_{n\to\infty}\left[\mathbb{E}\exp\left(i\sum_{j=1}^d\theta_j\frac{S_{\lfloor nt_j\rfloor}}{a_n}\right)\right]^{m_n}\\ &= \exp\left(-\sum_{\boldsymbol{\delta}\in\Lambda_d}\left|\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle\right|^\alpha\mathfrak{m}_{\alpha,\beta}(\boldsymbol{t},\boldsymbol{\delta})\right) = \mathbb{E}\exp\left(i\sum_{j=1}^d\theta_j\zeta_{\alpha,\beta}(t_j)\right).\end{aligned}$$

Therefore, it remains to prove (3.2). As a preparation, note that for $q>0$ fixed, by standard Poisson approximation for binomial distribution with parameter $(n,q/n)$, the point process

$$\sum_{i=1}^n\delta_{i/n}\mathbf{1}_{\left\{\eta_i^{(q/n)}=1\right\}}$$

converges in distribution to a Poisson point process on $(0,1)$ with intensity q. As a consequence,

$$\lim_{n\to\infty}\mathbb{P}_q(A_{n,q/n,\boldsymbol{t},\boldsymbol{\delta}}) = \mathbb{P}_q\left(\boldsymbol{N}(q\boldsymbol{t}) = \boldsymbol{\delta} \bmod 2\right), \text{ for all } q>0.$$

Moreover, the above convergence is uniform on any neighborhood of zero, that is,

$$\lim_{n\to\infty}\sup_{q\in[0,\epsilon]}\frac{\mathbb{P}_q(A_{n,q/n,\boldsymbol{t},\boldsymbol{\delta}})}{\mathbb{P}_q\left(\boldsymbol{N}(q\boldsymbol{t})=\boldsymbol{\delta} \bmod 2\right)} = 1, \text{ for all } \epsilon>0.$$

This can be checked by writing explicitly the expressions for the two probabilities. For the sake of simplicity we write only for $d=2, t_1<t_2$ and $\delta_1=\delta_2=1$:

$$\begin{aligned}\mathbb{P}_q(A_{n,q/n,(t_1,t_2),(1,1)}) &= \frac12\left(1-\left(1-\frac{2q}{n}\right)^{\lfloor nt_1\rfloor}\right)\cdot\frac12\left(1+\left(1-\frac{2q}{n}\right)^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}\right)\\ &\to \frac12\left(1-e^{-2qt_1}\right)\cdot\frac12\left(1+e^{-2q(t_2-t_1)}\right) = \mathbb{P}_q(N(qt_1)\text{ odd}, N(qt_2)\text{ odd}),\end{aligned}$$

and the uniform convergence is readily checked.

We first establish a lower bound for Φ_n in (3.2). If we restrict expectation to $q\in[\epsilon/n,\epsilon^{-1}/n]$ for $\epsilon\in(0,1)$ instead of $q\in[0,1]$, then it follows that

$$\begin{aligned}\Phi_n \ge \Phi_{n,\epsilon} &:= \int_\epsilon^{\epsilon^{-1}}\left(1-\phi_{\mathcal{X}}\left(\frac{\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle}{a_n n^{-1/\alpha'}q^{1/\alpha'}}\right)\right)\mathbb{P}_q(A_{n,q/n,\boldsymbol{t},\boldsymbol{\delta}})\left(\frac{q}{n}\right)^{-\rho}L(n/q)\frac{dq}{n}\\ &\sim \int_\epsilon^{\epsilon^{-1}}\sigma_{\mathcal{X}}^\alpha\left|\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle\right|^\alpha(a_n n^{-1/\alpha'}q^{1/\alpha'})^{-\alpha}\mathbb{P}_q(\boldsymbol{N}(q\boldsymbol{t})=\boldsymbol{\delta} \bmod 2)\left(\frac{q}{n}\right)^{-\rho}L(n/q)\frac{dq}{n} \qquad (3.3)\\ &= \frac{\left|\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle\right|^\alpha}{m_n}\frac{\beta}{\Gamma(1-\beta)\mathsf{C}_\alpha}\int_\epsilon^{\epsilon^{-1}}\frac{L(n/q)}{L(n)}q^{-\gamma-\rho}\mathbb{P}_q(\boldsymbol{N}(q\boldsymbol{t})=\boldsymbol{\delta} \bmod 2)dq\\ &\sim \frac{\left|\langle\boldsymbol{\theta},\boldsymbol{\delta}\rangle\right|^\alpha}{m_n}\frac{\beta}{\Gamma(1-\beta)\mathsf{C}_\alpha}\int_\epsilon^{\epsilon^{-1}}q^{-\gamma-\rho}\mathbb{P}_q(\boldsymbol{N}(q\boldsymbol{t})=\boldsymbol{\delta} \bmod 2)dq,\end{aligned}$$

which is the same as the right-hand side of (3.2) by taking $\epsilon\downarrow 0$. In the second line above we need $a_n n^{-1/\alpha'}\to\infty$ as $n\to\infty$, which is the same as our assumption on m_n in (1.11). Note that according to the uniform convergence theorem of the slowly varying functions (see, for example, Bingham et al. (1987)), we need to restrict to a compact interval $[\epsilon,\epsilon^{-1}]$ bounded away from 0 to have the uniform convergence in the second and the fourth steps above.

It remains to show that $\lim_{\delta\downarrow 0}\limsup_{n\to\infty} m_n(\Phi_n - \Phi_{n,\epsilon}) = 0$, and we need to work with the intervals $[0,\epsilon/n]$ and $[\epsilon^{-1}/n, 1]$. This part can be done, and a similar treatment shows up in the proof of Theorem 1.2 (more precisely, see (4.5) and (4.6)). Since Theorem 1.2 is a stronger result than Theorem 1.1, we provide full details therein and omit the rest of the proof here.

4. Limit theorems for point processes

We first prove Theorem 1.2 in Section 4.2. It leads to a second proof of Theorem 1.1 in Section 4.3, and also the convergence of the so-called Karlin random sup-measures introduced in Durieu and Wang (2018) in Section 4.4.

4.1. *A preparation.* We start by proving a weaker version of Theorem 1.2. Recall that

$$a_n = \left(C_{\mathcal{X}}\frac{\Gamma(1-\beta)}{\beta}\cdot n^\beta m_n L(n)\right)^{1/\alpha}.$$

We shall provide two versions in the following proposition, the second with the alternating signs $(-1)^{\tau_{i,j}^{(q_i)}}$ taken into account but the first not. For each $\ell\in\mathbb{N}$, let $U_{\ell,1:Q_{\beta,\ell}} < \cdots < U_{\ell,Q_{\beta,\ell}:Q_{\beta,\ell}}$ denote the order statistics of $\{U_{\ell,j}\}_{j=1,\dots,Q_{\beta,\ell}}$.

Proposition 4.1. *Under the assumption of Theorem 1.2, we have*

$$\widetilde{\xi}_n := \sum_{i=1}^{m_n}\sum_{j=1}^{n}\eta_{i,j}^{(q_i)}\delta_{\left(\mathcal{X}_i/(a_n q_i^{1/\alpha'}),j/n\right)} \Rightarrow \xi := \sum_{\ell=1}^{\infty}\sum_{j=1}^{Q_{\beta,\ell}}\delta_{\left(\varepsilon_\ell\Gamma_\ell^{-1/\alpha},U_{\ell,j}\right)}, \tag{4.1}$$

where ξ is as in (1.13), and

$$\widehat{\xi}_n := \sum_{i=1}^{m_n}\sum_{j=1}^{n}\eta_{i,j}^{(q_i)}\delta_{\left((-1)^{\tau_{i,j}^{(q_i)}+1}\mathcal{X}_i/(a_n q_i^{1/\alpha'}),j/n\right)} \Rightarrow \widehat{\xi} := \sum_{\ell=1}^{\infty}\sum_{j=1}^{Q_{\beta,\ell}}\delta_{\left((-1)^{j+1}\varepsilon_\ell\Gamma_\ell^{-1/\alpha},U_{\ell,j:Q_{\beta,\ell}}\right)}. \tag{4.2}$$

Throughout, we let $\widetilde{V}_\alpha$ denote a symmetrized α-Pareto random variable (symmetric and $\mathbb{P}(|\widetilde{V}_\alpha| > x) = x^{-\alpha}, x\ge 1$). The following is the key step.

Proposition 4.2. *Introduce*

$$\Omega_n(x) := \left\{\frac{|\mathcal{X}|}{a_n q^{1/\alpha'}} > x, \tau_n^{(q)} \neq 0\right\}.$$

Then,

$$\mathbb{P}\left(\Omega_n(x)\right) \sim \frac{1}{m_n}x^{-\alpha},\ \textit{for all } x>0, \tag{4.3}$$

and

$$\mathcal{L}\left(\tau_n^{(q)}, nq, \frac{\mathcal{X}}{a_n q^{1/\alpha'}}\,\middle|\,\Omega_n(x)\right) \rightsquigarrow \mathcal{L}\left(Q_\beta, G(Q_\beta-\beta), x\widetilde{V}_\alpha\right),\ \textit{for all } x>0, \tag{4.4}$$

where on the right-hand side $G(Q_\beta-\beta)$ is a Gamma random variable with random parameter $Q_\beta-\beta$, and $\widetilde{V}_\alpha$ a symmetrized α-Pareto random variable, independent from the first two.

The convergence (4.4) reads as the weak convergence of the conditional law of the random vector $(\tau_n^{(q)}, nq, \mathcal{X}/(a_n q^{1/\alpha'}))$ given $\Omega_n(x)$ to the law of the random vector $(Q_\beta, G(Q_\beta-\beta), x\widetilde{V}_\alpha)$.

Remark 4.3. The convergence to $G(Q_\beta-\beta)$ is not needed in our proof. Nevertheless, it has a probability density in closed form that can be derived as follows. Notice that $G(Q_\beta-\beta) \overset{d}{=} G(1-\beta) + \sum_{j=1}^{Q_\beta-1} G_j(1)$, where $G(1-\beta)$ is Gamma with parameter $1-\beta$, $\{G_j(1)\}_{j\in\mathbb{N}}$ are standard exponential random variables and all these random variables and Q_β are independent. Recall also the identity that $\mathbb{E}z^{Q_\beta} = 1-(1-z)^\beta$. Then, it follows that

$$\begin{aligned}\mathbb{E}e^{-\theta G(Q_\beta-\beta)} &= \mathbb{E}e^{-\theta G(1-\beta)}\mathbb{E}\left(\left(\mathbb{E}e^{-\theta G(1)}\right)^{Q_\beta-1}\right)\\ &= (1+\theta)^{\beta-1}\mathbb{E}\left((1+\theta)^{-Q_\beta}\right)(1+\theta) = (1+\theta)^\beta-\theta^\beta.\end{aligned}$$

This is the Laplace transform of the probability density function

$$\frac{(1-e^{-x})\beta x^{-\beta-1}}{\Gamma(1-\beta)}, \quad x\ge 0.$$

(See Prudnikov et al. (1992, 2.2.4.2).)

Proof of Proposition 4.2: Write

$$\begin{aligned}\mathbb{P}(\Omega_n(x)) &= \mathbb{P}\left(\tau_n^{(q)}\neq 0, \frac{|\mathcal{X}|}{a_n q^{1/\alpha'}}>x\right) = \mathbb{E}\left(\mathbb{E}\left((1-(1-q)^n)\,\mathbf{1}_{\{|\mathcal{X}|>a_nq^{1/\alpha'}x\}}\;\Big|\; q\right)\right)\\ &= \mathbb{E}\left((1-(1-q)^n)\,\overline{F}_{|\mathcal{X}|}\left(a_nq^{1/\alpha'}x\right)\right) = \Psi_{n,\delta}(x)+\Psi_{n,\delta,1}(x)+\Psi_{n,\delta,2}(x),\end{aligned} \tag{4.5}$$

with, for $\delta\in(0,1)$,

$$\begin{aligned}\Psi_{n,\delta}(x) &:= \mathbb{E}\left((1-(1-q)^n)\,\overline{F}_{|\mathcal{X}|}\left(a_nq^{1/\alpha'}x\right); q\in[\delta/n,\delta^{-1}/n]\right),\\ \Psi_{n,\delta,1}(x) &:= \mathbb{E}\left((1-(1-q)^n)\,\overline{F}_{|\mathcal{X}|}\left(a_nq^{1/\alpha'}x\right); q>\delta^{-1}/n\right),\\ \Psi_{n,\delta,2}(x) &:= \mathbb{E}\left((1-(1-q)^n)\,\overline{F}_{|\mathcal{X}|}\left(a_nq^{1/\alpha'}x\right); q<\delta/n\right).\end{aligned}$$

First, for all $\delta\in(0,1)$,

$$\begin{aligned}\Psi_{n,\delta}(x) &= \int_\delta^{\delta^{-1}}\left(\frac{q}{n}\right)^{-\rho}L(n/q)\left(1-\left(1-\frac{q}{n}\right)^n\right)\overline{F}_{|\mathcal{X}|}\left(a_n\left(\frac{q}{n}\right)^{1/\alpha'}x\right)\frac{dq}{n}\\ &\sim C_{\mathcal{X}}a_n^{-\alpha}x^{-\alpha}n^{\rho+\gamma-1}L(n)\int_\delta^{\delta^{-1}}q^{-\rho}\frac{L(n/q)}{L(n)}\left(1-\left(1-\frac{q}{n}\right)^n\right)q^{-\gamma}dq\\ &\sim \frac{\beta}{\Gamma(1-\beta)}\frac{x^{-\alpha}}{m_n}\int_\delta^{\delta^{-1}}(1-e^{-q})q^{-\gamma-\rho}dq.\end{aligned}$$

In the second line, we applied $\overline{F}_{|\mathcal{X}|}(a_n(q/n)^{1/\alpha'}x)\sim C_{\mathcal{X}}(a_nx)^{-\alpha}(q/n)^{-\gamma}$ uniformly in $q\in[\delta,\delta^{-1}]$, and for this purpose we shall need $a_nn^{-1/\alpha'}\to\infty$, or equivalently $n^{\beta-\gamma}m_nL(n)\to\infty$, which is our standing condition (1.11). Restricted to the same interval we also have $L(n/q)/L(n)\to 1$ uniformly in q. Moreover, we used the fact that $\lim_{n\to\infty}(1-q/n)^n = e^{-q}$ uniformly over any compact interval $[0,C]$, $C>0$. Recalling that $\beta=\gamma+\rho-1$, we see that

$$\lim_{\delta\downarrow 0}\int_\delta^{\delta^{-1}}(1-e^{-q})q^{-\gamma-\rho}dq = \int_0^\infty(1-e^{-q})q^{-\beta-1}dq = \frac{\Gamma(1-\beta)}{\beta}.$$

Thus $\liminf_{n\to\infty}\mathbb{P}(\Omega_n(x))\ge\lim_{n\to\infty}\Psi_{n,\delta}(x)$ and this lower bound is tight as it becomes the desired limit in (4.3) as $\delta\downarrow 0$.

For the upper bound, it remains to show that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} m_n\Psi_{n,\delta,i}(x) = 0, \quad i=1,2. \tag{4.6}$$

We first show (4.6) with $i = 1$. This time we use

$$\Psi_{n,\delta,1}(x) \le \int_{\delta^{-1}/n}^{1} \overline{F}_{|\mathcal{X}|}\left(a_n q^{1/\alpha'} x\right) q^{-\rho} L(1/q) dq.$$

For n large enough, the condition on m_n (1.11) and the lower bound δ^{-1}/n on q guarantee that $a_n q^{1/\alpha'}$ is bounded away from 0. Hence by (1.9), $\overline{F}_{|\mathcal{X}|}(a_n q^{1/\alpha'} x) \le (1+\delta) C_{\mathcal{X}} (a_n q^{1/\alpha'} x)^{-\alpha}$, for all q in the range of the integral. Therefore, the above is bounded by, for n large enough,

$$\begin{aligned}(1+\delta) C_{\mathcal{X}} \int_{\delta^{-1}/n}^{1} (a_n q^{1/\alpha'} x)^{-\alpha} q^{-\rho} L(1/q) dq &= \frac{(1+\delta) C_{\mathcal{X}}}{(a_n x)^{\alpha}} \int_{1}^{\delta n} q^{\gamma+\rho-2} L(q) dq \\ &\sim \frac{(1+\delta) C_{\mathcal{X}}}{(a_n x)^{\alpha}} \beta^{-1} (\delta n)^{\beta} L(\delta n) \sim \frac{1}{m_n} \frac{1+\delta}{\Gamma(1-\beta)} \delta^{\beta} x^{-\alpha}.\end{aligned}$$

In the second step we invoked Karamata's theorem. Now (4.6) with $i = 1$ follows.

Next we show (4.6) with $i = 2$. Pick D_δ be such that for all $x > D_\delta$,

$$\sup_{x > D_\delta} \frac{\overline{F}_{|\mathcal{X}|}(x)}{C_{\mathcal{X}} x^{-\alpha}} < 1 + \delta. \tag{4.7}$$

Write $D_{n,\delta,x} = (D_\delta/(a_n x))^{\alpha'}$. One checks readily that the convergence $\lim_{n\to\infty} n D_{n,\delta,x} = 0$ is the same as $\lim_{n\to\infty} n^{\beta-\gamma} m_n L(n) = \infty$. We decompose further the integral area with respect to $q \in [0, \delta/n]$ into $[0, D_{n,\delta,x}]$ and $[D_{n,\delta,x}, \delta/n]$, and write respectively $\Psi_{n,\delta,2}(x) = \Psi_{n,\delta,2,1}(x) + \Psi_{n,\delta,2,2}(x)$. Then,

$$\begin{aligned}\Psi_{n,\delta,2,2}(x) &= \int_{nD_{n,\delta,x}}^{\delta} \left(\frac{q}{n}\right)^{-\rho} L(n/q) \left(1 - \left(1 - \frac{q}{n}\right)^n\right) \overline{F}_{|\mathcal{X}|}\left(a_n \left(\frac{q}{n}\right)^{1/\alpha'} x\right) \frac{dq}{n} \\ &\le (1+\delta) C_{\mathcal{X}} a_n^{-\alpha} x^{-\alpha} \int_{nD_{n,\delta,x}}^{\delta} \left(\frac{q}{n}\right)^{-\rho} L(n/q) \left(1 - \left(1 - \frac{q}{n}\right)^n\right) \left(\frac{q}{n}\right)^{-\gamma} \frac{dq}{n} \\ &\le (1+\delta) C_{\mathcal{X}} x^{-\alpha} \frac{n^{\beta} L(n)}{a_n^{\alpha}} \int_0^{\delta} \frac{L(n/q)}{L(n)} \left(1 - \left(1 - \frac{q}{n}\right)^n\right) q^{-\gamma-\rho} dq.\end{aligned}$$

In the first inequality above we applied (4.7), and in the second we used $\beta + 1 = \rho + \gamma$ and extended the lower bound of the integral region to zero. Then, for n large enough, so that $L(n/q)/L(n) < (1+\delta) q^{-\delta}$ for all $q \in (0, \delta)$ (Potter's bound (Resnick, 2007)), the above is bounded by, for $\delta \in (0, 2 - (\gamma + \rho))$,

$$\frac{1}{m_n} \frac{(1+\delta)^2 \beta x^{-\alpha}}{\Gamma(1-\beta)} \int_0^{\delta} \left(1 - \left(1 - \frac{q}{n}\right)^n\right) q^{-\gamma-\rho-\delta} dq \sim \frac{1}{m_n} \frac{(1+\delta)^2 \beta x^{-\alpha}}{\Gamma(1-\beta)} \int_0^{\delta} (1 - e^{-q}) q^{-\gamma-\rho-\delta} dq.$$

The right-hand side above has the expression $R_\delta(x)/m_n$ with $\lim_{\delta\downarrow 0} R_\delta(x) = 0$. Next, for $\delta > 0$ small enough,

$$\begin{aligned}\Psi_{n,\delta,2,1} &\le \int_0^{nD_{n,\delta,x}} \left(1 - \left(1 - \frac{q}{n}\right)^n\right) \left(\frac{q}{n}\right)^{-\rho} L(n/q) \frac{dq}{n} \\ &= L(n) n^{\rho-1} \int_0^{nD_{n,\delta,x}} \left(1 - \left(1 - \frac{q}{n}\right)^n\right) \frac{L(n/q)}{L(n)} q^{-\rho} dq \\ &\le (1+\delta) L(n) n^{\rho-1} \int_0^{nD_{n,\delta,x}} (1 - e^{-q}) q^{-\rho-\delta} dq \sim \frac{1+\delta}{2-\rho-\delta} L(n) n^{\rho-1} \left(n \left(\frac{D_\delta}{a_n x}\right)^{\alpha'}\right)^{2-\rho-\delta},\end{aligned}$$

where in the second inequality above we used Potter's bound again, for n large enough. We want to show the above is asymptotically of a smaller order than m_n^{-1}, or equivalently, dropping the dependence on ρ, D_δ, x,

$$(m_n L(n))^{1-(2-\rho-\delta)/\gamma} n^{1-\delta-\beta(2-\rho-\delta)/\gamma} = (m_n L(n))^{(\gamma+\rho+\delta-2)/\gamma} n^{(\gamma(1-\delta)-\beta(2-\rho-\delta))/\gamma} \to 0.$$

Indeed, since $\rho+\gamma \in (1,2)$ and $\rho < 1$, one could take $\delta > 0$ small enough (precisely, $\rho+\delta < 1, \gamma+\rho+\delta < 2$) so that

$$n^{\frac{\gamma(1-\delta)-\beta(2-\rho-\delta)}{\gamma+\delta-(2-\rho)}} m_n L(n) = n^{\frac{\beta(2-\rho-\delta)-\gamma(1-\delta)}{2-(\gamma+\rho+\delta)}} m_n L(n) \geq n^{\beta-\gamma} m_n L(n) \to \infty,$$

where the last step is our standing assumption. Combining the above we have proved (4.6) with $i=2$, and hence (4.3).

Similarly, one can show that

$$\begin{aligned}\mathbb{P}\left(nq \in (a,b), \tau_n^{(q)} = k, \frac{|\mathcal{X}|}{a_n q^{1/\alpha'}} > x\right) &= \int_{a/n}^{b/n} \binom{n}{k} q^k (1-q)^{n-k} \overline{F}_{|\mathcal{X}|}\left(a_n q^{1/\alpha'} x\right) q^{-\rho} L(1/q) dq \\ &\sim \frac{1}{m_n} \frac{\beta x^{-\alpha}}{\Gamma(1-\beta)} \int_a^b \frac{n^k}{k!} \left(\frac{q}{n}\right)^{k-\rho} \left(1-\frac{q}{n}\right)^n (n^{\beta-\gamma} q^\gamma)^{-1} \frac{dq}{n} \\ &\sim \frac{1}{m_n} \frac{\beta x^{-\alpha}}{\Gamma(1-\beta)} \int_a^b \frac{e^{-q}}{k!} q^{k-1-\beta} dq.\end{aligned}$$

The asymptotic equivalence above follows from the dominated convergence theorem and is much simpler than before. We omit the details. So, we have

$$\begin{aligned}\mathbb{P}\left(nq \in (a,b), \tau_n^{(q)} = k, \frac{|\mathcal{X}|}{a_n q^{1/\alpha'}} > x\right) &\sim \frac{x^{-\alpha}}{m_n} \frac{\beta}{\Gamma(1-\beta)} \frac{\Gamma(k-\beta)}{\Gamma(k+1)} \mathbb{P}(G(k-\beta) \in (a,b)) \\ &= \frac{x^{-\alpha}}{m_n} \mathbb{P}(Q_\beta = k, G(k-\beta) \in (a,b)),\end{aligned}$$

where $G(k-\beta)$ is a Gamma random variable with parameter $k-\beta$, independent from Q_β. The desired (4.4) then follows. $\square$

Proof of Proposition 4.1: The second convergence (4.2) can be proved in exactly the same way as (4.1), and the only difference is the alternating signs in both the discrete-time aggregated model and the limit point process. Therefore, we prove only (4.1) for the sake of notational simplicity.

We prove by computing the Laplace transform. Let $f(x,y)$ be a bounded and continuous function from $\mathbb{R} \times [0,1]$ to $\mathbb{R}_+$ such that $f(x,y) = 0$ for all $|x| \leq \kappa$ for some $\kappa > 0$. Then,

$$\begin{aligned}\mathbb{E} e^{-\widetilde{\xi}_n(f)} &= \mathbb{E}\exp\left(-\sum_{i=1}^{m_n}\sum_{j=1}^n f\left(\mathcal{X}_i/(a_n q_i^{1/\alpha'}), j/n\right) \eta_{i,j}^{(q_i)}\right) \\ &= \left(\mathbb{E}\exp\left(-\sum_{j=1}^n f\left(\mathcal{X}/(a_n q^{1/\alpha'}), j/n\right)\eta_j^{(q)}\right)\right)^{m_n} = \left(\mathbb{P}(\Omega_n(\kappa))\Psi_n(\kappa) + 1 - \mathbb{P}(\Omega_n(\kappa))\right)^{m_n},\end{aligned}$$

with

$$\Psi_n(\kappa) := \mathbb{E}\left(\exp\left(-\sum_{j=1}^n f\left(\mathcal{X}/(a_n q^{1/\alpha'}), j/n\right)\eta_j^{(q)}\right) \,\middle|\, \Omega_n(\kappa)\right).$$

Then, by Proposition 4.2, writing $\Omega_{n,\ell}(\kappa) := \{|\mathcal{X}|/(a_n q^{1/\alpha'}) > \kappa, \tau_n^{(q)} = \ell\}$,

$$\Psi_n(\kappa) = \sum_{\ell=1}^{\infty} \mathbb{E}\left(\exp\left(-\sum_{j=1}^{n} f\left(\mathcal{X}/(a_n q^{1/\alpha'}), j/n\right)\eta_j^{(q)}\right) \,\middle|\, \Omega_{n,\ell}(\kappa)\right)\mathbb{P}\left(\Omega_{n,\ell}(\kappa) \mid \Omega_n(\kappa)\right)$$

$$\to \sum_{\ell=1}^{\infty} \mathbb{E}\exp\left(-\sum_{j=1}^{\ell} f(\kappa\widetilde{V}_\alpha, U_j)\right)\mathbb{P}(Q_\beta = \ell) = \mathbb{E}\exp\left(-\sum_{j=1}^{Q_\beta} f(\kappa\widetilde{V}_\alpha, U_j)\right) =: \Psi(\kappa).$$

The convergence above follows from the observation that given $\Omega_{n,\ell}(\kappa)$, $\{\eta_j^{(q)}\}_{j=1,\dots,n}$ is exchangeable; from this we derive that since $\sum_{j=1}^n \eta_j^{(q)} = \ell$, the law of $\{j/n\}_{j=1,\dots,n,\eta_j^{(q)}=1}$ follows the law of ℓ-sampling without replacement from $\{1,\dots,n\}$, which has the limit as $\{U_j\}_{j=1,\dots,\ell}$, and their independence from $\widetilde{V}_\alpha$ follows from the conditional independence of $\{\eta_j^{(q)}\}_{j=1,\dots,n}$ from $\mathcal{X}/(a_n q^{1/\alpha'})$. Therefore, it follows that, recalling $\mathbb{P}(\Omega_n(\kappa)) \sim \kappa^{-\alpha}/m_n$ in (4.3),

$$\begin{aligned}\mathbb{E}e^{-\widetilde{\xi}_n(f)} &= (1 - \mathbb{P}(\Omega_n(\kappa))(1 - \Psi_n(\kappa)))^{m_n}\\ &\to \exp\left(-\lim_{n\to\infty} m_n \mathbb{P}(\Omega_n(\kappa))(1 - \Psi(\kappa))\right) = \exp\left(-\kappa^{-\alpha}(1 - \Psi(\kappa))\right).\end{aligned}$$

At the same time, let N_κ denote a Poisson random variable with intensity $\kappa^{-\alpha}$. Then,

$$\mathbb{E}e^{-\xi(f)} = \mathbb{E}\exp\left(-\sum_{i=1}^{N_\kappa}\sum_{j=1}^{Q_{\beta,i}} f(\widetilde{V}_{\alpha,i}, U_{i,j})\right) = \mathbb{E}\left(\Psi(\kappa)^{N_\kappa}\right) = e^{-\kappa^{-\alpha}(1-\Psi(\kappa))},$$

where $(\widetilde{V}_{\alpha,i}, Q_{\beta,i}, \{U_{i,j}\}_{j\in\mathbb{N}}), i \in \mathbb{N}$ are i.i.d. copies of $(\widetilde{V}_\alpha, Q_\beta, \{U_j\}_{j\in\mathbb{N}})$. This completes the proof. $\square$

4.2. *Proof of Theorem 1.2.* Set

$$\widetilde{X}_{n,j} := \frac{1}{a_n}\sum_{i=1}^{m_n} \frac{\mathcal{X}_i \eta_{i,j}^{(q_i)}}{q_i^{1/\alpha'}}.$$

Recall the notations around (1.7). Recall also that $\rho + \gamma = \beta + 1 \in (1,2)$, $\gamma = \alpha/\alpha' > 0$ and $\rho < 1$. Introduce for $\epsilon > 0$,

$$\widetilde{X}_{n,j,\epsilon} := \frac{1}{a_n}\sum_{i=1}^{m_n} \frac{\mathcal{X}_i \eta_{i,j}^{(q_i)}}{q_i^{1/\alpha'}} \mathbf{1}_{\left\{|\mathcal{X}_i| > a_n q_i^{1/\alpha'}\epsilon\right\}}.$$

The idea of the proof is to compare

$$\xi_{n,\epsilon} := \sum_{j=1}^{n} \delta_{(\widetilde{X}_{n,j,\epsilon}, j/n)} \quad \text{and} \quad \widetilde{\xi}_{n,\epsilon} := \sum_{\substack{i=1,\dots,m_n\\ |\mathcal{X}_i| > a_n q_i^{1/\alpha'}\epsilon}} \sum_{j=1}^{n} \eta_{i,j}^{(q_i)} \delta_{\left(\mathcal{X}_i/(a_n q_i^{1/\alpha'}), j/n\right)}. \tag{4.8}$$

We have seen in Proposition 4.1 that $\widetilde{\xi}_{n,\epsilon}$ converges to the desired point process ξ in (1.13) restricted to $([-\infty, -\epsilon] \cup [\epsilon, \infty]) \times [0,1]$. Introduce also

$$\widehat{C}_{n,\epsilon}(i) := \left\{ j = 1,\dots,n : \frac{|\mathcal{X}_i|}{q_i^{1/\alpha'}}\eta_{i,j}^{(q_i)} > a_n\epsilon \right\} \quad \text{and} \quad \widehat{C}_{n,\epsilon} := \bigcup_{i=1}^{m_n} \widehat{C}_{n,\epsilon}(i),$$

and furthermore

$$\epsilon_n := n^{-\beta_0/\alpha}, n \in \mathbb{N}, \tag{4.9}$$

for any $\beta_0 \in (0,\beta)$. We begin by analyzing $\widetilde{X}_{n,j} - \widetilde{X}_{n,j,\epsilon}$, which is the same as $Z_{n,j,\epsilon,\epsilon_n} + W_{n,j,\epsilon_n}$ with

$$Z_{n,j,\epsilon,\epsilon_n} := \sum_{i=1}^{m_n} \frac{\mathcal{X}_i}{a_n q_i^{1/\alpha'}} \eta_{i,j}^{(q_i)} \mathbf{1}_{\left\{|\mathcal{X}_i|/\left(a_n q_i^{1/\alpha'}\right) \in [\epsilon_n,\epsilon]\right\}},$$
$$W_{n,j,\epsilon_n} := \sum_{i=1}^{m_n} \frac{\mathcal{X}_i}{a_n q_i^{1/\alpha'}} \eta_{i,j}^{(q_i)} \mathbf{1}_{\left\{|\mathcal{X}_i|/\left(a_n q_i^{1/\alpha'}\right) < \epsilon_n\right\}}, \quad j = 1,\dots,n.$$

Lemma 4.4. *We have, for $r \equiv r_{\beta,\beta_0} := \lfloor 1/(\beta-\beta_0) \rfloor + 1$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{j=1,\dots,n} |Z_{n,j,\epsilon,\epsilon_n}| \ge r\epsilon \right) = 0, \text{ for all } \epsilon > 0, \tag{4.10}$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{j\in \widehat{C}_{n,\epsilon}} |Z_{n,j,\epsilon,\epsilon_n}| > 0 \right) = 0. \tag{4.11}$$

Lemma 4.5. *We have*

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{j=1,\dots,n} |W_{n,j,\epsilon_n}| > \lambda \right) = 0, \text{ for all } \lambda > 0. \tag{4.12}$$

Proof of Lemma 4.4: We shall need

$$\mathbb{P}\left(\frac{|\mathcal{X}|}{a_n q^{1/\alpha'}} > \epsilon_n, \eta_1^{(q)} = 1 \right) \le C(a_n\epsilon_n)^{-\alpha}, \text{ for all } n \in \mathbb{N}. \tag{4.13}$$

Here and below, we let C denote a positive constant that may change from line to line. To see the above, we write the probability on the left-hand side of (4.13) as $\int_0^1 q^{1-\rho} L(1/q) \overline{F}_{|\mathcal{X}|}(a_n q^{1/\alpha'}\epsilon_n) dq$, and let d_n be such that

$$d_n \downarrow 0, \quad d_n(a_n\epsilon_n)^{\alpha'} \to \infty \quad \text{and} \quad d_n^{2-\rho} L(1/d_n)(a_n\epsilon_n)^{\alpha} \to 0. \tag{4.14}$$

(One readily checks that such a sequence exists since $(a_n\epsilon_n)^{-\alpha'} \ll (a_n\epsilon_n)^{-\alpha/(2-\rho)}$, which is equivalent to $\alpha' > \alpha/(2-\rho)$, or $2-\rho-\gamma > 0$.) Decompose the integral into $\int_0^{d_n}$ and $\int_{d_n}^1$, we bound the first by $\int_0^{d_n} q^{1-\rho} L(1/q) dq \sim (2-\rho)^{-1} d_n^{2-\rho} L(1/d_n)$, and the second by

$$C \int_{d_n}^1 q^{1-\rho} L(1/q)(a_n\epsilon_n q^{1/\alpha'})^{-\alpha} dq = C(a_n\epsilon_n)^{-\alpha} \int_{d_n}^1 q^{1-\rho-\gamma} L(1/q) dq \sim C(a_n\epsilon_n)^{-\alpha}.$$

Note that in the above, we need $d_n(a_n\epsilon_n)^{\alpha'} \to \infty$ (the second condition in (4.14)), and the third condition in (4.14) now implies that the integral over $[d_n,1]$ is dominant. We have proved (4.13).

Now, to prove (4.10), it suffices to prove

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{j=1,\dots,n} \sum_{i=1}^{m_n} \eta_{i,j}^{(q_i)} \mathbf{1}_{\left\{|\mathcal{X}_i| > a_n q_i^{1/\alpha'}\epsilon_n\right\}} \ge r \right) = 0.$$

In words, with probability going to zero, at some location j there are more than r different indices i such that $|\mathcal{X}_i|$ is large and also $\eta_{i,j}^{(q_i)} = 1$ (in the complement of this event, the largest possible value of $|Z_{n,j,\epsilon,\epsilon_n}|$ is $(r-1)\epsilon_n$, for all j). An upper bound of the probability of interest above is then

$$n\binom{m_n}{r}\left(\mathbb{P}\left(\frac{|\mathcal{X}|}{a_n q^{1/\alpha'}} > \epsilon_n, \eta_1^{(q)} = 1 \right) \right)^r \le C n m_n^r (a_n\epsilon_n)^{-\alpha r}.$$

We see that our choices of $\beta_0 \in (0,\beta)$ and r entail that the right-hand side above decays to zero. We have thus proved (4.10).

Next, we prove (4.11). By a similar argument as above, we have

$$\mathbb{P}\left(\max_{j\in\widehat{C}_{n,\epsilon}}\sum_{i=1}^{m_n}\eta_{i,j}^{(q_i)}\mathbf{1}_{\left\{|\mathcal{X}_i|>a_nq_i^{1/\alpha'}\epsilon_n\right\}}>0\right)\le\mathbb{P}\left(|\widehat{C}_{n,\epsilon}|>K\right)+Km_n\mathbb{P}\left(\frac{|\mathcal{X}|}{a_nq^{1/\alpha'}}>\epsilon_n,\eta_1^{(q)}=1\right).$$

The second term on the right-hand side above is bounded from above by $CKm_n(a_n\epsilon_n)^{-\alpha}\to 0$, for all $K>0$ fixed. So we have

$$\limsup_{n\to\infty}\mathbb{P}\left(\max_{j\in\widehat{C}_{n,\epsilon}}\sum_{i=1}^{m_n}\eta_{i,j}^{(q_i)}\mathbf{1}_{\left\{|\mathcal{X}_i|>a_nq_i^{1/\alpha'}\epsilon_n\right\}}>0\right)\le\limsup_{n\to\infty}\mathbb{P}\left(|\widehat{C}_{n,\epsilon}|>K\right),$$

where the right-hand side tends to zero by taking $K\to\infty$. Indeed, first notice that by Proposition 4.1, $|\widehat{C}_{n,\epsilon}|\le\sum_{i=1}^{m_n}|\widehat{C}_{n,\epsilon}(i)|\Rightarrow\sum_{i=1}^{N_\epsilon}Q_{\beta,i}$, where N_ϵ is a Poisson random variable with parameter $\epsilon^{-\alpha}$, and $\{Q_{\beta,i}\}_{i\in\mathbb{N}}$ are i.i.d. random variables independent from N_ϵ. Therefore,

$$\limsup_{n\to\infty}\mathbb{P}\left(|\widehat{C}_{n,\epsilon}|>K\right)\le\mathbb{P}\left(\sum_{i=1}^{N_\epsilon}Q_{\beta,i}>K\right).$$

(This inequality is actually an equality, as later on we shall see that $|\widehat{C}_{n,\epsilon}|=\sum_{i=1}^{m_n}|\widehat{C}_{n,\epsilon}(i)|$ with probability tending to one; i.e., $\lim_{n\to\infty}\mathbb{P}(\mathcal{E}_{n,1,\epsilon}^c)=0$ in the proof of Theorem 1.2.) It thus follows that

$$\lim_{K\to\infty}\lim_{n\to\infty}\mathbb{P}\left(|\widehat{C}_{n,\epsilon}|>K\right)=0,\ \text{for all } \epsilon>0. \tag{4.15}$$

We have proved (4.11). $\square$

Proof of Lemma 4.5: Now we prove (4.12). Write

$$W_{n,1,\epsilon_n}=\sum_{i=1}^{m_n}V_{n,i,\epsilon_n}\quad\text{with}\quad V_{n,i,\epsilon_n}:=\frac{\mathcal{X}_i}{a_nq_i^{1/\alpha'}}\eta_{i,1}^{(q_i)}\mathbf{1}_{\left\{|\mathcal{X}_i|/\left(a_nq_i^{1/\alpha'}\right)<\epsilon_n\right\}}.$$

Observe that $|V_{n,i,\epsilon_n}|\le\epsilon_n$ and write $w_n:=m_n\mathbb{E}V_{n,1,\epsilon_n}^2$. By union bound first and then the Bernstein inequality (Boucheron et al., 2013, (2.10)) , we have

$$\mathbb{P}\left(\max_{j=1,\dots,n}|W_{n,j,\epsilon_n}|>\lambda\right)\le n\mathbb{P}(|W_{n,1,\epsilon_n}|>\lambda)\le 2n\exp\left(-\frac{\lambda^2}{2(w_n+\epsilon_n\lambda/3)}\right), \tag{4.16}$$

for all $\lambda>0$ and $n\in\mathbb{N}$. We shall compute at the end

$$w_n\le\begin{cases}C\dfrac{m_n}{a_n^\alpha}\epsilon_n^{2-\alpha}, & \text{if } \alpha\in(0,2),\\[2ex] C\dfrac{m_n}{a_n^2}(1+\log(a_n\epsilon_n)_+), & \text{if } \alpha=2,\\[2ex] C\dfrac{m_n}{a_n^2}, & \text{if } \alpha>2.\end{cases} \tag{4.17}$$

Then, by (4.16) and our choice of ϵ_n, it suffices to check that $w_n\to 0$ at a polynomial rate. This is true for $\alpha\in(0,2)$, and for $\alpha\ge 2$ an additional assumption on m_n is needed. Indeed, with $\alpha=2$ the $\log(a_n\epsilon_n)$ might be problematic if m_n grows at an exponential rate, while any polynomial growth of m_n would cause no problem; and with $\alpha>2$,

$$\frac{m_n}{a_n^2}=C\frac{a_n^{\alpha-2}}{n^\beta L(n)}=C\frac{m_n^{1-2/\alpha}}{(n^\beta L(n))^{2/\alpha}}\to 0$$

at a polynomial rate is guaranteed by $m_n\le Cn^\kappa$ for any $\kappa<2\beta/(\alpha-2)$. Therefore, the desired (4.12) follows from (4.16) and (4.17).

It remains to prove (4.17). We have

$$w_n = m_n \mathbb{E}\left(\left(\frac{\mathcal{X}}{a_n q^{1/\alpha'}}\right)^2 \eta^{(q)} \mathbf{1}_{\{|\mathcal{X}|<a_n\epsilon_n q^{1/\alpha'}\}}\right)$$
$$= \frac{m_n}{a_n^2}\int_0^1 q^{1-\rho-2/\alpha'} L(1/q)\mathbb{E}\left(\mathcal{X}^2 \mathbf{1}_{\{|\mathcal{X}|<a_n\epsilon_n q^{1/\alpha'}\}}\right) dq. \tag{4.18}$$

Now the discussions shall depend on the values of $\alpha > 0$ in three cases.

(i) Assume $\alpha < 2$. Introduce a parameter $d_n = (a_n\epsilon_n)^{-\alpha'} \downarrow 0$ (we no longer need the same constraints on d_n as in (4.14) before as we only need an upper bound now). Again decompose the integral in (4.18) into two parts on $\int_0^{d_n}$ and $\int_{d_n}^1$, respectively. Applying Karamata's theorem on the expectation, the second part (with the factor m_n/a_n^2 in front) can be bounded by,

$$C\frac{m_n}{a_n^2}\int_{d_n}^1 q^{1-\rho-2/\alpha'}L(1/q)(a_n\epsilon_n q^{1/\alpha'})^{2-\alpha}dq = C\frac{m_n\epsilon_n^{2-\alpha}}{a_n^\alpha}\int_{d_n}^1 q^{1-\rho-\gamma}L(1/q)dq.$$

The first part can be bounded by

$$m_n\epsilon_n^2\int_0^{d_n} q^{1-\rho}L(1/q)dq \le C m_n\epsilon_n^2 d_n^{2-\rho}L(1/d_n)$$
$$= C\frac{m_n\epsilon_n^{2-\alpha}}{a_n^\alpha}(a_n\epsilon_n)^{\alpha-\alpha'(2-\rho)}L(1/d_n) = o\left(\frac{m_n\epsilon_n^{2-\alpha}}{a_n^\alpha}\right). \tag{4.19}$$

(Note that $\alpha-\alpha'(2-\rho) = \alpha'(\gamma+\rho-2) < 0$.)

(ii) If $\alpha > 2$, then

$$w_n \le \mathbb{E}\mathcal{X}^2\frac{m_n}{a_n^2}\int_0^1 q^{1-\rho-2/\alpha'}L(1/q)dq \le C\frac{m_n}{a_n^2}.$$

(iii) If $\alpha = 2$, under the assumption $\mathbb{P}(|\mathcal{X}| > x) \sim C_{\mathcal{X}}x^{-2}$, there exists a constant C such that

$$\mathbb{E}\left(|\mathcal{X}|^2\mathbf{1}_{\{|\mathcal{X}|<x\}}\right) \le 1 + C(\log x)_+,\ \text{for all } x > 0. \tag{4.20}$$

Then, (4.18) with the integrals restricted to $[0, d_n]$ (we use the same bound as in (4.19)) and $[d_n, 1]$ (we use the bound (4.20) above) are bounded from above by respectively

$$C\frac{m_n}{a_n^2}(a_n\epsilon_n)^{2-\alpha'(2-\rho)}L(1/d_n) \quad \text{and} \quad C\frac{m_n}{a_n^2}(1+(\log(a_n\epsilon_n))_+).$$

Again the part over $[d_n, 1]$ is dominant. We have thus proved (4.17). □

Proof of Theorem 1.2: Consider a Lipschitz continuous and bounded non-negative function $f(x, y)$ such that $f(x, y) = 0$ for all $x \in [-\kappa, \kappa]$, with Lipschitz constant C_f. Let $r = r_{\beta,\beta_0} = \lfloor 1/(\beta-\beta_0)\rfloor + 1$ as in Lemma 4.4, and $\epsilon \in (0, \kappa/(r+1))$. Introduce

$$\mathcal{E}_{n,1,\epsilon} := \left\{\left\{\widehat{C}_{n,\epsilon}(i)\right\}_{i=1,\dots,m_n} \text{ are all disjoint }\right\},$$
$$\mathcal{E}_{n,2,\epsilon} := \left\{\max_{j=1,\dots,n}\left|\widetilde{X}_{n,j} - \widetilde{X}_{n,j,\epsilon}\right| \le (r+1)\epsilon\right\},$$
$$\mathcal{E}_{n,3,\epsilon,K} := \left\{|\widehat{C}_{n,\epsilon}| \le K\right\},$$
$$\mathcal{E}_{n,4,\epsilon,\lambda} := \left\{\max_{j\in\widehat{C}_{n,\epsilon}}\left|\widetilde{X}_{n,j} - \widetilde{X}_{n,j,\epsilon}\right| \le \lambda\right\},$$

and $\mathcal{E}_{n,\epsilon,K,\lambda} := \mathcal{E}_{n,1,\epsilon} \cap \mathcal{E}_{n,2,\epsilon} \cap \mathcal{E}_{n,3,\epsilon,K} \cap \mathcal{E}_{n,4,\epsilon,\lambda}$. Recall $\xi_{n,\epsilon}$ and $\widetilde{\xi}_{n,\epsilon}$ in (4.8). The key relation in the approximation is for all $K > 0$,

$$e^{-\widetilde{\xi}_{n,\epsilon}(f)} e^{-\lambda K C_f} \le e^{-\xi_n(f)} \le e^{-\widetilde{\xi}_{n,\epsilon}(f)} e^{\lambda K C_f}, \text{ restricted to } \mathcal{E}_{n,\epsilon,K,\lambda}. \tag{4.21}$$

We prove the upper-bound part only as the lower-bound part is similar. Restricted to $\mathcal{E}_{n,\epsilon,K,\lambda}$, we have

$$\begin{aligned} e^{-\xi_n(f)} &= \exp\left(-\sum_{j\in\widehat{C}_{n,\epsilon}} f\left(\widetilde{X}_{n,j}, j/n\right)\right) \le \exp\left(-\sum_{j\in\widehat{C}_{n,\epsilon}} f\left(\widetilde{X}_{n,j,\epsilon}, j/n\right)\right) e^{\lambda K C_f} \\ &= e^{-\xi_{n,\epsilon}(f)} e^{\lambda K C_f} = e^{-\widetilde{\xi}_{n,\epsilon}(f)} e^{\lambda K C_f}, \end{aligned}$$

where we used the restrictions to $\mathcal{E}_{n,2,\epsilon}$ in the first equality (since for $j \notin \widehat{C}_{n,\epsilon}$, $\widetilde{X}_{n,j} = \widetilde{X}_{n,j} - \widetilde{X}_{n,j,\epsilon}$, which is small when restricted to $\mathcal{E}_{n,2,\epsilon}$), to $\mathcal{E}_{n,3,\epsilon,K} \cap \mathcal{E}_{n,4,\epsilon,\lambda}$ in the first inequality by Lipschitz continuity, and to $\mathcal{E}_{n,1,\epsilon}$ in the third equality, respectively. (In the third equality, we used the observation that restricted to the event $\mathcal{E}_{n,1,\epsilon}$, $\xi_{n,\epsilon} = \widetilde{\xi}_{n,\epsilon}$. Indeed, on the event $\mathcal{E}_{n,1,\epsilon}$ if $\widetilde{X}_{n,j,\epsilon} \neq 0$ for some j, then necessarily $\widetilde{X}_{n,j,\epsilon} = \mathcal{X}_i \eta_{i,j}^{(q_i)}/(a_n q_i^{1/\alpha'})$ for a unique index $i \in \{1, \ldots, m_n\}$ and for all other i', $|\mathcal{X}_{i'}|\eta_{i',j}^{(q_{i'})} \le a_n q_{i'}^{1/\alpha'}\epsilon$.)

Recall our choice of ϵ_n in (4.9). Then, the upper bound in (4.21) becomes

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}e^{-\xi_n(f)} &\le \limsup_{n\to\infty} \mathbb{E}\left(e^{-\widetilde{\xi}_{n,\epsilon}(f)+\lambda K C_f} \mathbf{1}_{\mathcal{E}_{n,\epsilon,K,\lambda}}\right) + \limsup_{n\to\infty} \mathbb{E}\left(e^{-\xi_n(f)} \mathbf{1}_{\mathcal{E}^c_{n,\epsilon,K,\lambda}}\right) \\ &\le \limsup_{n\to\infty} \mathbb{E}\left(e^{-\widetilde{\xi}_{n,\epsilon}(f)+\lambda K C_f}\right) + \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right) \\ &= \mathbb{E}e^{-\xi(f)} \cdot e^{\lambda K C_f} + \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right). \end{aligned}$$

In the last step we used first $\widetilde{\xi}_{n,\epsilon}(f) = \widetilde{\xi}_n(f)$ thanks to the assumption that $f(x,y) = 0$ for $x \in [-\kappa, \kappa]$, and then $\lim_{n\to\infty} \mathbb{E}e^{-\widetilde{\xi}_n(f)} = \mathbb{E}e^{-\xi(f)}$ by Proposition 4.1. A similar argument yields the lower bound

$$\begin{aligned} \liminf_{n\to\infty} \mathbb{E}e^{-\xi_n(f)} &\ge \liminf_{n\to\infty} \mathbb{E}\left(e^{-\widetilde{\xi}_{n,\epsilon}(f)-\lambda K C_f} \mathbf{1}_{\mathcal{E}_{n,\epsilon,K,\lambda}}\right) \\ &\ge \liminf_{n\to\infty} \mathbb{E}\left(e^{-\widetilde{\xi}_{n,\epsilon}(f)-\lambda K C_f}\right) - \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right) \\ &= \mathbb{E}e^{-\xi(f)} \cdot e^{-\lambda K C_f} - \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right). \end{aligned}$$

Combining these two bounds gives

$$\begin{aligned} \mathbb{E}e^{-\xi(f)} e^{-\lambda K C_f} - \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right) &\le \liminf_{n\to\infty} \mathbb{E}e^{-\xi_n(f)} \\ &\le \limsup_{n\to\infty} \mathbb{E}e^{-\xi_n(f)} \le \mathbb{E}e^{-\xi(f)} e^{\lambda K C_f} + \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right). \end{aligned}$$

Now, the desired convergence $\lim_{n\to\infty} \mathbb{E}e^{-\xi_n(f)} = \mathbb{E}e^{-\xi(f)}$ follows by first taking $\lambda \downarrow 0$ and then $K \to \infty$, combined with the following facts:

$$\limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,\epsilon,K,\lambda}\right) \le \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,3,\epsilon,K}\right), \text{ for all } K, \lambda > 0,$$

and

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{E}^c_{n,3,\epsilon,K}\right) = 0.$$

To see the above we examine each of the four events separately.

(i) $\lim_{n\to\infty}\mathbb{P}(\mathcal{E}^c_{n,1,\epsilon})=0$. Asymptotically, there are N_ϵ (a Poisson random variable with mean $\epsilon^{-\alpha}$) number of $\widehat C_{n,\epsilon}(i)$ that are non-empty. Therefore, it suffices to show that

$$\lim_{n\to\infty}\mathbb{P}\left(\widehat C_{n,\epsilon}(1)\cap\widehat C_{n,\epsilon}(2)\neq\emptyset\;\middle|\;\widehat C_{n,\epsilon}(i)\neq\emptyset, i=1,2\right)=0. \tag{4.22}$$

Again, we can restrict to the event $|\widehat C_{n,\epsilon}(i)|\le K_0, i=1,2$ for $K_0\in\mathbb{N}$, and it is clear that

$$\lim_{n\to\infty}\mathbb{P}\left(\widehat C_{n,\epsilon}(1)\cap\widehat C_{n,\epsilon}(2)\neq\emptyset, |\widehat C_{n,\epsilon}(i)|\le K_0, i=1,2\right)=0,$$

and by (4.15)

$$\lim_{K_0\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|\widehat C_{n,\epsilon}(i)|>K_0,\ \text{for } i=1 \text{ or } 2\right)=0.$$

The desired (4.22) then follows.

(ii) $\lim_{n\to\infty}\mathbb{P}(\mathcal{E}^c_{n,2,\epsilon})=0$. This follows from (4.10) and (4.12), and the identity that $\widetilde X_{n,j}-\widetilde X_{n,j,\epsilon}=Z_{n,j,\epsilon,\epsilon_n}+W_{n,j,\epsilon_n}$.

(iii) $\lim_{K\to\infty}\lim_{n\to\infty}\mathbb{P}(\mathcal{E}^c_{n,3,\epsilon,K})=0$. We already proved this in (4.15).

(iv) $\lim_{n\to\infty}\mathbb{P}(\mathcal{E}^c_{n,4,\epsilon,\lambda})=0$. To see this, use the relation

$$\mathbb{P}\left(\mathcal{E}^c_{n,4,\epsilon,\lambda}\right)\le\mathbb{P}\left(\max_{j\in\widehat C_{n,\epsilon}}|Z_{n,j,\epsilon,\epsilon_n}|>0\right)+\mathbb{P}\left(\max_{j=1,\dots,n}|W_{n,j,\epsilon_n}|>\lambda\right).$$

Then recall (4.11) and (4.12).

We have completed the proof. □

4.3. *A second proof of Theorem 1.1.* In the case of i.i.d. random variables with regularly-varying tails of tail index $\alpha\in(0,2)$, it is a classical result that once the point-process convergence is established, the functional central limit theorem holds (Resnick, 1986, proof of Proposition 3.4). Here we can also obtain another proof of Theorem 1.1 following Proposition 4.2. However, as mentioned in Remark 1.3, the tightness is hard for Karlin stable processes. We only manage to prove the convergence of finite-dimensional distributions.

The proof consists of an approximation argument. Let $T_{2,\epsilon}$ be as in Resnick (1986, proof of Proposition 3.4). This is a mapping from $\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\}\times[0,1])$ to $D([0,1])$, with, for any $\zeta=\sum_i\delta_{(y_i,u_i)}\in\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\}\times[0,1])$,

$$[T_{2,\epsilon}\zeta](t):=\sum_i y_i\mathbf{1}_{\{u_i\le t,|y_i|>\epsilon\}}, t\in[0,1].$$

Thus, by continuous mapping theorem applied to Proposition 4.1, $T_{2,\epsilon}\widehat\xi_n\Rightarrow T_{2,\epsilon}\widehat\xi$ ($T_{2,\epsilon}$ is almost surely continuous with respect to law induced by $\widehat\xi$), which is the same as (compare with (1.3))

$$\left\{\frac{1}{a_n}\sum_{i=1}^{m_n}\frac{\mathcal{X}_i}{q_i^{1/\alpha'}}\mathbf{1}_{\left\{\tau^{(q_i)}_{i,\lfloor nt\rfloor}\ \text{odd}\right\}}\mathbf{1}_{\left\{|\mathcal{X}_i|>a_nq_i^{1/\alpha'}\epsilon\right\}}\right\}_{t\in[0,1]}\Rightarrow\left\{\sum_{\ell=1}^\infty\frac{\varepsilon_\ell}{\Gamma_\ell^{1/\alpha}}\mathbf{1}_{\left\{\sum_{j=1}^{Q_{\beta,\ell}}\mathbf{1}_{\{U_{\ell,j}\le t\}}\ \text{odd}\right\}}\mathbf{1}_{\left\{\Gamma_\ell^{-1/\alpha}>\epsilon\right\}}\right\}_{t\in[0,1]}$$

in $D([0,1])$. The above implies the convergence of finite-dimensional distribution of the truncated process, and it remains to show that for every $t\in[0,1]$,

$$\lim_{\epsilon\downarrow0}\limsup_{n\to\infty}\mathbb{P}\left(\left|\frac{1}{a_n}\sum_{i=1}^{m_n}\frac{\mathcal{X}_i}{q_i^{1/\alpha'}}\mathbf{1}_{\left\{|\mathcal{X}_i|\le a_nq_i^{1/\alpha'}\epsilon\right\}}\mathbf{1}_{\left\{\tau^{(q_i)}_{i,\lfloor nt\rfloor}\ \text{odd}\right\}}\right|>\lambda\right)=0,\ \text{for all } \lambda>0.$$

(See Dehling et al. (2009, Theorem 2).) It suffices to prove for a fixed t, and without loss of generality we take $t = 1$. In this case the above follows from Chebychev inequality and, for all $\epsilon > 0$,

$$\limsup_{n\to\infty} v_{n,\epsilon} \le C\epsilon^{2-\alpha} \quad \text{with} \quad v_{n,\epsilon} := m_n \mathbb{E}\left(\left(\frac{\mathcal{X}}{a_n q^{1/\alpha'}}\right)^2 \mathbf{1}_{\{|\mathcal{X}|\le a_n q^{1/\alpha'}\epsilon\}} \mathbf{1}_{\left\{\tau_n^{(q)} \text{ odd}\right\}}\right). \tag{4.23}$$

We first compute $v_{n,\epsilon}$, with the expectation restricted to $q \in [1/n, 1]$. An upper bound is then (bounding the second indicator function by 1), for n large enough,

$$\frac{m_n}{a_n^2}\int_{1/n}^1 q^{-\rho-2/\alpha'} L(1/q)\mathbb{E}_q\left(\mathcal{X}^2 \mathbf{1}_{\{|\mathcal{X}|\le a_n q^{1/\alpha'}\epsilon\}}\right) dq \le \frac{Cm_n\epsilon^{2-\alpha}}{a_n^\alpha}\int_{1/n}^1 q^{-\rho-\gamma}L(1/q)dq \le C\epsilon^{2-\alpha}.$$

(More precisely, $\epsilon > 0$ is fixed, C can be taken independent of ϵ, while the above holds only for all $n > n_{C,\epsilon}$ for some $n_{C,\epsilon}$.) For $v_{n,\epsilon}$ with the expectation restricted to $q \in [0, 1/n]$, note that then $\mathbb{P}_q\left(\tau_n^{(q)} \text{ odd}\right) = (1-(1-2q)^n)/2$ and

$$\sup_{q\in[0,1/n]} \frac{(1-(1-2q)^n)}{qn} = 2.$$

Therefore,

$$\begin{aligned}
&\frac{m_n}{a_n^2}\int_0^{1/n} \mathbb{E}_q\left(\mathcal{X}^2\mathbf{1}_{\{|\mathcal{X}|\le a_n q^{1/\alpha'}\epsilon\}}\right)\mathbb{P}_q\left(\tau_{\lfloor nt\rfloor}^{(q)} \text{ odd}\right) q^{-\rho-2/\alpha'}L(1/q)dq \\
&\le \frac{m_n}{a_n^2}\int_0^{1/n} \mathbb{E}_q\left(\mathcal{X}^2\mathbf{1}_{\{|\mathcal{X}|\le a_n q^{1/\alpha'}\epsilon\}}\right) nq^{1-\rho-2/\alpha'}L(1/q)dq \\
&\le \frac{Cm_n n}{a_n^2}\int_0^{1/n}\left(a_n q^{1/\alpha'}\epsilon\right)^{2-\alpha} q^{1-\rho-2/\alpha'}L(1/q)dq = \frac{Cm_n n}{a_n^\alpha}\epsilon^{2-\alpha}\int_n^\infty q^{\beta-2}L(q)dq \\
&\le \frac{Cm_n n}{a_n^\alpha} n^{\beta-1}L(n)\epsilon^{2-\alpha} = C\epsilon^{2-\alpha}.
\end{aligned}$$

We have thus proved (4.23).

Remark 4.6. If we want to enhance the result to a functional central limit theorem in $D([0,1])$, a sufficient condition would be

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]}\left|\frac{1}{a_n}\sum_{i=1}^{m_n}\frac{\mathcal{X}_i}{q_i^{1/\alpha'}}\mathbf{1}_{\{|\mathcal{X}_i|\le a_n q_i^{1/\alpha'}\epsilon\}}\mathbf{1}_{\left\{\tau_{i,\lfloor nt\rfloor}^{(q_i)} \text{ odd}\right\}}\right| > \lambda\right) = 0, \text{ for all } \lambda > 0.$$

Whether the above is true remains an open question. This is closely related to the tightness issues in Remark 1.3.

4.4. *A limit theorem for Karlin random sup-measures.* Now we explain how Theorem 1.2 entails the convergence of random sup-measures. Random sup-measures provide a natural framework to characterize scaling limits of extremes, although they are not commonly used yet in the literature. For background on random sup-measures, see O'Brien et al. (1990); Vervaat (1997); Molchanov (2017). For the sake of simplicity, we shall treat random sup-measures as α-Fréchet max-stable set-indexed process $\{\mathcal{M}_{\alpha,\beta}(I)\}_{I\in\mathcal{I}}$, with $\mathcal{I}$ the collection of all open sets of $[0,1]$, denoted by

$$\mathcal{M}_{\alpha,\beta}(I) := \sup_{\ell\in\mathbb{N}}\frac{1}{\Gamma_\ell^{1/\alpha}}\mathbf{1}_{\left\{\left(\bigcup_{j=1}^{Q_{\beta,\ell}}\{U_{\ell,j}\}\right)\cap I\neq\emptyset\right\}}, I \subset \mathcal{I},$$

and prove the convergence of finite-dimensional distributions of the set-indexed processes (for max-stable processes, see de Haan (1984); Kabluchko (2009); Stoev (2010)). For $\mathcal{M}_{\alpha,\beta}$, it has the following multivariate α-Fréchet finite-dimensional distributions (although we do not need to work with the explicit formula):

$$\mathbb{P}\left(\mathcal{M}_{\alpha,\beta}(I_1) \le x_1, \dots, \mathcal{M}_{\alpha,\beta}(I_d) \le x_d\right) = \exp\left(-\mathbb{E}\left(\max_{k=1,\dots,d} \frac{\mathbf{1}_{\left\{\mathcal{C}_{Q_\beta} \cap I_k \neq \emptyset\right\}}}{x_k^\alpha}\right)\right),$$

for all $I_1, \dots, I_d \in \mathcal{I}, x_1, \dots, x_d > 0$.

The following result on the convergence of max-stable processes can be strengthened immediately to convergence of random sup-measures (which is defined for *all* subsets of $[0,1]$). We just mention that Karlin random sup-measures are translation-invariant and β/α-self-similar, and they are a special case of the recently introduced Choquet random sup-measures (Molchanov and Strokorb, 2016). We refer to Durieu and Wang (2018) for more results on the Karlin random sup-measures.

Introduce

$$M_n(I) := \max_{j/n \in I} \frac{1}{a_n} \left| \sum_{i=1}^{m_n} \frac{\mathcal{X}_i}{q_i^{1/\alpha'}} \eta_{i,j}^{(q_j)} \right|, \quad I \subset \mathcal{I}, n \in \mathbb{N},$$

Corollary 4.7. *Under the assumption of Theorem 1.2,*

$$\{M_n(I)\}_{I \in \mathcal{I}} \xrightarrow{f.d.d.} \{\mathcal{M}_{\alpha,\beta}(I)\}_{I \in \mathcal{I}}.$$

Proof: By definition, it suffices to show

$$\lim_{n\to\infty} \mathbb{P}\left(M_n(I_1) \le x_1, \dots, M_n(I_d) \le x_d\right) = \mathbb{P}\left(\mathcal{M}_{\alpha,\beta}(I_1) \le x_1, \dots, \mathcal{M}_{\alpha,\beta}(I_d) \le x_d\right), \tag{4.24}$$

for all $d \in \mathbb{N}, x_i > 0, I_i \in \mathcal{I}, i = 1, \dots, d$. Now, Theorem 1.2 implies that, ignoring the signs of the values and working with point processes in $\mathfrak{M}_p((0,\infty] \times [0,1])$,

$$\xi_n^* := \sum_{j=1}^n \delta_{\left(\left|\sum_{i=1}^{m_n} \mathcal{X}_i \eta_{i,j}^{(q_i)}/(a_n q_i^{1/\alpha'})\right|, j/n\right)} \Rightarrow \xi^* := \sum_{\ell=1}^\infty \sum_{j=1}^{Q_{\beta,j}} \delta_{\left(\Gamma_\ell^{-1/\alpha}, U_\ell\right)}.$$

The above then implies in particular, with $B := \bigcup_{k=1}^d \left((x_k, \infty] \times I_k\right)$,

$$\lim_{n\to\infty} \mathbb{P}\left(\xi_n^*(B) = 0\right) = \mathbb{P}(\xi^*(B) = 0).$$

The above is exactly the desired convergence in (4.24). This completes the proof. $\square$

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