ALEA, Lat. Am. J. Probab. Math. Stat. **20**, 1247–1270 (2023)
DOI: 10.30757/ALEA.v20-46

Joint functional convergence of partial sums and maxima for moving averages with weakly dependent heavy-tailed innovations and random coefficients

Danijel Krizmanić

Danijel Krizmanić, Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: dkrizmanic@math.uniri.hr
URL: https://www.math.uniri.hr/∼dkrizmanic

Abstract. For moving average processes with random coefficients and heavy-tailed innovations that are weakly dependent in the sense of strong mixing and local dependence condition D' we study joint functional convergence of partial sums and maxima. Under the assumption that all partial sums of the series of coefficients are a.s. bounded between zero and the sum of the series we derive a functional limit theorem in the space of $\mathbb{R}^2$–valued càdlàg functions on $[0,1]$ with the Skorokhod weak M_2 topology.

1. Introduction

It is known that the joint partial sums and maxima processes constructed from i.i.d. regularly varying random variables with the tail index $\alpha \in (0,2)$ converge weakly in the space $D([0,1],\mathbb{R}^2)$ of $\mathbb{R}^2$–valued càdlàg functions on $[0,1]$ with the Skorokhod J_1 topology, with the limit consisting of a stable Lévy process and an extremal process, see Chow and Teugels (1979) and Resnick (1986).

The joint functional convergence holds also in the weakly dependent case. Anderson and Turkman (1991, 1995) studied weak convergence of the joint partial sums and maxima processes in the case when the underlying random variables are strongly mixing, and in the heavy-tailed case, under Leadbetter's D and D' dependence conditions familiar from extreme value theory. Conditions D and D' are quite restrictive, since they exclude m-dependent sequences.

Recently, Krizmanić (2020) showed that for a regularly varying sequence of dependent random variables with index $\alpha \in (0,2)$, for which clusters of high-threshold excesses can be broken down into asymptotically independent blocks, the joint stochastic processes of partial sums and maxima converge in the space $D([0,1],\mathbb{R}^2)$ endowed with the Skorokhod weak M_1 topology under the condition that all extremes within each cluster of big values have the same sign. This topology is weaker than the more commonly used Skorokhod J_1 topology, the latter being appropriate when

Received by the editors October 24th, 2022; accepted June 9th, 2023.

2010 *Mathematics Subject Classification.* 60F17, 60G52, 60G70.

Key words and phrases. Functional limit theorem, regular variation, M_2 topology, moving average process, extremal process, stable Lévy process.

This work has been supported in part by University of Rijeka research grants uniri-prirod-18-9 and uniri-pr-prirod-19-16 and by Croatian Science Foundation under the project IP-2019-04-1239.

there is no clustering of extremes. This result extends the functional limit theorem obtained by Krizmanić (2018b) in the special case of linear processes with i.i.d. regularly varying innovations and all (deterministic) coefficients of the same sign. Even when these coefficients are not of the same sign, but all partial sums of the series of coefficients are bounded between zero and the sum of the series, joint functional convergence still holds, but with respect to the weaker Skorokhod M_2 topology, see Krizmanić (2018a).

In this paper we study joint functional convergence of partial sums and maxima for linear or moving averages processes with weakly dependent innovations and random coefficients. In proving the joint functional convergence for these processes we will rely on already established marginal functional convergence for partial sums when the innovations are weakly dependent in the sense of strong mixing and local dependence condition D' (see Krizmanić, 2022a) and for partial maxima of linear processes with i.i.d. innovations (see Krizmanić, 2022b).

We proceed by stating the problem precisely. Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of regularly varying random variables with index of regular variation $\alpha \in (0,2)$. This means that

$$\mathrm{P}(|Z_i| > x) = x^{-\alpha}L(x), \qquad x > 0, \tag{1.1}$$

where L is a slowly varying function at ∞. Let (a_n) be a sequence of positive real numbers such that

$$n\,\mathrm{P}(|Z_1| > a_n) \to 1, \tag{1.2}$$

as $n \to \infty$. Then regular variation of Z_i can be expressed in terms of vague convergence of measures on $\mathbb{E} = \overline{\mathbb{R}} \setminus \{0\}$:

$$n\,\mathrm{P}(a_n^{-1}Z_i \in \cdot\,) \xrightarrow{v} \mu(\,\cdot\,) \qquad \text{as } n \to \infty, \tag{1.3}$$

where μ is a measure on $\mathbb{E}$ given by

$$\mu(\mathrm{d}x) = \big(p\,1_{(0,\infty)}(x) + r\,1_{(-\infty,0)}(x)\big)\,\alpha|x|^{-\alpha-1}\,\mathrm{d}x, \tag{1.4}$$

with

$$p = \lim_{x\to\infty}\frac{\mathrm{P}(Z_i > x)}{\mathrm{P}(|Z_i| > x)} \qquad \text{and} \qquad r = \lim_{x\to\infty}\frac{\mathrm{P}(Z_i < -x)}{\mathrm{P}(|Z_i| > x)}. \tag{1.5}$$

We study the moving average process with random coefficients, defined by

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \qquad i \in \mathbb{Z}, \tag{1.6}$$

where $(C_i)_{i\geq 0}$ is a sequence of random variables independent of (Z_i) such that the above series is a.s. convergent. One well-known sufficient condition for that is

$$\sum_{j=0}^{\infty} \mathrm{E}|C_j|^{\delta} < \infty \qquad \text{for some } \delta < \alpha,\ 0 < \delta \leq 1. \tag{1.7}$$

The moment condition (1.7), stationarity of the sequence (Z_i) and $\mathrm{E}|Z_1|^{\beta} < \infty$ for every $\beta \in (0,\alpha)$ (which follows from the regular variation property and Karamata's theorem) imply the a.s. convergence of the series in (1.6), since

$$\mathrm{E}|X_i|^{\delta} \leq \sum_{j=0}^{\infty} \mathrm{E}|C_j|^{\delta}\mathrm{E}|Z_{i-j}|^{\delta} = \mathrm{E}|Z_1|^{\delta}\sum_{j=0}^{\infty}\mathrm{E}|C_j|^{\delta} < \infty.$$

Condition (1.7) implies also the a.s. convergence of the series $\sum_{j=0}^{\infty} C_j$. Another condition that assures the a.s. convergence of the series in the definition of moving average processes with

$$\begin{aligned} \mathrm{E}(Z_1) = 0, &\qquad \text{if } \alpha \in (1,2),\\ Z_1 \text{ is symmetric}, &\qquad \text{if } \alpha = 1, \end{aligned}$$

and a.s. bounded coefficients can be deduced from the results in Astrauskas (1983):

$$\sum_{j=0}^{\infty} c_j^{\alpha} L(c_j^{-1}) < \infty,$$

where (c_j) is a sequence of positive real numbers such that $|C_j| \le c_j$ a.s. for all j (c.f. Balan et al., 2016).

If the innovations Z_i's are independent, Krizmanić (2019) derived a (marginal) functional limit theorem for the partial sum stochastic process

$$V_n(t) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i, \qquad t \in [0,1], \tag{1.8}$$

in the space $D([0,1],\mathbb{R})$ of real–valued right continuous functions on $[0,1]$ with left limits, endowed with the Skorohod M_2 topology, under some usual regularity conditions and the assumption that all partial sums of the series $C = \sum_{i=0}^{\infty} C_i$ are a.s. bounded between zero and the sum of the series, i.e.

$$0 \le \sum_{i=0}^{s} C_i \Big/ \sum_{i=0}^{\infty} C_i \le 1 \ \text{ a.s.} \qquad \text{for every } s = 0,1,2\ldots. \tag{1.9}$$

More precisely,

$$V_n(\,\cdot\,) \xrightarrow{d} \widetilde{C} V(\,\cdot\,) \qquad \text{as } n \to \infty, \tag{1.10}$$

in $D([0,1],\mathbb{R})$ endowed with the M_2 topology, where V is an α–stable Lévy process with characteristic triple $(0,\mu,b)$, with μ as in (1.4),

$$b = \begin{cases} 0, & \alpha = 1, \\ (p-r)\frac{\alpha}{1-\alpha}, & \alpha \in (0,1) \cup (1,2), \end{cases}$$

and $\widetilde{C}$ is a random variable, independent of V, such that $\widetilde{C} \stackrel{d}{=} C$. When the sequence of coefficients (C_j) is deterministic, relation (1.10) reduces to

$$V_n(\,\cdot\,) \xrightarrow{d} C V(\,\cdot\,) \qquad \text{as } n \to \infty$$

(see Proposition 3.2 in Krizmanić, 2019). Simplifying notation, we sometimes omit brackets and write $V_n \xrightarrow{d} CV$. This functional convergence, as shown by Avram and Taqqu (1992), can not be strengthened to the Skorokhod J_1 convergence on $D([0,1],\mathbb{R})$, and it also fails in the M_1 topology even for finite order moving averages with coefficients of both signs (but if all coefficients are nonnegative then the M_1 convergence actually holds).

Further, (marginal) functional convergence of partial maxima processes in the i.i.d. case was obtained in Krizmanić (2022b), i.e. under some standard moment conditions on the sequence of coefficients (C_i) it holds that

$$M_n(\,\cdot\,) \xrightarrow{d} M(\,\cdot\,) \qquad \text{as } n \to \infty, \tag{1.11}$$

in $D([0,1],\mathbb{R})$ endowed with the Skorokhod M_1 topology, where

$$M_n(t) = \begin{cases} \dfrac{1}{a_n} \bigvee_{i=1}^{\lfloor nt \rfloor} X_i, & t \in \Big[\dfrac{1}{n}, 1\Big], \\ \dfrac{X_1}{a_n}, & t \in \Big[0, \dfrac{1}{n}\Big\rangle, \end{cases} \tag{1.12}$$

is the corresponding partial maximum process and $M = C^{(1)} W^{(1)} \vee C^{(2)} W^{(2)}$, where $W^{(1)}$ is an extremal process with exponent measure $\mu_+(\mathrm{d}x) = p\alpha x^{-\alpha-1}\mathrm{d}x$ for $x > 0$, $W^{(2)}$ is an extremal process with exponent measure $\mu_-(\mathrm{d}x) = r\alpha x^{-\alpha-1}\mathrm{d}x$ for $x > 0$, and $(C^{(1)}, C^{(2)})$ is a two dimensional random vector, independent of $(W^{(1)}, W^{(2)})$, such that $(C^{(1)}, C^{(2)}) \stackrel{d}{=} (C_+, C_-)$, with

$$C_+ = \max\{C_j \vee 0 : j \geq 0\} \qquad \text{and} \qquad C_- = \max\{-C_j \vee 0 : j \geq 0\}. \tag{1.13}$$

Recently, in Krizmanić (2022a) functional convergence in (1.10) was extended to the case when the innovations Z_i are weakly dependent, in the sense that (Z_i) is a strongly mixing sequence which satisfies the local dependence condition D' as is given in Davis (1983):

$$\lim_{k\to\infty} \limsup_{n\to\infty} \; n \sum_{i=1}^{\lfloor n/k \rfloor} \mathrm{P}\left(\frac{|Z_0|}{a_n} > x, \frac{|Z_i|}{a_n} > x \right) = 0 \qquad \text{for all } x > 0. \tag{1.14}$$

For instance, a process which is an instantaneous function of a stationary Gaussian process with covariance function r_n behaving like $r_n \log n \to 0$ as $n \to \infty$ satisfies condition (1.14) (see Davis, 1983). Other examples of time series that satisfy (1.14), related to stochastic volatility models and ARMAX processes, can be found in Davis and Mikosch (2009) and Ferreira and Canto e Castro (2008). This condition, together with the strong mixing property, assures that, as in the i.i.d. case, the extremes of the sequence (Z_i) are isolated. Recall here that a sequence (ξ_n) is strongly mixing if $\alpha(n) \to 0$ as $n \to \infty$, where

$$\alpha(n) = \sup\{|\,\mathrm{P}(A \cap B) - \mathrm{P}(A)\,\mathrm{P}(B)| : A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{n}^{\infty}\}$$

and $\mathcal{F}_k^l = \sigma(\{\xi_i : k \leq i \leq l\})$ for $-\infty \leq k \leq l \leq \infty$. For some related results on limit theory for moving averages with random coefficients we refer to Hult and Samorodnitsky (2008) and Kulik (2006).

Our aim in this paper is to join the convergence relations (1.10) and (1.11) into a single one, or more precisely, to find sufficient conditions on moving average processes with weakly dependent innovations and random coefficients such that, with respect to some Skorokhod topology on $D([0,1], \mathbb{R}^2)$

$$L_n(\,\cdot\,) := (V_n(\,\cdot\,), M_n(\,\cdot\,)) \xrightarrow{d} (\widetilde{C}V(\,\cdot\,), M(\,\cdot\,)) \qquad \text{as } n \to \infty. \tag{1.15}$$

Note that if we prove (1.15) then it will follow directly that the functional convergence of partial maxima processes in (1.11) holds also in the case when the innovations are weakly dependent.

The main results of this paper deal with establishing relation (1.15) for finite order moving averages in Theorem 4.1, and then in the general case for infinite order moving average processes in Theorem 5.1. For Z_1 we assume the already mentioned standard regularity conditions:

$$\mathrm{E}Z_1 = 0, \quad \text{if} \quad \alpha \in (1,2), \tag{1.16}$$

$$Z_1 \text{ is symmetric}, \quad \text{if} \quad \alpha = 1. \tag{1.17}$$

In the case $\alpha \in [1,2)$ we will need to assume the following condition to deal with small jumps:

$$\lim_{u\downarrow 0} \limsup_{n\to\infty} \; \mathrm{P}\left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left(\frac{Z_i}{a_n} 1_{\left\{\frac{|Z_i|}{a_n} \leq u\right\}} - \mathrm{E}\left(\frac{Z_i}{a_n} 1_{\left\{\frac{|Z_i|}{a_n} \leq u\right\}} \right) \right) \right| > \epsilon \right] = 0 \tag{1.18}$$

for all $\epsilon > 0$. This condition holds if the sequence (Z_i) is ρ-mixing at a certain rate (see Lemma 4.8 in Tyran-Kamińska, 2010a). In case $\alpha \in (0,1)$ it is a simple consequence of regular variation and Karamata's theorem. Similar conditions are standardly used in the limit theory for partial sums, see Avram and Taqqu (1992); Basrak et al. (2012); Durrett and Resnick (1978); Tyran-Kamińska (2010a). For infinite order moving averages, beside condition (1.7) we will require also some other moment conditions, which will be specified latter in Section 5.

Since the stochastic processes V_n and M_n converge (separately) in the space $D([0,1], \mathbb{R})$ equipped with the M_2 topology, for the convergence in relation (1.15) we will use the weak M_2 topology. Since for partial maxima processes, functional convergence actually holds in the stronger M_1 topology, it is possible to obtain also a joint convergence of L_n in the M_2 topology on the first coordinate and in the M_1 topology on the second coordinate, see Remark 4.3 below. In general, functional M_1 convergence of partial sum processes fails to hold, as for instance, in the case of moving averages with i.i.d. heavy-tailed innovations Z_i and deterministic coefficients $C_0 = 1$, $C_1 = -1$, $C_2 = 1$ and $C_i = 0$ for $i \geq 3$:

$$X_i = Z_i - Z_{i-1} + Z_{i-2}, \qquad i \in \mathbb{Z}.$$

This shows that in general functional convergence of L_n does not hold in the (weak) M_1 topology.

The paper is organized as follows. In Section 2 we recall definitions of weak and strong Skorohod M_1 and M_2 topologies. In Section 3 we introduce some basic notions about joint regular variation and point processes, and obtain some auxiliary results that will be used in Section 4 in establishing the limiting relation (1.15) for finite order moving average processes with weakly dependent innovations and random coefficients. Then in Section 5 we extend this result to infinite order moving average processes.

2. Skorokhod M_1 and M_2 topologies

We start with the definition of the Skorokhod weak M_2 topology in a general space $D([0,1],\mathbb{R}^d)$ of $\mathbb{R}^d$–valued càdlàg functions on $[0,1]$. It is standardly defined using completed graphs and their parametric representations.

For $x \in D([0,1],\mathbb{R}^d)$ the completed (thick) graph of x is the set

$$G_x = \{(t,z) \in [0,1] \times \mathbb{R}^d : z \in [[x(t-), x(t)]]\},$$

where $x(t-)$ is the left limit of x at t and $[[a,b]]$ is the product segment, i.e. $[[a,b]] = [a_1,b_1] \times [a_2,b_2] \ldots \times [a_d,b_d]$ for $a = (a_1,a_2,\ldots,a_d), b = (b_1,b_2,\ldots,b_d) \in \mathbb{R}^d$. We define an order on the graph G_x by saying that $(t_1,z_1) \leq (t_2,z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x_j(t_1-) - z_{1j}| \leq |x_j(t_2-) - z_{2j}|$ for all $j = 1,2,\ldots,d$. The relation $\leq$ induces only a partial order on the graph G_x. A weak M_2 parametric representation of the graph G_x is a continuous function (r,u) mapping $[0,1]$ into G_x, such that r is nondecreasing with $r(0) = 0$, $r(1) = 1$ and $u(1) = x(1)$ (r is the time component and u the spatial component). Denote by $\Pi_w^{M_2}(x)$ the set of weak M_2 parametric representations of the graph G_x. For $x_1, x_2 \in D([0,1],\mathbb{R}^d)$ define

$$d_w(x_1,x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i,u_i) \in \Pi_w^{M_2}(x_i), i = 1,2\},$$

where $\|x\|_{[0,1]} = \sup\{\|x(t)\| : t \in [0,1]\}$ for $x \colon [0,1] \to \mathbb{R}^k$, with $\|\cdot\|$ denoting the max-norm on $\mathbb{R}^k$. Now we define the weak M_2 topology sequentially by saying that a sequence $(x_n)_n$ converges to x in $D([0,1],\mathbb{R}^d)$ in the weak Skorokhod M_2 (or shortly WM_2) topology if $d_w(x_n,x) \to 0$ as $n \to \infty$.

If we replace above the graph G_x with the completed (thin) graph

$$\Gamma_x = \{(t,z) \in [0,1] \times \mathbb{R}^d : z = \lambda x(t-) + (1-\lambda)x(t) \text{ for some } \lambda \in [0,1]\},$$

and a weak M_2 parametric representation with a strong M_2 parametric representation (i.e. a continuous function (r,u) mapping $[0,1]$ onto Γ_x such that r is nondecreasing), then we obtain the standard (or strong) M_2 topology. This topology is stronger than the weak M_2 topology, but they coincide if $d = 1$.

Note that in M_2 parametric representations (r,u) we required that only the time component r is nondecreasing. If we also require that the spatial component u is nondecreasing, then we obtain M_1 parametric representations and (weak and strong) Skorokhod M_1 topologies, which are stronger than the corresponding M_2 topologies.

Often the following characterization of the M_2 topology with the Hausdorff metric on the spaces of graphs is useful. For $x_1, x_2 \in D([0,1],\mathbb{R}^d)$, the M_2 distance between x_1 and x_2 is given by

$$d_{M_2}(x_1,x_2) = \bigg(\sup_{a \in \Gamma_{x_1}} \inf_{b \in \Gamma_{x_2}} d(a,b)\bigg) \vee \bigg(\sup_{a \in \Gamma_{x_2}} \inf_{b \in \Gamma_{x_1}} d(a,b)\bigg),$$

where d is the metric induced by the maximum norm on $\mathbb{R}^{d+1}$. The metric d_{M_2} induces the strong M_2 topology. The weak M_2 topology on $D([0,1],\mathbb{R}^d)$ coincides with the (product) topology induced by the metric

$$d_p^{M_2}(x_1,x_2) = \max_{j=1,\ldots,d} d_{M_2}(x_{1j},x_{2j}) \tag{2.1}$$

for $x_i = (x_{i1},\ldots,x_{id}) \in D([0,1],\mathbb{R}^d)$, $i=1,2$. For detailed discussion of the strong and weak M_2 topologies we refer to Whitt (2002), sections 12.10–12.11. Denote by $D_{\uparrow}([0,1],\mathbb{R}^d)$ the subspace of functions x in $D([0,1],\mathbb{R}^d)$ for which the coordinate functions x_i are non-decreasing for all $i=1,\ldots,d$. For simplicity of notation let $D^d \equiv D([0,1],\mathbb{R}^d)$ and $D_{\uparrow}^d \equiv D_{\uparrow}([0,1],\mathbb{R}^d)$.

Similar to relation (2.1) for the weak M_2 topology, the weak M_1 topology on D^d coincides with the topology induced by the metric

$$d_p^{M_1}(x_1,x_2) = \max_{j=1,\ldots,d} d_{M_1}(x_{1j},x_{2j}) \tag{2.2}$$

for $x_i = (x_{i1},\ldots,x_{id}) \in D^d$, $i=1,2$ (see Whitt, 2002, Theorem 12.5.2). Here d_{M_1} denotes the M_1 metric on D^1, defined by

$$d_{M_1}(y_1,y_2) = \inf\{\|r_1-r_2\|_{[0,1]} \vee \|u_1-u_2\|_{[0,1]} : (r_i,u_i) \in \Pi^{M_1}(y_i), i=1,2\}$$

for $y_1,y_2 \in D^1$, where $\Pi^{M_1}(y)$ is the set of M_1 parametric representations of the completed graph Γ_y, i.e. continuous nondecreasing functions (r,u) mapping $[0,1]$ onto Γ_y.

In the next section we will use the following three lemmas. The first one is about preservation of weak M_1 convergence of stochastic processes under transformations that add certain càdlàg functions to the first components of the underlying processes. This results is a simple consequence of M_1 continuity of addition, the continuous mapping theorem and Slutsky's theorem. It can be proven similarly as Lemma 1 in Krizmanić Krizmanić (2018a) (for the M_2 convergence). The remaining two lemmas deal with M_1 continuity of multiplication and maximum of two càdlàg functions. The first one is based on Theorem 13.3.2 in Whitt (2002), and the second one follows easily from the fact that for monotone functions M_1 convergence is equivalent to point-wise convergence in a dense subset of $[0,1]$ including 0 and 1 (cf. Whitt, 2002, Corollary 12.5.1). Denote by $\mathrm{Disc}(x)$ the set of discontinuity points of $x \in D^1$.

Lemma 2.1. *Let (A_n,B_n,C_n), $n=0,1,2,\ldots$, be stochastic processes in D^3 such that, as $n\to\infty$,*

$$(A_n,B_n,C_n) \xrightarrow{d} (A_0,B_0,C_0) \tag{2.3}$$

in D^3 with the weak M_1 topology. Suppose x_n, $n=0,1,2,\ldots$, are elements of D^1 with x_0 being continuous, such that, as $n\to\infty$,

$$x_n(t) \to x_0(t)$$

uniformly in t. Then

$$(A_n+x_n,B_n,C_n) \xrightarrow{d} (A_0+x_0,B_0,C_0)$$

in D^3 with the weak M_1 topology.

Lemma 2.2. *Suppose that $x_n \to x$ and $y_n \to y$ in D^1 with the M_1 topology. If for each $t \in \mathrm{Disc}(x)\cap \mathrm{Disc}(y)$, $x(t)$, $x(t-)$, $y(t)$ and $y(t-)$ are all nonnegative and $[x(t)-x(t-)][y(t)-y(t-)] \geq 0$, then $x_ny_n \to xy$ in D^1 with the M_1 topology, where $(xy)(t)=x(t)y(t)$ for $t\in[0,1]$.*

Lemma 2.3. *The function $h\colon D_{\uparrow}^2 \to D_{\uparrow}^1$ defined by $h(x,y)=x\vee y$, where*

$$(x\vee y)(t) = x(t)\vee y(t), \qquad t\in[0,1],$$

is continuous when $D_{\uparrow}^2$ is endowed with the weak M_1 topology and $D_{\uparrow}^1$ is endowed with the standard M_1 topology.

3. Joint regular variation, point processes and sum-maximum functional

We say that a strictly stationary sequence of random variables $(\xi_n)_{n\in\mathbb{Z}}$ is (jointly) regularly varying with index $\alpha > 0$ if for any nonnegative integer k the k-dimensional random vector $\xi = (\xi_1, \dots, \xi_k)$ is multivariate regularly varying with index α, i.e. there exists a random vector Θ on the unit sphere $\mathbb{S}^{k-1} = \{x \in \mathbb{R}^k : \|x\| = 1\}$ such that for every $u > 0$, as $x \to \infty$,

$$\frac{\mathrm{P}(\|\xi\| > ux,\, \xi/\|\xi\| \in \cdot\,)}{\mathrm{P}(\|\xi\| > x)} \xrightarrow{w} u^{-\alpha}\,\mathrm{P}(\Theta \in \cdot\,), \tag{3.1}$$

where the arrow "$\xrightarrow{w}$" denotes the weak convergence of finite measures. There is a convenient characterization of joint regular variation due to Basrak and Segers (2009): it is necessary and sufficient that there exists a process $(Y_n)_{n\in\mathbb{Z}}$ with $\mathrm{P}(|Y_0| > y) = y^{-\alpha}$ for $y \geq 1$ such that, as $x \to \infty$,

$$\big((x^{-1}\,\xi_n)_{n\in\mathbb{Z}}\,\big|\,|\xi_0| > x\big) \xrightarrow{\text{fidi}} (Y_n)_{n\in\mathbb{Z}}, \tag{3.2}$$

where "$\xrightarrow{\text{fidi}}$" denotes convergence of finite-dimensional distributions. The process (Y_n) is called the tail process of (ξ_n).

Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of regularly varying random variables with index $\alpha \in (0, 2)$, such that the local dependence condition D' and conditions (1.16) and (1.17) hold. If $\alpha \in [1, 2)$, also suppose that condition (1.18) holds. Condition D' and strong mixing imply that (Z_i) is jointly regularly varying with the tail process (Y_i) being the same as in the i.i.d. case, that is, $Y_i = 0$ for $i \neq 0$, and Y_0 as described above (Basrak et al., 2012, Example 4.1). This in particular means that (Y_i) has no two values of the opposite sign.

Define the time-space point processes

$$N_n = \sum_{i=1}^{n} \delta_{(i/n,\, Z_i/a_n)} \qquad \text{for all } n \in \mathbb{N},$$

with a_n as in (1.2). The point process convergence for the sequence (N_n) on the space $[0, 1] \times \mathbb{E}$ was obtained by Basrak and Tafro (2016) under joint regular variation and the following two weak dependence conditions.

Condition 3.1. There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every nonnegative continuous function f on $[0, 1] \times \mathbb{E}$ with compact support, denoting $k_n = \lfloor n/r_n \rfloor$, as $n \to \infty$,

$$\mathrm{E}\bigg[\exp\bigg\{-\sum_{i=1}^{n} f\Big(\frac{i}{n}, \frac{Z_i}{a_n}\Big)\bigg\}\bigg] - \prod_{k=1}^{k_n} \mathrm{E}\bigg[\exp\bigg\{-\sum_{i=1}^{r_n} f\Big(\frac{kr_n}{n}, \frac{Z_i}{a_n}\Big)\bigg\}\bigg] \to 0. \tag{3.3}$$

Condition 3.2. There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every $u > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\Big(\max_{m\leq|i|\leq r_n} |Z_i| > ua_n \,\Big|\, |Z_0| > ua_n\Big) = 0. \tag{3.4}$$

Condition 3.1 is implied by the strong mixing property (see Krizmanić, 2010, 2016). Condition 3.2 follows from condition D', for the latter implies

$$\lim_{n\to\infty} n \sum_{i=1}^{r_n} \mathrm{P}\left(\frac{|Z_0|}{a_n} > u, \frac{|Z_i|}{a_n} > u\right) = 0 \quad \text{for all } u > 0,$$

for any sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$. Therefore, in our case, by Theorem 3.1 in Basrak and Tafro (2016), as $n \to \infty$,

$$N_n \xrightarrow{d} N = \sum_i \sum_j \delta_{(T_i, P_i\eta_{ij})} \tag{3.5}$$

in $[0,1]\times\mathbb{E}$, where $\sum_{i=1}^{\infty}\delta_{(T_i,P_i)}$ is a Poisson process on $[0,1]\times(0,\infty)$ with intensity measure $Leb\times\nu$ where $\nu(\mathrm{d}x)=\theta\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,\mathrm{d}x$ with θ being the extremal index of the sequence (Z_i), and $(\sum_{j=1}^{\infty}\delta_{\eta_{ij}})_i$ is an i.i.d. sequence of point processes in $\mathbb{E}$ independent of $\sum_i\delta_{(T_i,P_i)}$ and with common distribution equal to the distribution of $\sum_j\delta_{\widetilde{Y}_j/L(\widetilde{Y})}$, where $L(\widetilde{Y})=\sup_{j\in\mathbb{Z}}|\widetilde{Y}_j|$ and $\sum_j\delta_{\widetilde{Y}_j}$ is distributed as $(\sum_{j\in\mathbb{Z}}\delta_{Y_j}\mid\sup_{i\le-1}|Y_i|\le 1)$. Condition D' and strong mixing imply that the extremes of the sequence (Z_i) are isolated, i.e. $\theta=1$ (see Leadbetter and Rootzén, 1988, page 439, and Leadbetter et al., 1983, Theorem 3.4.1). Taking into account the form of the tail process (Y_i) it holds that $N=\sum_i\delta_{(T_i,P_i\eta_{i0})}$ with $|\eta_{i0}|=1$. Further, by (1.5) and (3.2) we obtain

$$\begin{aligned}\mathrm{P}(\eta_{i0}=1) &= \mathrm{P}(Y_0>0)=\mathrm{P}(Y_0>1)=\lim_{x\to\infty}\mathrm{P}\big(x^{-1}Z_0>1\,\big|\,|Z_0|>x\big)\\ &= \lim_{x\to\infty}\frac{\mathrm{P}(Z_0>x)}{\mathrm{P}(|Z_0|>x)}=p,\end{aligned}$$

and similarly $\mathrm{P}(\eta_{i0}=-1)=r$. Hence, denoting $Q_i=\eta_{i0}$, the limiting point process in relation (3.5) reduces to

$$N=\sum_i\delta_{(T_i,P_iQ_i)},\tag{3.6}$$

with $\mathrm{P}(Q_i=1)=p$ and $\mathrm{P}(Q_i=-1)=r$. Since the sequence (Q_i) is independent of the Poisson process $\sum_{i=1}^{\infty}\delta_{(T_i,P_i)}$, an application of Proposition 5.2 and Proposition 5.3 in Resnick (2007) yields that N is a Poisson process with intensity measure $Leb\times\nu'$ where

$$\begin{aligned}\nu'(\mathrm{d}x) &= \{\mathrm{E}[Q_1^{\alpha}1_{\{Q_1>0\}}]1_{(0,\infty)}(x)+\mathrm{E}[(-Q_1)^{\alpha}1_{\{Q_1<0\}}]1_{(-\infty,0)}(x)\}\alpha|x|^{-\alpha-1}\,\mathrm{d}x\\ &= (p1_{(0,\infty)}(x)+r1_{(-\infty,0)}(x))\alpha|x|^{-\alpha-1}\,\mathrm{d}x\\ &= \mu(\mathrm{d}x).\end{aligned}$$

Fix $0<u<\infty$ and define the sum-maximum functional

$$\Phi^{(u)}\colon\mathbf{M}_p([0,1]\times\mathbb{E})\to D^1\times D^2_{\uparrow}$$

by

$$\Phi^{(u)}\Big(\sum_i\delta_{(t_i,x_i)}\Big)(t)=\Big(\sum_{t_i\le t}x_i\,1_{\{u<|x_i|<\infty\}},\ \bigvee_{t_i\le t}|x_i|1_{\{x_i>0\}},\ \bigvee_{t_i\le t}|x_i|1_{\{x_i<0\}}\Big),\quad t\in[0,1]$$

(with the convention $\vee\emptyset=0$), where the space $\mathbf{M}_p([0,1]\times\mathbb{E})$ of Radon point measures on $[0,1]\times\mathbb{E}$ is equipped with the vague topology (see Resnick, 1987, Chapter 3). Let $\Lambda=\Lambda_1\cap\Lambda_2$, where

$$\Lambda_1=\{\eta\in\mathbf{M}_p([0,1]\times\mathbb{E}):\eta(\{0,1\}\times\mathbb{E})=0=\eta([0,1]\times\{\pm\infty,\pm u\})\},$$

$$\Lambda_2=\{\eta\in\mathbf{M}_p([0,1]\times\mathbb{E}):\eta(\{t\}\times(u,\infty])\cdot\eta(\{t\}\times[-\infty,-u))=0\ \text{for all } t\in[0,1]\}.$$

Observe that the elements of Λ_2 have the property that atoms in $[0,1]\times\mathbb{E}_u$ with the same time coordinate are all on the same side of the time axis, where $\mathbb{E}_u=\{x\in\mathbb{E}:|x|>u\}$.

Lemma 3.3. *The maximum functional $\Phi^{(u)}\colon\mathbf{M}_p([0,1]\times\mathbb{E})\to D^1\times D^2_{\uparrow}$ is continuous on the set Λ when $D^1\times D^2_{\uparrow}$ is endowed with the weak M_1 topology.*

Proof: Take an arbitrary $\eta\in\Lambda$ and suppose that $\eta_n\xrightarrow{v}\eta$ as $n\to\infty$ in $\mathbf{M}_p([0,1]\times\mathbb{E})$. We need to show that $\Phi^{(u)}(\eta_n)\to\Phi^{(u)}(\eta)$ in $D^1\times D^2_{\uparrow}$ according to the weak M_1 topology. By Theorem 12.5.2 in Whitt (2002), it suffices to prove that, as $n\to\infty$,

$$d_p^{M_1}(\Phi^{(u)}(\eta_n),\Phi^{(u)}(\eta))=\max_{k=1,2,3}d_{M_1}(\Phi_k^{(u)}(\eta_n),\Phi_k^{(u)}(\eta))\to 0.$$

Following, with small modifications, the lines in the proof of Lemma 3.2 in Basrak et al. (2012) we obtain $d_{M_1}(\Phi_1^{(u)}(\eta_n), \Phi_1^{(u)}(\eta)) \to 0$ as $n \to \infty$. Since Proposition 3.1 in Krizmanić (2022b) implies

$$\max_{k=2,3} d_{M_1}(\Phi_k^{(u)}(\eta_n), \Phi_k^{(u)}(\eta)) \to 0 \qquad \text{as } n \to \infty,$$

we conclude that $\Phi^{(u)}$ is continuous at η. □

In the next result we show that a particular stochastic process $\widetilde{L}_n$, constructed carefully from the sequence (Z_i), converges to the limiting process in relation (1.15) in $D^1 \times D^1_\uparrow$ with the weak M_1 topology. Later, as our main result, we will show that the M_2 distance between processes L_n and $\widetilde{L}_n$ is asymptotically negligible (as n tends to infinity), which will imply the desired convergence in (1.15).

Proposition 3.4. *Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of regularly varying random variables satisfying (1.1) and (1.5) with $\alpha \in (0,2)$, such that the local dependence condition D' and conditions (1.16) and (1.17) hold. If $\alpha \in [1,2)$, also suppose that condition (1.18) holds. Let $(C_i)_{i\geq 0}$ be a sequence of random variables independent of (Z_i) such that the series defying the linear process*

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \qquad i \in \mathbb{Z},$$

is a.s. convergent. Let

$$\widetilde{V}_n(t) := \sum_{i=1}^{\lfloor nt \rfloor} \frac{CZ_i}{a_n} \quad \text{and} \quad \widetilde{M}_n(t) := \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{|Z_i|}{a_n}(C_+ 1_{\{Z_i>0\}} + C_- 1_{\{Z_i<0\}}), \qquad t \in [0,1],$$

with $C = \sum_{i=0}^{\infty} C_i$, and C_+ and C_- defined in (1.13). Then, as $n \to \infty$,

$$\widetilde{L}_n(\,\cdot\,) := (\widetilde{V}_n(\,\cdot\,), \widetilde{M}_n(\,\cdot\,)) \xrightarrow{d} (C^{(0)}V(\,\cdot\,), C^{(1)}M^{(1)}(\,\cdot\,) \vee C^{(2)}M^{(2)}(\,\cdot\,))$$

in $D^1 \times D^1_\uparrow$ with the weak M_1 topology, where V is an α–stable Lévy process with characteristic triple $(0, \mu, b)$, with μ as in (1.4),

$$b = \begin{cases} 0, & \alpha = 1, \\ (p-r)\frac{\alpha}{1-\alpha}, & \alpha \in (0,1) \cup (1,2), \end{cases}$$

$M^{(1)}$ and $M^{(2)}$ are extremal processes with exponent measures $p\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,dx$ and $r\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,dx$ respectively, with p and r defined in (1.5), and $(C^{(0)}, C^{(1)}, C^{(2)})$ is a random vector, independent of $(V, M^{(1)}, M^{(2)})$, such that $(C^{(0)}, C^{(1)}, C^{(2)}) \stackrel{d}{=} (C, C_+, C_-)$.

Before the proof of the proposition recall here some basic facts on Lévy processes and extremal processes. The distribution of a Lévy process V is characterized by its characteristic triple (i.e. the characteristic triple of the infinitely divisible distribution of $V(1)$). The characteristic function of $V(1)$ and the characteristic triple (a, ρ, c) are related in the following way:

$$\mathrm{E}[e^{izV(1)}] = \exp\biggl(-\frac{1}{2}az^2 + icz + \int_{\mathbb{R}} \bigl(e^{izx} - 1 - izx1_{[-1,1]}(x)\bigr)\,\rho(\mathrm{d}x)\biggr)$$

for $z \in \mathbb{R}$, where $a \geq 0$, $c \in \mathbb{R}$ are constants, and ρ is a measure on $\mathbb{R}$ satisfying

$$\rho(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} (|x|^2 \wedge 1)\,\rho(\mathrm{d}x) < \infty.$$

We refer to Sato (1999) for a textbook treatment of Lévy processes. The distribution of an nonnegative extremal process W is characterized by its exponent measure ρ in the following way:

$$\mathrm{P}(W(t) \leq x) = e^{-t\rho(x,\infty)}$$

for $t > 0$ and $x > 0$, where ρ is a measure on $(0, \infty)$ satisfying $\rho(\delta, \infty) < \infty$ for any $\delta > 0$ (see Resnick, 2007, page 161). If ρ is a null measure, we suppose W is a zero process.

Proof of Proposition 3.4: Take an arbitrary $0 < u < 1$, and consider

$$\Phi^{(u)}(N_n)(\,\cdot\,) = \bigg(\sum_{i/n \leq \,\cdot} \frac{Z_i}{a_n} 1_{\big\{ \frac{|Z_i|}{a_n} > u \big\}}, \bigvee_{i/n \leq \,\cdot} \frac{|Z_i|}{a_n} 1_{\{Z_i > 0\}}, \bigvee_{i/n \leq \,\cdot} \frac{|Z_i|}{a_n} 1_{\{Z_i < 0\}} \bigg).$$

Since the limiting point process N is a Poisson process, it is almost surely contained in the set Λ (see Resnick, 2007, page 221). Therefore, since by Lemma 3.3 the sum-maximum functional $\Phi^{(u)}$ is continuous on the set Λ, the continuous mapping theorem applied to the convergence in (3.5) yields $\Phi^{(u)}(N_n) \xrightarrow{d} \Phi^{(u)}(N)$ in $D^1 \times D^2_{\uparrow}$ under the weak M_1 topology, i.e.

$$\begin{aligned} &\bigg(\sum_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{Z_i}{a_n} 1_{\big\{ \frac{|Z_i|}{a_n} > u \big\}}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i > 0\}}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i < 0\}} \bigg) \\ &\qquad \xrightarrow{d} \bigg(\sum_{T_i \leq \,\cdot} P_i Q_i 1_{\{|P_i Q_i| > u\}}, \bigvee_{T_i \leq \,\cdot} |P_i Q_i| 1_{\{P_i Q_i > 0\}}, \bigvee_{T_i \leq \,\cdot} |P_i Q_i| 1_{\{P_i Q_i < 0\}} \bigg). \end{aligned} \tag{3.7}$$

Since $P_i > 0$ and $|Q_i| = 1$ for all i, the limiting process in (3.7) reduces to

$$\bigg(\sum_{T_i \leq \,\cdot} P_i Q_i 1_{\{P_i > u\}}, \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i > 0\}}, \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i < 0\}} \bigg).$$

Relation (1.3) implies, as $n \to \infty$,

$$\begin{aligned} \lfloor nt \rfloor \mathrm{E}\bigg(\frac{Z_1}{a_n} 1_{\big\{ u < \frac{|Z_1|}{a_n} \leq 1 \big\}} \bigg) &= \frac{\lfloor nt \rfloor}{n} \int_{u < |x| \leq 1} x \, n \, \mathrm{P}\left(\frac{Z_1}{a_n} \in \mathrm{d}x \right) \\ &\to t \int_{u < |x| \leq 1} x \mu(\mathrm{d}x) \end{aligned} \tag{3.8}$$

for every $t \in [0,1]$, and this convergence is uniform in t. Therefore an application of Lemma 2.1 to (3.7) and (3.8) yields, as $n \to \infty$,

$$\begin{aligned} L_n^{(u)}(\,\cdot\,) &:= \bigg(\sum_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{Z_i}{a_n} 1_{\big\{ \frac{|Z_i|}{a_n} > u \big\}} - \lfloor n\,\cdot \rfloor b_n^{(u)}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i > 0\}}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i < 0\}} \bigg) \\ &\xrightarrow{d} L^{(u)}(\,\cdot\,) := \bigg(\sum_{T_i \leq \,\cdot} P_i Q_i 1_{\{P_i > u\}} - (\,\cdot\,) b^{(u)}, \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i > 0\}}, \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i < 0\}} \bigg) \end{aligned} \tag{3.9}$$

in $D^1 \times D^2_{\uparrow}$ under the weak M_1 topology, where

$$b_n^{(u)} = \mathrm{E}\bigg(\frac{Z_1}{a_n} 1_{\big\{ u < \frac{|Z_1|}{a_n} \leq 1 \big\}} \bigg) \qquad \text{and} \qquad b^{(u)} = \int_{u < |x| \leq 1} x \mu(\mathrm{d}x).$$

Since $N = \sum_i \delta_{(T_i, P_i Q_i)}$ is a Poisson process with intensity measure $Leb \times \mu$, by the Itô representation of a Lévy process (see Resnick, 2007, pages 150–157; and Sato, 1999, Theorem 14.3 and Theorem 19.2), there exists an α–stable Lévy process $V^{(0)}(\,\cdot\,)$ with characteristic triple $(0, \mu, 0)$ such that

$$\sup_{t \in [0,1]} |L^{(u)1}(t) - V^{(0)}(t)| \to 0$$

almost surely as $u \to 0$, where $L^{(u)1}$ is the first component of the process $L^{(u)}$. Since uniform convergence implies Skorohod M_1 convergence, we get

$$d_{M_1}(L^{(u)1}, V) \to 0 \tag{3.10}$$

almost surely as $u \to 0$. Let

$$L^{(0)}(\,\cdot\,) := \bigg(V^{(0)}(\,\cdot\,), \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i>0\}}, \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i<0\}} \bigg).$$

Then from (3.10) we obtain

$$d_p^{M_1}(L^{(u)}, L^{(0)}) \to 0$$

almost surely as $u \to 0$. Since almost sure convergence implies convergence in distribution, we have, as $u \to 0$,

$$L^{(u)}(\,\cdot\,) \xrightarrow{d} L^{(0)}(\,\cdot\,) \tag{3.11}$$

in $D^1 \times D^2_{\uparrow}$ endowed with the weak M_1 topology. By Proposition 5.2 and Proposition 5.3 in Resnick (2007), the process

$$\sum_i \delta_{(T_i, P_i 1_{\{Q_1>0\}})}$$

is a Poisson process with intensity measure $Leb \times \nu_1$, where

$$\nu_1(\mathrm{d}x) = \mathrm{E}[(1_{\{Q_1>0\}})^\alpha] \alpha x^{-\alpha-1} 1_{(0,\infty)}(x)\,\mathrm{d}x = p\alpha x^{-\alpha-1} 1_{(0,\infty)}(x)\,\mathrm{d}x.$$

Therefore the process

$$M^{(1)}(\,\cdot\,) := \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i>0\}}$$

is an extremal process with exponent measure ν_1 (see Resnick, 2007, page 161). Similarly, the process

$$\sum_i \delta_{(T_i, P_i 1_{\{Q_1<0\}})}$$

is a Poisson process with intensity measure $Leb \times \nu_2$, where

$$\nu_2(\mathrm{d}x) = \mathrm{E}[(1_{\{Q_1<0\}})^\alpha] \alpha x^{-\alpha-1} 1_{(0,\infty)}(x)\,\mathrm{d}x = r\alpha x^{-\alpha-1} 1_{(0,\infty)}(x)\,\mathrm{d}x,$$

and thus the process

$$M^{(2)}(\,\cdot\,) := \bigvee_{T_i \leq \,\cdot} P_i 1_{\{Q_i<0\}}$$

is an extremal process with exponent measure ν_2.

Let

$$L_n^{(0)}(\,\cdot\,) := \bigg(\sum_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{Z_i}{a_n} - \lfloor n\,\cdot \rfloor \mathrm{E}\Big(\frac{Z_1}{a_n} 1_{\left\{\frac{|Z_1|}{a_n} \leq 1\right\}} \Big), \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i>0\}}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i<0\}} \bigg).$$

If we show that

$$\lim_{u \to 0} \limsup_{n \to \infty} \mathrm{P}(d_p^{M_1}(L_n^{(0)}, L_n^{(u)}) > \epsilon) = 0$$

for every $\epsilon > 0$, from (3.9) and (3.11) by a generalization of Slutsky's theorem (see Theorem 3.5 in Resnick, 2007) it will follow that $L_n^{(0)} \xrightarrow{d} L^{(0)}$ as $n \to \infty$, in $D^1 \times D^2_{\uparrow}$ with the weak M_1 topology.

Recalling the definitions, the fact that the metric $d_p^{M_1}$ is bounded above by the uniform metric (see Theorem 12.10.3 in Whitt, 2002), and using stationarity and Markov's inequality we obtain

$$\begin{aligned} \mathrm{P}(d_p^{M_1}(L_n^{(0)}, L_n^{(u)}) > \epsilon) &\leq \mathrm{P}\left(\sup_{t\in[0,1]} \|L_n^{(0)}(t) - L_n^{(u)}(t)\| > \epsilon \right) \\ &= \mathrm{P}\left[\sup_{t\in[0,1]} \left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{Z_i}{a_n} 1_{\left\{ \frac{|Z_i|}{a_n} \leq u \right\}} - \lfloor nt \rfloor \mathrm{E}\Big(\frac{Z_1}{a_n} 1_{\left\{ \frac{|Z_1|}{a_n} \leq u \right\}} \Big) \right| > \epsilon \right] \\ &= \mathrm{P}\left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left(\frac{Z_i}{a_n} 1_{\left\{ \frac{|Z_i|}{a_n} \leq u \right\}} - \mathrm{E}\Big(\frac{Z_1}{a_n} 1_{\left\{ \frac{|Z_1|}{a_n} \leq u \right\}} \Big) \right) \right| > \epsilon \right] \end{aligned}$$

Therefore we have to show

$$\lim_{u\to 0} \limsup_{n\to\infty} \mathrm{P}\left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left(\frac{Z_i}{a_n} 1_{\left\{ \frac{|Z_i|}{a_n} \leq u \right\}} - \mathrm{E}\Big(\frac{Z_1}{a_n} 1_{\left\{ \frac{|Z_1|}{a_n} \leq u \right\}} \Big) \right) \right| > \epsilon \right]. \tag{3.12}$$

For $\alpha \in [1,2)$ this relation is simply condition (1.18). In the case $\alpha \in (0,1)$, let

$$I(u,n,\epsilon) = \mathrm{P}\left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left(\frac{Z_i}{a_n} 1_{\left\{ \frac{|Z_i|}{a_n} \leq u \right\}} - \mathrm{E}\left(\frac{Z_1}{a_n} 1_{\left\{ \frac{|Z_1|}{a_n} \leq u \right\}} \right) \right) \right| > \epsilon \right].$$

Using stationarity and Chebyshev's inequality we get the bound

$$\begin{aligned} I(u,n,\epsilon) &\leq \epsilon^{-1} \mathrm{E}\left[\sum_{i=1}^{n} \left| \frac{Z_i}{a_n} 1_{\left\{ \frac{|Z_i|}{a_n} \leq u \right\}} - \mathrm{E}\left(\frac{Z_1}{a_n} 1_{\left\{ \frac{|Z_1|}{a_n} \leq u \right\}} \right) \right| \right] \leq \frac{2n}{\epsilon} \mathrm{E}\left(\frac{|Z_1|}{a_n} 1_{\left\{ \frac{|Z_1|}{a_n} \leq u \right\}} \right) \\ &= \frac{2u}{\epsilon} \cdot n\,\mathrm{P}(|Z_1| > a_n) \cdot \frac{\mathrm{P}(|Z_1| > ua_n)}{\mathrm{P}(|Z_1| > a_n)} \cdot \frac{\mathrm{E}(|Z_1| 1_{\{|Z_1| \leq ua_n\}})}{ua_n \mathrm{P}(|Z_1| > ua_n)}. \end{aligned}$$

Since the random variable Z_1 is regularly varying with index α, it holds that

$$\lim_{n\to\infty} \frac{\mathrm{P}(|Z_1| > ua_n)}{\mathrm{P}(|Z_1| > a_n)} = u^{-\alpha}.$$

By Karamata's theorem

$$\lim_{n\to\infty} \frac{\mathrm{E}(|Z_1| 1_{\{|Z_1| \leq ua_n\}})}{ua_n \mathrm{P}(|Z_1| > ua_n)} = \frac{\alpha}{1-\alpha},$$

and therefore taking into account relation (1.2), we get

$$\limsup_{n\to\infty} I(u,n,\epsilon) \leq u^{1-\alpha} \frac{\alpha}{\epsilon(1-\alpha)}.$$

Since in this case $1-\alpha > 0$, letting $u \to 0$ we obtain

$$\lim_{u\to 0} \limsup_{n\to\infty} I(u,n,\epsilon) = 0.$$

Hence we conclude

$$L_n^{(0)}(\,\cdot\,) \xrightarrow{d} L^{(0)}(\,\cdot\,) \qquad \text{as } n \to \infty, \tag{3.13}$$

in $D^1 \times D^2_{\uparrow}$ with the weak M_1 topology.

By Karamata's theorem, as $n \to \infty$,

$$n\,\mathrm{E}\Big(\frac{Z_1}{a_n} 1_{\{|Z_1| \leq a_n\}} \Big) \to (p-r) \frac{\alpha}{1-\alpha}, \qquad \text{if } \alpha \in (0,1),$$

$$n\,\mathrm{E}\Big(\frac{Z_1}{a_n} 1_{\{|Z_1| > a_n\}} \Big) \to (p-r) \frac{\alpha}{\alpha-1}, \qquad \text{if } \alpha \in (1,2),$$

with p and r as in (1.5). Therefore conditions (1.16) and (1.17), and Lemma 2.1, applied to the convergence in (3.13), yield

$$L_n^*(\,\cdot\,) \xrightarrow{d} L(\,\cdot\,) \qquad \text{as } n \to \infty \tag{3.14}$$

in $D^1 \times D_{\uparrow}^2$ with the weak M_1 topology, where

$$L_n^*(\,\cdot\,) := \bigg(\sum_{i=1}^{\lfloor n\,\cdot\,\rfloor} \frac{Z_i}{a_n}, \bigvee_{i=1}^{\lfloor n\,\cdot\,\rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i>0\}}, \bigvee_{i=1}^{\lfloor n\,\cdot\,\rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i<0\}} \bigg)$$

and $L := (V, M^{(1)}, M^{(2)})$, with

$$V(t) = \begin{cases} V^{(0)}(t), & \alpha = 1, \\ V^{(0)}(t) + (p-r)\frac{\alpha}{1-\alpha}, & \alpha \in (0,1) \cup (1,2), \end{cases}$$

being an α–stable Lévy process with characteristic triple $(0, \mu, 0)$ if $\alpha = 1$ and $(0, \mu, (p-r)\alpha/(1-\alpha))$ if $\alpha \in (0,1) \cup (1,2)$. Note that in case $\alpha = 1$ it holds that $L_n^* = L_n^{(0)}$ (since Z_1 is symmetric).

It is known that D^1 equipped with the Skorokhod J_1 topology is a Polish space, i.e. metrizable as a complete separable metric space (see Billingsley, 1968, Section 14), and therefore the same holds for the M_1 topology, since it is topologically complete (see Whitt, 2002, Section 12.8) and separability remains preserved in the weaker topology. Since the space $D_{\uparrow}^1$ is a closed subspace of D^1 (cf. Whitt, 2002, Lemma 13.2.3), it is also Polish. Further, the space $D^1 \times D_{\uparrow}^2$ equipped with the weak M_1 topology is separable as a direct product of three separable topological spaces. It is also topologically complete since the product metric $d_p^{M_1}$ inherits the completeness of the component metrics. Hence $D^1 \times D_{\uparrow}^2$ with the weak M_1 topology is also a Polish space. This allows us to apply Corollary 5.18 in Kallenberg (1997) to conclude that there exists a random vector $(C^{(0)}, C^{(1)}, C^{(2)})$, independent of $(V, M^{(1)}, M^{(2)})$, such that

$$(C^{(0)}, C^{(1)}, C^{(2)}) \stackrel{d}{=} (C, C_+, C_-). \tag{3.15}$$

This, relation (3.14), the fact that (C, C_+, C_-) is independent of L_n^*, and Theorem 3.29 in Kallenberg Kallenberg (1997), imply that, as $n \to \infty$,

$$(B^0, B^1, B^2, L_n^{*1}, L_n^{*2}, L_n^{*3}) \xrightarrow{d} (B^{(0)}, B^{(1)}, B^{(2)}, V, M^{(1)}, M^{(2)}) \tag{3.16}$$

in $D^1 \times D_{\uparrow}^2 \times D^1 \times D_{\uparrow}^2$ with the product M_1 topology, where L_n^{*i} is the i–th component of L_n^* $(i = 1,2,3)$, $B^0(t) = C$, $B^1(t) = C_+$, $B^2(t) = C_-$, $B^{(0)}(t) = C^{(0)}$, $B^{(1)}(t) = C^{(1)}$ and $B^{(2)}(t) = C^{(2)}$ for $t \in [0,1]$.

Let $g \colon D^1 \times D_{\uparrow}^2 \times D^1 \times D_{\uparrow}^2 \to D^1 \times D_{\uparrow}^2$ be a function defined by

$$g(x) = (x_1 x_4, x_2 x_5, x_3 x_6), \qquad x = (x_1, \ldots, x_6) \in D^1 \times D_{\uparrow}^2 \times D^1 \times D_{\uparrow}^2.$$

Denote by $\widetilde{D}_{1,2,3}$ the set of all functions in $D^1 \times D_{\uparrow}^2 \times D^1 \times D_{\uparrow}^2$ for which the first three component functions have no discontinuity points, that is

$$\widetilde{D}_{1,2,3} = \{(x_1, \ldots, x_6) \in D^1 \times D_{\uparrow}^2 \times D^1 \times D_{\uparrow}^2 : \mathrm{Disc}(x_i) = \emptyset,\ i = 1,2,3\}.$$

By Lemma 2.2 the function g is continuous on the set $\widetilde{D}_{1,2,3}$ in the weak M_1 topology, and hence $\mathrm{Disc}(g) \subseteq \widetilde{D}_{1,2,3}^c$. Denoting $\widetilde{D}_1 = \{u \in D_{\uparrow}^1 : \mathrm{Disc}(u) = \emptyset\}$ we obtain

$$\begin{aligned} \mathrm{P}[(B^{(0)}, B^{(1)}, B^{(2)}, V, M^{(1)}, M^{(2)}) \in \mathrm{Disc}(g)] \ &\leq \ \mathrm{P}[(B^{(0)}, B^{(1)}, B^{(2)}, V, M^{(1)}, M^{(2)}) \in \widetilde{D}_{1,2,3}^c] \\ &\leq \ \mathrm{P}[\{B^{(0)} \in \widetilde{D}_1^c\} \cup \{B^{(1)} \in \widetilde{D}_1^c\} \cup \{B^{(2)} \in \widetilde{D}_1^c\}] = 0, \end{aligned}$$

where the last equality holds since $B^{(0)}$, $B^{(1)}$ and $B^{(2)}$ have no discontinuity points. This allows us to apply the continuous mapping theorem to relation (3.16) yielding $g(B^0, B^1, B^2, L_n^{*1}, L_n^{*2}, L_n^{*3}) \xrightarrow{d} g(B^{(0)}, B^{(1)}, B^{(2)}, V, M^{(1)}, M^{(2)})$ as $n \to \infty$, that is

$$\bigg(\sum_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{CZ_i}{a_n}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{C_+|Z_i|}{a_n} 1_{\{Z_i>0\}}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{C_-|Z_i|}{a_n} 1_{\{Z_i<0\}} \bigg)$$

$$\xrightarrow{d} (C^{(0)}V(\,\cdot\,), C^{(1)}M^{(1)}(\,\cdot\,), C^{(2)}M^{(2)}(\,\cdot\,)) \tag{3.17}$$

in $D^1 \times D^2_{\uparrow}$ with the weak M_1 topology.

Lemma 2.3 implies the function $f \colon D^1 \times D^2_{\uparrow} \to D^1 \times D^1_{\uparrow}$, defined by

$$f(x,y,z) = (x, y \vee z), \qquad (x,y,z) \in D^1 \times D^2_{\uparrow},$$

is continuous when $D^1 \times D^2_{\uparrow}$ and $D^1 \times D^1_{\uparrow}$ are endowed with the weak M_1 topology. Therefore from (3.17), by an application of the continuous mapping theorem, we obtain, as $n \to \infty$,

$$\bigg(\sum_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{CZ_i}{a_n}, \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{C_+|Z_i|}{a_n} 1_{\{Z_i>0\}} \vee \bigvee_{i=1}^{\lfloor n\,\cdot \rfloor} \frac{C_-|Z_i|}{a_n} 1_{\{Z_i<0\}} \bigg) \xrightarrow{d} (C^{(0)}V(\,\cdot\,), C^{(1)}M^{(1)}(\,\cdot\,) \vee C^{(2)}M^{(2)}(\,\cdot\,)),$$

that is, $\widetilde{L}_n \xrightarrow{d} (C^{(0)}V, C^{(1)}M^{(1)} \vee C^{(2)}M^{(2)})$ in $D^1 \times D^1_{\uparrow}$ with the weak M_1 topology, which completes the proof. □

4. Finite order moving average processes

Fix $q \in \mathbb{N}$ and let $C_0, C_1, \ldots, C_q$ be random variables satisfying

$$0 \le \sum_{i=0}^{s} C_i \bigg/ \sum_{i=0}^{q} C_i \le 1 \quad \text{a.s.} \qquad \text{for every } s = 0, 1, \ldots, q. \tag{4.1}$$

In this case C, C_+ and C_- reduce to

$$C = \sum_{i=0}^{q} C_i, \quad C_+ = \max_{0 \le i \le q} (C_i \vee 0) \quad \text{and} \quad C_- = \max_{0 \le i \le q} (-C_i \vee 0).$$

Condition (4.1) implies that C, $\sum_{i=0}^{s} C_i$ and $\sum_{i=s}^{q} C_i$ are a.s. of the same sign for every $s = 0, 1, \ldots, q$. If the C_j's are all nonnegative or all nonpositive, then condition (4.1) is trivially satisfied. Recall that $L_n = (V_n, M_n)$, where

$$V_n(t) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i, \qquad t \in [0,1],$$

and

$$M_n(t) = \begin{cases} \dfrac{1}{a_n} \displaystyle\bigvee_{i=1}^{\lfloor nt \rfloor} X_i, & t \in \Big[\dfrac{1}{n}, 1\Big], \\[2ex] \dfrac{X_1}{a_n}, & t \in \Big[0, \dfrac{1}{n}\Big\rangle, \end{cases}$$

are the partial sum and partial maxima processes constructed from moving average processes

$$X_i = \sum_{j=0}^{q} C_j Z_{i-j}, \qquad i \in \mathbb{Z},$$

with the normalizing sequence (a_n) as in (1.2).

Theorem 4.1. *Let $(Z_i)_{i\in\mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of regularly varying random variables satisfying (1.1) and (1.5) with $\alpha \in (0,2)$, such that the local dependence condition D' and conditions (1.16) and (1.17) hold. If $\alpha \in [1,2)$, also suppose that condition (1.18) holds. Let $C_0,\dots,C_q$ be random variables, independent of (Z_i), such that condition (4.1) holds. Then, as $n\to\infty$,*

$$L_n(\,\cdot\,) \xrightarrow{d} (C^{(0)}V(\,\cdot\,), C^{(1)}M^{(1)}(\,\cdot\,) \vee C^{(2)}M^{(2)}(\,\cdot\,))$$

in $D^1 \times D^1_{\uparrow}$ with the weak M_2 topology, where V is an α–stable Lévy process with characteristic triple $(0,\mu,b)$, with μ as in (1.4),

$$b = \begin{cases} 0, & \alpha = 1,\\ (p-r)\frac{\alpha}{1-\alpha}, & \alpha \in (0,1)\cup(1,2), \end{cases}$$

$M^{(1)}$ and $M^{(2)}$ are extremal processes with exponent measures $p\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,dx$ and $r\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,dx$ respectively, with p and r defined in (1.5), and $(C^{(0)},C^{(1)},C^{(2)})$ is a random vector, independent of $(V,M^{(1)},M^{(2)})$, such that $(C^{(0)},C^{(1)},C^{(2)}) \stackrel{d}{=} (C,C_+,C_-)$.

Proof: By Proposition 3.4, $\widetilde{L}_n = (\widetilde{V}_n,\widetilde{M}_n) \xrightarrow{d} (C^{(0)}V, C^{(1)}M^{(1)} \vee C^{(2)}M^{(2)})$ as $n\to\infty$, in $D^1\times D^1_{\uparrow}$ with the weak M_1 topology, where

$$\widetilde{V}_n(t) = \sum_{i=1}^{\lfloor nt\rfloor} \frac{CZ_i}{a_n} \quad \text{and} \quad \widetilde{M}_n(t) = \bigvee_{i=1}^{\lfloor nt\rfloor} \frac{|Z_i|}{a_n}(C_+1_{\{Z_i>0\}} + C_-1_{\{Z_i<0\}}), \qquad t\in[0,1].$$

Since M_1 convergence implies M_2 convergence, we have

$$\widetilde{L}_n(\,\cdot\,) = (\widetilde{V}_n(\,\cdot\,),\widetilde{M}_n(\,\cdot\,)) \xrightarrow{d} (C^{(0)}V(\,\cdot\,), C^{(1)}M^{(1)}(\,\cdot\,)\vee C^{(2)}M^{(2)}(\,\cdot\,)) \tag{4.2}$$

in $D^1\times D^1_{\uparrow}$ with the weak M_2 topology as well.

If we show that

$$\lim_{n\to\infty} \mathrm{P}[d_p^{M_2}(\widetilde{L}_n, L_n) > \delta] = 0$$

for any $\delta>0$, then from (4.2) by an application of Slutsky's theorem (see Theorem 3.4 in Resnick, 2007) it will follow that $L_n \xrightarrow{d} (C^{(0)}V, C^{(1)}M^{(1)}\vee C^{(2)}M^{(2)})$ as $n\to\infty$, in $D^1\times D^1_{\uparrow}$ with the weak M_2 topology. By the definition of the metric $d_p^{M_2}$ in (2.1) it suffices to show that

$$\lim_{n\to\infty} \mathrm{P}[d_{M_2}(\widetilde{V}_n, V_n) > \delta] = 0 \tag{4.3}$$

and

$$\lim_{n\to\infty} \mathrm{P}[d_{M_2}(\widetilde{M}_n, M_n) > \delta] = 0. \tag{4.4}$$

Relation (4.3) was established in the proof of Theorem 2.1 in Krizmanić (2022a). As for relation (4.4), it holds in a special case when the innovations Z_i are i.i.d. (see the proof of Theorem 3.3 in Krizmanić, 2022b). It remains to show that it also holds in the weak dependence case, i.e. when the independence property is replaced by the local dependence condition D' in (1.14). Krizmanić (2022b) showed that

$$\{d_{M_2}(\widetilde{M}_n, M_n) > \delta\} \subseteq H_{n,1}\cup H_{n,2}\cup H_{n,3}, \tag{4.5}$$

where

$$
\begin{aligned}
H_{n,1} &= \left\{ \exists\, l \in \{-q,\ldots,q\} \cup \{n-q+1,\ldots,n\} \text{ such that } \frac{C_*|Z_l|}{a_n} > \frac{\delta}{4(q+1)} \right\}, \\
H_{n,2} &= \Big\{ \exists\, k \in \{1,\ldots,n\} \text{ and } \exists\, l \in \{k-q,\ldots,k+q\} \setminus \{k\} \text{ such that} \\
&\qquad \frac{C_*|Z_k|}{a_n} > \frac{\delta}{4(q+1)} \text{ and } \frac{C_*|Z_l|}{a_n} > \frac{\delta}{4(q+1)} \Big\}, \\
H_{n,3} &= \Big\{ \exists\, k \in \{1,\ldots,n\},\ \exists\, j \in \{1,\ldots,n\} \setminus \{k,\ldots,k+q\},\ \exists\, l_1 \in \{0,\ldots,q\} \\
&\qquad \text{and } \exists\, l \in \{0,\ldots,q\} \setminus \{l_1\} \text{ such that } \frac{C_*|Z_k|}{a_n} > \frac{\delta}{4(q+1)}, \\
&\qquad \frac{C_*|Z_{j-l_1}|}{a_n} > \frac{\delta}{4(q+1)} \text{ and } \frac{C_*|Z_{j-l}|}{a_n} > \frac{\delta}{4(q+1)} \Big\},
\end{aligned}
$$

with $C_* := C_+ \vee C_-$, and the independence property was used there only in establishing

$$
\lim_{n\to\infty} \mathrm{P}(H_{n,2}) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{P}(H_{n,3}) = 0.
$$

Now we are going to show the last two limiting relations still hold under condition D'. For an arbitrary $M > 0$ it holds that

$$
\begin{aligned}
\mathrm{P}(H_{n,2} \cap \{C_* \le M\}) &= \sum_{k=1}^{n} \sum_{\substack{l=k-q \\ l\neq k}}^{k+q} \mathrm{P}\left(\frac{C_*|Z_k|}{a_n} > \frac{\delta}{4(q+1)},\, \frac{C_*|Z_l|}{a_n} > \frac{\delta}{4(q+1)},\, C_* \le M \right) \\
&\le 2n \sum_{i=1}^{q} \mathrm{P}\left(\frac{|Z_0|}{a_n} > \frac{\delta}{4(q+1)M},\, \frac{|Z_i|}{a_n} > \frac{\delta}{4(q+1)M} \right) \\
&\le 2n \sum_{i=1}^{\lfloor n/k \rfloor} \mathrm{P}\left(\frac{|Z_0|}{a_n} > \frac{\delta}{4(q+1)M},\, \frac{|Z_i|}{a_n} > \frac{\delta}{4(q+1)M} \right)
\end{aligned}
$$

for all positive integers k such that $k \le n/q$. By letting $n \to \infty$, and then $k \to \infty$, we see that condition D' yields that $\mathrm{P}(H_{n,2} \cap \{C \le M\}) \to 0$ as $n \to \infty$. Hence

$$
\limsup_{n\to\infty} \mathrm{P}(H_{n,2}) \le \limsup_{n\to\infty} \mathrm{P}(H_{n,2} \cap \{C_* > M\}) \le \mathrm{P}(C_* > M),
$$

and letting $M \to \infty$ we conclude

$$
\lim_{n\to\infty} \mathrm{P}(H_{n,2}) = 0. \tag{4.6}
$$

Note that

$$
\begin{aligned}
H_{n,3} \subseteq \Big\{ &\exists\, s \in \{1-q,\ldots,n\} \text{ and } \exists\, s_1 \in \{s-q,\ldots,s+q\} \setminus \{s\} \\
&\text{such that } \frac{C_*|Z_s|}{a_n} > \frac{\delta}{4(q+1)} \text{ and } \frac{C_*|Z_{s_1}|}{a_n} > \frac{\delta}{4(q+1)} \Big\},
\end{aligned}
$$

which implies

$$\mathrm{P}(H_{n,3} \cap \{C_* \le M\}) = \sum_{s=1-q}^{n} \sum_{\substack{s_1=s-q \\ s_1 \neq s}}^{s+q} \mathrm{P}\left(\frac{C_*|Z_s|}{a_n} > \frac{\delta}{4(q+1)}, \frac{C_*|Z_{s_1}|}{a_n} > \frac{\delta}{4(q+1)}, C_* \le M\right)$$

$$\le 2[n-(1-q)+1] \sum_{i=1}^{q} \mathrm{P}\left(\frac{|Z_0|}{a_n} > \frac{\delta}{4(q+1)M}, \frac{|Z_i|}{a_n} > \frac{\delta}{4(q+1)M}\right)$$

$$\le 4n \sum_{i=1}^{\lfloor n/k \rfloor} \mathrm{P}\left(\frac{|Z_0|}{a_n} > \frac{\delta}{4(q+1)M}, \frac{|Z_i|}{a_n} > \frac{\delta}{4(q+1)M}\right)$$

for arbitrary $M > 0$, $n \ge q$ and positive integers k such that $k \le n/q$. Hence, similar as in (4.6), we obtain

$$\lim_{n\to\infty} \mathrm{P}(H_{n,3}) = 0. \tag{4.7}$$

Further, by stationarity and the regular variation property it holds that

$$\limsup_{n\to\infty} \mathrm{P}(H_{n,1} \cap \{C_* \le M\}) \le (3q+1) \limsup_{n\to\infty} \mathrm{P}\left(\frac{|Z_1|}{a_n} > \frac{\delta}{4(q+1)M}\right) = 0$$

for arbitrary $M > 0$, and this, as before, implies

$$\lim_{n\to\infty} \mathrm{P}(H_{n,1}) = 0. \tag{4.8}$$

Now, from (4.5)–(4.8) we obtain (4.4), and therefore finally conclude that $L_n \xrightarrow{d} (C^{(0)}V, C^{(1)}M^{(1)} \vee C^{(2)}M^{(2)})$ as $n \to \infty$, in $D^1 \times D^1_{\uparrow}$ with the weak M_2 topology. □

Remark 4.2. From the proof of Proposition 3.4 it follows that the components of the limiting process $(C^{(0)}V, C^{(1)}M^{(1)} \vee C^{(2)}M^{(2)})$ can be expressed as functionals of the limiting point process $N = \sum_i \delta_{(T_i, P_i Q_i)}$ from relation (3.6), that is

$$V(t) = \lim_{u\to 0}\left(\sum_{T_i \le t} P_i Q_i 1_{\{P_i > u\}} - t\int_{u<|x|\le 1} x\mu(\mathrm{d}x)\right) + (p-r)\frac{\alpha}{1-\alpha}1_{\{\alpha \neq 1\}},$$

where the limit holds almost surely uniformly on $[0,1]$, and

$$M^{(1)}(t) = \bigvee_{T_i \le t} P_i 1_{\{Q_i > 0\}}, \qquad M^{(2)}(t) = \bigvee_{T_i \le t} P_i 1_{\{Q_i < 0\}}.$$

Remark 4.3. Theorem 4.1 establishes functional convergence of the joint stochastic process L_n of partial sums and maxima in the space $D^1 \times D^1_{\uparrow}$ endowed with the weak M_2 topology induced by the metric $d_p^{M_2}$ given in (2.1). Since for the second coordinate of L_n, i.e. the partial maxima process M_n, functional convergence holds also in the stronger M_1 topology, it is possible to obtain a sort of joint convergence of L_n in the M_2 topology on the first coordinate and in the M_1 topology on the second coordinate.

Precisely, by Remark 12.8.1 in Whitt (2002) the following metric is a complete metric topologically equivalent to d_{M_1}:

$$d^*_{M_1}(x_1, x_2) = d_{M_2}(x_1, x_2) + \lambda(\widehat{\omega}(x_1, \cdot), \widehat{\omega}(x_2, \cdot)),$$

where λ is the Lévy metric on a space of distributions

$$\lambda(F_1, F_2) = \inf\{\epsilon > 0 : F_2(x-\epsilon) - \epsilon \le F_1(x) \le F_2(x+\epsilon) + \epsilon \ \text{ for all } x\}$$

and

$$\widehat{\omega}(x, z) = \begin{cases} \omega(x, e^z), & z < 0, \\ \omega(x, 1), & z \ge 0, \end{cases}$$

with $\omega(x,\rho) = \sup_{0\leq t\leq 1} \omega(x,t,\rho)$ and

$$\omega(x,t,\rho) = \sup_{0\vee(t-\rho)\leq t_1<t_2<t_3\leq(t+\rho)\wedge 1} ||x(t_2) - [x(t_1), x(t_3)]||$$

where $\rho > 0$ and $\|z - A\|$ denotes the distance between a point z and a subset $A \subseteq \mathbb{R}$. Since $\widetilde{M}_n$ and M_n are nondecreasing, for $t_1 < t_2 < t_3$ it holds that $\|\widetilde{M}_n(t_2) - [\widetilde{M}_n(t_1), \widetilde{M}_n(t_3)]\| = 0$, which implies $\omega(\widetilde{M}_n, \rho) = 0$ for all $\rho > 0$, and similarly $\omega(M_n, \rho) = 0$. Therefore $\lambda(\widetilde{M}_n, M_n) = 0$, and $d^*_{M_1}(\widetilde{M}_n, M_n) = d_{M_2}(\widetilde{M}_n, M_n)$.

Now from (4.4) we obtain

$$\lim_{n\to\infty} \mathrm{P}[d^*_{M_1}(\widetilde{M}_n, M_n) > \delta] = 0$$

for any $\delta > 0$, which allows us to conclude that L_n converges in distribution to $(C^{(0)}V, C^{(1)}M^{(1)} \vee C^{(2)}M^{(2)})$ in the topology induced by the metric

$$d_p{}^*(x,y) = \max\{d_{M_2}(x_1,y_1), d^*_{M_1}(x_2,y_2)\}$$

for $x = (x_1, x_2), y = (y_1, y_2) \in D^2$, that is, in the M_2 topology on the first coordinate and in the M_1 topology on the second coordinate.

Remark 4.4. In the case of deterministic coefficients of the same sign, functional convergence of L_n in Theorem 4.1 can be obtained also by an application of Theorem 3.4 in Krizmanić (2020) with some appropriate modifications due to different centering and normalizing sequences used. In this case the convergence actually holds in the weak M_1 topology.

5. Infinite order moving average processes

When dealing with infinite order moving averages the standard idea is to approximate them by a sequence of finite order moving averages, for which the weak convergence holds, and to show that the error of approximation is negligible in the limit. In our case, we will approximate them by finite order moving averages for which Theorem 4.1 holds, and then we will show that the error of approximation is negligible with respect to the uniform metric.

Theorem 5.1. *Let (X_i) be a moving average process defined by*

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \qquad i \in \mathbb{Z},$$

where $(Z_i)_{i\in\mathbb{Z}}$ is a strictly stationary and strongly mixing sequence of regularly varying random variables satisfying (1.1) and (1.5) with $\alpha \in (0,2)$, such that the local dependence condition D' and conditions (1.16) and (1.17) hold. Let $(C_i)_{i\geq 0}$ be a sequence of random variables,independent of (Z_i), satisfying conditions (1.7) and (1.9). If $\alpha \in (0,1)$ suppose further

$$\sum_{i=0}^{\infty} \mathrm{E}|C_i|^{\gamma} < \infty \qquad \textit{for some } \gamma \in (\alpha, 1), \tag{5.1}$$

while if $\alpha \in [1,2)$ suppose condition (1.18) holds,

$$\limsup_{n\to\infty} \sup_{j\geq 0} \mathrm{E}\bigg[\max_{1\leq l\leq n} \bigg| \frac{1}{a_n} \sum_{i=1}^{l} Z_{i-j} 1_{\{|Z_{i-j}|\leq a_n\}} \bigg|^r \bigg] < \infty \qquad \textit{for some } r \geq 1, \tag{5.2}$$

and

$$\sum_{j=0}^{\infty} \mathrm{E}|C_j| < \infty. \tag{5.3}$$

Then, as $n \to \infty$,

$$L_n(\,\cdot\,) \xrightarrow{d} (C^{(0)}V(\,\cdot\,), C^{(1)}M^{(1)}(\,\cdot\,) \vee C^{(2)}M^{(2)}(\,\cdot\,)) \tag{5.4}$$

in $D^1 \times D^1_{\uparrow}$ *with the weak* M_2 *topology, where* V *is an* α*–stable Lévy process with characteristic triple* $(0, \mu, b)$*, with* μ *as in* (1.4),

$$b = \begin{cases} 0, & \alpha = 1, \\ (p-r)\frac{\alpha}{1-\alpha}, & \alpha \in (0,1) \cup (1,2), \end{cases}$$

$M^{(1)}$ *and* $M^{(2)}$ *are extremal processes with exponent measures* $p\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,dx$ *and* $r\alpha x^{-\alpha-1}1_{(0,\infty)}(x)\,dx$ *respectively, with* p *and* r *defined in* (1.5)*, and* $(C^{(0)}, C^{(1)}, C^{(2)})$ *is a random vector, independent of* $(V, M^{(1)}, M^{(2)})$*, such that* $(C^{(0)}, C^{(1)}, C^{(2)}) \stackrel{d}{=} (C, C_+, C_-)$*, where* $C = \sum_{i=0}^{\infty} C_i$ *and* C_+, C_- *as defined in* (1.13).

Proof: For $q \in \mathbb{N}$ define

$$X_i^q = \sum_{j=0}^{q-1} C_j Z_{i-j} + C^q Z_{i-q}, \qquad i \in \mathbb{Z},$$

where $C^q = \sum_{j=q}^{\infty} C_j$, and let

$$V_{n,q}(t) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i^q}{a_n} \quad \text{and} \quad M_{n,q}(t) = \frac{X_1^q}{a_n} 1_{[0,1/n\rangle}(t) + \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i^q}{a_n} 1_{[1/n,1]}(t), \qquad t \in [0,1].$$

Condition (1.9) implies that coefficients $C_0, \ldots, C_{q-1}, C^q$ satisfy condition (4.1), and hence an application of Theorem 4.1 to the finite order moving average process $(X_i^q)_i$ yields that

$$L_{n,q}(\,\cdot\,) := (V_{n,q}(\,\cdot\,), M_{n,q}(\,\cdot\,)) \xrightarrow{d} L^q(\,\cdot\,) \qquad \text{as } n \to \infty, \tag{5.5}$$

in $D^1 \times D^1_{\uparrow}$ with the weak M_2 topology, with $L^q = (C^{(0)}V, C_q^{(1)}M^{(1)} \vee C_q^{(2)}M^{(2)})$, where V is an α–stable Lévy process and $M^{(1)}, M^{(2)}$ are extremal processes as in Theorem 4.1, and $(C^{(0)}, C_q^{(1)}, C_q^{(2)})$ is a random vector, independent of $(V, M^{(1)}, M^{(2)})$, such that $(C^{(0)}, C_q^{(1)}, C_q^{(2)}) \stackrel{d}{=} (C, C_{+,q}, C_{-,q})$, with

$$C_{+,q} = \max\{C_j \vee 0 : j = 0, \ldots, q-1\} \vee (C^q \vee 0)$$

and

$$C_{-,q} = \max\{-C_j \vee 0 : j = 0, \ldots, q-1\} \vee (-C^q \vee 0).$$

Since $\sum_{j=0}^{\infty} |C_j| < \infty$ a.s. (which follows from condition (1.7)), it is straightforward to obtain

$$C_{+,q} \to C_+ \quad \text{and} \quad C_{-,q} \to C_-$$

almost surely as $q \to \infty$. Therefore

$$\|(B, B_{+,q}, B_{-,q}) - (B, B_+, B_-)\|_{[0,1]} \to 0$$

almost surely as $q \to \infty$, where $B(t) = C$, $B_{+,q}(t) = C_{+,q}$, $B_{-,q}(t) = C_{-,q}$, $B_+(t) = C_+$ and $B_-(t) = C_-$ for $t \in [0,1]$. Since uniform convergence implies Skorokhod M_1 convergence, it follows that $d_p^{M_1}((B, B_{+,q}, B_{-,q}), (B, B_+, B_-)) \to 0$ almost surely as $q \to \infty$, and hence, since almost sure convergence implies convergence in distribution, we have

$$(B, B_{+,q}, B_{-,q}) \xrightarrow{d} (B, B_+, B_-) \qquad \text{as } q \to \infty,$$

in $D^1 \times D^2_{\uparrow}$ with the weak M_1 topology. An application of Theorem 3.29 and Corollary 5.18 in Kallenberg (1997) yields

$$(B^{(0)}, B_q^{(1)}, B_q^{(2)}, V, M^{(1)}, M^{(2)}) \xrightarrow{d} (B^{(0)}, B^{(1)}, B^{(2)}, V, M^{(1)}, M^{(2)}) \tag{5.6}$$

in $D^1 \times D^2_{\uparrow} \times D^1 \times D^2_{\uparrow}$ with the product M_1 topology, where $B^{(0)}(t) = C^{(0)}$, $B^{(1)}_q(t) = C^{(1)}_q$, $B^{(2)}_q(t) = C^{(2)}_q$, $B^{(1)}(t) = C^{(1)}$ and $B^{(2)}(t) = C^{(2)}$ for $t \in [0,1]$, and $(C^{(0)}, C^{(1)}, C^{(2)})$ is a random vector, independent of $(V, M^{(1)}, M^{(2)})$, such that $(C^{(0)}, C^{(1)}, C^{(2)}) \stackrel{d}{=} (C, C_+, C_-)$.

Now, as in the proof of Proposition 3.4, an application of the continuous mapping theorem to the convergence relation in (5.6), with the function g given by

$$g(x) = (x_1 x_4, x_2 x_5, x_3 x_6), \qquad x = (x_1, \ldots, x_6) \in D^1 \times D^2_{\uparrow} \times D^1 \times D^2_{\uparrow},$$

yields

$$g(B^{(0)}, B^{(1)}_q, B^{(2)}_q, V, M^{(1)}, M^{(2)}) \xrightarrow{d} g(B^{(0)}, B^{(1)}, B^{(2)}, V, M^{(1)}, M^{(2)}),$$

as $q \to \infty$, that is

$$L^q = (C^{(0)}, C^{(1)}_q M^{(1)} \vee C^{(2)}_q M^{(2)}) \xrightarrow{d} L := (C^{(0)} V, C^{(1)} M^{(1)} \vee C^{(2)} M^{(2)}) \tag{5.7}$$

in $D^1 \times D^1_{\uparrow}$ with the weak M_1 topology, and then also with the weak M_2 topology.

If we show that for every $\epsilon > 0$

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathrm{P}[d_p^{M_2}(L_n, L_{n,q}) > \epsilon] = 0, \tag{5.8}$$

then from relations (5.5) and (5.7) by a generalization of Slutsky's theorem (Resnick, 2007, Theorem 3.5) it will follow that $L_n(\,\cdot\,) \xrightarrow{d} L(\,\cdot\,)$ in $D^1 \times D^1_{\uparrow}$ with the weak M_2 topology. By the definition of the metric $d_p^{M_2}$ it suffices to show that

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathrm{P}[d_{M_2}(V_n, V_{n,q}) > \epsilon] = 0 \tag{5.9}$$

and

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathrm{P}[d_{M_2}(M_n, M_{n,q}) > \epsilon] = 0. \tag{5.10}$$

Since the metric d_{M_2} is bounded above by the uniform metric,

$$\mathrm{P}[d_{M_2}(M_n, M_{n,q}) > \epsilon] \le \mathrm{P}\left(\sup_{t\in[0,1]} |M_n(t) - M_{n,q}(t)| > \epsilon \right),$$

and hence by using the same arguments as in the proof of Theorem 3.4 Krizmanić (2022b), conditions (1.7), (5.1) and (5.3) imply

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\sup_{t\in[0,1]} |M_n(t) - M_{n,q}(t)| > \epsilon \right) = 0,$$

and hence relation (5.10) holds. Note that

$$\mathrm{P}[d_{M_2}(V_n, V_{n,q}) > \epsilon] \le \mathrm{P}\left(\sup_{t\in[0,1]} |V_n(t) - V_{n,q}(t)| > \epsilon \right) \le \mathrm{P}\left(\sum_{i=1}^{n} \frac{|X_i - X_i^q|}{a_n} > \epsilon \right).$$

In the case $\alpha \in (0,1)$ by repeating the arguments from the proof of Theorem 3.4 in Krizmanić (2022b), from conditions (1.7) and (5.1) we obtain

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\sum_{i=1}^{n} \frac{|X_i - X_i^q|}{a_n} > \epsilon \right) = 0,$$

and relation (5.9) holds.

In the case $\alpha \in [1,2)$ define $Z^{\le}_{n,j} = a_n^{-1} Z_j 1_{\{|Z_j| \le a_n\}}$ and $Z^{>}_{n,j} = a_n^{-1} Z_j 1_{\{|Z_j| > a_n\}}$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. Let

$$\widetilde{C}_j = \begin{cases} C_j, & \text{if } j \ge q+1, \\ C_q - C^q, & \text{if } j = q. \end{cases}$$

and observe that

$$V_n(t) - V_{n,q}(t) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{a_n} \bigg(\sum_{j=q}^{\infty} C_j Z_{i-j} - C^q Z_{i-q} \bigg) = \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{\leq} + \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{>}.$$

Therefore

$$\begin{aligned} \mathrm{P}[d_{M_2}(V_n, V_{n,q}) > \epsilon] &\leq \mathrm{P} \bigg(\sup_{t \in [0,1]} |V_n(t) - V_{n,q}(t)| > \epsilon \bigg) \\ &\leq \mathrm{P} \bigg(\max_{1 \leq l \leq n} \bigg| \sum_{i=1}^{l} \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{\leq} \bigg| > \frac{\epsilon}{2} \bigg) + \mathrm{P} \bigg(\max_{1 \leq l \leq n} \bigg| \sum_{i=1}^{l} \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{>} \bigg| > \frac{\epsilon}{2} \bigg) \\ &=: I_1 + I_2. \end{aligned} \tag{5.11}$$

Now following the arguments from the proof of Theorem 3.1 in Krizmanić (2022a) we obtain, by Hölder's and Markov's inequalities and the fact that the sequence $(C_i)_{i \geq 0}$ is independent of (Z_i),

$$I_1 \leq \mathrm{E} \bigg(\sum_{j=q}^{\infty} |\widetilde{C}_j| \bigg) + \frac{2^r}{\epsilon^r} \sum_{j=q}^{\infty} \mathrm{E}|\widetilde{C}_j| \cdot \sup_{k \geq q} \mathrm{E} \bigg(\max_{1 \leq l \leq n} \bigg| \sum_{i=1}^{l} Z_{n,i-k}^{\leq} \bigg|^r \bigg),$$

with r as in (5.2). Since $\sum_{j=q}^{\infty} |\widetilde{C}_j| \leq 2 \sum_{j=q}^{\infty} |C_j|$, condition (5.2) implies that there exists a positive constant D_1 such that for all $q \in \mathbb{N}$ it holds that

$$\limsup_{n \to \infty} I_1 \leq D_1 \sum_{j=q}^{\infty} \mathrm{E}|C_j|. \tag{5.12}$$

In order to estimate I_2 we consider separately the cases $\alpha \in (1,2)$ and $\alpha = 1$. Assume first $\alpha \in (1,2)$. Applying Markov's inequality, the fact that the sequence $(C_i)_{i \geq 0}$ is independent of (Z_i) and the stationarity of the sequence (Z_i) we obtain

$$\begin{aligned} I_2 &\leq \mathrm{P} \bigg(\sum_{i=1}^{n} \bigg| \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{>} \bigg| > \frac{\epsilon}{2} \bigg) \leq \frac{2}{\epsilon} \mathrm{E} \bigg(\sum_{i=1}^{n} \bigg| \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{>} \bigg| \bigg) \\ &\leq \frac{2n}{\epsilon a_n} \sum_{j=q}^{\infty} \mathrm{E}|\widetilde{C}_j| \cdot \mathrm{E} \Big(|Z_1| 1_{\{|Z_1| > a_n\}} \Big) \end{aligned} \tag{5.13}$$

Since by Karamata's theorem

$$\lim_{n \to \infty} \frac{n}{a_n} \mathrm{E} \Big(|Z_1| 1_{\{|Z_1| > a_n\}} \Big) = \frac{\alpha}{\alpha - 1},$$

from (5.13) we conclude that there exists a positive constant D_2 such that

$$\limsup_{n \to \infty} I_2 \leq D_2 \sum_{j=q}^{\infty} \mathrm{E}|C_j|. \tag{5.14}$$

Now assume $\alpha = 1$. Markov's inequality implies

$$I_2 \leq \frac{2^\delta}{\epsilon^\delta} \mathrm{E} \bigg(\sum_{i=1}^{n} \bigg| \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{>} \bigg| \bigg)^{\delta},$$

with δ as in relation (1.7). Since $\delta < 1$, a double application of the triangle inequality $|\sum_{i=1}^{\infty} a_i|^s \leq \sum_{i=1}^{\infty} |a_i|^s$ with $s \in (0,1]$ yields

$$I_2 \leq \frac{2^\delta}{\epsilon^\delta} \sum_{i=1}^{n} \mathrm{E}\bigg(\bigg| \sum_{j=q}^{\infty} \widetilde{C}_j Z_{n,i-j}^{>} \bigg|^\delta \bigg) \leq \frac{2^\delta}{\epsilon^\delta a_n^\delta} \sum_{i=1}^{n} \sum_{j=q}^{\infty} \mathrm{E}\bigg(\bigg| \widetilde{C}_j Z_{i-j} 1_{\{|Z_{i-j}| > a_n\}} \bigg|^\delta \bigg).$$

Using again the fact that (C_i) is independent of (Z_i) and the stationarity of (Z_i) we obtain

$$I_2 \leq \frac{2^\delta n}{\epsilon^\delta a_n^\delta} \mathrm{E}\Big(|Z_1|^\delta 1_{\{|Z_1| > a_n\}} \Big) \sum_{j=q}^{\infty} \mathrm{E}|\widetilde{C}_j|^\delta.$$

From this, since by Karamata's theorem

$$\lim_{n\to\infty} \frac{n}{a_n^\delta} \mathrm{E}\Big(|Z_1|^\delta 1_{\{|Z_1| > a_n\}} \Big) = \frac{1}{1-\delta},$$

it follows that there exists a positive constant D_3 such that

$$\limsup_{n\to\infty} I_2 \leq D_3 \sum_{j=q}^{\infty} \mathrm{E}|C_j|^\delta.$$

This together with (5.11), (5.12) and (5.14) shows that

$$\limsup_{n\to\infty} \mathrm{P}[d_{M_2}(V_n, V_{n,q}) > \epsilon] \leq D_1 \sum_{j=q}^{\infty} \mathrm{E}|C_j| + (D_2 + D_3) \sum_{j=q}^{\infty} \mathrm{E}|C_j|^s,$$

where

$$s = \begin{cases} \delta, & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha \in (1,2). \end{cases}$$

Finally, the dominated convergence theorem and conditions (1.7) and (5.3) imply relation (5.9) for $\alpha \in [1,2)$. Therefore we conclude that $L_n(\,\cdot\,) \xrightarrow{d} L(\,\cdot\,)$ in $D^1 \times D^1_{\uparrow}$ with the weak M_2 topology. $\square$

Remark 5.2. Condition (5.2) holds with $r = 2$ when the sequence (Z_i) is an i.i.d. or ρ–mixing sequence with $\sum_{i=1}^{\infty} \rho(2^i) < \infty$, where

$$\rho(n) = \sup\{|\mathrm{corr}(f,g)| : f \in \mathrm{L}^2(\mathcal{F}_1^k), g \in \mathrm{L}^2(\mathcal{F}_{k+n}^{\infty}), k = 1, 2, \ldots\}$$

(see Tyran-Kamińska, 2010b).

In the case when the sequence of coefficients (C_j) is deterministic, conditions (5.1) and (5.3) can be dropped since they are implied by (1.7), but this in general does not hold when the coefficients are random (see Remark 3.1 in Krizmanić, 2022a).

References

Anderson, C. W. and Turkman, K. F. The joint limiting distribution of sums and maxima of stationary sequences. *J. Appl. Probab.*, **28** (1), 33–44 (1991). MR1090445.

Anderson, C. W. and Turkman, K. F. Sums and maxima of stationary sequences with heavy tailed distributions. *Sankhyā Ser. A*, **57** (1), 1–10 (1995). MR1392626.

Astrauskas, A. Limit theorems for sums of linearly generated random variables. *Litovsk. Mat. Sb.*, **23** (2), 3–12 (1983). MR706002.

Avram, F. and Taqqu, M. S. Weak convergence of sums of moving averages in the α-stable domain of attraction. *Ann. Probab.*, **20** (1), 483–503 (1992). MR1143432.

Balan, R., Jakubowski, A., and Louhichi, S. Functional convergence of linear processes with heavy-tailed innovations. *J. Theoret. Probab.*, **29** (2), 491–526 (2016). MR3500408.

Basrak, B., Krizmanić, D., and Segers, J. A functional limit theorem for dependent sequences with infinite variance stable limits. *Ann. Probab.*, **40** (5), 2008–2033 (2012). MR3025708.

Basrak, B. and Segers, J. Regularly varying multivariate time series. *Stochastic Process. Appl.*, **119** (4), 1055–1080 (2009). MR2508565.

Basrak, B. and Tafro, A. A complete convergence theorem for stationary regularly varying multivariate time series. *Extremes*, **19** (3), 549–560 (2016). MR3535966.

Billingsley, P. *Convergence of probability measures.* John Wiley & Sons, Inc., New York-London-Sydney (1968). MR233396.

Chow, T. L. and Teugels, J. L. The sum and the maximum of i.i.d. random variables. In *Proceedings of the Second Prague Symposium on Asymptotic Statistics (Hradec Králové, 1978)*, pp. 81–92. North-Holland, Amsterdam-New York (1979). MR571177.

Davis, R. A. Stable limits for partial sums of dependent random variables. *Ann. Probab.*, **11** (2), 262–269 (1983). MR690127.

Davis, R. A. and Mikosch, T. Extremes of Stochastic Volatility Models. In *Handbook of Financial Time Series*, pp. 355–364. Springer (2009). DOI: 10.1007/978-3-540-71297-8.

Durrett, R. and Resnick, S. I. Functional limit theorems for dependent variables. *Ann. Probab.*, **6** (5), 829–846 (1978). MR503954.

Ferreira, M. and Canto e Castro, L. Tail and dependence behavior of levels that persist for a fixed period of time. *Extremes*, **11** (2), 113–133 (2008). MR2412121.

Hult, H. and Samorodnitsky, G. Tail probabilities for infinite series of regularly varying random vectors. *Bernoulli*, **14** (3), 838–864 (2008). MR2537814.

Kallenberg, O. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York (1997). ISBN 0-387-94957-7. MR1464694.

Krizmanić, D. *Functional limit theorems for weakly dependent regularly varying time series.* Ph.D. thesis, University of Zagreb, Croatia (2010). Available at `https://www.math.uniri.hr/~dkrizmanic/DKthesis.pdf`.

Krizmanić, D. Functional weak convergence of partial maxima processes. *Extremes*, **19** (1), 7–23 (2016). MR3454028.

Krizmanić, D. Joint functional convergence of partial sums and maxima for linear processes. *Lith. Math. J.*, **58** (4), 457–479 (2018a). MR3881926.

Krizmanić, D. A note on joint functional convergence of partial sum and maxima for linear processes. *Statist. Probab. Lett.*, **138**, 42–46 (2018b). MR3788717.

Krizmanić, D. Functional convergence for moving averages with heavy tails and random coefficients. *ALEA Lat. Am. J. Probab. Math. Stat.*, **16** (1), 729–757 (2019). MR3949276.

Krizmanić, D. On joint weak convergence of partial sum and maxima processes. *Stochastics*, **92** (6), 876–899 (2020). MR4139088.

Krizmanić, D. A functional limit theorem for moving averages with weakly dependent heavy-tailed innovations. *Braz. J. Probab. Stat.*, **36** (1), 138–156 (2022a). MR4377126.

Krizmanić, D. Maxima of linear processes with heavy-tailed innovations and random coefficients. *J. Time Series Anal.*, **43** (2), 238–262 (2022b). MR4400293.

Kulik, R. Limit theorems for moving averages with random coefficients and heavy-tailed noise. *J. Appl. Probab.*, **43** (1), 245–256 (2006). MR2225064.

Leadbetter, M. R., Lindgren, G., and Rootzén, H. *Extremes and related properties of random sequences and processes.* Springer Series in Statistics. Springer-Verlag, New York-Berlin (1983). ISBN 0-387-90731-9. MR691492.

Leadbetter, M. R. and Rootzén, H. Extremal theory for stochastic processes. *Ann. Probab.*, **16** (2), 431–478 (1988). MR929071.

Resnick, S. I. Point processes, regular variation and weak convergence. *Adv. in Appl. Probab.*, **18** (1), 66–138 (1986). MR827332.

Resnick, S. I. *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust*. Springer-Verlag, New York (1987). ISBN 0-387-96481-9. MR900810.

Resnick, S. I. *Heavy-tail phenomena. Probabilistic and statistical modeling.* Springer Series in Operations Research and Financial Engineering. Springer, New York (2007). ISBN 978-0-387-24272-9; 0-387-24272-4. MR2271424.

Sato, K.-i. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (1999). ISBN 0-521-55302-4. MR1739520.

Tyran-Kamińska, M. Convergence to Lévy stable processes under some weak dependence conditions. *Stochastic Process. Appl.*, **120** (9), 1629–1650 (2010a). MR2673968.

Tyran-Kamińska, M. Functional limit theorems for linear processes in the domain of attraction of stable laws. *Statist. Probab. Lett.*, **80** (11-12), 975–981 (2010b). MR2638967.

Whitt, W. *Stochastic-process limits. An introduction to stochastic-process limits and their application to queues.* Springer Series in Operations Research. Springer-Verlag, New York (2002). ISBN 0-387-95358-2. MR1876437.